

SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 1 Volumes 1-3



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Reflect on This



What do you see when you look in the mirror? This module takes a peek at some concepts in both physics and geometry—through the looking glass.

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Reflect on This

Introduction

A kaleidoscope creates fascinating designs that change as it is rotated. Two mirrors hinged together may be used to model the effects of a kaleidoscope. To make such a model, complete Parts a–e below.

- a. Place the reflective sides of two mirrors face to face. Tape one set of edges together to make a hinge.
- b. Cut small pieces of confetti from colored paper.
- c. Overlap a sheet of white paper with a sheet of colored paper. Half of each sheet should remain visible.
- d. Position the hinged mirrors across the two sheets of paper as shown in Figure 1. The distance from *A* to *B* should be approximately equal to the distance from *A* to *C*.

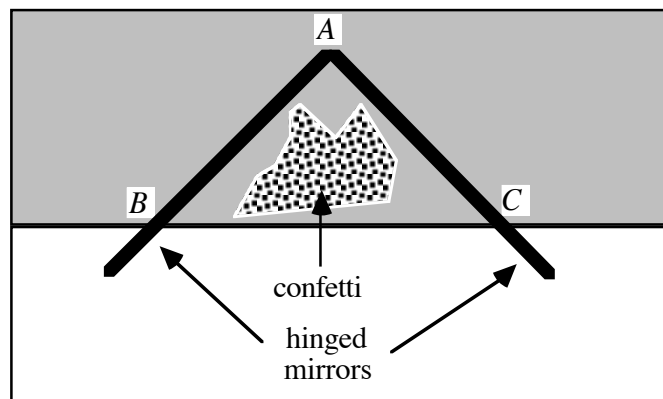


Figure 1: Kaleidoscope model

- e.
 1. Open and close the mirrors, keeping the distance from *A* to *B* approximately equal to the distance from *A* to *C*. Observe the patterns formed by the confetti and colored paper.
 2. Make a record of the patterns formed by the colored paper.
 2. Rearrange the confetti, then open and close the mirrors again. Observe the new patterns formed.

Discussion

- a. Describe the patterns created by the colored paper in your kaleidoscope.
- b. What happened as you opened and closed the hinged mirrors?
- c. What is the relationship between the size of the hinge angle and the number of reflections seen

Activity 1

Opening and closing hinged mirrors causes multiple reflections. These reflections form patterns and shapes similar to those you created with a model kaleidoscope.

Mathematics Note

A **polygon** is a union of coplanar segments intersecting only at endpoints. At most two segments intersect at any one endpoint and each segment intersects exactly two other segments. Each segment in a polygon is a **side**; each endpoint is a **vertex** (plural **vertices**).

In Figure 2, for example, points C , G , and O are vertices, while \overline{AB} , \overline{EF} , and \overline{MN} are sides.

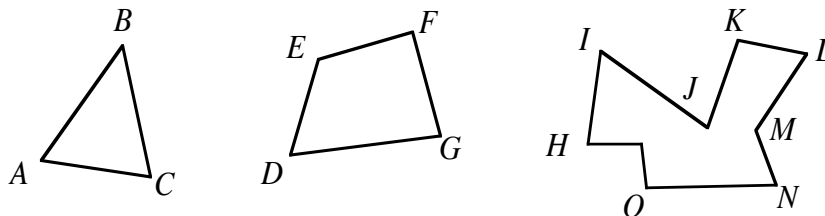


Figure 2: Three polygons

Exploration

In this exploration, you examine the polygons formed by reflections in hinged mirrors and investigate the relationship between these polygons and the hinge angle.

- Reconstruct the kaleidoscope from the introduction (without the confetti).
- Begin with the mirrors completely open. Slowly close them until a triangle is formed by the colored paper and its reflections. As shown in Figure 3, the mirrors form a hinge angle at the center of the triangle. Draw this angle and determine its measure.

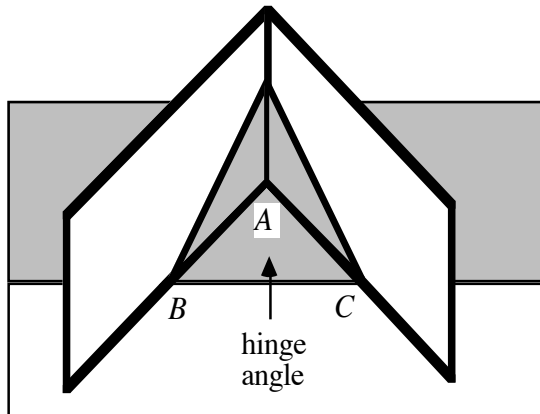


Figure 3: Triangle with hinge angle

- c. Continue to close the mirrors, keeping the distance from A to B equal to the distance from A to C. Other polygons will appear, such as a quadrilateral, a pentagon, and a hexagon. Record the measure of the hinge angle for each polygon that appears, up to a decagon, in Table 1.

Table 1: Polygon information

Polygon	No. of Sides	Measure of Hinge Angle
triangle	3	120°
quadrilateral		
pentagon		
hexagon		
heptagon		$51\frac{3}{7}^\circ$
octagon		
nonagon		
decagon	10	

- d. Use the patterns you observe in the table to describe a relationship between the number of sides of a polygon and the measure of the hinge angle.

Discussion

- a. What appears to occur as the measure of the hinge angle becomes smaller?

Mathematics Note

A polygon is **equiangular** if its interior angles are congruent. A polygon is **equilateral** if its sides are congruent. A polygon is **regular** if its sides are congruent and its interior angles are congruent.

For example, Figure 4 shows two regular polygons, a square and an equilateral triangle.

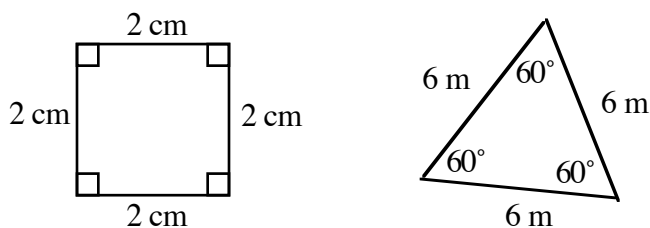


Figure 4: Two regular polygons

An angle formed by two rays drawn from the center of a circle is a **central angle**. If a regular polygon is inscribed in a circle (all vertices of the polygon lie on the circle), then the central angle formed by rays drawn from the center of the circle to two consecutive vertices of the polygon divide the polygon into congruent isosceles triangles.

For example, Figure 5 shows equilateral triangle ABC inscribed in a circle with center O . In this case, $\angle AOB$, $\angle AOC$, and $\angle BOC$ are central angles of ΔABC .

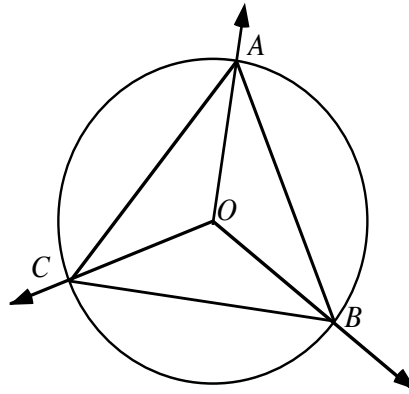


Figure 5: Inscribed equilateral triangle with central angles

- b. Are the polygons formed by the reflections in the exploration always regular polygons? Why or why not?
- c.
 1. What relationship exists between the number of sides of a regular polygon and the measure of its central angle?
 2. Express this relationship as an algebraic formula.
- d. How many central angles are there in any regular polygon?
- e. Identify the congruent isosceles triangles in Figure 5.
- f. What is the relationship between the vertex angles of the isosceles triangles and the central angles of the polygon?
- g. What is the relationship between the base angles of the isosceles triangles and the interior angles of the polygon?

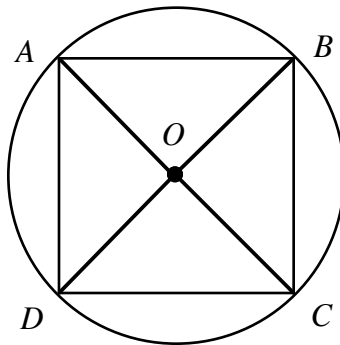
Assignment

- 1.1 Determine the measure of the central angle for a regular polygon with:
 - a. 12 sides
 - b. 18 sides
 - c. 21 sides
 - d. n sides.
- 1.2 Find the number of sides in a regular polygon with a central angle measure of:
 - a. 40°
 - b. 20°
 - c. 24°
 - d. 0°

- 1.3** Explain how to draw each of the following figures using only a ruler and two hinged mirrors.
- two perpendicular lines
 - an angle with measure 120°
 - a regular hexagon with sides 5 cm long

* * * * *

- 1.4** Determine the measure of a central angle for each of the following:
- a regular pentagon
 - a regular octagon
 - a regular decagon
 - a regular n -gon
- 1.5** The following diagram shows square $ABCD$ inscribed in a circle with center O . In this case, the measure of central angle AOB is 90° , while the measure of angle OAB is 45° .



What is the measure of angle OAB if the inscribed polygon is:

- a regular pentagon?
 - a regular octagon?
 - a regular decagon?
 - a regular n -gon?
- 1.6**
- Find the measure of the interior angles in each of the following:
 - a regular pentagon
 - a regular octagon
 - a regular decagon
 - a regular n -gon
 - Determine the sum of the measures of the interior angles for each regular polygon in Part **a**.

- 1.7 As mentioned in the mathematics note, the central angles of a regular polygon divide the polygon into congruent isosceles triangles. Describe how you could use the area of these triangles to find the area of each of the following:
- a. a regular pentagon
 - b. a regular octagon
 - c. a regular decagon
 - d. a regular n -gon

* * * * *

Activity 2

When you look into your model kaleidoscope, the images you see are caused by light rays reflecting from the confetti, striking the hinged mirrors, then bouncing off the mirrors into your eyes. As you change the positions of the mirrors, the paths of the light rays also change. In this activity, you explore the paths that light rays follow in a reflection.

Exploration

- a. Draw an x -axis and a y -axis on a sheet of graph paper with the origin $(0,0)$ near the center of the sheet. Label the origin as point A and a point on the positive x -axis as point B .
- b. As shown in Figure 6, place a mirror tightly against one edge of a block of fiberboard and perpendicular to the broad face of the block.

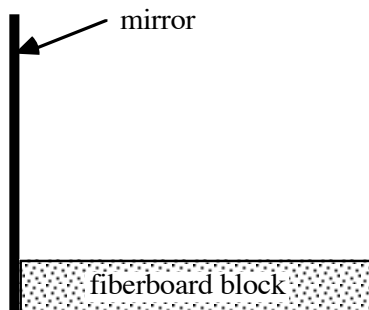


Figure 6: Light-ray experiment (side view)

- c. Carefully fold the graph paper along the x -axis. Place the folded sheet on top of the fiberboard block with the x -axis tightly against the mirror, as shown in Figure 7.

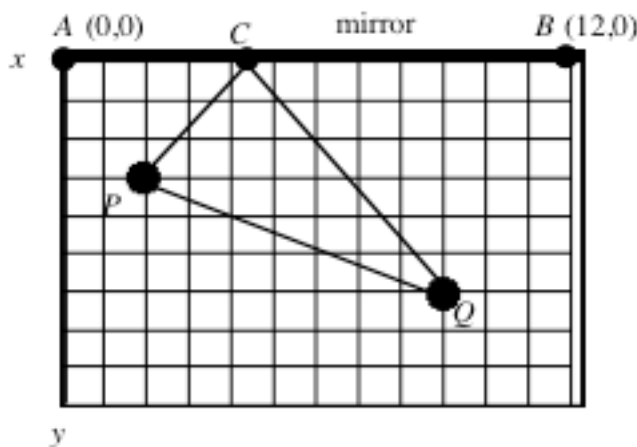


Figure 7: Light-ray experiment (top view)

- d. Choose two points that fit the description below.
1. The x -coordinates are at least 3 units apart.
 2. The y -coordinates are different and between -3 and -7 .
- e.
1. Label the point with the lesser x -coordinate as P and the other point as Q .
 2. Place one pushpin through P and another through Q .
 3. Sight through P toward the mirror with one eye closed. Looking at the reflection of Q , move your head until P aligns with the image of Q .
 4. Mark a point C on the x -axis where P , C , and the reflection of Q are collinear (on the same line).
- f. Place a rubber band around the pushpins. Using the point of a pencil, pull the side of the rubber band closer to the mirror until the pencil is at C . Check the position of C again to make sure P , C , and the reflection of Q are collinear. The side of the rubber band you stretched shows the path that light rays follow from Q to the mirror to P .
- g.
1. Remove your graph paper from the fiberboard block and unfold it. Draw \overline{PC} and \overline{CQ} . These segments show the path of light as it travels from the pushpin at Q to your eye at P by reflecting off the mirror.
 2. Measure $\angle QCB$ and $\angle PCA$. Record these angle measures. **Note:** Save your graph paper for use in the assignment and in Activity 3.

Science Note

Light rays reflecting off a flat surface form rays and angles as shown in Figure 8.

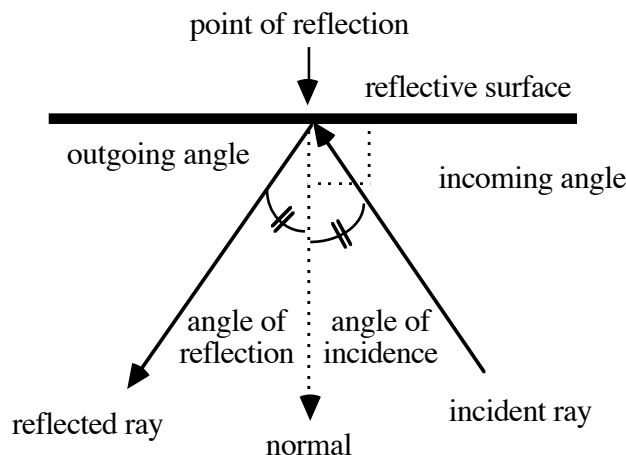


Figure 8: A reflected light ray

The **incident ray** models the path that a light ray follows from the object to the reflective surface, while the **reflected ray** models the path of the light ray away from the reflective surface.

The **incoming angle** is the angle between the reflective surface and the incident (incoming) ray. The **outgoing angle** is the angle between the surface and the reflected (outgoing) ray.

The **normal** of the reflecting surface is the perpendicular line to the surface at the **point of reflection**. The **angle of incidence** is the angle formed by the normal and the incident ray, while the **angle of reflection** is the angle formed by the normal and the reflected ray.

Discussion

- Using terms from the Science Note, describe the path traveled by a light ray that passes through Q and strikes the mirror before reaching your eye.
- Make a conjecture about the relationship between the measures of the incoming angle and the outgoing angle.
- Compare the angle measures you recorded with others in your group. Do any of these measurements contradict your conjecture? If so, do you still believe that your conjecture is correct? Explain your response.
- Describe several ways in which an error in measurement could occur in the exploration. How might these errors affect your conclusions?
- Suggest another method for confirming your conjecture in Part c.

Mathematics Note

If the sum of the measures of two angles is 90° , then the two angles are **complementary**. In Figure 9, for example, $\angle BAC$ and $\angle HIG$ are complementary angles because $30^\circ + 60^\circ = 90^\circ$.

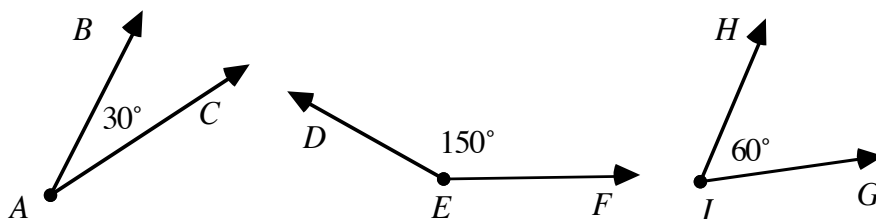


Figure 9: Three angles

If the sum of the measures of two angles is 180° , then the two angles are **supplementary**. In Figure 9, $\angle BAC$ and $\angle DEF$ are supplementary angles because $30^\circ + 150^\circ = 180^\circ$.

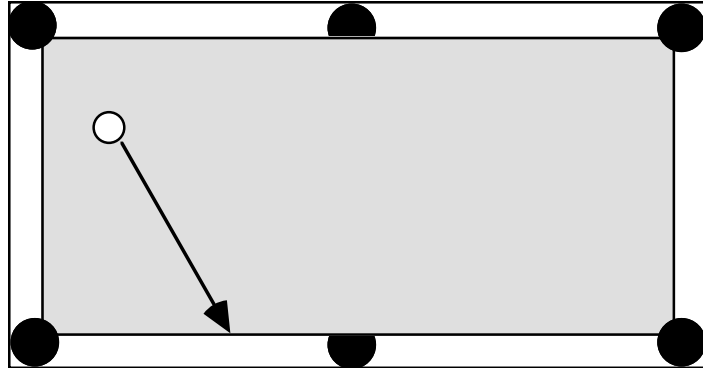
- f. Identify the pairs of angles in Figure 8 that are complementary.

Assignment

- 2.1 Use your graph from the exploration to complete Parts a–c below.
1. Draw and label a normal on the graph at the point where the light rays were reflected.
 2. Measure and label the angle of incidence and the angle of reflection.
- b. Use the graph to make a conjecture about the relationship between each of the following pairs of angles:
1. the angle of incidence and the angle of reflection
 2. the incoming angle and the angle of incidence
 3. the outgoing angle and the angle of reflection.
- c. Write an argument to convince a classmate that your conjectures are correct.
- 2.2
- a. Use a protractor to draw a light ray bouncing off a mirror with an angle of incidence of 55° .
 - b. Draw and label the normal to the surface at the point of reflection.
 - c. Label all angles and give their measures.
- 2.3
- a. Prepare a labeled diagram that shows what happens when a ray of sunlight reflects off the surface of a watch.
 - b. Write a summary of the mathematical ideas represented in the diagram.

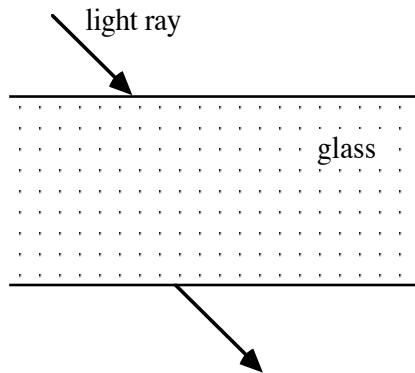
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- 2.4 Using mathematical terms, describe what happens when a driver adjusts the side mirror on a car.
- 2.5 Reflections occur not only in mirrors, but also in other situations. In the game of pool, players use a cue stick to strike a white ball (the cue ball). When a player hits the cue ball correctly, it strikes one of the other balls and knocks it into one of six holes, or pockets, on the table. In a bank shot, a ball bounces off a side rail before falling into a pocket. The path of a ball bouncing off a side rail is like the path of light reflecting off a mirror.



Suppose a pool ball is hit along the path shown in the diagram. Will the ball fall into the upper right corner pocket after a single bounce off the side rail? Explain your response.

- 2.6 Light rays passing through glass are bent towards the normal. This bending of light rays is called **refraction**. For example, a stick in water appears bent because of refraction. The light ray enters and exits the glass as shown below.



- Copy the diagram above and draw the path of light from the entry point to the exit point.
- Draw and label the normals at the entry and exit points.
- In your drawing from Parts **a** and **b**, identify and label all pairs of angles that appear to be congruent.

* * * * *

Research Project

1. Describe how to determine the length of the smallest mirror that would allow every member of your family to see a complete reflection from head to toe. Include a diagram with your report.
 2. Write a report about periscopes that answers the following questions.
 - a. What is a periscope?
 - b. In what types of situations are periscopes useful?
 - c. How does a periscope work? (Include a diagram in your report.)
 - d. How can you make a simple periscope using a tube and two mirrors? (Include a model with your report.)
-
-

Activity 3

When you look in a mirror, your reflection sometimes appears to be behind the mirror's surface. This illusion, often called a **virtual image**, is created by light rays reflecting off the mirror. In this activity, you explore the relationship between the position of an object and the apparent position of its reflection in a mirror.

Exploration 1

- a. Unfold your sheet of graph paper from the exploration in Activity 2 and lay it flat. Position a tinted plastic reflector along the x -axis with points P and Q on your side of the reflector.

Mathematics Note

The reflection of an object is its **image**. The object itself is the **preimage**. If point A is the preimage, then the image of a point A can be represented as A' (read "A prime").

- b.
 1. Find the reflection of point Q in the reflector.
 2. Place your pencil behind the reflector and mark the point on the graph paper where the reflection of Q appears to be. This point is the image of Q . Label it Q' .
 3. Remove the reflector and draw $\overline{PQ'}$. As shown in Figure 10, points P , C , and Q' should be collinear.

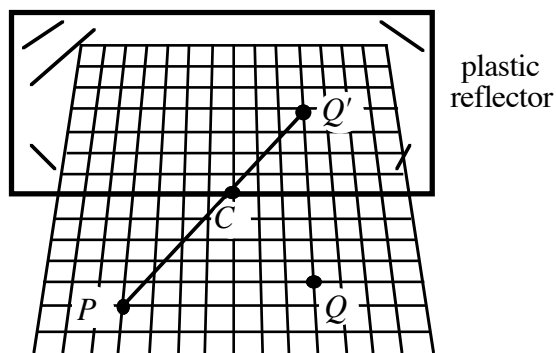


Figure 10: Finding Q'

- c.
 1. Use the plastic reflector to mark the image of point P on your graph paper. Label this point P' .
 2. Remove the reflector and draw $\overline{QP'}$.
- d. The **distance from a point to a line** is the distance along a path perpendicular to the line. Measure the distance from the x -axis to each of the following points:
 1. Q and Q'
 2. P and P' .
- e. Make a conjecture about the relationship between the distance from an object to the mirror line and the distance from its image to the mirror line.

Discussion 1

- a. In Activity 2, you pushed pins through two layers of folded graph paper at points P and Q . How do the locations of the pinholes in the second layer compare with the locations of P' and Q' ?
- b. How is the distance from Q to the x -axis related to the distance from Q' to the x -axis?
- c.
 1. Where do $\overline{PQ'}$ and $\overline{QP'}$ intersect?
 2. What does this tell you about the paths traveled by reflected light?
- d. Does the class data support the conjecture you made in Part e of Exploration 1? Explain your response.

Exploration 2

In this exploration, you use technology to model the process described in Exploration 1. Most geometry utilities have reflection tools for finding the images of objects. To use this feature, you must first define a mirror line and the points to be reflected.

- a.
 1. Construct a mirror line segment, \overline{AB} .
 2. Construct two points, P and Q , on the same side of \overline{AB} .
 3. Reflect points P and Q in \overline{AB} and label the images P' and Q' , respectively.
 4. Construct $\overline{PQ'}$ and $\overline{QP'}$.
 5. Construct a point C at the intersection of \overline{AB} and $\overline{PQ'}$.
 6. Measure $\angle PCA$ and $\angle QCB$.
 7. Construct $\overline{PP'}$ and $\overline{QQ'}$.
 8. Label the intersection of \overline{AB} and $\overline{PP'}$ as point D and the intersection of \overline{AB} and $\overline{QQ'}$ as point E . Your construction should now resemble the one in Figure 11 below.

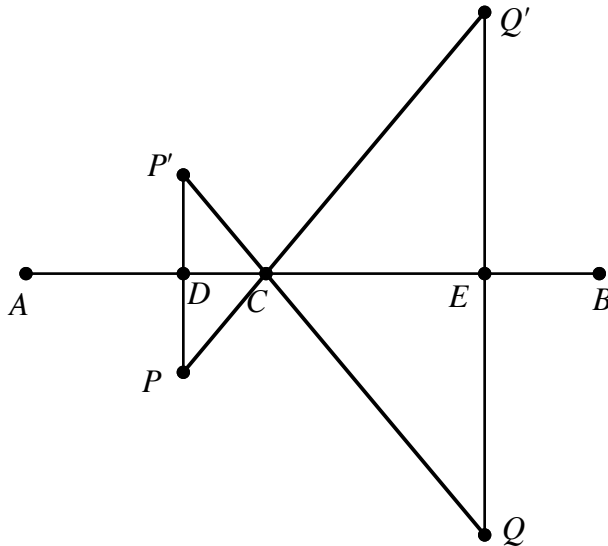


Figure 11: Reflection modeled on a geometry utility

9. Measure $\angle PDA$ and $\angle QEB$.
 10. Measure the distances from \overline{AB} to points P , P' , Q , and Q' .
 11. Measure \overline{EQ} and $\overline{EQ'}$.
- b. Use your measurements in Part a to make a conjecture about the relationship between \overline{AB} and $\overline{QQ'}$.

- c. Select and move various points on your construction. Note any relationships you observe among the measurements of angles and segments as you move the points.
- d. Complete Steps 1–3 below using your construction from Part a.
 1. Construct a point S anywhere on \overline{AB} .
 2. Measure the total distance traveled when moving from Q to S and then from S to P .
 3. By moving S along \overline{AB} , find the point where the total distance traveled in Step 2 is shortest.

Mathematics Note

The **perpendicular bisector** of a segment is the line that is perpendicular (\perp) to the segment and divides the segment into two congruent parts.

In Figure 12, for example, line m is the perpendicular bisector of $\overline{CC'}$ because line m and $\overline{CC'}$ are perpendicular and $\overline{CM} \cong \overline{C'M}$.

A **reflection** in a line is a pairing of points in a plane so that the **line of reflection** (or **mirror line**) is the perpendicular bisector of every segment connecting a point in the preimage to its corresponding point in the image. Every point on the line of reflection is its own image.

In Figure 12, parallelogram $C'D'E'F'$ is the image of parallelogram $CDEF$. Line m is the line of reflection since it is the perpendicular bisector of each segment joining a point in the preimage to its corresponding point in the image.

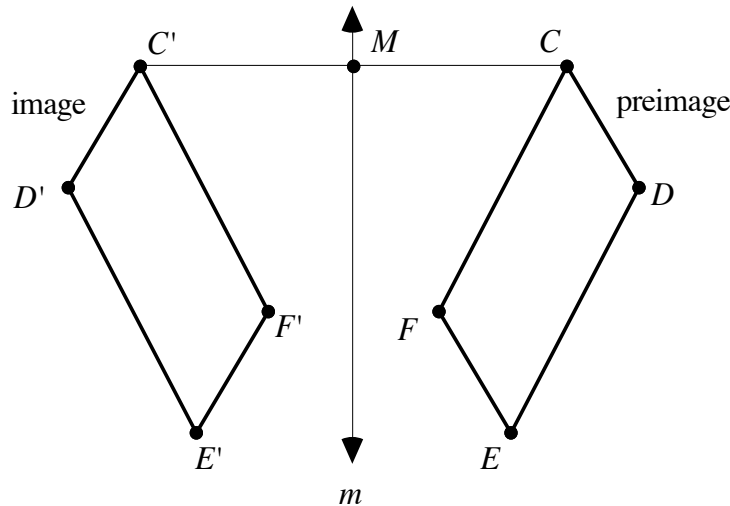


Figure 12: Reflection of parallelogram $CDEF$ in line m

Discussion 2

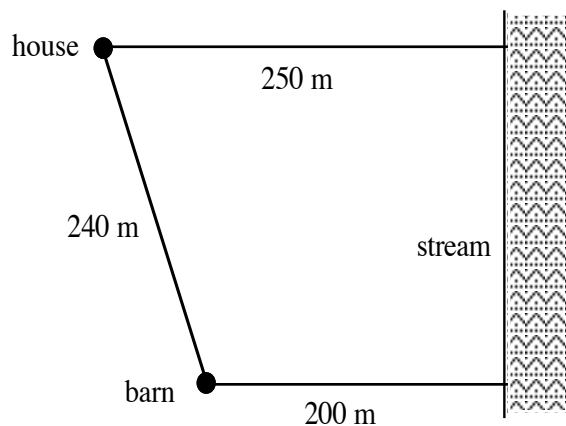
- a.
 1. Make a conjecture about the relationship between \overline{AB} and $\overline{PP'}$ in Figure 11.
 2. Do your measurements in Exploration 2 support this conjecture?
- b. What conjecture can you make about the path which results in the shortest total distance in Part c of Exploration 2 and the point of reflection?
- c. Imagine that \overline{AB} in Figure 11 is on the x -axis.
 1. What does the measure of $\angle QEB$ tell you about the x -coordinates of points Q and Q' ?
 2. What do the lengths of \overline{EQ} and $\overline{EQ'}$ tell you about the y -coordinates of Q and Q' ?
 3. Which segments would you measure to compare the y -coordinates of points P and P' ?
- d.
 1. Describe the relationship between the coordinates of point Q and its image Q' .
 2. Describe the relationship between the coordinates of point P and its image P' .
 3. Are these relationships also true for each preimage and image in Exploration 1?
- e. What generalization might you make about the relationship between the coordinates of a point in the preimage and the coordinates of its image under a reflection in each of the following lines:
 1. the x -axis?
 2. the y -axis?
- f. Many mathematical ideas are based on observed patterns. When conjectures based on patterns are stated in a general form and proved, they become accepted mathematical **theorems**.
 1. In Activity 2, you made a conjecture about the relationship between the measures of the incoming and outgoing angles. Do your measurements in Exploration 2 support this conjecture?
 2. If so, could this conjecture now become a theorem?

Assignment

- 3.1**
- Plot three points on a coordinate grid so that one point has all positive coordinates, one point has all negative coordinates, and one point has one positive and one negative coordinate. Label these points A , B , and C .
 - Plot the reflections of A , B and C in the y -axis. Label the image points A' , B' , and C' , respectively.
 - List the coordinates of each point and its image.
 - Consider a point D with coordinates (x,y) . If D is reflected in the y -axis, what are the coordinates of D' ?

- 3.2** Use your graph paper from the Exploration in Activity 2 to complete Parts **a** and **b** below.
- Describe the shortest path from Q' to P .
 - How does the length of this path compare to the length of the path that light travels from Q to P ? Explain your response.

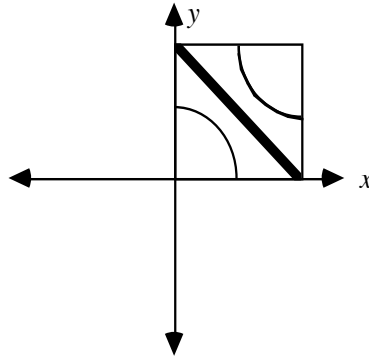
- 3.3** A family wants to build the shortest possible trail from their house to the stream and then to the barn. The distances between these locations are shown in the diagram below.



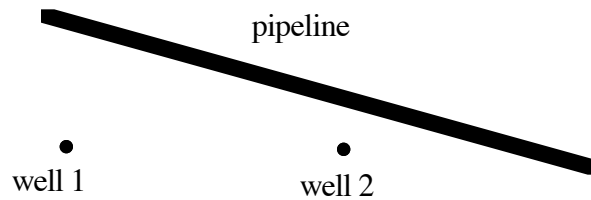
- Make a scale drawing of this situation.
- Use your drawing to determine where the trail should reach the stream to make the shortest trail from the house to the stream to the barn.
- Explain how you used the properties of reflections to determine your solution.

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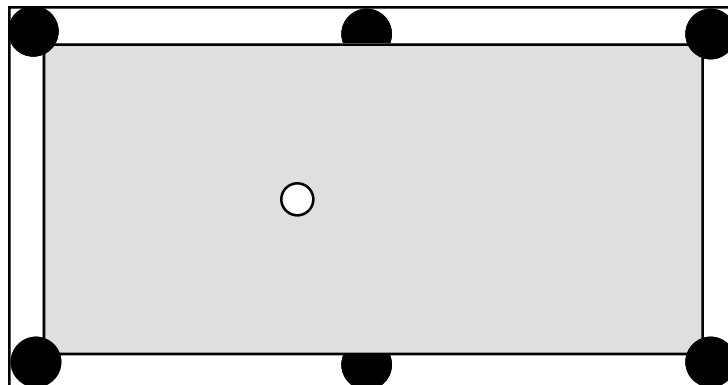
- 3.4** The diagram below shows a square tile with its lower left-hand corner at the origin of an xy -coordinate system. Sketch a copy of this diagram on a sheet of graph paper.



- Sketch the image of the tile reflected in the x -axis.
 - Sketch the image of your response to Part **a** reflected in the y -axis.
- 3.5** The diagram below shows a section of an oil pipeline along with two nearby wells. To transport oil from the wells to the pipeline, the company plans to build a new pumping station somewhere on the line. If the company wants to minimize the amount of pipe needed to connect the two wells to the pipeline, where should it locate the pumping station?



- 3.6** The diagram below shows the top of a pool table along with a cue ball.

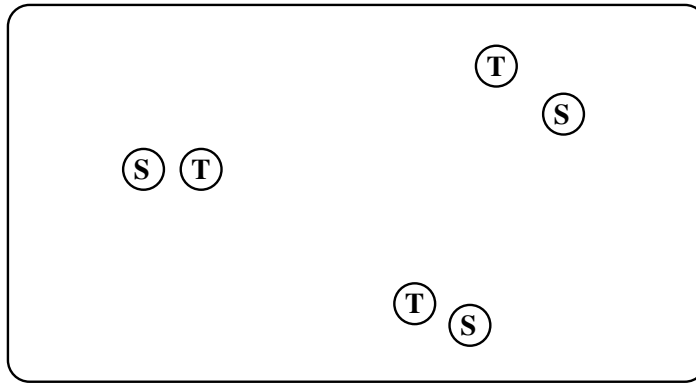


- Trace a path that banks the cue ball off exactly one side and places it in a pocket.
- Describe how you determined the path you chose in Part **a**.

- c. Find another path to the same pocket, using a different side of the table to bank the ball.

3.7 In a hockey game between the Saints and the Tornados, six players are positioned as shown in the diagram below. Each circle that contains a letter S represents a member of the Saints; each circle that contains a T represents a member of the Tornados.

Draw all the paths that show a successful pass from the Saint on the left to one of the other two Saints. Each pass must reflect off one of the boards (sides). Explain why your pass is possible.



* * * * *

Activity 4

In the previous activities, you examined how light reflects off a single flat mirror. The kaleidoscope you built in the introduction, however, used two mirrors. In this situation, light reflects off one mirror, then the other. In the following exploration, you investigate the virtual images produced by light reflecting off two mirrors.

Exploration 1

- a. On a sheet of graph paper, draw two perpendicular line segments and a point *Q*, as shown in Figure 13 below.

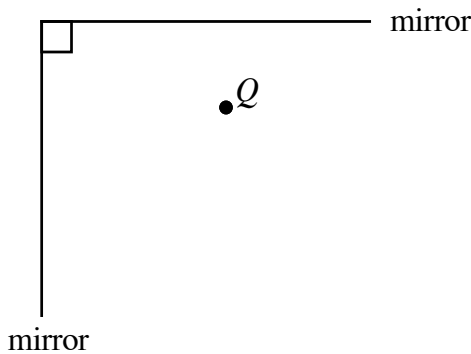


Figure 13: Two perpendicular mirrors

- b. If hinged mirrors like those used in Activity 1 were placed along the line segments, how many images of point Q would you see in the mirrors? Record your prediction.
- c. Use two hinged mirrors to test your prediction in Part b. Record how many images of point Q you actually observe.
- d. To discover why this number of images occurs, use your diagram from Part a to complete Steps 1–6 below.
 1. Reflect Q in one of the line segments and label the image Q_1' .
 2. Reflect Q in the other line segment and label this image Q_2' .
 3. Place the hinged mirrors on the segments again. Observe that two of the virtual images of Q in the mirrors correspond to the two images of Q found in Steps 1 and 2.
 4. Remove the hinged mirrors. Reflect Q_1' in the other line segment and label its image Q'' (read “Q double-prime”).
 5. Repeat Step 4 using Q_2' to locate Q_2'' . What do you observe about the positions of Q'' and Q_2'' ?
 6. Place the hinged mirrors back on the line segments and observe the virtual image that corresponds to Q'' . **Note:** Save your diagram for use in Exploration 2.

Discussion 1

- a. How did the number of images you observed in Part c of Exploration 1 compare with your prediction in Part b?
- b. Explain why the two mirrors in Exploration 1 produce three virtual images.
- c. In Exploration 1, you labeled the image of a single reflection Q' , and the image of a double reflection Q'' . How would you label the image of a triple reflection?
- d. How do your observations in Part d of Exploration 1 help explain the patterns you saw in the kaleidoscope?

Exploration 2

In this exploration, you use the properties of single reflections to investigate the path of light in a double reflection.

- a. Add a point E to your diagram from Exploration 1, as shown in Figure 14 below. In this diagram, point E represents the location of your eye.

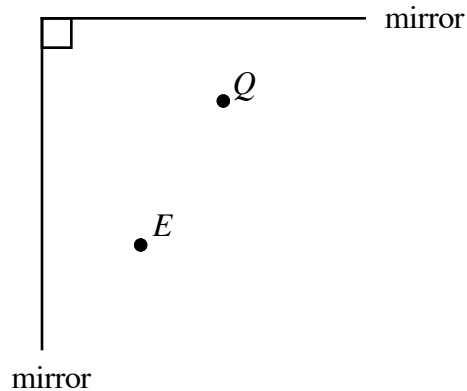


Figure 14: Perpendicular mirrors with points E and Q

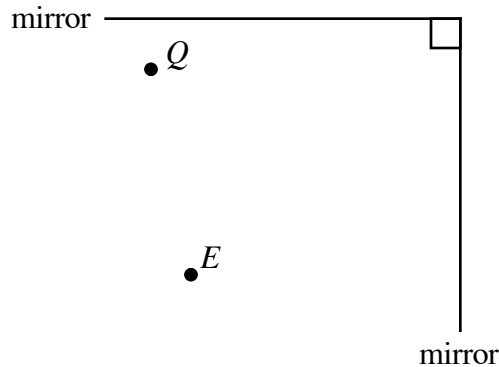
- b. To find the path light travels in a double reflection from Q to E , complete Steps 1–4 below.
 1. Use a straightedge to connect E and Q'' .
 2. Label the point where $\overline{EQ''}$ intersects the mirror line segment as C_2 .
 3. Use a straightedge to connect C_2 and Q' .
 4. Label the point where $\overline{C_2Q'}$ intersects the other mirror line segment as C_1 .
- c. Draw the path from Q to C_1 , then to C_2 , and finally to E . **Note:** Save your diagram for use in the assignment.

Discussion 2

- a. What does the path drawn in Part c of Exploration 2 represent?
- b. What do the points C_1 and C_2 represent?
- c. Using light rays, explain why the virtual image that corresponds to Q'' appears where it does in the mirror.

Assignment

- 4.1** The diagram below shows two perpendicular mirrors and the points Q and E . On a copy of this diagram, draw the path light travels in a double reflection from Q to E .

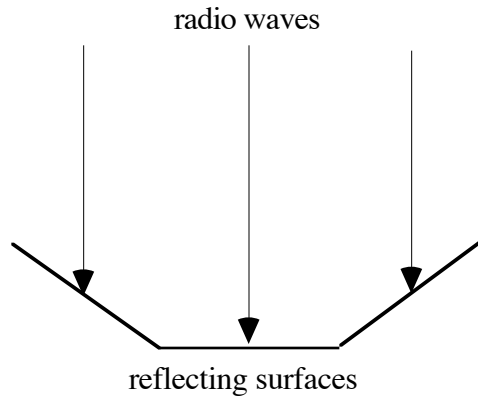


- 4.2** Use the diagram of two perpendicular mirrors you created in Exploration 2 to complete Parts **a–e** below.
- Label the intersection of the two mirror line segments as point $Z(0,0)$ and graph two more points: $A(0,-4)$ and $B(4,0)$.
 - Use a protractor to measure $\angle ZC_2C_1$ to the nearest degree. Record the measurement on your graph paper and label the angle as either outgoing or incoming.
 - Explain how the measurements of the remaining angles can be determined by knowing only the measure of $\angle ZC_2C_1$.
 - Label the remaining three angles as incoming or outgoing and record their measures.
 - Calculate the sum of the measures of the two incoming and two outgoing angles.
 - Use the sum of the four angles to make a conjecture about the path of a light ray in a double reflection or the path of a pool ball in a double bank shot.
- 4.3** To model a pair of mirrors, draw two perpendicular segments on a sheet of graph paper as in Exploration 2. Label the intersection of the two segments point $Z(0,0)$ and graph two more points: $A(0,-4)$ and $B(4,0)$. Find the path that light would travel from point $O(4,-1)$ to point $E(3,-7)$ if it reflected off both mirrors. Label all points, angles, and segments that correspond to those found in Exploration 2. Identify the incoming and outgoing angles and give their measures.

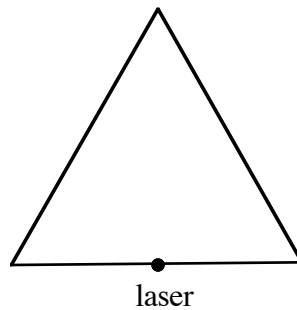
- 4.4**
- a. Use a protractor to draw two line segments representing mirrors hinged at 70° on a sheet of blank paper.
 - b. Label two points between the mirrors: point O , representing an object, and point E , representing the perspective of an eye.
 - c. Find the locations of the images of O . Label these points O' and O'' .
 - d. Sketch the path of light from O to E in a double reflection.
 - e. Measure one of the incoming or outgoing angles and record this measurement on your drawing.
 - f. Determine the measures of the remaining three angles and record these measures on your drawing. Label all angles either as incoming or outgoing.
 - g. Calculate the sum of the measures of all four angles. Does this sum support the conjecture you made in Problem 4.2?
- 4.5** A full-size pool table measures approximately $126 \text{ cm} \times 255 \text{ cm}$.
- a. Make a scale drawing of a pool table, including the six pockets.
 - b. Mark the location of a ball somewhere on your scale drawing, then draw the path of a shot that requires the ball to strike at least two side rails before reaching a pocket.
 - c. Label the appropriate points (with measurements in centimeters), so that a pool player could:
 1. locate the starting position of the ball on an actual pool table
 2. locate the point where the ball must hit the first side rail in order to make the shot.
 - d. If possible, test your calculations on a real pool table, and write an account of your test.

* * * * *

- 4.6** Some large radio telescopes use several reflecting surfaces positioned in special ways. The diagram below shows three such surfaces in a radio telescope built in Arecibo, Puerto Rico, in 1963. On a copy of this diagram, draw the path of each reflected radio wave and explain why these paths occur.



- 4.7** The diagram below shows three hinged mirrors positioned in the shape of an equilateral triangular prism. A hole in the center of one mirror allows a laser beam to enter the prism.

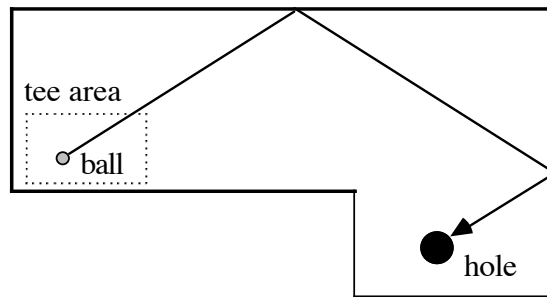


- Draw a path for the laser beam that allows it to reflect off each of the other two mirrors exactly once and pass back through its starting point.
- Determine the measures of every incoming and outgoing angle and the measures of every angle of incidence and angle of reflection.
- Identify the shapes formed by the lines of reflection. How are these shapes related to the original triangle?

* * * * *

Summary Assessment

1. Most miniature golf courses have holes requiring a player to hit the ball off at least one wall to score a hole-in-one. An example is shown in the diagram below.



Design and draw some miniature golf holes according to the following three rules.

- The drawing must be to scale, with all dimensions indicated.
- Only line segments may be used for walls.
- A tee area must be provided.

As you design holes, sketch some possible paths for a hole-in-one. In your final drawing for each hole, however, do not reveal your winning strategy. Design one of each of the following types of holes.

- a. A hole that looks simple, but where a hole-in-one is probably impossible.
 - b. A hole that looks difficult, but has a simple path for the ball.
 - c. A hole that has many possible paths.
 - d. A hole that requires a player to bank the ball off exactly three walls to get a hole-in-one.
2. To test your designs, trade drawings with a classmate. Try to sketch the path of a hole-in-one in each of your classmate's drawings.
 3. After your designs have been tested, present one of them to the rest of the class. Use mathematical ideas and the language of this module to explain your design.

Module Summary

- A **polygon** is a union of coplanar segments intersecting only at endpoints. At most two segments intersect at any one endpoint and each segment intersects exactly two other segments. Each segment in a polygon is a **side**; each endpoint is a **vertex** (plural **vertices**).
- A polygon is **equiangular** if its interior angles are congruent. A polygon is **equilateral** if its sides are congruent. A polygon is **regular** if its sides are congruent and its interior angles are congruent.
- An angle formed by two rays drawn from the center of a circle is a **central angle**.
- If a regular polygon is inscribed in a circle (all vertices of the polygon lie on the circle), then the central angle formed by rays drawn from the center of the circle to two consecutive vertices of the polygon divide the polygon into congruent isosceles triangles. The measure of a central angle of a regular polygon with n sides is:

$$\frac{360^\circ}{n}$$

- The measure of each **interior angle** of a regular polygon can be determined using the following formula, where n represents the number of sides:

$$180^\circ - \frac{360^\circ}{n}$$

- The sum of the measures of the interior angles of a regular polygon can be determined using the following formula, where n represents the number of sides:

$$\left(180^\circ - \frac{360^\circ}{n}\right)(n) = 180^\circ n - 360^\circ$$

- The **incident ray** models the path that a light ray follows from the object to the reflective surface, while the **reflected ray** models the path of the light ray away from the reflective surface.
- The **incoming angle** is the angle between the reflective surface and the incident (incoming) ray. The **outgoing angle** is the angle between the surface and the reflected (outgoing) ray.
- The **normal** of the reflecting surface is the perpendicular line to the surface at the **point of reflection**. The **angle of incidence** is the angle formed by the normal and the incident ray, while the **angle of reflection** is the angle formed by the normal and the reflected ray.

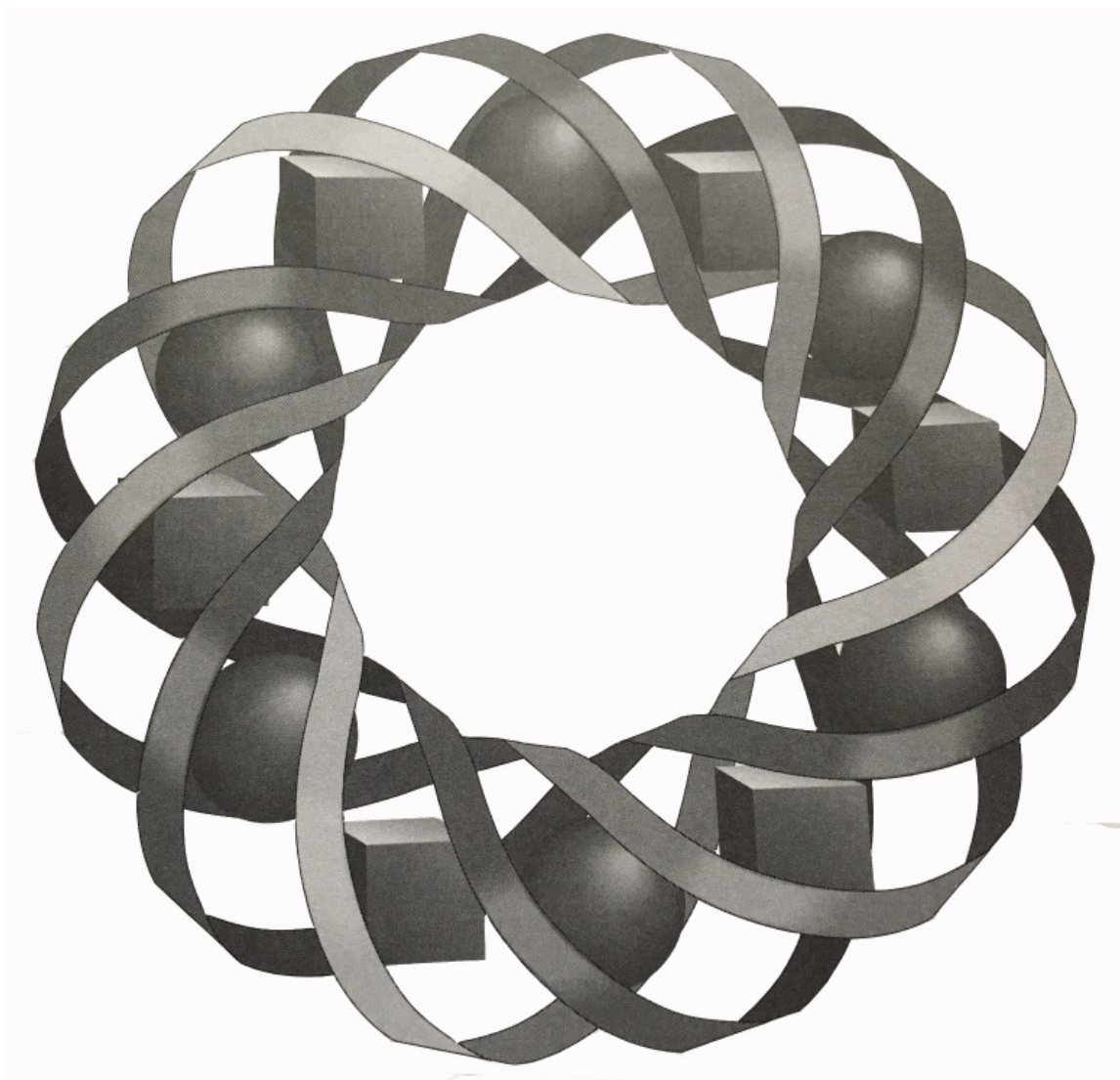
- If the sum of the measures of two angles is 90° , then the two angles are **complementary**.
- If the sum of the measures of two angles is 180° , then the two angles are **supplementary**.
- The incoming angle and the angle of incidence are **complementary**, in other words, the sum of their measures is 90° . The same is true for the outgoing angle and the angle of reflection.
- The **distance from a point to a line** is the distance along a path perpendicular to the line.
- The reflection of an object is its **image**. The object itself is the **preimage**. If point A is the preimage, then the image of a point A can be represented as A' (read “A prime”).
- The **perpendicular bisector** of a segment is the line that is perpendicular (\perp) to the segment and divides the segment into two congruent parts.
- A **reflection** in a line is a pairing of points in a plane so that the **line of reflection** is the perpendicular bisector of each segment connecting a preimage point to its corresponding image point. Every point on the line of reflection is its own image.
- When conjectures based on patterns are generalized and shown to be true for all cases, they become accepted mathematical **theorems**.

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So You Want to Buy a Car



Choosing a new car is never easy. In this module, you'll use mathematics and technology to analyze some common concerns of today's car buyers.

Ted Dreith • Anne Merrifield • Dean Preble • Deanna Turley



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So You Want to Buy a Car

Introduction

In today's world, buying a car is no simple task. Before making a final decision, you must consider numerous manufacturers, dozens of styles, and countless options. Some buyers analyze a car's handling and performance. Others focus more on fuel economy and resale value. Which of these characteristics are most important to you?

Exploration

- a. Make a list of the 10 factors you consider most important when buying a new car.
- b. Share your list with others in the class. Identify any similarities and differences among the lists.
- c.
 1. As a class, decide on a method that can be used to determine a class list of the "Top 10" factors to consider when buying a car.
 2. Use this method to determine the Top 10 factors.

Discussion

- a. Do you think that the Top 10 adequately represents the opinions of all members of your class?
- b. What are the benefits or limitations of the strategy used to generate the Top 10?

Activity 1

In the introduction, you determined a class list of the 10 most important factors to consider when buying a new car. Unless you have some way to organize all that information, comparing 10 factors for several different cars can be a difficult task.

Exploration

In this exploration, you use a spreadsheet to help organize data and create a graph of the results.

- a. Rank the 10 factors from your class list in order of importance to you. Use 1 to indicate the most important and 10 to indicate the least important.

- b. Collect and organize the rank numbers from some other students in the class. Enter the data in a spreadsheet. Table 1 below shows a sample arrangement.

Table 1: Sample spreadsheet arrangement

	Student X	Student Y	Student Z
Factor A	6	3	1
Factor B	7	5	4
Factor C	2	2	8
Factor D	9	9	2

Technology Note

Like a table, a spreadsheet consists of rows and columns. As shown in Figure 1, rows are typically labeled with numbers, while columns are labeled with letters. Each individual section or **cell** in a spreadsheet can be identified by the column and row that contain it.

In Figure 1, for example, the cell containing the number 8.5 is referred to as B2. The cell containing the number 14.3 is referred to as A3.

	A	B	C
1			
2		8.5	
3	14.3		

Figure 1: Portion of a spreadsheet

- c. Use the following steps, along with the data you collected in the spreadsheet from Part b, to rank the 10 factors in order of importance.
1. Consider the first factor in the spreadsheet. Write a formula using individual cell notation to find the sum of the rank numbers.
 2. Label a new column for the sum of the rank numbers. Use the formula created in Step 1 to calculate the sum of the rank numbers for the first factor and display it in the first cell of the new column.
 3. Most spreadsheets have a built-in function that allows you to compute the sum of the numbers in a group of cells. Use this function to find the sum of the rank numbers for the second factor and display it in the second cell of the new column.
 4. Find and display the sums of the rank numbers for the remaining factors in the corresponding cells of the new column.
 5. Rank the factors in order of importance according to their sums.

- d. Most spreadsheets also have a built-in function that allows you to find the mean of the numbers in a group of cells.
 - 1. Use this function to find the means of the rank numbers for the 10 factors and display them in another column.
 - 2. Rank the factors in order of importance according to their means. Compare this rank order to the order from Part c.
- e. The information stored in spreadsheets can also be displayed graphically. Use the spreadsheet to create two different bar graphs: one to display the sums of the rank numbers for the 10 factors and another to display the means of the rank numbers.
- f. Spreadsheets are often able to sort entries in increasing or decreasing order, either alphabetically or numerically. When sorting data, it is important to make sure that all the data moves together. Use your spreadsheet to complete Steps 1–3 below.
 - 1. Sort the factors alphabetically.
 - 2. Sort the sums or means of the rank numbers numerically.
 - 3. Sort the factors in the order that you feel is most useful. Create a bar graph to display the sorted information.

Discussion

- a. Describe some of the benefits of using a spreadsheet to help analyze data.
- b. What problems did you encounter in using the spreadsheet?
- c. Compare the two bar graphs you created in Part e of the exploration. What differences or similarities do you observe?
- d. How can sorting the data in a table or graph help you interpret it?

Assignment

- 1.1** An automobile dealership has space for two new cars in its showroom. There are four different models available: a convertible, a sports utility vehicle, a luxury car, and a pickup truck. Since the sales staff cannot agree on their two choices, they decide to rank each model in order of preference from 1 to 4, with 1 being the most preferred. The rank numbers for each model are shown in the table below.

Salesperson	Convertible	Utility	Luxury	Truck
A	1	4	3	2
B	3	2	4	1
C	1	3	4	2
D	2	1	3	4

After compiling their rankings, the staff asks you to help them analyze the data and recommend which two models they should order for the showroom.

- Use a spreadsheet to analyze the data using the sums and means of the rank numbers.
 - Display the results graphically.
 - Which two cars would you recommend for the showroom? Write a short summary to support your recommendation.
- 1.2** The data in the following table shows the value of new car exports, in millions of dollars, from the United States to Japan, Canada, and other countries from 1989 to 1992.

Year	Japan	Canada	Other
1989	327	6824	2274
1990	531	6232	2465
1991	496	6195	3070
1992	694	5928	5091

Source: U.S. Bureau of the Census.

- Enter the data from the table in a spreadsheet.
- Use the spreadsheet to display the total value of new car exports for the United States per year.
- Create a graph that displays the totals.
- Describe any trends you observe in new car exports.

* * * * *

- 1.3** Imagine that you and three other students have been chosen to ride the space shuttle to an international space station. Due to the shuttle’s strict limits on cargo, you can take only a few personal items with you.
- a. To help plan your trip, the space center sends you a list of 10 items that previous astronauts have taken with them into space. This list includes a camera, a medal, a photograph, a stuffed animal, a favorite food, a favorite article of clothing, a compact disc, a book, a hand-held video game, and a portable computer. Enter this information in a spreadsheet.
 - b.
 1. If you could take some of these items with you on the journey, which ones would you choose? Rank each item from 1 to 10, with 1 indicating your first choice.
 2. Collect the rankings from three of your classmates.
 3. Enter the information from Steps 1 and 2 in a spreadsheet.
 - c. Find and display the sums and means of the rank numbers for the 10 items.
 - d. Create two bar graphs: one to display the sums of the rank numbers and the other to display the means of the rank numbers. Compare the two graphs.
 - e. Sort the names of the 10 items alphabetically. How might this information be used?
 - f. Sort the sums or means of the rank numbers numerically. How might this information be used?
 - g. Decide which sorting technique creates the most useful form of the data. Construct a bar graph to display this information.

* * * * *

Activity 2

The year 1992 was an unusual one in automotive history. For the first time, three major publications—*Car and Driver*, *Motor Trend*, and *Automobile Magazine*—chose the same domestic “Car of the Year.”

When making their selections, automotive magazines use both **subjective** and **objective** types of information. For example, a car’s comfort level is based on the passenger’s opinion; therefore, it is a subjective factor. A car’s weight, however, is not based on opinion. Weight is an objective factor.

Each magazine used its own selection process and mix of considerations. The editors of *Motor Trend*, for example, rated eight cars on eight different factors. The six subjective factors were styling and design, quality control, occupant

comfort and convenience, ride and drive, chassis dynamics, and dollar value/market significance. The two objective factors were handling and performance.

A car's handling rating included the results of three tests: braking, maneuverability, and cornering. The braking test measured the distance required to bring a car traveling at 60 mph to a stop. The maneuverability test measured the speed that could be maintained while driving through a slalom course.

The cornering test measured the force required to put a car into a skid. In this case, the unit of measure for force is the *g*, where 1 *g* equals the force of gravity on an object at rest. Cars that receive higher values handle corners better than those that receive lower values. *Motor Trend's* test results for eight cars are shown in Table 2.

Table 2: Handling

Model	Braking (ft)	Slalom (mph)	Skid (g)
Cadillac Eldorado Touring Coupe	149	61.7	0.78
Cadillac Seville Touring Sedan	144	65.3	0.80
Ford Taurus SHO	135	63.3	0.80
Mercury Sable LS	142	61.8	0.79
Oldsmobile Achieva SCX	134	67.7	0.86
Oldsmobile 88 Royale LS	137	65.1	0.79
Pontiac Bonneville SSEi	140	65.4	0.80
Pontiac Grand Am GT	141	67.2	0.86

Each car's performance rating combined the results of two fuel economy tests and two acceleration tests. Fuel economy was measured in miles per gallon (mpg) during both city driving and highway driving.

The results of the acceleration tests were reported in terms of the time required to reach 60 mph and to travel a quarter mile. *Motor Trend's* test results for the same eight cars are shown in Table 3.

Table 3: Performance

Model	Fuel Economy (mpg)		Acceleration Time (sec)	
	City	Highway	0-60 mph	1/4-mile
Cadillac Eldorado Touring Coupe	16	25	8.3	16.2
Cadillac Seville Touring Sedan	16	25	8.4	16.4
Ford Taurus SHO	18	26	6.8	15.2
Mercury Sable LS	18	28	9.4	17.0
Oldsmobile Achieva SCX	21	31	8.0	16.2
Oldsmobile 88 Royale LS	18	28	9.2	16.8
Pontiac Bonneville SSEi	16	25	8.1	16.1
Pontiac Grand Am GT	21	31	7.5	15.9

Exploration

In this exploration, you use bar graphs to analyze the data in Tables 2 and 3.

Based on your analysis, you then determine your own “Car of the Year.”

- a.
 1. Draw separate bar graphs to display the results of each handling test in Table 2.
 2. For each graph, decide whether a longer bar or a shorter bar indicates a better test result. In the skid test, for example, cars that require more force before skidding handle corners better than cars that require less force. Therefore, a longer bar means a better test result.
- b.
 1. Draw one bar graph to display the test results for both city and highway fuel economy in Table 3.
 2. Draw one bar graph to display the results of the two acceleration tests in Table 3.
 3. For each graph, decide whether a longer bar or a shorter bar indicates a better result. Explain your reasoning.
- c. Use the test results in Table 2 to rank the eight cars on handling.
- d. Use the test results in Table 3 to rank the eight cars on performance.
- e. Rank the eight cars on combined handling and performance where both categories are equally important. Use this information to choose your own 1992 Car of the Year.

Discussion

- a. Describe the method you used to rank cars in each category.
- b. How did you combine the rankings in the two categories to select a Car of the Year?
- c. Table 2 contains data on three different handling tests, while Table 3 contains the results of four different performance tests. Did you give each of these seven tests equal consideration in making your selection?
- d. *Motor Trend's* 1992 Car of the Year was the Cadillac Seville Touring Sedan. Do your results agree with this selection?
- e. How might the editors of *Motor Trend* justify their choice?

Assignment

- 2.1**
- Use technology to determine the amount of fuel each car in Table 3 will consume during:
 - 400 miles of city driving
 - 400 miles of highway driving.
 - Create a bar graph that compares the amount of fuel consumed by 400 miles of city driving with the amount consumed by 400 miles of highway driving.
- 2.2**
- Use technology to determine the fuel costs for each car in Table 3 for an 800-mile trip, including 250 miles of city driving. Use the current price of fuel in your area to determine these costs.
 - Create a bar graph that compares the fuel costs for the eight cars.
- 2.3**
- The slalom course used in the maneuverability test was 600 ft long. Use the information in Table 2 to create a bar graph that compares the number of seconds each car required to complete this course. Assume that each car's speed is constant for the entire length of the course.

* * * * *

- 2.4**
- Nutritionists generally recommend that you limit your intake of fat, sodium, and cholesterol. The following table contains some nutritional information for hamburgers sold at four fast-food restaurants.

Burger	Calories	Fat (g)	Cholesterol (mg)	Sodium (mg)	Protein (g)
A	500	26	100	890	25
B	584	34	73	733	26
C	630	39	90	850	27
D	730	47	80	1110	28

- Draw a bar graph to display the sodium and cholesterol content of the four hamburgers in the table.
- In your graph from Part a, decide whether a longer bar or a shorter bar indicates a better hamburger. Explain your reasoning.
- If separate bar graphs were drawn for each remaining category in the table, would a better hamburger be represented by a longer bar or a shorter bar?
- Use the information in the table to decide which restaurant provides the most nutritious hamburger. Justify your choice.
- Your selection in Part d was based on objective data. What subjective information about each hamburger might also help you decide where to eat?

* * * * *

Research Project

The cost of owning a vehicle involves more than just the sticker price. If you finance your purchase, for example, your interest payments increase the cost of the car. Additional costs result from operating and maintaining the vehicle.

In this project, you must determine the cost per mile to operate a vehicle of your choice for the first five years of ownership. Your report should include the purchase price and the criteria you used to choose the vehicle. If applicable, your report also should describe the down payment, the annual interest rate, the number and size of the monthly payments, the total amount of interest paid, and the total cost of the vehicle (including interest).

When determining operating and maintenance expenses, consider each of the following items:

- the number of miles the vehicle is driven annually
 - the cost to insure the vehicle
 - the cost to license the vehicle
 - the cost of replacement tires
 - the cost of regular oil changes, tune-ups, and other scheduled repairs
 - the cost of unscheduled repairs
 - the vehicle's loss in value over the five-year period.
-

Activity 3

When designing a new car, engineers must often consider the relationship between two different quantities. For example, the relationship between fuel economy and weight can help determine what materials are used in a vehicle.

Mathematics Note

A **scatterplot** is a type of graph in which the corresponding values of two quantities are treated as ordered pairs, (x,y) . These ordered pairs are plotted as points on a rectangular coordinate system. A scatterplot can be a useful tool for examining the relationship between two quantities.

For example, Table 4 shows the weight and highway fuel economy of eight cars.

Table 4: Highway fuel economy and weight for eight cars

Car Model	Fuel Economy (mpg)	Weight (pounds)
Cadillac Eldorado Touring Coupe	25	3612
Cadillac Seville Touring Sedan	25	3707
Ford Taurus SHO	26	3309
Mercury Sable LS	28	3191
Oldsmobile Achieva SCX	31	2816
Oldsmobile 88 Royale LS	28	3468
Pontiac Bonneville SSEi	25	3691
Pontiac Grand Am GT	31	2804

A scatterplot of the data in Table 4 appears in Figure 2.

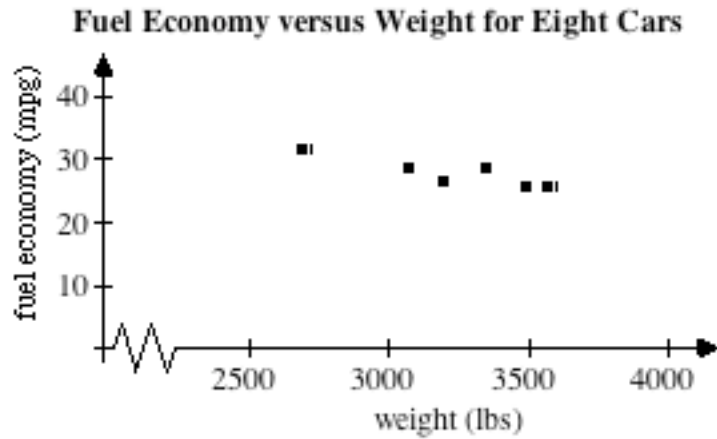


Figure 2: Scatterplot of fuel economy versus weight

The pattern of the points in the scatterplot implies that there is a relationship between weight and fuel economy. As the weight of a vehicle increases, the fuel economy tends to decrease.

Exploration

In this exploration, you use scatterplots to analyze the relationships between quantities.

- a. As a car is driven through an obstacle course, a scorekeeper records the distance it has traveled at some specified times.
1. Sketch a graph that you believe illustrates the relationship between the time the car has been on the course and the distance it has traveled. Label the axes of the graph.
 2. Table 5 shows the data collected during one car's trip through the obstacle course. The distance traveled is reported in meters (m) and the time is given in seconds (sec). Use technology to create a scatterplot of the data and compare the scatterplot with your sketch.

Table 5: Distance traveled and time on obstacle course

Time (sec)	Distance (m)	Time (sec)	Distance (m)
1.0	27	4.0	113
1.5	43	4.5	114
2.0	50	5.0	143
2.5	52	6.0	158
3.0	80	7.0	188
3.5	94	8.0	215

3. Describe the relationship between time and distance traveled.
 4. If possible, use the scatterplot to estimate the distance the car will have traveled in 9 sec.
- b. To collect information on traffic flow, a group of commuters records the number of red lights they encounter as they travel to work.
1. Table 6 shows the data collected by the commuters. The distance traveled is reported in kilometers (km). Use technology to create a scatterplot of the data.

Table 6: Distance traveled and red lights encountered

Distance (km)	Red Lights	Distance (km)	Red Lights
1.0	3	4.0	8
1.5	7	4.5	2
2.0	1	5.0	4
2.5	11	6.0	7
3.0	4	7.0	3
3.5	1	8.0	5

2. Describe the relationship between distance traveled and the number of red lights encountered.
3. If possible, use the scatterplot to estimate the number of red lights a commuter will encounter when traveling 9 km.

- c. As you may have observed during the research project, the trade-in value of a vehicle changes as the vehicle ages.
1. Table 7 shows one car's trade-in value in dollars over a period of 9 years. Use technology to create a scatterplot of the data.

Table 7: Trade-in value and age of car

Age (years)	Dollar Value	Age (years)	Dollar Value
1.0	12,500	4.0	7600
1.5	11,700	5.0	6500
2.0	11,000	6.0	6000
2.5	10,300	7.0	5200
3.0	9400	8.0	4500
3.5	8100	9.0	4000

2. Describe the relationship between age in years and trade-in value.
3. If possible, use the scatterplot to estimate the car's trade-in value when it is 10 years old.

Discussion

- a. Describe the three scatterplots you created in the exploration.
- b. Would these graphs show the same relationships if the quantities graphed on the x -axis and the y -axis were interchanged?

Mathematics Note

Two quantities are said to have a **positive association** if, as the values of one quantity increase (or decrease), the values of the other quantity also increase (or decrease). When data in a positive association is graphed in a scatterplot, the pattern formed by the data points slants upward from left to right. As the x -values increase (or decrease), so will the y -values.

Two quantities are said to have a **negative association** if, as the values of one quantity increase, the values of the other quantity decrease. When data in a negative association is graphed in a scatterplot, the pattern formed by the data points slants downward from left to right. As the x -values increase, the y -values decrease.

If two quantities have neither a positive nor a negative association, their scatterplot will show no trend.

These three types of associations are illustrated in Figure 3.

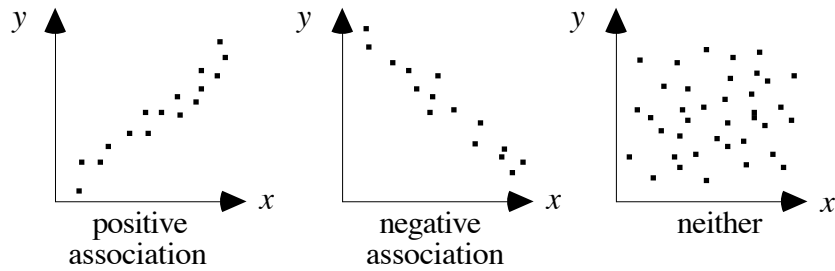


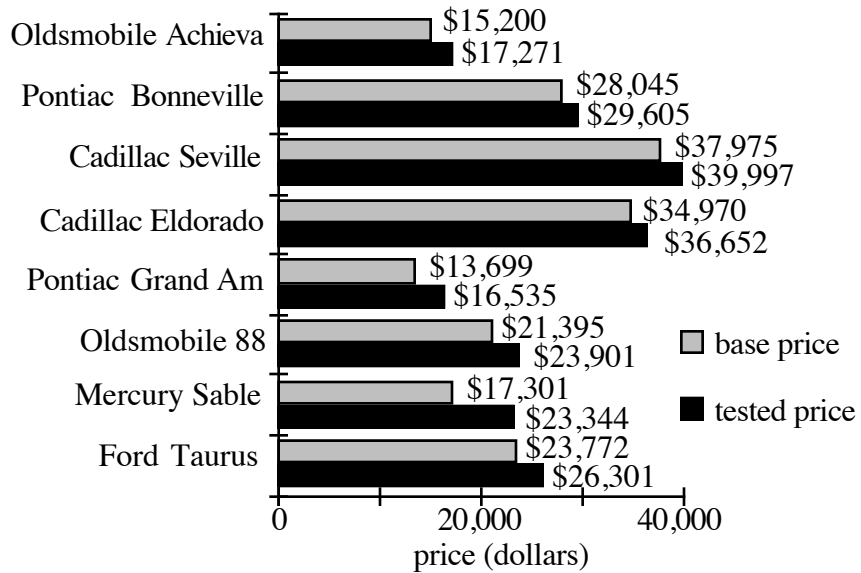
Figure 3: Three types of associations

- c. Describe the relationship indicated in each graph from the exploration as a positive association, a negative association, or neither.
- d. When can scatterplots reasonably be used to make predictions?

Assignment

- 3.1 The following bar graph shows the base price and tested price of each car nominated for *Motor Trend's* 1992 Car of the Year. (The base price of a car includes no added options. The tested price is the price of the actual model inspected by the magazine, including any options.)

Base Price and Tested Price of Nominated Cars



- a. Use a graphing utility to create a scatterplot that compares base price and tested price for these cars.
- b. What type of association exists between base price and tested price? Explain your response.
- c. Why might a manufacturer supply an automotive magazine with a car that features more options than its base model?

- 3.2** The table below shows the weight in pounds of each car nominated for *Motor Trend's* 1992 Car of the Year.

Model	Weight (pounds)
Cadillac Eldorado Touring Coupe	3612
Cadillac Seville Touring Sedan	3707
Ford Taurus SHO	3309
Mercury Sable LS	3191
Oldsmobile Achieva SCX	2816
Oldsmobile 88 Royale LS	3468
Pontiac Bonneville SSEi	3691
Pontiac Grand Am GT	2804

- Do you think that price and weight would have a positive association, a negative association, or neither?
 - To graph the data for price and weight, would you use base price, tested price, or both? Explain your reasoning.
 - Use a graphing utility to create a scatterplot of price versus weight. Does the scatterplot support your response to Part **a**?
 - If possible, use your scatterplot to estimate the cost of a car that weighs 4000 lb. If it is not possible to make a reasonable estimate, explain why.
- 3.3** Use your scatterplot from Problem **3.2** to decide whether each of the following statements is true or false. If the statement is true, justify your decision. If the statement is false, use an example to show why.
- Lighter cars are less expensive than heavier cars.
 - More expensive cars weigh less than less expensive cars.
 - As the weight of a car increases, the price of a car decreases.
 - As the price of a car increases, the weight of a car increases.
- 3.4** The table below shows the weight and the results of two acceleration tests for eight cars. Use a scatterplot of this data to determine if lighter cars accelerate better than heavier cars.

Car Model	Time (sec)		Weight (lb)
	0–60 mph	1/4 mile	
Cadillac Eldorado	8.3	16.2	3612
Cadillac Seville	8.4	16.4	3707
Ford Taurus	6.8	15.2	3309
Mercury Sable	9.4	17.0	3191
Oldsmobile Achieva	8.0	16.2	2816
Oldsmobile 88	9.2	16.8	3468
Pontiac Bonneville	8.1	16.1	3691
Pontiac Grand Am	7.5	15.9	2804

- 3.5** Identify two quantities related to cars (such as speed, weight, or fuel economy) that illustrate each type of association listed below. Use examples that have not been previously described in this module and explain why you believe the two quantities are associated in this way.
- a positive association
 - a negative association
 - neither a positive nor a negative association

* * * * *

- 3.6** The following table shows the calories and fat contained in a single serving of some common meats.

Meat Product	Fat (grams)	Calories
chicken breast	2	120
pork tenderloin	5	159
turkey (light meat)	8	176
sirloin steak	8	190
salmon	9	180
hamburger	17	250
beef hot dogs	26	284

Source: U.S. Department of Agriculture.

- Create a scatterplot that compares calories in each meat product versus the number of grams of fat.
- Describe the type of association that exists between these two quantities.

* * * * *

Activity 4

Car buyers who trade in their old vehicles for newer models often consider resale value when selecting their new cars. A car's resale value often declines as it gets older. This decrease in value over time is called **depreciation**. The rate of depreciation for cars varies from model to model. A car with a low rate of depreciation might rank high with some consumers because they would receive a trade-in allowance closer to the car's original price.

Tables 8–11 show the high and low trade-in values for the 1986 models of four of the cars examined in previous activities. Notice that the high and low values given for 1986 are equal; this was the manufacturer’s suggested retail price for a car with standard equipment.

Table 8: Cadillac Eldorado

Year	Low value (dollars)	High value (dollars)
1986	26,756	26,756
1987	17,475	21,925
1988	14,775	18,700
1989	12,200	15,675
1990	10,175	13,375
1991	8,250	11,250
1992	6,300	8,775
1993	5,675	8,125

Table 9: Cadillac Seville

Year	Low value (dollars)	High value (dollars)
1986	24,251	24,251
1987	16,275	20,550
1988	13,725	17,475
1989	11,900	15,300
1990	9,550	12,600
1991	7,925	10,775
1992	6,525	9,075
1993	6,125	8,675

Table 10: Oldsmobile 88

Year	Low value (dollars)	High value (dollars)
1986	12,760	12,760
1987	8,550	10,925
1988	8,225	10,525
1989	6,975	8,975
1990	5,825	7,875
1991	4,600	6,500
1992	3,475	5,125
1993	2,500	4,025

Table 11: Pontiac Bonneville

Year	Low value (dollars)	High value (dollars)
1986	10,249	10,249
1987	7,075	9,100
1988	6,475	8,375
1989	5,425	7,125
1990	4,500	6,325
1991	3,250	4,850
1992	2,100	3,400
1993	1,625	2,900

Exploration

In this exploration, you use a spreadsheet to compare the rates of depreciation for four vehicles.

- a.
 1. Table 8 contains the trade-in data for a 1986 Cadillac Eldorado. Enter this information in the appropriate columns of a spreadsheet with the headings shown below.

Year	High	Low	Mean

2. Calculate the mean of the low and high values for each year and display it in the appropriate column of the spreadsheet.

- b.
 1. Create a scatterplot of the data that represents the year on the horizontal axis and the mean value on the vertical axis.
 2. Describe the shape of the graph.
 3. Does the mean value of this car appear to increase or decrease over time?
 4. Use the graph to estimate the mean value of the car in 1995.
- c. Repeat Parts **a** and **b** for the other three cars.
- d. Compare the four scatterplots. What similarities or differences do you observe?
- e.
 1. Add another column to each spreadsheet. Use this column to display the decrease in mean value from year to year for each car.
 2. Create a scatterplot that represents a car's age in years on the x -axis and its decrease in mean value on the y -axis.
- f. Determine the total depreciation in the mean value of each car over the period from 1986 to 1993.
- g. Imagine that all four cars are driven 70,000 miles over the period from 1986 to 1993. Compare the depreciation per mile for the four cars.

Note: Save the spreadsheets you created in this exploration for use in the assignment.

Discussion

- a.
 1. In Part **e** of the exploration, why are there no spreadsheet entries for decrease in mean value in 1986?
 2. How are the data points for the different cars represented on the scatterplot?
 3. Is the decrease in mean value the same from year to year for any car? How can you tell?
 4. In the scatterplot for the Oldsmobile 88, the y -value for year 3 is greater than the y -value for year 2. Does this mean that the car increased in value? Explain your response.
- b. Which car had the greatest decrease in mean value between 1990 and 1991? Did you use your spreadsheets or your graphs to answer this question? Is one easier to use than the other?
- c.
 1. In the exploration, you expressed the amount of depreciation in terms of dollars. How else could you describe a car's loss in value over time?
 2. Why might this be a more reasonable way to compare the depreciation of two different vehicles?

Mathematics Note

The **percent increase** in a quantity can be found by expressing the increase as a fractional part of the original value, then expressing that fraction as a percentage.

Similarly, the **percent decrease** in a quantity can be found by expressing the decrease as a fractional part of the original value, then expressing that fraction as a percentage.

For example, consider a car whose fuel economy increased from 25 mpg to 28 mpg in its most recent model year. The percent increase in fuel economy can be calculated as shown below:

$$\frac{\text{increase}}{\text{original value}} = \frac{28 - 25}{25} = \frac{3}{25} = 0.12 = 12\%$$

- d.
 1. Which car showed the smallest decrease in mean value over the period from 1986 to 1993?
 2. Did this car also have the least percent decrease in mean value?
- e. If a buyer ranked depreciation as the most important factor in choosing a car, which of the four vehicles would you recommend? Explain your response.

Assignment

- 4.1 From Table 8, the mean value of a 1986 Cadillac Eldorado in 1987 was \$19,700. Its mean value in 1988 was \$16,738.
- a.
 1. What percentage of the car's 1987 value is represented by its 1988 value?
 2. The Cadillac Eldorado lost \$2,962 of its mean value from 1987 to 1988. What percentage of the 1987 value is represented by this decrease?
 - b. Which of the two percentages calculated in Part a represents the percent decrease in value? Explain your response.
 - c. Add the two percentages you found in Part a. What does this sum represent?

- 4.2**
- Using your spreadsheets from the exploration, calculate and display the percent decrease in the mean value of each car after each year.
 - Create a graph that shows the percent decrease in mean value each year for all four cars.
 - Describe any patterns you observe in the data.
 - Which model(s) had the greatest percent decrease for a single year? Between what years did this decrease occur? Did the same car have the greatest amount of depreciation for that year?
 - Describe a situation in which the car that had the greatest amount of depreciation was not the car that had the greatest percent decrease for that year.
 - Which is the better method for comparing losses in value, amount of depreciation or percent decrease? Explain your reasoning.
- 4.3** In Part **f** of the exploration, you determined the total depreciation in the mean value of each car from 1986 to 1993.
- Find the total percent decrease in mean value for each car.
 - Rank the cars using both the total amount of depreciation and the total percent decrease.
 - Use your rankings to determine which car best holds its value over time.

* * * * *

- 4.4** The Shure Shine Company is planning to use the following advertisement to market its shampoo.



- The original bottle contained 200 milliliters (mL) of shampoo. What is the percent increase in the volume of the bottle?
- Do you think that the advertisement should use percent increase to market the shampoo? Explain your response.

- 4.5** The money that an investment earns is referred to as interest. Investments can earn interest in two ways. Using simple interest, interest is paid only on the original amount of money invested.
- a. An investment of \$1000 at an annual interest rate of 10% for 1 year would earn $\$1000 \cdot 0.10 = \100 in simple interest. What is the percent increase in the original investment after 1 year?
 - b. After 5 years, the same investment would earn \$500 in simple interest. What is the percent increase in the original investment after 5 years?

- 4.6** In some types of investments, the interest earned is periodically added to the amount originally invested. In these situations, the interest itself also earns interest. This is known as compound interest.

For example, when \$1,000 is invested at an annual interest rate of 10%, the interest earned after 1 year is $0.10 \cdot \$1,000 = \100 . If this amount is added to the original investment, the total value of the investment during the second year is $\$100 + \$1,000$ or \$1,100.

The interest earned on this amount after the second year is $0.10 \cdot \$1,100 = \110 . The value of the investment during the third year is $\$110 + \$1,100$ or \$1,210.

- a. Use a spreadsheet to find the value of this investment at the end of each year for 15 years.
- b. Determine the percent increase in the investment's value after 15 years.
- c. Explain why the percent increase after 15 years is not $15 \cdot 10\% = 150\%$.

* * * * *

Summary Assessment

1. Not all cars lose value over time. After decreasing in value for the first decade of use, some cars become collector's items. The value of these vintage cars then starts to increase. For example, consider the following three vintage cars.
 - A 1967 Jaguar XE-6 worth \$22,000 is expected to increase in value by 5% each year.
 - A 1948 Mercedes sedan worth \$31,000 is expected to increase in value by 4% each year.
 - A 1924 Ford Model A worth \$10,000 is expected to increase in value by 8% each year.

Imagine that an investor plans to buy one of these three cars and keep it for either 10 or 20 years. After 10 or 20 years, the investor would then sell the car for a profit.

 - a. To help this investor decide which car to purchase, determine the value of each car after 10 years and after 20 years.
 - b. For each time period, compare each car's increase in value to its percent increase.
 - c. In either time period, does the car that shows the greatest increase in value also show the greatest percent increase?
2. Would you recommend the same car for both a 10-year investment and a 20-year investment? Explain your response.
3. Write a report to the investor that explains which car you would recommend for each investment period. Use appropriate data and at least one graph to support your selections.

Module Summary

- **Subjective** information is based on opinion.
- **Objective** information is based on measurable results.
- A **scatterplot** is a type of graph in which the corresponding values of two quantities are treated as ordered pairs, (x,y) . These ordered pairs are plotted as points on a rectangular coordinate system.
- Two quantities are said to have a **positive association** if, as the values of one quantity increase (or decrease), the values of the other quantity also increase (or decrease). When data in a positive association is graphed in a scatterplot, the pattern formed by the data points slants upward from left to right. As the x -values increase (or decrease), so will the y -values.
- Two quantities are said to have a **negative association** if, as the values of one quantity increase, the values of the other quantity decrease. When data in a negative association is graphed in a scatterplot, the pattern formed by the data points slants downward from left to right. As the x -values increase, the y -values decrease.
- If two quantities have neither a positive nor a negative association, their scatterplot will show no trend.
- The decrease in value of a car or other property over time is **depreciation**.
- The **percent decrease** in a quantity can be found by expressing the decrease as a fractional part of the original value, then expressing that fraction as a percentage.
- The **percent increase** in a quantity can be found by expressing the increase as a fractional part of the original value, then expressing that fraction as a percentage.

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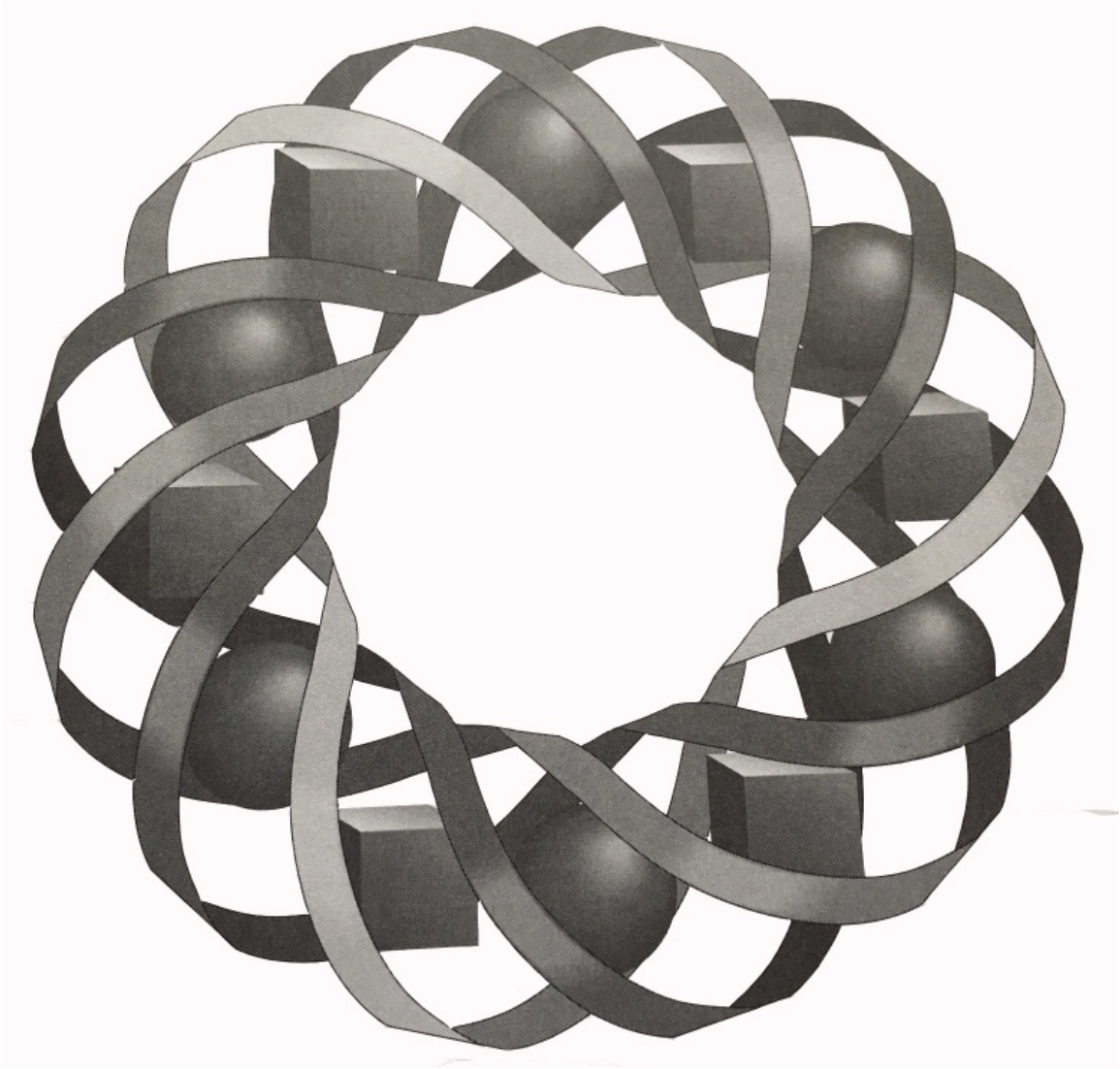
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U.S. Bureau of the Census. *Statistical Abstract of the United States: 1993*. Washington, DC: U.S. Government Printing Office, 1993.

Weaver, L., ed. *N.A.D.A. Official Used Car Guide*. McLean, VA: National Automobile Dealers Used Car Guide Co., 1992.

Other useful resources include: *Automobile Magazine*, *Car and Driver*, *Consumer Reports*, *Fodor's Used Car Guide*, and *Road and Track*.

Yesterday's Food Is Walking and Talking Today



You are what you eat, give or take a few calories. But how those calories get used depends largely on what you do. In this module, you examine how some daily activities—like walking, talking, or doing the backstroke—affect your dietary needs.

John Carter • Janet Higgins • Paul Swenson • Steve Yockim



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Yesterday's Food Is Walking and Talking Today

Introduction

By eating, you provide your body with fuel in the form of calories. In this module, you examine the number of calories the body needs to perform some typical activities.

Exploration

A **calorie** is a measure of heat energy. One way to determine the number of calories in a food is to measure the amount of heat released when that food is burned. In the following exploration, you use this technique to estimate the number of calories in a nut.

Note: You must wear eye protection throughout this experiment. Put on a pair of goggles *before* beginning the exploration.

- a. Arrange the equipment as shown in Figure 1 below. The bottom of the water can should be no more than 5 cm above the nut.

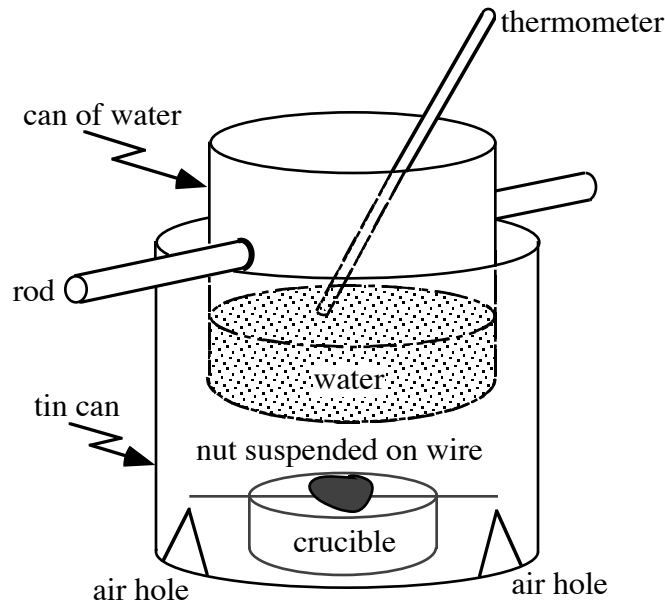


Figure 1: Experiment for estimating calories in a nut

- b. Make a table with headings like those in Table 1 and record the types of nuts you plan to test.

Table 1: Data for calorie experiment

Type of Nut	Initial Temperature (°C)	Maximum Temperature (°C)	Initial Mass (g)	Final Mass (g)	Volume of Water (mL)

- c. Measure and record each of the following:
1. the initial temperature of the water in degrees Celsius (°C)
 2. the volume of the water in milliliters (mL).
 3. the total mass of the nut, crucible, and wire in grams (g).
- d.
1. Ignite the nut. As the nut burns, observe the change in water temperature.
 2. Allow the nut to burn completely. Record the maximum temperature of the water.
- e. Determine the total mass of the ash, crucible, and wire. The difference between this value and the mass measured in Part c is the change in mass (the number of grams that burned).
- f. Repeat Parts c–e for several different types of nuts.

Science Note

A **calorie (cal)** is the amount of energy required to raise the temperature of one milliliter (1 mL) of water one degree Celsius (1°C).

A **kilocalorie (kcal)**, 1000 cal, is the amount of energy needed to raise the temperature of one liter (1 L) of water 1°C.

A **dietary calorie** (typically referred to as a Calorie with a capital C) is equal to 1 kcal. The calorie-per-gram rating on most food labels refers to dietary calories.

- g. Create a table with headings like those in Table 2 below.

Table 2: Kilocalories in different kinds of nuts

Type of Nut	Change in Temperature (°C)	Change in Mass (g)	Volume of Water (mL)	Calories per Gram	Kcal per Gram

- h. Use the data you recorded in Table 1 and the definitions described in the science note to complete Table 2. In this case, the number of calories per gram in each type of nut can be calculated using the following formula:

$$\text{calories per gram} = \frac{\text{volume of water (mL)} \cdot \text{change in temperature (°C)}}{\text{change in mass (g)}}$$

Discussion

- a. Do all types of nuts appear to contain about the same number of kilocalories per gram?
- b. Compare your results with the Calorie-per-gram rating on a package of nuts.
- c. If a nut failed to burn completely, would the experiment produce faulty data? Explain your response.
- d. What types of experimental errors might have affected your data?
- e. How would using a different quantity of water change the results of the experiment?

Activity 1

The human body uses calories at varying rates, depending on the level of activity. Calorie consumption also depends on a person's size, physical condition, and other factors. Table 3 shows the number of kilocalories used per minute during a variety of activities.

Table 3: Kilocalories used per minute per kilogram of body mass

Activity	$\frac{\text{kcal}}{\text{min} \cdot \text{kg}}$	Activity	$\frac{\text{kcal}}{\text{min} \cdot \text{kg}}$	Activity	$\frac{\text{kcal}}{\text{min} \cdot \text{kg}}$
archery	0.065	drawing	0.036	painting	0.034
badminton	0.097	eating	0.023	racquetball	0.178
basketball	0.138	football	0.132	running	
card playing	0.025	golf	0.085	7.2 min/km	0.135
carpentry	0.052	gymnastics	0.066	5.6 min/km	0.193
circuit training		jumping rope	0.162	5.0 min/km	0.208
Universal	0.116	judo	0.195	3.7 min/km	0.252
Nautilus	0.092	lying at ease	0.022	sitting quietly	0.021
free weights	0.086	music playing		stock clerking	0.054
cycling		cello	0.041	swimming	
slow	0.064	drums	0.066	backstroke	0.169
medium	0.100	flute	0.035	crawl	0.156
fast	0.169	organ	0.053	table tennis	0.068
dancing		piano	0.040	typing	0.027
aerobic	0.135	trumpet	0.031	walking	0.080
normal	0.075	woodwind	0.032	writing	0.029

Sources: McArdle, et al., *Exercise Physiology*; Sharkey, *Physiology of Fitness*.
 [Reprinted, by permission, from B.J. Sharkey, 1979, *Physiology of Fitness*, 1st ed. (Champaign, IL: Human Kinetics Publishers), 350–353.]

Discussion 1

- a. What types of activities burn calories at a high rate?
- b. What types of activities burn calories at a low rate?
- c. When would your body burn no calories at all?
- d. The units for the values in Table 3 are

$$\frac{\text{kcal}}{\text{min} \cdot \text{kg}}$$

Describe how these units can help you determine the number of kilocalories a 60-kg person burns while lying at ease for 30 min.

- e. A 60-kg person playing basketball for x minutes burns 500 kcal. This can be represented by the equation $500 = 8.28x$.
- Both sides of this equation must have the same units: kilocalories. Explain why the expression $8.28x$ represents kilocalories.
 - Describe how to determine x , the number of minutes spent playing basketball.

Mathematics Note

An equation of the form $y = mx$ can be solved for x by dividing each side of the equation by m .

For example, if $10 = 7x$, divide each side of the equation by 7. This results in the equation

$$\frac{10}{7} = x$$

The solution, therefore, is $10/7$.

Exploration

- Choose one activity from Table 3 that requires a high amount of energy, one that requires a moderate amount of energy, and one that requires a low amount of energy.
- Create a table with headings like those Table 4 below. Record the names of your chosen activities.

Table 4: Time required to burn kilocalories for three activities

Energy Required	High	Moderate	Low
Activity			
Time to Burn 100 kcal			
Time to Burn 200 kcal			
Time to Burn 300 kcal			
Time to Burn 400 kcal			
Time to Burn 500 kcal			

- For each activity you chose, calculate the time required for a 60-kg person to burn 100 kcal, 200 kcal, 300 kcal, 400 kcal, and 500 kcal. Record the answers in your table.
- Graph a scatterplot of the data in Table 4 on a single set of axes. Represent time on the x -axis and energy used on the y -axis.

Mathematics Note

An equation of the form $y = mx$ represents a line that passes through the origin where m is the **slope** of the line. The slope is the ratio of the change in vertical distance to the change in horizontal distance between any two points on the line.

The slope of a line containing two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $y_2 - y_1$ is the change in the vertical distance and $x_2 - x_1$ is the change in the horizontal distance. If $x_1 = x_2$, the line is vertical and has no slope.

For example, the graph in Figure 2 shows the energy used by a 80-kg person playing cards. This graph can be represented by an equation of the form $y = mx$, where y represents energy used in kilocalories and x represents time in minutes.

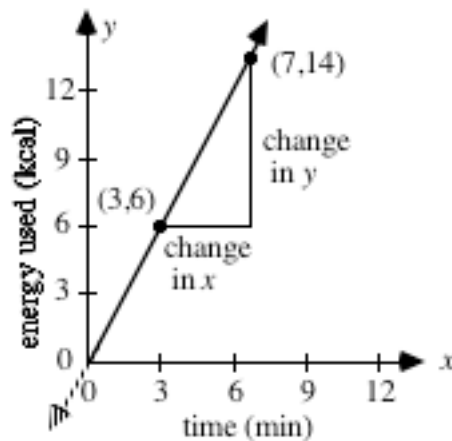


Figure 2: The slope of a line

Using the points $(7,14)$ and $(3,6)$, the change in the vertical distance is $14 - 6$; the change in the horizontal distance is $7 - 3$. Therefore, the slope of the line is:

$$m = \frac{14 - 6}{7 - 3} = \frac{8}{4} = 2$$

Since the line passes through the origin and its slope is 2, the equation of the line is $y = 2x$.

- e. For each activity in Table 4, find an equation that expresses y in terms of x , where y represents energy used in kilocalories and x represents time in minutes.
- f. Graph your equations from Part e on the same set of axes as the scatterplots from Part d.

Discussion 2

- a. What common point is contained by all the graphs in Part **f** of the exploration?
- b. What type of activities have graphs that are closest to vertical?
- c. What type of activities have graphs that are closest to horizontal?
- d. In Figure 2, the line $y = 2x$ is used to represent the energy consumed over time by a 80-kg person playing cards.
 1. Describe what a vertical change of 6 and a horizontal change of 3 means in terms of playing cards.
 2. In this situation, the slope of the line $y = 2x$ is the measure of a rate. Describe this rate.
- e.
 1. Identify the slope of each equation you wrote in Part **e** of the exploration.
 2. Explain how the values in Table 4 can be used to determine the slope of each equation.
 3. Describe the rates represented by the slopes of these equations.

Mathematics Note

A **relation** between two variables is a set of ordered pairs of the form (x,y) .

The **domain** of a relation is the set of first elements in the ordered pairs (the x -values). The **range** of a relation is the set of second elements in the ordered pairs (the y -values).

A **function** is a relation in which each element of the domain is paired with an element of the range and each element of the domain occurs in only one ordered pair. A function may be described by a rule or equation.

In mathematics, functions usually involve domains and ranges that are sets of real numbers. When a function is written without specifying the domain, you can assume that the domain comes from the set of real numbers. The range can be determined by finding the y -value that corresponds with each number in the domain.

For example, consider the function defined by the equation $y = 2x + 1$. In this case, the domain is the set of all real numbers. Each x -value is paired with only one y -value: a number that is 1 greater than twice the x -value. The range is also the set of all real numbers.

- f.
 1. What is the domain of the relation described by each equation you wrote in Part e of the exploration?
 2. What is the range of each relation?
- g.
 1. When the equation $y = 2x$ is used to represent the energy consumed over time by a 80-kg person playing cards, what values of the domain make sense?
 2. What values of the range correspond with these values of the domain?

Assignment

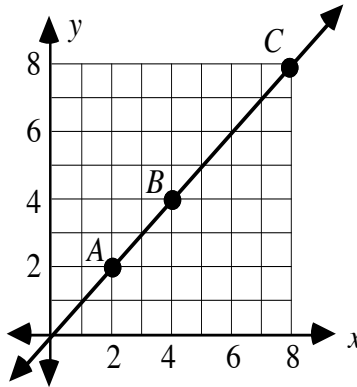
- 1.1
 - a. How many kilocalories does a 57-kg person use playing racquetball for 23 min?
 - b. Tristin has a mass of 61 kg. If he plays football for 2 hr, how many kilocalories will he burn?
 - c. Write an equation that describes the number of kilocalories expended by a 58-kg person while playing golf for x minutes.
 - d. Identify the domain and range of the relation you described in Part c.
- 1.2
 - a. Sigrid has a mass of 58 kg. While practicing judo, she used 300 kcal of energy. How many minutes did she practice?
 - b. Miguel has a mass of 72 kg. His breakfast contained 320 kcal. How long will it take Miguel to use these kilocalories at his aerobics class?
 - c. A 60-kg person uses y kilocalories while typing. Write an equation that expresses the amount of time, x , spent typing.
- 1.3
 - a. While running for 30 min, Alexi burns 500 kcal. Draw and label a graph to represent this situation. Use the graph to estimate how many kilocalories Alexi had burned after 17 min of running.
 - b. While typing for 30 min, David burns 50 kcal. Draw and label a graph to represent this situation. Use the graph to estimate how many kilocalories David had used after 12 min of typing.
 - c. While swimming, Alexi burns 150 kcal in 12 min and David burns 300 kcal in 28 min. Draw one graph to represent this situation. Use the graph to determine who would burn more kilocalories in 1 hr of swimming. Explain your reasoning.
 - d. Which quantity represents the range of the relations described in Parts a–c: the number of minutes or the number of kilocalories? Explain your response.

- 1.4** a. Determine the slope for each change in vertical and horizontal distance shown in the table below.

Change in Vertical Distance	Change in Horizontal Distance	Slope
8	4	
4	8	
-8	4	
3	-12	

- b. On a single set of axes, sketch the graphs of the lines that pass through the origin with the slopes in Part a.
- 1.5** Find the slope of the line through each of the following pairs of points.
- $(3, 7)$ and $(12, 3)$
 - $(5, 2)$ and $(6, -4)$
 - $(12, -8)$ and $(10, 4)$
 - $(-7, -2)$ and $(-3, 2)$
 - $\left(\frac{9}{7}, \frac{2}{5}\right)$ and $\left(-\frac{5}{7}, -\frac{3}{5}\right)$
- 1.6** The slopes of the lines you graphed in the exploration describe rates of energy usage in kilocalories per minute. Give an example of a rate which corresponds with each of the following:
- a large positive value for slope
 - a small positive value for slope
 - a negative value for slope.

- 1.7 Use the following graph to complete Parts a–c.



- a. List the coordinates of points A , B , and C .
 - b. Find the change in vertical distance, the change in horizontal distance, and the slope (m) between:
 1. points A and B
 2. points B and C
 3. points A and C .
 - c. Does the pair of points used to calculate the slope of a line affect the value of the slope? Explain your response.
- 1.8
- a. Select one activity from Table 3. Write an equation that describes the energy used over time by a 50-kg person performing this activity, where y represents energy in kilocalories and x represents time in minutes.
 - b. Graph the equation.
 - c. Identify the domain and range in this setting.
 - d. Find the slope of the line and describe what it represents in this situation.
 - e. Determine the number of kilocalories used when performing this activity for:
 1. 30 min
 2. 100 min
 - f. Describe how the slope of the line can be determined from your responses in Part e.

- 1.9**
- Draw the graph of a horizontal line and label the coordinates of any two points on the line.
 - Determine the slope of the line.
 - Draw the graph of a vertical line and label the coordinates of any two points on the line.
 - Determine the slope of the line.

Mathematics Note

A horizontal line has a slope of 0 and an equation of the form $y = c$. All the points on a horizontal line have the same y -coordinate, c .

A vertical line has no slope and an equation of the form $x = c$. All the points on a vertical line have the same x -coordinate, c .

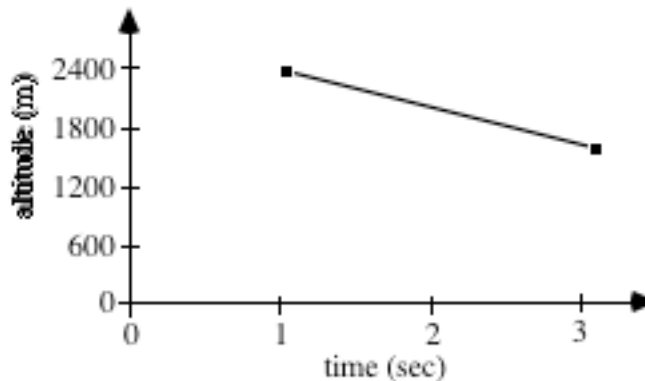
- Write the equation of the horizontal line in Part **a**.
 - Identify the domain and range for this relation.
- Write the equation of the vertical line in Part **c**.
 - Identify the domain and range for this relation.

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- 1.10** Sam's fast-food lunch contained 865 kcal. If his mass is 75 kg, how long would Sam have to jump rope to burn this amount of energy?
- 1.11**
- A sumo wrestler has a mass of 227 kg. Write an equation that describes the energy he uses over time while lying at ease. Let y represent number of kilocalories and x represent time in minutes.
 - Sam's mass is 75 kg. Write an equation that describes the energy he uses over time while lying at ease. As in Part **a**, let y represent number of kilocalories and x represent time in minutes.
 - Graph the two equations you wrote in Parts **a** and **b** on the same set of axes.
 - Which line appears closer to vertical? What does this observation indicate in this setting?

1.12 The slope of a line can be used to describe the average rate of change in one quantity with respect to another. For example, one familiar rate of change is speed, often expressed in kilometers per hour.

- a. Rolando is climbing a mountain. At 10:00 A.M., he stood at an elevation of 1000 m. By 2:00 P.M., he had reached an elevation of 1220 m. Determine Rolando's average rate of change in elevation in meters per hour.
- b. In 1980, the city of Tucson, Arizona, had a population of 534,000. By 1990, its population had grown to 655,000. Find Tucson's average rate of change in population in people per year.
- c. In 1950, the price of a pair of jeans was about \$8.00. By 1990, the price had risen to about \$32.00. Find the average rate of change in the price of jeans in dollars per year.
- d. The graph below shows the change in a parachutist's altitude (in meters) during an interval of 2 sec. Use the graph to estimate the parachutist's average rate of change in altitude in meters per second.



- e. Identify the domain and range for the graph in Part d.

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Activity 2

Many dietary specialists think of breakfast as the most important meal of the day. In this activity, you use linear equations to help plan an adequate breakfast for an active morning. Table 5 shows the number of kilocalories per serving in some typical breakfast foods.

Table 5: Kilocalories per serving for common breakfast foods

Food	kcal/item	Food	kcal/item
Toast, white	80	Croissant, egg, bacon, cheese	386
Toast, wheat	70	Biscuit, bacon, egg, cheese	483
Doughnut, plain	240	Biscuit with sausage	330
Cereal, with sugar	180	Cherry pie	260
Cereal, plain	120	Egg with muffin	340
Apple	60	English muffin with butter	186
Banana	80	Hotcakes with butter	500
Grapefruit	60	French toast	400
Orange juice	120	Fries	360
Egg, fried	120	Omelet	290
Egg, scrambled	80	Sausage, one patty	200
Egg, substitute	90	Milk, 2%	112
Milk, whole	160	Milk, chocolate	192
Yogurt, plain	120	Peanut butter and jam	500
Coffee	0	Soda pop	144

Sources: Gebhardt and Matthews, *Nutritive Value of Foods*; McArdle, et al., *Exercise Physiology*; Page and Raper, *Food and Your Weight*.

Exploration

- a. Use the information in Tables 3 and 5 to design a breakfast that will supply a 62-kg person with the number of kilocalories necessary to play racquetball for 1 hr.
- b. Make a table that shows the kilocalories remaining from the meal at the end of each 5-min interval of racquetball.
- c. Create a scatterplot of the data from Part b. Let the time in minutes be the domain of the relation and the energy remaining be the range.
- d. Determine the slope of the data.
- e. Determine the coordinates of the point where the data intersects the y-axis.

Mathematics Note

The y -coordinate of the point where a line intersects the y -axis is known as the **y -intercept**.

The equation of a line with slope m and y -intercept b can be written in the form $y = mx + b$. This is the **slope-intercept** form of the equation of a line.

For example, Figure 3 shows a graph of the equation $y = -2x + 4$. This line has a slope of -2 and a y -intercept of 4 . It intersects the y -axis at the point $(0,4)$.

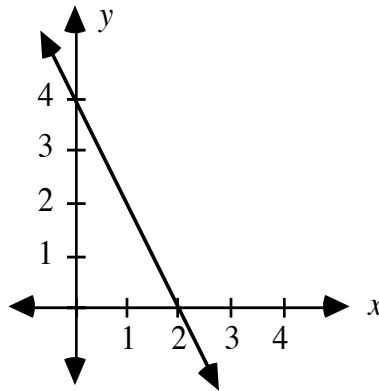


Figure 3: Graph of $y = -2x + 4$

- f. On the same set of axes as the scatterplot in Part **c**, graph the equation $y = mx + b$, where m is the slope of data and b is the y -intercept.
- g. Repeat Parts **b–f** for a 62-kg person who ate a breakfast of 800 kcal.

Discussion

- a. In Part **f** of the exploration, how did the scatterplot compare with the graph of the equation in the form $y = mx + b$?
- b. Describe what the variables y and x represent in the exploration.
- c. What information besides the slope do you need to identify a specific line?
- d. Compare the slopes of the equations in Parts **f** and **g** of the exploration.

Mathematics Note

Two lines that have equal slopes are **parallel**.

As shown in Figure 4, for example, the graphs of the lines $y = 2x$ and $y = 2x + 3$ are parallel.

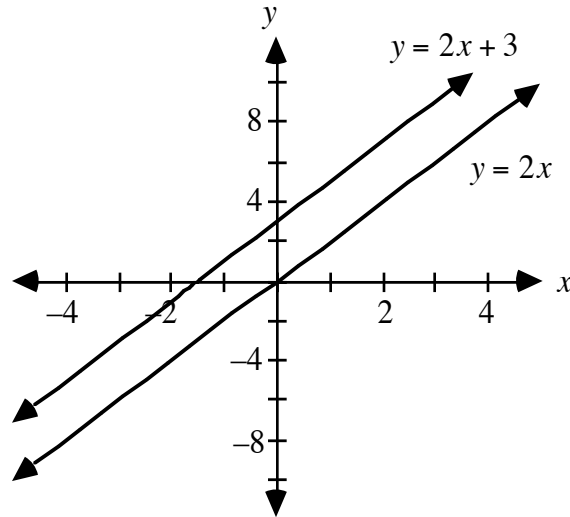


Figure 4: Graphs of $y = 2x$ and $y = 2x + 3$

Although vertical lines have no slope, any two vertical lines are also parallel.

- e. Given the equation of a line in the form $y = mx + b$, describe how to determine its slope and y-intercept.

Assignment

- 2.1 Imagine that a 50-kg person plans to dance aerobically for 1 hr.
- Use the information in Tables 3 and 5 to design a breakfast that will provide enough energy for this activity.
 - Write an equation in slope-intercept form that describes the number of kilocalories remaining from the meal in Part a at the end of each minute of dancing.
 - Write an equation in slope-intercept form that describes the number of kilocalories remaining from a 600-kcal meal at the end of each minute of dancing.
 - Sketch the graphs of both equations on a single set of axes.

- 2.2**
- Identify the y -intercept of a nonvertical line that passes through the origin $(0,0)$.
 - Write an equation for the line with a slope of 3 and a y -intercept of 4.
 - Write an equation for the line that crosses the y -axis at $(0,5)$ and has a slope of -2 .
 - Write an equation for the line with a slope of $7/3$ and a y -intercept of $2/5$.
 - Write an equation for the line that crosses the y -axis at $(0,-3)$ and has a slope of $2/5$.

- 2.3** The following two equations were rewritten in slope-intercept form by solving for y in terms of x .

$$\begin{array}{rcl}
 y - 5 = 7x & & y + 5x = 7x \\
 y - 5 + 5 = 7x + 5 & & y + 5x + (-5x) = 7x + (-5x) \\
 y = 7x + 5 & & y = 7x + (-5x) \\
 & & y = 2x
 \end{array}$$

Use similar methods to solve each of the following equations for y .

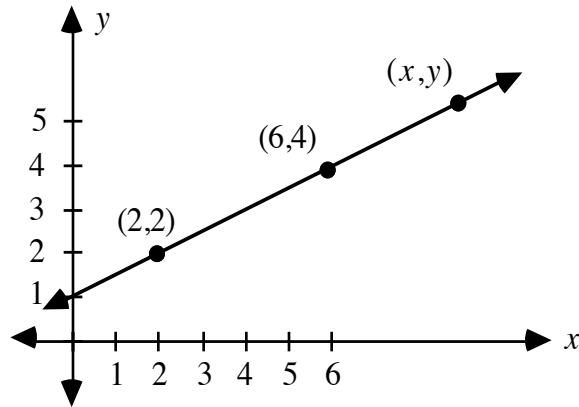
- $y + 3 = 2x$
 - $y - 5 = 3x + 2$
 - $y + 6x = 2x - 7$
 - $3x + 4y = 7$
 - $2x - 3y = 6$
- 2.4** The following two equations were rewritten in slope-intercept form by multiplying both sides of each equation by the same quantity.

$$\begin{array}{rcl}
 \frac{y}{3} = 4x & & 4y = 16x \\
 3 \cdot \frac{y}{3} = 3 \cdot 4x & & \frac{1}{4} \cdot 4y = \frac{1}{4} \cdot 16x \\
 y = 12x & & y = 4x
 \end{array}$$

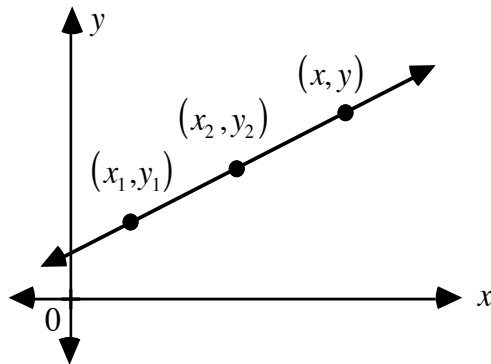
Use a similar method to solve each of the following equations for y .

- $\frac{y}{-5} = 2x$
- $7y = 4x$
- $-2y = 8x$

- 2.5** As shown in the graph below, the points with coordinates $(2,2)$, $(6,4)$, and (x,y) are on the same line. In this case, the coordinates (x,y) represent any point on the line.



- Calculate the slope of the line using the points $(2,2)$ and $(6,4)$.
 - Calculate the slope of the line using the points $(2,2)$ and (x,y) .
 - Write a mathematical equation that describes the relationship between the two slopes calculated in Parts **a** and **b**.
- 2.6** As shown in the following graph, the points with coordinates (x,y) , (x_1,y_1) , and (x_2,y_2) are on the same line. In this case, the coordinates (x,y) represent any point on the line.



- Write a representation for the slope of the line using the points (x_1,y_1) , and (x_2,y_2) .
- Write a representation for the slope of the line using the points (x,y) and (x_1,y_1) .
- What relationship exists between the two representations you wrote in Parts **a** and **b**?
- Write a mathematical equation that describes this relationship.

Mathematics Note

The equation of a line that passes through the point (x_1, y_1) and has a slope of m can be written in the form: $y - y_1 = m(x - x_1)$. This is the **point-slope form** of the equation of a line.

For example, the point-slope equation of a line that passes through the point $(2, -4)$ and has slope of -7 is $y - (-4) = -7(x - 2)$.

- 2.7** **a.** Write an equation in point-slope form for each of the following lines.
1. The line that passes through $(5, 8)$ and has a slope of -3 .
 2. The line that passes through $(2, 10)$ and $(12, 5)$.
 3. The line that passes through the origin and has a slope of $2/3$.
 4. The line that passes through $(-6, 4)$ and $(-2, 5)$.
- b.** Rewrite each equation from Part **a** in slope-intercept form.

* * * * *

- 2.8** Some equations do not represent lines. Using appropriate technology, graph the following equations and identify which ones are linear.

- a. $y = 2x + 3$
- b. $y = x^3 + 2$
- c. $y = 3x^2$
- d. $y = \frac{4}{x}$
- e. $y = -\frac{1}{3}x - 2$
- f. $y = \sqrt{x}$
- g. $y = 5x - 2$

- 2.9** Ordered pairs do not always involve integer values. Even when ordered pairs contain decimal values, however, the slope and equation of a line can still be found using the methods described in this activity.

- a. Find the slope of the line through each of the following pairs of points:
 1. $(6.4, 8.2)$ and $(1.9, 7.5)$
 2. $(9.00, 0.12)$ and $(5.00, -0.46)$
- b. Write equations in point-slope form for the lines in Part **a**.
- c. Rewrite each equation from Part **b** in slope-intercept form.

* * * * *

Activity 3

In the previous activities, you examined nutritional information using a number of different methods, including graphs.

Exploration

After breakfast, Maurice and Janet enjoy a leisurely ride on their bicycles. As Janet rides, the number of kilocalories remaining from her meal can be described by the equation $y = -6x + 500$, where x represents time in minutes.

Since Maurice is heavier than Janet, he uses more energy while cycling. The equation that describes the number of kilocalories remaining from his breakfast is $y = -9x + 710$.

- Describe what the slope and y -intercept of each equation represents in this situation.
- Graph both equations on the same set of axes.
- Estimate the coordinates of the point common to both lines.
- Describe what the coordinates of the common point represent in terms of kilocalories and time.

Mathematics Note

Two lines with different slopes that lie in the same plane always intersect in a single point. The coordinates of the point of intersection satisfy the equations of both lines.

For example, Figure 5 shows the graphs of $y = 3x$ and $y = 0.5x$. The two lines intersect at $(0,0)$. Since $0 = 3(0)$ and $0 = 0.5(0)$, the coordinates of the point of intersection satisfy both equations.

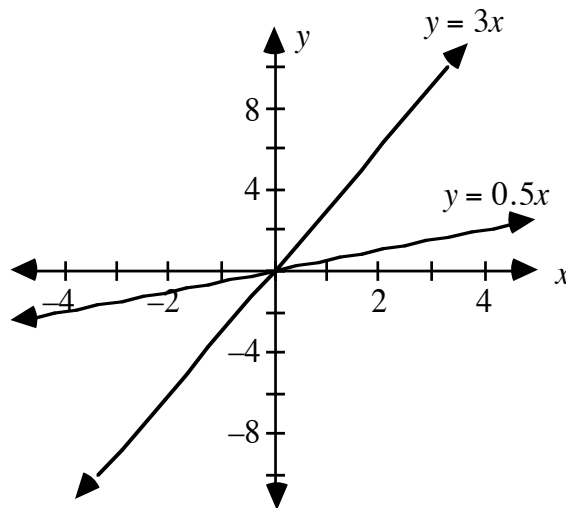


Figure 5: Graphs of $y = 3x$ and $y = 0.5x$

Discussion

- a. What information do the two equations in the exploration give about Janet and Maurice?
- b. What would have to be true about these two equations if their graphs did not intersect?
- c. Describe how to determine when Maurice and Janet had the same number of kilocalories remaining from breakfast.
- d. What advantages are there to writing the equation of a line in slope-intercept form?

Mathematics Note

The **distributive property** of multiplication over addition is a key connection between the two operations. It can be described as follows:

$$a(b + c) = ab + ac \text{ or } (b + c)a = ba + ca$$

For example,

$$\begin{array}{ccc} 2(3 + 4) = 2 \cdot 3 + 2 \cdot 4 & \text{or} & (3 + 4)2 = 3 \cdot 2 + 4 \cdot 2 \\ = 14 & & = 14 \end{array}$$

- e. The following two equations were rewritten in slope-intercept form using the distributive property:

$$\begin{array}{ll} y = 2(x + 3) & y = -5(x - 3) \\ y = 2 \cdot x + 2 \cdot 3 & y = -5 \cdot x - (-5) \cdot 3 \\ y = 2x + 6 & y = -5x - (-15) \\ & y = -5x + 15 \end{array}$$

Use the distributive property to write each of the following equations in slope-intercept form:

1. $y = 5(x - 4)$
 2. $y = -3(x + 2)$
 3. $y = -7(x - 5)$
- f. Using the distributive property, describe how to change an equation in point-slope form, $y - y_1 = m(x - x_1)$, to an equation in slope-intercept form, $y = mx + b$.

Assignment

- 3.1** It may require several steps to change some equations to slope-intercept form. For example, consider the following two equations:

$$\begin{array}{l} y - 3 = 2(x - 5) \\ y - 3 = 2x - 10 \\ y = 2x - 10 + 3 \\ y = 2x - 7 \end{array} \qquad \begin{array}{l} \frac{y}{3} = x - 1 \\ 3\left(\frac{y}{3}\right) = 3(x - 1) \\ y = 3x - 3 \end{array}$$

Use similar methods to rewrite each of the following equations in slope-intercept form.

- a. $y - 5 = 2(x - 4)$
 - b. $y + 2 = 2x + 11$
 - c. $y + 7 = 2(8x - 3)$
 - d. $\frac{y}{5} = x + 2$
 - e. $2y = 6x + 4$
 - f. $\frac{y}{3} = 2x - 6$
- 3.2**
- a. The equation $y - 700 = -16(x + 5)$, where x represents time in minutes, describes the number of kilocalories remaining from Ricardo's breakfast during a cross-country race. Write this equation in slope-intercept form and sketch its graph. What does the slope represent in this situation?
 - b. During the Olympic decathlon competition, Perry burned kilocalories at a rate described by the equation $y - 94 = 2(x + 3)$, where x represents time in minutes. Write this equation in slope-intercept form and sketch its graph. What does the y -intercept represent in this situation?
 - c. Kelly is spending a rainy summer afternoon reading a mystery novel. While reading, she uses kilocalories at a rate described by the equation $3y - 150 = 5(x - 12)$, where x represents time in minutes. What can you tell about Kelly's energy usage from this equation?

- 3.3** In Olympic competition, wrestlers must compete in specific classes according to their mass in kilograms. Two athletes, Norjar and Ruiz, plan to wrestle in the same class. On June 12, Norjar had a mass of 78 kg. By June 16, his mass had decreased to 77 kg. Ruiz's mass on June 14 was 62 kg. By June 18, it had risen to 64 kg.
- Write the dates and masses for Norjar as two ordered pairs. What is the slope of the line containing these points?
 - Write an equation for the line describing Norjar's change in mass in slope-intercept form.
 - Determine an equation that describes Ruiz's change in mass.
 - Graph the two equations from Parts **b** and **c** on the same set of axes.
 - What does the point common to both lines represent in this situation?
 - What are the coordinates of this common point?
 - If the mass of each wrestler continues to change at its previous rate, on what date will their masses be equal?
 - What is the value of their masses on the date the masses are equal?

- 3.4**
- Determine the slope of the line that passes through the points (4,8) and (6,14).
 - Use the point (4,8) and the slope from Part **a** to write an equation of the line in point-slope form.
 - Use the point (6,14) and the slope from Part **a** to write an equation of the line in point-slope form.
 - Are the two equations you wrote in Parts **b** and **c** equivalent? Justify your response.

- 3.5** Imagine that you are the project director for the next space shuttle flight. For an experiment on energy usage in space, mission specialists Kimberly and Manuel must have the same mass on launch day. To reach this target mass, Manuel has increased his daily activities, while Kimberly has increased her consumption of calories. As project director, you have received the data in the table below.

Astronaut	Day 5	Day 10	Day 15	Day 20
Kimberly	no data	63.0 kg	64.5 kg	66.0 kg
Manuel	74.0 kg	no data	72.0 kg	71.0 kg

If the mass of both astronauts continues to change at the rate for the previous 16 days, on what day can the launch proceed?

- 3.6** Four other astronauts also have been training for the launch described in Problem **3.5**. They are willing to change their caloric intake as necessary to substitute for Manuel or Kimberly. As project director, you must select one of them as an alternate. The table below shows the mass of each astronaut at some time during the previous 16 days.

Astronaut	Mass
Britte	69.0 kg on day 5
Kwasi	72.5 kg on day 15
Sergei	78.0 kg on day 20
Yukawa	62.5 kg on day 10

- a.** Using the mass of Manuel and Kimberly and the launch day found in Problem **3.5**, write an equation for each potential alternate that describes the change in mass required to meet the mission restrictions.
- b.** As project director, which astronaut would you select to serve as an alternate for this launch? Explain your decision.
- 3.7** Rolf and Tanya meet every morning to exercise. While warming up, Tanya uses 100 kcal of energy. She then burns 4.4 kcal/min during her walk. Rolf uses 60 kcal of energy during his warm-up and burns 15 kcal/min while running.
- a.** The equation $y = 4.4x + 100$, where x represents time in minutes, describes Tanya's energy usage in kilocalories. Write an equation that describes Rolf's energy usage in kilocalories.
- b.** Graph both equations in Part **a** on the same set of axes. Estimate the coordinates of the point of intersection and describe what these coordinates represent in this situation.
- 3.8** **a.** Find the coordinates of the point of intersection for each of the following pairs of equations.
1. $y = 4x - 2$, $y = 2x - 6$
 2. $y = 3(x + 5)$, $y = x - 9$
 3. $y = \frac{1}{2}(x + 4)$, $y = 2(x - 3)$
 4. $(y - 2) = 2(x - 4)$, $(y - 5) = (x - 6)$
- b.** Check each solution by verifying that the coordinates satisfy both equations.

* * * * *

- 3.9** Write an equation in slope-intercept form for the line that passes through the given point with the given slope.
- $(-4, 3)$, $m = -1$
 - $(3, -1)$, $m = 0$
 - $(-4, -2)$, $m = 1/2$
- 3.10** Denali is climbing a cliff 300 m high. After 30 min, she has moved 75 m up the cliff.
- Assuming that Denali continues to climb at a constant rate, write an equation that describes her distance from the bottom in meters in terms of time in hours.
 - Identify the slope of the equation in Part a and describe what it represents in this situation.
 - How long will it take Denali to reach the top of the cliff?
- 3.11** Find the coordinates of the point common to each pair of lines below.
- $$\begin{cases} y = x - 3 \\ 4x + y = 32 \end{cases}$$
 - $$\begin{cases} -3x + 6y = 4 \\ 2x + y = 4 \end{cases}$$
 - $$\begin{cases} 3x + y = 13 \\ 2x - 4y = 18 \end{cases}$$

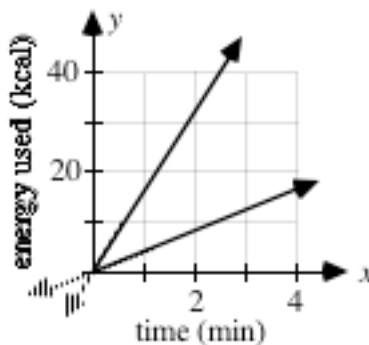
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Summary Assessment

Rick has a mass of 66 kg. Before sitting down at his desk, he always eats breakfast and completes a morning workout. After eating a large breakfast, Rick exercises by running at the fast rate of 3.7 min/km. After eating a moderate breakfast, he exercises by bicycling at a slow rate.

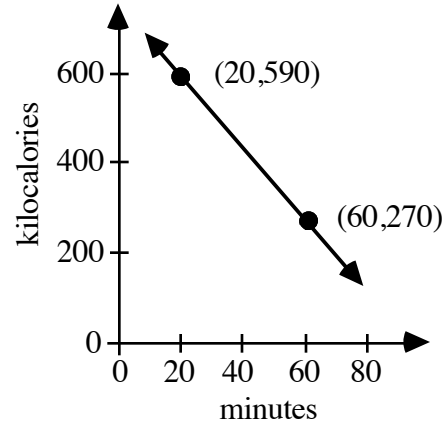
For Rick, a large breakfast consists of hotcakes with butter, an omelet, a cup of 2% milk, and a glass of orange juice. This provides a total of 1,022 kcal. A moderate breakfast consists of an omelet, a cup of 2% milk, and a glass of orange juice, for a total of 522 kcal.

1. On a single set of axes, create a graph of each of the following:
 - a. the kilocalories remaining from the large breakfast while Rick runs at the fast rate of 3.7 min/km
 - b. the kilocalories remaining from the moderate breakfast while Rick bicycles at a slow rate
2.
 - a. Estimate the coordinates of the intersection of the two lines in Problem 1.
 - b. Describe what these coordinates represent in this situation.
3. After a certain number of minutes of running or biking, Rick will have the same number of kilocalories remaining from either breakfast. Determine how long Rick can perform his job as a writer using these remaining kilocalories.
4. The graph below shows Rick's energy usage in kilocalories while running fast and cycling slowly.



- a. Use the concepts of slope and rate to explain which line represents energy used while cycling and which line represents energy used while running.

- b. Describe how the graph can be used to approximate the slope of each line.
- c. Identify the domain and range of the relations that describe Rick's energy usage during both activities.
5. The following graph shows the number of kilocalories remaining after an entirely different breakfast and exercise routine for Rick.



- a. Write an equation in point-slope form that describes the kilocalories remaining from breakfast as Rick exercises.
- b. Rewrite the equation in Part a so that it shows the number of kilocalories in Rick's breakfast.
- c. Identify an exercise that would result in this rate of energy use for Rick.

Module Summary

- A **calorie** is the amount of energy required to raise the temperature of 1 mL of water 1°C.
- A **kilocalorie (kcal)**, 1000 cal, is the amount of energy needed to raise the temperature of 1 L of water 1°C.
- A **dietary calorie** (normally referred to as a **Calorie** with a capital C) is equal to 1 kcal. The calorie-per-gram rating on most food labels measures dietary calories.
- To solve an equation of the form $y = mx$ for x , divide each side of the equation by m .
- An equation of the form $y = mx$ represents a line that passes through the origin where m is the **slope** of the line. The slope is the ratio of the change in vertical distance to the change in horizontal distance between any two points on the line.
- The slope of a line containing two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $y_2 - y_1$ is the change in the vertical distance and $x_2 - x_1$ is the change in the horizontal distance. If $x_1 = x_2$, the line is vertical and has no slope.

- A **relation** between two variables is a set of ordered pairs of the form (x, y) .
The **domain** of a relation is the set of first elements in the ordered pairs (the x -values). The **range** of a relation is the set of second elements in the ordered pairs (the y -values).
- A **function** is a relation in which each element of the domain is paired with an element of the range and each element of the domain occurs in only one ordered pair. A function may be described by a rule or equation.
- A horizontal line has a slope of 0 and an equation of the form $y = c$. All the points on a horizontal line have the same y -coordinate, c .
- A vertical line has no slope and an equation of the form $x = c$. All the points on a vertical line have the same x -coordinate, c .
- The y -coordinate of the point where a line intersects the y -axis is known as the **y -intercept**.
- The equation of a line with slope m and y -intercept b can be written in the form $y = mx + b$. This is the **slope-intercept** form of the equation of a line.

- Two lines that have equal slopes are **parallel**. Although vertical lines have no slope, any two vertical lines are also parallel.
- The equation of a line that passes through the point (x_1, y_1) and has a slope of m can be written in the form: $y - y_1 = m(x - x_1)$. This is the **point-slope form** of the equation of a line.
- Two lines with different slopes that lie in the same plane always intersect in a single point. The coordinates of the point of intersection satisfy the equations of both lines.
- The **distributive property** of multiplication over addition is a key connection between the two operations. It can be described as follows:

$$a(b + c) = ab + ac \text{ or } (b + c)a = ba + ca$$

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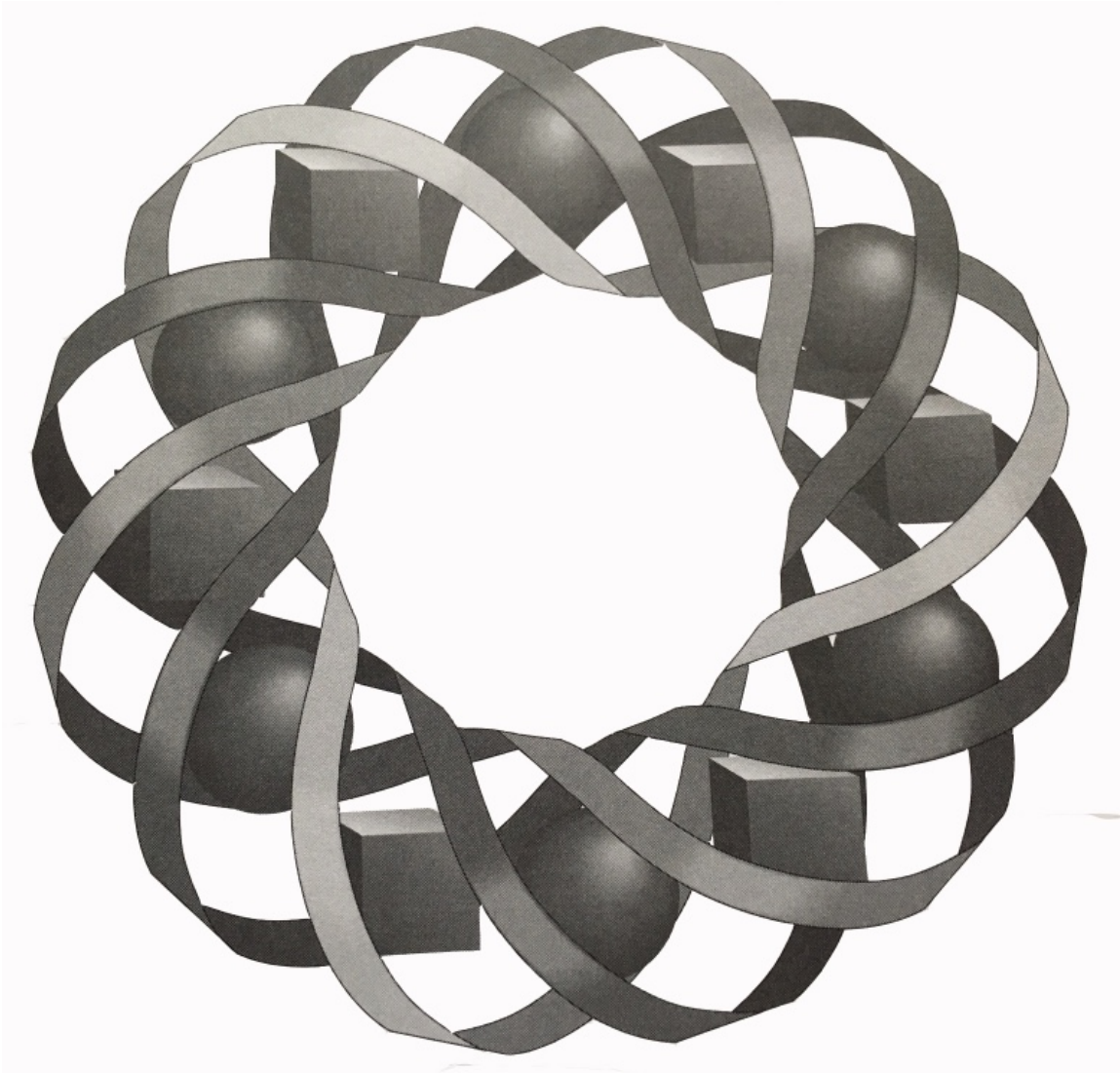
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A New Look at Boxing



What do a box of cereal and a take-out pizza have in common? What are templates, tilings, and tessellations? And how can you catch an apothem without a net? This module answers all of these questions—and more.

Bill Chalgren • Darlene Pugh • Teri Willard • Lisa Wood



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A New Look at Boxing

Introduction

Gloria watches the clock: tick, tick, tick. Time always seems to drag when she's waiting for lunch. Finally, the bell rings. It's pizza time!

Friends, pop, breadsticks, and pizzas make Little Cheesers the most popular lunch-time hangout at Gloria's school. There's only one problem: time flies when she's having fun, and lunch is only 30 minutes long. As she heads for the door, Gloria asks for a box. She still has one slice of pizza left. On the way back to school, Gloria finishes her meal. Without a second thought, she tosses the box into the trash.

The cashier used a full-size box for Gloria's leftovers. A box big enough to hold a whole pizza may not be the best container for a single slice. If the box had been the right shape and size, less material would have been wasted. In this module, you will look at some different properties of boxes and explore how much material it takes to make them.

Mathematics Note

A **prism** is a three-dimensional figure determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.

For example, Figure **1a** shows a cereal box that, like many pizza boxes, is a prism with rectangular faces. Figure **1b** shows a box for a candy bar in the shape of a triangular prism.

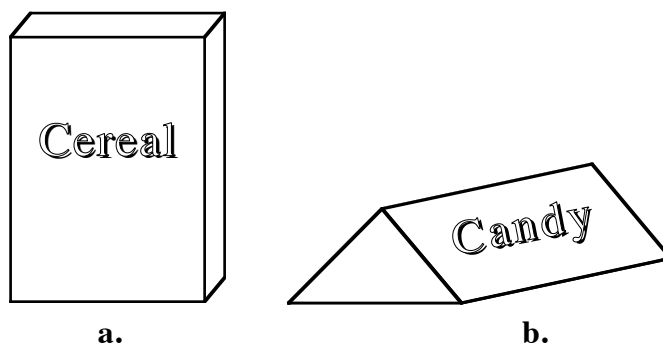


Figure 1: Two boxes

Discussion

- Explain why a cereal box is a prism.
- Which faces of the prism in Figure **1b** are the bases?
- Describe several examples of other objects that are prisms.
- Is a tennis-ball container a prism? Explain your response.

Activity 1

The cardboard for the cereal box in Figure 1 was cut from a pattern. What does this pattern look like? In this activity, you examine a box and the pattern used to create it.

Exploration

Use a box similar to the cereal box in Figure 1 to complete the following steps.

- Estimate the total area, in square centimeters, of all the sides of the box.
- On a sheet of grid paper, make a scale drawing of the pattern you think the manufacturer used to create the box. Use dotted line segments to indicate folds. Cut out your paper pattern and fold it into a box.
- Without tearing the cardboard, take the box apart and lay it flat. The box probably was constructed using glued tabs.

A pattern with tabs is a **template**. A pattern without tabs is a **net**. Figure 2 shows both a net and a template for a cube.

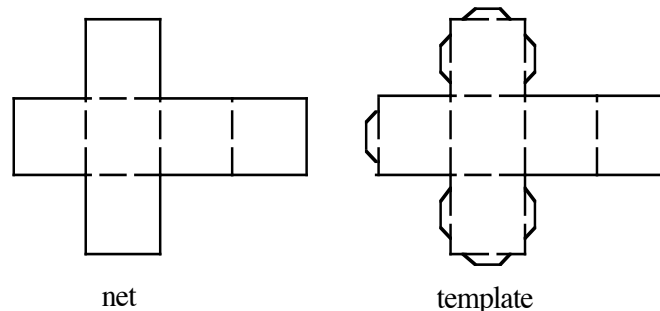


Figure 2: Cube net and template

- Find the area of the net for the box. Compare this area to the estimate you made in Part a.
- Imagine that the template for the box was cut from a rectangle of cardboard. Record the dimensions of the smallest rectangle that will enclose the template.

Discussion

- Does your folded box from Part b of the exploration resemble the original box?
- How does your paper pattern from Part b of the exploration compare to the template for the box?
- Without making any calculations, compare the area of the net to the area of the template.

- d. Why might a box manufacturer be interested in the smallest rectangle that will enclose a template?
- e. How do the areas of the net and the template compare to the actual surface area of the box?

Mathematics Note

The **surface area** of a prism is the sum of the areas of its bases and lateral faces.

For example, the surface area of the cube in Figure 3 below is $9 + 9 + 9 + 9 + 9 + 9 = 54 \text{ cm}^2$. The area of the net in Figure 3 is also 54 cm^2 .

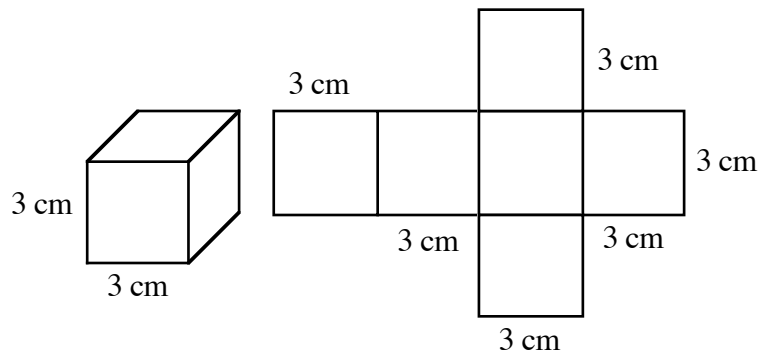
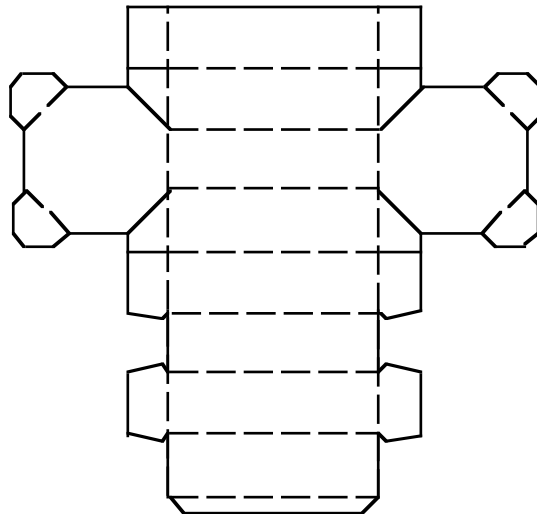


Figure 3: A cube and its net

- f. Are all containers prisms? Use examples to support your answer.

Assignment

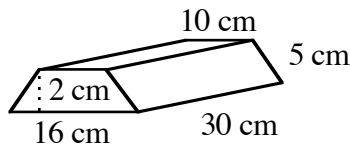
- 1.1 The template shown below can be used to create an octagonal prism.
 - a. Draw the corresponding net for the prism.
 - b. Make a sketch of the three-dimensional figure.



- 1.2** In Part e of the exploration, you recorded the dimensions of the smallest rectangle that would enclose the template for the box.
- Find the area of this rectangle in square centimeters (cm^2).
 - Find the area of the template.
 - When the template is cut from the rectangle in Part a, what percentage of the cardboard is wasted?
 - Imagine that a box manufacturer makes 500,000 boxes by cutting templates from cardboard rectangles. If cardboard costs 14 cents per square meter, what is the cost of the wasted cardboard? Describe any assumptions you make in solving this problem.
- 1.3**
- Make a sketch of a container that is not a prism but has faces that are polygons.
 - Draw a possible template for this container.
 - Explain why this container is not a prism.
- 1.4** A sugar company would like to sell packages of 100 sugar cubes. Each cube is approximately 1 cm on a side.
- Draw a net for a container which could be used to package 100 sugar cubes.
 - Explain how you could position several copies of your net so that the containers could be produced efficiently.

* * * * *

- 1.5** You are in charge of repainting the red background on all the stop signs in a large city. Before you can order paint, you must determine the area of a stop sign. Draw a regular octagon and describe how to find its area.
- 1.6** Gold bullion is often molded into blocks. As shown in the diagram below, the cross section of one type of block is an isosceles trapezoid. The trapezoid has a height of 2 cm and bases of 10 and 16 cm.



- Draw a net that could be used to create a model of the block.
- Use the net to find the surface area of the block.

* * * * *

Activity 2

When trying to minimize waste in box construction, you must consider more than just the shape that encloses the template. How those shapes fit together is also important. For example, a box template may be enclosed by a rectangle. To improve efficiency, a manufacturer might place these rectangles edge to edge on a large sheet of cardboard and cut many templates at once.

Figure 4 shows one way in which smaller rectangles can be arranged on a larger rectangle of cardboard. To cover an even larger sheet without gaps or overlaps, more of the smaller rectangles could be added.

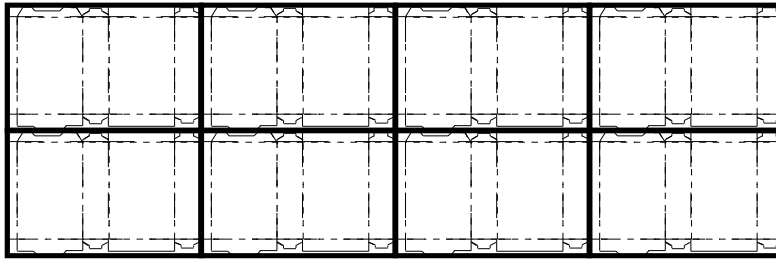


Figure 4: Cereal box templates

Mathematics Note

When a shape is repeated to form a pattern that covers an entire plane without gaps or overlaps, it **tessellates** or **tiles** the plane. The pattern that covers the plane is a **tessellation** or **tiling**.

For example, Figure 5 shows two tessellations.

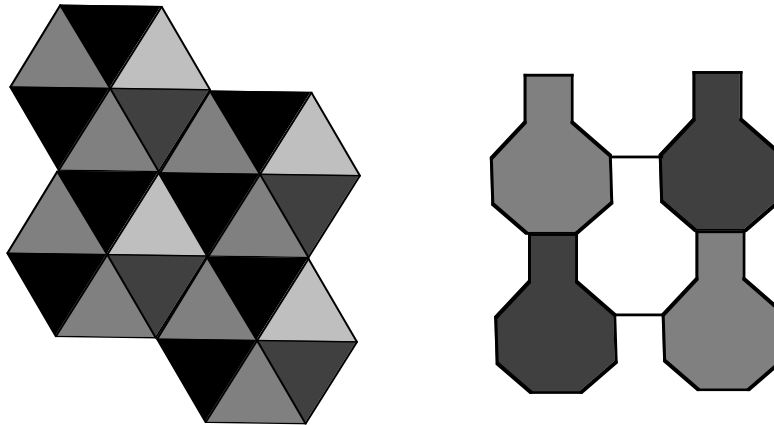


Figure 5: Two tessellations

Exploration

Installing floor tiles is similar to covering a plane with regular polygons. Since squares fit together easily without gaps or overlaps, many floor tiles are square. However, it is also possible to tile a floor with other polygons.

- Cut out templates of all the regular polygons, other than squares, with up to 12 sides. Determine which of these regular polygons can tile a plane. Use drawings to record your results.
- On a sheet of paper, extend the sides of an equilateral triangle to form three equal exterior angles. In Figure 6 below, $\angle 1$, $\angle 2$, and $\angle 3$ are the exterior angles and $\angle 4$, $\angle 5$, and $\angle 6$ are the interior angles.

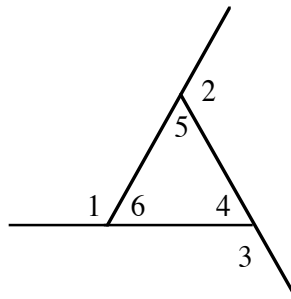


Figure 6: An equilateral triangle with exterior angles

- Without measuring, determine the sum of the measures of the exterior angles. Hint: Start at the vertex of $\angle 1$ as shown in Figure 6 and visualize walking around the polygon (triangle) until you return to the starting point facing in the same direction. How many degrees did you turn during the walk?
- Record the measure of a single exterior angle in a spreadsheet with headings like those in Table 1 below.
- Use your response to Step 2 to determine the measure of an interior angle of a regular polygon and record the result in Table 1.
- Determine the number of these regular polygons that would “fit” at one vertex and record the result in Table 1.

Table 1: Measures of angles of regular polygons

No. of Sides in Polygon	Measure of Exterior Angle (x)	Measure of Interior Angle (m)	No. of Polygons that “Fit” at One Vertex ($360^\circ/m$)
3			
4			
\vdots			

- Repeat Part b for all regular polygons with up to 12 sides.

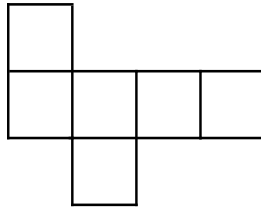
Discussion

- a. How can you use Table 1 to determine which regular polygons tessellate a plane?
- b. Identify the regular polygons that tessellate a plane. Explain your answer.
- c. Will the method of finding the sum of the measures of the exterior angles described in Part b of the exploration work for any regular polygon? Explain your response.
- d. What is the sum of the measures of the exterior angles of any regular polygon?
- e. What is the measure of an exterior angle of a regular polygon with n sides?
- f. What is the measure of an interior angle of a regular polygon with n sides?
- g. If you knew the number of degrees in an interior angle of a regular polygon, how could you determine the number of sides in the polygon?

Assignment

- 2.1 Consider a regular polygon with 24 sides.
 - a. Find the measure of an interior angle of the polygon.
 - b. Find the sum of the measures of the interior angles of the polygon.
- 2.2 Find the measure of an interior angle of a regular polygon with 102 sides.
- 2.3 Is there a regular polygon with more than six sides that will tile a plane? Explain your response.
- 2.4 Describe one of your tilings from Part a of the exploration so that someone unfamiliar with regular polygons and tessellations could reconstruct the pattern.
- 2.5 A box manufacturer wishes to cut templates from a roll of cardboard 400 cm wide and 6000 m long. The dimensions of the smallest rectangle that will enclose the template are 22.4 cm by 35.5 cm. How many templates can be cut from one roll?

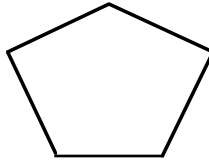
- 2.6** a. The net shown below is just one of several possible nets of a cube. Draw three different cube nets.



- b. On a sheet of grid paper, find a cube net that tessellates a plane. Use at least six copies of the net to show that the pattern may be extended in all directions.

* * * * *

- 2.7** Although a regular pentagon will not tile the plane, there are pentagons that do form tessellations. The pentagon in the following diagram is equilateral but not equiangular.



- a. Trace this pentagon and determine whether it will tessellate the plane.
- b. Design a pentagon different from the one above that tessellates the plane.
- 2.8** Do you think that all quadrilaterals will tessellate a plane? Explain your response.
- 2.9** a. Find the measures of an interior angle and an exterior angle of a 16-sided regular polygon.
- b. Calculate the sum of the interior angles and the sum of the exterior angles of a 16-sided regular polygon.
- c. If the measure of an interior angle of a regular polygon is 170° , how many sides does the polygon contain?

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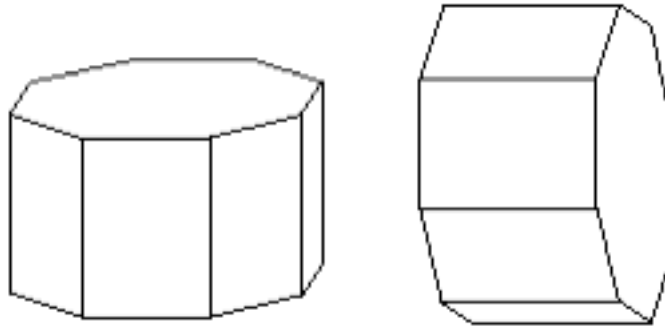
Research Project

Regular polygons can be arranged in many interesting patterns.

- a. Find at least six ways in which a combination of regular polygons will tessellate a plane. In each tiling, use at least two different regular polygons.
 - b. Make a careful drawing of each of your patterns.
 - c. Describe each pattern in a few sentences.
 - d. Find at least two examples of tilings that use polygons other than squares and rectangles in your home or community.
-

Activity 3

Not all prisms have rectangular bases. Some have bases that are triangular. Others, such as the octagonal prisms shown in Figure 7, have bases that are regular polygons.



In this activity, you use your knowledge of the area of triangles and squares to develop a method for finding the area of regular polygons with five or more sides.

Exploration

- a. Use a geometry utility to construct a regular pentagon by completing the following steps.
 1. Construct a circle. Place five points on the circle. Use segments to connect the center of the circle to each of the five points. Each angle formed by two adjacent radii is a **central angle**.
 2. Drag the points on the circle until the measures of all the central angles are equal.
 3. Connect the points on the circle to form a regular pentagon.

- b. Create a table with headings like those in Table 2 below.

Table 2: Triangles in regular polygons

Polygon	No. of Triangles	Apothem (a)	Length of Side (s)	Area of Polygon
pentagon	5			
heptagon				
decagon				
n -gon				

- c. Measure the perpendicular distance from the center of the polygon to one side. This distance is the length of the **apothem**. In Figure 7, for example, \overline{AG} is the apothem. Record this measure in the appropriate column in Table 2.

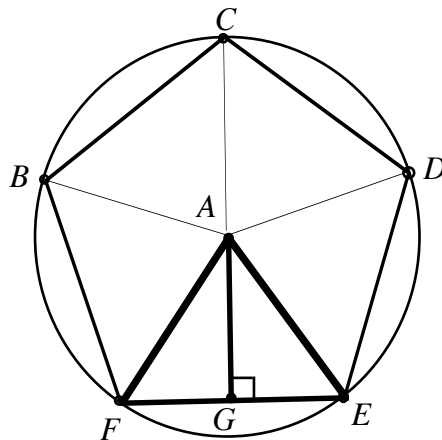


Figure 7: Constructing a regular pentagon

- d. Measure the length of one side of the polygon. Record this length in the appropriate column of Table 2.
- e.
1. Your construction of a polygon contains congruent triangles. Create a formula using the length of the apothem to find the area of one of these triangles.
 2. Use the area of one congruent triangle to find the total area of the polygon. Record this area in the appropriate column of Table 2.
- f. Use the geometry utility to find the area of the polygon. Compare this value to the one you determined in Part e.
- g. Repeat Parts a–f for a regular heptagon and a regular decagon.
- h. Create a formula for finding the area of a regular n -gon. Enter it in the appropriate cell of Table 2.

Discussion

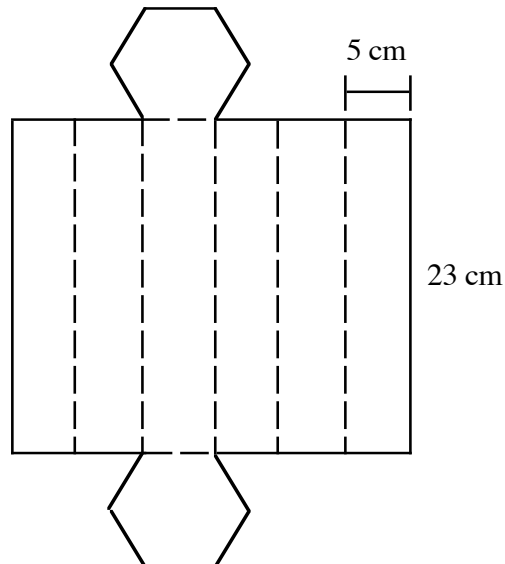
- a. As the number of sides of a polygon increases, what happens to the shape of the polygon?
- b. How does the measure of the central angle of a polygon affect the shape of the polygon?
- c. How do the areas of the polygons found using your formula compare to the areas of the same polygons found using the geometry utility?
- d. The area of a regular polygon with n sides, apothem a , and side length s can be described by the following equation:

$$\text{Area} = \left(\frac{1}{2}as\right)n$$

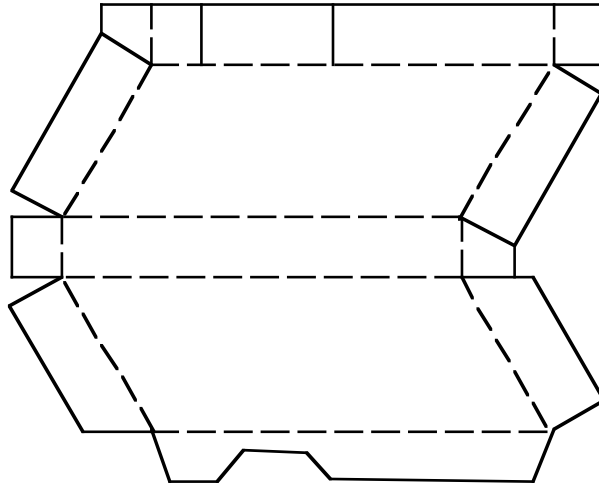
Is this equation equivalent to the formula you developed in Part **h** of the exploration? Explain your response.

Assignment

- 3.1 The floor of Greg's new hot tub is shaped like a regular hexagon. He wants to install a tile floor. What measurements should he make?
- 3.2 Write a formula for finding the area of a regular polygon in terms of the apothem and the perimeter.
- 3.3 Use the net below to find the surface area of the corresponding hexagonal box.



- 3.4** A product engineer has created the template below for a cardboard box.



To hold down production costs, you must design a shape that encloses the template and minimizes waste. Prepare a presentation for the product engineer that includes:

- a. the area of the template and the area of the shape
- b. a sketch of the tessellation of the shape
- c. the percentage of cardboard wasted when the template is cut from the shape.

* * * * *

- 3.5** Find a box that is not a rectangular prism. Carefully unfold the box and lay its template flat.

- a. Draw the smallest shape that encloses the template and also tessellates a plane.
- b. Sketch the tessellation of the shape.
- c. Find the area of the shape from Part a.
- d. Calculate the percentage of cardboard wasted when the template is cut from the shape.

- 3.6** A soccer ball can be modeled with 12 regular pentagons and 20 regular hexagons, each with a side length of approximately 4.5 cm. Find the approximate surface area of a soccer ball. Hint: You may need to draw a sample hexagon and pentagon and measure their apothems.

* * * * *

Summary Assessment

You have been asked to design and build an efficient box for one slice of pizza. The specifications and costs are listed below.

- A whole pizza has a diameter of 30 cm. Each pizza is cut into 8 equal slices.
- Cardboard costs 22 cents per square meter.
- The template must assemble into a closed container.
- The shape enclosing the template should both tessellate the plane and minimize waste.

Prepare a report describing your container, including the following:

1. a template
2. the area of the template
3. the shape that encloses the template and tessellates a plane
4. a sketch of the tessellation
5. the cost to make one template
6. the percentage of cardboard wasted in making one template
7. an attractive advertising logo sketched on a template
8. a model of the container.

Module Summary

- A **prism** is a solid determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.
- A **template** is a two-dimensional pattern with tabs that can be folded to make a three-dimensional solid.
- A **net** is a two-dimensional pattern without tabs that can be folded to make a three-dimensional solid.
- The **surface area** of a prism is the sum of the areas of its bases and lateral faces.
- When a shape is repeated to form a pattern that covers an entire plane without gaps or overlaps, it **tessellates** or **tiles** the plane. The pattern that covers the plane is a **tessellation** or **tiling**.
- The measure of an exterior angle of a regular polygon with n sides is $360^\circ/n$.
- The measure of an interior angle of a regular polygon with n sides is:
$$180^\circ - \frac{360^\circ}{n}$$
- The **apothem** is the segment whose measure is the perpendicular distance from the center of a regular polygon to one of its sides.
- The area of a regular polygon with n sides, apothem a , and side length s can be described by the following equation:

$$\text{Area} = \left(\frac{1}{2}as\right)n$$

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What Will We Do When the Well Runs Dry?



The availability of fresh, clean water affects us personally, locally, and globally. In this module, you'll use volume, rates of change, and linear models to assess individual water use.

Kyle Boyce • Karen Longhart • Mike Lundin • Karen Umbaugh



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What Will We Do When the Well Runs Dry?

Introduction

What is a drink of fresh, clean water worth to you? Imagine a time when citizens pay \$1.00 for a glass of water and \$10.00 for a bath. At those prices, public fountains will disappear and green lawns will become rare luxuries. If this sounds far-fetched, think about the millions of people who already buy bottled water. Although humans can only survive a few days without water, its availability is often taken for granted.

Discussion

- a. Do you think that water will become as expensive as the introduction suggests? Why or why not?
- b. Is there a water shortage in your area?
- c.
 1. What causes water shortages?
 2. Can these causes be avoided?
- d.
 1. Estimate the amount of water you use each day.
 2. How could you measure your daily water use?

Activity 1

How can you estimate the amount of water you use in one day? Before confronting any of the issues involved with water use and conservation, you must first understand volume.

Mathematics Note

A **prism** is a three-dimensional figure determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.

Prisms are named by the polygonal shape of the two bases. The **height** of a prism is the perpendicular distance between the bases.

For example, Figure 1 shows a trapezoidal prism and a triangular prism.

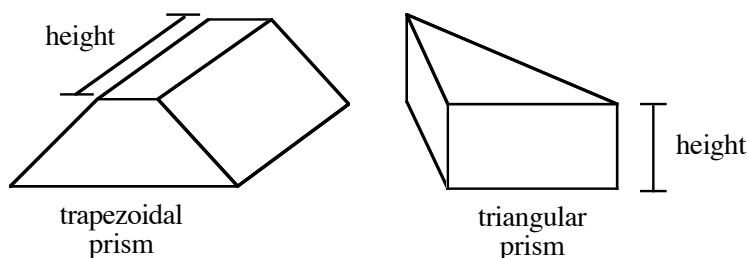


Figure 1: Two prisms

The amount of space occupied by an object is its **volume**. Volume is measured in units such as cubic centimeters (cm^3) or liters (L).

The volume (V) of a prism may be found by multiplying the area of the polygonal base (B) by the height (h): $V = Bh$.

For example, if the area of the base of a triangular prism is 120 cm^2 and its height is 30 cm, then its volume is $V = Bh = (120)(30) = 3600 \text{ cm}^3$.

Exploration 1

In this exploration, you determine the volume of a cube.

- a. On a sheet of cardboard, draw a net for a cube with $10 \text{ cm} \times 10 \text{ cm}$ faces.
- b. Cut out, fold, and tape the net to form a cube.
- c. Calculate the volume of the cube in each of the following units:
 1. cubic centimeters (cm^3)
 2. cubic decimeters (dm^3).
- d. Determine a relationship between cubic centimeters and cubic decimeters.
- e. Estimate the number of liters (L) of water which the cube will hold.
- f. Since it would not be practical to pour water into your cardboard cube, check your estimate by completing Steps 1–3 below.
 1. Open one face of the cube.
 2. Fill the cube with rice. **Note:** Make sure that the edges of the cube are securely taped. To prevent spills, you may wish to place the cube inside a bucket while pouring rice.
 3. Use a 1-L container to measure the amount of rice in the cube.
- g. Determine a relationship between liters and each of the following units:
 1. cubic centimeters
 2. cubic decimeters.

Discussion 1

- a.
 1. Is the cube you created in Exploration 1 a prism? Why or why not?
 2. How can you distinguish between the bases of a cube and its lateral faces?
- b.
 1. How many bases does a prism have?
 2. How many lateral faces does a prism have?
 3. How many lateral edges are there in a prism?
 4. What is the total number of edges in a prism?
- c. Describe the relationship between each of the following:
 1. a cubic centimeter and a cubic decimeter
 2. a liter and a cubic centimeter
 3. a liter and a cubic decimeter.

Exploration 2

In Exploration 1, you estimated then calculated the volume of a prism. In many communities, however, water supplies are stored in reservoirs that do not have polygonal bases. In this exploration, you examine a method for estimating the volume of objects that are not prisms.

- a. On a 10 cm \times 10 cm sheet of graph paper, draw a closed geometric figure that is not a polygon, such as the one shown in Figure 2 below.



Figure 2: A closed figure

- b. To estimate the area of the figure, complete Steps 1–4 below.
 1. Determine the area of each square on the graph paper.
 2. Count the number of whole squares in the figure.
 3. Count the number of partial squares in the figure and divide this number by 2.
 4. Find the sum of your answers to Steps 2 and 3. Multiply this sum by the area of one square determined in Step 1.

- c. Using bases shaped like your geometric figure from Part **a**, make a three-dimensional solid 10 cm high, such as the one shown in Figure 3. Tape the edges of the solid as securely as possible.

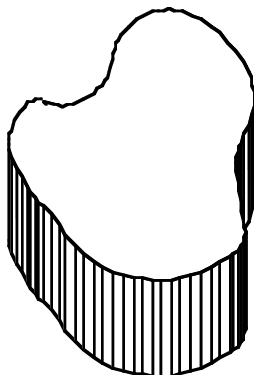


Figure 3: A three-dimensional solid

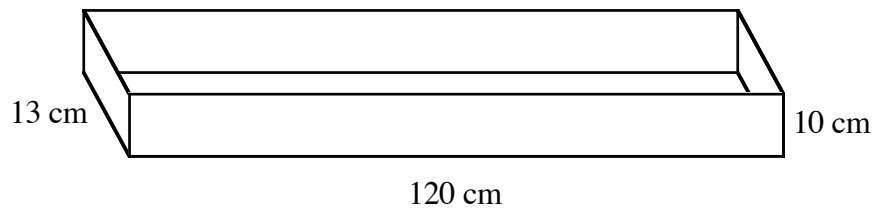
- d. Use the estimated area of the figure found in Part **b** to estimate the volume of the solid in each of the following units:
1. cubic centimeters
 2. liters.
- e. Check your estimate from Part **d** by opening your solid, filling it with rice, then measuring the amount of rice in the solid. **Note:** To prevent spills, you may wish to place the solid inside a bucket while pouring rice.

Discussion 2

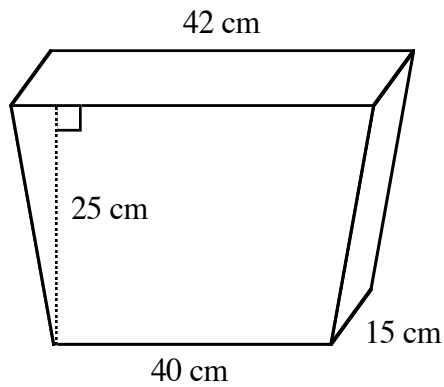
- a. In Part **b** of Exploration 2, you estimated the area of a figure by counting squares. What other methods could you use to determine the area of a figure that is not a polygon?
- b.
1. How did you determine the volume of the solid in Part **d** of Exploration 2?
 2. How did this value compare to the amount of rice that filled the solid?
- c. Describe how you could calculate the volume of a prism with bases shaped like each of the following:
1. triangles, rectangles, or trapezoids
 2. polygons other than triangles, rectangles, or trapezoids.

Assignment

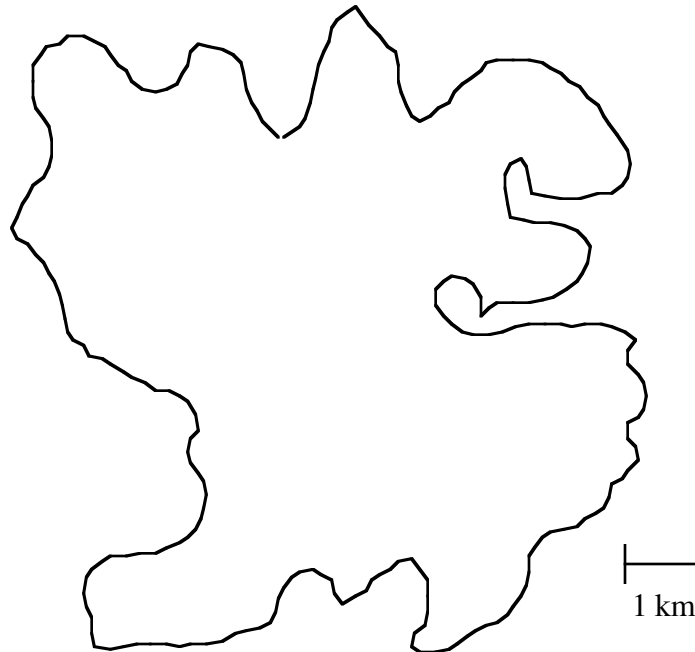
- 1.1** José has a bathtub that holds 250 L of water. Thaddeus has a tub that has inside measurements of $5 \text{ dm} \times 3 \text{ dm} \times 15 \text{ dm}$. Which tub holds more water? Justify your response.
- 1.2** A forced-air heating duct has dimensions $2 \text{ m} \times 6 \text{ dm} \times 50 \text{ cm}$.
- Make a scale drawing of this duct in centimeters.
 - Determine its volume in cubic centimeters.
- 1.3** Determine the volume of each of the following objects:
- a water trough



- a toilet tank



- 1.4** A Canadian town of 50,000 draws its water from the lake pictured below. The average depth of the lake is 15 m.



- a. Estimate the volume of water in the lake and describe how you determined your estimate.
- b. The average person in this town uses 380 L of water per day. If the lake is not replenished, how long will the water last?

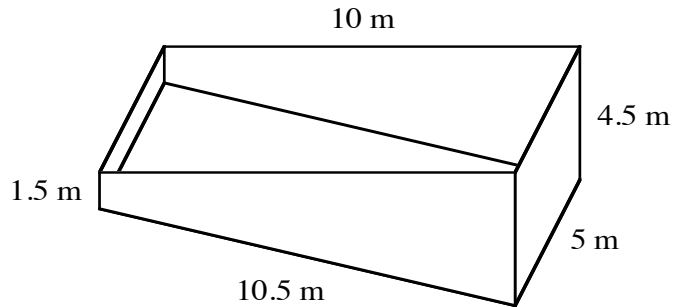
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- 1.5** The municipal reservoir for a U.S. city of 6 million people lies in a V-shaped valley. One end of the reservoir is dammed, while the other is faced by a steep rock wall. The body of water itself is 6 km long, 2 km wide, and has an average depth of 400 m.

- a. Make a scale drawing of this reservoir in kilometers.
- b. Determine its volume in liters.
- c. The average person in this city uses 420 L of water per day. Assuming that the reservoir is not refilled, how long can it supply the city with water?

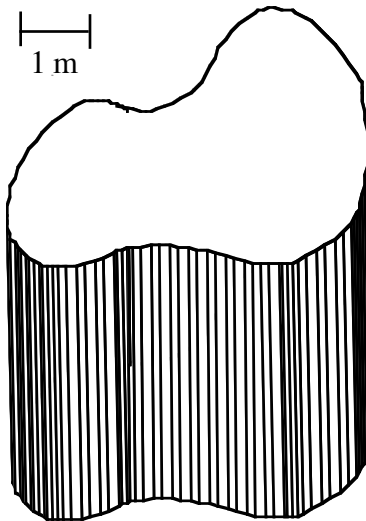
- 1.6** Although cubic centimeters are generally used to measure the volume of solid materials, health professionals also use these units to measure liquid volumes. In this setting, 1 cm^3 is referred to as 1 cc. What is the relationship between cubic centimeters and milliliters?

- 1.7 The following diagram shows the dimensions of a swimming pool with rectangular ends. How many liters of water does this pool hold?



- 1.8 Construction companies often measure volume using cubic yards. How many cubic yards of concrete are needed to pave a driveway with the following dimensions: 20 ft \times 15 ft \times 6 in?

- 1.9 The diagram below shows a small fish pond.



Determine the volume of the pond in each of the following units:

- a. cubic meters
- b. liters.

* * * * *

Activity 2

How much water does a leaky faucet waste? How long does it take to fill a bathtub? How long would it take a broken water main to flood a basement? The answers to these questions depend partly on the water's rate of flow. In this activity, you use graphs to relate the rate of flow to the slope of a line. This activity extends the notions of lines found in "Yesterday's Food is Walking and Talking Today" to model a rate of flow.

Exploration

In this exploration, you measure the rate at which water flows through a funnel.

Note: To prevent damage to papers, books, and electronic equipment, rice is used to simulate water.

- a. Use cardboard to make a sturdy funnel with a volume of at least 2 L and an opening at the narrow end approximately 2 cm in diameter.
- b.
 1. Hold the funnel over a bucket and place one hand under the narrow end to block the flow of rice. Pour 2.0 L of rice into the funnel.
 2. Simultaneously remove your hand from the narrow end of the funnel and start a timer. Determine the time (to the nearest 0.1 sec) required for the funnel to empty completely. Record the time in a table with headings like those in Table 1 below.

Table 1: Time data

Funnel Opening (cm)	Time for Trial 1	Time for Trial 2	Time for Trial 3
2			
3			
4			

- c. Repeat Part b for two more trials.
- d. To complete Table 1, repeat Parts a–c using funnels with openings of 3 cm and 4 cm.

Mathematics Note

A **rate** compares the change in one quantity to the change in another quantity.

For example, the rate of flowing water compares a change in volume to a change in time. A rate of flow of 20 liters per minute may be written as 20 L/min

- e. Calculate the average time (to the nearest 0.1 sec) required for each size of funnel to empty.
- f. Use these average times to determine the rate at which rice flowed from each funnel in the following units.
 - 1. liters per second
 - 2. liters per minute
 - 3. liters per hour
- g. Use the rates from Part f to predict the volume (to the nearest 0.1 L) that could flow through each funnel in 1 min, 2 min, 5 min, and 10 min. Record your predictions in a table with headings like those in Table 2.

Table 2: Time vs. volume data

Time (min)	Volume (L)		
	2-cm Funnel	3-cm Funnel	4-cm Funnel
1			
2			
5			
10			

- h.
 - 1. Make a scatterplot of the predictions in Table 2 for the 2-cm funnel. Let x represent the time in minutes and y represent the volume in liters.
 - 2. Find a line that fits the points as closely as possible. Draw the line on the same set of axes as the scatterplot.
 - 3. Determine the slope of the line.
- i. Use your line to estimate the volume that could flow through the funnel in 7 min and in 12 min. **Note:** Save your work in this exploration for use in the assignment.

Discussion

- a. Compare the data you collected in Table 1 with that of others in the class. What may have caused the differences you observe?
- b. What relationship did you observe between the size of the funnel opening and the rate of flow?
- c. Do you think that the rates determined in Part f of the exploration are reliable? Explain your response.
- d. Describe the relationship between the rate of flow and the slope of the line you found in Part h of the exploration.
- e. If a leaky faucet drips once every second and each drip has a volume of 0.50 mL, what volume of water will leak from the faucet in 1 day?

Assignment

- 2.1** Use your line from Part **h** of the exploration to complete the following.
- Identify the y -intercept of the line.
 - What does the y -intercept represent in terms of the exploration?
 - Write an equation of the line in the form $y = mx + b$, where y represents volume in liters and x represents time in minutes.
 - Use your equation to predict the volume that could flow through the funnel in 7 min and in 12 min.
 - Compare these predictions with the ones you made in Part **i** of the exploration.
- 2.2**
- Make a scatterplot of the data in Table **2** for the 3-cm funnel.
 - Find a line that fits the data as closely as possible. Draw the line on the same set of axes as the scatterplot.
 - Write an equation of the line in the form $y = mx + b$, where y represents volume in liters and x represents time in minutes.
 - Repeat Part **a** for the data for the 4-cm funnel.
- 2.3** Graph the equations found in Problem **2.1b** and Problem **2.2a** and **b** on the same coordinate system. Describe any similarities or differences you observe and explain why they occur in terms of the exploration.
- 2.4** The table below shows the average time required to empty a 2.0-L water bottle using openings of different sizes.

Bottle Opening (cm)	Time to Empty (sec)
0.6	78.0
1.3	18.0
2.5	5.0

- Use this data to determine the rate at which water flowed through each opening in the following units.
 - liters per second
 - liters per minute
 - liters per hour
- Write an equation of the form $y = mx + b$ to describe the flow of water through each opening, where y represents volume in liters and x represents time in minutes.

* * * * *

- 2.5** The All School Club is a service organization at Larry's school. When Larry became president of the club, he decided to start a membership drive. The campaign hopes to sign 4 new members per week. After 10 weeks, this should bring the club's total membership to 375 students. The following table shows the membership during the first five weeks of the campaign.

Week	Total Membership
1	335
2	339
3	343
4	347
5	351

- a.
 1. Draw a scatterplot of this data. Let x represent the week and y represent the total membership.
 2. Draw a line that fits the data as closely as possible.
 - b.
 1. Is the slope of the line positive or negative?
 2. What does the slope indicate in terms of the membership drive?
 - c.
 1. Identify the y -intercept of the line.
 2. What does the y -intercept represent in terms of the membership drive?
- 2.6** Crickets make chirping sounds by rubbing their wings together. For some crickets, the relationship between the number of chirps per minute and the air temperature is very close to a line. When the air temperature is 20°C , these crickets chirp 124 times per minute. When the temperature is 26.6°C , they chirp 172 times per minute.
- a. Graph this information on a scatterplot. Let x represent the temperature in degrees Celsius and y represent the number of chirps per minute.
 - b. Draw a line that fits the data as closely as possible.
 - c. Identify the y -intercept of this line.
 - d. Determine the slope of the line.
 - e. Write an equation of the line in the form $y = mx + b$.
 - f. Predict the temperature at which these crickets make 150 chirps per minute.

- 2.7 Lumber is typically sold in units called board feet. The table below shows the number of board feet contained in lengths of three common dimensions of lumber: 1×4 , 2×4 , and 2×12 . (A 2×4 is approximately 2 in. thick and 4 in. wide.)

Dimensions	Length				
	5 ft	10 ft	20 ft	30 ft	40 ft
1×4	$1\frac{2}{3}$	$3\frac{1}{3}$	$6\frac{2}{3}$	10	$13\frac{1}{3}$
2×4	$3\frac{1}{3}$	$6\frac{2}{3}$	$13\frac{1}{3}$	20	$26\frac{2}{3}$
2×12	10	20	40	60	80

- On a single coordinate system, create a scatterplot that shows the number of board feet in a piece of lumber versus its length for each of the following dimensions:
 - 1×4
 - 2×4
 - 2×12 .
- Describe the relationship between the length of a piece of lumber and the number of board feet it contains.
- Draw a line that fits each scatterplot as closely as possible.
- Write an equation for each line from Part c in the form $y = mx + b$.
- Find the dimensions, including the length, of a piece of lumber that contains 1 board foot. Describe how you determined your response.

Activity 3

An aquifer is a water-filled layer of sand or gravel—a sort of underground deposit of water. The High Plains Aquifer, which underlies parts of eight states, is one of the largest known. It contains as much water as Lake Huron: about 4.24 quadrillion liters. For this reason, some geologists call it the “sixth Great Lake.”

The geographical boundaries of the aquifer are shown in Figure 4.



Figure 4: The High Plains Aquifer

Source: Dugan and Schild, *Water-Level Changes in the High Plains Aquifer*

This region accounts for nearly 15% of the grain, 25% of the cotton, and almost 40% of the beef produced in the United States. Much of this production is due at least in part to the availability of water from the High Plains Aquifer.

Although rain and snow help to replenish the aquifer, scientists predict that, in some areas, it may be depleted in less than 100 years. To prevent this, many farmers are planting crops that require less water, while others are irrigating in ways that conserve the resource. Without irrigation, however, acres of productive farms would return to the original prairie.

Exploration

To monitor changes in the aquifer, researchers drilled wells in Chase County, Nebraska. They measured the distance from the surface of the ground to the water. As shown in Figure 5, high water readings were taken in June, while low water readings were taken in September.

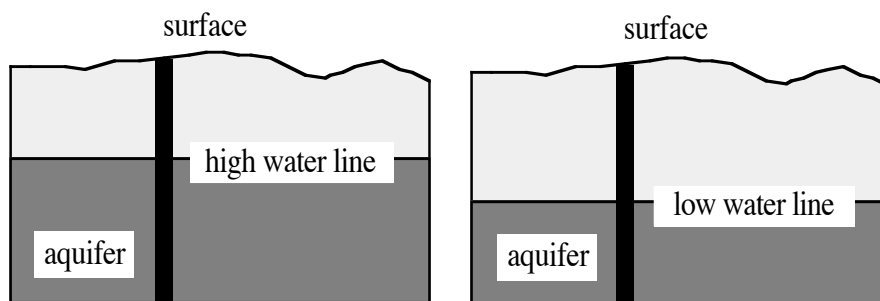


Figure 5: High and low water lines

Table 3 shows the distances from the surface to the high and low water lines from 1964 to 1978. The information in Table 3 seems to indicate that some sort of change is occurring in the aquifer. In this exploration, you determine whether or not this data can be reasonably modeled by a line.

Table 3: Distance from surface to water line (in meters)

Year	High	Low	Year	High	Low
1964	17.1	18.1	1972	19.7	21.3
1965	17.2	18.1	1973	19.8	22.3
1966	17.1	18.2	1974	20.1	22.9
1967	17.4	18.2	1975	20.6	23.8
1968	17.7	19.4	1976	21.3	26.0
1969	18.1	19.8	1977	22.3	26.5
1970	18.9	20.7	1978	22.7	27.6
1971	19.4	21.3			

Source: Dugan and Schild, *Water-Level Changes in the High Plains Aquifer*.

- a. Enter the distances to the high water level in a spreadsheet with headings like those in Table 4. Note that the first column is headed “Years after 1964.” This means that 1964 corresponds with 0, 1965 corresponds with 1, and so on.

Table 4: Distances to high water level after 1964

Years after 1964	Distance (m)
0	17.1
1	17.2
⋮	⋮
14	22.7

- b. Create a scatterplot of the data from Part a. Let y represent the distance and x represent the number of years after 1964.
- c. Mathematical models often use graphs or equations to describe relationships that arise in real-world situations. A mathematical model that consists of a line or its equation is a **linear model**.
1. Draw a linear model that fits the data from Part a as closely as possible.
 2. Find the equation of the line.
- d.
1. Add a third column to your spreadsheet with the heading “Predicted Distance.”
 2. Use the equation from Part c to predict the distance to the high water level for each year in Table 3.
 3. Enter these predicted values in the appropriate column of the spreadsheet.

Mathematics Note

As shown in Figure 6, a linear model may not exactly fit every data point. However, even when a line does not fit every point, it can still provide a reasonable model of the data.

A **residual** is the difference between an observed value and the predicted value. In Figure 6, the residual for each data point is the difference between the y -coordinate of the data point and the corresponding y -value of the model. Since data points may be located above or below the line, the values of residuals may be positive or negative.

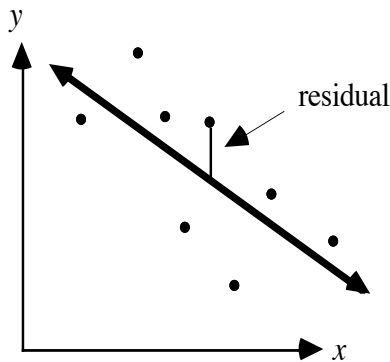


Figure 6: A linear model

The **absolute value of a residual** is a measure of the distance from the data point to the linear model. In general, the smaller the sum of the absolute values of the residuals, the more closely a line approximates the data.

- e.
 1. Add a fourth column to your spreadsheet with the heading “Residuals.”
 2. Using your predicted distances from Part d, find and enter the residual for each data point.
 3. Determine the sum of the residuals.
- f.
 1. Add a fifth column to your spreadsheet with the heading “Absolute Value of the Residuals.”
 2. Calculate and record the absolute value of each residual.
 3. Determine the sum of the absolute values of the residuals.
- g. Divide the sum of the absolute values of the residuals by the number of data points. This is the average distance from each data point to the model.

Discussion

- a. Does the line you drew in Part **c** of the exploration fit the data exactly? Explain your response.
- b. How could you use the average distance from each data point to the model to determine if a model fits reasonably well?
- c. Does your line appear to be a good model for the data?
- d. If the sum of the absolute values of the residuals is 0, then a linear model fits the data perfectly. Is this also true for the sum of the residuals? Use an example to justify your response.
- e. Describe the slope of your linear model. What might this slope indicate about the aquifer in Chase County?
- f. Do you think that water-level data has been recorded for enough years to support your response to Part **e** of the discussion? Why or why not?
- g. What practices might be implemented to ensure that future generations can continue to draw water from the High Plains Aquifer?
- h. Is there an aquifer in your area? If so, where would you go to find more information about it?

Assignment

- 3.1
 - a. Create a scatterplot like the one in the exploration for the distances to the low water levels in Table 3.
 - b. Draw a line that closely approximates the data.
 - c. Write an equation of the line in the form $y = mx + b$. Describe the method you used to find this equation.
 - d. Use your equation to predict the distances to the low water levels of the aquifer from 1964 to 1978.
 - e.
 1. Find the residual of each point in the scatterplot.
 2. Calculate the sum of the absolute values of the residuals.
 - f. Explain whether or not your line is a good model for the data.

- 3.2** Imagine that you have a leaky pipe under the kitchen sink. To catch the water, you place a coffee can under the leak.

The following table shows the time that the can has been in place and the total mass of the can and water.

Time (min)	Total Mass (g)
5	160
10	350
15	470
20	570
25	790

If y represents the total mass in grams and x represents the time in minutes, which of the following equations more closely approximates this data? Support your choice by determining the sum of the absolute values of the residuals for each model.

$$y = 30x + 16$$

$$y = 33x + 15$$

- 3.3** When an inflated balloon is placed in a freezer, its volume decreases as the air inside it grows colder. When the balloon is removed from the freezer, its volume increases as it warms.

The following table shows some data comparing the temperature of the air in the balloon to its volume.

Temperature (°C)	Volume (mL)
10	500
20	520
30	531
40	558

- Make a scatterplot of this data. Let y represent the volume in milliliters and x represent the temperature in degrees Celsius.
- Draw a line that closely approximates the data.
- Write an equation of the line in Part **b** in slope-intercept form.
- Find the sum of the absolute values of the residuals.
 - Describe what the calculation from Part **d1** means in relationship to the linear model.
- Predict the volume of the balloon at an air temperature of 100 °C.

* * * * *

- 3.4** Jason is raising hamsters to sell to pet stores. As shown in the following table, he has 64 hamsters after 5 months.

Month (x)	Hamsters (y)
1	2
2	6
3	18
4	35
5	64

The local pet supplier buys hamsters only in lots of 200. In order to predict when he will have enough hamsters to sell, Jason decides to create a graph of the data in the table.

- Make a scatterplot of Jason's data.
 - Draw a line that approximates the data.
 - Write an equation of the line in slope-intercept form.
 - Find the sum of the absolute values of the residuals.
 - Do you think Jason should use a linear model to predict when he will have 200 hamsters? If so, predict the time. If not, explain why not.
- 3.5** Over the past 70 years, Olympic swimmers have lowered the winning time in the women's 400-meter freestyle by more than 2 min. The table below shows the winning times in each race from 1924 to 1988, rounded to the nearest 0.01 min.

Year	Time	Year	Time	Year	Time
1924	6.04	1952	5.20	1972	4.32
1928	5.71	1956	4.91	1976	4.17
1932	5.48	1960	4.84	1980	4.15
1936	5.44	1964	4.72	1984	4.12
1948	5.30	1968	4.53	1988	4.06

- Graph this information on a scatterplot. Let x represent the number of years after 1920 and y represent time in minutes.
 - Draw a line that fits the data as closely as possible.
- Write an equation of the line in the form $y = mx + b$.
- Determine how well your model fits the data.
- Use your model to predict the winning time in the 400-meter freestyle in each of the following years:
 - 2000
 - 2080
- Do you think that the predictions you made in Part e are realistic? Explain your response.

- 3.6** Another method for determining how well a linear model fits a specific data point uses **percent error**. The percent error is the absolute value of the difference between the estimated value and the measured value, divided by the measured value, and expressed as a percentage:

$$\text{percent error} = \left| \frac{\text{estimated} - \text{measured}}{\text{measured}} \right| \cdot 100$$

For example, if a data point has coordinates (10,24) and the corresponding point on the linear model has coordinates (10,21), then the percent error can be calculated as follows:

$$\left| \frac{21 - 24}{24} \right| \cdot 100 = 12.5\%$$

If the measured value is 0, then percent error cannot be calculated.

- a. Using the predicted distances to the high water levels from Part **d** of the exploration, find the percent error for your model for each of the following years:
 1. 1964
 2. 1970
 3. 1975.
- b. The equation $y = 0.7x + 17$ is one possible model for the distances to the low water levels in Table **3**. Find the percent error for this model for each of the following years:
 1. 1965
 2. 1970
 3. 1975.
- c. Do you think that percent error would be a good measure of fit for a set of data? Why or why not?

* * * * *

Research Project

On average, each resident of the United States uses about 420 L of water per day. How does your daily water use compare to this value? In the following research project, you analyze your own personal water use.

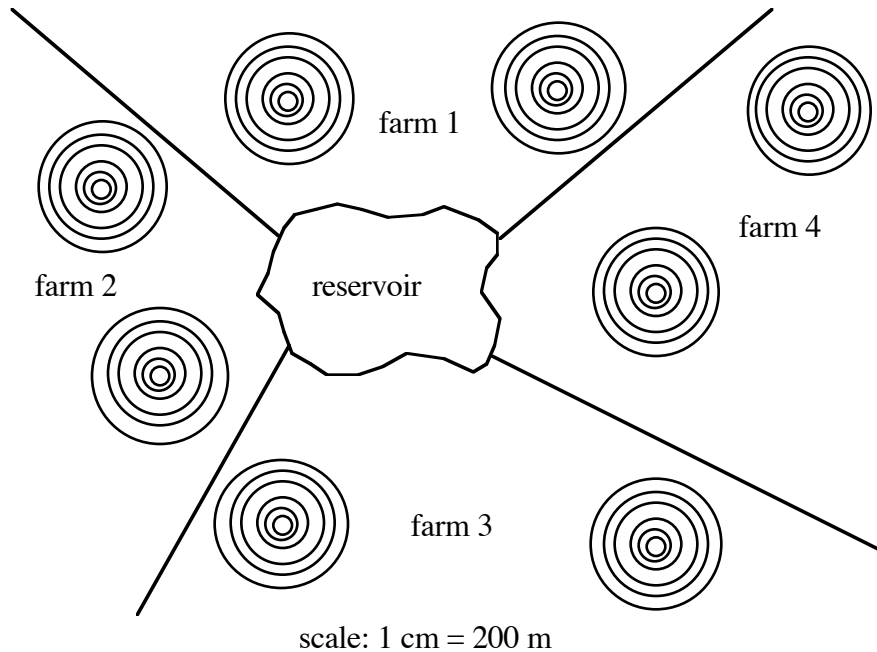
Create a table with headings like the one below. To complete the table, you will need to develop some innovative ways to measure water use. For example, how can you determine the volume of water used in a shower or bath? And how much water does it take to flush a toilet?

Use	Rate of Use	No. of Uses or Time Used	Daily Volume
washing machine			
dishwasher			
bathroom sink			
kitchen sink			
toilet			
shower or bath			
other			
other			
Total			

Your report should include a description of the methods you used to determine each measurement, a comparison of your daily water usage to the national average, and a discussion of any differences you observe.

Summary Assessment

1. As shown in the diagram below, four farms share irrigation water from the same reservoir. The average depth of the reservoir is 8.0 m. Each farmer has 2 center-pivot irrigation sprinklers. Each sprinkler can pump more than 4000 L of water per minute.



- a. During the 3-month growing season, the farmers plan to irrigate their fields for 16 hours a day, every day. Assuming that the reservoir is not refilled during the growing season, do they have enough water for their plan? Justify your response, showing all calculations.
- b. If the farmers do not modify their plan, for how many days can the reservoir supply them with water?
- c. What is the maximum number of hours per day the farmers can operate the sprinklers and still irrigate for the full 3 months?

2. The table below shows the volume of water flowing from a garden hose over time.

Time (min)	Volume (L)
0.00	0.0
1.00	15.0
2.00	29.0
3.00	48.0
4.00	65.0
5.00	73.0
6.00	94.0

- Draw a scatterplot of the data. Let y represent volume in liters and x represent time in minutes.
- Draw a line on the scatterplot that closely models the data points.
- Write an equation for the line in the form $y = mx + b$.
- Determine the average rate of flow in each of the following units:
 - liters per minute
 - liters per second
 - liters per hour
- How is the rate of flow in liters per minute related to the graph in Part **b**? Include mathematical terms and concepts in your response.
- Assuming that the rate of flow remains constant, determine the volume of water which will flow from the hose in:
 - 15 min
 - 2 hr.
- Find the absolute value of the residual for each data point.
 - Determine the sum of the absolute values of the residuals.
 - Describe what this sum tells you about your model.
- Another possible model for this data is the line $y = 14x + 2$. Compare the sum of the absolute values of the residuals found using this equation to the sum you calculated in Part **g**. Use the comparison to determine which equation is the better model.

Module Summary

- A **prism** is a three-dimensional figure determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.
- Prisms are named by the polygonal shape of the two bases. The **height** of a prism is the perpendicular distance between the bases.
- The amount of space occupied by an object is its **volume**. Volume is measured in units such as cubic centimeters (cm^3) or liters (L).
- The volume (V) of a prism may be found by multiplying the area of the polygonal base (B) by the height (h): $V = Bh$.
- A **rate** compares the change in one quantity to the change in another quantity.
- Mathematical models often use graphs or equations to describe relationships that arise in real-world situations. A mathematical model that consists of a line or its equation is a **linear model**.
- The difference between the y -coordinate of a data point and the corresponding y -value of a linear model is a **residual**. Since data points may be located above or below the line, the values of residuals may be positive or negative.
- The **absolute value of a residual** is a measure of the distance from the data point to the linear model. In general, the smaller the sum of the absolute values of the residuals, the more closely a line approximates the data.

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Skeeters Are Overrunning the World



How does the size of a population change over time? In this module, you use a simple model to shed light on some complicated issues.

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Skeeters Are Overrunning the World

Introduction

At last count, more than 5.4 billion people inhabited the Earth. If each of us laid head to toe, we would make a chain of humanity long enough to wrap around the equator 250 times.

How many people can live on Earth without destroying the environment? How many people can our planet successfully feed? Several organizations studying the ever-increasing human population are concerned with just these questions.

While foretelling the future is never a sure bet, you can gather information about the past and present, find any existing patterns, and use these patterns to make predictions.

Graphs can be useful tools for determining patterns. For example, Figure 1 shows the world population since 1650.

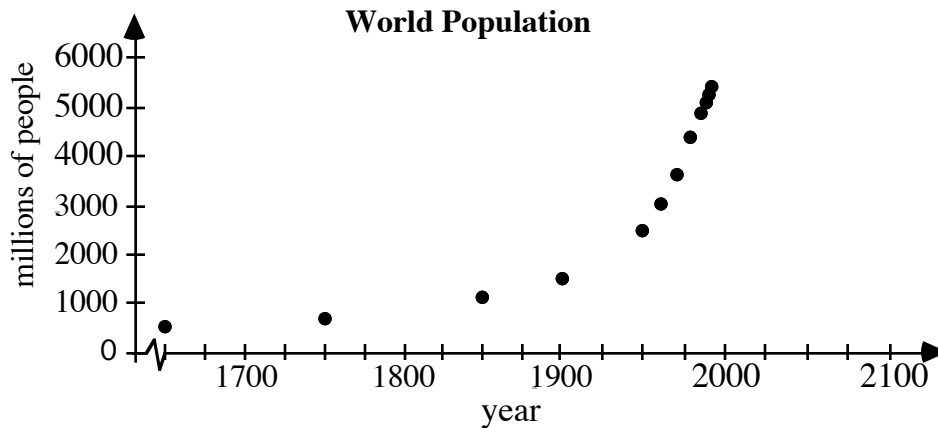


Figure 1: World population from 1650 A.D.

Discussion 1

- Describe any pattern you see in the world's population since 1650.
- Based on the pattern you find, predict what you think the world population will be in the year 2075.
- A prediction is only as good as the information and assumptions on which it is based. The human population has not always grown as rapidly as it has in the past 40 years. Describe any current events that may alter how fast the human population will increase.

Exploration

Statistics like those shown in Figure 1, along with an appropriate mathematical model, allow researchers to make forecasts about population trends. For example, scientists at the United Nations predict a world population of at least 8.2 billion by the year 2020.

To help make predictions in real-world situations, researchers often use experiments known as **simulations**. The results of the simulations are gathered and analyzed. This data is then compared with known information about the actual population. If the results seem questionable, the simulation may be revised. This modeling process can be summarized by the following five steps:

- creating a model
- translating the model into mathematics
- using the mathematics
- relating the results to the real-world situation
- revising the model.

In the following exploration, you investigate this modeling process using a population of Skeeters.

- a. Obtain a large, flat container with a lid, a sack of Skeeters, and several sheets of graph paper.
- b. Before beginning the simulation, read Steps 1–7 below and predict how you think the number of Skeeters will change.
 1. Place two Skeeters in your container. This is the initial population.
 2. After closing the lid, shake the container.
 3. Open the lid and count the number of Skeeters with the marked side up.
 4. Skeeters reproduce asexually (by themselves). Reproduction is triggered when the marked side of a Skeeter is exposed to light. Add one Skeeter to the container for each mark counted.
 5. Record the total number of Skeeters now in the container. This is the end of one “shake.”

The end of each shake represents the end of one time period. The number of Skeeters present at the end of a shake is the total population at that time. (Remember that at shake 0, the number of Skeeters was 2.)

6. Design a method of recording and organizing your data.
7. Repeat Parts 2–5 for 15 shakes.

- c.
 1. Create a scatterplot to display the data you recorded. Represent the shake number on the x -axis. Select a scale for each axis that will allow you to make predictions for shake numbers through 20.
 2. Describe any patterns you see in your data.
- d.
 1. Use the pattern described in Part c to predict the number of Skeeters after shake 20.
 2. How large a box would be necessary to hold this population? Explain how you came to this conclusion.
 3. Predict how many shakes it would take for the Skeeter population to reach 1000. Describe how you reached your prediction.

Discussion 2

- a. Discuss any similarities or differences you observe between your scatterplot and those of your classmates.
- b. How did the number of Skeeters in your population change during the exploration?
- c.
 1. Consider your scatterplot as describing the change in the population of Skeeters over time. Use this idea to explain the shape of the graph.
 2. How do the graphs obtained in the exploration compare to the linear graphs explored in previous modules?
- d.
 1. What other types of living creatures might show the same pattern of population growth as the Skeeters?
 2. What limitations might this simulation have in modeling a real-world population?

Activity 1

Within any population, there are differences in appearance and behavior due to genetics and environment. In this activity, you investigate some Skeeter populations with different growth characteristics.

Exploration

In this exploration, each color of Skeeter has its own growth characteristics and initial population. Table 1 shows a list of these characteristics for each color.

Table 1: Skeeter growth characteristics

Color	Growth Characteristics	Initial Population
green	For every green Skeeter with or without a mark showing, add 2 green Skeeters.	1 green
yellow	For every yellow Skeeter with or without a mark showing, add 1 yellow Skeeter.	1 yellow
orange	For every orange Skeeter with a mark showing, add 1 orange Skeeter.	1 orange
red	For every red Skeeter with a mark showing, add 1 red Skeeter.	2 red
purple	For every purple Skeeter with a mark showing, add 1 purple Skeeter.	5 purple

- a. Consider the information given in Table 1.
 1. Predict what will happen to the population of green Skeeters for the first 3 shakes.
 2. Predict which population will be largest after 10 shakes.
- b. Obtain a large, flat container with a lid, a sack of Skeeters of different colors, and a sheet of graph paper. Place the initial population of each color of Skeeters (indicated in Table 1) in the box.
- c. Place the lid on the container and shake it.
- d. At the end of each shake, use the growth characteristics from Table 1 to add the appropriate number of Skeeters of each color.
- e. Record the total number of Skeeters of each color at the end of each shake. (Record the initial population as the number at shake 0.)
- f. Repeat Parts c–e for 10 shakes.
- g. After 10 shakes, graph the data for each Skeeter population on the same coordinate system, using different colors to indicate the different populations. **Note:** Save your data for the orange, red, and purple populations for Activities 2 and 3.

Discussion

- a. Describe the relationship between the numbers of yellow Skeeters at the end of two consecutive shakes.
- b.
 1. Describe the relationship between the number of yellow Skeeters at the end of a shake and the shake number.
 2. Restate this relationship as a mathematical equation.
- c. Does your equation from Part **b** describe the population of yellow Skeeters after any shake? Explain your response.
- d. For which colors of Skeeters is the relationship between shake number and population a function? Recall that a set of ordered pairs (x,y) is a function if every value of x is paired with a value of y and every value of x occurs in only one ordered pair.
- e. In the relations you graphed in Part **g** of the exploration, which values represent the domain and which values represent the range?

Mathematics Note

The **growth rate** of a population from one time period to the next is the percent increase or decrease in the population between the two time periods.

For example, Table 2 shows the population of wild horses on an island over three years.

Table 2: A horse population

Year	Total Population
1992	15
1993	18
1994	24

The growth rate of the horse population from 1993 to 1994 is:

$$\frac{24 - 18}{18} \approx 0.33 = 33\% \text{ per year}$$

- f. Is the growth rate constant from shake to shake for each population of Skeeters in the exploration? Explain your response.

Assignment

- 1.1 a. Complete the following table for the yellow Skeeter population.

Shake Number	Total Population	Expanded Notation	Exponential Notation
0	1	$1 \cdot 1$	$1 \cdot 2^0$
1	2	$1 \cdot 2$	$1 \cdot 2^1$
2	4	$1 \cdot 2 \cdot 2$	$1 \cdot 2^2$
\vdots	\vdots	\vdots	\vdots
10			

- b. Write an equation that relates shake number to the total population after that shake.
- c. Is the equation you wrote in Part **b** a function? Explain your response.
- d. 1. Describe how to predict the total population of yellow Skeeters for shake numbers greater than 10.
2. Predict the total population of yellow Skeeters for shake 20.
- e. 1. By what factor is the population of yellow Skeeters increased after each shake?
2. Explain how this factor relates to the equation you wrote in Part **b**.
- f. 1. What is the growth rate for the population of yellow Skeeters?
2. Explain how this growth rate relates to the equation in Part **b**.
- 1.2 Create a table like the one in Problem 1.1 for your data for the green Skeeter population.
- a. Use the table to determine how the shake number relates to the total population after that shake. Write an equation for finding the population based on the shake number.
- b. Predict the shake number at which the population of green Skeeters will be close to the population of yellow Skeeters at shake 24.
- c. 1. By what factor is the population of green Skeeters increased after each shake?
2. Explain how this factor relates to the equation you wrote in Part **a**.
- d. 1. What is the growth rate for the green Skeeter population?
2. How does the growth rate relate to the equation in Part **a**?

Mathematics Note

When the values for a variable depend on the outcome of another variable, that variable is **dependent**.

When the values for a variable do not depend on the outcome of another variable, that variable is **independent**.

In a savings account, for example, the amount of time for which money remains in the account determines the amount of interest earned. In this situation, time is the independent variable, while interest earned is the dependent variable.

When drawing a graph, the values for the independent variable are plotted along the horizontal axis. The values for the dependent variable are plotted along the vertical axis.

- 1.3**
- a.** Compare the equations you wrote in Problems **1.1b** and **1.2a**. Describe any similarities or differences you observe.
 - b.** Which quantity represents the dependent variable in these equations, the population or the shake number? Explain your response.
 - c.** Rewrite each equation in terms of a dependent variable (y) and an independent variable (x).
- 1.4** Consider some Skeeters whose population growth can be modeled by the equation $y = 5^x$, where y represents the total population after a shake and x represents the shake number.
- a.** Use this equation to predict the population after 10 shakes.
 - b.** How does the population after each shake compare to the population before the shake?

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- 1.5** In January of 1990, Maridee deposited \$5000 in a savings account. At the end of each year, the interest earned was added to the account. The following table shows the account balance, after interest was added, during the five years from 1990 to 1994.

Year	Account Balance
1990	\$5000.00
1991	\$5250.00
1992	\$5512.50
1993	\$5788.13
1994	\$6077.53

- a. Create a scatterplot of the information in the table.
 - b. What is the growth rate in the account per year?
 - c. Use the growth rate determined in Part **b** to calculate the account balance at the end of 1995.
 - d. In what year will the account balance have doubled Maridee's original deposit?
- 1.6** One type of Skeeter produces 3 offspring after every shake, whether the marked side is showing or not.
- a. Using an initial population of 1 of these Skeeters, create a table that shows the shake number and total population for the first 6 shakes.
 - b. Write an equation in the form $y = 1 \cdot b^x$ that models the growth in this population.
 - c. Determine the total population after 10 shakes.
 - d. Does this population have a constant growth rate? If so, calculate this growth rate. If not, determine the growth rates between consecutive shakes for the first 6 shakes.

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Activity 2

In Activity 1 you looked at Skeeter populations that doubled or tripled after each shake. What happens if the ratio of consecutive populations is not an integer value?

Discussion 1

- a. Recall the orange Skeeter population from Activity 1. After each shake, only the Skeeters with the marked side showing produced offspring. If you shook a box containing 10 of these Skeeters, how many would you expect to land with the marked side up?
- b. How does the probability of a Skeeter landing with the marked side up compare with the probability of a tossed coin landing heads up?
- c. What growth rate would you expect to find between consecutive shakes of the orange Skeeter population?

Exploration

- a. 1. Create a spreadsheet with headings like those in Table 3 below.

Table 3: Orange Skeeter population and growth rate

Shake	Expected Population	Actual Population	Actual Growth Rate (from Previous Shake)
0	1	1	
⋮	⋮	⋮	
10			

2. Use the growth rate determined in Part c of Discussion 1 to calculate the expected population after each shake. Record this in the appropriate column of the table.
 3. Enter the actual data for the orange Skeeter population obtained in Activity 1 in the appropriate column of the spreadsheet.
 4. Use the spreadsheet to calculate the actual growth rates between consecutive shakes. Record these values in the appropriate column.
- b. On the same set of axes, create scatterplots of the expected data and the actual data for the orange Skeeter population.
 - c. 1. Use the growth rate from Part c of Discussion 1 to write an equation that describes the orange Skeeter population after x shakes.
2. Sketch a graph of the equation on the same set of axes as the scatterplots from Part b.

Discussion 2

- a. In Part **b** of the exploration, how does the graph of the actual data compare with the graph of the expected data?
- b. How well does the equation from Part **c** of the exploration model the actual data for the orange Skeeter population?

Assignment

- 2.1** Imagine that you have a container of 20 Skeeters. After shaking the container, you add one Skeeter for every Skeeter with its marked side up.
- a. How many Skeeters would you expect to add at the end of the first shake? What would you expect the total population to be after the first shake?
 - b. How many Skeeters would you expect to add at the end of the second shake? What would you expect the total population to be after the second shake?
 - c. If you had p Skeeters before a shake, how many would you expect to add after the shake?
 - d. What is the growth rate for this population?
 - e. Write an equation that describes the total expected population after a shake if the population before the shake is p .
 - f. Use your equation from Part **e** to describe the number of Skeeters you would expect to add after the next shake.
 - g. Describe the formula you would use in a spreadsheet to find the total expected population after any shake.
- 2.2** When Skeeters are shaken in a container, is the probability of a Skeeter landing marked side up always $1/2$? Explain your response.
- 2.3** In 1990, the population of Tanzania was approximately 27,000,000 people. The expected growth rate is 3.5% per year.
- a. Calculate the expected population in each of the 10 years after 1990.
 - b. Make a scatterplot of the expected population data from Part **a**.
 - c. How do you think the expected values for Tanzania's population will compare with the actual values? Explain your response.

* * * * *

- 2.4** Chauncy’s parents have decided to offer him a weekly allowance. During the first year, he will receive \$10 per week. In each of the following years, they have given him the choice of either a \$7 raise or a 40% increase in his weekly allowance.

If Chauncy plans on living at home for the next 5 years, which proposed increase should he choose? Explain your response.

- 2.5** In each year between 1990 and 1995, Sue earned a gross salary of \$30,000. In 1990, she paid \$4338 in federal income taxes. In 1991, she paid \$3905 in federal income taxes.
- Calculate the growth rate in Sue’s federal income taxes between 1990 and 1991.
 - Use this rate to predict the taxes Sue can expect to pay in 1995.
 - Describe some possible limitations in using this model to predict Sue’s taxes.

* * * * *

Activity 3

How does the initial population size influence future Skeeter populations? In this activity, you use technology to model the growth of three Skeeter populations from Activity 1.

Discussion 1

- Compare your data for the orange, red, and purple Skeeter populations from Activity 1. Explain any similarities or differences you see.
- What effect, if any, does the initial population appear to have on the growth of each population?

Exploration

- Create a spreadsheet with headings like those in Table 4 below. Use initial populations of 1 orange Skeeter, 2 red Skeeters, and 5 purple Skeeters.

Table 4: Three Skeeter populations

Shake No.	Orange	Red	Purple
0	1	2	5
⋮	⋮	⋮	⋮

- b. Using an expected growth rate of 0.5 Skeeters per shake, generate a table of values for each population for 20 shakes.
- c. On the same set of axes, create a scatterplot of the expected data for each population for the first 5 shakes.
- d. On another set of axes, create a scatterplot of the expected data for each population for 20 shakes.

Discussion 2

- a. How do the growth rates you observed in the red, purple, and orange Skeeter populations in Activity 1 compare to the expected growth rate of 0.5?
- b. Compare the population data you collected in Activity 1 with the expected data generated by the spreadsheet. What similarities or differences do you see?
- c.
 - 1. In Part c of the exploration, what similarities or differences do you see in the scatterplots for the three populations?
 - 2. Is it possible to determine the initial size of each population by looking at the graph?
- d. Describe the mathematical operations used to calculate the population of purple Skeeters in your spreadsheet.
- e. If a current population (p) of Skeeters has a growth rate of r , explain why the equation for the total population (T) after the next shake can be expressed as $T = p(1 + r)$.
- f.
 - 1. If the population before a given shake is $p(1 + r)$, what expression can be used to describe the total population after that shake?
 - 2. Describe a relationship between initial population (p), growth rate (r), shake number (n), and total population (T) after that shake number.

Assignment

- 3.1** How closely does the relationship found in Part **f** of Discussion **2** model the data collected in Activity **1** for the orange and red Skeeter populations? Explain your response.
- 3.2** One equation that can be used to determine the total Skeeter population (T) given an initial population p , a growth rate r , and a shake number n , is

$$T = p(1 + r)^n$$

This is one example of an **exponential equation**. The general form of an exponential equation is $y = ab^x$.

- What would each variable in the general form of an exponential equation represent in terms of a Skeeter population?
 - What effect would a change in the initial population have on the equation?
 - Use appropriate technology to create a graph that shows the growth of a Skeeter population with an initial size of 4 and a growth rate of 150% per shake.
- 3.3** Explain why a constant growth rate of 0.5 per shake produces little change in population size from one shake to the next when the shake number is small, but results in greater increases in the size of each successive population after many shakes.

Mathematics Note

The exponential equation $y = a \cdot b^x$ can be used to describe a pattern of **exponential growth**. If this equation describes population growth, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the growth rate). The independent variable x represents a time period.

Considering a population of Skeeters, for example, the independent variable x represents the shake number at which the population is counted. The dependent variable y represents the total population. A population of Skeeters with an initial population of 6 and a growth rate of 0.5% per shake can be modeled by the equation $y = 6 \cdot 1.005^x$.

- 3.4** Two of your classmates have used a spreadsheet to model the growth of a population of Skeeters. The initial population was 7. After shake 8, the total population was 2,734,375. What growth rate did they use? Hint: Substitute the appropriate values into an equation of the form $y = a \cdot b^x$ to find b ; then use this value to determine the growth rate.

- 3.5 a. The relation $y = 3 \cdot 2^x$ models the growth in a population of Skeeters. Describe the values of the domain and range in this setting.
- b. Is this relation a function? Explain your response.
- 3.6 a. The two equations below represent two different populations of Skeeters. Graph both equations on the same set of axes. When will these populations be approximately the same size?
1. $y = 10 \cdot 1.5^x$
 2. $y = 1 \cdot 2^x$
- b. Find the size of each population when $x = 0$. What do these values represent?
- 3.7 The table below shows the population of Skeeters in a container after each of 5 shakes. Write an equation which could be used to describe this data. (Your equation may not describe the data exactly.)

Shake Number	Population Total
0	2
1	7
2	25
3	86
4	300
5	1050

* * * * *

- 3.8 The following table shows the growth in a savings account with an initial deposit of \$4000.

Year	Account Balance
1990	\$4000.00
1991	\$4160.00
1992	\$4326.40
1993	\$4498.50
1994	\$4679.40

- a. Write an equation in the form $y = a \cdot b^x$ to describe this data.
 - b. Using your equation, what is the value of y when $x = 0$?
 - c. What is the value of x when the initial deposit of \$4000 has doubled?
- 3.9 In 1987, Vincent Van Gogh's painting *Irises* was auctioned for \$53.9 million. Assume that the painting's value grew exponentially since 1889. If the painting initially sold for \$50, by what percentage did its value increase each year?

* * * * *

Research Project

Imagine your own population of creatures with a growth rate and initial population different from those used in the explorations.

- Use a spreadsheet to simulate the growth in this population for at least 10 time periods. Display your simulated data in both a table and a graph.
 - Find an equation that models the growth in the population over time.
 - Write a story about your population. Include a description of the growth rate and explain the consequences this rate will have on the population over time.
-

Activity 4

In 1991, the world's human population was approximately 5.3 billion. This total was increasing by about 250,000 per day, or 3 people every second. At this rate, over 1 billion people will have been added to the Earth's population by the end of the decade.

As a citizen of Earth and the United States, you have some questions to consider when making future personal and political decisions. What is the growth rate of the world population? How does this rate compare with the growth rate of the U.S. population? Is a growth rate of zero desirable? If so, how can zero population growth be obtained?

Exploration

Table 5 shows the population of the United States at 10-year intervals from 1790 to 1990.

Table 5: U.S. population from 1790 to 1990

Year of Census	Number of People	Year of Census	Number of People	Year of Census	Number of People
1790	3,929,214	1860	31,443,321	1930	122,775,046
1800	5,308,483	1870	39,818,449	1940	131,669,275
1810	7,239,881	1880	50,155,783	1950	150,697,361
1820	9,638,453	1890	62,947,714	1960	179,323,175
1830	12,866,020	1900	75,994,575	1970	203,302,031
1840	17,069,453	1910	91,972,266	1980	226,545,805
1850	23,191,876	1920	105,710,620	1990	248,709,873

Source: U.S. Bureau of the Census.

- a. Identify any patterns you see in the data in Table 5.
- b. In the previous activities, you calculated growth rates for Skeeter populations between consecutive shakes. Use a similar technique to calculate the growth rates for the U.S. population for each 10-year period.

Discussion

- a. What historical events may have affected U.S. population growth during the past 200 years?
- b. How would you find a representative growth rate for the U.S. population for the 200-year period from 1790 to 1990?
- c. If a population with a growth rate of 0 has a birth rate of 0.05 per year, what is the death rate?
- d. Consider a population with a growth rate of -0.05 per year and an initial population of 65. What equation could you use to model this population?
- e. How does a negative growth rate affect the total population over time?

Assignment

- 4.1 By describing the patterns found in the U.S. population data and displaying these patterns in graphs or equations, you can make predictions about the future.
 - a. Predict what the U.S. population would have been in 1990 if the growth rate had remained unchanged since 1790. Compare this number to the actual population in 1990.
 - b. Describe how you could use either the data in Table 5 or an exponential equation to predict the U.S. population in the year 2040.
 - c. If the future growth rate remains the same as it was from 1980 to 1990, predict the U.S. population in the year 2040.
- 4.2
 - a. Select a growth rate that produces a decreasing population in each successive 10-year period.
 - b. Use this growth rate and the U.S. population in 1990 to predict the U.S. population in 2040.
 - c. What conditions might cause a population to decrease?
- 4.3 From 1920 to 1930, the U.S. population grew by approximately 16%
 - a. Use this growth rate to estimate the U.S. population in 1940 and 1950.
 - b. Compare your estimates with the actual values for the U.S. population given in Table 5.

- 4.4** The table below shows the growth rates, per year, in the populations of three cities. Use this data to predict the population of each city in the year 2040. What do you notice about the predicted populations for these cities?

City	1992 Population	Annual Growth Rate
New York	14,628,000	0.00%
London	9,168,000	-1.00%
Lagos	8,487,000	4.80%

Source: U.S. Bureau of the Census.

- 4.5** How does a growth rate of 0 affect total population numbers over time?

* * * * *

- 4.6** In 1930, the U.S. national debt totaled \$16,185,000,000. By 1940, it had risen to \$42,968,000,000.

- Calculate the percent increase in the national debt over the decade from 1930 to 1940.
- What would the national debt have been in 1990 if the growth rate from 1930 to 1940 had remained unchanged?
- Since 1930, there have been periods of increase and decrease in the growth rate of the national debt. The actual debt in 1990 was about \$3,233,000,000,000.
 - In 1980, the national debt was \$907,700,000,000. Calculate its growth rate from 1980 to 1990.
 - What will the national debt be in 2040 if the growth rate from 1980 to 1990 remains unchanged?

- 4.7** The enrollment at Eagle Canyon High School has been increasing steadily in the past few years. The present high school building is designed for a maximum of 1000 students. Within the next 5 years, the school board wants to build an addition that will increase the high school's capacity by 200 students.

- Last year's enrollment at Eagle Canyon High was 947 students. This year's enrollment is 958 students. If this annual growth rate remains unchanged, will the new addition be needed within 5 years? Explain your response.
- Assume that the growth rate in the student population remains unchanged. Even with the completion of the high school addition, will the school still have enough capacity for the next 20 years? Explain your response.

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Summary Assessment

The table below contains data on the population, growth rate, land area, and population density for four nations. Use this information to complete Problems 1–4.

Nation	1993 Population	Annual Growth Rate	Land Area (km²)	Density (people/km²)
Canada	27,700,000	1.0%	8,968,000	3.1
China	1,177,600,000	1.4%	9,327,000	126.3
Hungary	10,324,000	−0.3%	92,000	112.2
India	903,159,000	2.1%	2,972,000	303.9

Source: U.S. Bureau of the Census.

1. For each nation listed in the table, write an exponential equation that describes its population growth.
2. Assuming that the given growth rates remain unchanged, predict the years in which the 1993 populations of Canada and Hungary will have doubled.
3. Although India has a smaller population than China, it is growing at a faster rate.
 - a. Determine the year in which India's population will surpass that of China.
 - b. What will India's population density be at that time?
4. If these four nations continue to grow at their current rates until 2093, which nation will have the highest population density? Support your conclusion with figures and graphs.

Module Summary

- A **simulation** is an experiment conducted to investigate real-world situations.
- The **growth rate** of a population from one time period to the next is the percent increase or decrease in the population between the two time periods.
- When the values for a variable do not depend on the outcome of another variable, that variable is **independent**.
- When the values for a variable depend on the outcome of another variable, that variable is **dependent**.
- When drawing a graph, the values for the independent variable are plotted along the horizontal axis. The values for the dependent variable are plotted along the vertical axis.
- The exponential equation $y = a \cdot b^x$ can be used to describe a pattern of **exponential growth**. If this equation describes population growth, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the growth rate). The independent variable x represents a time period.

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Oil: Black Gold



Oil spills can have disastrous effects on the environment. When planning their response, clean-up crews must estimate both the volume of the spill and the area it covers. In this module, you investigate direct and inverse relationships in the context of oil spills.

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Oil: Black Gold

Introduction

Large oil spills consistently make international headlines. According to the April 13, 1992, *U.S. News & World Report*, “Some 920,000 barrels of oil—roughly 1 out of every 10 barrels produced [in Russia]—are spilled every day in Russia.”

The barrel is the international standard for measuring crude oil. Since 1 barrel of oil equals 42 U.S. gallons, 920,000 barrels of oil represents about 39 million gallons.

Discussion 1

- a. Describe the number of gallons mentioned above in more familiar terms. For example, how many times would this much oil fill up your classroom?
- b. How is the environment affected when oil is spilled into a body of water?
- c. What methods are used to clean up oil spills?

Exploration

What happens when crude oil hits water? What shape does the spill make? How large an area will the spill cover? In the following exploration, you simulate the effects of an oil spill on water.

- a. Pour water into a shallow container to a depth of approximately 2 cm.
- b. Use a medicine dropper to add one drop of oil to the water. Observe how the oil and water interact.
- c. Place a second drop of oil on the center of the first drop and note the changes that occur in the oil slick.
- d. Quickly add several more drops of oil to the center of the existing slick. Record the number of drops added and note any changes that occur.

Discussion 2

- a. In Part **b** of the exploration, how is the volume of oil in the drop related to the volume of oil in the slick?
- b. Describe the geometric properties of the oil slick after the additional drops of oil were added.
- c. How thick do you think the oil slick is?

- d. How could you estimate the area covered by the surface of the oil slick?
- e. When an oil spill occurs in the real world, what natural factors might affect the shape of the slick?

Activity 1

In order to reduce oil’s harmful effects on the environment, clean-up efforts typically begin as soon as possible after a spill. To help plan their work, clean-up crews need to know both the area of the surface covered by the spill and the volume of oil. Since precise answers are seldom available, working approximations are used.

Mathematics Note

A **simple closed curve** in a plane is a curve with no endpoints that does not intersect itself. Figure 1 shows some examples of simple closed curves.

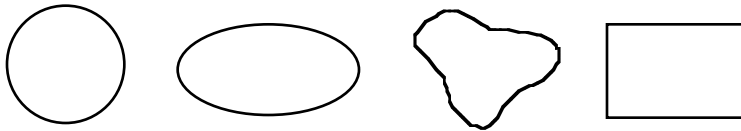


Figure 1: Four simple closed curves

Exploration

Figure 2 shows the shapes of two oil slicks. In the following exploration, you investigate two different methods for estimating the area and volume of these slicks.

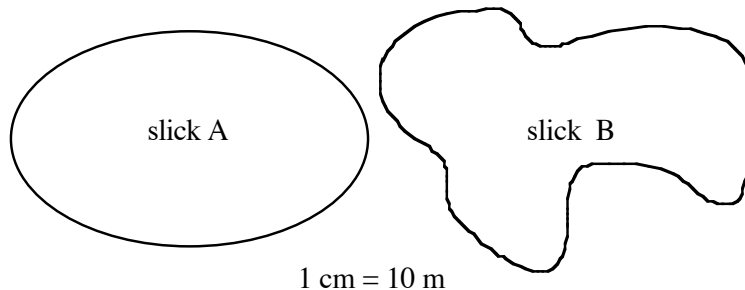


Figure 2: Two oil slicks

- a. In the module “What Will We Do When the Well Runs Dry,” you estimated the area of irregular shapes by counting squares on a sheet of graph paper.
 1. Use this method and a centimeter grid to estimate the area of each slick in Figure 2.
 2. The oil in each slick is 1 mm deep. Use the formula for the volume of a cylinder to determine the volume of each slick.

Mathematics Note

A **cylinder** is a three-dimensional solid with bases that are congruent, simple closed curves (nonpolygons) in parallel planes. For example, Figure 3 shows three different cylinders.

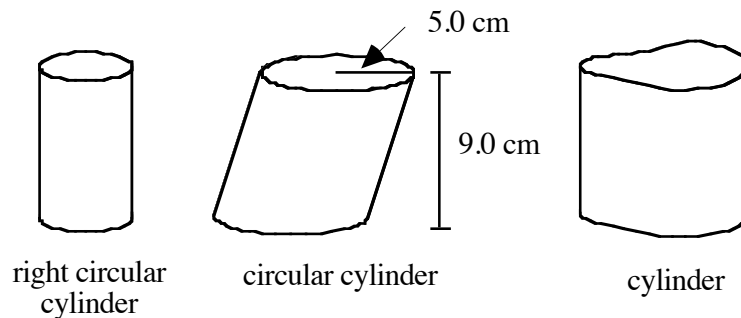


Figure 3: Three cylinders

The volume (V) of a cylinder can be found by multiplying the area of its base (B) by its height (h): $V = B \cdot h$.

For example, to find the volume of the circular cylinder in Figure 3, you must first calculate the area of the base. Since the base is a circle with a radius of 5.0 cm, its area is $\pi(r^2) = \pi(5 \text{ cm})^2 \approx 79 \text{ cm}^2$. Since the height of the cylinder is 9.0 cm, its volume is $79 \text{ cm}^2(9 \text{ cm}) \approx 710 \text{ cm}^3$.

- b. Another method for estimating the area of irregular shapes uses the formula for the area of a circle with a radius r , $A = \pi r^2$. Complete Steps 1–5 below for both slicks in Figure 2.
1. Locate a point at the approximate center of the slick.
 2. Measure the distance from the center to several random points on the edge of the slick.
 3. Find the mean of the distances from Step 2. This mean is the approximate radius of a circle with comparable area.
 4. Use the mean radius found in Step 3 and the formula for the area of a circle to estimate the area of the slick.
 5. Assuming that the oil is 1 mm deep, determine the volume of the slick.

Discussion

- a. When using a grid to estimate area, why do you divide the number of partially covered squares by 2?
- b. How could you modify the grid used in Part a of the exploration to obtain more accurate estimates of area?

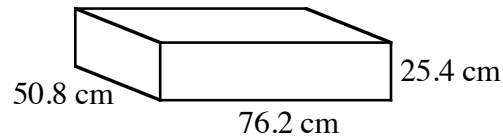
- c. How might an environmental engineer find the area covered by the surface of an oil spill?
- d.
 1. Compare the two estimates you obtained for the volume of each slick.
 2. If you were in charge of cleaning up slick B, which estimate would you use?

Assignment

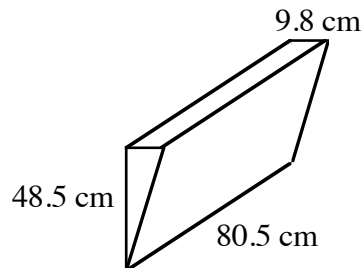
1.1 The formula for the volume of a prism is $V = B \cdot h$, where V is the volume, B is the area of the base, and h is the height.

a. Determine the volume of each fuel tank below in cubic centimeters.

1. This truck fuel tank is a rectangular prism.



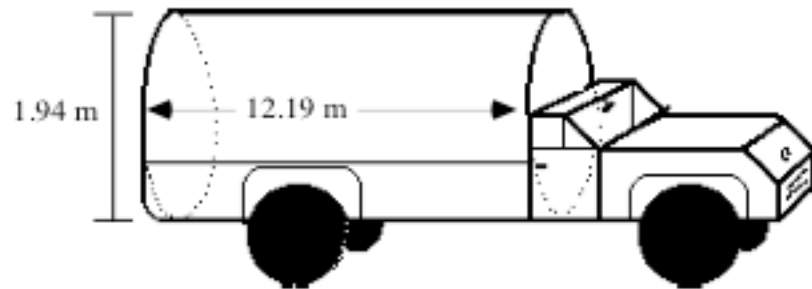
2. This boat fuel tank is a triangular prism in which the bases are right triangles.



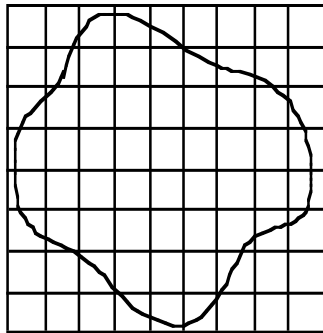
b. How many liters of fuel will each tank hold?

1.2 Paper is usually purchased in reams. One ream contains 500 sheets of paper and is about 5.2 cm thick. Find the volume of one sheet of letter-size paper in cubic centimeters.

1.3 The tank on the oil truck in the figure below is a circular cylinder. Determine the volume of the tank.



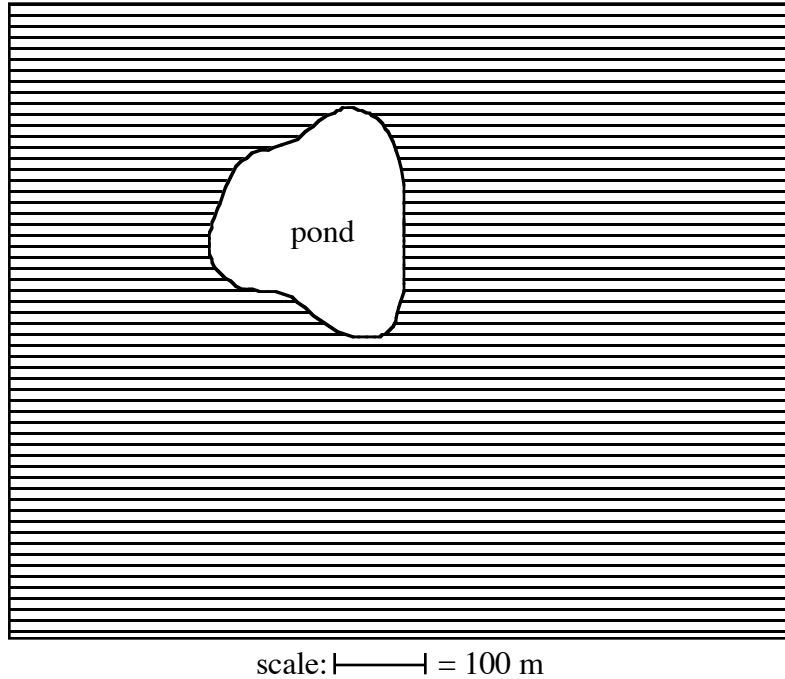
- 1.4** Imagine that some highly refined oil is spilled into a calm body of water. Under these conditions, the oil can spread to a very thin film, approximately $2.5 \cdot 10^{-3}$ cm thick.
- What type of three-dimensional figure could be used to describe the oil slick?
 - The mean radius of the slick is 405 m. Determine the volume of oil in the spill in liters.
- 1.5**
- Estimate the area that would be covered by a spill of 1 barrel of highly refined oil in a calm body of water. (There are approximately 3.8 L in 1 gal.) Describe how you determined your estimate.
 - Do you think that an actual spill of 1 barrel of oil will spread as much as you estimated in Part **a**? Why or why not?
- 1.6** The following diagram shows an aerial view of an oil slick. The grid superimposed on the photograph has squares that measure 1 km on each side.



- Estimate the area covered by the slick.
 - Determine the volume of oil involved in the spill if the slick is 0.05 cm thick. Record your answer in barrels. (There are 42 gal in 1 bbl.)
- 1.7** In December of 1989, an explosion on an Iranian supertanker spilled 19 million gallons of crude oil into the Atlantic Ocean. The oil slick covered an area of about 260 km^2 .
- Approximately how thick was this oil slick?
 - How does the thickness of this spill compare to the thickness of a spill of highly refined oil?

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- 1.8 The federal government assures wheat farmers of a certain base price per bushel if they farm a limited acreage. In order to determine if a farmer is in compliance, regulators take an aerial photograph of the farm. The diagram below shows an aerial view of a wheat field with a pond. This farm has been allotted a maximum of 200 acres of wheat. (One acre is approximately 4047 m^2 .) Is this farmer in compliance?



- 1.9 Use an ordinary soft-drink can to complete Parts **a–c** below.
- Measure the height and diameter of the can.
 - Use these measurements to find the volume of the can in cubic centimeters.
 - Convert the volume of the can in cubic centimeters to milliliters. How does this value compare to the volume printed on the can?

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Activity 2

A **simulation** of a real-world event involves creating a similar, but more simplified, model. In the introduction, for example, you simulated an oil spill on the ocean using a few drops of oil in a pan of water. In this activity, you simulate oil spills on land by placing drops of oil on sheets of paper.

Exploration

In this exploration, you simulate spills involving eight different volumes of oil.

Note: Save your data, observations, and calculations for the assignment in this activity.

- a. Obtain a small amount of oil, a medicine dropper, a ruler, eight sheets of absorbent paper, and some paper towels.
- b. Spread the eight sheets of absorbent paper on a flat, nonabsorbent surface. Arrange the sheets so that they do not touch each other. Number the sheets from 1 through 8 and place a pencil dot in the middle of each sheet. Be careful not to fold or wrinkle the paper.
- c. Carefully place 8 drops of oil on the pencil dot on sheet 8. Make the size of each drop as consistent as possible.

Continue creating oil spills of different volumes by placing 7 drops on sheet 7, 6 drops on sheet 6, and so on.

- d. Without disturbing the sheets of paper, observe each spill and record your observations. Describe the general relationship between the volume of oil (number of drops) and the shape and area of the spill.
- e. Determine the mean radius of each spill to the nearest 0.1 cm. Start with sheet 1 and continue in numerical order to sheet 8.
- f.
 1. Determine the area covered by each spill and record these values in Table 1 below.

Table 1: Volume and area of oil spills

Volume (drops)	Area (cm ²)
0	0
1	
2	
⋮	
8	

2. Create a scatterplot of the data in Table 1. Represent the area covered by the spill on the y -axis and the volume of oil in drops on the x -axis.
3. Determine the equation of a line that reasonably approximates the data. The line should have the same y -intercept as the scatterplot.

- g.** Use your equation to predict the area of an oil spill with each of the following volumes:
 1. 0.04 drops
 2. 0.5 drops
 3. 2.5 drops
 4. 25 drops (≈ 1 mL)
 5. 25,000 drops (≈ 1 L)
- h.** For each spill in Table 1, calculate and record the ratio of the area covered to the volume (number of drops).
- i.** Dispose of the oil-soaked paper as directed by your teacher.

Discussion

- a.** What observations did you make concerning the spread of the oil, the shape of the spill, and the area covered by the spill?
- b.** What problems did you encounter in measuring the area covered by the oil spills?
- c.**
 1. How does the precision of your measurement of the mean radius affect the accuracy of the calculated area?
 2. What implications does your response have for measuring actual oil spills?
- d.**
 1. Describe the area you predicted for an oil spill of 1 L (approximately 25,000 drops) in Part **g** of the exploration.
 2. Do you think that this prediction is realistic? Explain your response.
- e.** Is it reasonable to assume that 0 drops of oil produce an oil spill with a surface area of 0 cm^2 ? Explain your response.
- f.**
 1. Describe any pattern you observed in the ratios of surface area to volume determined in Part **h** of the exploration.
 2. What method might you use to find a single number m to represent all these ratios?
 3. If this value of m were used to write an equation $y = mx$, what would each variable represent in terms of the oil spills?
 4. How does this equation compare to the one you found in Part **f** of the exploration?
 5. Could this equation be used to accurately predict the area of an oil spill of 500 bbl on the ground? If not, how could you change the model to improve the prediction?

Mathematics Note

One quantity is **directly proportional** to another when the ratio of the two quantities is constant (the same). The constant is the **constant of proportionality** and the ratio is a **direct proportion**.

A direct proportion can be described by a linear equation of the form $y = mx$, where m is the constant of proportionality. The graph of a direct proportion always contains the origin since $y = 0$ whenever $x = 0$.

For example, in the direct proportion $y = 2x$, the constant of proportionality is 2. A graph of $y = 2x$ is shown in Figure 4.

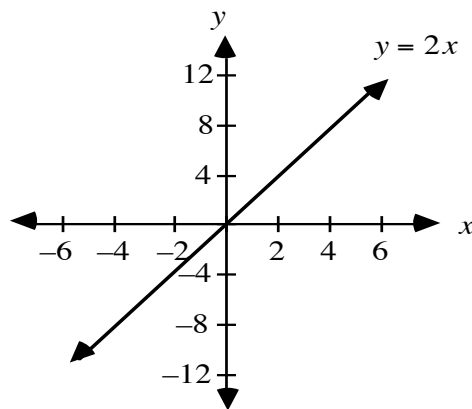


Figure 4: Graph of a direct proportion

- g. Given the direct proportion $y = mx$, describe how the values of y change as:
 - 1. the values of x increase when $m > 0$
 - 2. the values of x increase when $m < 0$
- h. Are an oil spill's area and volume directly proportional? Explain your response.
- i. If you know any point with coordinates (p, q) on the graph of a direct proportion, what is the slope of the line?

Assignment

2.1 Determine if x and y are directly proportional in each of the following relationships. Defend your responses.

a.

x	1	3	4	6	25
y	3	9	12	18	75

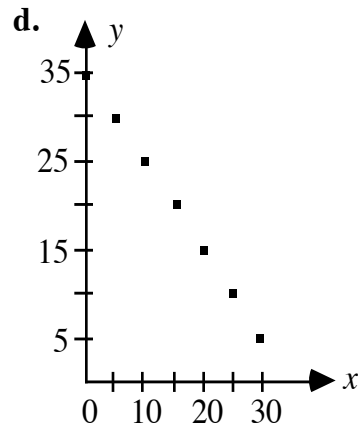
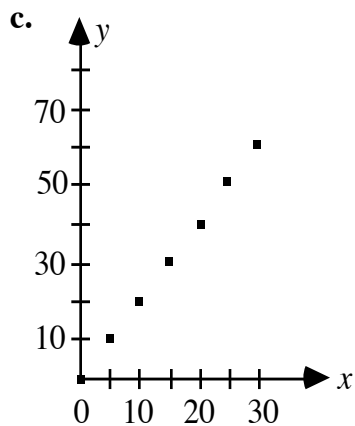
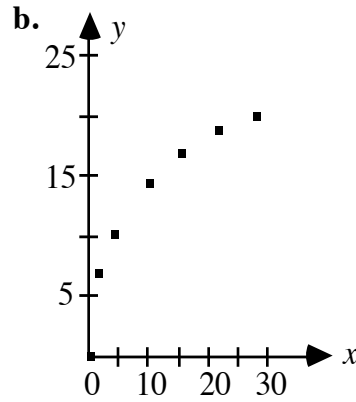
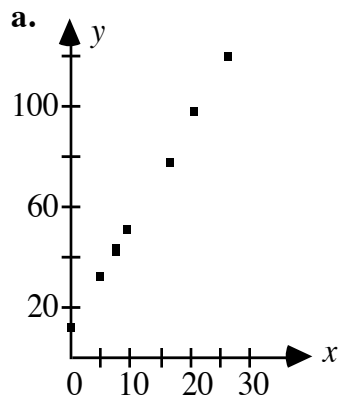
b.

x	1	3	4	6	25
y	8	14	17	23	80

c.

x	0	3	4	5	7
y	0	10	17	26	50

2.2 For each of the scatterplots determine if they appear to represent a direct proportion. Defend your answer.



- 2.3** a. Graph each of the following equations of direct proportions on the same coordinate system, with $0 \leq x \leq 10$. Compare the graphs.
1. $y = 0.5x$
 2. $y = 5x$
 3. $y = 1x$
 4. $y = -0.25x$
 5. $y = 1.75x$
 6. $y = -3.5x$
- b. In your own words, describe the characteristics of the graph of a direct proportion.

- 2.4** Use your data from the exploration to answer the following questions.
- a. Is the mean radius of an oil spill directly proportional to the volume of the spill?
 - b. Is the square of the mean radius directly proportional to the volume?

- 2.5** When oil is spilled on a sheet of absorbent paper, the spill can be modeled by a cylinder. In this situation, the base of the cylinder is the shape of the spill, while the height of the cylinder is the thickness of the paper.

The volume of a cylinder is determined by multiplying the area of the base by the height. Is the volume of the spill directly proportional to the area covered by the spill? Explain your response.

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- 2.6** In a circle, the circumference (C) is directly proportional to the diameter (d). The constant of proportionality is π .
- a. Write an equation for this direct proportion.
 - b. Is the relationship for each of the following a direct proportion? If so, write an equation for the proportion and identify the constant of proportionality. If not, explain why not.
 1. the circumference of a circle and its radius
 2. the area of a circle and its radius

- 2.7** The relationship between temperature measured in degrees Fahrenheit (F) and temperature measured in degrees Celsius (C) is:

$$C = \frac{5}{9}F - \frac{160}{9}$$

Explain why this relationship is not a direct proportion.

- 2.8** Hailstones are formed when raindrops are caught in updrafts and carried into high clouds containing very cold air. The radius of a hailstone is directly proportional to the amount of time it remains in the high cloud.
- After remaining in a high cloud for 10 sec, a hailstone has a radius of about 1.3 cm. What was the radius of the hailstone after 1 sec?
 - Write an equation for this direct proportion.
 - How long would a hailstone have to remain in high clouds to reach a radius greater than 2.5 cm?
- 2.9** In 1980, per capita personal income in the United States was \$9948. Ten years later, per capita income rose to \$18,699.
- What was the total increase in per capita personal income from 1980 to 1990?
 - Assume that the yearly increases in per capita income and the number of years after 1980 are directly proportional.
 - What was the yearly increase in per capita personal income between 1980 and 1990?
 - Write an equation for the direct proportion between yearly increases in per capita income and number of years after 1980.
 - Predict the per capita personal income in 1994.
 - Write an equation that could be used to model the per capita income for any year after 1980.
 - Does this equation define a direct proportion? Explain your response.
 - The actual per capita personal income in 1994 was \$21,846. What limitations might the model in Part **c** have for predicting per capita personal income?

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Activity 3

As soon as a quantity of oil is spilled, it starts to spread. If not contained, the resulting slick can cover a very large area. As the oil continues to spread, the depth of the slick decreases. In the following exploration, you investigate the relationship between the depth of a spill and the area it covers.

Exploration

When liquid is poured into a cylindrical container, the surface of the liquid takes the same shape as the base of the container. In this exploration, you use a fixed amount of water to represent the volume of an oil spill. The spread of the spill is simulated by pouring the water into several containers with different base areas.

Note: Save your work, including the spreadsheet, for use in the assignment.

- a. Obtain a cylindrical container from your teacher. Determine and record its base area (B) in square centimeters.
- b. Pour 200 mL of water into the container. Measure and record the height (h) of the water in centimeters.
- c. Calculate $B \cdot h$, where B is the base area and h is the height of the water in the container. Label the product with the appropriate units and record the result.
- d.
 1. Collect the class data for the different containers.
 2. Enter the data in a spreadsheet.
 3. Sort the data so that the base areas appear in ascending order (from least to greatest).
- e. Create a scatterplot of the height of the water versus the base area.

Discussion

- a.
 1. What should be true of each value for $B \cdot h$ calculated in Part c of the exploration?
 2. Do the class values support this conclusion? Explain your response.
- b. Describe the graph obtained in Part e of the exploration.
- c. Consider a right circular cylinder and a right triangular prism with the same base areas. Each contains an equal volume of water. Does the height of the water depend on the shapes of the bases?
- d.
 1. In the exploration, what happened to the height of the water as it was poured into containers with larger base areas?
 2. Is the height directly proportional to the area of the base? Explain your response.
- e. Do you think that examining the heights of liquid in a series of containers with increasing base areas provides a good model of the spread of an oil spill? Explain your response.

Assignment

- 3.1** In the exploration, you poured 200 mL of water into containers with different base areas.
- Let B represent the base area of a container and h represent the height of water in the container. Write an equation that describes the relationship of B and h to 200 mL.
 - Solve this equation for h .
 - Graph the equation on the same set of axes as the scatterplot from Part **e** of the exploration.
 - Which appears to be the better model of the experiment in the exploration—the scatterplot or the graph of the equation? Defend your choice.
 - Why can there be no negative values for h ?
- 3.2**
- Solve the equation you wrote in Problem **3.1a** for B .
 - Predict the area covered by an oil spill of 200 mL if it spreads to a thickness of $2.5 \cdot 10^{-3}$ cm.
 - If the spill is circular, what is its diameter?
- 3.3** Consider a spill of 100 bbl of highly refined oil on a calm body of water.
- Write an equation that models this spill in terms of B , h , and 100 bbl.
 - How many cubic centimeters of oil are there in 100 bbl? (There are approximately 160 L in 1 bbl; 1 cm^3 contains 1 mL.)
 - Rewrite your equation in Part **a** by replacing 100 bbl with its equivalent in cubic centimeters.
 - Determine the area (in square kilometers) covered by a 100-bbl spill that spreads to a thickness of $2.5 \cdot 10^{-3}$ cm.
 - If the spill is circular, what is its diameter?

- 3.4
- Solve the following equation for y : $20 = x \cdot y$.
 - Choose at least five different values for x . Find the corresponding y -values and organize these results in a table.
 - As the values of x increase, what happens to the corresponding values of y ? Is this consistent with what you observed in the exploration? Explain your response.

Mathematics Note

One quantity is **inversely proportional** to another when the product of the two quantities is constant. An inverse proportion can be described by an equation of the form $xy = k$, where k is the **constant of proportionality**. The equation of an inverse proportion can also be written in the form $y = k/x$.

For example, the inverse proportion $xy = 2$ can be written as $y = 2/x$. The graph of $y = 2/x$ is shown in Figure 5.

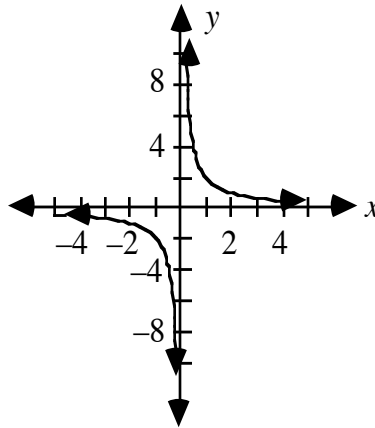


Figure 5: Graph of an inverse proportion

- 3.5
- Solve each of the following inverse proportions for y in terms of x and identify the constants of proportionality.
 - $x \cdot y = 20$
 - $x \cdot y = 10$
 - $y \cdot x = 5$
 - $0.5 = y \cdot x$
 - Using a graphing utility, graph the equations from Part **a** on the same set of axes, where $-10 \leq x \leq 10$. What appears to be true about each of the graphs?

- c. 1. As the x -values approach 0, what happens to the corresponding y -values?
- 2. As the x -values increase from 100 to 1000 to 10,000, what happens to the corresponding y -values?
- 3. As the x -values decrease from -100 to -1000 to $-10,000$, what happens to the corresponding y -values?
- d. How does the constant of proportionality affect the graph?
- e. What does the constant of proportionality represent in the equation you wrote in Problem 3.1?

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- 3.6 a. Compare the shapes of the graphs of a direct proportion and an inverse proportion.
- b. Compare the equations of a direct proportion and an inverse proportion.

3.7 Consider the set of ordered pairs shown in the following table.

x	y
1	100
2	50
4	25
5	20
10	10
20	5
25	4

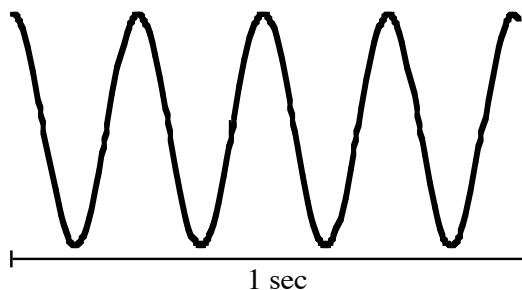
- a. Create a scatterplot of this data.
- b. As the values of x become large, what happens to the values of y ?
- c. As the values of x become small, what happens to the values of y ?
- d. Is the relationship between x and y a direct proportion, an inverse proportion, or neither? Justify your conclusion.
- e. Write an equation that describes this relationship and identify the constant of proportionality, if one exists.
- f. Describe a real-world situation that might generate these ordered pairs.

- 3.8** At 0 °C, 32.0 g of oxygen gas occupies a volume of 22.4 L with a pressure of 1.0 atmosphere (atm). As the volume is decreased at constant temperature, the pressure changes as shown in the following table.

Volume (L)	Pressure (atm)
22.4	1.00
17.1	1.31
11.2	2.00
5.60	4.00
2.24	10.0

- Create a scatterplot of the data.
 - At constant temperature, are pressure and volume directly proportional or inversely proportional? Explain your response.
 - Write an equation that describes the relationship and identify the constant of proportionality.
 - Graph this equation on the same set of axes as the scatterplot from Part **a**.
 - Predict the pressure on 32.0 g of oxygen gas when the volume is 20.5 L.
- 3.9** Two characteristics of waves are wavelength and frequency. **Wavelength** is the distance between two consecutive peaks or troughs in a wave. **Frequency** is the number of wavelengths that pass a given point in a certain amount of time.

The standard unit of frequency is the **hertz (Hz)**. One hertz is equal to 1 cycle per second. For example, the diagram below shows a wave with a frequency of 4 cycles per second, or 4 Hz.



The table below shows the wavelengths and frequencies of some different forms of electromagnetic radiation.

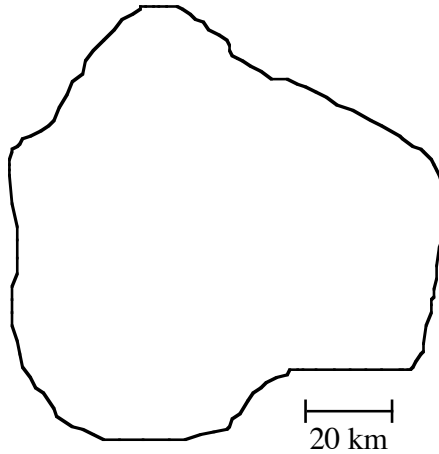
Electromagnetic Radiation	Wavelength (m)	Frequency (Hz)
gamma rays	1.0×10^{-12}	3.0×10^{20}
X rays	1.0×10^{-10}	3.0×10^{18}
red light	7.0×10^{-7}	4.29×10^{14}
microwaves	1.0×10^{-2}	3.0×10^{10}
radio waves	1.0×10^2	3.0×10^6

- Is the relationship between wavelength and frequency a direct proportion or an inverse proportion? Explain your response.
- Write an equation that describes this relationship.
- Identify the constant of proportionality.
- Violet light has a wavelength of about 4.0×10^{-7} m. Use your equation from Part **b** to determine the frequency of violet light.

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Summary Assessment

1. The diagram below shows the shape of an oil slick that spread to a thickness of $2.5 \cdot 10^{-3}$ cm.



Estimate the volume of oil in the spill in gallons.

2. a. Assuming that the spilled oil spreads to a thickness of $2.5 \cdot 10^{-3}$, complete the following table.

Area of Slick (km^2)	Volume of Spill (gal)
5000	
7500	
10,000	
25,000	

- b. Determine the type of relationship formed by the data collected in the table. Create a graph that displays this relationship.
- c. Find an equation that models the graph from Part **b**.
- d. Use this equation to predict the number of gallons of oil that would create a slick of $71,000 \text{ km}^2$.

3. Assuming constant pressure, the time required to fill an oil tank is inversely proportional to the square of the diameter of the hose used to fill it. The table below shows the diameters of four hoses and the corresponding times to fill the tank.

Diameter (cm)	Square of Diameter	Time (min)
2		36
3		16
4		9
6		4

- Complete the column for the squares of the diameters.
- Create a scatterplot of the time versus the square of the diameter.
- Find an equation that represents this inverse proportion.
- Predict how long it would take to fill the tank using a hose with a diameter of 10 cm.
- Predict the diameter of a pipe that could fill the tank in 30 min.

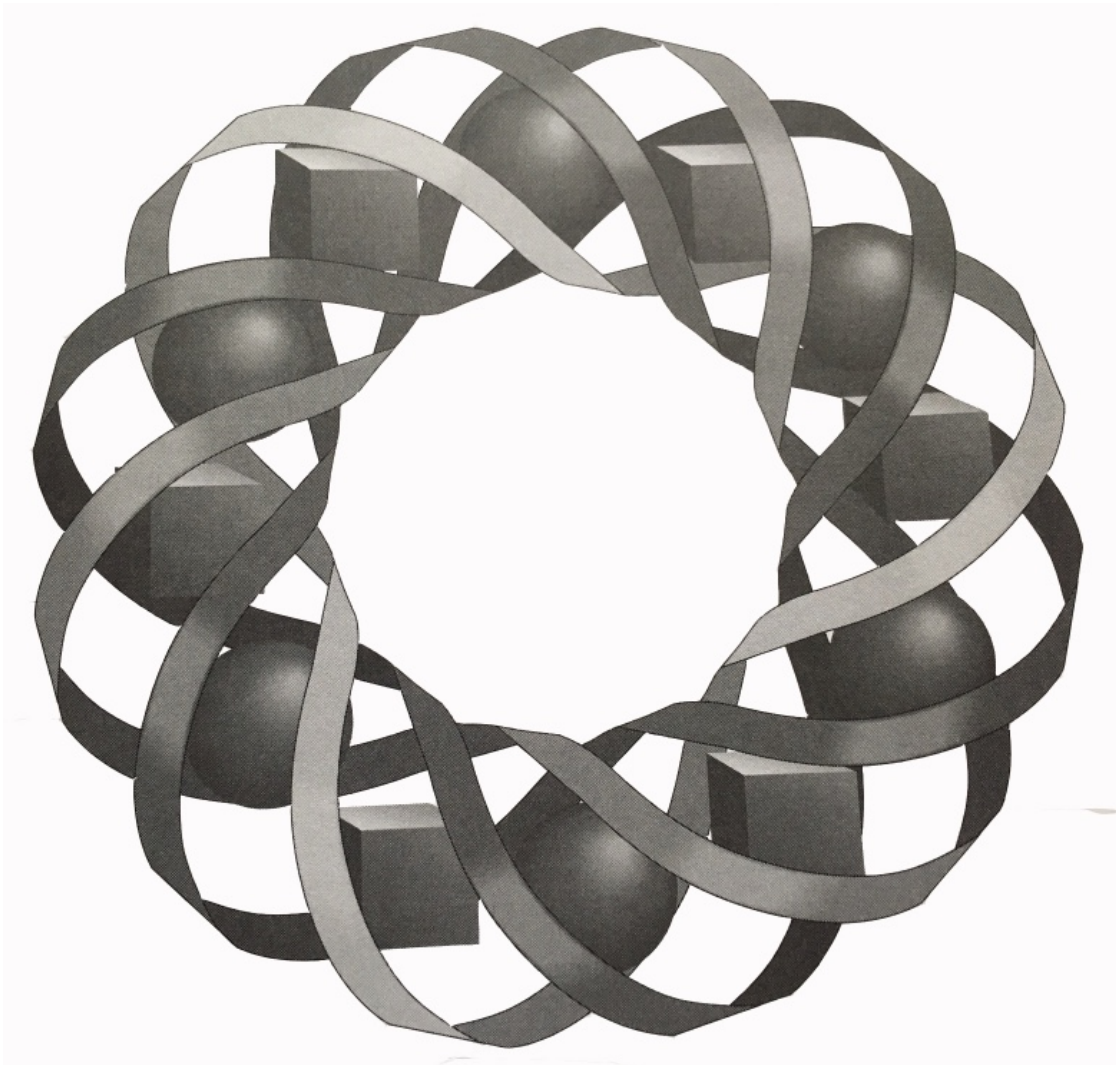
Module Summary

- A **simple closed curve** in a plane is a curve with no endpoints that does not intersect itself.
- A **cylinder** is a three-dimensional solid with bases that are congruent simple closed curves (non-polygons) in parallel planes.
- The volume (V) of a cylinder can be found by multiplying the area of its base (B) by its height (h): $V = B \cdot h$.
- One quantity is **directly proportional** to another when the ratio of the two quantities is constant (the same). The constant is the **constant of proportionality** and the ratio is a **direct proportion**.
- A direct proportion can be described by an equation of the form $y = mx$, where m is the constant of proportionality. The graph of a direct proportion always contains the origin since $y = 0$ whenever $x = 0$.
- One quantity is **inversely proportional** to another when the product of the two quantities is constant.
- An inverse proportion can be described by an equation of the form $xy = k$, where k is the **constant of proportionality**. The equation of an inverse proportion can also be written in the form $y = k/x$.

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I'm Not So Sure Anymore



Are you betting on the lottery to change your life? The odds of hitting that big jackpot are not as good as you might think. In this module, you'll explore the mathematical probability of waking up a winner.

Ted Dreith • Anne Merrifield • Dean Preble • Deanna Turley



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I'm Not So Sure Anymore

Introduction

Sarah looked at the clock. The drawing was five minutes away. She clutched the week's lottery ticket in both hands, trying to convince herself that it was a winner. She knew that the probability of winning the jackpot was extremely low. Still, she had hope. After all, she'd bought her ticket at the Yummy Mart, where three other winning tickets had been sold, and she'd chosen lucky numbers, numbers that had shown up in many previous drawings.

As the numbers appeared on the television screen, Sarah moved to the edge of her chair. Her first number matched. So did the second. Her heart raced. The next two numbers also matched. As the last number was being drawn, the lights flickered and the screen went blank. A power outage!

Was the power restored in time for Sarah to see the fifth number? Did she win? And what were the rules of the game she was playing? Although you can't answer all of these questions, you may be able to determine the probability that Sarah is holding the winning ticket.

Exploration

In this exploration, you investigate a simple lottery game in which the prizes are small. In the Apple Lottery, players win red, green, or yellow apples. A sample ticket is shown in Figure 1.

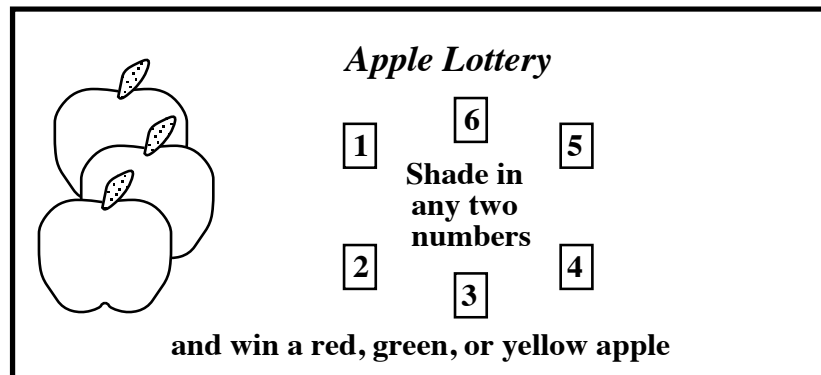


Figure 1: A ticket for the Apple Lottery

All lottery games have rules, often printed on the back of the ticket. Figure 2 shows the back of an Apple Lottery ticket.

<p>How to Play</p> <ul style="list-style-type: none">• Shade two different numbers from 1 to 6 on the front of the ticket. <p>How to Win</p> <p>Apple Lottery officials randomly draw two different numbers from 1 to 6.</p> <ul style="list-style-type: none">• Win a yellow apple by matching 2 numbers.• Win a green apple by matching 1 number.• Win a red apple by matching 0 numbers.

Figure 2: The back of an Apple Lottery ticket

- a. When numbers are chosen at random, there is no way to predict which numbers will be chosen. Suggest a method that Apple Lottery officials might use to draw two different numbers from 1 to 6 at random.
- b. In the Apple Lottery, there are three different **events**—winning a red apple, winning a green apple, and winning a yellow apple.

Predict the number of each type of apple you would win if you played the Apple Lottery 1000 times. Record your predictions.
- c. Pick two numbers from 1 to 6 that you would shade on an Apple Lottery ticket. Record these numbers.
- d. Recall that a simulation is a model of a real-world occurrence. The results of a simulation are often used to make predictions.

Use the following steps to simulate the Apple Lottery officials drawing two numbers.

 1. Place 6 objects in a container. Each object should be marked with a different number from 1 to 6.
 2. Shake the container. Without looking, draw two of the numbered objects.
- e. Compare the numbers drawn in Part **d** with the two numbers you picked in Part **c**. Record the type of apple won.
- f. Repeat Parts **d** and **e** 24 more times.

Mathematics Note

One way of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event can be calculated using the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

For example, suppose that you counted 60 heads in 100 trials of a coin toss. The experimental probability of obtaining a head on any one toss is:

$$\frac{60}{100} = \frac{3}{5}$$

- g.** Using your results from Part **f**, determine the experimental probability of each of the following events:
1. winning a red apple
 2. winning a yellow apple
 3. winning a green apple
 4. winning an apple of any color.
- h.** Combine your results from Part **f** with those of the rest of the class. Use the combined results to determine the experimental probability of each of the following events:
1. winning a red apple
 2. winning a yellow apple
 3. winning a green apple
 4. winning an apple of any color.
- i.**
1. Predict how many apples of each color you would win in 1000 games of the Apple Lottery.
 2. Compare your response to the prediction made in Part **b** of the exploration.

Discussion

- a. Compare the experimental probabilities found in Parts **g** and **h** of the exploration. Which do you believe give better estimates of the true chances of winning? Explain your response.
- b. Explain why the experimental probability of winning an apple in the Apple Lottery is 1.
- c.
 1. What other methods could you use to generate two random numbers for the Apple Lottery?
 2. Do these methods guarantee that two different numbers will be generated?
 3. What problem may occur when using a simulation that can generate identical numbers?
 4. How could you modify the simulation to ensure that the two numbers are different?
- d. What advantages might there be in using technology to simulate the Apple Lottery?

Activity 1

When using experimental probability to make predictions, a large number of trials provides a better estimate of the true likelihood of an event. In the following exploration, you use technology to help you simulate the results of many games of the Apple Lottery.

Exploration

- a. Select two numbers from 1 to 6 for a new Apple Lottery ticket. Record these numbers.
- b. When Apple Lottery officials draw two numbers, these may or may not match your numbers. The first column in Table **1** lists all the possible pairs of numbers in the Apple Lottery. This is the **sample space** for the lottery.

Make a copy of Table 1. For each possible pair of numbers, record the number of matching digits and the color of the apple you would win with your ticket.

Table 1: Apple Lottery sample space

Pair of Numbers	Number of Matching Digits	Color of Apple Won
1, 2		
1, 3		
1, 4		
1, 5		
1, 6		
2, 3		
2, 4		
2, 5		
2, 6		
3, 4		
3, 5		
3, 6		
4, 5		
4, 6		
5, 6		

- c. Use technology to simulate the Apple Lottery by completing the following steps.
 1. Randomly generate the first number.
 2. Randomly generate the second number.
 3. If the second number is the same as the first, generate another number. Repeat until you obtain a number different from the first.
- d. Use Table 1 to determine which apple you won.
- e. Repeat Parts c and d 99 more times, recording the number of times you won each color of apple.
- f. Use the results of your 100 trials to determine the experimental probability of each of the following events:
 1. winning a red apple
 2. winning a yellow apple
 3. winning a green apple

Mathematics Note

The set of all possible outcomes for an experiment is the **sample space**.

An **event** is a subset of the sample space.

If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

For example, the sample space for tossing two fair coins can be represented as $\{HH, TH, HT, TT\}$, where H stands for head and T stands for tail. The event of getting one tail when tossing two coins consists of 2 outcomes: TH and HT. Because the total number of outcomes in the sample space is 4, the theoretical probability of getting one tail is:

$$\frac{2}{4} = \frac{1}{2}$$

- g.** Use the information in Table 1 to determine the theoretical probability of each of the following events:
1. winning a red apple
 2. winning a yellow apple
 3. winning a green apple
 4. winning an apple of any color

Note: Save your results for use in Problem 3.1.

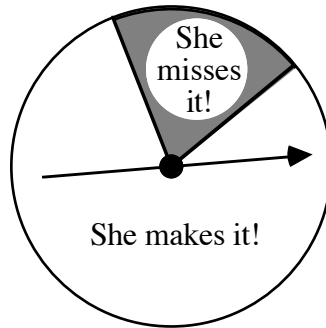
Discussion

- a.** Compare the theoretical probabilities of the events in the Apple Lottery with the experimental probabilities you determined in Part **f** of the exploration.
- b.** Using the combined class results for Parts **c–e** of the exploration, determine the experimental probability of each of the following events:
 1. winning a red apple
 2. winning a yellow apple
 3. winning a green apple
- c.** Compare the experimental probabilities determined using the class results with their corresponding theoretical probabilities.
- d.** How does the pair of numbers you selected for your Apple Lottery ticket affect the theoretical probability of winning each type of apple?

Assignment

- 1.1 Explain why picking a new ticket in the Apple Lottery does not change the theoretical probability of winning a particular type of apple.
- 1.2 Judging from the theoretical probabilities, how many apples of each color do you think you would win after playing the Apple Lottery 1000 times?
- 1.3 Time has expired at the divisional basketball championship. The game is tied and Charrette is at the foul line. During the season, she made 4 out of every 5 of her free throws.

One way to simulate this situation is to use a spinner, as shown in the diagram below.



- a. What does one spin represent in this simulation?
 - b. Use a pencil and a paper clip to simulate the arrow in the spinner. On a copy of the diagram above, spin the paper clip to determine whether Charrette makes or misses the free throw. If the paper clip lands on a boundary segment, spin again.
 - c. Record the results of 30 trials. Find the experimental probability of Charrette making the free throw.
 - d. Compare the experimental and theoretical probabilities of Charrette making the free throw.
- 1.4 Describe the sample space for each of the following situations.
- a. At the end of the school year, a student receives a letter grade for a science class.
 - b. Two ordinary dice are rolled and the numbers added. For example, if one die shows a 6 and the other shows a 2, the result is 8.

- 1.5** Camie has asked Alicia to play a game of cards. This game involves two piles of four cards each. The first pile contains the ace, king, queen, and jack of diamonds. The second pile contains the ace, king, queen, and jack of spades. The object of the game is to select a card from the first pile, then match it with a card from the second pile.
- Determine the sample space for this game.
 - What is the theoretical probability of obtaining a winning combination of cards?
 - Does this game seem fair? Explain your response.
 - If all eight cards were shuffled together in one pile, would the sample space and theoretical probabilities remain the same? Explain your response.

* * * * *

- 1.6** A football coach wants to know the probability of winning the coin toss in the next three games if the team captain calls heads each time.
- One player decides to use a simulation to predict the probability of this event. The table below shows the results of 250 trials of tossing three coins.

Three Heads	Two Heads	One Head	No Heads
29	109	91	21

Use these results to determine the experimental probability of getting three heads.

- Another player tells the coach that he can determine the probability of getting three heads using the following sample space:

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Use this sample space to determine the theoretical probability of getting three heads.

- Do the results of the simulation agree with the theoretical probability? Explain your response.

- 1.7** In another version of the Apple Lottery, players choose two different numbers from 1 to 5. After lottery officials randomly select two different numbers from 1 to 5, prizes are awarded as in the original Apple Lottery.
- a. Determine the sample space for this version of the Apple Lottery.
 - b. What is the theoretical probability of winning a red apple? a yellow apple? a green apple?
 - c. Devise a way to simulate playing this game 100 times.
 1. Describe your simulation.
 2. Run the simulation 100 times and record the results.
 3. Using this data, what is the experimental probability of winning a red apple? a yellow apple? a green apple?
 - d. Do the results of the simulation agree with the theoretical probabilities you calculated in Part **b**? Explain your response.
 - e. In which version of the Apple Lottery are you more likely to win a yellow apple? Explain your response.

* * * * *

Research Project

A new game called the Match Lottery has the same rules as the Apple Lottery, except that the second number drawn need not be different from the first. In the Match Lottery, for example, players may select the numbers 3, 3.

Write a report on the Match Lottery that includes the following:

- a. a description of the sample space
 - b. an explanation of how the difference in rules affects its simulation
 - c. the results of at least 100 trials in which the lottery ticket chosen has two different numbers
 - d. the results of at least 100 trials in which the lottery ticket chosen has two identical numbers
 - e. the experimental probabilities of winning each prize
 - f. the theoretical probabilities of winning each prize
 - g. an explanation of whether or not the results of the simulations support the theoretical probabilities
 - h. an explanation of whether or not any tickets have a better chance of winning a yellow apple than other tickets.
-

Activity 2

In order to calculate the probability of winning a lottery, you must determine the number of outcomes in the sample space. While the sample space for the Apple Lottery is relatively small, other lotteries may have more than a million possible outcomes.

Exploration

By counting the number of outcomes in small sample spaces, you may observe some patterns that can help you determine the size of large sample spaces.

In one simple lottery, for example, players select 1 number from the set $\{1, 2, 3\}$. The sample space for this lottery contains 3 singles:

1	2	3
---	---	---

If players must select 2 different numbers from the set $\{1, 2, 3\}$, the sample space contains 3 pairs:

1, 2	1, 3	2, 3
------	------	------

If players must select 3 different numbers from this set, the sample space contains 1 triple:

1, 2, 3

The numbers of outcomes in these sample spaces are recorded in the third row of Table 2 below. (Because this lottery only has three available numbers, there are no quadruples or quintuples.)

Table 2: Size of sample space for different lotteries

Numbers for the Lottery	No. of Singles	No. of Pairs	No. of Triples	No. of Quadruples	No. of Quintuples
$\{1\}$	1	0	0	0	0
$\{1, 2\}$	2	1	0	0	0
$\{1, 2, 3\}$	3	3	1	0	0
$\{1, 2, 3, 4\}$					
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$\{1, 2, 3, \dots, 50\}$	50				

- a. Create a spreadsheet with headings like those in Table 2.

- b. Consider a lottery in which players pick from the set $\{1, 2, 3, 4\}$. As you complete Steps 1–4, record your responses in the spreadsheet.
1. If players select 1 number from the set, how many singles are there in the sample space?
 2. If players select 2 different numbers from the set, how many pairs are there in the sample space?
 3. If players select 3 different numbers from the set, how many triples are there in the sample space?
 4. If players select 4 different numbers from the set, how many quadruples are there in the sample space?
- c. Continue to determine the size of the sample spaces for other sets of lottery numbers. As you work, look for patterns that will allow you to quickly fill in all the cells of the spreadsheet. **Note:** Save your completed table for use later in this module.

Discussion

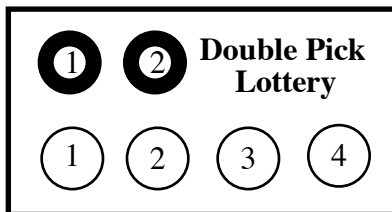
- a. What patterns do you observe in the spreadsheet?
- b. How did you complete the spreadsheet?
- c. What does the number in each cell represent?
- d. How would you use the spreadsheet to determine the theoretical probability of winning a lottery in which 3 different numbers are picked from the set $\{1, 2, 3, \dots, 20\}$?

Assignment

- 2.1 Using your spreadsheet, determine the number of quadruples you can select from 40 available numbers.
- 2.2 In one state lottery, players choose 4 different numbers from a set of 24. To win the jackpot, a player must match all 4 numbers.
- a. Use your spreadsheet to determine the probability of winning the jackpot with one ticket.
 - b. What is the probability of winning the jackpot with 10 different tickets for the same drawing?

- 2.3** In the Double Pick Lottery, players pick a number from 1 to 4 from a white panel and a number from 1 to 2 on a black panel. Lottery officials randomly draw one ball from a container of four white balls and one ball from a container of two black balls. To win a prize, players must match the numbers on both the white and black balls.

A ticket for the Double Pick Lottery is shown below.



- a. List the sample space for this game and determine the number of possible outcomes.
- b. Describe how to determine the size of the sample space using the spreadsheet created in the exploration.
- 2.4** Another popular lottery game involves picking five numbers from 1 to 35 on a white panel and one number from 1 to 35 on a black panel. Lottery officials draw five balls from a container of 35 white balls and one ball from a container of 35 black balls. To win the lottery, players must match all five white balls and the black ball.
- a. Determine the size of the sample space when selecting five numbers from a set of 35.
- b. Determine the size of the sample space for selecting one number from a set of 35.
- c. Using the method you described in Problem **2.3b**, determine the size of the sample space for this game.
- d. Determine the theoretical probability of matching the five white balls and one black ball.
- 2.5** Determine the number of different groups of 5 that there are in your math class.
- * * * * *
- 2.6** In Lottery A, officials randomly draw six numbers from the set $\{1, 2, 3, \dots, 20\}$. In Lottery B, officials randomly draw eight numbers from the set $\{1, 2, 3, \dots, 18\}$.
- Josh thinks that he is more likely to match all the numbers in Lottery B than in Lottery A because the set of available numbers is smaller. Do you agree with Josh? Explain your response.
- 2.7** A drawer contains an assortment of 5 different pairs of gloves (a total of 10 single gloves). If you randomly select two gloves from the drawer, what are your chances of getting a matching pair? Explain your response.

- 2.8** In one state lottery, players pick five numbers from the set $\{1, 2, 3, \dots, 45\}$ and one number from the set $\{1, 2, 3, \dots, 45\}$.
- Determine the size of the sample space for randomly selecting five numbers from the set $\{1, 2, 3, \dots, 45\}$.
 - Determine the size of the sample space for randomly selecting one number from the set $\{1, 2, 3, \dots, 45\}$.
 - Determine the size of the sample space for making both selections in Parts **a** and **b**.
 - Determine the theoretical probability that a player will pick the same six numbers as the lottery officials on a single ticket.
- 2.9** In another lottery game, officials randomly select four numbers from the set $\{1, 2, 3, \dots, 12\}$ and a fifth number from a different set. The probability that a player will pick the same five numbers as lottery officials on a single ticket is $1/2970$. How many numbers are there in the second set?

* * * * *

Activity 3

In many lotteries, the cost of playing is relatively small, while the potential winnings could be very large. Typically, the probability of winning a large prize with any one ticket is low. Will playing the game many times increase your chances of winning? In the following activity, you learn about expected lottery values.

Exploration

The Apple Lottery has decided to change its prizes. In the new version of the game, players that match neither of the two numbers win nothing, players that match exactly one of the numbers receive \$1.00, and players that match both of the numbers receive \$3.00.

The cost of a New Apple Lottery ticket is \$1.00. In the following exploration, you examine how much a player might expect to win at this game.

- Create a spreadsheet with headings like those in Table 3 below.

Table 3: Experimental results for New Apple Lottery

Event	Prize	No. of Wins	Total Winnings
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum	20	

- b. Play the New Apple Lottery 20 times. Determine the number of times you won each prize and enter your results in the appropriate column of the spreadsheet.
- c. Determine the total winnings for each row in the spreadsheet.
- d. Find the sum of the winnings for all three events.
- e. Calculate the mean amount won per game.
- f. Determine the experimental probability of winning each prize in the New Apple Lottery. Record these probabilities in a spreadsheet with headings like those in Table 4 below.

Table 4: Experimental probabilities for New Apple Lottery

Event	Prize	Experimental Probability	Expected Winnings
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum		

- g.
 1. Multiply the value of each prize by its experimental probability and enter the product in the expected winnings column of Table 4.
 2. Find the sum of the experimental probabilities and the sum of the expected winnings for the three events.
 3. Compare the sum of the expected winnings to the mean amount won per game calculated in Part e.

Discussion

- a. Compare the mean amount you won per game with others in the class.
- b. Why does the sum of the expected winnings determined in Part g equal the mean amount won per game?
- c. On average, how much do you think you would win or lose by playing the New Apple Lottery?

Mathematics Note

The mean value of an experiment is the **expected value**. Expected value can be calculated by adding the products of the value of each event and its corresponding theoretical probability.

For example, consider a game in which players predict heads or tails, then flip a coin. If the prediction matches the result of the coin toss, the player wins \$1.00. If the prediction does not match, the player wins \$0.00. The products of the value of each event and its corresponding theoretical probability are shown in Figure 3.

Event	Value	Theoretical Probability	Product
match	\$1.00	1/2	\$0.50
no match	\$0.00	1/2	\$0.00
	Sum	1	\$0.50

Figure 3: Expected value of a coin game

Since the sum of the products is \$0.50, the expected value of the game is \$0.50.

A **fair game** is one in which the expected value equals the cost of playing. For example, if you paid \$0.50 to play the coin game described above, the game would be mathematically fair.

- d. If the coin game described in the mathematics note cost \$1.00 to play, how might the prizes be changed to make it a fair game?
- e. Judging from your experimental results, do you believe that the New Apple Lottery is a fair game? Explain your response.
- f. Why do you think most lotteries are not fair games?

Assignment

- 3.1 In order to determine if the New Apple Lottery is a fair game, it must be analyzed using expected value.
 - a. Create and complete a spreadsheet with headings like those in the table below.

Event	Value	Theoretical Probability	Product
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum		

- b. Explain why the New Apple Lottery is not a fair game.
- c. Change the values of the prizes to make the lottery a fair game.
- d. Are there other prize values that make this a fair game? Explain your response. Hint: Use the spreadsheet to help you examine possible prize values.

- 3.2** In one state lottery, players select five numbers from 1 through 37. Tickets cost \$1.00 each. The table below shows the values of the prizes in this lottery.

No. of Matches	Prize	Probability of Winning with One Ticket
5	\$20,000.00	$\frac{1}{435,897}$
4	\$200.00	$\frac{160}{435,897}$
3	\$5.00	$\frac{4960}{435,897}$

- a. If a player buys one ticket, what is the expected value of the game? Describe how you determined your response.
- b. Can players make the lottery a fair game by buying more than one ticket? Explain your response.
- 3.3** In a carnival game, players pay 10 cents for one roll of a 20-sided die. Each side of the die shows a different number from 1 to 20. The number rolled is the value of the prize in cents.
- a. Find the theoretical probability of rolling each of the following:
1. 20
 2. 15
- b. Use expected value to determine if this is a fair game.
- 3.4** A basketball player has an 80% chance of making a free throw. Determine the number of shots you would expect the player to make in:
- a. 30 attempts
 - b. 500 attempts
 - c. n attempts.
- 3.5** Imagine that you have purchased one ticket for a benefit raffle. A total of 750 tickets have been sold at \$2 each. From these tickets, one winner will be chosen at random. The prize is worth \$300.
- a. Determine the probability that you will win the \$300 prize.
 - b. What is the expected value of this raffle for one ticket?
 - c. Is the raffle a fair game? Explain your response.

- 3.6** A typical roulette wheel has 38 compartments, each of which has an equal chance of being selected during one spin of the wheel.
- If a player wins \$30 for selecting the right compartment, what is the expected value of the game?
 - In order for this to be a fair game, how much should it cost to play? Explain your response.

* * * * *

- 3.7** As part of its annual fund drive, the local hospital sells 400 raffle tickets for \$5.00 each. From these tickets, one winner will be chosen at random to receive a prize of \$100.00. If you buy one ticket, what is the expected value for the raffle? Explain your response.

- 3.8** In the Red and Blue Lottery, players choose four numbers from 1 to 20 on a red panel and one number from 1 to 20 on a blue panel. Tickets cost \$1.00 each. The table below shows the probability of winning the two smaller prizes in the game.

Number of Matches	Prize	Probability of Winning with One Ticket
four red, one blue		$\frac{1}{96,900}$
three red, one blue	\$200.00	$\frac{16}{24,225}$
two red, one blue	\$5.00	$\frac{12}{1615}$

- Describe how to determine the probability of matching four numbers on the red panel and one number on the blue panel.
- If the expected value for one play is approximately \$0.70, what is the value of the prize for matching four red numbers and one blue number?
- If the lottery commission sells 100,000 tickets for one game, how much money can it expect to make?

* * * * *

Summary Assessment

Your state legislature has decided to start a new lottery. Earnings from the game will fund the construction of an amusement park. In order to attract an innovative and appealing design, the governor has agreed to pay 10% of all lottery profits to the creator of the new game.

Design a new lottery for your state. Your proposal to the state gaming commission should include:

- a description of how to play the lottery and the cost to play (you may wish to include a sketch of a sample lottery ticket)
- a list of prizes and a description of how each prize is won
- the theoretical probability of winning each prize
- the expected value of the lottery for one ticket and for 1 million tickets
- the amount you expect to earn for one ticket and for 1 million tickets
- a summary of the experimental results obtained from a simulation of your lottery.

Show all calculations and explain how your experimental results support the theoretical probabilities and expected values.

Module Summary

- One way of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event can be calculated using the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

- The set of all possible outcomes for an experiment is the **sample space**.
- An **event** is a subset of the sample space.
- If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

- The mean value of an experiment is the **expected value**. Expected value can be calculated by adding the products of the value of each event and its corresponding theoretical probability.
- A **fair game** is one in which the expected value equals the cost of playing.

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Are You Just A Small Giant?



How do you measure up against the world's tallest person? In this module, you'll examine some human—and inhuman—proportions and investigate the concept of similarity.

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Are You Just a Small Giant?

Introduction

According to the 1990 *Guinness Book of World Records*, Robert Wadlow (1918–1940) was the world’s tallest human. Wadlow’s measurements at the time of his death are listed in Table 1.

Table 1: Robert Wadlow’s measurements at death

Height	272 cm
Weight	1950 N
Mass	199 kg
Shoe Length	47 cm
Hand Length	32 cm
Arm Span	289 cm
Ring Size	25

How does your size and shape compare with Wadlow’s? If Robert Wadlow were alive today, would the two of you look similar? The answer to this question depends on the definition of *similar*.

Mathematics Note

Two ratios, a/b ($b \neq 0$) and c/d ($d \neq 0$), are **proportional**, or **in proportion**, if:

$$\frac{a}{b} = \frac{c}{d}$$

When two such ratios are proportional, it is also true that a/c and b/d are proportional, where $c \neq 0$ and $d \neq 0$.

In mathematics, two objects are **similar** if they have the same shape and the ratios of corresponding lengths are proportional. The ratio of corresponding sides is the **scale factor**.

For example, Figure 1 shows two similar triangles, ABC and DEF .

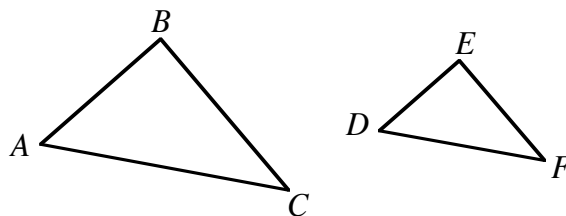


Figure 1: Two similar triangles

Since ABC and DEF are similar, the ratios of corresponding sides are proportional and equal to the scale factor. In this case, the scale factor is 1.5.

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 1.5$$

When two triangles are similar, the measures of the corresponding angles also are equal.

Discussion

- a. When a photograph is enlarged, is the new image similar to the original? Explain your response.
- b. Explain why a photographic image of a person is not similar to the actual person.
- c.
 1. How are scale factors used in scale drawings?
 2. Do scale drawings always produce similar figures?
- d. Describe the equation that results when the following proportion is solved for y .

$$\frac{y}{x} = \frac{a}{b}$$

- e. In the module “Oil: Black Gold,” you learned that a direct proportion can be described by a linear equation of the form $y = mx$, where m is the constant of proportionality.

Does the equation you described in Part **d** represent a direct proportion? If so, identify the constant of proportionality. If not, explain why not.

Activity 1

Are Robert Wadlow’s dimensions proportional to those of other people? In this activity, you compare Wadlow’s measurements at the time of his death with the measurements of other humans.

Exploration

In this exploration, you use proportions to investigate similarity in scale drawings. You also use proportions to determine if two people are similar.

Figure 2 shows a scale drawing of Robert Wadlow and his father.

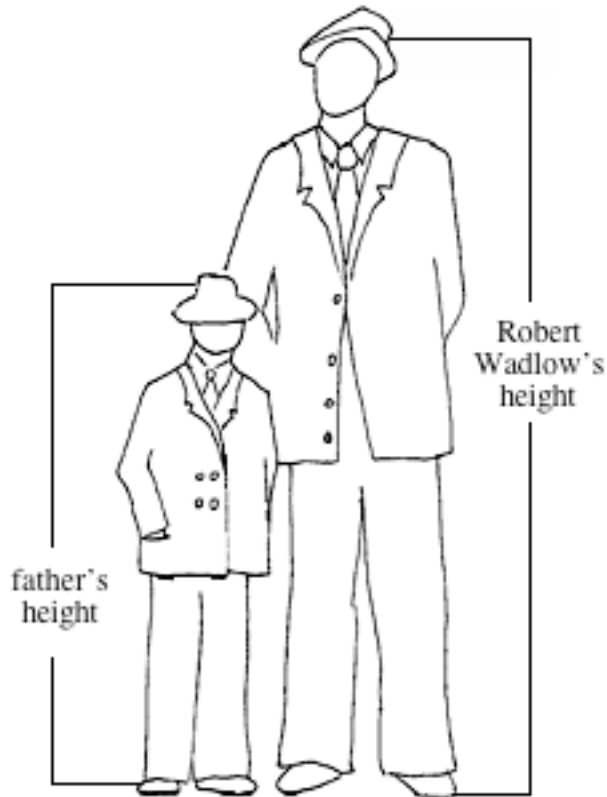


Figure 2: Scale drawing of Robert Wadlow and his father

- a. Obtain a second scale drawing of Robert Wadlow and his father from your teacher.
 1. Measure Robert Wadlow's height, in centimeters, in both drawings.
 2. Determine the ratio of these two measurements.
- b.
 1. Measure the height, in centimeters, of Wadlow's father in both drawings.
 2. Determine the ratio of these two measurements.
- c.
 1. Do your results in Parts **a** and **b** indicate that the measurements in the two drawings are proportional?
 2. Are the two drawings similar? Explain your response.

- d. Provide additional evidence for (or against) your response to Part **c1** by taking a third measurement on each drawing and determining the ratio of the measurements.
- e.
 1. Write the ratio of Wadlow's actual height to the height of his image in Figure 2.
 2. Use this ratio and the height of Wadlow's father in Figure 2 to write a proportion that can be used to determine the father's actual height.
 3. Determine the father's actual height.
- f. Repeat Part **e** using the scale drawing on the template.
- g. Compare the height you determined in Part **f** with the height you determined in Part **e3**.
- h.
 1. Measure your height and shoe length in centimeters.
 2. Use these measurements to determine if you are similar to Robert Wadlow at the time of his death.
- i.
 1. Use your height and shoe length and the shoe length of a classmate to predict the height of that classmate.
 2. Compare the predicted height with your classmate's actual height.
 3. What do your results indicate about the two of you?

Discussion

- a. Can you use your measurements from Part **h** of the exploration to make predictions about the measurements of other people? Explain your response.
- b. If two polygons are similar, their corresponding sides are proportional. What is the relationship between their corresponding angles?
- c. In the equation below, why can't b and d be equal to 0?

$$\frac{a}{b} = \frac{c}{d}$$
- d. Describe how a photocopier that enlarges or reduces preserves similarity using the following terms: *proportional*, *similar*, and *scale factor*.

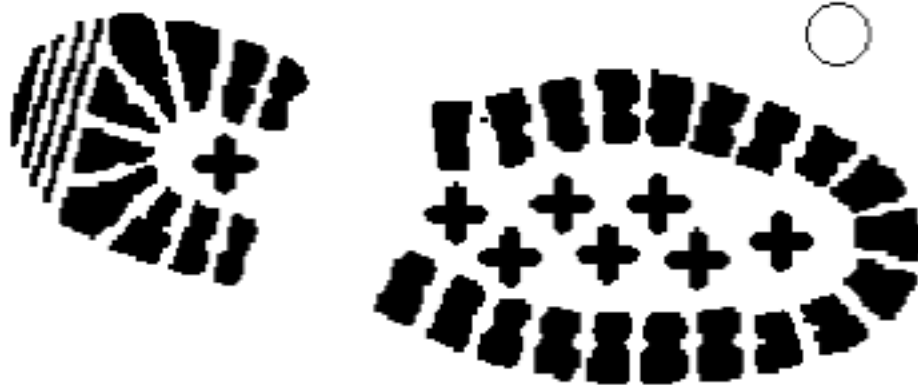
Assignment

- 1.1** Use Robert Wadlow's shoe length and height to estimate each of the following:
- the height of a similar person with a shoe length of 40 cm
 - the shoe length of a similar person with a height of 165 cm.
- 1.2**
- A family of five all have proportional shoe lengths and heights. The shoe length and height, in centimeters, of one member of the family can be written as the ordered pair (21,126). Use these dimensions to complete the ordered pairs for the rest of the family.
 - brother: (12, __)
 - mother: (24, __)
 - sister: (__, 108)
 - father: (__, 216)
 - In the following table, the ratios of corresponding lengths are expressed in the form "row:column." Use the ordered pairs from Part **a** to complete a copy of the table.

Family Member	father	mother	brother	sister
father	1:1	3:2		
mother				
brother			1:1	2:3
sister				

- 1.3** A person similar to you has a thumb length of 6.5 cm. Use your thumb length and height to estimate this person's height.
- 1.4** A newborn baby is 46 cm long, with a head circumference of 33 cm. Are you similar to this baby? Explain your response.

- 1.5 During a criminal investigation, a detective photographed the print of a suspect's shoe next to the outline of a penny, as shown below.



- Use this photograph to determine the length of the footprint.
- Predict the suspect's height if the suspect is similar to Robert Wadlow.
- What other useful predictions might the detective make from this footprint?

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- 1.6 A basketball player's height is 2.5 m. Assuming that the player's shape is similar to yours and that height is proportional to wrist circumference, determine the circumference of the player's wrist.
- 1.7
- Using a word-processing program, Theo can keyboard 2000 words in 50 min. He has 90 min available to enter a 3500-word essay in the computer. Will he be able to finish in time?
 - To check the spelling in a document, the word processor requires 3 min per 1000 words. How much time will it take to check the spelling in Theo's essay?
 - Will Theo need more than 90 min to finish his essay if he also plans to check the spelling? Explain your response.
- 1.8 When an object is caught in a whirlpool, its speed is inversely proportional to its distance from the whirlpool's center. If an object's speed is 7.8 cm/sec at a distance of 900 cm from the center, what is its speed at a distance of 10 cm from the center?

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Activity 2

In Activity 1, you used scale factors to predict lengths in similar figures. In this activity, scale factors will be used to predict areas of similar figures.

Exploration

All squares are similar. How is the scale factor for two squares related to their areas? To answer this question, complete Parts **a–e**.

- a. Table 2 lists the side lengths of nine different squares. Draw each of these squares on a sheet of centimeter graph paper.

Table 2: Side lengths of squares

Square	Side Length (cm)
A	1.0
B	2.0
C	3.0
D	4.0
E	5.0
F	6.0
G	1.5
H	2.5
I	1.2

- b. How do you think scale factors might be used to predict the areas of these squares?
- c. Complete Table 3 using the formula for the area of a square.

Table 3: Side lengths of squares

Square	Side Length (cm)	Area (cm ²)
A	1	
B	2	
C	3	
D	4	
E	5	
F	6	
G	1.5	
H	2.5	
I	1.2	
J	z	
K	y	

- d. Use the information in Table 3 to complete Table 4.

Table 4: Scale factors and ratios of areas of squares

Squares	Ratio of Side Lengths (Scale Factor)	Ratio of Areas
A to B		
E to A		
B to D		
E to C		
I to A		
G to A		
J to F		
J to K		

Mathematics Note

To raise a fraction to a power n , where n is a non-negative integer, both the numerator and the denominator may be raised to the indicated power. In general,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

where $b \neq 0$.

For example, the fraction $3/4$ can be raised to the third power as follows:

$$\left(\frac{3}{4}\right)^3 = \frac{3^3}{4^3} = \frac{27}{64}$$

- e. Use the information in Table 4 to describe the relationship between the scale factor for two squares and the ratio of their areas.
- f. Predict whether the relationship between scale factor and area for squares is also true for similar triangles.

- g. To test your prediction in Part f, complete Steps 1–4 below.
- Using a geometry utility, construct a triangle ABC .
 - Connect the midpoints of the sides of triangle ABC to form triangle IGH . Your construction should now resemble the one shown in Figure 3.

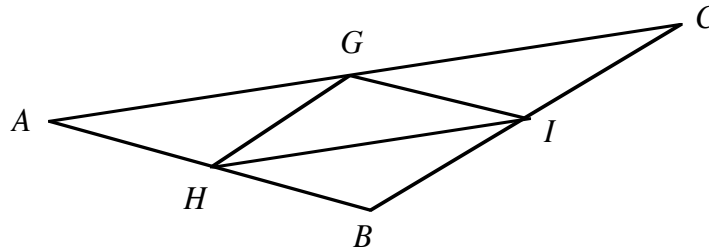


Figure 3: Triangles ABC and IGH

- Using the lengths of corresponding sides and the measures of corresponding angles, prove that triangles ABC and IGH are similar.
- Determine the relationship between the scale factor of the two triangles and the ratio of their areas.

Discussion

- Why are all squares similar?
- When the side length of a square is doubled, what happens to the area of the square?
- When the side lengths of a triangle are tripled, what happens to the area of the triangle?

Mathematics Note

A **square root** of a non-negative number a is a number s such that $s^2 = a$.

For example, since $5^2 = 25$, the number 5 is a square root of 25. Because $(-5)^2 = 25$, the number -5 is also a square root of 25.

The positive square root of a number is its **principal square root**. The principal square root of a is usually denoted by \sqrt{a} , although it may also be written as $^2\sqrt{a}$. For example, $^2\sqrt{25} = \sqrt{25} = 5$.

In general, an n th root of a non-negative number a is a number s such that $s^n = a$. The non-negative n th root of a is denoted as $^n\sqrt{a}$.

The n th root of a fraction can be found by taking the n th root of the numerator and dividing it by the n th root of the denominator. In general,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

where $b \neq 0$.

For example, the principal square root of $9/16$ can be found as follows:

$$\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{\sqrt{9}}{4} = \frac{3}{4}$$

- d. If the area of a square is $a \text{ cm}^2$, what is the side length of the square?
- e. Consider two squares, one with an area of 49 cm^2 and another with an area of 1 cm^2 . What is the scale factor for these squares?
- f. In Table 4, you recorded the ratio of the area of square J to the area of square K. What is the square root of this ratio?
- g. Given the area of square A and the ratio of the side length of square A to the side length of square B, how could you determine the area of square B?
- h. Do you think that the relationship between scale factor and the ratio of areas discovered in the exploration is true for all similar figures? Explain your response.

Mathematics Note

The ratio of the areas of two similar objects is the square of the ratio of the lengths of corresponding sides (the scale factor).

For example, if the scale factor for two similar objects is $4/5$, the ratio of their areas is:

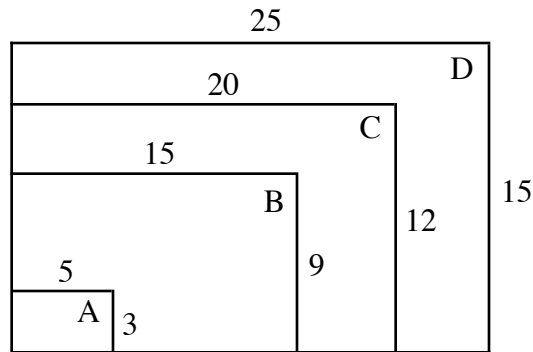
$$\left(\frac{4}{5}\right)^2 = \frac{4^2}{5^2} = \frac{16}{25}$$

Assignment

- 2.1** A figure is enlarged until its area is 225 times the area of the original figure. By what number was each length in the original figure multiplied?
- 2.2** A circle with a radius of 3 cm is enlarged by a scale factor of 10. How many times the area of the original circle is the area of the larger circle?
- 2.3** A rectangle with an area of 80 m^2 is enlarged by a scale factor of 3. Find the area of the larger rectangle.
- 2.4** Zino's Pizzeria charges \$11.30 for a pizza with a diameter of 30 cm and \$18.95 for a pizza with a diameter of 41 cm.
- The two pizzas are similar. Determine the scale factor when comparing the larger pizza to the smaller one.
 - Find the ratio of their areas.
 - Use the ratio of the areas and the price of the smaller pizza to determine a corresponding price for the larger pizza.
 - Decide which pizza is the better buy and explain your reasoning.
- 2.5** A professional basketball player 215 cm tall has a footprint with an area of 468 cm^2 . This basketball player and Nelson have similar bodies. If the area of Nelson's footprint is 325 cm^2 , how tall is Nelson?
- 2.6** The size of a television screen is typically reported in terms of the length of its diagonal. For example, if a store advertises a 50-cm screen, this means that the length of the screen's diagonal is 50 cm.
- What is the ratio of the area of a 63-cm screen to the area of a 33-cm screen?
 - What is the length of the diagonal for a screen with twice the area of a 33-cm screen?
- 2.7**
- Determine the area of a right triangle with sides that measure 3 cm, 4 cm, and 5 cm.
 - Use your answer to Part **a** to predict the area of a right triangle with sides that measure 18 cm, 24 cm, and 30 cm.
 - Verify your prediction using the formula for the area of a triangle.

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- 2.8** Andreas and Jonalynn installed new carpeting in two of their bedrooms. The floors in both rooms are rectangles. The smaller of the two rooms is 3.2 m wide and 4.1 m long. The ratio of the widths of the two rooms is 1.2, while the ratio of the lengths is 1.6.
- Find the length and width of the larger room.
 - Can the ratios of the widths and the lengths of the two rooms be used to determine the ratio of the areas? Explain your response.
 - Find the areas of both rooms.
 - What is the ratio of the areas of the two rooms? Does this ratio verify your response to Part **b**? Why or why not?
- 2.9** Damion's Turf Service charges \$50.00 to clean a section of artificial turf 10 yd wide and 20 yd long. Excluding the end zones, a football field is 50 yd wide and 100 yd long. How much should the company charge to clean a football field? Explain your response.
- 2.10** The figure below shows four rectangles and their dimensions in centimeters.



- Compare the perimeters and areas of these rectangles.
- Identify the rectangles that are similar and explain how you determined your response.

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Activity 3

In previous activities, you used scale factors to predict lengths and areas for similar figures. In this activity, scale factors are used to predict volumes of similar objects.

Exploration

- a. 1. Using a set of unit cubes, create a cube with an edge length of 2 units. Determine the volume of this cube and record it in the appropriate cell of Table 5.

Table 5: Edge length and volume of cubes

Cube	Edge Length	Volume
A	1	1
B	2	
C		27
N	n	

2. Determine the edge length of a cube with a volume of 27 units³ and record it in the appropriate cell of Table 5.
3. Determine the volume of a cube with an edge length of n and record it in the appropriate cell of Table 5.
- b. 1. Use the information in Table 5 to complete Table 6.

Table 6: Ratios of edge lengths and volumes

Cubes	Ratio of Edge Lengths (scale factor)	Ratio of Volumes
B to A	$2/1$	
B to C	$2/3$	
C to A		$27/1$
	a/b	

2. Describe the relationship between the ratio of the edge lengths (scale factor) for two cubes and the ratio of their volumes.

Mathematics Note

A **cube root** of a number a is a number b such that $b^3 = a$. Since $2^3 = 8$, for example, 2 is the cube root of 8.

The cube root of a is denoted by $\sqrt[3]{a}$. For example, $\sqrt[3]{-8} = -2$ and

$$\sqrt[3]{\frac{64}{125}} = \frac{\sqrt[3]{64}}{\sqrt[3]{125}} = \frac{4}{5}$$

Discussion

- a.
 1. What is the ratio of volumes for two cubes with volumes of 125 cm^3 and 8 cm^3 ?
 2. What is the ratio of the edge lengths for these cubes?
- b. In general, what is the relationship between the scale factor and the ratio of volumes for two cubes?
- c. If a cube is enlarged until its volume is 64 times the original volume, by what scale factor has each edge length been multiplied?
- d. If the volume of a cube is $d \text{ cm}^3$, what is the length of one edge?
- e. Do you think that the relationship you described in Part **b** is true for all similar figures? Explain your response.

Mathematics Note

If the scale factor of two similar figures is a/b , then the ratio of their volumes is:

$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

For example, Figure 4 shows two cubes with edge lengths of 2 cm and 3 cm, respectively.

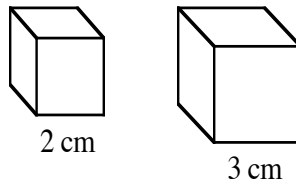


Figure 4: Two cubes

Since the ratio of the edge lengths (scale factor) is $2/3$, the ratio of the volumes is:

$$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$$

Assignment

- 3.1** Consider two spheres: the smaller has a volume of 1 m^3 , the larger has a volume of 343 m^3 . What is the scale factor for these spheres?
- 3.2** Two eggs are similar in shape. One egg is twice as long as the other. What is the ratio of their volumes?

Science Note

The **density** of a substance is the ratio of its mass to its volume. For example, the density of water is 1 g/cm^3 .

Since the density of any pure substance is a constant, the mass of the substance is directly proportional to its volume. In other words, as the volume increases, so does the mass.

- 3.3** A steel sphere with a diameter of 4 cm has a volume of 33.5 cm^3 . Its mass is 261 g.
- What is the density of the sphere?
 - What is the density of a steel sphere with a diameter of 6 cm?
 - What is the scale factor between the 6-cm sphere and the 4-cm sphere?
 - What is the ratio of the volume of the 6-cm sphere to the volume of the 4-cm sphere?
 - How does the ratio of their volumes compare to:
 - the ratio of their masses?
 - the scale factor?
- 3.4** The two similar fish shown below are drawn to scale. The smaller fish is 25 cm long and has a mass of 0.75 kg. Assuming that the densities of the two fish are the same, use scale factor to estimate the length and mass of the larger fish.



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- 3.5** A baker's favorite pie recipe calls for 6 cups of apples to make a 9-inch pie. How many cups of apples should he use to make a similar 8-inch pie?
- 3.6** The circumference of one egg is three times the circumference of another egg. If the two eggs are similar, what is the ratio of the mass of the larger egg to the mass of the smaller egg?

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Activity 4

In Activity 2, you discovered that when a square's side length is doubled, its area is quadrupled. In this activity, you examine the relationship between length and area from a graphical point of view.

Exploration

The area of a square is related to the length of its sides. The area of a shoe print is related to the length of the shoe. How are these two relationships similar? How are they different?

- a. Figure 5 shows a scale drawing of a child's shoe print. Each square in the drawing represents an area of 1 cm^2 . Use this drawing to estimate the actual length and area of the shoe print.

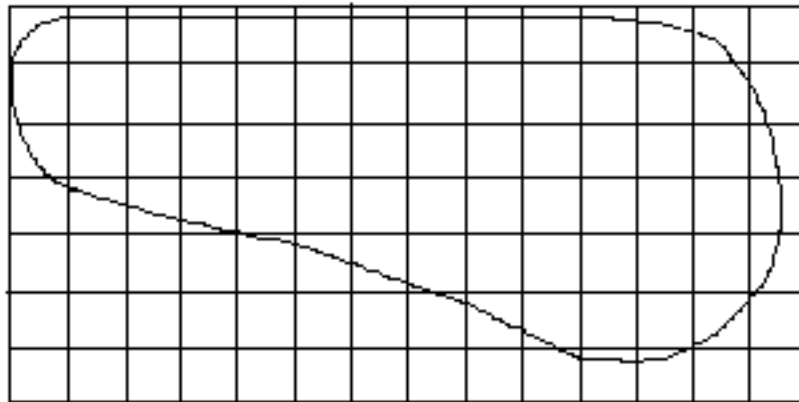


Figure 5: A child's shoe print

- b. Draw your own shoe print on centimeter graph paper. Measure its length and estimate its area.
- c.
1. Collect and organize this information for the entire class. To this data set, add the child's data from Part a as well as a shoe length of 0 cm and the corresponding area.
 2. Create a scatterplot of the class data. Let y represent area and x represent shoe length.
- d.
1. Graph the formula for the area of a square on the same coordinate system as the scatterplot in Part c. Let y represent area and x represent side length.
 2. Describe any similarities or differences you observe in the two graphs.

Mathematics Note

An equation of the form $y = ax^b$ is a **power equation**.

For example, the formula for the area of a square, $y = x^2$, is a power equation in which $a = 1$ and $b = 2$.

- e. Find an equation of the form $y = ax^2$ that models the scatterplot in Part c by varying the value of a until the graph of the equation reasonably approximates the data points.
- f. One way to determine if another power equation fits the data better than your equation from Part e is to compare the residuals.
 1. Enter the class data in a spreadsheet with the following headings:

Length (x)	Area (y)	Predicted Area	Absolute Value of Residual
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2. Use your model from Part e to determine the predicted area for each value of x .
 3. Determine the absolute value of each residual.
 4. Find the sum of the absolute values of the residuals.
- g. Vary the value of a in your model of the form $y = ax^2$ to determine the equation that minimizes the sum of the absolute values of the residuals. Record this equation.
 - h.
 1. Graph your equation from Part g on the same coordinate system as the scatterplot from Part c and print a copy of the resulting graph.
 2. Mark the point on the scatterplot that represents your shoe length and area.
 3. Mark the point on the curve that represents the ordered pair (x,y) , where x is your shoe length and y is the predicted area. Connect this point to the point in Step 2.

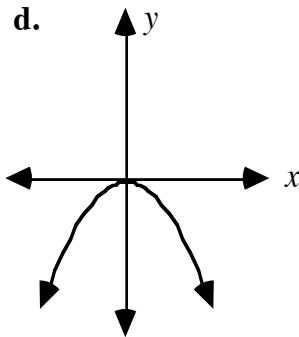
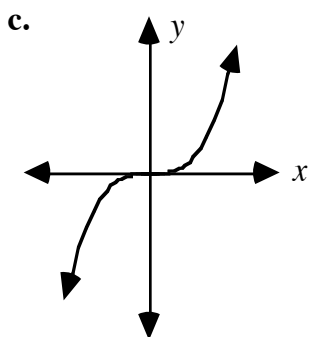
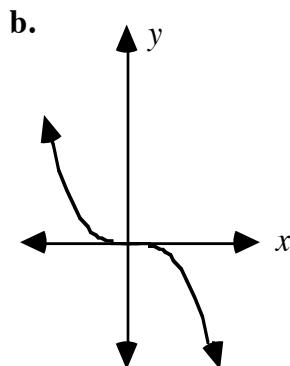
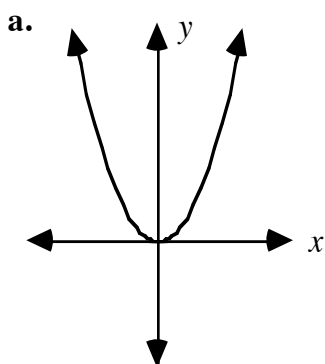
Discussion

- a. Describe the general relationship between shoe length and the area of a shoe print.
- b. Compare your equation in Part g, along with its corresponding sum of the absolute values of the residuals, with those obtained by others in the class.
- c. How well does your equation model the data for your own shoe length and area?

- d. How does the value of a affect the graph of the equation $y = ax^2$?
- e. If the lengths of two similar shoe prints have a scale factor of 3, what is the ratio of their areas?

Assignment

- 4.1** In the exploration, you graphed equations of the form $y = ax^b$ where $b = 2$ and both the domain and range were limited to positive numbers. To observe how different values of a and b affect the graph of $y = ax^b$, use a graphing utility to complete Parts **a** and **b** below.
- a. Graph at least three examples of each of the following. Let the domain be the set of real numbers from -10 to 10 .
 1. $y = ax^2$ where a is positive
 2. $y = ax^2$ where a is negative
 - b. Repeat Part **a** for $y = ax^3$.
 - c. Repeat Part **a** for $y = ax^b$ where b is greater than 3.
 - d. Describe how the values of a and b appear to affect the graph of $y = ax^b$.
- 4.2** Each of the following graphs was generated by an equation of the form $y = ax^b$. For each one, determine two equations with different values of b that produce graphs with roughly the same general shape.



- 4.3 Assuming that Robert Wadlow is similar to you, estimate the area of his shoe print.
- 4.4 Neva is 170 cm tall. Estimate her shoe length and the area of her shoe print if she is similar to you.
- 4.5 The table below shows some data on the length and mass of bird eggs.

Type of Egg	Length (cm)	Mass (g)
hummingbird	1.3	0.5
black swift	2.5	3.5
dove	3.18	6.41
partridge	3.0	8.7
Arctic tern	4.2	18
grebe	4.3	19.7
Louisiana egret	4.5	27.5
very small chicken	5.2	44.8
mallard duck	6.17	80
very large chicken	6.5	85
great black-backed gull	7.62	111
Canada goose	8.9	197
condor	11.0	270
ostrich	17.0	1400

- Make a scatterplot of this data. Let y represent mass in grams and x represent length in centimeters.
- Find an equation of the form $y = ax^3$ that models the data.
- Which type of egg fits the curve least well? Explain your response.
- The egg of a bald eagle is 7.3 cm long. Use your equation to predict the mass of this egg.
- Write a paragraph summarizing the relationship between the length and mass of bird eggs.

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- 4.6 The table below shows the data collected as an object fell through the air.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.00	0.000	0.35	0.539
0.05	0.000	0.40	0.717
0.10	0.017	0.45	0.920
0.15	0.072	0.50	1.144
0.20	0.152	0.55	1.393
0.25	0.256	0.60	1.668
0.30	0.387	0.65	1.889

- a. Create a scatterplot of this data. Let y represent distance in meters and x represent time in seconds.
- b. Use the process described in Parts **e–g** of the exploration to find an equation that models the data.
- c. Graph this equation on the same coordinate system as in Part **a**.
- d. Use your model to predict how far the object will fall in 2 sec.

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Activity 5

When you stand upright, the soles of your feet place a certain amount of pressure on the ground. In this activity, you explore the differences in pressure created by wearing shoes with flat soles, shoes with small heels, or no shoes at all.

Science Note

Although the kilogram is often referred to as a metric unit of weight, it is actually a unit of mass. Weight is a force determined by gravity. One metric unit of force is the **newton (N)**. On the surface of the earth, the weight of an object in newtons is its mass in kilograms multiplied by 9.8 m/sec^2 (the acceleration due to gravity).

For example, the weight in newtons of a 70-kg person can be calculated as follows:

$$70 \text{ kg} \cdot \frac{9.8 \text{ m}}{\text{sec}^2} = \frac{686 \text{ kg} \cdot \text{m}}{\text{sec}^2} = 686 \text{ N}$$

Exploration

- a.
 - 1. Place your shoeless foot on a sheet of centimeter graph paper and trace around it. Use the tracing to estimate the area of your footprint without shoes.
 - 2. Estimate the area of your shoe print when wearing shoes with flat soles. Assume that the bottom of your shoe makes complete contact with the ground.
 - 3. Estimate the area of your shoe print when wearing a shoe with small heels.
- b. Compare the three areas determined in Part **a**.
- c. Determine your weight in newtons.

- d. Find the pressure, in newtons per square centimeter, that your feet place on the ground in each of the following situations. Hint: Since your weight is distributed over both feet, divide this weight by the area of two footprints.
1. while not wearing shoes
 2. while wearing shoes with flat soles
 3. while wearing shoes with small heels
- e. Compare the three pressures determined in Part d.

Discussion

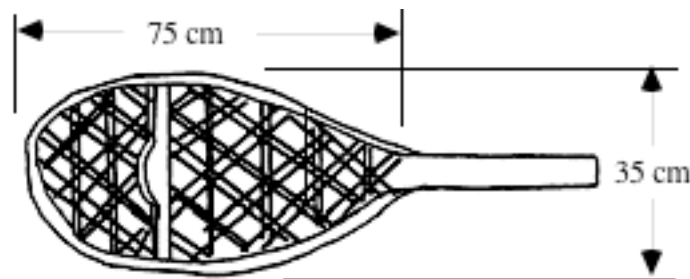
- a. When designing shoes for specific purposes, manufacturers often consider the amount of pressure that the foot places on the ground. For example, the sole on a running shoe typically has a greater area than the sole on a casual shoe.

Describe the purposes of some different types of shoes and the approximate area of the sole for each type.

- b. Why do you think that high heels (and other shoes with small heel areas) are banned in some buildings?
- c. Why does the frame of a bed with a mattress and box spring differ from the frame of a waterbed?

Assignment

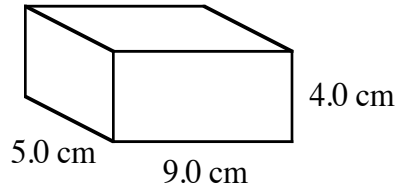
- 5.1 Walking involves the smooth transfer of weight from one foot to the other. At the beginning of each step, about half your weight rests on the heel of the forward shoe.
- a. Arlis wears shoes with a heel area of 0.75 cm^2 each. Sketch the print of one heel.
 - b. Arlis weighs 500 N. Estimate the pressure that the heel of her forward shoe places on the ground at the beginning of each step.
- 5.2 The diagram below shows the shape and dimensions of a typical snowshoe. Use this diagram to explain why snowshoes make it easier to stand or walk on snow.



- 5.3 Assume that you and Robert Wadlow are similar. Determine the pressure he places on the ground when standing upright. How does this amount differ from the pressure you place on the ground?

* * * * *

- 5.4 The diagram below shows a block of lead in the shape of a rectangular prism. The density of lead is 11.34 g/cm^3 .



- Calculate the pressure, in newtons per square centimeter, that the block exerts on the floor.
 - Suppose that the lead block is balanced on top of a wooden cube with an edge length of 2.5 cm. Assuming that the cube adds no significant mass to this situation, determine the pressure that the combination of block and cube exerts on the floor.
 - Explain any differences you observe in the pressures calculated in Parts **a** and **b**.
- 5.5 When placed upright on its base, a cylinder exerts a pressure of 0.50 N/cm^2 on the floor. The radius of the base is 35 cm. What is the mass of the cylinder?
- 5.6 Gravity on the planet Mars is about $1/3$ the gravity on Earth. How much pressure would you place on the Martian surface when standing upright?

* * * * *

Activity 6

The **femur**, or thigh bone, extends from the hip to the knee. In humans, two femurs (one in each leg) support the weight of the body. A horse, however, has four related bones to bear its weight. Figure 6 shows scale drawings of both a human femur and the femur of a horse.

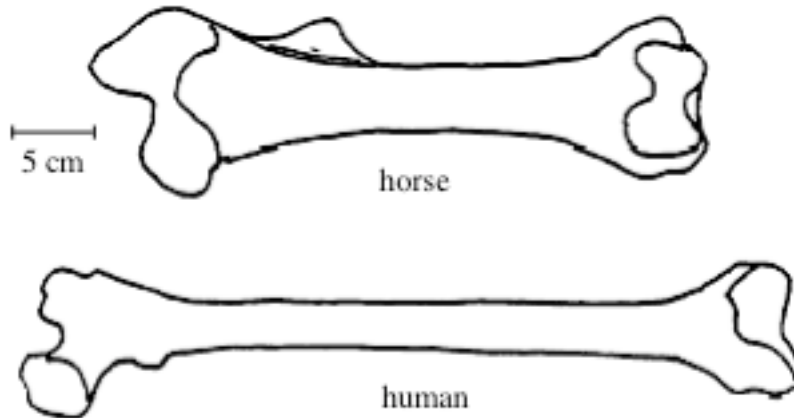


Figure 6: Two femurs

How much weight can these bones support without breaking? In this activity, you build two model femurs and use them to investigate one limit to biological growth.

Exploration

- a. Use paper cylinders to model each femur in Figure 6. The diameter of each cylinder should equal the smallest diameter of each bone.
Write the corresponding length, diameter, and circumference on each model. Compare the two models for similarity and record your observations.
- b. As shown in Figure 7, a **cross section** is the surface found by slicing an object perpendicular to its length.

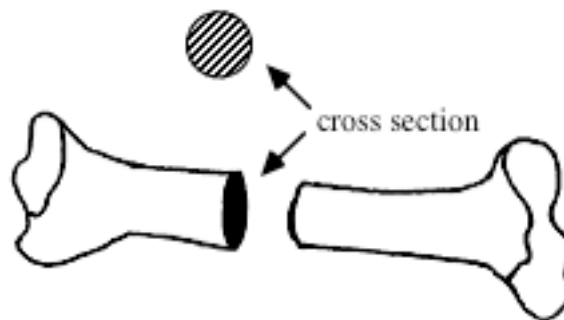


Figure 7: A cross section of bone

1. Using centimeter graph paper, draw an accurate cross section of each femur. Estimate the area of each cross section in square centimeters.
2. Calculate the cross-sectional area of each femur and compare these values with your estimates.
3. Write the calculated cross-sectional area on each model.
4. The cross-sectional area of a bone is a good indicator of relative

strength. Compare the strength of the femurs shown in Figure 6.

- c. The person whose femur appears in Figure 6 was 180 cm tall and weighed 600 N. Imagine a similar person who is twice as tall (a giant).
1. Use a strip of paper to make a cross-sectional model of the giant's femur, as shown in Figure 8.

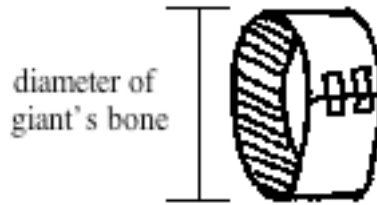


Figure 8: Paper model of cross section of bone

2. Write the diameter, circumference, and cross-sectional area on your model.
 3. Would the giant's bone have to support the same weight per square unit as the bone of the person who weighed 600 N? Explain your response.
 4. Write a paragraph describing the femurs of the 600-N human and the giant, including the dimensions of the bones and the pressures on the cross-sectional areas.
 5. The relationship between volume and area creates a tension on bone structure that affects the biological growth of animals. How might this tension affect the giant?
- d. A vertical compression test measures how much pressure a bone can withstand before crushing. In one test, a human femur withstood a pressure of approximately 1200 N/cm^2 .

Considering the results of this test, do you think that there is a limit on the size of humans? Explain your response.

- e. To investigate this question further, create a spreadsheet with columns as in Table 7. Complete the spreadsheet using the data from Parts a and b of this exploration.

Table 7: Scale factor and human body structure

Scale Factor	Body Height (cm)	Femur Diameter (cm)	Cross-sectional Area of Femur (cm^2)	Body Weight (N)	Pressure on Femur (N/cm^2)
1	180			600	
2					
3					
⋮					
30					

- f. Considering the information in your spreadsheet, do you think that there is a limit on the size of humans? Explain your response.
- g. Use the data in your spreadsheet to create the following:
 1. a scatterplot of femur diameter versus scale factor
 2. a scatterplot of the cross-sectional area of a femur versus scale factor
 3. a scatterplot of body weight versus scale factor.
- h.
 1. Predict the shape of a scatterplot of pressure on the femur versus scale factor.
 2. Create a scatterplot of pressure on the femur versus scale factor. How does the shape of the graph compare with your prediction?

Discussion

- a. What did you discover about the femurs of humans and horses?
- b. Is it possible for a human to grow taller than Robert Wadlow? Explain your response.
- c. Compare the three graphs you created in Part **g** of the exploration.
- d. Describe the relationship between pressure on the femur and scale factor.

Assignment

- 6.1 The person whose femur is shown in Figure 6 weighed 600 N and was 180 cm tall. Assuming that your body is similar, use your height to estimate the pressure on your femur in newtons per square centimeter.
 - 6.2 Assume that a human femur can withstand a pressure of approximately 1200 N/cm^2 . Use your response to Problem 6.1 to determine the maximum weight that your femur can support.
 - 6.3 Describe some other factors that might limit the maximum height of humans.
- * * * * *
- 6.4 The mean height of male gorillas is approximately 1.8 m. Their mean mass is about 200 kg.
 - a. Determine the mean weight, in newtons, of male gorillas.
 - b. A gorilla's femur can support a maximum of 10 times the mean body weight. What is this maximum weight?
 - c. In one version of the story, the gorilla King Kong was supposed to be about 9.8 m tall. Is it possible for a gorilla like King Kong to exist? Explain your response.

6.5 The femur of a Tyrannosaurus in a Montana museum is 103 cm long and has a diameter of 25 cm. Scientists estimate that this dinosaur weighed between 35,000 N and 62,000 N. Which bone is subject to greater pressure: the femur of this Tyrannosaurus or the femur of a 600-N person? Explain your response.

* * * * *

Summary Assessment

Do you think that human siblings are similar? Use the data in the following table (or collect data from your own family) to support your response.

Sibling	Length of Foot (cm)	Circumference of Foot (cm)	Area of Footprint (cm^2)	Body Weight (N)
A	15.0	35	61	93
B	19.5	45	101	208
C	23.0	54	149	340
D	30.0	70	251	756

In justifying your position, include examples of each of the following:

- proportionality
- scale factors
- linear equations of the form $y = ax$
- power equations of the forms $y = ax^2$ and $y = ax^3$
- square roots
- cube roots

Your report should also include predictions made using scale factors, graphs, and equations and use residuals to determine how well an equation models a data set.

Module Summary

- Two ratios, a/b ($b \neq 0$) and c/d ($d \neq 0$), are **proportional**, or **in proportion**, if

$$\frac{a}{b} = \frac{c}{d}$$

When two such ratios are proportional, it is also true that

$$\frac{a}{c} = \frac{b}{d}$$

where $c \neq 0$ and $d \neq 0$.

- Two objects are **similar** if they have the same shape and the ratios of corresponding lengths are proportional. The ratio of corresponding sides is the **scale factor**.
- To raise a fraction to a power n , where n is a non-negative integer, both the numerator and the denominator may be raised to the indicated power. In general,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

where $b \neq 0$.

- A **square root** of a non-negative number a is a number s such that $s^2 = a$.
- The positive square root of a number is its **principal square root**. The principal square root of a is usually denoted by \sqrt{a} , although it may also be written as ${}^2\sqrt{a}$.
- A **cube root** of a number a is a number b such that $b^3 = a$. The cube root of a is denoted by ${}^3\sqrt{a}$.
- In general, the n th root of a non-negative number a is a number s such that $s^n = a$. The non-negative n th root of a is denoted as ${}^n\sqrt{a}$.
- The n th root of a fraction can be found by taking the n th root of the numerator and dividing it by the n th root of the denominator. In general,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

where $b \neq 0$.

- An equation of the form $y = ax^b$ is a **power equation**.

- The ratio of the areas of two similar objects is the square of the ratio of the lengths of corresponding sides (the scale factor).
- If the scale factor of two similar figures is a/b , then the ratio of their areas is:

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

while the ratio of their volumes is:

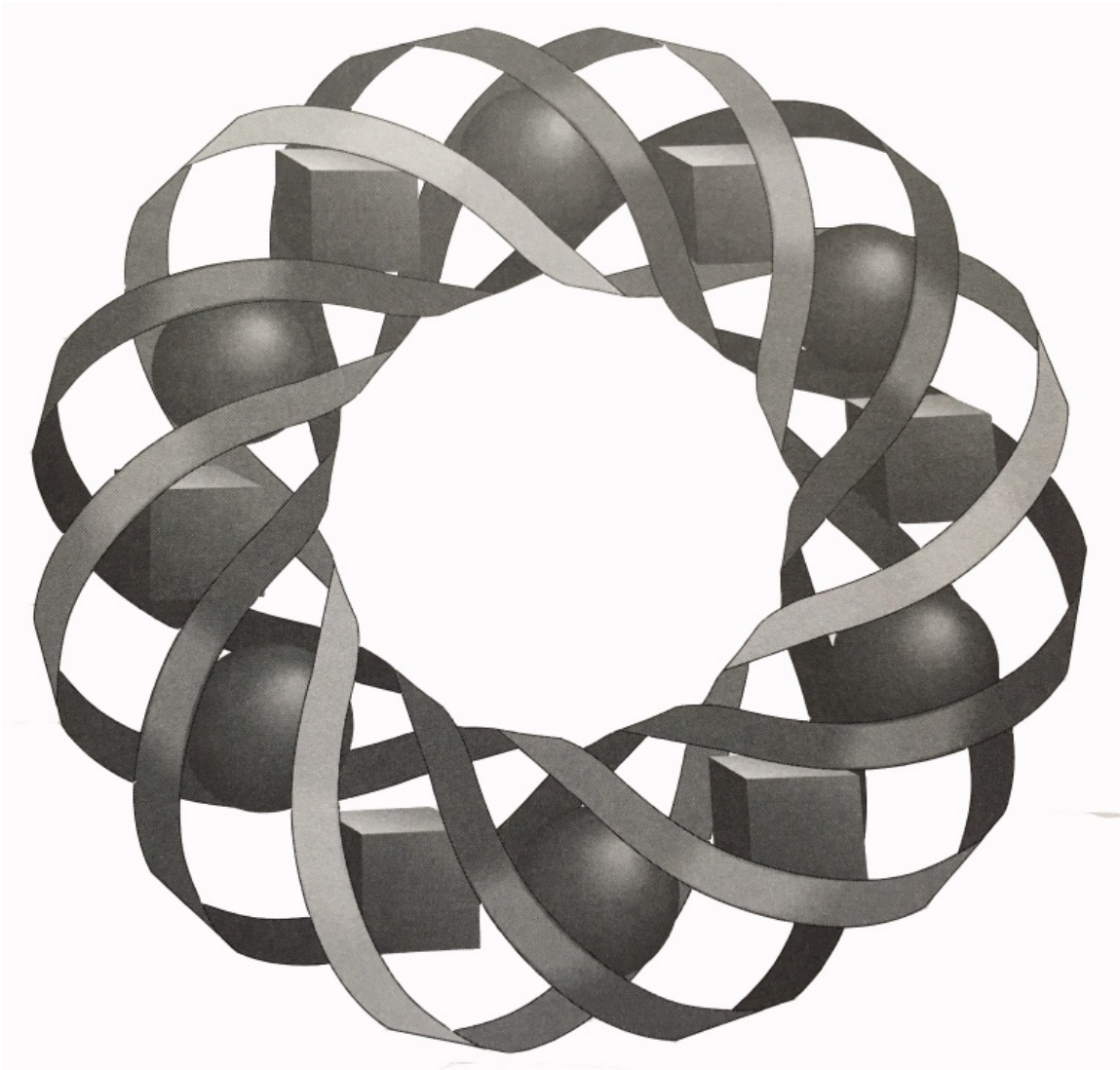
$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

- The **density** of a substance is the ratio of its mass to its volume.
- One metric unit of force is the **newton (N)**. The weight of an object in newtons is its mass in kilograms multiplied by 9.8 m/sec^2 (the acceleration due to gravity).

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AIDS: The Preventable Epidemic



A recent congressional study reported that AIDS and HIV infections among teenagers rose an alarming 70% between 1990 and 1992. Are teens playing Russian roulette with their sex lives?

Shirley Bagwell • Patricia Bean • Karen Longhart • Michael Trudnowski



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AIDS: The Preventable Epidemic

Introduction

Acquired Immune Deficiency Syndrome (AIDS) is a fatal, communicable disease. By the end of 1992, approximately 200,000 cases of AIDS had been diagnosed in the United States. The AIDS epidemic represents a particular danger to America's youth. People under age 25 account for at least a quarter of the estimated 40,000 newly infected Americans each year, and a congressional study reported that AIDS and Human Immunodeficiency Virus (HIV) infections among teenagers rose an alarming 70% between 1990 and 1992. Dr. Lloyd Kold, director of the Centers for Disease Control's Division of Adolescent and School Health, noted that "Many teens are playing Russian roulette with their sex lives."

As the population of AIDS patients continues to grow, our health care system has been struggling to provide hospital space, medication, and other services. In the United States, the yearly cost of treating one AIDS victim is about \$38,000 (as of July 1992). The projected lifetime cost is at least \$102,000. While insurance companies and the federal government are the initial sources of payment, higher taxes and insurance premiums may ultimately shift the financial burden to the American public.

Discussion

- a. Why should the AIDS epidemic be a concern for everyone?
- b. What steps may be taken to help prevent the spread of HIV?
- c. Approximately 20% of American teenagers have had four or more sexual partners by their senior year of high school. This group is particularly susceptible to HIV infection. What do these numbers mean in terms of your school population?

Activity 1

Due to the unusual nature of the AIDS virus, health professionals did not immediately recognize the manner of transmission. In 1987, several massive educational programs began to inform people about the dangers of AIDS.

With improved screening of the nation's blood supply, the number of new cases of AIDS among recipients of blood transfusions has decreased substantially. In addition, those who engaged in risky behavior began to make wiser choices.

Consequently, the number of new cases in two particularly high-risk populations—intravenous drug users and homosexuals—also began to level off. However, because of a mistaken sense of security, the number of AIDS cases among heterosexuals is on the rise.

Exploration 1

In this exploration, you use a model to examine the spread of infectious diseases.

- a. Pour one cup of red beans into a flat box. Each red bean represents a healthy, disease-free individual.
- b. Place one white bean in the container. This bean represents an individual with an infectious disease.
- c. Gently shake the container. Replace every red bean that is within 1 mm of the white bean with a white bean. These beans represent individuals who came into contact with the infectious individual and have contracted the disease. (You can use the edge of a dime to check the distance between beans.)
- d. Count the number of white beans in the container. Write this number in the appropriate column of Table 1.

Table 1: Simulated spread of a disease

Shake No. (x)	No. of White Beans (y)
0	1
1	
2	
\vdots	
6	

- e. Repeat Parts **c** and **d** for six shakes.
- f.
 1. Create a scatterplot of your data in Table 1. **Note:** Save this scatterplot for use in Exploration 2.
 2. Describe any patterns you observe in the scatterplot.

- g. Table 2 shows the total U.S. AIDS cases reported for people ages 13–29 from 1980 to 1991.

Table 2: Total U.S. AIDS cases reported for ages 13-29

Year	No. of Cases	Year	No. of Cases
1980	18	1986	7897
1981	79	1987	13,307
1982	301	1988	19,998
1983	886	1989	27,999
1984	2064	1990	37,022
1985	4296	1991	48,007

Source: Centers for Disease Control, 1995.

Create a scatterplot of the data in Table 2. Let y represent the number of AIDS cases and x represent the number of years after 1980 (for 1980, $x = 0$). **Note:** Save this scatterplot for use in Exploration 2.

- h. Compare the two scatterplots you created in Parts f and g.

Discussion 1

- a.
1. Compare your scatterplot from Part f of Exploration 1 with those of others in the class. Describe any differences or similarities you observe.
 2. As the shake number increases, does your scatterplot level off? If so, explain why.
 3. How does this scatterplot compare with the scatterplot of actual AIDS cases?
- b. Do you think that the scatterplot of AIDS cases will level off in the years after 1991? Explain your response.
- c.
1. Can the data in Tables 1 and 2 be modeled well by linear equations? Why or why not?
 2. Can the data in Tables 1 and 2 be modeled well by exponential equations? Why or why not?
- d.
1. How does the data collected during the simulated spread of an infectious disease resemble the actual spread of HIV?
 2. How does the simulation differ from the actual spread of HIV?

Exploration 2

In this exploration, you use exponential equations to model the data in Tables 1 and 2.

Mathematics Note

The equation $y = a \cdot b^x$ describes a pattern of **exponential growth**. When describing population growth, a is the initial population and x is the number of time periods. The value of b is the sum of two percentages: 100 (representing the initial population) and r , the growth rate from one time period to the next.

For example, consider the sample data in Table 3.

Table 3: Sample population data

Year	Population
1940	250
1950	450
1960	810
1970	1458
1980	2624
1990	4724

A scatterplot of this data is shown in Figure 1 below.

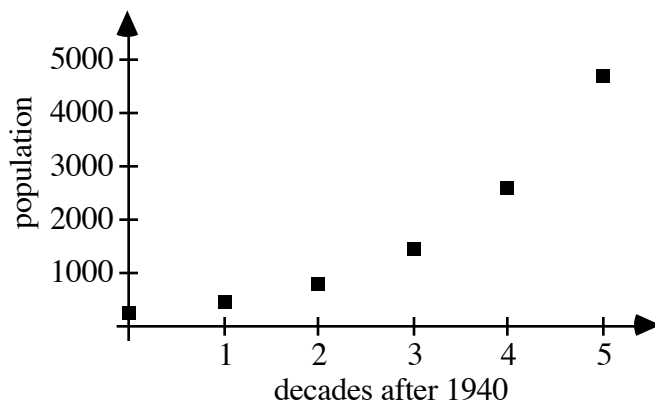


Figure 1: Scatterplot of population data

Because this data shows a pattern of exponential growth, it can be modeled by an exponential equation. In this case, the initial population (a) is 250. The growth rate (r) from one decade to the next can be found by determining the percent increase from 1940 to 1950:

$$\frac{450 - 250}{250} = 0.8 = 80\%$$

Since $r = 0.8$, $b = 1 + 0.8 = 1.8$. Therefore, the data can be modeled by the exponential equation $y = 250(1.8)^x$, where y is the population at the end of each decade, a is the initial population, and x is the number of decades after 1940.

- a.
 1. Determine an exponential equation that models the data for the simulation in Table 1.
 2. Graph this equation on the same set of axes as the scatterplot from Part f of Exploration 1.
 3. Determine the sum of the absolute values of the residuals for this exponential model.
- b.
 1. Determine an exponential equation that models the number of AIDS cases in Table 2.
 2. Graph this equation on the same set of axes as the scatterplot from Part g of Exploration 1.
 3. Determine the sum of the absolute values of the residuals for this exponential model.

Discussion 2

- a.
 1. In Part a of Exploration 2, you found an exponential model for the data from the simulation. Describe how you determined this model.
 2. Does your equation appear to provide a good model for the data? Explain your response.
- b.
 1. In Part a of Exploration 2, you found an exponential model for the number of AIDS cases. Describe how you determined this model.
 2. Does your equation appear to provide a good model for this data? Explain your response.

Assignment

- 1.1 The growth of a population can be modeled by the equation $y = a(1.5)^x$, where x represents time in years. When $x = 1$, the population is 210. What is the size of the initial population?
- 1.2 The following table lists the total number of AIDS cases reported in the United States from 1980 to 1985.

Year	No. of Cases
1980	94
1981	417
1982	1601
1983	4720
1984	10,996
1985	22,890

Source: Centers for Disease Control, 1995.

- Create a scatterplot of the data. Let y represent the number of AIDS cases; x represent the number of years after 1980 (for 1980, $x = 0$).
- Find an exponential equation that models the data.
- Does your equation appear to be a good model of the data? Explain your response.
- Using your equation from Part **b**, predict the total number of AIDS cases reported in each year from 1986 to 1991.
- Do your predictions seem reasonable? Explain your response.

1.3 The following table lists the total number of AIDS cases reported in the United States from 1986 to 1991.

Year	No. of Cases
1986	42,085
1987	70,918
1988	106,580
1989	149,223
1990	197,315
1991	255,864

Source: Centers for Disease Control, 1995.

- Compare this data with the predictions you made in Problem **1.2d**.
- Why do you think the actual numbers of reported cases differ from your predictions?
- Create a scatterplot of the AIDS data from 1980 to 1991. Let y represent the number of AIDS cases and x represent the number of years after 1980 (for 1980, $x = 0$).
- Find an exponential equation that models the data.
- Does your equation appear to be a good model of the data? Explain your response.

1.4 The following table shows estimates for the total number of U.S. children and adolescents orphaned by AIDS from 1983 to 1994.

Year	No. of Orphans	Year	No. of Orphans
1983	300	1989	10,000
1984	700	1990	14,400
1985	1500	1991	18,500
1986	2900	1992	23,900
1987	4500	1993	31,100
1988	6800	1994	38,400

Source: D. Michaels and C. Levine, "Estimates of the Number of Motherless Youth Orphaned by AIDS in the United States."

- a. Make a scatterplot of this data. Let y represent the number of children orphaned by AIDS and x represent the number of years after 1983 (for 1983, $x = 0$).
- b. Find an exponential equation that models the data.
- c. Determine the sum of the absolute values of the residuals for your exponential model.
- d. Do you think that your equation is a good model for the data? Justify your response.

1.5 The following table shows a dramatic decrease in the population of the Blackfeet tribe due to an epidemic of smallpox.

Year	Population
1822	2800
1838	600
1841	200

Source: J. F. Decker, "Depopulation of the Northern Plains Natives."

- a. Find an exponential equation that models the data.
- b. Do you think that three data points are enough to determine an accurate model? Explain your response.

* * * * *

1.6 The table below shows the amount of money in an investment account from 1988 to 1994.

Year	Amount
1988	\$500
1989	\$550
1990	\$605
1991	\$666
1992	\$732
1993	\$805
1994	\$886

- a. Make a scatterplot of the data. Let y represent the amount of money and x represent the number of years after 1988 (for 1988, $x = 0$).
- b. Find an exponential equation that models the data.
- c. Use your model to predict the amount of money in the account in each year from 1995 to 2000.
- d. Do you think that your model is a good one? Justify your response.

- 1.7 The table below shows the number of deaths due to AIDS in the United States per 10 million people for selected years from 1981 to 1985.

Year	Deaths per 10 million
1981	6
1983	64
1984	148
1985	292

Source: U.S. Bureau of the Census.

- Make a scatterplot of the data. Let y represent the number of deaths per 10 million people and x represent the number of years after 1981 (for 1981, $x = 0$).
- Find an exponential equation that models the data.
- Do you think that your equation is a good model for the data? Justify your response.
- Use your model to estimate the number of deaths due to AIDS per 10 million people in each of the following years:
 - 1982
 - 1987
- In 1982, there were approximately 20 deaths due to AIDS per 10 million people. In 1987, there were approximately 672 deaths due to AIDS per 10 million people. Did your model provide reasonable estimates for either one of these statistics?

* * * * *

Activity 2

There are several ways in which a person can be exposed to HIV. The Centers for Disease Control (CDC) records the source of exposure for each AIDS-related death or illness in categories similar to those shown in Table 4 below.

Table 4: Categories of Exposure

Category of Exposure	No. of Cases Reported through June 1995
heterosexual contact —any kind of sexual intercourse between members of the opposite sex	64,676
intravenous drug use —drug use involving needles	149,718
receipt of blood —receiving blood transfusions for transplants and surgical procedures or for hemophilia and coagulation disorder (two diseases requiring blood transfusions)	18,451
male homosexual contact (there are no known cases of women infecting other women with HIV through homosexual contact)	275,259
other —cases in which the type of exposure does not belong to any of the above categories or has not been determined	52

When a person is diagnosed as HIV positive, the patient may report one or more potential sources of exposure. For example, a person may have had a blood transfusion and also used intravenous drugs. This person would be listed under two categories of exposure: receipt of blood and intravenous drug use.

Organizing this data for thousands of cases is a very complicated task. In this activity, you explore ways to organize large amounts of information.

Exploration

Data like that collected by the CDC about AIDS is done by conducting surveys. In this activity, you organize the data you collected in the survey of childhood diseases.

- a. Combine the data you collected in the childhood disease survey with the information collected by your classmates.

Use the combined data to complete Table 5 below, placing an X in the appropriate cell for each individual who has had a particular disease.

Table 5: Results of childhood disease survey

Individual	Disease				
	Mumps	Measles	Strep Throat	Chicken Pox	Whooping Cough
1					
2					
3					
⋮					
50					

- b. Use the information in Table 5 to answer the following questions.
1. How many people in the survey have had strep throat?
 2. How many people have had chicken pox?
 3. How many people have had strep throat and chicken pox?
 4. How many people have had mumps or chicken pox?
 5. How many people have had strep throat, mumps, and chicken pox?
 6. How many people have had whooping cough, but did not have strep throat?
 7. How many people have had whooping cough and measles?

Mathematics Note

Venn diagrams are mathematical models that show relationships among different sets of data. Each circle in a Venn diagram represents a set of data.

The shaded region in Figure 1a shows the **intersection** of set A with set B, or the set of elements common to both A and B. The intersection of A and B can be written as $A \cap B$.

The shaded region in Figure 1b shows the **union** of sets A, B, and C. This union contains all the elements in sets A, B, or C. It can be written as $(A \cup B) \cup C$ or $A \cup B \cup C$. Parentheses are not necessary to show order.

Figure 1c shows two **disjoint sets**, or sets that have no elements in common. The intersection of disjoint sets is the **empty set**, \emptyset . This can be written as $A \cap B = \emptyset$.

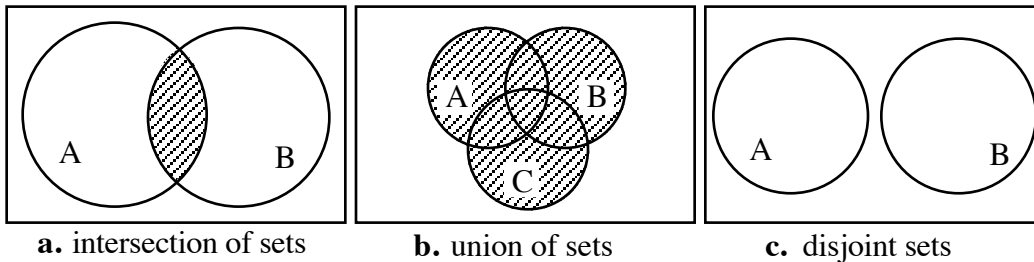


Figure 1: Venn diagrams

Venn diagrams can provide a useful tool for displaying relationships among groups. For example, the Venn diagram in Figure 2 shows some data on the incidence of two diseases in a school of 100 students. Fifty of the students have had chicken pox and 18 have had mumps. Eight of the them have had both; while 40 have had neither disease.

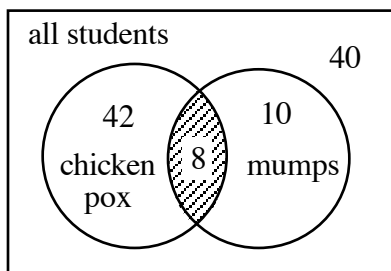


Figure 2: Incidence of chicken pox and mumps

Venn diagrams also can be helpful when you must determine the probability of a particular event. For example, suppose you choose one student at random from the group who has had either chicken pox or mumps. The probability that this student has had only the mumps can be determined as follows: From Figure 2, the number of students who have had either disease is $42 + 8 + 10 = 60$, while the number who have had mumps only is 10. Therefore, the probability that a randomly selected student has had mumps only is $10/60 \approx 0.17 = 17\%$.

- c. Draw a Venn diagram to represent each of the following groups in your survey results. Shade the appropriate region in the diagram and identify each shaded region as either an intersection of sets, a union of sets, or an empty set.
1. the number of people who have had strep throat and chicken pox
 2. the number of people who have had mumps or chicken pox
 3. the number of people who have had strep throat, mumps, and chicken pox
 4. the number of people who have had whooping cough, but did not have strep throat
 5. the number of people who have had whooping cough and measles
- d.
1. Determine the probability that a person chosen at random from the survey group has had both strep throat and chicken pox.
 2. Determine the probability that a person chosen at random from the survey group has had both whooping cough and measles.

Discussion

- a. How are the childhood diseases named in the survey similar to AIDS? How are they different?
- b. Which provides a better way to organize the information in the exploration—a table or a Venn diagram?
- c. If you had to organize the disease data for everyone in the United States, would you use a table or a Venn diagram? Explain your response.
- d. How could Venn diagrams have helped you to answer the questions in Part **b** of the exploration?
- e. How can Venn diagrams help you determine the probabilities of events?

Assignment

- 2.1 The table below contains data for two of the five categories of HIV exposure listed in Table 4, along with the number of cases who fall into both categories.

Category of Exposure	No. of Cases Reported through June 1995
intravenous drug use	149,718
receipt of blood	18,451
intravenous drug use and receipt of blood	2969

Source: Centers for Disease Control, 1995.

- Use a Venn diagram to organize this data.
 - If an AIDS patient is selected at random from the two exposure categories in the table, what is the probability that this patient was exposed to HIV only by intravenous drug use?
- 2.2 The following table contains data for three of the five categories of HIV exposure listed in Table 4, along with the number of cases who fall into more than one of the categories.

Category of Exposure	No. of Cases Reported through June 1995
intravenous drug use (set A)	149,718
male homosexual contact (set B)	275,259
receipt of blood (set C)	18,451
male homosexual contact and intravenous drug use	31,024
male homosexual contact and receipt of blood	4086
intravenous drug use and receipt of blood	2969
all three of these categories	690

Source: Centers for Disease Control, 1995.

- Use a Venn diagram to organize this data.
- What is the total number of AIDS cases represented in the table?
- If an AIDS patient is selected at random from the three exposure categories in the table, what is the probability that this patient was infected by both intravenous drug use and homosexual contact?

- 2.3 The table below contains data for three of the five categories of HIV exposure listed in Table 4, along with the number of cases who fall into more than one of the categories.

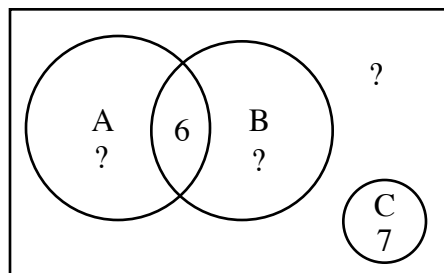
Category of Exposure	No. of Cases Reported through June 1995
heterosexual contact	64,676
intravenous drug use	149,718
male homosexual contact	275,259
male homosexual contact and intravenous drug use	31,024
male homosexual contact and heterosexual contact	9345
intravenous drug use and heterosexual contact	22,856
all three of these categories	3304

Source: Centers for Disease Control, 1995.

- Use a Venn diagram to organize this data.
- If an AIDS patient is selected at random from the three exposure categories in the table, what is the probability that this patient was exposed to the virus both by intravenous drug use and heterosexual contact?

* * * * *

- 2.4 In a group of 458 students, 228 are taking mathematics, 166 are taking business, and 7 are taking consumer decision-making. In the incomplete Venn diagram below, students enrolled in mathematics are represented by set A, those in business are represented by set B, and those in consumer decision-making are represented by set C.



- Draw a complete copy of this Venn diagram.
- What is the probability that a student chosen at random from this group:
 - is taking both mathematics and business?
 - is not taking mathematics, business, or consumer decision-making?
 - is taking all three classes?

2.5 Consider the following six definitions of polygons.

- A quadrilateral is any four-sided figure.
 - A trapezoid is a quadrilateral in which at least one pair of opposite sides is parallel.
 - A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.
 - A rhombus is a parallelogram with four congruent sides.
 - A rectangle is a parallelogram with four right angles.
 - A square is a parallelogram that is both a rhombus and a rectangle.
- a. Sketch a trapezoid.
 - b. Sketch a parallelogram that is neither a rhombus nor a rectangle.
 - c. Sketch a rhombus that is not a rectangle.
 - d. Draw a Venn diagram that shows the relationships among these six types of polygons.

2.6 The following table shows the distribution of students among three clubs at Canterville High School.

Name of Club	No. of Students
Pep Club	150
Spanish Club	58
Science Club	29
Spanish and Pep Clubs	23
Spanish and Science Clubs	15
Pep and Science Clubs	11
all three clubs	5

- a. Organize this data using a Venn diagram.
- b. The total student population at Canterville High is 230. Modify your Venn diagram to reflect this total.
- c. If a student is selected at random from the group who participates in at least one of the three clubs, what is the probability that the student is a member of both the Spanish Club and the Science Club?
- d. What is the probability that a student chosen at random from the high school is in none of the three clubs?

* * * * *

Research Project

Obtain the most recent data for U.S. AIDS cases and exposure categories. You may use the National AIDS Hotline (1-800-342-AIDS), a physician, medical clinic, or local health department as sources. You may also obtain information from the Centers for Disease Control's National Aids Clearinghouse at the following electronic address:

<http://cdcnac.aspensys.com:86/aidsinfo.html>

- a. Compare the most recent data with the data presented in this module.
 1. What changes do you notice?
 2. Which category seems to be growing the fastest?
 - b. Write at least two problems involving this data that can be solved with the help of a Venn diagram.
-

Activity 3

The study of disease and epidemics often involves probability. Most laboratory tests, for example, are not 100% accurate. Similarly, few diseases infect all members of a population and medical treatments are not effective for every patient. In this activity, you learn more about probability and investigate another way to help organize information.

Exploration

In this exploration, you play a game called "Epidemic."

- a. The rules for Epidemic are listed below. Read Steps 1-5 before beginning the exploration.
 1. Select three students as players. Identify these students as player 1, player 2, and player 3.
 2. Without revealing their choices to the class, each of these players randomly chooses one outcome from the appropriate list.
 - Player 1: whooping cough, measles, mumps, healthy
 - Player 2: measles, healthy
 - Player 3: strep throat, chicken pox, healthy
 3. Each remaining member of the class predicts the outcomes chosen by the three players. For example, one student might make the following prediction: "Player 1 has mumps, player 2 is healthy, and player 3 has chicken pox."

4. The three players then reveal their choices.
 5. Record the numbers of correct and incorrect predictions for the entire class.
- b. Play Epidemic 10 times.

Mathematics Note

A **tree diagram** is a mathematical model that shows all the possible outcomes for a series of events or decisions. Each line segment in a tree diagram is a **branch**. Each branch may be assigned a probability.

For example, Figure 3 shows a tree diagram for Epidemic. Notice how the number of branches at a given location in the diagram corresponds with the number of choices available to that player.

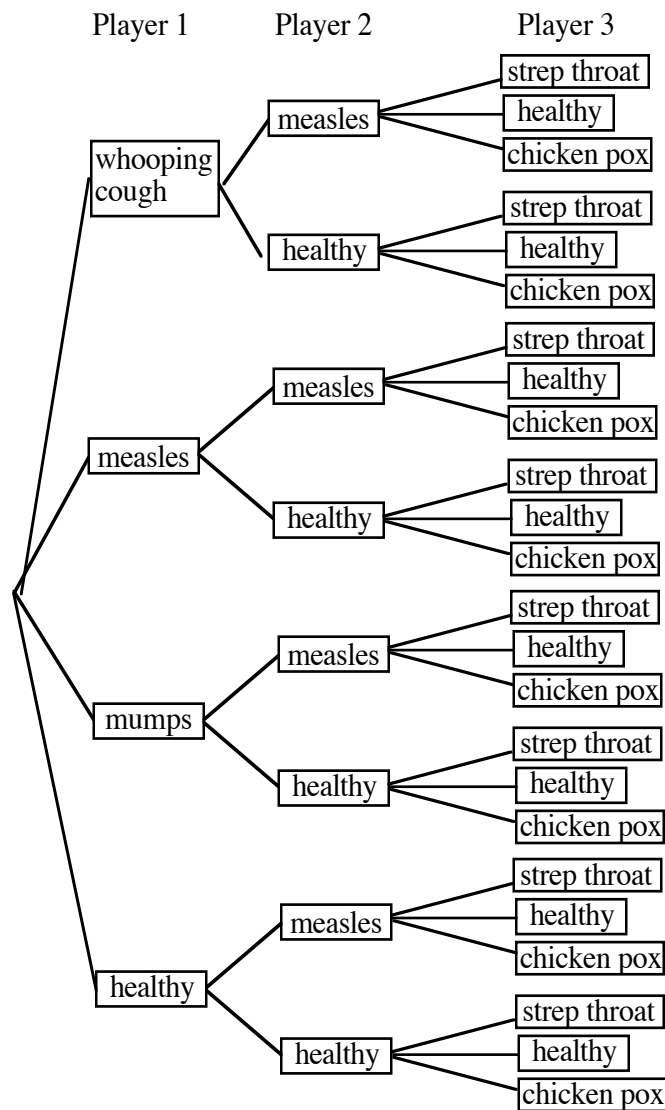


Figure 3: A tree diagram for Epidemic

- c.
 1. List all the possible outcomes for one round of Epidemic.
 2. Determine the size of the sample space for one round of Epidemic.
 3. Describe how your strategy in Step 1 ensures that you have listed all the possibilities.
- d. In one round of Epidemic, what is the probability that:
 1. player 1 has the measles?
 2. at least one player is healthy?

Mathematics Note

The **fundamental counting principle** provides a method for determining the total number of outcomes for a series of events or decisions. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways in which the two events can occur is $m \cdot n$.

In Epidemic, for example, player 1 has three choices and player 2 has two choices. Considering these players, the number of possible outcomes is $3 \cdot 2 = 6$.

- e. Use the fundamental counting principle to verify the size of the sample space you determined in Part c of the exploration.
- f.
 1. How many outcomes are possible in a game like Epidemic in which three players each have three choices?
 2. What is the probability of each of these outcomes?
- g.
 1. How many outcomes are possible in a game like Epidemic in which player 1 has four choices, player 2 has two choices, player 3 has three choices, and player 4 has five choices?
 2. The selection “healthy” is a possible choice for players 1, 2, and 3, but not for player 4. In one round of this game, what is the probability that at least one player chooses “healthy”?

Discussion

- a. What is the probability of making a correct prediction in Epidemic?
- b. If a class of 30 students plays Epidemic, how many would you expect to make a correct prediction in one round of the game?
- c. In one round of Epidemic, is it possible for all three players to choose “healthy”? Is it very likely? Explain your responses.
- d. In one round of Epidemic, is it possible for all three players to choose a disease? Is it very likely? Explain your responses.

- e. How can a tree diagram help you calculate the probability of a particular outcome?
- f. How can the fundamental counting principle help you calculate the probability of a particular outcome?

Assignment

- 3.1** One version of Epidemic involves only two players. Player 1 chooses randomly from either AIDS or hepatitis, while player 2 chooses randomly from AIDS, hepatitis, polio, or healthy.
- a. Draw a tree diagram that shows all the possible outcomes for one round of this game.
 - b. List all the possible outcomes for one round.
 - c. Determine the size of the sample space for this game.
 - d. In one round of this game, what is the probability that:
 - 1. both players choose “AIDS”?
 - 2. one player chooses “healthy”?
- 3.2** Consider a game like Epidemic in which each of three players has two choices. What is the size of the sample space for this game? Describe how you determined your response.
- 3.3** In another game like Epidemic, player 1 has five choices, player 2 has k choices, and player 3 has n choices. Write a formula that can be used to determine the number of possible outcomes for this game.
- 3.4** Statistics show that condom use is not 100% effective in preventing the exchange of body fluids. When used correctly, approximately 2 out of 100 condoms are ineffective. However, since condoms are not always used correctly, about 10 out of 100 are actually ineffective.
- a. If a condom is used correctly, what is the probability that it is effective?
 - b. What is the actual probability that a condom is effective?
 - c. Is there a significant difference between these two probabilities? Explain your reasoning.
- * * * * *
- 3.5** Although Shane plans to take art during third period, he has not yet selected the classes for the rest of his schedule. He has a choice of 2 classes each for first and second periods, a choice of 5 classes each for fourth and fifth periods, and a choice of 3 classes for sixth period. How many different six-period schedules are possible for him?

3.6 When Cheyenne arrived at the university for a three-week agricultural camp, she received three posters: one featured cattle, one showed a wheat field, and one displayed a barnyard scene. After meeting her two neighbors in the dormitory, she agreed to the following plan: the three students would display a different poster on each door so that, at any one time, all three posters would be displayed. They agreed to exchange posters after the first week, then again after the second week. In addition, they decided that no one of them would display a particular poster more than once.

- a. Draw a tree diagram to represent all the possible poster displays during the three-week camp.
- b. What is the probability that Cheyenne will display the posters in the following order: cattle, wheat field, barnyard?

* * * * *

Summary Assessment

1. The following table lists the total number of U.S. AIDS cases for all persons aged 13 or older for selected years between 1983 and 1987.

Year	No. of Cases
1983	4589
1984	10,750
1985	22,399
1987	69,592

Source: Centers for Disease Control, 1995.

- a. Make a scatterplot of this data. Let y represent the number of AIDS cases and x represent the number of years after 1983.
 - b. Determine an exponential equation that models the data.
 - c.
 1. Use your model to estimate the number of AIDS cases in 1986.
 2. The actual number of AIDS cases in 1986 was 41,256. Does your estimate seem reasonable? Explain your response.
 - d.
 1. Use your model to predict when the number of AIDS cases in the United States will exceed 500,000.
 2. Does this prediction seem reasonable? Explain your response.
2. The ninth-grade class at Piedmont High School has 472 students. To evaluate class participation in extracurricular activities, the principal requests rosters from the athletics, music, and drama departments.

The athletics roster lists 217 students, while the music roster lists 312 students. There were 92 students named on both the athletics and music rosters, 45 on both the athletics and drama rosters, and 87 on both the music and drama rosters. Only 15 members of the class were listed on all three rosters, while 30 students did not appear on any of the rosters.

- a. Draw a Venn diagram to illustrate the involvement of this ninth-grade class in extracurricular activities.
- b. How many members of the class participate in drama?
- c. What is the probability that a student chosen at random from this class is involved only in athletics?
- d. How many members of the class are involved in athletics or music? Describe how you determined your response.

3. A benefit carnival features a game in which a stack of three cards is turned face down. The cards are numbered 1, 2, and 3. To play the game, you shuffle the cards, then turn them over one at a time. To win the game, the first card turned over must be greater than 1, the second card turned over also must be greater than 1, and the third card turned over must be 1.

Using what you know about tree diagrams and probability, decide whether or not this is a good game for the carnival and explain the reasons for your decision.

Module Summary

- The equation $y = a \cdot b^x$ describes a pattern of **exponential growth**. When describing population growth, a is the initial population and x is the number of time periods. The value of b is the sum of two percentages: 100 (representing the initial population) and r , the growth rate from one time period to the next.
- **Venn diagrams** are mathematical models that show relationships among different sets of data. Each circle in a Venn diagram represents a set of data.
- The **intersection** of set A with set B is the set of elements common to both A and B. This can be written as $A \cap B$.
- The **union** of sets A, B, and C is the set of all elements in sets A, B, or C. This can be written as $(A \cup B) \cup C$ or $A \cup B \cup C$. Parentheses are not necessary to show order.
- Two **disjoint sets** have no elements in common. The intersection of disjoint sets is called the **empty set**, \emptyset . This can be written as $A \cap B = \emptyset$.
- A **tree diagram** is a mathematical model that shows all the possible outcomes for a series of events or decisions. Each line segment in a tree diagram is a **branch**. Each branch may be assigned a probability.
- The **fundamental counting principle** provides a method for determining the total number of outcomes for a series of events or decisions. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways in which the two events can occur is $m \cdot n$.

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For more information on AIDS and HIV, call the Centers for Disease Control (CDC) National AIDS Hotline: 1-800-342-2437.

Going in Circuits



After leaving for work in the morning, a bus driver must make six stops before returning home. There are more than 500 ways to complete this route. Which ones are best? In this module, you explore some ways to find an efficient route without testing all the possibilities.

John Carter • Janet Higgins • Paul Swenson • Steve Yockim



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Going in Circuits

Introduction

The map in Figure 1 shows the floor plan of William R. Hamilton High School.

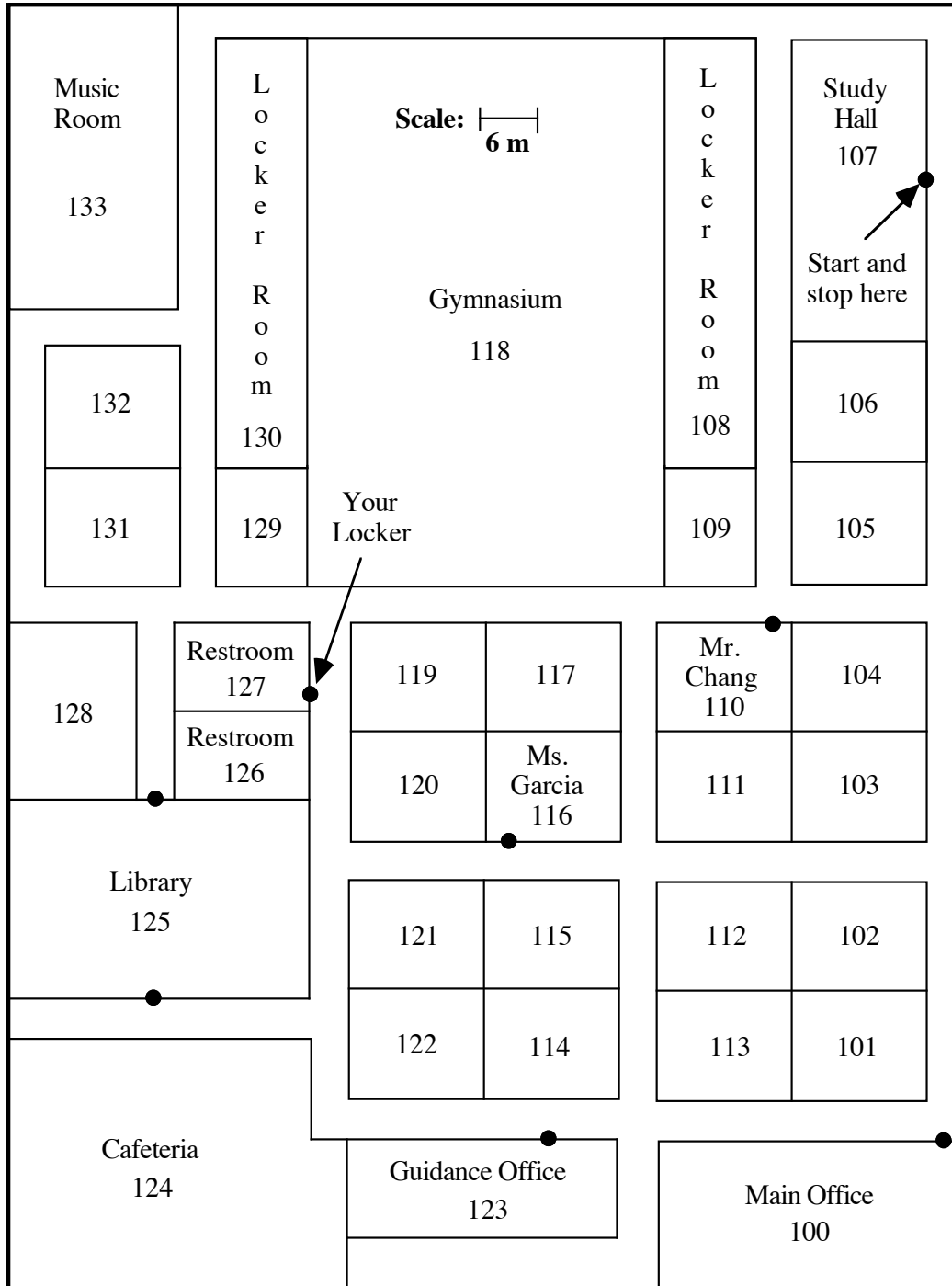


Figure 1: Floor plan of Hamilton High School

As a new student at Hamilton High, you have a lot to do. In order to complete your first day on schedule, you must visit six different rooms during a 55-min study hall. The following list describes each room, the task involved, and the time required to complete the task.

- Main Office: fill out a new student information card (7 min)
- Guidance Office: make an appointment to talk to a counselor (1 min)
- Room 116: question Ms. Garcia about a science experiment (10 min)
- Room 110: talk to Mr. Chang about taking French (10 min)
- Library: find a book for an English assignment (15 min)
- Locker: pick up a notebook (1 min).

You may visit the rooms in any order. However, your route must satisfy the following requirements.

- You must start and stop in Study Hall, Room 107.
- You must visit all the rooms and accomplish all the tasks.
- You must return to Study Hall before the end of the class period (55 min).

How can you find an efficient route to all these rooms and back to Study Hall? A situation such as this one, in which an appropriate solution makes good use of time or other resources, involves **optimization**.

Activity 1

Before beginning to solve an optimization problem, it can help to organize the available information in a table or graph.

By representing each destination as a point, one possible route can be described using the diagram shown in Figure 2. This route leaves Study Hall, visits the locker first, then eventually returns to Study Hall. Although the diagram is not drawn to scale, it does show the sequence in which the rooms are visited.

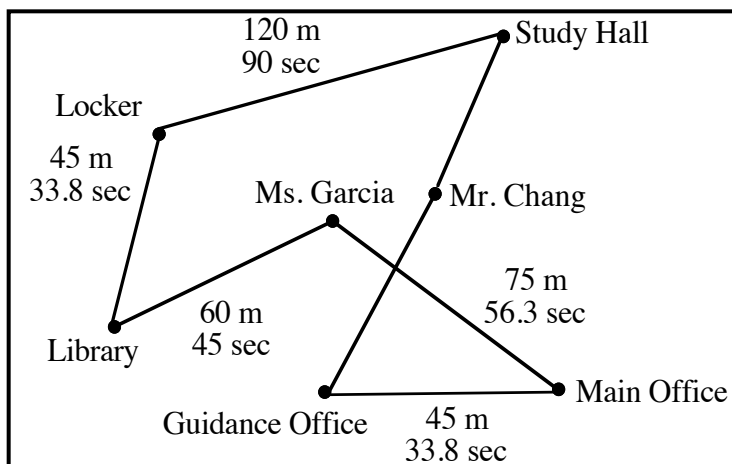


Figure 2: Possible route for completing study hall tasks

Exploration

Figure 2 also shows the distances between some of the stops and the time required to walk those distances. The distances were determined by measuring the hallways on the map in Figure 1, then using the map's scale to convert these distances to meters. The corresponding walking times were calculated using a rate of 4.8 km/hr.

- a.
 1. Use Figure 1 to find the distances along the hallways from the Guidance Office to Mr. Chang's room and from Mr. Chang's room to the Study Hall.
 2. Using a walking speed of 4.8 km/hr, calculate the walking times for these trips.
 3. Determine the walking time for the entire route.
 4. Calculate the total time required to walk the route and complete all the tasks.

Mathematics Note

A **graph** is a set of points or **vertices** (plural of **vertex**) and the edges connecting them. For example, Figure 3 shows a graph with five vertices: *A*, *B*, *C*, *D*, and *E*. This graph has seven edges. The edge connecting vertices *A* and *D* does not intersect the edge connecting vertices *B* and *E*.

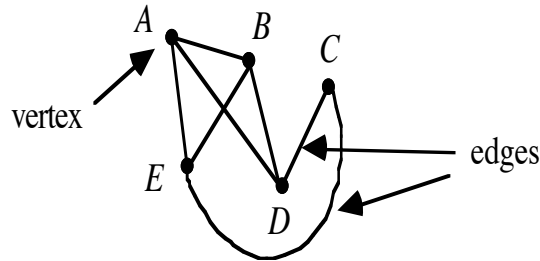


Figure 3: A graph with five vertices and seven edges

In a **weighted graph**, each edge is assigned a numerical value. For example, the numerical value on each edge in Figure 4a represents the flight time between the corresponding pair of cities. In Figure 4b, the numbers represent the airfares between two cities.

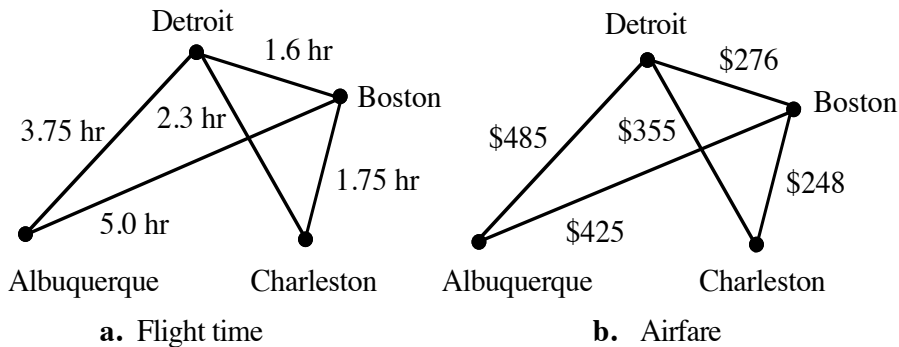


Figure 4: Two weighted graphs

- b. Find an acceptable route through Hamilton High School that takes less time than the one shown in Figure 2.
1. Display the route as a graph.
 2. Weight the edges of the graph with the corresponding walking times.
 3. Find the total time required to complete the route.

Discussion

- a. Which route in the exploration do you think is the better one? Explain your response.
- b. What characteristics do you think describe the best possible route?
- c. How could you make sure that you found the best possible route?

Mathematics Note

A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices, however, can occur more than once in a path.

In Figure 5, for example, $A-B-D-C-E$ represents a path in which no vertex is repeated, while $B-E-A-B-D$ represents a path in which one vertex (B) is repeated.

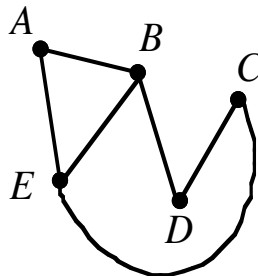


Figure 5: A graph with five vertices

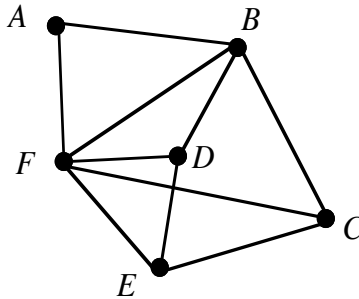
A **circuit** is a path that starts and ends with the same vertex in which no intermediate vertex is repeated. For example, $E-B-D-C-E$ and $A-B-E-A$ in Figure 5 are circuits.

A **Hamiltonian circuit** is a circuit in which every vertex in the graph is visited exactly once. For example, $A-B-D-C-E-A$ in Figure 5 is a Hamiltonian circuit.

- d. What do the vertices of the graphs in the exploration represent?
- e. What do the edges of the graphs in the exploration represent?
- f. Explain why your graph from Part b of the exploration is a Hamiltonian circuit.
- g. Why is $A-B-D-C-E-B-A$ in Figure 5 not a Hamiltonian circuit?

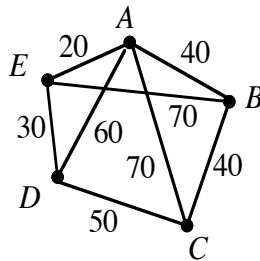
Assignment

- 1.1 The diagram below shows a graph with six vertices: A , B , C , D , E , and F .



Use this graph to describe an example, if possible, of each of the following:

- a path
 - a circuit that is not Hamiltonian circuit
 - a path that is not a circuit
 - a circuit that is not a path
 - a Hamiltonian circuit.
- 1.2 In the weighted graph below, the vertices represent cities and the numbers on the edges represent the time in minutes required to drive from one city to the next.

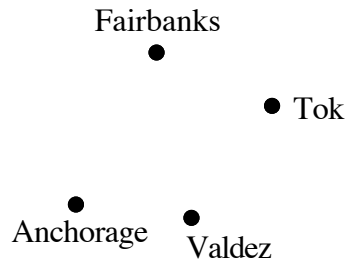


- Find the path from A to D that uses the minimum amount of time.
- Identify one Hamiltonian circuit in the graph and determine the total time required for this route.
- Identify another Hamiltonian circuit in the graph that results in a different total time.
- Which of the circuits from Parts **b** and **c** makes more efficient use of time?

- 1.3 The following chart shows the distances in kilometers between some Alaskan cities.

	Valdez	Tok	Fairbanks
Anchorage	478	520	581
Fairbanks	584	333	
Tok	409		

- a. The vertices in the diagram below represent the relative locations of the cities. On a copy of this diagram, draw all the possible edges between these vertices.

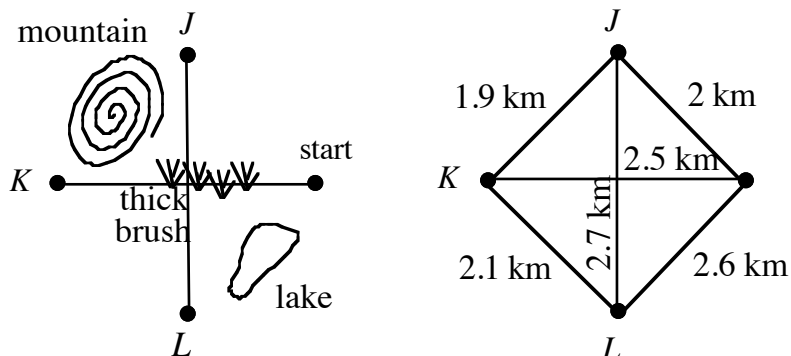


- b. Weight your graph from Part a using the distances between cities.
 c. What is the shortest path on the graph?
 d. Name one circuit in the graph that is not a Hamiltonian circuit.
 e. Identify two Hamiltonian circuits that result in different total distances.

* * * * *

- 1.4 A scoutmaster has designed an orienteering competition for six scouts. Each scout will be assigned a different course. All six courses, however, start at the same point, visit three locations, then return to the starting point. The scouts must use maps and compasses to complete the assigned course in as little time as possible.

The diagram on the left shows the relative positions of the three locations and the starting point. It also shows the presence of some obstacles on the course, including a mountain, thick brush, and a lake. The weighted graph on the right shows the distances between the locations.



- a. Each of the six possible routes is a Hamiltonian circuit. Identify the six routes.
- b. Over easy terrain, the scouts can travel at an average speed of 9.6 km/hr. Over rugged terrain, such as mountains or thick brush, the average speed is 7 km/hr.

If you were one of the six scouts participating in the competition, which route would you want to be assigned? Explain your response.

* * * * *

Activity 2

In the previous activity, you described two different routes through Hamilton High School. To determine the best route in this situation, you might be tempted to consider all the possible Hamiltonian circuits. However, this would mean taking the time to examine over 500 circuits.

In this activity, you investigate a method for quickly determining the number of possible solutions to a problem.

Exploration

Imagine that you are the manager of the most popular band in the nation. The members of the band are in Miami, Florida, and must fly to Los Angeles, California, to sign a recording contract.

There are two flights leaving Miami—one to Chicago, Illinois, and one to Kansas City, Missouri. From both Chicago and Kansas City, there are three flights going west—one to Denver, Colorado; one to Salt Lake City, Utah; and one to Las Vegas, Nevada. There are direct flights to Los Angeles from Denver, Salt Lake City, and Las Vegas.

- a. A **directed graph** or **digraph** is a graph in which a direction is indicated on each edge. Draw a directed graph that shows the relative positions of the cities named above and the flights between them.
- b. Draw a tree diagram to illustrate all possible routes from Miami to Los Angeles.
- c. How many different routes are there from Miami to Los Angeles?

Discussion

- a. Describe how you determined the number of different routes from Miami to Los Angeles.
- b. Would your method work well if several more cities were involved?
- c. The **fundamental counting principle** provides a method for determining the total number of ways in which a task can be performed. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways that the two events can occur is $m \cdot n$.

Describe how the fundamental counting principle can be used to determine the number of different routes from Miami to Los Angeles.

Assignment

- 2.1** As the manager of a band, you are planning a five-city tour to promote the band's new album. The tour will begin and end in Miami, with stops in Seattle, Washington; Kansas City, Missouri; Chicago, Illinois; and New York City. The band will visit each city only once and the order of the cities is not important.
- a. Determine the number of possible routes that the band can take on the tour.
 - b. Draw a tree diagram to verify your answer from Part a.
 - c. Describe the process you used to create the tree diagram in Part b.

Mathematics Note

An **algorithm** is a step-by-step process used to accomplish a task.

In a **brute force algorithm**, every potential solution to a problem is examined. For example, suppose that you wanted to determine the number of possible routes from your home to school. To solve this problem with a brute force algorithm, you could use a tree diagram to list every route.

- 2.2** On the first day of a new semester, your principal greets you with a list of the seven classes in your school day.
- a. In how many different ways could these seven classes be arranged?
 - b. Why is a brute force algorithm not appropriate in this situation?

Mathematics Note

Factorial notation is often used to simplify the representation of the product of the positive integers from 1 to n . If n is a positive integer, then **n factorial** (denoted by $n!$) can be expressed as

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

Zero factorial, or $0!$, is defined as 1.

For example, the number of possible arrangements of the letters A, B, and C can be found using the fundamental counting principle as follows: $3 \cdot 2 \cdot 1 = 6$. Using factorial notation, this can be represented as $3!$

- 2.3** As part of the publicity for their tour, the band will appear on the cover of *Singer's Circuit* magazine. Before choosing a photo, the magazine editors want to look at samples with the four band members arranged in different orders.
- If the band members stand side by side, how many different arrangements are possible?
 - Use factorial notation to represent the number of possible arrangements.
- 2.4** During its concerts, the band plays 10 songs in a set before taking a break.
- In how many different orders can 10 songs be played?
 - If the band wants to play its current hit song first, how does this affect the number of possible orders for a 10-song set?
 - How would your answer to Part **b** change if the band chose to play its hit song fourth in the set?
 - If the band wants to play its hit song first and its shortest song second, how does this affect the number of possible orders for a 10-song set?
 - In how many different orders can n songs be played?
- 2.5** The band is planning several tours during the summer concert season. All of these tours begin and end in the band's hometown.
- Use the fundamental counting principle to determine the number of possible tours involving each of the following numbers of cities:
 - 2 cities (the band's hometown plus one other city)
 - 3 cities (the band's hometown plus two other cities)
 - 4 cities (the band's hometown plus three other cities)
 - 10 cities (the band's hometown plus nine other cities)
 - Use the pattern you observe in your responses to Part **a** to develop a formula for calculating the number of tours possible for n cities if all tours begin and end in the band's hometown.

- 2.6** **a.** Determine the number of possible routes through Hamilton High School that begin and end in Study Hall and visit all six destinations described in Activity 1.

- b.** Explain why it is not feasible to list all the possible routes.

* * * * *

- 2.7** As captain of the softball team, Corrine must determine the order in which the team’s nine players will bat. Use the fundamental counting principle to determine the number of different batting orders that Corrine could choose.

- 2.8** **a.** Find the value of $7!$

- b.** Find the value of $9!/7!$

- c.** Explain why it is possible to write $200!$ as $200(199!)$.

- d.** One way to find the value of $100!/95!$ is to calculate $100!$ and divide it by $95!$

Describe another way to calculate the value of $100!/95!$

- e.** Simplify the following expression:

$$\frac{n!}{(n-1)!}$$

- 2.9** On his first day at Hamilton High School, Marcus forgets the combination to his locker. In an attempt to jog his memory, he tries a few different combinations.

- a.** The combination to the lock consists of a sequence of 3 different numbers. Each of these numbers can be chosen from 40 different numbers. How many different combinations are possible?

- b.** If it takes 15 seconds to try each combination, how long would it take Marcus to try them all?

* * * * *

Activity 3

In Activity 2, you identified the number of possible routes for your band’s upcoming tour. As manager, however, you must also consider the time and expense associated with the tour. Selecting the shortest route possible, for example, might save both time and money.

Exploration

The stops on your concert tour include Miami, Seattle, Chicago, New York, and Kansas City. Table 1 below shows the distances in kilometers between these cities.

Table 1: Distances between cities (in kilometers)

	Seattle	Kansas City	Chicago	New York
Miami	4400	1997	1912	1757
New York	3875	1765	1147	
Chicago	2795	667		
Kansas City	2424			

- Without using a brute force algorithm, plan a tour that starts in Miami, ends in Miami, visits each of the other cities exactly once, and travels as few kilometers as you think possible.

Mathematics Note

In a **greedy algorithm**, the choice made at each step is the best of all remaining choices. One greedy algorithm is the **nearest neighbor algorithm**. The steps for the nearest neighbor algorithm are described below.

- Starting with any vertex, draw an edge to its nearest vertex. In Figure 6, for example, vertex A is connected to its nearest vertex, B .



Figure 6: Connecting the first vertex to the nearest vertex

- Continue this process from the second vertex, drawing an edge to the next nearest vertex not yet visited, and so on, until all vertices have been visited. For example, Figure 7 shows edges drawn from B to C and from C to D .

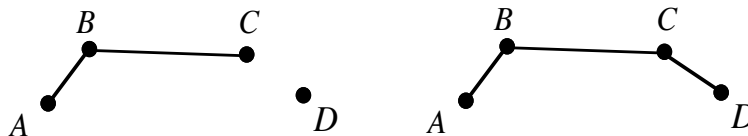


Figure 7: Continuing the nearest neighbor algorithm

- To complete a Hamiltonian circuit, return to the original vertex. For example, Figure 8 shows an edge drawn from D back to A .

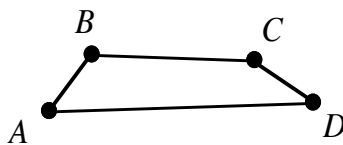


Figure 8: Completing a Hamiltonian circuit

- b. Use the nearest neighbor algorithm to draw weighted Hamiltonian circuits that start and stop in each of the five cities in Table 1.
- c.
 1. Compare the graphs of the five circuits.
 2. Determine which weighted graph represents the tour with the shortest total distance.
- d. Compare the shortest tour found using the nearest neighbor algorithm to the tour you identified in Part a.

Discussion

- a.
 1. What advantages are there to using the nearest neighbor algorithm to find a short tour?
 2. What advantages are there to using a brute force algorithm to find a short tour?
- b. Describe a situation in which the nearest neighbor algorithm would provide an appropriate method for solving a problem.
- c. Describe a situation in which a brute force algorithm would provide an appropriate method for solving a problem.
- d. In Part b of the exploration, you drew Hamiltonian circuits that started and stopped in each of five cities. Which of these circuits could you use to plan a tour that starts and stops in Miami? Explain your response.

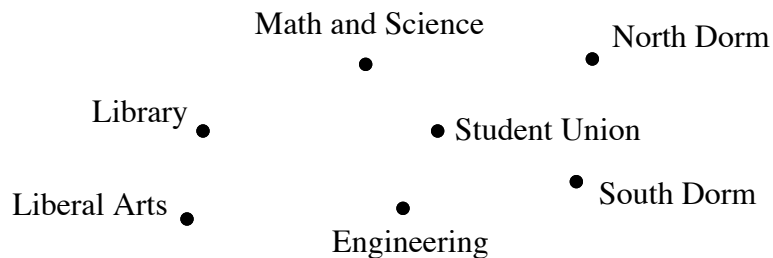
Assignment

- 3.1 The band would like to add Los Angeles, California, to the tour in the exploration. The following table shows the distances in kilometers between Los Angeles and the other five cities.

	Los Angeles
Miami	3764
New York	3945
Chicago	2808
Kansas City	2182
Seattle	1543

- a. Use the nearest neighbor algorithm to draw weighted Hamiltonian circuits that start and stop in each city on the tour.
- b. Determine which circuit results in the shortest total distance.
- c. Using the nearest neighbor algorithm, how many different routes did you have to check?
- d. If you used a brute force algorithm to select the shortest tour, how many routes would you have to check?

- 3.2**
- About how many seconds did it take you to generate and calculate the total distance of one circuit in Problem 3.1?
 - Using your response to Problem 3.1d, about how long would it take you to check all the possible tours?
- 3.3** One of the band members suggests that this summer's tour begin with a send-off concert in your town, visit 24 other cities, then end with a homecoming concert in your town.
- Determine the number of possible 25-city tours that begin and end in the same town.
 - Using the time you estimated in Problem 3.2, determine the total time needed to check all the possible tour routes.
- 3.4** Katherine works in the admissions office at Applegate University. As part of her duties, she leads prospective students on a walking tour of the campus. The diagram below shows the relative positions of the campus buildings.



- If she visits each of the seven buildings exactly once, in how many different ways can Katherine organize the walking tour?
- Since new students live in South Dorm, Katherine likes to begin and end the tour there. How many ways are there to organize this tour?
- The table below shows the distances in meters between buildings on campus. Use these distances and the nearest neighbor algorithm to graph a possible route for the walking tour you described in Part b.

	Student Union	South Dorm	North Dorm	Math/Science	Library	Liberal Arts
Engineering	20	40	55	40	50	45
Liberal Arts	55	90	100	60	20	
Library	50	85	90	40		
Math/Science	25	50	50			
North Dorm	40	30				
South Dorm	35					

Mathematics Note

The nearest neighbor algorithm is just one of many greedy algorithms that can be used to solve optimization problems. In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each stage, regardless of starting and stopping points.

Individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that is not Hamiltonian, then the next best action is taken. When a Hamiltonian circuit is found, the algorithm is complete.

For example, Figure 9 shows the steps used to draw a Hamiltonian circuit with the cheapest link algorithm.

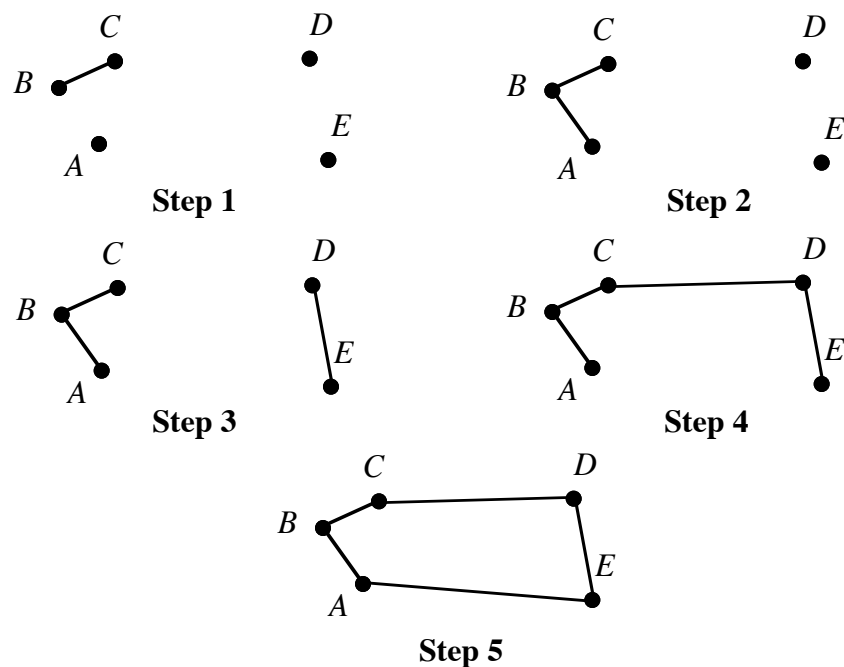


Figure 9: Steps in cheapest link algorithm

In Step 3, notice that *D* and *E* are connected even though these points are farther apart than *C* and *A*. This occurs because drawing an edge between *A* and *C* would complete a circuit that is not Hamiltonian.

- 3.5**
- Use the cheapest link algorithm to determine a possible route for the walking tour in Problem 3.4b.
 - How does this tour compare with the one you identified in Problem 3.4c?

- 3.6**
- a. Use the cheapest link algorithm to find a tour of the six cities in Problem **3.1**.
 - b. Calculate the distance covered by the tour.
 - c. Compare this distance with the distance found using the nearest neighbor algorithm.

* * * * *

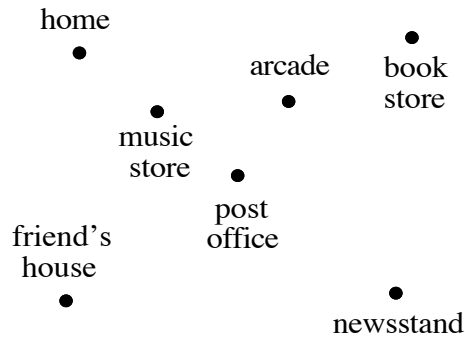
3.7 As the result of a stellar performance on the state math exam, your class has been awarded a trip to four interesting places in your state. Your class may choose both the places and the route, but the trip must start and end in your town.

- a. Choose four destinations and create a table that shows the distances from each one to the others, including your town.
- b. Determine the number of possible routes for your trip.
- c. Use the nearest neighbor algorithm to plan your route. Determine the total distance traveled.
- d. Use the cheapest link algorithm to plan your route. Determine the total distance traveled.
- e. Is the shorter of the two routes necessarily the shortest possible route? Explain your response.
- f. Use a brute force algorithm to support your response to Part e.

* * * * *

Summary Assessment

At 4:30 P.M., Jack and Jill leave their house to run some errands for their parents. They must stop by a friend's house, the music store, the book store, the newsstand, and the post office. As a reward, their parents have given them enough money to play video games at the arcade. The relative positions of these destinations are shown in the diagram below.



Each errand should take 5 min to complete and Jack and Jill must be home by 6:00 P.M. for dinner. The following table shows the walking time in minutes between the locations.

	Friend's House	Newsstand	Book Store	Post Office	Arcade	Music Store
Home	5.5	9.5	6.5	5.5	4.0	2.5
Music Store	3.0	6.5	6.0	2.5	2.5	
Arcade	5.0	5.5	3.5	2.0		
Post Office	3.0	4.0	5.5			
Book Store	8.5	5.5				
Newsstand	3.5					

1. Select a route that allows Jack and Jill to spend more time at the arcade. Draw a weighted graph to represent this route and indicate the order in which the edges were chosen.
2. The games at the arcade last about 5 min each. Using the route you selected in Problem 1, how many games can Jack and Jill play?
3. Identify the algorithm you used to select a route. If you devised an algorithm of your own, describe its steps so that any student in the class could repeat your results.
4. If you had to find the quickest possible route that visited all these destinations, how many circuits would you have to check?

Module Summary

- In an **optimization** problem, the goal is to make the best use of time or other resources.
- A **graph** is a set of **vertices** (plural of **vertex**) and **edges**. Each edge connects two vertices.
- In a **weighted graph**, each edge is assigned a numerical value.
- A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices, however, can occur more than once in a path.
- A **circuit** is a path that starts and ends with the same vertex in which no intermediate vertex is repeated.
- A **Hamiltonian circuit** is a circuit in which every vertex in the graph is visited exactly once.
- A **directed graph** or **digraph** is a graph in which a direction is indicated on each edge.
- The **fundamental counting principle** provides a method for determining the total number of ways in which a task can be performed. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways that the two events can occur is $m \cdot n$.
- An **algorithm** is a step-by-step process used to accomplish a task.
- In a **brute force algorithm**, every potential solution to a problem is examined.
- **Factorial notation** is often used to simplify the representation of the product of the positive integers from 1 to n . If n is a positive integer, then **n factorial** (denoted by $n!$) can be expressed as

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

- **Zero factorial**, or $0!$, is defined as 1.
- In a **greedy algorithm**, the choice made at each step is the best of all remaining choices.
- To use the **nearest neighbor algorithm**, start with any vertex and draw an edge to its nearest vertex. Continue this process from the second vertex to the next nearest vertex not yet visited, and so on, until all vertices have been visited. To complete a Hamiltonian circuit, return to the original vertex.
- In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each stage, regardless of starting and stopping points. Individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that is not Hamiltonian, then the next best action is taken. When a Hamiltonian circuit is formed, the algorithm is complete.

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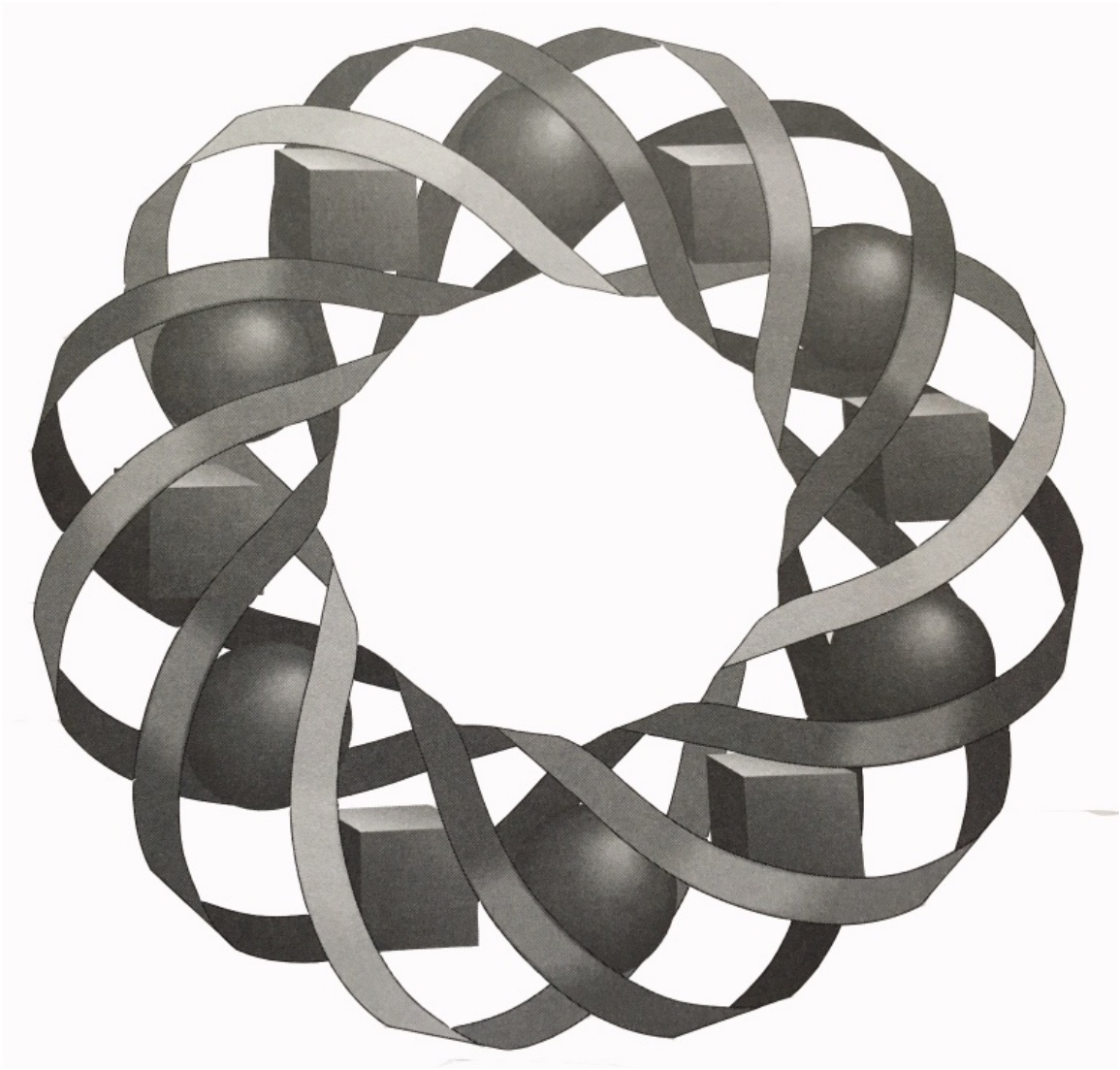
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One Step Beyond



How much did you earn last week? How much was that long-distance phone call? And what did it cost to send that package to Pawtucket? Every time you cash a paycheck, dial a telephone, or mail a postcard, you're stepping off into the unknown.

Bill Chalgren • Bonnie Eichenberger • Paul Swenson • Karen Umbaugh



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One Step Beyond

Introduction

Shelly wants to open a new business: “Shelly’s Shirt Shop by the Sea.” As part of her business plan, Shelly must consider the cost of paying employees, the cost of telephone service, and the cost of shipping her products. Each of these costs is an example of a **step function**. In this module, you examine the step functions which Shelly encounters as she analyzes the cost of doing business.

Activity 1

Before hiring any employees, Shelly must determine exactly how she will pay them for their work. She plans to offer a wage of \$6.00 per hour, but has not decided how to pay workers for parts of hours. She is considering one of three different methods, all based on multiples of 15 minutes—rounding down, rounding to the nearest, or rounding up. These three rounding methods are described in Table 1.

Table 1: Rounding methods for employee hours

Rounding Method	Rounding Down	Rounding to the Nearest	Rounding Up
Description	rounds to previous multiple of 15	rounds to nearest multiple of 15	rounds to next multiple of 15
For 17 min of Work	paid for 15 min	paid for 15 min	paid for 30 min
For 27 min of Work	paid for 15 min	paid for 30 min	paid for 30 min

Exploration

In this exploration, you create two different graphs of an employee’s pay when time is rounded down to the previous multiple of 15 minutes.

- a.
 1. Obtain a copy of the employee’s pay template from your teacher. Each ordered pair on the template represents a number of minutes worked and the corresponding amount earned. Complete the ordered pairs for each minute in the period from 0 min to 79 min.

- Cut out each square on the template, creating a set of 80 squares each labeled with a different ordered pair. Figure 1 shows the ordered pairs for 17 min and 53 min.

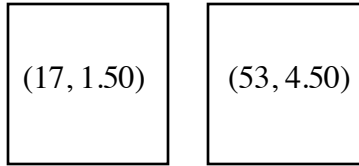


Figure 1: Two paper squares

- On a sheet of freezer paper, create a two-dimensional coordinate system with the origin located near the lower left-hand corner of the paper. Represent an employee's pay in dollars on the y -axis and time in minutes on the x -axis.
- As shown in Figure 2, use the paper squares to plot the corresponding ordered pairs on the coordinate system from Step 3.

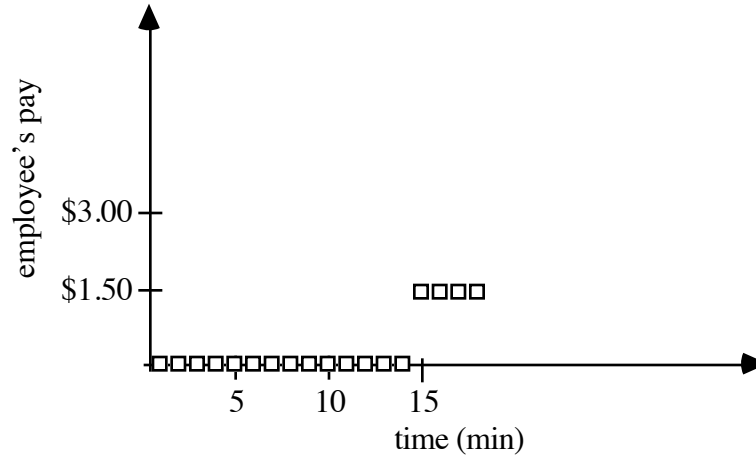


Figure 2: Points plotted on coordinate axes

- Another way of representing an employee's pay when rounding down is to graph the 15-min intervals in the period from 0 min to 79 min.

Mathematics Note

An **interval** is the set of all real numbers between two fixed endpoints. Each of these endpoints may or may not be included in the interval.

When an interval is graphed on a number line, a closed circle indicates that the endpoint is included in the interval. An open circle indicates that the endpoint is not included in the interval.

For example, the graph in Figure 3 represents all real numbers between 15 and 30, including 15 but not including 30.

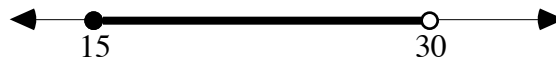


Figure 3: Graph of an interval

One way to describe an interval is with an **inequality**. For example, the interval in Figure 3 can be described by the inequality $15 \leq x < 30$.

A real-number interval also can be described using **interval notation**. In interval notation, a square bracket,] or [, indicates that the endpoint is included in the interval. A parenthesis,) or (, indicates that the endpoint is not included in the interval. For example, the interval shown in Figure 3 can be written as $[15, 30)$.

1. On a sheet of graph paper, create another two-dimensional coordinate system, representing pay in dollars on the y -axis and time in minutes on the x -axis.
2. Graph as many 15-min intervals as necessary to show the pay earned for working from 0 min to 79 min. Use open or closed circles to indicate whether or not each endpoint is included in an interval.

Discussion

- a. Describe the general shapes of the graphs created in the exploration.
- b. If you consider each paper square as representing a pixel on a screen, the graph you created in Part a of the exploration resembles the graphs created by some types of technology. Compare this graph to the graph created in Part b of the exploration.
- c. Does either graph show the exact time when an employee's pay jumps from one step to another? Explain your response.

Mathematics Note

A **relation** between two variables is a set of ordered pairs of the form (x,y) .

The **domain** of a relation is the set of first elements in the ordered pairs (the x -values). The **range** of a relation is the set of second elements in the ordered pairs (the y -values).

A **function** is a relation in which each element of the domain is paired with an element of the range and each element of the domain occurs in only one ordered pair. A function may be described by a rule or equation.

For example, consider a function that pairs each number of hours with a number of minutes. This function can be written as $m = 60h$, where h represents hours and m represents minutes. Figure 4 shows a graph of this function. In this case, both the domain and the range are limited to the set of non-negative real numbers.

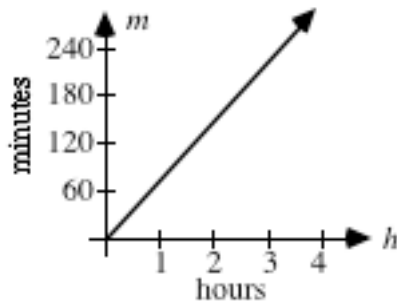


Figure 4: Graph of the function $m = 60h$

- d. Is your graph from Part a of the exploration a function? Explain your response.
- e. Given a graph on a coordinate plane with the domain represented on the horizontal axis and the range on the vertical axis, a **vertical line test** can sometimes be used to determine when the graph is not a function. If a vertical line can be drawn that intersects the graph in more than one point, this indicates that an element of the domain is paired with more than one element of the range. If this occurs, the graph does not represent a function.
- Does your graph from Part a of the exploration pass the vertical line test? Explain your response.
- f.
 1. How is exactly 15 min represented on your graph from Part b of the exploration?
 2. What does the length of each step represent?
 3. What does the vertical distance between successive steps represent?
- g. To determine pay for parts of hours, Shelly could round down using multiples of 10 minutes instead of multiples of 15. How would this affect employee earnings?

- h.** The symbol $+\infty$ can be used to indicate that an interval contains all the numbers greater than a particular value. Similarly, the symbol $-\infty$ can be used to indicate that an interval contains all the numbers less than a particular value. For example, the set of all numbers less than or equal to -2 could be written as $(-\infty, -2]$.


Using these symbols, how could you represent the domain of the function $m = 60h$ in Figure 4 as an interval?

Assignment

- 1.1** Shelly plans to pay her employees \$6.00 per hour. Assuming that she rounds parts of hours down to the previous multiple of 15 minutes, complete the following table.

Time	Wages Earned
14 min	
14 min, 20 sec	
14 min, 57 sec	
15 min	
15 min 2 sec	
16 min	

- 1.2** If Shelly rounds down to the previous multiple of 15 minutes, there is an interval for which an employee will earn \$0.
- Represent this period of time as an inequality.
 - Describe this period of time in a sentence.
 - Graph this period of time on a number line.
 - Express this interval using interval notation.
- 1.3** Some of Shelly's employees begin work at 7:00 A.M. The following table includes a column for the wages earned in each 15-min interval, beginning at 7:00 A.M., when parts of hours are rounded down to the previous multiple of 15.

Time Interval (min)	Interval Notation	Number-line Graph	Wages Earned
$0 \leq t < 15$			
	$[15, 30)$		\$1.50
$30 \leq t < 45$			
\vdots	\vdots	\vdots	\vdots

- Use your responses from Problem 1.2 to complete the first row.
- Continue to complete each row until the table includes 8:19 A.M.

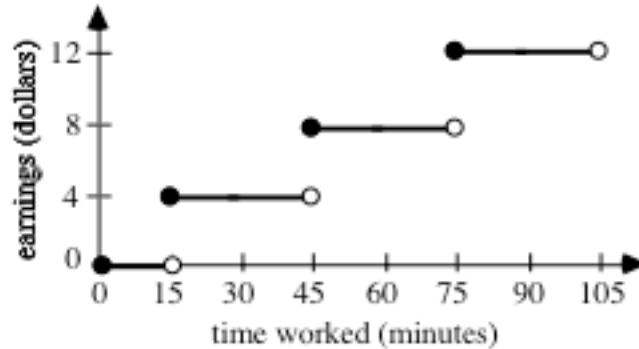
- 1.4** As shown in Table 1, rounding to the nearest multiple of 15 minutes and rounding up to the next multiple of 15 are two different methods for rounding an employee's time. Complete Parts **a** and **b** below using a wage of \$6.00 per hour.
- On a coordinate plane, create a graph of an employee's earnings from 0 min to 79 min when time is rounded to the nearest multiple of 15.
 - Create a graph as in Part **a** for an employee's earnings from 0 min to 79 min when time is rounded up to the next multiple of 15.

- 1.5** Assume that Shelly considers time in 15-min intervals. One of her employees has worked x min where x is in the interval $[0, 60]$.
- List the five different amounts this employee could earn for working x min at a wage of \$6.00 per hour.
 - If time is rounded to the nearest multiple of 15 minutes, determine the time interval that would pay \$3.00. Write this interval both as an inequality and using interval notation.
 - If time is rounded up to the next multiple of 15 minutes, determine the time interval that would pay \$3.00. Write the interval both as an inequality and using interval notation.
 - Complete the following table, listing the time intervals for which the two rounding methods pay each dollar amount identified in Part **a**.

Money Earned	Time Worked in Minutes	
	Rounding Up	Rounding to the Nearest
\$0.00	$x = 0$	$0 \leq x < 7.5$
	$0 < x \leq 15$	
⋮	⋮	⋮
\$6.00		

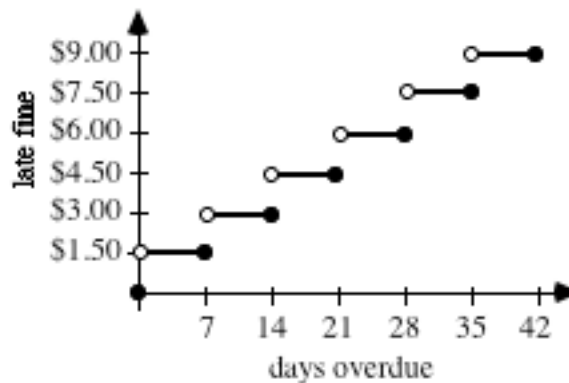
- Identify all the time intervals in the first hour of work for which the two rounding methods result in the same amount of pay.
- 1.6** Use your graphs for the three different rounding methods to help you answer the following questions.
- Using each rounding method, what is the minimum amount of time an employee can work to earn \$1.50?
 - Which of the three rounding methods favors employees? Explain your response.
 - Which of the three rounding methods favors the employer? Explain your response.
 - Which method might serve as a compromise for employees and employers? Explain your response.

- 1.7 Identify all the time intervals in the first hour of work for which all three rounding methods in Table 1 result in the same amount of pay.
- 1.8 The following graph shows another method which Shelly could use to pay her employees.



- a. Using this method, how much would Shelly pay her employees per hour?
- b. Describe this new payment method.
- * * * * *
- 1.9 A salesperson in a clothing store receives a commission of \$135 for each \$1000 in sales. The store “rounds down” when determining commissions. For example, if the amount of merchandise sold is less than \$1000, the salesperson gets no commission. If \$1200 worth of merchandise is sold, the salesperson gets \$135.
- a. How much commission would a salesperson receive for \$4800 in sales?
- b. Identify the intervals of sales for which a salesperson would earn from \$0 to \$540 in commissions. List the commission earned for each interval.
- c. Create a graph of commission earned versus merchandise sold for the intervals identified in Part b.

- 1.10 A library uses the schedule shown below to determine fines for overdue books.



- a. Describe the method the library uses for rounding the number of days overdue.
- b. Explain why the graph represents a function.
- c. Identify the domain and range of the function.
- d. Given a fine of \$4.50, determine the interval of days for which the book might have been overdue. Write this interval both as an inequality and using interval notation.

* * * * *

Activity 2

Shelly’s employees park their cars at Jeff’s Parking Lot, which is located next to their store. To accommodate shoppers and delivery trucks, parking is free at Jeff’s for the first hour. After that, the lot charges \$1.00 per hour. Parts of hours are rounded up to the next whole hour.

Exploration 1

In this exploration, you examine an equation Jeff can use to calculate parking charges.

- a. Table 2 shows the cost of parking in Jeff’s lot. Use the patterns you observe in the table to complete the bottom row.

Table 2: Cost of parking in Jeff’s lot

Hours Parked (interval notation)	Number of Hours Charged	Parking Cost
[0, 1)	0	\$0
[1, 2)	1	$1 \cdot \$1.00 = \1.00
[2, 3)	2	$2 \cdot \$1.00 = \2.00
⋮	⋮	⋮
[x, (x + 1))		

- b. Create a graph of parking cost versus hours parked using the domain [0, 8).
- c. Determine the parking costs for each of the following times.
 1. 1.5 hr
 2. 5.1 hr
 3. 7.9 hr

Mathematics Note

The **greatest integer function** assigns every element x in the domain to the greatest integer less than or equal to x . This function can be written as $y = [x]$.

Using the greatest integer function, the number 2.7, for example, would be assigned the value of 2. This can be written as $[2.7] = 2$. Similarly, $[1.9] = 1$, $[0.4] = 0$, and $[-3.4] = -4$.

- d. Evaluate $[x]$ for each of the following values of x :
1. 2.2
 2. 2.9
 3. -2.2
 4. -2.9
- e. Determine how the technology you use handles the greatest integer function.
- f. When y represents hours parked and x represents parking costs, the greatest integer function can be used to determine the charges for Jeff's Parking Lot.
1. Use technology to graph $y = [x]$.
 2. How does the graph of $y = [x]$ differ from your graph in Part **b**?

Discussion 1

- a. Describe how to use the equation $y = [x]$ to determine the cost of parking in Jeff's Lot for each of the following:
1. 7.5 hr
 2. 0.9 hr
- b. After the first hour, parking in Jeff's Lot is no longer free. How is this represented in your graphs from Parts **b** and **f** of Exploration 1?
- c. For what values of the domain does $y = [x]$ model the cost of parking at Jeff's Lot?

Exploration 2

As you observed in Exploration 1, the function $y = [x]$ can be used to model the parking costs at Jeff's Lot. In this exploration, you discover how greatest integer functions can be used to model other situations.

- a. 1. In the equation $y = a[x]$, the greatest integer function is multiplied by a constant a . Complete Table 3 for $a = 1$, $a = 3$, and $a = -2$.

Table 3: Multiplying $[x]$ by a constant

x	$y = 1[x]$	$y = 3[x]$	$y = -2[x]$
0			
0.4			
1.6			
2.4			

2. Graph $y = 1[x]$, $y = 3[x]$, and $y = -2[x]$ on the same set of axes.
3. Compare the three graphs.
- b. 1. In the equation $y = [x] + d$, a constant d is added to the greatest integer function. Complete Table 4 for $d = 0$, $d = 3$, and $d = -5$.

Table 4: Adding a constant to $[x]$

x	$y = [x] + 0$	$y = [x] + 3$	$y = [x] - 5$
0			
0.4			
1.6			
2.4			

2. Graph $y = [x] + 0$, $y = [x] + 3$, and $y = [x] - 5$ on the same set of axes.
3. Compare the three graphs.

- c. 1. In the equation $y = [bx]$, the variable x is multiplied by a constant b before the greatest integer function is applied. Complete Table 5 for $b = 1$ and $b = 0.5$.

Table 5: Multiplying x by a constant b in $[x]$

x	$y = [1x]$	$y = [0.5x]$
0		
0.4		
1.6		
2.4		
3.6		
4.2		

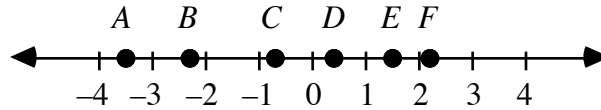
2. Graph $y = [1x]$ and $y = [0.5x]$ on the same set of axes.
 3. Compare the two graphs.

Discussion 2

- a. If $y = 3[x]$ models the cost to park x hours, what effect does the constant 3 have on the graph of $y = [x]$.
- b. Describe how the value of a affects the graph of $y = a[x]$.
- c. If $y = [x] + 3$ models the cost to park x hours, what effect does the constant 3 have on the graph of $y = [x]$.
- d. Describe how the value of d affects the graph of $y = [x] + d$.
- e. If $y = [0.5x]$ models the cost to park x hours, what effect does the constant 0.5 have on the graph of $y = [x]$.
- f. Describe how the value of b affects the graph of $y = [bx]$.
- g. The function $y = [x]$ describes the cost of parking x hours in Jeff's Parking Lot. The function $y = [0.5x]$ describes the cost of parking for x hours in a competitor's lot.
- Which lot would you prefer to park in?
 - How could you use the graphs of the functions to choose the better parking lot?

Assignment

- 2.1 Determine the value of the greatest integer less than or equal to each of the six points on the following graph.



- 2.2 Determine the value of y in each of the following equations:

- $y = [-2.3]$
- $y = [0]$
- $y = [1.5] + 0.5$
- $y = 3[(0.45) \cdot 10] + 7$

- 2.3 a. 1. Explain why the solution set of $[x] = 9$ can be represented as shown below.



- Express the solution set of $[x] = 9$ as an inequality.
 - Represent the solution set of each of the following equations as a graph on a number line and as an inequality.
 - $[x] = 5$
 - $[x] = -3$
 - $[x] = n$
- 2.4 Jeff is considering three different ways to change the parking fees at his lot. Write an equation that could be used to model each of the following situations. Use technology to verify that each equation correctly describes the fees.
- Jeff would charge \$1.00 per hour or any part of an hour. Using this method, it would cost \$1.00 to park for 20 min and \$3.00 to park for 2 hr 20 min.
 - Jeff would charge \$1.50 per hour or any part of an hour. Using this method, it would cost \$1.50 to park for 20 min and \$4.50 to park for 2 hr 20 min.
 - Jeff would charge \$0.75 for each full 30-min period. Using this method, there would be no cost to park 20 min. It would cost \$0.75 to park 45 min and \$6.75 to park 4 hr 35 min.

- 2.5**
- On a coordinate plane, draw a graph that shows values of x in the interval $[-4, 5.5)$ rounded to the nearest integer. Using this method of rounding, 2.4 is rounded to 2, while 2.6 is rounded to 3. Since 2.5 is exactly between 2 and 3, it is rounded to 3.
 - Describe the shape of the graph. Determine whether or not it represents a function and justify your response.
 - Write an equation of the form $y = [x + c]$ to represent the graph. Use technology to check your equation for several values of x .
 - Graph your equation from Part **c** for values of x in the interval $[-4, 5.5)$. Does your equation work for $x = -1.5$ and $x = -2.5$? Explain your response.
 - Determine how the rounding function on the technology you use rounds negative numbers such as -1.5 and -2.5 . Does your equation from Part **c** produce the same results?

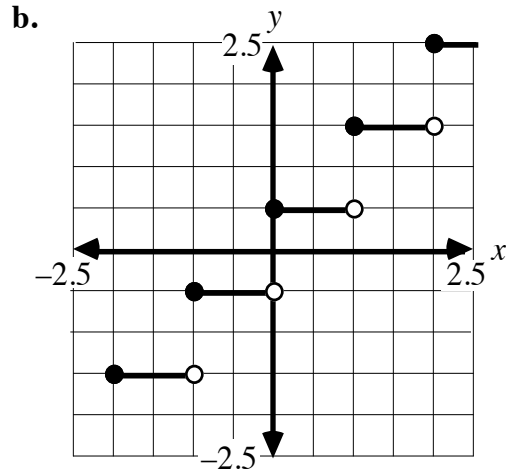
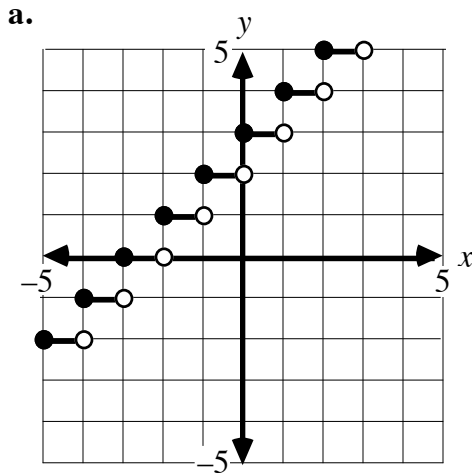
- 2.6**
- On a coordinate plane, draw a graph that shows values of x in the interval $(-5, 5]$ rounded up to the next integer. Using this method of rounding, 2.4, 2.5, and 2.6 are all rounded up to 3, while -2.4 , -2.5 , and -2.6 are all rounded up to -2 .
 - Graph $y = -[-x]$. Compare this graph to your graph in Part **a**.
 - What might you call the function given in Part **b**?

- 2.7**
- The table below shows four different functions. Complete the table using $-4, -3, -2, -1, 0, 1, 2, 3,$ and 4 for x .

x	$y = [x + 2]$	$y = [x] + 2$	$y = [x + 2.5]$	$y = [x] + 2.5$
-4				
-3				
-2				
-1				
0				
1				
2				
3				
4				

- Using the domain $[-10, 10]$, graph $y = [x + 2]$ and $y = [x] + 2$ on the same set of axes. Compare the two graphs.
- Do you think that $[x] + c = [x + c]$? Explain your response.

2.8 Write an equation for each step function graphed in Parts a and b.

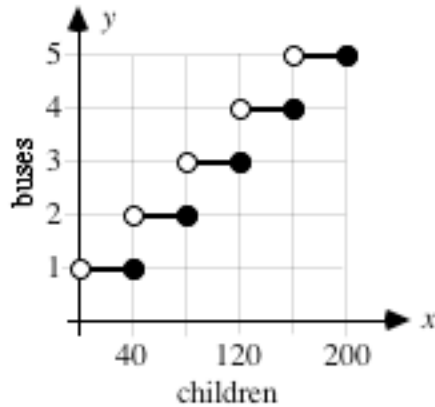


2.9 One teacher rounds student scores to the nearest integer, then applies the grading scale shown in the table below.

Grade	Score
A	90-100
B	80-89
C	70-79
D	60-69
F	0-59

- Why does this teacher round scores before assigning letter grades?
- Write an inequality that describes all the scores that correspond to a grade of A.
- Draw a graph to represent this teacher's grading system.
- Does your graph in Part c represent a function? Explain your response.
- Do you think that this method of rounding scores is fair? Explain your response.

- 2.10** a. While planning for the summer season, a camp director drew the following graph. This graph shows the number of buses needed to transport up to 200 children to camp.



1. Describe the domain for this graph.
 2. How many children can each bus carry?
 3. Write an equation that models the graph. Use technology to test your equation.
- b.**
1. The director of another camp plans to hire buses that can carry up to 79 children each. Write a step function that models this situation.
 2. Use this function to determine the number of buses required to transport 1543 students to camp.
- 2.11** Dana runs the concession stand at a concert hall. She is paid \$175 per week plus an additional \$45 for each \$300 in sales.
- a. Determine how much Dana earns for a week in which sales reach \$1000.
 - b. Draw a graph to represent Dana's earnings for weekly sales from \$0 to \$1000.
 - c. Write an equation to represent Dana's weekly earnings.

* * * * *

Activity 3

Shelly's budget for her new store includes telephone costs. Before selecting a long-distance company, she wants to compare the discounts they offer. In this activity, you use the greatest integer function to graph these discounts.

Exploration

Company A offers an initial evening rate of \$0.10 per min. After the first 5 min, the company cuts the rate to \$0.09 per min. After the next 5 min, the rate falls another penny to \$0.08 per min. This pattern continues for the next three multiples of 5 minutes, until a minimum rate of \$0.05 per min is reached.

When calculating the cost of a call, company A always rounds parts of minutes up to the next whole minute.

- a. Table 6 shows an incomplete list of the company's rates.

Table 6: Long-distance telephone rates

Interval of Minutes	Decrease in Initial Rate	Discounted Rate (Cost per Minute)	Cost of Call
(0, 1)	\$0.00	\$0.10	\$0.10
[1, 2)	\$0.00	\$0.10	\$0.20
[2, 3)	\$0.00	\$0.10	\$0.30
⋮	⋮	⋮	⋮
[7, 8)	\$0.01	\$0.09	\$0.77
⋮	⋮	⋮	⋮
		\$0.05	

Create a spreadsheet with headings like those in the table, then extend the spreadsheet until the discounted rate falls to \$0.05 per min.

- b.
1. Create a graph using the number of minutes as the domain and the discounted rate as the range. (Hint: Select at least one value from each interval of minutes in Table 6. Plot these values with the corresponding discounted rates.)
 2. Make a sketch of your graph. Make sure that your sketch clearly indicates the points where the rate changes. **Note:** Save your graph for use in Problem 3.2.
 3. After a certain number of minutes, the rate reaches the minimum of \$0.05 per minute. Extend your sketch to show what happens to the graph after this point is reached.

Discussion

- a. What type of association exists between the number of minutes and the discounted rate? Explain your response.
- b. Compare the general shape of the graph in Part **b** of the exploration with the graphs you created in previous activities.
- c. Explain why the graph of the phone company's rates is a function.
- d. How did you use the spreadsheet to determine the cost of a call that lasts a given number of minutes?

Assignment

- 3.1 Company B offers a different discount than Company A. The initial evening rate is \$0.10 per minute. After the first 10 min, the company cuts the rate to \$0.07 per min. After the next 10 min, the rate falls to \$0.04 per min. This is the minimum rate for company B.
 - a. Create a table like Table 6 to show company B's rates for evening calls up to 30 min long.
 - b. Using company B's rates, determine the total cost of a call for each minute from 1 min to 30 min.
 - c. Using company A's rates, determine the total cost of a call for each minute from 1 min to 30 min.
 - d. Compare the costs of calls made using each company.
- 3.2
 - a. Create a graph of the discounted rate versus the number of minutes for company B. Use a domain of [0 min, 30 min].
 - b. Compare this graph with the graph of company A's rates you created in the exploration.
 1. How does the vertical distance between steps compare?
 2. When do the rates for each company change?
 3. When are the companies' rates equal?
 - c. Why are these graphs meaningful only in the first quadrant? Explain your response.

* * * * *

- 3.3** Shelly's Shirt Shop charges by the number of shirts ordered. The following table shows the cost per shirt for orders from 1–47 shirts.

Number of Shirts	Cost per Shirt
[1, 11]	\$16.00
[12, 23]	\$14.50
[24, 35]	\$13.00
[36, 47]	\$11.50

- Describe the domain for the relationship in the table.
- Create a graph of the cost per shirt versus the number of shirts.
- Write an equation that models the cost per shirt for any number of shirts in the interval [1, 47].

* * * * *

Summary Assessment

1. The U.S. Postal Service uses the following chart to determine the cost of mailing letters first class (based on rates effective January 1, 1995). Using this chart, a letter that weighs 0.5 oz costs \$0.32 to mail, while one that weighs 3.3 oz costs \$1.01.

Weight up to and including (oz)	Cost
1	\$0.32
2	\$0.55
3	\$0.78
4	\$1.01
5	\$1.24
6	\$1.47
7	\$1.70
8	\$1.93
9	\$2.16
10	\$2.39
11	\$2.62

When a parcel's weight exceeds 11 oz, the post office uses a different chart to determine mailing costs, as shown below.

Weight up to and including (lb)	Cost
2	\$3.00
3	\$4.00
4	\$5.00
5	\$6.00

When a parcel's weight exceeds 5 lb, the cost depends on both weight and destination.

- a. In a paragraph, describe the cost of mailing letters that weigh up to 11 oz.
- b. Use inequalities to express the weights in each row of the chart for parcels that weigh more than 11 oz.
- c. Draw a graph that represents the cost of mailing letters or parcels that weigh up to 5 lb.
- d. Describe the advantages of each of the following ways of representing postal rates: charts, explanations, inequalities, and graphs.

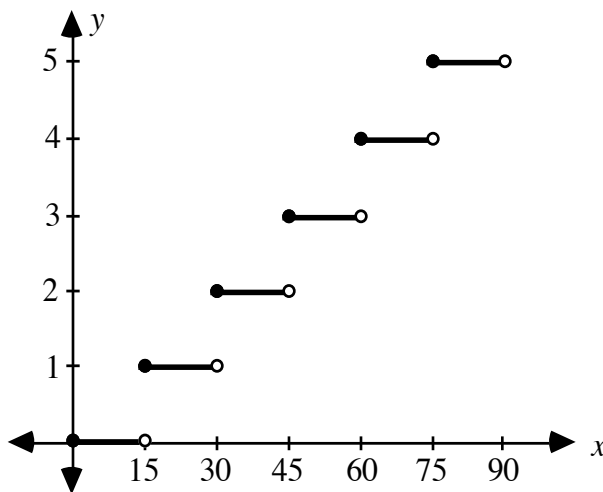
2. Another function that rounds numbers is the **integer part function**. The integer part function removes the decimal or fractional part of a number, leaving only the integer. This process is often referred to as **truncation**.

When $x = 7.2$, for example, the integer part function removes the decimal part, leaving the integer 7. When $x = 7.9$, the result is also 7.

- a. Describe how the integer part function is related to the functions $y = [x]$ and $y = -[-x]$.
- b. Find two intervals, one including positive numbers and one including negative numbers, for which the following functions give the same result:
 1. “integer part of x ” and “ x rounded to the nearest integer”
 2. $[x]$, “ x rounded to the nearest integer,” and “integer part of x ”

Module Summary

- An **interval** is the set of all real numbers between two fixed endpoints. Each of these endpoints may or may not be included in the interval.
- When an interval is graphed on a number line, a closed circle indicates that the endpoint is included in the interval. An open circle indicates that the endpoint is not included in the interval.
- A real-number interval can be described with an **inequality** or by using **interval notation**. In interval notation, a square bracket, $]$ or $[$, indicates that the endpoint is included in the interval. A parenthesis, $)$ or $($, indicates that the endpoint is not included in the interval.
- A **relation** between two variables is a set of ordered pairs of the form (x,y) .
- The **domain** of a relation is the set of first elements in the ordered pairs (the x -values). The **range** of a relation is the set of second elements in the ordered pairs (the y -values).
- A **function** is a relation in which each element of the domain is paired with an element of the range and each element of the domain occurs in only one ordered pair. A function may be described by a rule or equation.
- The graph below shows one example of a **step function**.



- The **greatest integer function** is a step function that assigns to the number x the greatest integer less than or equal to x . The function may be symbolized as $y = [x]$.

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From Rock Bands to Recursion



You've endured hours on the highway, rows of seats in an empty amphitheater, lagging ticket sales, crowded soft drink concessions, bouncing beach balls, and—finally—the notes of an electric guitar. Is this a pattern worth repeating?

Shirley Bagwell • Janet Higgins • Sandy Johnson • Terry Souhrada



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From Rock Bands to Recursion

Introduction

Since the time you first learned to count (1, 2, 3), skip-count (2, 4, 6, 8), and stack blocks, you've been investigating patterns of numbers. For example, if you continue to stack blocks as shown in Figure 1, the number of blocks in each row of a stack forms a pattern. The total number of blocks used to build each stack forms another pattern.

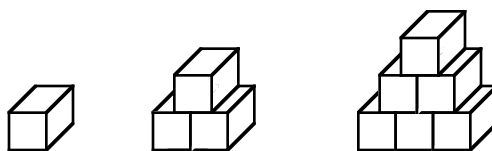


Figure 1: Stacks of blocks

In this module, you go on tour with the rock band Stellar Attraction. Along the way, you use technology and mathematical formulas to examine patterns in the sales of CDs and concert tickets. You also analyze the sequence of notes played by the lead guitarist.

Activity 1

In this activity, you investigate the basic number patterns formed by the number of performances during Stellar Attraction's first world tour.

Exploration

Before starting their world tour, the band's concert career was relatively short—only 6 performances. The upcoming schedule, however, contains 2 performances per week. Table 1 summarizes the band's career number of performances over the next 10-week period.

Table 1: Performances by Stellar Attraction

No. of Weeks on Tour	Total No. of Performances
1	8
2	10
3	12
4	14
⋮	⋮
10	26

- a. Complete Table 1 for each of the next 10 weeks.
- b. Record any patterns you observe in the table.
- c. Use the patterns you discovered to answer the following questions.
 1. After how many weeks will the band have given 24 performances?
 2. How many performances will they have given after 12 weeks?
 3. After 16 weeks, Stellar Attraction will have performed 38 times. What will be the total number of performances after 18 weeks?

Mathematics Note

A **sequence** is an ordered list. Each item in the list is a **term** of the sequence.

The terms of a sequence may be represented by symbols, such as $p_1, p_2, p_3, \dots, p_n$. These symbols are **subscripted variables**, and the natural numbers (1, 2, 3, ..., n) are the **subscripts**. The symbol p_1 (read “ p sub one”) represents the first term of the sequence, the symbol p_2 (read “ p sub two”) represents the second term of the sequence, and so on. The symbol p_n (read “ p sub n ”), represents the **general**, or **n th, term** of a sequence.

For example, consider the following ordered list of numbers 16, 18, 20, 22, 24, In this sequence, $p_1 = 16$, $p_2 = 18$, $p_3 = 20$, and so on.

- d.
 1. Create a spreadsheet with headings like those shown in Table 2 below. The entries in the right-hand column represent the total number of performances at the end of each week. They also represent the terms of a sequence.

Table 2: Performances spreadsheet

No. of Weeks (n)	Total No. of Performances (p_n)
1	8
2	10
3	12
4	14
\vdots	\vdots
50	

2. Use the spreadsheet to determine the total number of performances after each week for the next 50 weeks.

Mathematics Note

A **recursive formula** is a rule for finding any term in a sequence by using the preceding term(s). The process of using a recursive formula is known as **recursion**.

A sequence in which every term after the first is found by adding a constant value to the preceding term is an **arithmetic sequence**.

The recursive formula for calculating any term in an arithmetic sequence is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, \quad n > 1 \end{cases}$$

where a_1 is the first term, a_n is the n th term, a_{n-1} is the term preceding a_n , and d is the **common difference** between any two consecutive terms, $a_n - a_{n-1}$.

For example, consider the sequence in which the first term (a_1) is 27 and the common difference (d) between any two consecutive terms is 5. The recursive formula for this sequence is:

$$\begin{cases} a_1 = 27 \\ a_n = a_{n-1} + 5, \quad n > 1 \end{cases}$$

Using this formula, the first four terms of the sequence can be found as follows:

$$a_1 = 27$$

$$a_2 = a_{2-1} + d = a_1 + d = 27 + 5 = 32$$

$$a_3 = a_{3-1} + d = a_2 + 5 = 32 + 5 = 37$$

$$a_4 = a_{4-1} + 5 = a_3 + d = 37 + 5 = 42$$

- e. Find a recursive formula for the sequence in the right-hand column of Table 2.

Discussion

- a. How did you use the patterns you described in Part b of the exploration to complete Table 2?
- b. Use the sequence in the right-hand column of Table 2 to answer the following questions.
1. What is the value of n for p_{20} ?
 2. What is the value of p_{20} ?
 3. If $p_n = 20$, what is the value of n ?
 4. When $n = 8$, what is the value of p_{n-1} , the term before p_n ?
- c. How can you use subscript notation to express the fact that the 30th term of a sequence is 66?

- d. Explain why the sequence found in each column of Table 2 is an arithmetic sequence.

Assignment

- 1.1 For their next performance, Stellar Attraction must drive from St. Louis, Missouri, to Portland, Oregon: a distance of 3290 km. They can average 100 km per hour during the trip.

- a. Create a spreadsheet with headings like those in the table below. Let h represent the number of hours driven and k_h represent the number of kilometers remaining after each hour.

Hours Driven (h)	Kilometers Remaining (k_h)
1	3190
2	3090
3	2990
\vdots	\vdots

- b. During which hour does the band arrive in Portland?
- c. What is the value of k_5 ?
- d. How many kilometers remain after the band has driven for 13 hr? Express your answer using subscript notation.
- e. For how many hours has the band been driving when they are 1390 km from Portland? Express your answer in subscript notation.
- f. What is the value of k_{h-1} when $k_h = 2290$?
- g. Write a recursive formula that describes the pattern for k_h .
- 1.2 After arriving in Portland, the band finds that concert tickets have been selling well. The ticket agency sold 790 tickets on the first day. During the next 20 days, they sold an average of 213 tickets per day.
- a. Let n represent the number of days that tickets have been on sale and t_n represent the total number of tickets sold after n days. Express the first 5 terms of this sequence using subscript notation.
- b. Using the sequence from Part a, on what day did total sales exceed 2200 tickets? Express your answer using subscript notation.
- c. What is the value of t_{n+1} when $t_n = 2494$?
- d. Write a recursive formula that describes the pattern of ticket sales.
- e. To complete Parts a-d, you assumed that the pattern of ticket sales was an arithmetic sequence where $t_1 = 790$ and $t_{21} = 790 + 20(213)$. Do you think that this is a reasonable assumption?

1.3 In some arithmetic sequences, the numbers increase with each successive term. In others, the numbers decrease with each successive term. What can you say about the common differences used to form these sequences?

1.4 Consider the following recursive formula:

$$\begin{cases} t_1 = 9.0 \\ t_n = t_{n-1} + 0.5, \quad n > 1 \end{cases}$$

- a. What is the value of the first term of this sequence?
- b. What are the next four terms of this sequence?

1.5 Find a recursive formula for the arithmetic sequence 3, 7, 11, 15,

- 1.6**
- a. Create your own arithmetic sequence.
 - b. What is the common difference for your sequence?
 - c. Write a recursive formula for your sequence.

1.7 In a paragraph, compare the two sequences described by the formulas shown below.

$$\begin{cases} t_1 = 5 \\ t_n = t_{n-1} + 3, \quad n > 1 \end{cases} \text{ and } \begin{cases} r_1 = -4 \\ r_n = r_{n-1} + 3, \quad n > 1 \end{cases}$$

* * * * *

1.8 The *National Geographic* magazine is published 12 times per year. Each edition of the magazine is approximately 0.8 cm thick.

- a. Imagine that your parents started collecting *National Geographic* in the year you were born. How much shelf space would they need to display their collection at the end of this year?
- b. Write a recursive formula that describes the amount of shelf space needed at the end of a year. Let the first term of the sequence be the width of the shelf when you were in kindergarten.
- c. Use your recursive formula to predict how much shelf space will be required when you are 60 years old.

- 1.9**
- a. Melinda has \$51.00 and spends \$3.00 per week. Write a recursive formula to describe the amount of money m_n Melinda has at the beginning of week n .
 - b. Kris has \$11.00 and saves an additional \$2.00 per week. Write a recursive formula to describe the amount of money k_n Kris has at the beginning of week n .
 - c. When will Melinda and Kris have the same amount of money?

1.10 Consider the following recursive formula:

$$\begin{cases} t_1 = 1 \\ t_2 = 1 \\ t_n = t_{n-2} + t_{n-1} \text{ for } n > 2 \end{cases}$$

- Find t_3 .
- Generate the first 10 terms of the sequence.
- Create a scatterplot of the first 10 terms of the sequence versus the term number.
- Is the sequence an arithmetic sequence? Justify your response.

* * * * *

Activity 2

The band’s record company tracks the number of compact discs (CDs) that Stellar Attraction sells each week. During the week of June 10–16, the band sold 9050 copies. Sales projections indicate that the band can expect weekly sales to increase by an average of 2353 copies each week for the next year. These projections are shown in Table 3 below.

Table 3: Weekly CD sales

Week	Week No.	CD Sales
June 10–16	1	9050
June 17–23	2	11,403
June 24–30	3	13,756
July 1–7	4	16,109
⋮	⋮	⋮
December 23–29	29	
⋮	⋮	⋮
June 2–8	52	

If the projections are accurate, how many CDs will the band sell during the last week of December? When will the total number of copies sold exceed 1 million? Since the predicted values for weekly sales represent an arithmetic sequence, you could use a recursive formula to answer these questions. However, using a recursive formula can be time consuming. In this activity, you develop another type of formula that will allow you to respond more quickly.

Exploration

The two left-hand columns in Table 4 show the term number and terms for an arithmetic sequence. The two right-hand columns show expanded (and equivalent) forms of the term values. In this exploration, you use the patterns in this table to develop another type of formula for the sequence.

Table 4: Patterns within arithmetic sequences

Term Number (n)	Term (p_n)	Recursive Form of p_n	Another Form of p_n
1	27	27	$27 + (0 \cdot 2)$
2	29	$27 + 2$	$27 + (1 \cdot 2)$
3	31	$27 + 2 + 2$	$27 + (2 \cdot 2)$
\vdots			
7			

- Complete Table 4 for the first seven terms of the sequence. Record any patterns you discover.
- Write an expression that represents the number of 2s added to the first term to form the n th term.
- Use the pattern in the far right-hand column to write a formula for p_n in terms of n .
- Create a scatterplot of the data in the two left-hand columns of the table. Use the horizontal axis for the term number (n) and the vertical axis for the term value (p_n).
- Find an equation that models the scatterplot and graph it on the coordinate system from Part d.

Compare this equation to the one you wrote in Part c.

Discussion

- In the sequence shown in Table 4, how many 2s would you have to add to find p_{21} ?
 - How many 2s would have to be added to 27 to find p_n ?
- Describe the formula you wrote for p_n using n as a variable.
 - Use this formula to determine the values of p_{47} and p_{100} .

Mathematics Note

An **explicit formula** for calculating any specific term in an arithmetic sequence is:

$$a_n = a_1 + d(n - 1)$$

where a_n is the n th term, a_1 is the first term, and d is the common difference between any two consecutive terms, $a_n - a_{n-1}$.

For example, consider the arithmetic sequence 8, 14, 20, 26, In this sequence, the first term is 8 and the common difference is 6. The explicit formula for this sequence is $a_n = 8 + 6(n - 1)$. This formula can be used to find the 20th term of the sequence as follows:

$$\begin{aligned} a_{20} &= 8 + 6(20 - 1) \\ &= 8 + 6(19) \\ &= 122 \end{aligned}$$

The sum of the terms of an arithmetic sequence is an **arithmetic series**.

For example, consider the arithmetic sequence 8, 14, 20, 26. The corresponding arithmetic series is $8 + 14 + 20 + 26$, or 68.

- c. How does the graph of a linear equation differ from the graph of an arithmetic sequence?
- d. Do you think that the scatterplot of an arithmetic sequence can always be modeled by a linear equation in slope-intercept form, $y = mx + b$? Why or why not?
- e. Using the distributive property, the explicit formula for an arithmetic sequence, $a_n = a_1 + d(n - 1)$, can be written as $a_n = a_1 + dn - d$. Using the associative property, it can also be written as $a_n = dn + (a_1 - d)$.
Assuming that the equation $y = mx + b$ models the same arithmetic sequence, describe the relationship between each of the following:
 1. the n th term a_n and y
 2. the term number n and x
 3. the difference d and the slope m
 4. the first term a_1 and the y -intercept b
- f. In what types of situations would an explicit formula be easier to use than a recursive formula?
- g. Using the relationships described in Part e of the discussion, show that the explicit formula for an arithmetic sequence, $a_n = a_1 + d(n - 1)$, is equivalent to the linear equation of the form $y = mx + b$ that models the same sequence.

Assignment

- 2.1** As shown in Table 3, weekly CD sales for Stellar Attraction are expected to increase by an average of 2353 copies per week for the next 52 weeks after June 10–16.
- If this projection is accurate, how many CDs will be sold during the week of December 23–29?
 - When will the total number of CDs sold exceed 1 million?
- 2.2** Stellar Attraction is playing at the Jan-San Amphitheater. The amphitheater has 120 seats in the front row, 136 seats in the second row, and 152 seats in the third row. This pattern continues from row to row. The last row has 584 seats.
- Write an explicit formula to determine the number of seats in any row of the theater.
 - How many seats are in the 16th row?
 - How many rows are in the theater?
 - How many total seats are in the theater?
- 2.3** Imagine that you work as an usher at the Jan-San Amphitheater. Your starting wage is \$4.25 per hour. Periodically, you will receive a raise of \$0.10 per hour. Write an explicit formula to calculate your hourly wage after n raises.
- 2.4**
- Before the concert begins, the Jan-San Amphitheater contains 85 employees. The doors open to the public at 6:00 P.M. Between 6:00 P.M. and 6:15 P.M., an average of 9 people per second enter the theater. Let p_n represent the number of people in the theater after n seconds.
 - List the first 5 terms of the sequence.
 - Identify the common difference d .
 - Write an explicit formula of the form $a_n = a_1 + d(n - 1)$.
 - Determine the number of people in the amphitheater at 6:15 P.M.
 - When the concession stands open, they have a supply of 8500 L of soft drinks. Soft drink sales average 110 L per minute throughout the evening. Let l_n represent the number of liters remaining after n minutes.
 - List the first 5 terms of the sequence.
 - Identify the common difference d .
 - Write an explicit formula of the form $a_n = a_1 + d(n - 1)$.
 - Determine how long the supply of soft drinks will last.

2.5 A cashier starts the evening with \$50.00 in the drawer. During the next 45 minutes, an average of \$2.15 per minute is added to the drawer. Let c_n represent the number of dollars in the drawer at the end of n minutes.

a. Write a recursive formula for this sequence of the form

$$\begin{cases} c_1 = \text{first term} \\ c_n = c_{n-1} + d, \quad n > 1 \end{cases}$$

Explain what each term in the formula represents.

b. Write an explicit formula for this sequence of the form

$$c_n = c_1 + d(n - 1)$$

Explain what each term in the formula represents.

c. Write an equation of the form $y = mx + b$ that models this sequence. Explain what each term in the equation represents.

* * * * *

2.6 Brianna chairs the membership committee of a club with 33 members. To meet club goals, she plans to recruit 5 new members each month.

a. If Brianna meets her membership goals, how many members will the club have after the next 12 months?

b. Each club member receives a monthly newsletter. If each newsletter requires one stamp to mail, how many stamps will be needed during the next year?

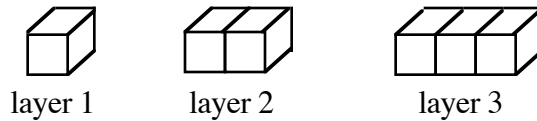
2.7 **a.** Create a scatterplot of the data in the following table.

n	t_n
1	7
2	9
3	11
4	13
5	15

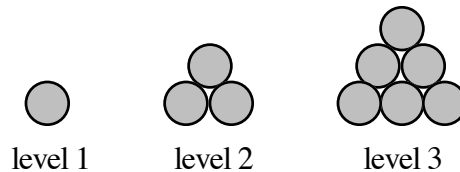
b. Write an equation for the line that fits the data. Graph this equation on your coordinate system from Part **a**. What does this graph tell you about an arithmetic sequence?

c. Write an explicit formula to determine t_n . How does this compare with the equation you wrote in Part **b**?

- 2.8** Raul works in a supermarket. He is stacking blocks of cheese for a dairy display. The following diagram shows the first three layers, top to bottom, in his display.



- a. Make a sketch of the next three layers in the display.
 - b. Let l_n represent the number of blocks in layer n . Find the first six terms of this sequence.
 - c. Is the sequence you wrote in Part **b** an arithmetic sequence? Justify your response.
 - d. Write a formula to describe this sequence.
 - e. Raul plans to build a display with seven layers. How many blocks of cheese will there be in the stack?
- 2.9** Another supermarket employee is creating a pyramid of oranges for a produce display. The following diagram shows the first three levels, top to bottom, in this display.



- a. Make a sketch of the next three levels in the display.
- b. Let l_n represent the number of oranges in level n . Find the first six terms of this sequence.
- c. Is the sequence you wrote in Part **b** an arithmetic sequence? Justify your response.
- d. Write both recursive and explicit formulas to describe this sequence.
- e. How many oranges are there in a 10-level display?

* * * * *

Activity 3

In Activities **1** and **2**, you examined how arithmetic sequences can be modeled by linear equations. However, just as there are many different types of equations, there are many different types of numerical sequences. In this activity, you explore a type of sequence that can be used to model a projected increase in sales.

Exploration

Before starting their next concert tour, Stellar Attraction releases a second CD. During the first week, 500 copies are sold. The band's manager predicts sales will double each week that the band is on tour.

- a. The band's concert tour will last 12 weeks. Table 5 shows the manager's predictions for both weekly sales and total sales during this period. Complete the table.

Table 5: Projected CD sales

Week	Weekly Sales	Total Sales
1	500	500
2	1000	1500
3	2000	3500
⋮	⋮	⋮
12		

- b. Record any patterns you observe in the table.

Mathematics Note

A **geometric sequence** is a sequence in which every term after the first is found by multiplying the preceding term by a constant value.

The recursive formula for calculating any term in a geometric sequence is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = g_{n-1}(r), n > 1 \end{cases}$$

where g_1 is the first term, g_n is the n th term, g_{n-1} is the term preceding g_n , and r is the **common ratio** between any two consecutive terms, g_n/g_{n-1} .

For example, consider the sequence in which the first term (g_1) is 4 and the common ratio (r) between any two consecutive terms is 5. The recursive formula for this sequence is:

$$\begin{cases} g_1 = 4 \\ g_n = g_{n-1}(5), n > 1 \end{cases}$$

Using this formula, the first four terms of the sequence can be found as follows:

$$\begin{aligned} g_1 &= 4 \\ g_2 &= g_{2-1}(r) = g_1(r) = 4(5) = 20 \\ g_3 &= g_{3-1}(r) = g_2(r) = 20(5) = 100 \\ g_4 &= g_{4-1}(r) = g_3(r) = 100(5) = 500 \end{aligned}$$

The sum of the terms of a geometric sequence is a **geometric series**.

For example, consider the geometric sequence 3, 12, 48, 192. The corresponding geometric series is $3 + 12 + 48 + 192$, or 255.

- c. In Table 5, the numbers in the “Weekly Sales” column form a geometric sequence.
1. Find the common ratio for this sequence.
 2. Write a recursive formula for this sequence.
 3. Determine the corresponding geometric series.

Discussion

- a. Describe the patterns you observed in the numbers in Table 5.
- b. Use these patterns to answer the following questions:
1. After 12 weeks, what are the predicted total sales?
 2. What are the predicted weekly sales for week 10?
 3. During what week are total sales predicted to exceed 1 million?
- c. Why do the numbers in the “Weekly Sales” column form a geometric sequence?
- d. Do the numbers in the “Total Sales” column also form a geometric sequence? Why or why not?
- e. If the band extends their tour for 12 more weeks, should they expect sales to continue the predicted pattern? Explain your response.

Assignment

- 3.1 Determine a recursive formula for each of the following geometric sequences:
- a. 4, 28, 196, 1372, ...
 - b. 9, 18.9, 39.69, ...
 - c. 144, 36, 9, $9/4$, ...
- 3.2
- a. Create a scatterplot of the first six terms of each geometric sequence in Problem 3.1. Let x represent the term number and y represent the value of the term.
 - b. Compare the graphs of the three sequences.

- 3.3** As soon as the concert tour ends, Stellar Attraction’s manager predicts that CD sales will begin to decline. Over the next 8 weeks, sales should fall by 75% each week.
- Use this information to extend Table 5 for weeks 13–20.
 - The numbers in the “Weekly Sales” column for weeks 12–20 form a geometric sequence. Write a recursive formula for this sequence. (The first term is the weekly sales for week 12.)
 - Does your formula work after week 18? Explain your response.
 - What are the predicted weekly sales for week 20?
 - What are the predicted total sales for week 20?

- 3.4** Following the concert at the Jan-San Amphitheater, an enterprising group of students offers to clean the arena at the following rate: \$0.01 for the first barrel of garbage, \$0.02 for the second barrel, and \$0.04 for the third barrel, with this doubling pattern continuing for each additional barrel.

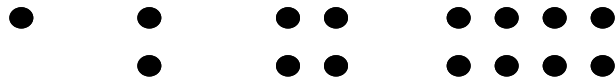
- Create a table with the following column headings. Complete this table for 20 barrels of garbage.

Barrel Number	Charge per Barrel	Total Charge
---------------	-------------------	--------------

- If the students want to earn at least \$500, how many barrels of garbage must they collect?
- The manager of the amphitheater has budgeted \$1000 to clean up after the concert. How many barrels of garbage must the students collect to exceed this budget?

* * * * *

- 3.5** Consider the pattern of dots shown in the following diagram.



- Draw the next picture in this pattern.
- Represent this pattern as a sequence.
- Is this sequence a geometric sequence? Explain your response.
- Write a recursive formula for the sequence.
- Graph the sequence as a scatterplot. Represent the term number on the x -axis and the value of the term on the y -axis.

- 3.6** As part of a holiday sales promotion, a clothing store plans to reduce the price of its \$20.00 shirts by 10% each week.
- Will the sale prices from week to week represent a geometric sequence? Explain your response.
 - The store originally paid \$12.00 for each shirt. In what week will the store begin to lose money on the sale items?
 - The store manager decides to stop the sale in the week before shirt prices fall below \$12.00. If you buy one shirt during each week of the sale, how much money will you spend?
- 3.7** Shahid and Yasmir borrowed money to buy their house. Their monthly payment includes the cost of the loan, insurance, and property taxes. During the first year of the loan, the monthly payment is \$350.00. In each year following the first, the monthly payment rises by 2%.
- Write a recursive formula that describes the monthly payment in any year.
 - Determine when the monthly payment will be more than \$500.00.
 - Shahid and Yasmir must pay a total of \$150,238.41 to pay off their loan. During what year will this occur?

* * * * *

Activity 4

As mentioned in Activity 3, Stellar Attraction’s manager predicted that the weekly sales of their CD would double each week. The predicted sales for the next 10 weeks form a geometric sequence. The two left-hand columns in Table 6 show the term number and terms for this sequence. The other three columns show expanded (and equivalent) forms of the term values.

Table 6: Patterns within geometric sequences

Term Number (n)	Term (t_n)	Recursive Form of t_n	Expanded Recursive Form of t_n	Explicit Form of t_n
1	500	500	500	500
2	1000	$500 \cdot 2$	$500 \cdot 2$	$500 \cdot 2^1$
3	2000	$1000 \cdot 2$	$500 \cdot 2 \cdot 2$	$500 \cdot 2^2$
4	4000	$2000 \cdot 2$	$500 \cdot 2 \cdot 2 \cdot 2$	$500 \cdot 2^3$
\vdots	\vdots	\vdots	\vdots	\vdots

In this activity, you use the patterns in this table to investigate explicit formulas for geometric sequences.

Exploration

- a. Develop an explicit formula for the sequence of terms, t_n , in Table 6.
- b. Create a three-column spreadsheet that duplicates the following three columns from Table 6.

Term Number (n)	Recursive Form of t_n	Explicit Form of t_n
---------------------	-------------------------	------------------------

- c.
 1. Extend the spreadsheet to at least $n = 10$.
 2. Compare the values in the two right-hand columns in your spreadsheet.
- d. Add a fourth column to the spreadsheet that determines the ratio of each term to its preceding term, t_n/t_{n-1} where $n > 1$.
- e. Create a scatterplot of the sequence. Let x represent the term number and y represent the value of each term.
- f. Find an equation that models the scatterplot and graph it on your coordinate system from Part e.

Compare this equation to the formula you wrote in Part a.

Discussion

- a.
 1. Using the explicit form of t_n described in Table 6, what power of 2 is multiplied by 500 to determine t_{19} ?
 2. How is the power of 2 in each row related to the value of n in that row?
- b.
 1. Describe a formula based on the value of n that could be used to calculate the value of t_n .
 2. Use this formula to calculate t_7 .
- c. What do you observe about the ratio found in Part d of the exploration?
- d. How is this ratio related to the formula described in Part b of the discussion?

Mathematics Note

An **explicit formula** for calculating any specific term in a geometric sequence is:

$$g_n = g_1 r^{n-1}$$

where g_n is the n th term, g_1 is the first term, and r is the common ratio between any two consecutive terms, g_n/g_{n-1} .

For example, consider the geometric sequence 6, 24, 96, 348, 1536, For this sequence, the first term is 6 and the common ratio 4. The explicit formula for this sequence is: $g_n = 6(4)^{n-1}$. Using this formula, the seventh term of the sequence can be found as follows:

$$g_7 = 6(4)^{7-1} = 6(4)^6 = 6(4096) = 24,576$$

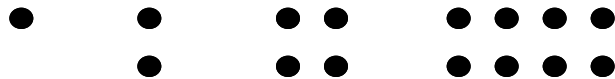
- e. Do you think that the scatterplot of a geometric sequence can always be modeled by an exponential equation? Why or why not?
- f. How does the graph of an exponential equation differ from the graph of a geometric sequence?
- g. Using the laws of exponents, the equation $y = ab^x$ can be rewritten as $y = a \cdot b^1 \cdot b^{x-1}$, which is equivalent to $y = (ab)b^{x-1}$. Assuming that this equation models the same geometric sequence as the explicit formula $g_n = g_1 r^{(n-1)}$, describe the relationship between each of the following:
 - 1. the n th term g_n and y
 - 2. n and x
 - 3. the common ratio r and b
 - 4. the first term g_1 and a
- h. When is an explicit formula for a geometric sequence easier to use than the recursive formula?
- i. Using the relationships described in Part g of the discussion, show that the explicit formula for a geometric sequence, $g_n = g_1 r^{(n-1)}$, is equivalent to the exponential equation of the form $y = ab^x$ that models the same sequence.

Assignment

- 4.1** Identify each of the following sequences as geometric or not geometric. For each geometric sequence, write an explicit formula and find the 10th term. Explain why each of the remaining sequences is not geometric.
- a. 0.5, 2.5, 12.5, 62.5, ...
 - b. 15, 150, 300, 900, ...
 - c. 4, -12, 36, -108, ...
 - d. 1000, 250, 62.5, 15.625, ...
- 4.2** Write the geometric series for each sequence below and find the corresponding sum.
- a. 1, 3, 9, 27, 81
 - b. 2, 5, 12.5, ..., 78.125
 - c. 0.5, 1, 2, ..., 64
 - d. 125, 25, 5, ..., 0.04
- 4.3** At one of Stellar Attraction's concerts, an exuberant fan hits a beach ball into the air. The ball falls onto the stage from a height of 24 m. The height of each bounce is approximately one-third the height of the preceding bounce. Predict the number of bounces the ball will take before coming to rest. Justify your response.
- 4.4** Describe the possible range of values for the common ratio in each of the following:
- a. an increasing geometric sequence
 - b. a decreasing geometric sequence.

* * * * *

- 4.5** Consider the pattern of dots shown in the following diagram.



- a. If this pattern is expressed as a geometric sequence, what are the values for t_1 and r ?
- b. Write an explicit formula that describes this pattern.
- c. Use your formula to determine the value of t_{14} .

Science Note

Sound is produced by vibrating objects. The frequency of a sound wave equals the frequency of the vibrating object.

The international unit of frequency is the **hertz (Hz)**, which represents one cycle per second. For example, a sound with a frequency of 220 Hz is produced by an object vibrating at 220 cycles per second.

- 4.6** When a guitar string vibrates, the guitar produces sound. The rate at which the string vibrates determines the note. A string vibrating at 220 Hz produces the note A immediately below middle C. The note A immediately above middle C vibrates at 440 Hz. The frequencies of consecutive A notes form a geometric sequence.

Note	Frequency
A	
A	
A	
A	440 Hz
middle C	
A	220 Hz
A	
A	
A	
A	

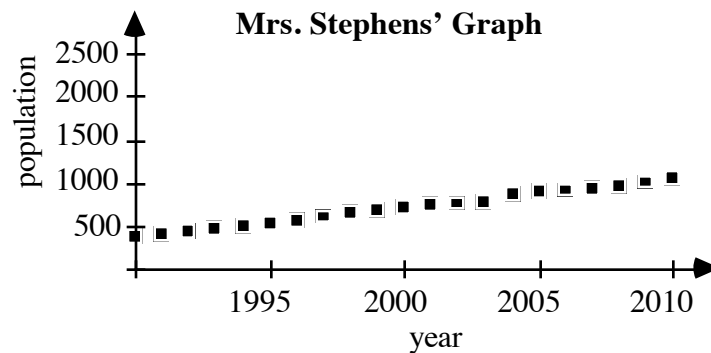
- a. Complete the table for all A notes above and below middle C.
 - b. Write a recursive formula that describes this sequence.
 - c. The average young person can hear sounds with frequencies from 20 Hz to 20,000 Hz. Which of the notes in the table in Part **a** could be heard by concert fans? Defend your response.
- 4.7** Annaborg has been offered two summer jobs, each for 12 weeks. The job at Plouvier’s Pottery pays \$5.00 per hour, with a 10% increase in the hourly wage every two weeks. The job at Brocklebank’s Bakery pays \$6.50 per hour, with a \$0.10 increase in the hourly wage every two weeks.
- a. Which type of sequence, arithmetic or geometric, best describes the wages offered by Plouvier’s Pottery? Justify your response.
 - b. Which type of sequence, arithmetic or geometric, best describes the wages offered by Brocklebank’s Bakery? Justify your response.
 - c. Write an explicit formula that describes the wages for each job.
 - d. If Annaborg wants to make as much money as possible, which job should she take? Justify your response.

* * * * *

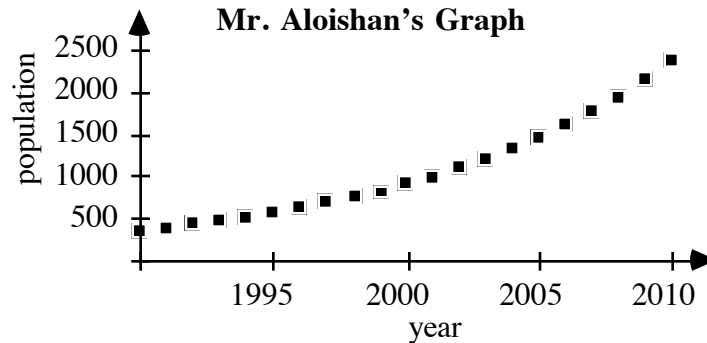
Summary Assessment

- In 1990, the village of Bone Gap had a population of 350 people. By the end of 1991, the population had grown to 385. Many long-time residents grew concerned about the future of their community. At a town meeting, two different predictions were made.

Mrs. Stephens presented the following graph and argued that residents should not be concerned about growth.



Mr. Aloishan, however, presented the graph below and argued that life in Bone Gap would change dramatically.



Based on these two graphs and the initial data, explain how the two residents reached their conclusions. Include formulas for each graph.

2. Imagine that the following two plans were proposed at the time that the U.S. national debt reached \$4 trillion.
- Plan A: Balance the budget to eliminate additional accumulation of debt. Reduce the debt by \$1.00 per second until the debt is canceled.
 - Plan B: Balance the budget to eliminate additional accumulation of debt. Reduce the debt by \$1.00 in the first year, \$2.00 in the second year, \$4.00 in the third year, and continue to double the reduction until the debt is canceled.

Use the ideas and tools in this module to analyze the two plans. Which one do you think is better? Defend your choice.

Module Summary

- A **sequence** is an ordered list. Each item in the list is a **term** of the sequence.
- The terms of a sequence may be represented by symbols, such as $p_1, p_2, p_3, \dots, p_n$. These symbols are **subscripted variables**, and the natural numbers (1, 2, 3, ..., n) are the **subscripts**. The symbol p_1 (read “ p sub one”) represents the first term of the sequence, the symbol p_2 (read “ p sub two”) represents the second term of the sequence, and so on. The symbol p_n (read “ p sub n ”), represents the **general**, or **n th, term** of a sequence.
- A **recursive formula** is a rule for finding any term in a sequence by using the preceding term(s). The process of using a recursive formula is known as **recursion**.
- A sequence in which every term after the first is found by adding a constant value to the preceding term is an **arithmetic sequence**.
- The recursive formula for calculating any term in an arithmetic sequence is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, n > 1 \end{cases}$$

where a_1 is the first term, a_n is the n th term, a_{n-1} is the term preceding a_n , and d is the **common difference** between any two consecutive terms, $a_n - a_{n-1}$.

- An **explicit formula** for calculating any specific term in an arithmetic sequence is:

$$a_n = a_1 + d(n - 1)$$

where a_n is the n th term, a_1 is the first term, and d is the common difference between any two consecutive terms, $a_n - a_{n-1}$.

- The sum of the terms of an arithmetic sequence is an **arithmetic series**.
- A **geometric sequence** is a sequence in which every term after the first is found by multiplying the preceding term by a constant value.
- The recursive formula for calculating any term in a geometric sequence is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = g_{n-1}(r), n > 1 \end{cases}$$

where g_1 is the first term, g_n is the n th term, g_{n-1} is the term preceding g_n , and r is the **common ratio** between any two consecutive terms, g_n/g_{n-1} .

- The sum of the terms of a geometric sequence is a **geometric series**.
- An **explicit formula** for calculating any specific term in a geometric sequence is:

$$g_n = g_1 r^{n-1}$$

where g_n is the n th term, g_1 is the first term, and r is the common ratio between any two consecutive terms, g_n/g_{n-1} .

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Under the Big Top But Above the Floor



It's carnival time at Dantzig High School. Before you play “Guess My Number” or “Roll-a-rama,” however, you'll need some practice. In this module, you'll develop the skills necessary to model and solve problems involving linear inequalities.

Bonnie Eichenberger • Paul Swenson • Teri Willard



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Under the Big Top but Above the Floor

Introduction

Every year at Dantzig High School, the mathematics club sponsors a school carnival. The club designs game booths to earn money. As booths compete to attract more customers, the atmosphere grows festive. Many students dress in crazy costumes. Balloons, tinsel, and streamers are everywhere, and the aromas of pizza, popcorn, and hot dogs waft through the gym.

The first game booth that you visit is “Guess My Number.” Inside the booth, Yvette stands next to a tumbling basket of numbers. Yvette picks a number from the basket. You must now guess the number by asking questions which can be answered by “yes” or “no.” If you guess correctly using seven or fewer questions, you win a prize.

Exploration

- a. To play Guess My Number with a classmate, each of you should select an integer in the interval $[-50, 50]$. Do not reveal your number to your classmate. Take turns asking yes-or-no questions until you discover each other’s number. Record the questions asked, the answers given, and the number of questions.

Play the game at least twice and consider some strategies you might use to guess correctly with the fewest questions.
- b. Play Guess My Number at least two more times, using only questions that include inequalities (for example, “Is your number less than 5?”). Record each question using inequality symbols. Record the responses on a number line.
- c. Play Guess My Number at least three more times, using questions that include inequalities.

Discussion

- a. If your opponent must choose an integer in the interval $[-50, 50]$, can you develop a strategy that guarantees you will know the number in seven questions or less?
- b. What do you think the minimum number of questions would be to determine an integer in the interval $[-75, 75]$?

Activity 1

The next booth you visit is “Guess My Location.” The object of this game is to determine the exact location of your opponent’s point on a grid. Pablo and Lisa are ahead of you at the booth. Watch them play while you wait for your turn. Figure 1 shows the beginning of their game.

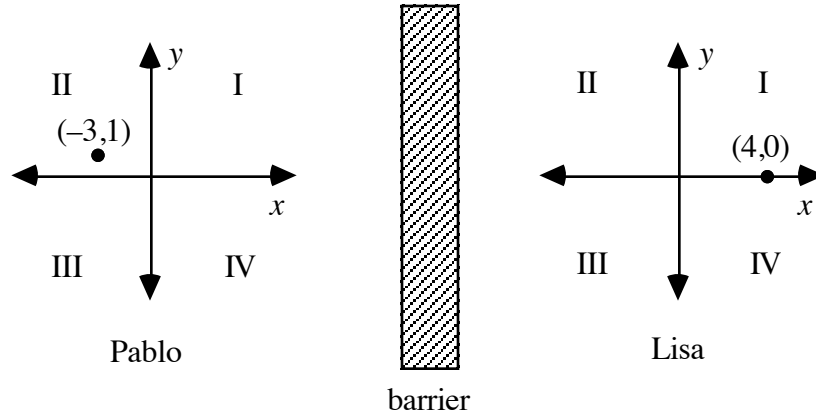


Figure 1: Pablo and Lisa play Guess My Location

Pablo asks, “Is $x > 0$?”

Lisa answers, “Yes.”

Pablo knows that Lisa’s point must be in quadrant I, quadrant IV, or on the positive x -axis.

Lisa asks, “Is $y \leq 0$?”

“No,” answers Pablo.

Lisa knows that Pablo’s point is above the x -axis.

It’s Pablo’s turn again. “Is $y \geq 0$?” he asks.

“Yes,” answers Lisa.

Pablo knows that Lisa’s point can only be in the first quadrant or on the positive x -axis.

Discussion 1

- a. The graph of the equation $y = -3$ is shown in Figure 2a. As shown by the shaded region in Figure 2b, the line separates the plane into two parts.

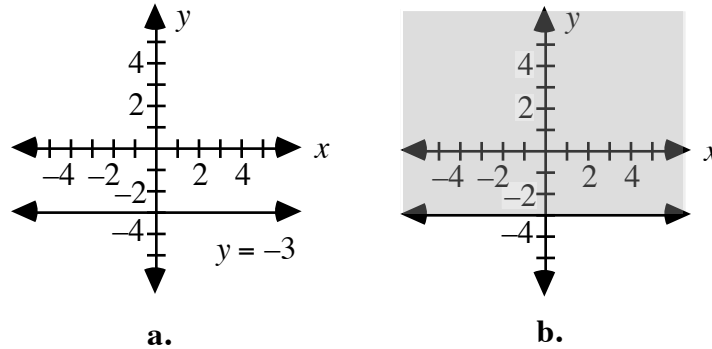


Figure 2: Graphs of $y = -3$ and the half-plane above $y = -3$

1. What must be true about all the points in the shaded region in Figure 2b?
 2. How can you use inequality notation to describe the points either on or above the line $y = -3$?
- b. The line $x = 5$ defines the boundary for the inequality $x < 5$.
1. Would a graph of $x < 5$ be shaded to the right or to the left of $x = 5$? Explain your response.
 2. Should the boundary be a part of the graph of $x < 5$? Explain your response.

Mathematics Note

The graph of a linear inequality is a shaded region that represents the **solution set** of the inequality. The solution set contains all the points, or solutions, that make the inequality true. The graph of a linear equation forms the **boundary line** for the region. A solid boundary indicates that the points on the line are part of the solution set. A dashed boundary indicates that the points on the line are not part of the solution set.

For example, Figure 3a shows the graph of $x \geq -2$ and Figure 3b shows the graph of $y < 3$.

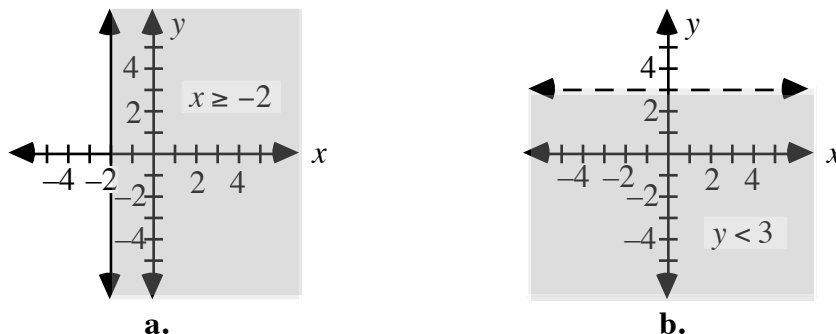


Figure 3: Graphs of $x \geq -2$ and $y < 3$

- c. Explain why the point $(2, -7)$ is in the solution set for $x \geq 2$ but not for $y < -7$.

Exploration 1

In this exploration, you play Guess My Location with a classmate. Read Parts a–d before beginning play.

- a. Each player makes a game sheet by drawing a coordinate system on a sheet of graph paper. Both game sheets should have the same scales on the x - and y -axes.

A book should be placed upright between the two game sheets so that neither player can see the other's graph.

- b. Both players choose and mark a point on their game sheets, without revealing the point's location to the opponent. The coordinates of the point must be integers.
- c. To gain information about the location of the opponent's point, players take turns asking yes-or-no questions involving inequalities.

Record each question you ask as an inequality. Record each answer you receive as the graph of an inequality on your coordinate system.

- d. The first player to identify the coordinates of the opponent's point wins the game. If a player guesses incorrectly, then that player loses the next turn.

Discussion 2

- a. Describe the strategies you used to play Guess My Location.
- b. Assume that your opponent's point is located in the third quadrant. What two mathematical statements would describe this region?

Mathematics Note

A **conjunction** combines two mathematical statements with the word *and*. A conjunction can be represented as the intersection of two sets.

As shown in Figure 4, for example, the conjunction $x > 0$ and $y > 0$ can be represented by the intersection of their solution sets.

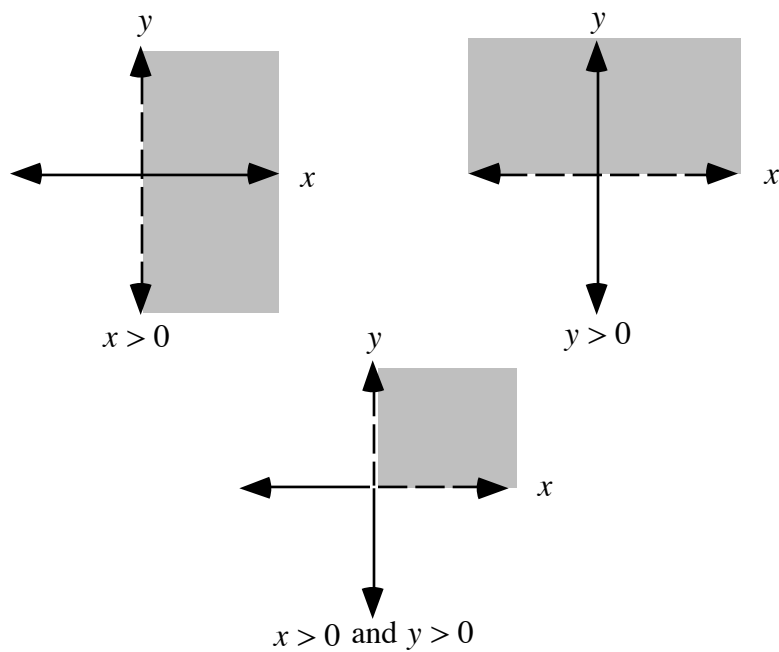


Figure 4: Intersection of two linear inequalities

- c. For all points in the first quadrant, the x - and y -coordinates are greater than 0. Does the conjunction shown in Figure 4 represent these points? Explain your response.
- d. Imagine that Pablo asked Lisa “Is $x > 0$ and $y < 0$?”
 1. If Lisa answers “No,” in what quadrant(s) could her point be located? Explain your response.
 2. If Lisa answers “Yes,” in what quadrant(s) could her point be located? Explain your response.

- e. What conjunction describes the intersection of the two inequalities shown in Figure 5?

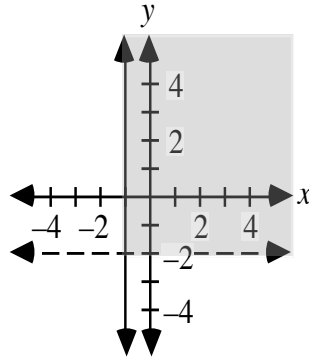


Figure 5: Two inequalities and their intersection

Exploration 2

Imagine that Pablo and Lisa replay the original game of Guess My Location shown in Figure 1. Pablo has chosen the point $(-3,1)$ and Lisa has chosen $(4,0)$. This time around, their questions use conjunctions.

“Is $x > 0$ and $y < 0$?” asks Pablo.

“No,” Lisa responds.

Pablo knows that the point is not in Quadrant IV.

Now it’s Lisa’s turn. She asks, “Is $x \leq 0$ and $y \geq 0$?”

Pablo answers, “Yes.”

Lisa knows that the point is not in Quadrant II.

Now Pablo asks, “Is $x \geq 0$ and $y \geq 0$?”

“Yes,” answers Lisa.

Following Pablo and Lisa’s example, play another game of Guess My Location. For this game, use the two additional rules described in Parts **a** and **b**.

- Each question must be a conjunction.
- One mathematical statement in the conjunction must refer only to x ; the other statement must refer only to y .

Discussion 3

- a. What advantages or disadvantages did you notice when using conjunctions to play Guess My Location?
- b. Consider Pablo's first conjunction, "Is $x > 0$ and $y < 0$?" The intersection of the two boundary lines for these inequalities, $x = 0$ and $y = 0$, is referred to as a **corner point**.

Is this corner point part of the solution set for the conjunction?
Explain your response.

- c. How could you verify that a point in the first quadrant is a solution to $x > 0$ and $y > 0$?
- d. Consider the points in the solution set of $x > 0$ and $y > 0$. The x -coordinates of these points must be positive real numbers. This condition on the x -coordinates is called a **constraint**.

Describe the constraints on the y -coordinates of the points in this solution set.

- e. How could you use conjunctions to describe the region shown in Figure 6?

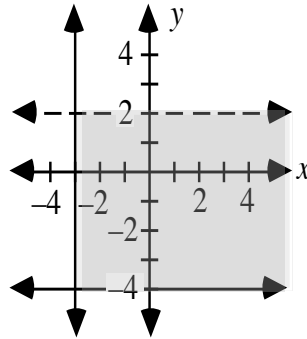
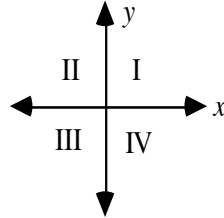


Figure 6: A region defined by conjunctions

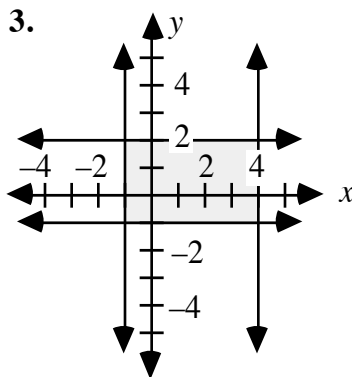
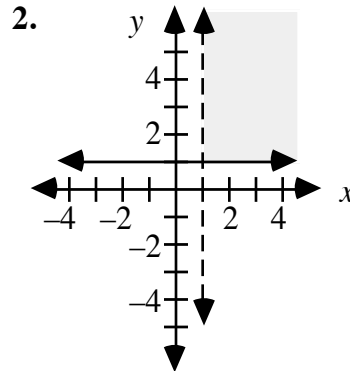
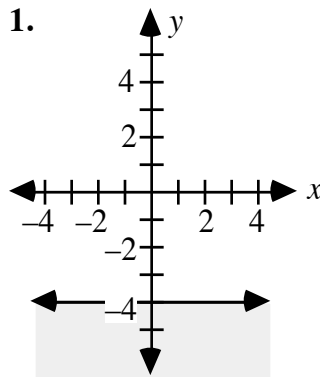
Assignment

- 1.1
 - a. Sketch a graph of the line $y = 4$.
 - b. Write an inequality to describe all the points located on and above the line.
 - c. Shade the region that corresponds to the solution set of the inequality.
 - d. Write the coordinates of at least three points included in the solution set.
 - e. Describe the constraints on the x -coordinates of the points in the solution set.
 - f. Describe the constraints on the y -coordinates of the points in the solution set.

- 1.2 The following graph shows the four quadrants of the coordinate plane. Use conjunctions of linear inequalities to describe each quadrant.
Note: The x - and y -axes are not included in any quadrant.



- 1.3
- On a coordinate plane, graph the region defined by the intersection of the linear inequalities $x \geq -4$ and $y \leq 2$.
 - Select a point in the intersection of the two inequalities that is not on a boundary. Verify that the point is a solution to both inequalities.
 - Select a point outside the shaded region. Verify that the point is not a solution for at least one of the inequalities.
 - Select a point on a boundary. Is this point a solution to both inequalities? Explain your response.
 - Identify the corner point of the conjunction.
- 1.4
- Use inequalities to describe the shaded region in each of the following graphs.



- Identify the coordinates of any corner points for each shaded region in Part a.

- 1.5** In a game of Guess My Location, your opponent chooses the point with coordinates $(0,-2)$. Using conjunctions, write two different questions about this point that will receive a “yes” answer.
- 1.6** In another game of Guess My Location, you ask the question, “Is $x = 5$ and $y > -2$?” Your opponent answers, “Yes.”
- On a coordinate plane, graph the set of all points that fit this information.
 - Describe the graph formed.
- 1.7** The tables in Parts **a** and **b** show some questions and answers from two different games of Guess My Location. For each game, graph the “smallest” region that contains all possible solutions and use linear inequalities to describe the region.

a.

Question	Answer
Is $x \geq 0$ and $y \geq 0$?	Yes
Is $x \leq 2$ and $y \geq 0$?	No
Is $x \geq 3$ and $y \geq 1$?	Yes
Is $x \geq 4$ and $y \leq 5$?	Yes

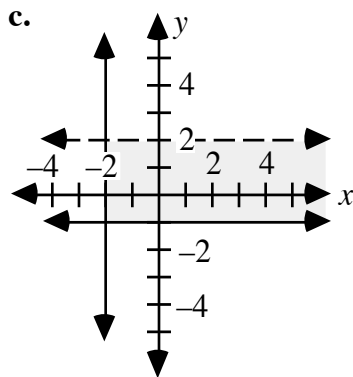
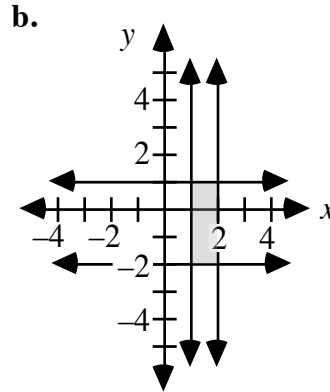
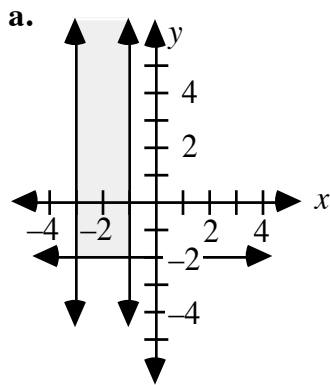
b.

Question	Answer
Is $x \geq 0$ and $y \leq 0$?	Yes
Is $x \geq 5$ and $y \geq -3$?	No
Is $x \leq 4$ and $y \geq -3$?	Yes

* * * * *

- 1.8** A cash register contains bills and coins. Using the horizontal axis to represent the value of the bills and the vertical axis to represent the value of the coins, draw a graph to describe each of the following statements:
- The cash register contains at least \$50.00 in bills and no more than \$25.00 in coins.
 - The cash register contains less than \$100.00 in bills and at least \$10.00 in coins.
 - The cash register contains more than \$75.00 in bills and no more than \$50.00 in coins.

1.9 Use linear inequalities to describe each shaded region in Parts a–c.



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Activity 2

Anton and Sharline are the school champions at Guess My Location. In this activity, you investigate some of the tools they use to analyze the game.

Exploration 1

Anton and Sharline are playing Guess My Location. Both players have selected points with x - and y -coordinates between -10 and 10 . After receiving an answer to his first question, Anton shades his game sheet as shown in Figure 7.

The points A , B , C , and D in Figure 7 lie at the intersections of the dashed line, the solid line, and the lines that define the right-hand and lower limits of the grid. The solid line through points A and C also passes through $(0, -3)$ and $(1, -1)$. The horizontal dashed line has a y -intercept of 7 .

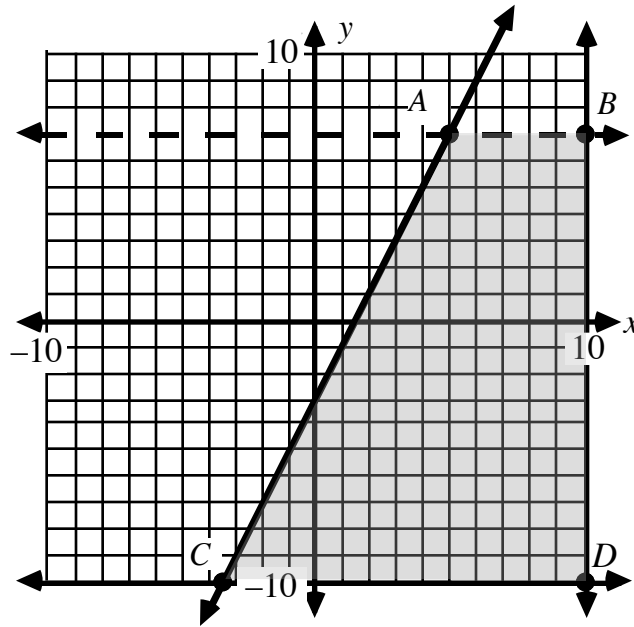


Figure 7: Anton's graph

- a. Determine the equation of each line that borders the shaded region in Anton's graph.

Mathematics Note

A **system of linear equations** is a set of two or more equations whose graphs are lines. A **solution** to a system of linear equations is a point where all the lines intersect. The coordinates of this point satisfy all the equations in the system.

For example, consider the following system of linear equations:

$$\begin{cases} y = 2x + 3 \\ y = x + 1 \end{cases}$$

As shown in Figure 8, the graphs of the two lines intersect at the point with coordinates $(-2, -1)$. By substitution, this point is a solution to the system since $-1 = 2(-2) + 3$ and $-1 = -2 + 1$.

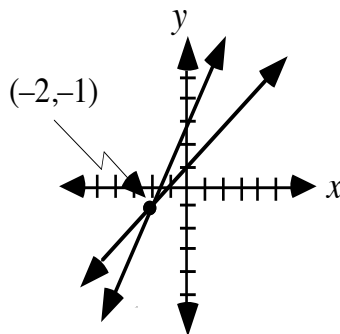


Figure 8: Solution to a system of linear equations

- b. In order to shade the region shown in Figure 7, what question might Anton have asked? What answer must Sharline have given?
- c. In Figure 7, point C is one of the corner points of the shaded region. Estimate the coordinates of point C as accurately as possible.

Discussion 1

- a. The shaded region in Figure 7 contains the location of Sharline's point. Anton used the limits of the grid, one solid line, and one dashed line to define the boundaries of the region.
 - 1. If Anton had wanted to define the same region using two dashed boundary lines, how could he have modified his question?
 - 2. If Anton had wanted to use two solid boundary lines, how could he have modified his question?
- b. What points are eliminated from a solution set when a dashed boundary line replaces a solid boundary line?
- c. The shaded region in Figure 7 shows the solution set for a system of linear inequalities. Which of the corner points are included in this solution set?
- d. The corner point C appears to have at least one coordinate that is not an integer. Did you find the exact coordinates of this point? Explain your response.
- e. The exact coordinates of a point where two lines intersect is the solution to the corresponding system of equations. To find these exact coordinates, you can use a method involving substitution.

For example, Table 1 lists the steps necessary to find the solution to one system of equations by substitution.

Table 1: Substituting to solve a system of equations

Consider the given system of equations.	$\begin{cases} y = 2x - 3 \\ y = 7 \end{cases}$
Since $y = 7$ in this case, substitute 7 for y .	$7 = 2x - 3$
Solve the resulting equation for x .	$5 = x$
Write the solution to the system in the form (x,y) .	$(5,7)$

To check the solution, you can substitute the coordinates of the point in each of the original equations in the system. If $x = 5$ and $y = 7$, then $y = 2x - 3 = 2(5) - 3 = 7$. Therefore, $(5,7)$ is a solution to the equation $y = 2x - 3$. Since $(5,7)$ is also a solution to the equation $y = 7$, then it is a solution to the system of equations.

1. Why are the coordinates of point C a solution to a system of linear equations?
2. Use the substitution method to check your estimate for the coordinates of point C .

Exploration 2

Figure 9 shows a graph of three linear equations. Point B is the intersection of $5x - 5y = 4$ and $3x + y = 8$.

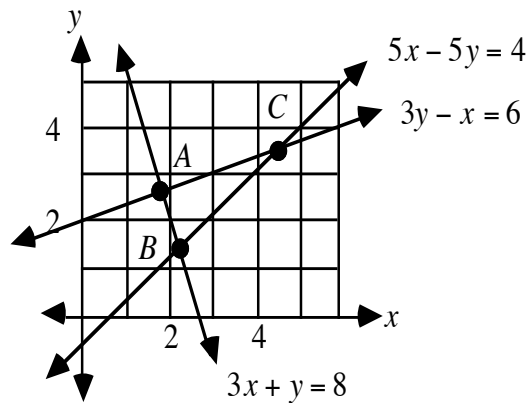


Figure 9: Three linear equations

- a. Estimate the coordinates of point B in Figure 9.

- b. Table 2 lists the steps necessary to find the solution to another system of equations by substitution.

Table 2: Substituting to solve a system of linear equations

Consider the given system of equations.	$\begin{cases} 2x + 3y = 6 \\ 9y - 4x = 13 \end{cases}$
Solve one of the equations for y .	$y = 2 - \frac{2}{3}x$
Substitute for y in the other equation.	$9\left(2 - \frac{2}{3}x\right) - 4x = 13$
Solve for x .	$x = \frac{1}{2}$
Substitute for x in one of the original equations.	$2\left(\frac{1}{2}\right) + 3y = 6$
Solve the resulting equation for y .	$y = \frac{5}{3}$
Write the solution to the system in the form (x,y) .	$\left(\frac{1}{2}, \frac{5}{3}\right)$

To check the solution, you can substitute the coordinates of the point in each of the original equations.

$$\begin{array}{rcl}
 2x + 3y = 6 & & 9y - 4x = 13 \\
 2\left(\frac{1}{2}\right) + 3\left(\frac{5}{3}\right) \stackrel{?}{=} 6 & \text{and} & 9\left(\frac{5}{3}\right) - 4\left(\frac{1}{2}\right) \stackrel{?}{=} 13 \\
 1 + 5 = 6 & & 15 - 2 = 13
 \end{array}$$

- Use the method described in Table 2 to find the exact coordinates of point B in Figure 9.
 - Compare your solution with the estimate you made in Part a.
 - Check your solution by substituting the coordinates of the point in each of the original equations.
- c. Repeat Parts a and b for points A and C in Figure 9.

Discussion 2

- Did your estimated coordinates for points A , B , and C support the solutions you found by solving systems of equations?
- Describe how to confirm the solution to a system of linear equations.
- What system of linear inequalities could be used to describe the region in Figure 9 with corner points A , B , and C ? Assume that the region includes the lines defining its boundaries.

Assignment

- 2.1** In another game of Guess My Location, Sharline asks, “Does $x = y + 4$ and $y = 1$?” Anton answers, “Yes.” Use the substitution method to find the coordinates of Anton’s point. Check your results by graphing.
- 2.2** Anton asks, “Does $y = 3x - 5$ and $6x = -2y - 6$?” Sharline answers, “Yes.” What are the coordinates of the point that satisfies this system of equations?
- 2.3** Graph each of the following systems of equations and use the graphs to estimate the solutions. Determine the solution to each system by substitution. Use your estimates to check your solutions.
- $y = x + 1$ and $y = 3x - 3$
 - $x + y = 10$ and $2x - 4y = 2$
 - $y = x + 7$ and $y = -2x + 6$
 - $3x + y = 10$ and $4x + 1 = y$
- 2.4**
- Graph the line $y + 3 = 2x$.
 - Write a linear inequality that describes the region below the line and including the line.
 - List the coordinates of two points that satisfy the inequality.
- 2.5**
- Graph the line $y + 2x = 1$ on the same set of axes you used in Problem **2.4**.
 - Write a linear inequality that describes the region above the line and including the line.
 - List the coordinates of two points that satisfy both this inequality and the one you wrote in Problem **2.4b**.
- 2.6**
- Identify the point of intersection of the lines $y - 2x + 3 = 0$ and $y + 2x = 1$.
 - Use substitution to verify your response to Part **a**.
- 2.7**
- Draw the region defined by the following system of inequalities: $x \geq 0$, $y \geq 0$, $x \leq 8$, and $y \leq 8$.
 - Describe the shape of this region.
 - Identify the corner points, or vertices, of the region.
 - Rewrite the system of inequalities in Part **a** so that the boundaries are not included in the region.
 - Write the equations of three different lines that divide the region in Part **a** into two parts with equal area.

2.8 Graph the region defined by each of the following inequalities:

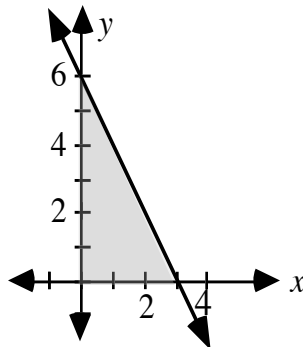
- a. $y > 4x - 3$
- b. $y \leq -2x + 7$
- c. $-3x + 2y \geq 6$

2.9 a. Graph the following system of inequalities:

$$\begin{cases} x + y \leq 6 \\ y \geq -2 \\ x \geq -2 \\ x \leq 4 \end{cases}$$

- b. Describe the shape of the region.
- c. List the coordinates of its vertices.

2.10 Use a system of linear inequalities to describe the shaded region in the following graph.



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2.11 Find the solution to each of the following systems of equations. Verify your solutions by substitution or graphing.

- a. $y = -4$ and $y = 2x + 10$
- b. $y = x - 2$ and $y = -x - 6$

2.12 Graph the region defined by each of the following systems of inequalities:

- a. $y > 4x - 3$ and $y \leq -2x + 7$
- b. $y \leq -2x + 7$ and $-3x + 2y \geq 6$

2.13 Jasmine has \$290.00 and saves \$5.00 per week. Alan has \$200.00 and saves \$8.00 per week.

- a. Write equations that describe the amount of money each person has after any week w .
- b. In how many weeks will the two have the same amount of money?

- 2.14** In a basketball game, a player made at least five baskets (2 points each), at least four free throws (1 point each), and scored a total of no more than 20 points. (Assume no three-point baskets were made.)
- Write a system of three inequalities to describe this situation.
 - Graph this system of inequalities. Use the horizontal axis to represent the number of baskets and the vertical axis to represent the number of free throws.
 - List the set of ordered pairs that satisfy this system of inequalities.

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Activity 3

Another booth at the carnival features a game called “Roll-a-rama.” This game offers many winning outcomes. In this activity, you examine the chances of winning at Roll-a-rama.

Exploration

In Roll-a-rama, players roll one white die and one red die. One roll of each die is considered a game. To win, the following must be true:

- the number on the red die must be at least 2 but no more than 5
 - the sum of the two dice must be less than or equal to 7.
- Play Roll-a-rama 10 times and record your results.
 - Determine the experimental probability of winning a game.
 - List all the possible rolls in Roll-a-rama.
 - Determine the theoretical probability of winning a game.

Discussion

- Are you more likely to win or lose when playing Roll-a-rama?
- Do you think it is possible to find a strategy that increases the probability of winning? Explain your response.
- If all the possible rolls were graphed on a coordinate plane where the roll on one die is represented on the x -axis and the roll of the other die is represented on the y -axis, what would the graph look like?

Mathematics Note

A **feasible region** is the graph of the solution set of a system of linear inequalities.

For example, consider the following system of linear inequalities: $0 \leq x \leq 6$, $0 \leq y \leq 4$, and $x + y \leq 8$. The shaded region in Figure 10 is the feasible region for this system.

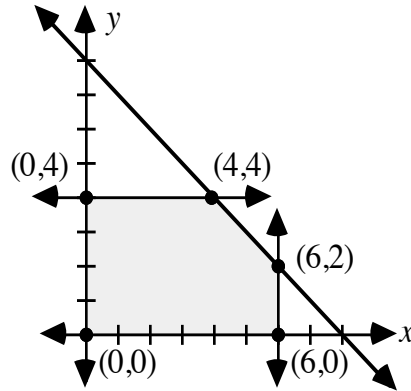


Figure 10: A feasible region

- d. The winning combinations for Roll-a-rama could be graphed as a feasible region.
1. Why might it be misleading to shade the feasible region for Roll-a-rama?
 2. How might you represent the feasible region for Roll-a-rama?

Assignment

- 3.1 You can identify the winning combinations in Roll-a-rama by graphing a system of linear inequalities on a coordinate plane. The inequalities that define the constraints on a winning roll can be written as follows:

$$\begin{cases} x \geq 1 \\ y \geq 2 \\ y \leq 5 \\ x + y \leq 7 \end{cases}$$

- a. What do x and y represent in these inequalities?
- b. Describe the rule of the game that corresponds with each constraint.
- c. Graph this system of inequalities on a set of coordinate axes and determine the feasible region.
- d. Identify the number of points in the feasible region that represent winning rolls. **Note:** Save this graph for use in Activity 4.

- 3.2** While playing Roll-a-rama, Sharline recorded each roll as an ordered pair in the form (white die, red die). In 10 games, she obtained the following rolls: (1,4), (3,2), (6,2), (4,4), (2,1), (4,3), (5,1), (3,3), (1,6), and (5,3). How many games did she win? Justify your response using your graph from Problem **3.1c**.
- 3.3**
- How many different losing combinations are there in Roll-a-rama?
 - List at least three of these losing combinations. Use your graph from Problem **3.1c** to explain why they do not represent winning rolls.
- 3.4** Is the point where the line $x + y = 7$ intersects the y -axis in the feasible region for Roll-a-rama? Explain your response.
- 3.5**
- What are the constraints on the region that includes all possible rolls in Roll-a-rama?
 - Write inequalities to describe these constraints.
- 3.6** Rewrite the rules for “Roll-a-rama” so that the probability of winning is the same as the probability of losing. To check your rules, play the new game several times and record your results.

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- 3.7** Graph the feasible region for each of the following systems of inequalities and label the vertices.

a.

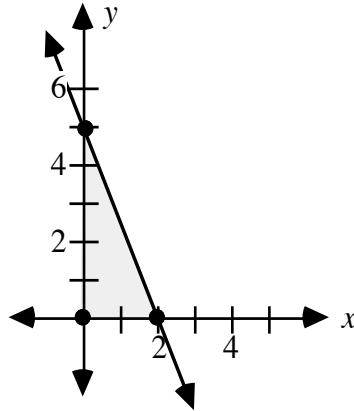
$$\begin{cases} w \geq 0 \\ t \geq 0 \\ 2w + 3t \leq 10 \end{cases}$$

b.

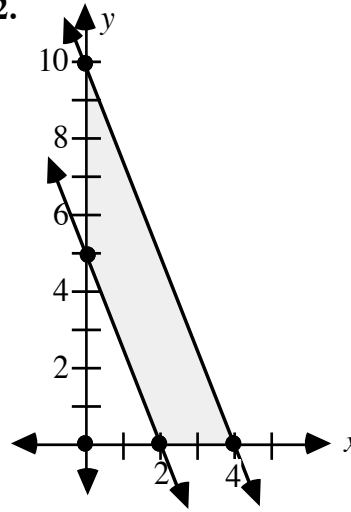
$$\begin{cases} x \geq -1 \\ y \leq 2 \\ y \geq x - 1 \end{cases}$$

- 3.8 a. Describe the constraints on the feasible region shown in each of the following graphs:

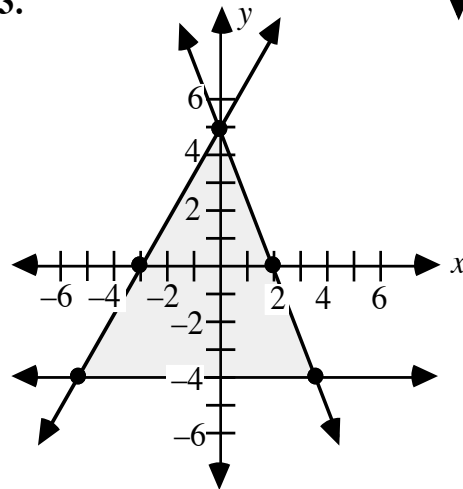
1.



2.



3.



- b. List the coordinates of the corner points for each feasible region in Part a.

- 3.9 A furniture manufacturer makes upholstered chairs and sofas. On average, each chair requires 4 hr to assemble and each sofa requires 8 hr to assemble. The company's carpenters can provide a total of no more than 124 hr of assembly work per day. Each chair also requires 2 hr to upholster, while each sofa requires 6 hr to upholster. The company's upholsterers can provide a total of no more than 72 hr of work per day.

- a. Write a system of inequalities that describes the numbers of chairs and sofas which the company can build and upholster in one day.
- b. Graph the feasible region described by the system you wrote in Part a and identify the coordinates of the corner points. Use the horizontal axis to represent the number of chairs and the vertical axis to represent the number of sofas. **Note:** Save your graph for use in Problem 4.3.

- 3.10** A group of friends would like to order two large pizzas with extra toppings and some soft drinks. Each pizza costs \$10.00. Each extra topping costs \$0.50, while each soft drink costs \$1.00. The group has \$30.00 to spend.
- Graph the feasible region that shows the possible numbers of toppings and soft drinks which the group can order.
 - Identify the coordinates of the corner points.

* * * * *

Activity 4

In Roll-a-rama, each winning player receives a second roll of the dice. If the player also wins on this second roll, a score is determined based on the number showing on each die. The score is calculated using the following rules:

- the number on the white die is multiplied by 2
- the number on the red die is multiplied by 3
- the two resulting numbers are added.

The object of the second roll is to obtain as high a score as possible. This score (S) can be determined by the equation $S = 2x + 3y$, where x represents the number on the white die and y represents the number on the red die. Since finding the best value of S is the objective, this is called an **objective equation**.

Exploration

- As noted in the previous activity, players roll one red die and one white die in Roll-a-Rama. To win,
 - the number on the red die must be at least 2 but no more than 5
 - the sum of the two dice must be less than or equal to 7.

The winning combinations can be described by a feasible region for a system of inequalities. On a copy of the Roll-a-Rama template, graph the lines that define this feasible region.

- b. Use the objective equation to find the value of S for each roll in the feasible region. Record this information in a table with headings like those in Table 3.

Table 3: Scores for each winning roll

Winning Roll (x,y)	Score (S)
(1,2)	8

- c.
1. Select two points in the feasible region with the same value for S . Draw the line that passes through these two points.
 2. Write the equation of this line in the form $y = mx + b$.
 3. Repeat Steps 1 and 2 for the remaining points in the feasible region.
- d.
1. Use the information in Table 3 to identify the maximum value for S .
 2. Substitute this value for S in the objective equation $S = 2x + 3y$. Solve the equation for y .
 3. Graph this equation on the template.
- e. In Roll-a-rama, you want to find the greatest value of the objective equation. In other situations, you may want to find the least value of an objective equation. Repeat Part d for the minimum value for S .

Discussion

- a. Where are the maximum and minimum values for S located on your graph from the exploration?

Mathematics Note

In problems which involve a system of linear inequalities (and where the corner points are part of the feasible region), the **corner principle** provides a method of finding the maximum or minimum values for an objective equation. According to this principle, the maximum and minimum values of an objective equation occur at corner points of the feasible region.

- b.
1. What similarities did you observe among the lines graphed in Parts c–e of the exploration?
 2. How are the equations of the lines related to the equation for S ?
- c. What must a player roll to obtain the greatest possible score?

Assignment

- 4.1** Sharline and Anton are designing a new game for the Dantzig High carnival. They have already selected two different prizes from a catalog, and need only to decide how many of each to buy. The small prizes cost \$0.25 cents each; the large ones cost \$0.75 each.

According to carnival guidelines, the game may offer no more than 100 prizes. To make the game attractive to players, Sharline and Anton want to buy at least 10 of the large prizes and no more than 60 of the small prizes.

- a. Write a linear inequality for each constraint in this situation. Let x represent the number of small prizes and y represent the number of large prizes.
 - b. Graph your system of linear inequalities on a coordinate plane. Identify the coordinates of the vertices.
 - c. List the coordinates of three points in the feasible region. What does each point mean in terms of buying prizes?
 - d. Sharline and Anton have decided to charge \$1.00 to play the game and give each player a prize. Since they have no other expenses, their **profit** is the money they take in minus the cost of the prizes.
 1. What is their profit on each small prize?
 2. What is their profit on each large prize?
 - e. Since Sharline and Anton wish to maximize profit, write an equation that describes the amount of profit (P) for each point (x,y) in the feasible region.
 - f. Use the corner principle to find the coordinates of the point that maximizes P . Assuming that 100 people play the game, how many of each prize should they buy?
- 4.2** Imagine that you are responsible for managing the concession stand at the carnival. Your budget for purchasing pizza and canned soft drinks is \$115.00. A six-pack of soft drinks costs \$3.00. A 12-serving pizza costs \$10.00. The number of soft drinks you order should not exceed the number of slices of pizza.
- a. The concession stand sells pizza for \$1.50 per slice, while soft drinks are sold for \$1.00 each. Assuming that the concession stand sells everything you purchase, write an equation that describes the profit from pizza and soft-drink sales.
 - b. How many six-packs and how many pizzas should you order? Explain your response.

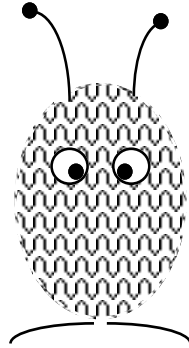
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- 4.3** The furniture company described in Problem **3.9** earns a profit of \$50.00 on each chair and \$80.00 on each sofa.
- Write an equation that describes the company's profit from sales of sofas and chairs.
 - Using your graph from Problem **3.9**, determine the number of sofas and chairs the company should make each day to maximize profit.
- 4.4** Imagine that you are a production manager at an electronics company. Your company makes two types of calculators: a scientific calculator and a graphing calculator.
- Each model uses the same plastic case and the same circuits. However, the graphing calculator requires 20 circuits while the scientific calculator requires only 10. The company has 240 plastic cases and 3200 circuits in stock. Graph the system of inequalities that represents these constraints.
 - The profit on a scientific calculator is \$9.00, while the profit on a graphing calculator is \$15.00. Write an equation that describes the company's profit from calculator sales.
 - How many of each type of calculator should the company produce to maximize profit using the stock on hand?

* * * * *

Summary Assessment

Instead of having a game booth, the Dantzig High Pep Club has decided to sell “Spirit Animals” at the carnival. The following illustration shows a Spirit Animal.



Spirit Animals are made from pompons and come in two sizes: small and large. Small animals require 1 pompon, while large animals require 3 pompons. The club has 270 pompons.

Seven members of the club have agreed to work for 30 min each making animals. It takes 1 min to make a small animal and 1.5 min to make a large animal.

The club plans to sell large animals for \$0.90 each and small animals for \$0.65 each. The club paid \$0.05 for each pompon. Your job is to help the Pep Club maximize their profit.

1. Choose variables to represent the numbers of each size of Spirit Animal the club can make.
2. The project is limited by the number of pompons available. Consider how many pompons are needed for each size of animal. Write the linear inequalities for these constraints.
3. The number of minutes that members can work on the project is also limited. Consider how much time is needed to make each size of animal. Write the linear inequalities for these constraints.
4. Graph the linear inequalities from Problems 2 and 3 on a coordinate plane and shade the feasible region.
5. What is the profit on each size of animal?
6. Assuming that club members sell every animal they make, write an equation to describe the club’s profit.
7. How many of each size of Spirit Animal should the Pep Club make to maximize profit?
8. Describe the steps you used to find the maximum profit.

Module Summary

- The graph of a linear inequality is a shaded region that represents the **solution set** of the inequality. The solution set contains all the points, or solutions, that make the inequality true. The graph of a linear equation forms the **boundary line** for the region. A solid boundary indicates that the points on the line are part of the solution set. A dashed boundary indicates that the points on the line are not part of the solution set.
- A **conjunction** combines two mathematical statements with the word *and*. A conjunction can be represented as the intersection of two sets.
- A **system of linear equations** is a set of two or more equations whose graphs are lines. A **solution** to a system of linear equations is a point where all the lines intersect. The coordinates of this point satisfy all the equations in the system.
- The **constraints** on a problem are the conditions that limit the number of possible solutions.
- A **feasible region** is the graph of the solution set of a system of linear inequalities.
- The intersection of two or more boundary lines forms a **corner point** (or vertex) of the feasible region.
- In problems which involve a system of linear equalities (and where the corner points are part of the feasible region), the **corner principle** provides a method of finding the maximum or minimum values for an objective equation. According to this principle, the maximum and minimum values of an objective equation occur at corner points of the feasible region.

Selected References

Farlow, S. J., and G. Haggard. *Finite Mathematics and Its Applications*. New York: Random House, 1988.

Under the Big Top But Above the Floor



It's carnival time at Dantzig High School. Before you play “Guess My Number” or “Roll-a-rama,” however, you'll need some practice. In this module, you'll develop the skills necessary to model and solve problems involving linear inequalities.

Bonnie Eichenberger • Paul Swenson • Teri Willard



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Digging into 3-D

Introduction

On September 12, 1857, about 300 km off the coast of South Carolina, disaster struck the *S.S. Central America*. During a hurricane, the ship sank with 578 passengers and three tons of gold aboard. Although the loss of human life can never be given a dollar value, the lost gold was worth nearly \$40 million at the time. In today's market, the gold's value would be in the hundreds of millions of dollars.

The Columbus America Discovery Group was formed in 1985 for the purpose of finding sunken treasure. For three summers, the group used side-scan sonar to search for the wreck of the *Central America*. The ship was eventually located at a depth of 2500 m. Using a custom-designed, remote-operated vehicle, the *Nemo*, the group recovered 1.4 tons of gold and other artifacts.

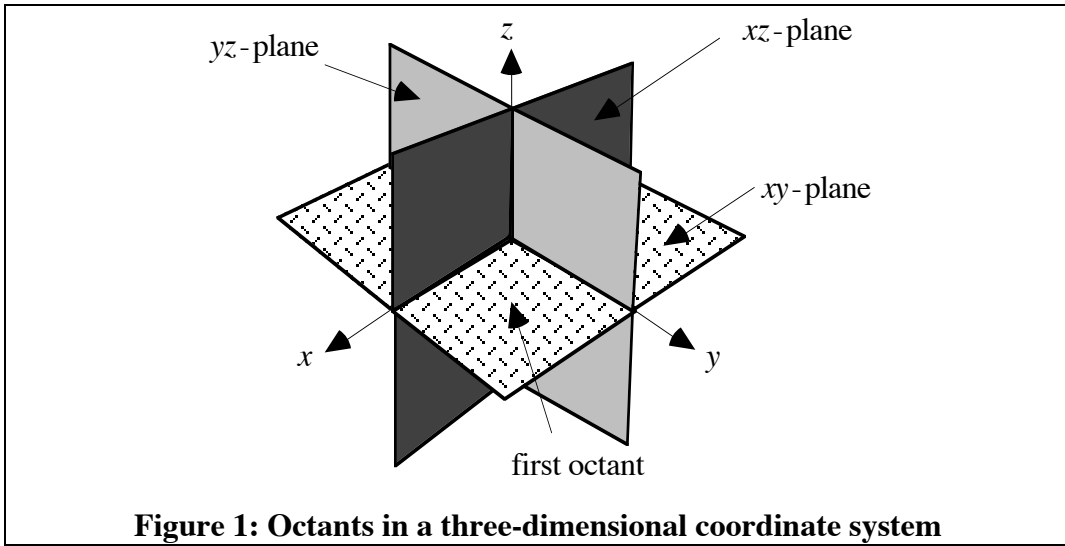
Three-dimensional mappings of the underwater site played a vital role in recovering the treasure. In this module, you examine the basics of three-dimensional graphing.

Activity 1

To describe the locations of artifacts on the sunken ship, the computer used by the Columbus America Discovery Group required three values: latitude, longitude, and depth. In some situations, however, latitude and longitude may not provide the most useful method of describing a location. In this activity, you use distances in three dimensions to describe the location of an object in your classroom.

Mathematics Note

A **three-dimensional coordinate system** in which three planes subdivide space into eight **octants** is shown in Figure 1. Each octant is bounded by the xy -plane, the yz -plane, and the xz -plane. The octant which includes positive x -, y -, and z -values is referred to as the **first octant**.



Exploration

- a. Use the templates provided by your teacher to build a model of a three-dimensional coordinate system.
 1. Tape or glue the templates to four pieces of cardboard and cut slots as shown in Figure 2.

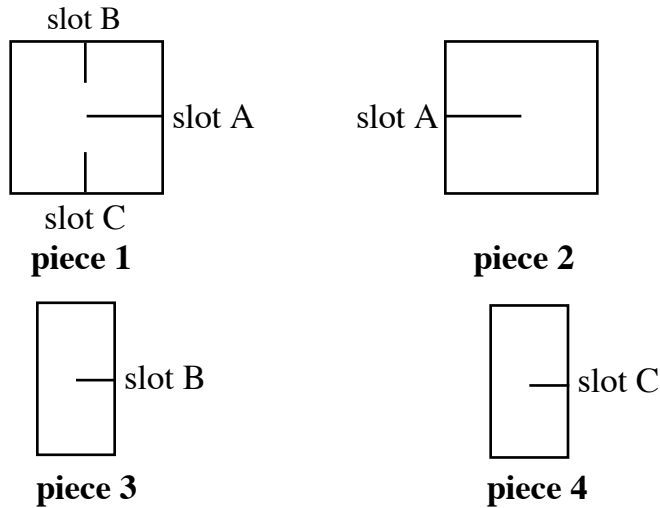


Figure 2: Cutouts for three-dimensional coordinate system

2. As shown in Figure 3a, slide pieces 1 and 2 together along slot A. Add pieces 3 and 4 by sliding them into the corresponding slots. Your model should now resemble the diagram in Figure 3b.

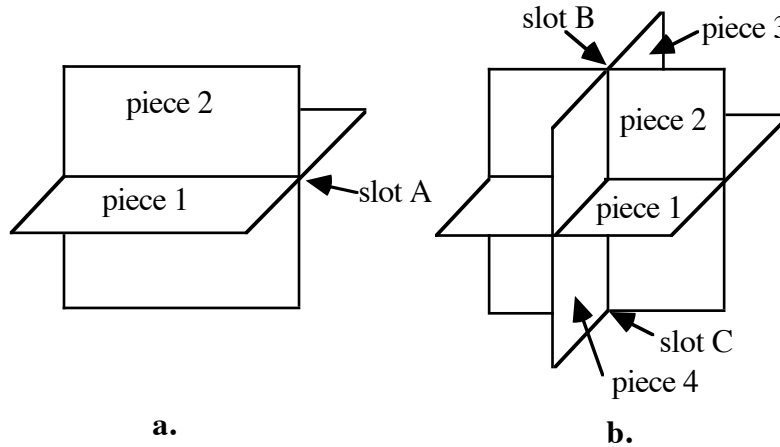


Figure 3: Assembly of three-dimensional coordinate system

- b. Select a corner of your classroom (or some other room in the shape of a rectangular prism). Let the intersection of one wall and the floor represent the x -axis in a three-dimensional coordinate system. Let the intersection of the other wall and the floor represent the y -axis.

The intersection of the two walls represents the z -axis. The intersection of the three axes (represented by the intersection of the two walls and the floor) is the origin.

1. Let 1 m be the unit of measure on each axis in your coordinate system. Use tape to mark the units for each axis.
 2. Choose one part of some object in the room and designate it as the location of a point. Identify the location of the point in terms of the x -axis, the y -axis, and the z -axis.
- c. The location of a point on the model coordinate system from Part a can be represented using a piece of graph paper.

For example, consider the location of the point shown in Figure 4. This point is 4 m from one wall, 5 m from the adjacent wall, and 3 m from the floor.

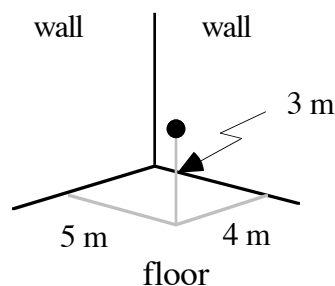


Figure 4: Location of a point in a room

This point's location can be modeled by folding a piece of graph paper as shown in Figure 5.

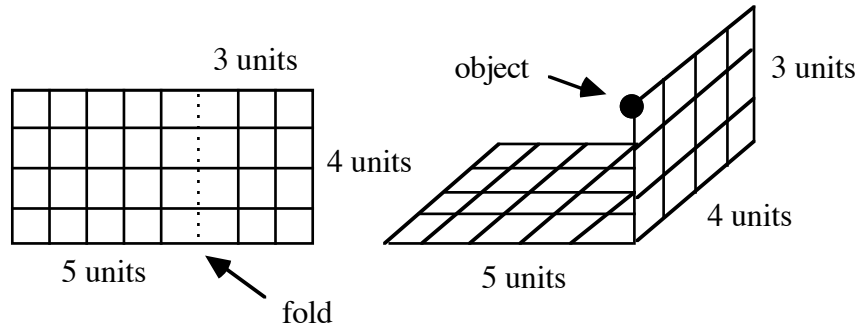


Figure 5: Graph-paper model

1. Use the process described above to model the location of your point with a folded piece of graph paper.
 2. To represent this location on your three-dimensional coordinate system from Part a, place the appropriate corner of the folded paper at the origin.
- d. Ask a classmate to identify the object you selected in Part b using the location of your point on the model coordinate system.

Mathematics Note

Three-dimensional coordinate systems are typically represented by graphs like the ones shown in Figure 6. The x - and y -axes determine the xy -plane. The z -axis is used to help depict points above, below, or on the xy -plane. The x - and z -axes determine the xz -plane, while the y - and z -axes determine the yz -plane.

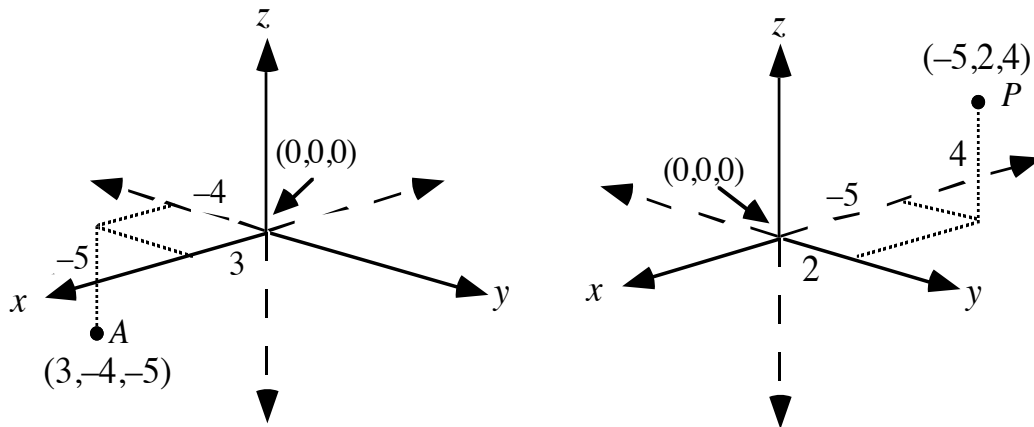


Figure 6: Three-dimensional graphs

The coordinates of a point in this type of three-dimensional coordinate system form an **ordered triple**, (x,y,z) . For example, the coordinates of point A in Figure 6 are $(3,-4,-5)$. In other words, the x -coordinate of point A is 3, the y -coordinate is -4 , and the z -coordinate is -5 , which indicates that point A is located 5 units below the xy -plane. Similarly, the coordinates of point P are $(-5,2,4)$, which means that its x -coordinate is -5 , its y -coordinate is 2, and it is located 4 units above the xy -plane.

The **origin** is the point where the three axes intersect. The origin has coordinates $(0,0,0)$ because at that point $x = 0$, $y = 0$, and $z = 0$.

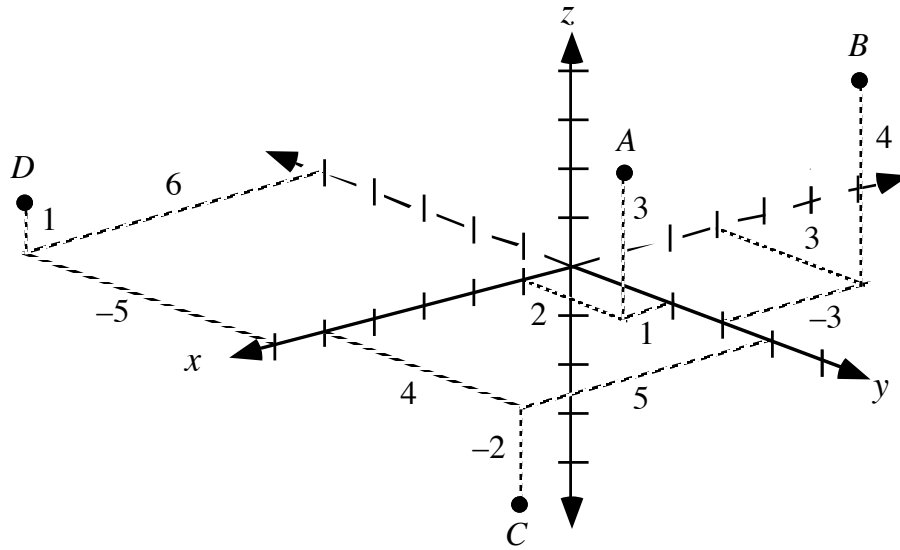
Discussion

- a. Describe how you determined the coordinates of the point you selected in Part **b** of the exploration.
- b. What do all the points on the floor of the room have in common?
- c. How would you model the position of points located below the floor of your classroom?
- d.
 1. In what octant are the coordinates for every point in your classroom?
 2. Are the coordinates for a point located in an adjacent room also in this octant? Explain your response.
- e. Use your model of a three-dimensional coordinate system to describe the coordinates of points in each of the eight octants.
- f. Compare an octant in space to a quadrant in a plane.
- g. As long as the axes are labeled as in Figure 6, does it make any difference in what order the x -, y - and z -coordinates are used to plot a point? Explain your response.
- h. How could you describe the precise location of a large object in your classroom?

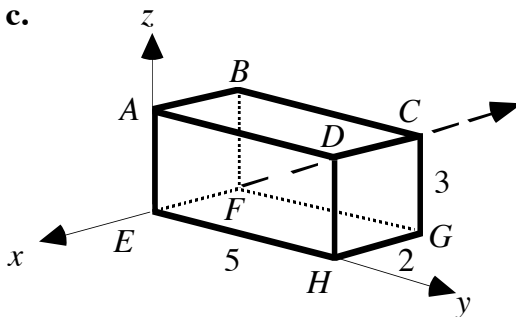
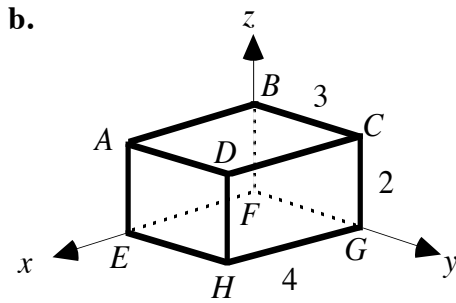
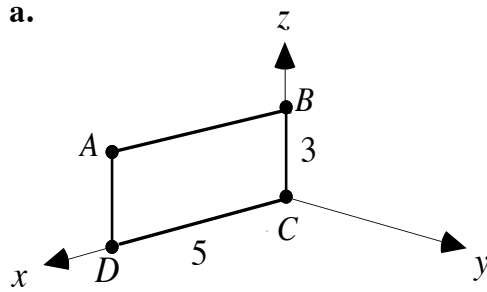
Assignment

- 1.1
 - a. Is the z -coordinate of a point above the xy -plane positive or negative?
 - b. What is the x -coordinate of a point on the yz -plane?
 - c. If the y -coordinate of a point is positive, what is its relationship to the xz -plane?

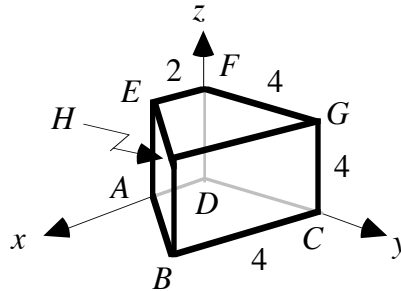
- 1.2 Identify the coordinates of points A , B , C and D in the graph below.



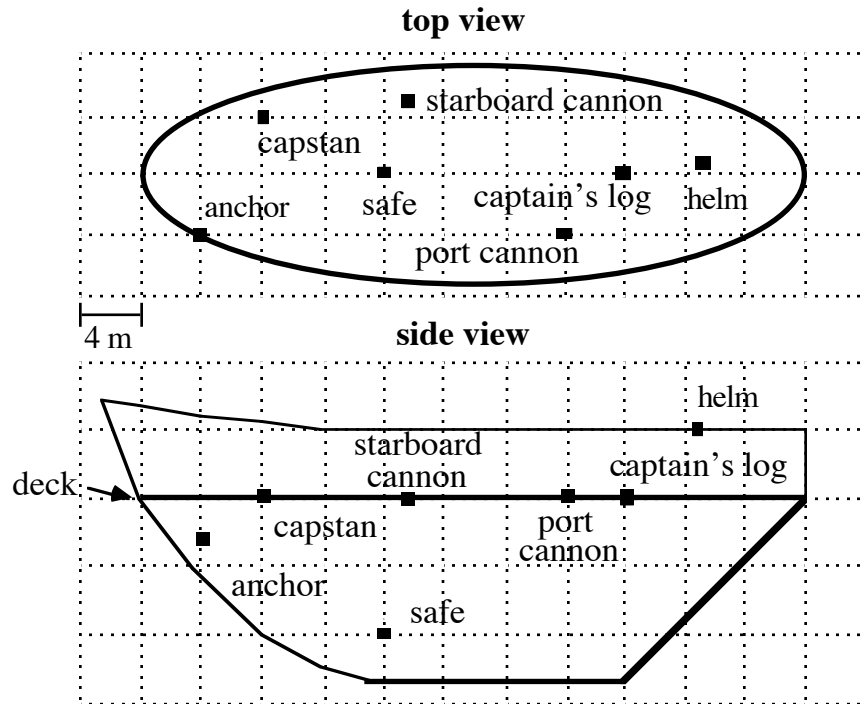
- 1.3 Describe the location of a point with coordinates $(-5, -4, -6)$ on a three-dimensional coordinate system.
- 1.4 Identify the coordinates of each labeled vertex in Parts a–c.



- 1.5** Use a coordinate system like the one shown in Problem 1.2 to sketch the graph of each point below.
- $A(4,5,3)$
 - $B(2,0,5)$
 - $C(0,5,0)$
 - $D(2,3,-2)$
 - $E(4,1,-5)$
- 1.6** Complete Parts **a–c** to make a sketch of a three-dimensional figure.
- Plot each of the following ordered triples and label each as a vertex:
 $A(2,2,0)$, $B(6,2,0)$, $C(6,6,0)$, $D(2,6,0)$, $E(4,4,5)$
 - Use segments to connect each vertex with every other vertex. Use solid segments to represent the edges that can be seen from your point of view. Use dotted segments to depict any unseen edges.
 - What kind of three-dimensional figure did you draw?
- 1.7** List the coordinates of the vertices of the following figure.



- 1.8** A team of archaeologists has discovered the wreck of the Spanish galleon *Isabella*. The diagrams below show both top and side views of the *Isabella*, along with the locations of several objects the team wants to bring to the surface.



- a. To help the scientists create a map of each artifact's location, complete Steps 1–3 below.
 1. Select and label a point on the deck as the origin of a three-dimensional coordinate system with its x - and y -axes on the deck.
 2. Select a scale for the axes.
 3. Identify the coordinates of each artifact as an ordered triple.
- b. Could you have selected a different point as the origin of your coordinate system? Why might one point be a better choice for the origin than another?

* * * * *

- 1.9** Describe the set of all points in space that are:
- a. 3 cm from a point
 - b. 3 cm from a segment
 - c. equidistant from two parallel lines
 - d. 3 cm from a plane.

- 1.10** Consider a room shaped like a rectangular prism. Describe the set of all points in the room that are:
- a. 180 cm from a point at the center of the floor
 - b. equidistant from the four corners of the ceiling.
- 1.11** While examining a wreck, a team of divers collected information about each artifact on the ship. As described in Parts **a–c** below, they recorded this data as ordered triples in which each unit represents a distance of 1 m. Make a sketch of each artifact on a three-dimensional coordinate system.
- a. a mast with its endpoints at (1,3,0) and (0,6,8)
 - b. a storage box with its vertices at (5,2,0), (7,2,0), (5,3,0), (7,3,0), (5,2,1), (7,2,1), (5,3,1), and (7,3,1)
 - c. a cannonball whose center is located at (8,8,1)

* * * * *

Research Project

On a typical two-dimensional map, differences in elevation can be hard to visualize. For this reason, hikers and others often use topographic maps, which show elevation on contour lines. Although this feature makes it easier to identify the elevation of a specific location, forming a mental picture of the terrain can still be difficult.

When a clear representation of the landscape is needed, mapmakers create relief maps. Relief maps show the features of the terrain in three dimensions. In this research project, you make a relief map of a region of your choice. (The area you select should have identifiable differences in elevations.)

- a. Create a two-dimensional map of your area on graph paper. For example, Figure 7, shows a map of Montana on graph paper.

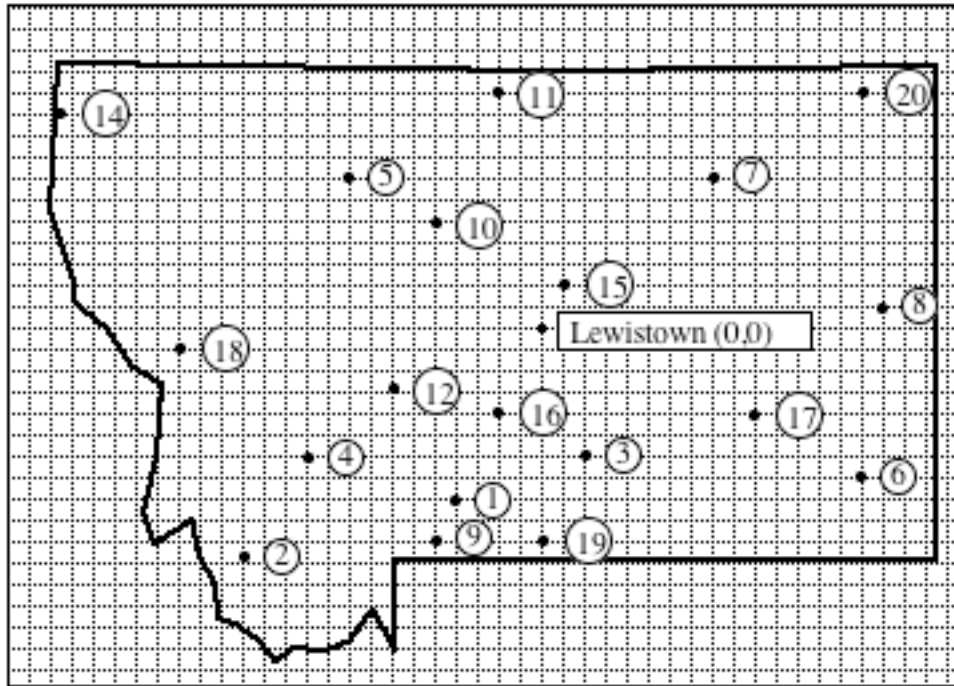


Figure 7: Map of Montana on graph paper

- b. Identify several locations of interest on your map and number them as illustrated in Figure 7. In this example, the numbered locations represent important historical sites.
- c. Determine the elevation of each site selected in Part **b**. In Figure 7, for example, the elevation of site 12 (Helena, the state capital) is 1267 m above sea level. Record these values in a table.
- d. Select a location to represent the origin.
- e. Tape the map onto a piece of cardboard. To select a scale for your three-dimensional map, complete the following steps.
 1. Place the map and cardboard on a table. To determine the thickness of the cardboard, push a toothpick into the cardboard until it hits the table. Mark the toothpick where it intersects the top of the cardboard.
 2. Measure the distance from this mark to the top of the toothpick in millimeters.
 3. Use the elevations you recorded in Part **c** to identify the highest site on the map.
 4. Use the distance from Step 2 and the elevation from Step 3 to create a scale that relates the elevation of a site in meters to the length of the toothpick in millimeters. Round the scale to the nearest meter.

5. Push a toothpick into the map at the location of the highest site. Using your scale, the elevation of this site is represented by the length of the toothpick above the cardboard.
- f. Gather enough toothpicks to identify all the sites on the map.
1. Mark each toothpick with the thickness of the cardboard.
 2. Use proportions to calculate the number of millimeters above the mark needed to represent the elevation of each site. (How would you show a site with an elevation below sea level?)
 3. Cut each toothpick to the appropriate length and push it into the map at the corresponding site.
- g. Write a paragraph identifying one of the sites by its location on your map and by its latitude, longitude, and elevation. In a second paragraph, explain the historical significance of that site.
-

Activity 2

In this activity, you investigate another kind of three-dimensional map: a **surface plot**. Surface plots are similar to the underwater maps used by the Columbus America Discovery Group. Their computer systems used sonar to obtain ordered triples. In the following exploration, you use a mechanical process to produce surface plots.

Mathematics Note

A **surface plot** is a three-dimensional graph used to display the surface of an object. Figure 8 shows one example of a surface plot.

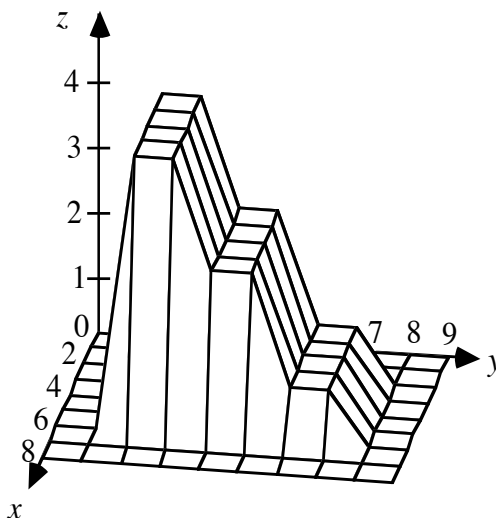


Figure 8: Surface plot

Exploration

Obtain a box with a lid, some stiff wire cut into equal lengths, and a sheet of centimeter grid paper. The box simulates the cargo hold of a sunken ship. Inside the ship is an artifact of unknown size and shape. Before you can recover the artifact, you must first determine its size and shape.

- a.
 1. Cover the lid of the box with centimeter graph paper.
 2. Place the origin of a three-dimensional coordinate system in one corner of the graph paper. Draw the x - and y -axes as shown in Figure 9.

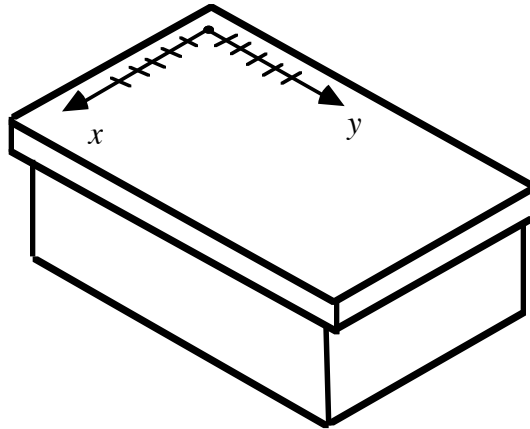


Figure 9: Placement of coordinate system

- b.
 1. At each point where the grid lines intersect (the **lattice points**), press a piece of wire through the lid until it strikes either the artifact or the bottom of the box. Leave each wire in the box.
 2. Measure the distance from the lid to the top of each wire in millimeters. Record your measurements in a data table or **matrix** like the one in Figure 10. Each entry in the table represents an ordered triple that describes the location of the top of a wire in a three-dimensional coordinate system.

	0	1	2	3	...	$y-1$	y
0							
1							
2							
3							
\vdots							
$x-1$							
x							

Figure 10: Data table

Mathematics Note

A **matrix** is a rectangular arrangement of data used for storing information in an organized fashion. Matrices can be used to create surface plots. For example, the matrix in Figure 11a was used to create the surface plot in Figure 8. The entries in this matrix represent z -values in the ordered triples (x,y,z) .

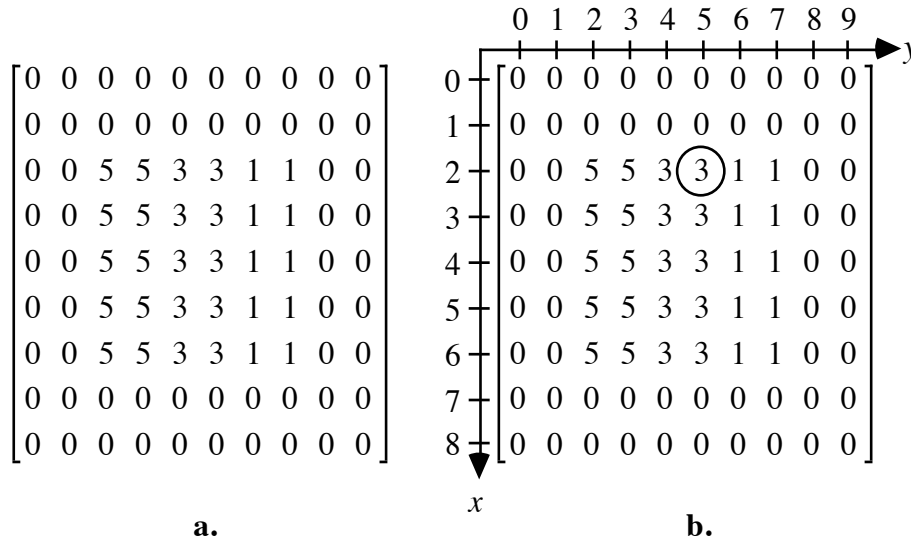


Figure 11: Matrix used to create a surface plot

The manner in which data is entered into a matrix may vary for different types of technology. In this module, the x - and y -coordinates are related to the matrix as illustrated in Figure 11b. For example, the circled entry in this matrix represents the ordered triple $(2,5,3)$.

- Use a symbolic manipulator to make a surface plot of the artifact in your box. Print a copy of the surface plot.
- On the basis of the surface plot, what do you think is in the box?
- “Recover” your artifact by removing the lid of the box.

Discussion

- How does the object described by your surface plot compare with the actual artifact?
- What is the relationship between the tops of the wires and the surface of the artifact?
- What is the relationship between the numbers in the data table and the surface plot?

Assignment

2.1 For each of the following data tables, sketch a surface plot on a three-dimensional coordinate system.

a.

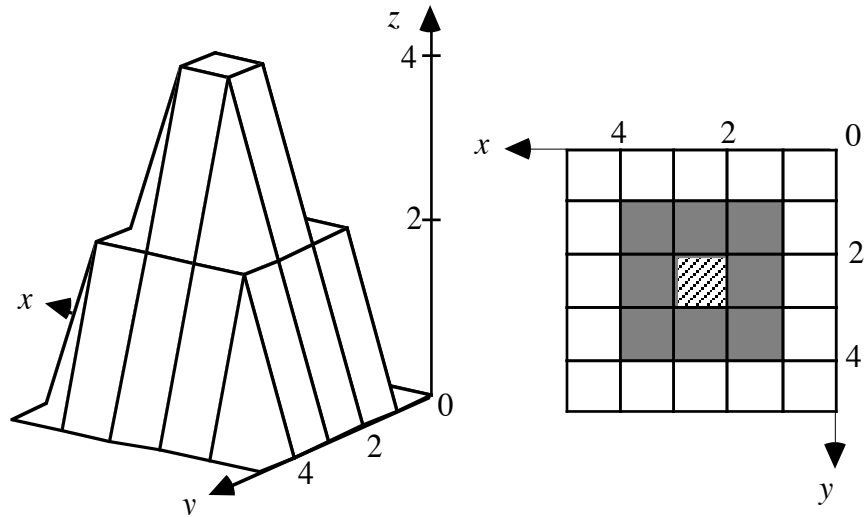
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 7 & 7 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 7 & 7 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b.

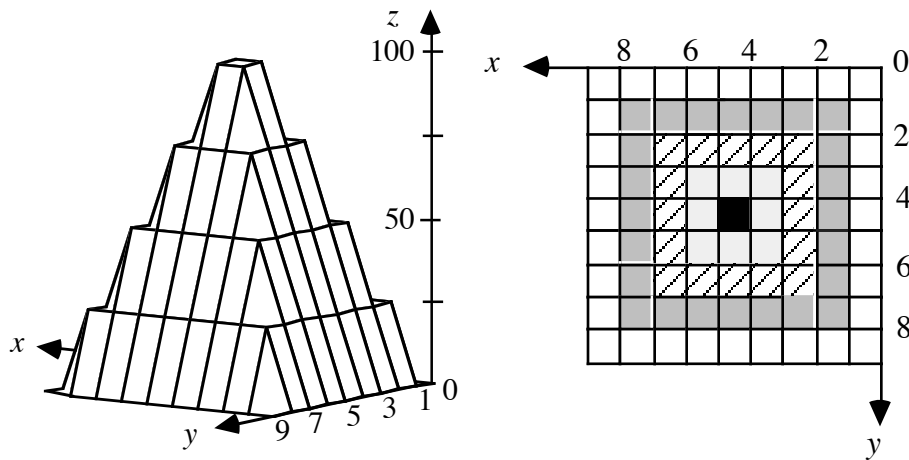
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 5 & 5 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 5 & 5 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.2 Complete a matrix for each of the following surface plots. In each case, the figure on the left shows the object as viewed from a perspective slightly to one side of it. The figure on the right shows the object as viewed from a perspective directly above it. The different shadings represent different elevations.

a.



b.



2.3 Create a surface plot that resembles a river flowing through a section of a deep canyon.

2.4 How would the data table in Problem 2.1a be affected in each of the following situations?

- The surface plot is moved 2 units upward.
- The height of the surface plot is tripled.
- The surface plot is reflected in the xy -plane.

2.5 Describe the surface plot that occurs when all the entries in a data table are equal.

* * * * *

2.6 a. On a three-dimensional coordinate system, sketch a surface plot of the object represented by the following matrix.

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

b. Determine the angle that the surface plot makes with the xy -plane.

2.7 In a one-dimensional coordinate system, the graph of the equation $x = 3$ is a point. In a two-dimensional coordinate system, the graph of this equation is a line. Describe the graph of each of the following in a three-dimensional coordinate system.

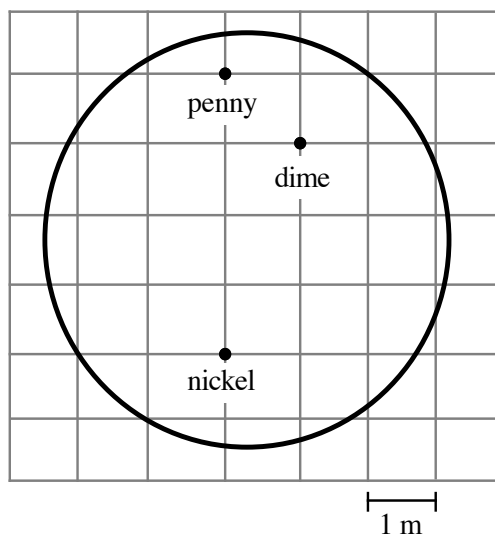
a. $x = 3$

b. $y = 3$

c. $z = 3$

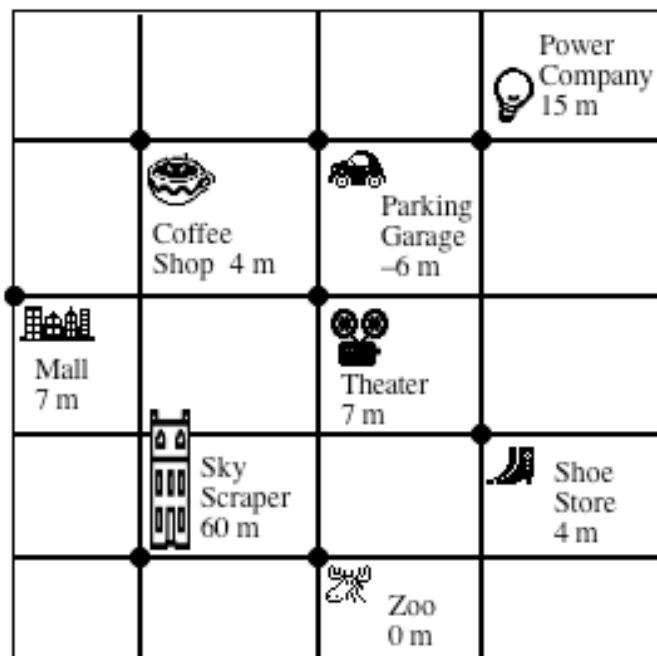
Summary Assessment

1. At the local wishing well, you throw in three coins—a dime, a nickel, and a penny. They land in the well as shown in the following diagram.



- a. Label a point in the diagram as the origin and record the coordinates of each coin.
 - b. The depth of the water in the well is 0.5 m. If the surface of the water is the xy -plane, identify the location of each coin as an ordered triple.
2. Imagine that you have a front-row seat at AquaLand's dolphin show. The star performer at AquaLand is Daphne. The pool is a rectangular prism 80 m long, 30 m wide, and 10 m deep.
- Your seat is at water level at the center of the long side of the pool. The announcer stands by the corner of the pool to your left, on the opposite side from your seat. The trainer stands on a platform whose base is at the center of the side of the pool on your right. The platform extends 1 m from the edge of the pool and 2 m above the water. At the center of the bottom of the pool is a drain.
- a. Make a sketch of the pool on a three-dimensional coordinate system. Let the surface of the pool be located in the xy -plane, with the narrator at the origin.
 - b. Identify the coordinates of the narrator, the trainer, and the drain.

- c. The starting position for Daphne's routine is the point $(25,50,-6)$. Use X to represent this point on your drawing.
- d. On the surface of the water, directly above Daphne's starting position, is a plastic hoop. What are the coordinates of the hoop's position?
3. The figure below shows the heights of eight structures in a city. One of these, the parking garage, is located underground.



- a. Use coordinates to identify the height and location of each structure.
- b. Organize this information in a matrix.
4. An architect's new home contains a rectangular room 8 m long and 6 m wide. The ceiling is 4 m high.
- a. Make a sketch of the room on a three-dimensional coordinate system.
- b. The electrician wants to install a light at the center of the ceiling. What are the coordinates of the point where the light should be placed?
- c. The door to the room is located in one of the 8-m walls. The door is 1 m wide and its hinges are 1 m from the adjacent 6-m wall. The door opens to the inside, toward the 6-m wall. The carpenter wants to install a doorstop to keep the door from hitting the wall. Where should the carpenter place the doorstop? Describe this location as an ordered triple.
- d. The thermostat control is located 1.5 m above the floor on one wall of the room, directly opposite the door. Describe the thermostat's location as an ordered triple.

Module Summary

- In one type of **three-dimensional coordinate system**, three planes subdivide space into eight **octants**. Each octant is bounded by the xy -plane, the yz -plane, and the xz -plane. The octant which includes positive x -, y -, and z -values is referred to as the **first octant**.
- The coordinates of a point in this type of three-dimensional coordinate system form an **ordered triple**.
- The **origin** in this three-dimensional coordinate system is the point where the three axes intersect. The origin has coordinates $(0,0,0)$.
- A **lattice point** is a point whose coordinates are integer values.
- A **surface plot** is a three-dimensional graph used to display the surface of an object.
- A **matrix** is a rectangular arrangement of data used for storing information in an organized fashion.

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