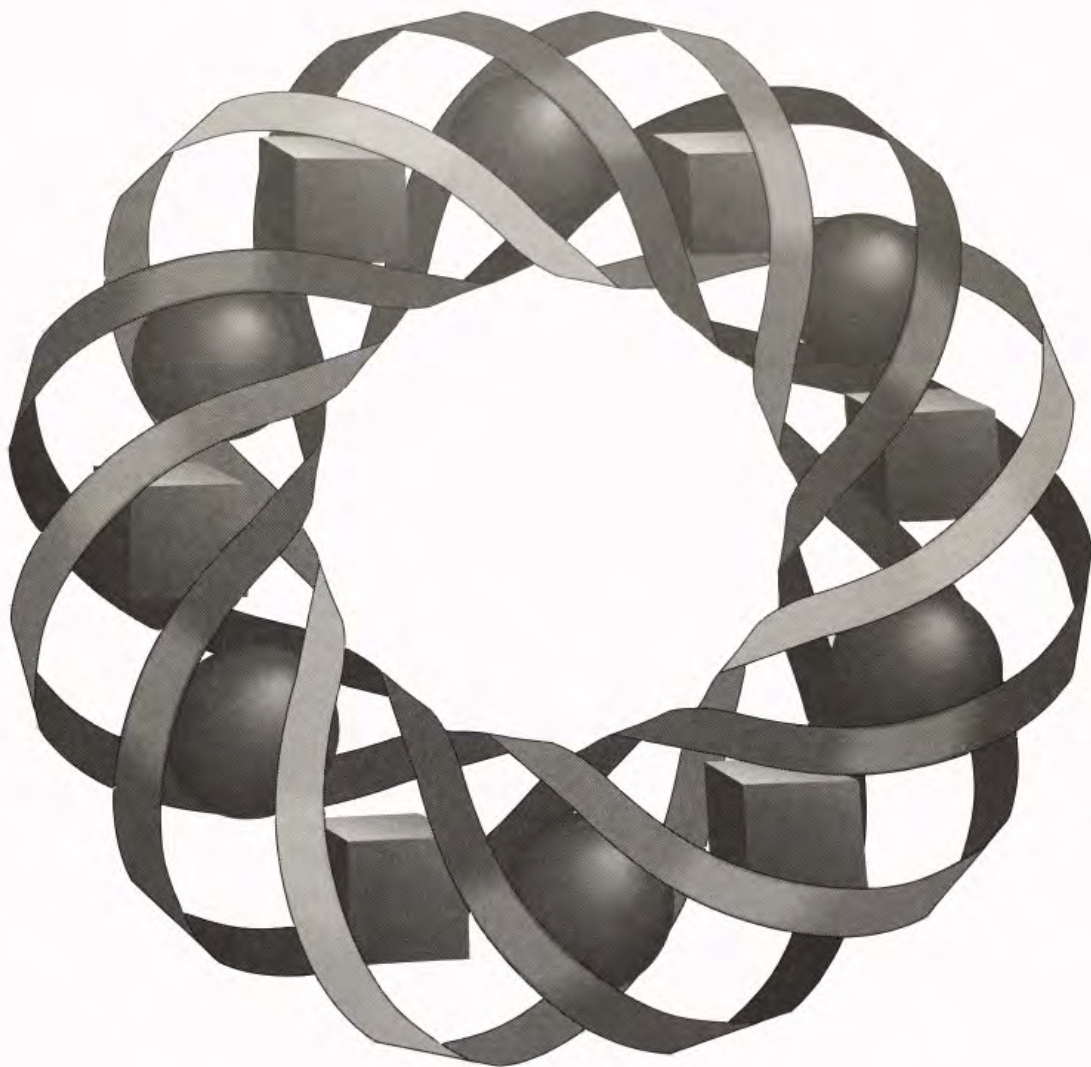


SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 2 Volumes 1-3



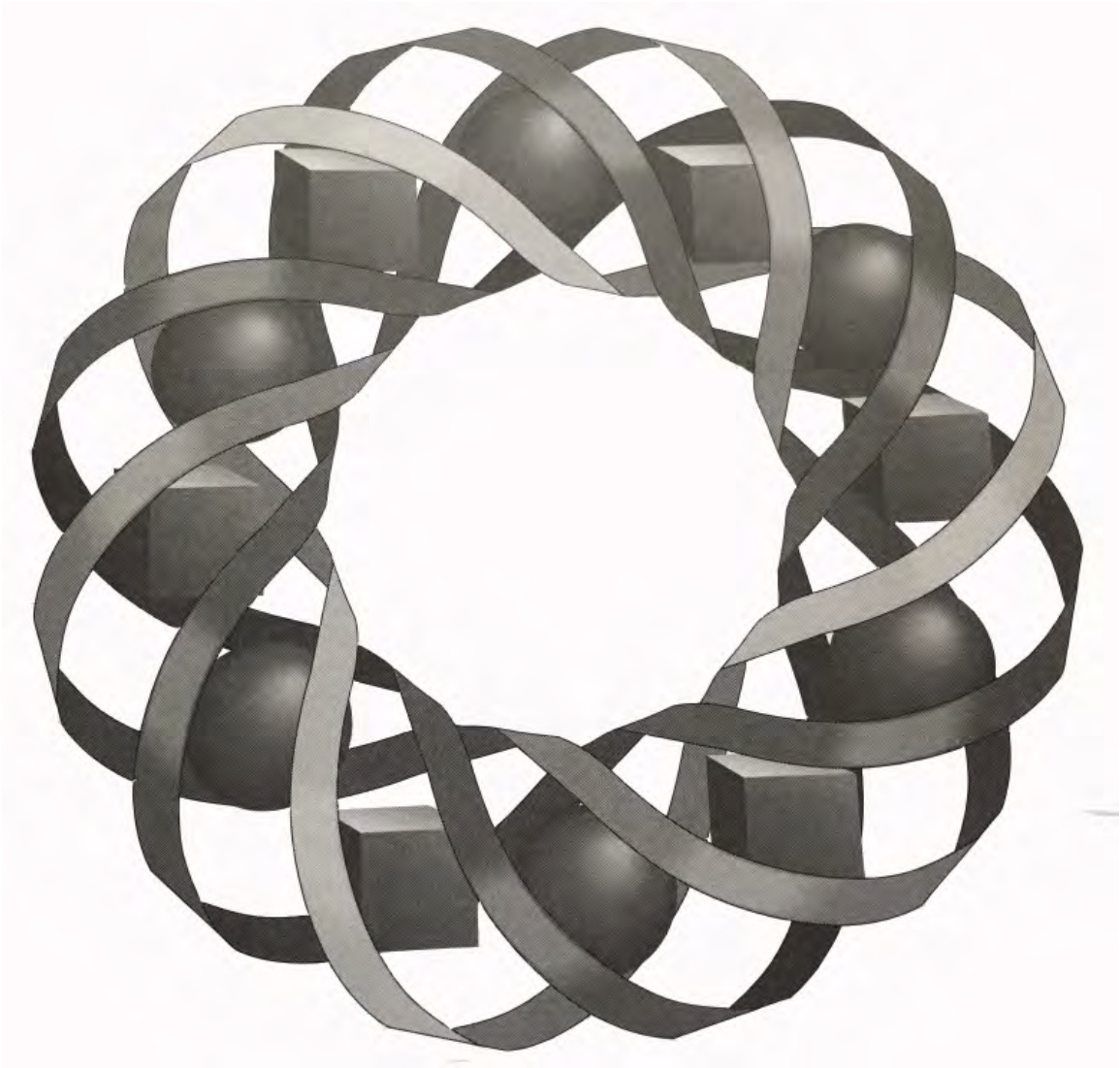
© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Table of Contents

1	Marvelous Matrices	1
2	A New Angle on an Old Pyramid	29
3	When to Deviate from a Mean Task	65
4	Who Gets What and Why	95
5	What Are My Child's Chances?	119
6	There's No Place Like Home	147
7	Making Concessions	179
8	Crazy Cartoons	207
9	Hurry! Hurry! Step Right Up!	245
10	Atomic Clocks Are Ticking	269
11	And the Survey Says . . .	293
12	Traditional Design	317
13	If the Shoe Fits . . .	349
14	Take It to the Limit	373
15	Algorithmic Thinking	403



Marvelous Matrices



How do businesses—large and small—keep track of their sales, inventory, and profits? In this module, you explore some of the ways that matrices are used to store and analyze information.

Kyle Boyce • Pete Stabio



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Marvelous Matrices

Introduction

Welcome to the Age of Information. Office workers and company presidents, homeowners and rent payers, consumers and manufacturers—all are bombarded daily by mountains of data. Somehow, some way, you have to make sensible decisions based on that information. Those decisions range from simple day-to-day choices to those with major personal or financial impact.

A typical business manager juggles information like monthly sales, production costs, company inventory, and employee productivity. A **matrix** is one mathematical tool used to organize the information needed to make good decisions. In this module, you look at how a small business grapples with the same problems that confront corporate giants such as General Motors and Motorola.

Activity 1

Family Snack, a branch of the Family Corporation, sells nuts, beef jerky, and jam. These three products are sold separately, as shown in Figure 1, as well as in assortment packages. Because they cannot be divided into smaller products, these three items are considered **simple components**.



Figure 1: Simple components

Family Snack's assortment packages contain different combinations of the three simple components. The Snack Pack contains 2 boxes of nuts and 6 pieces of jerky, the Gift Pack contains 3 jars of jam and 2 Snack Packs, and the Family Pack contains 3 jars of jam and 3 Gift Packs. These **composite products** are shown in Figure 2.

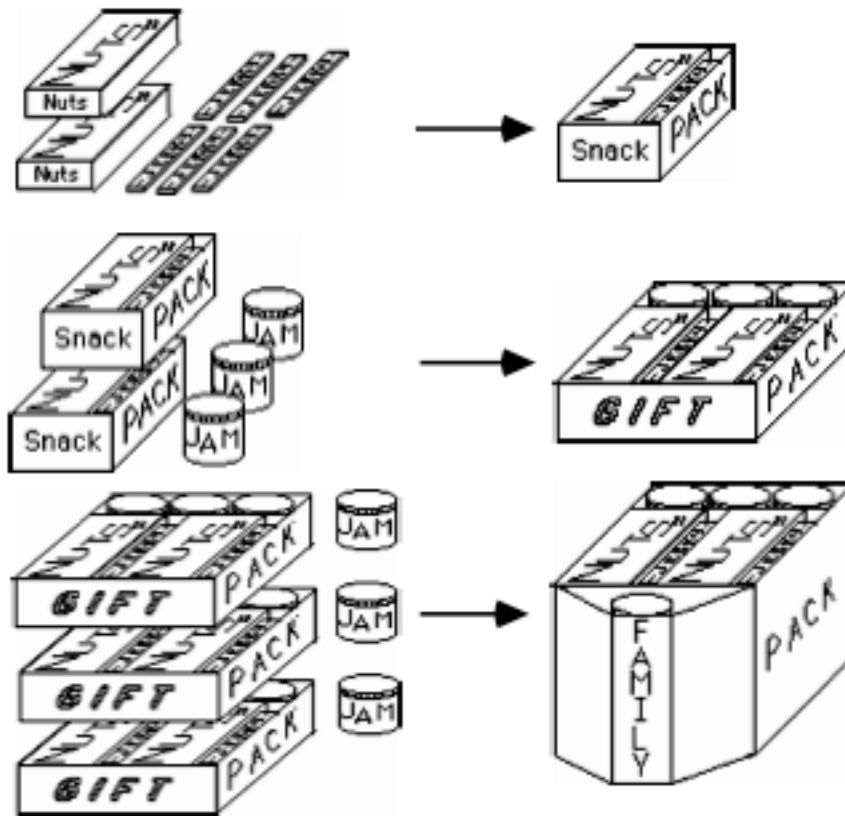


Figure 2: Composite products

Family Snack has received an order for 8 boxes of nuts, 20 pieces of jerky, 25 jars of jam, 10 Snack Packs, 5 Gift Packs, and 40 Family Packs. How many boxes of nuts, pieces of jerky, and jars of jam are needed to fill this order? In the following exploration, you use matrices to help answer this question.

Exploration

In this exploration, you discover how graphs and matrices can help you organize data and make informed decisions.

Business Note

One way to represent a company's product line is with a **requirement graph**. A requirement graph is a tree diagram that shows each product's components at each level of its assembly. A complete requirement graph has simple components at the end of each branch.

For example, consider a company that sells mathematics supplies. This company's simple components are graph paper, rulers, compasses, and protractors. It also sells two composite products: the Euclid Set and the Complete Set. The Euclid Set contains 10 rulers and 5 compasses, while the Complete Set contains 2 Euclid Sets, 10 packs of graph paper, and 10 protractors.

A requirement graph for this product line is shown in Figure 3.

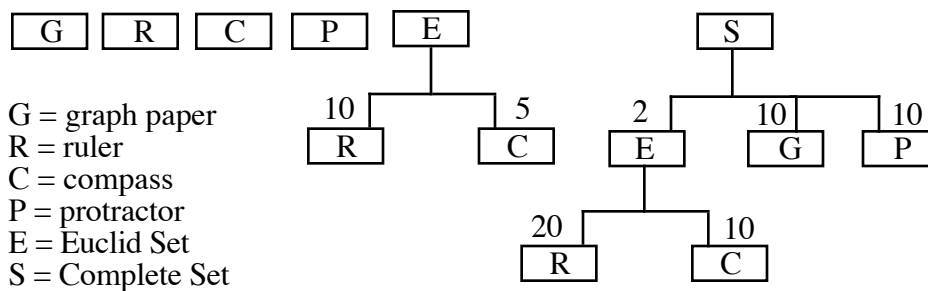


Figure 3: A requirement graph

- a. Draw a requirement graph for Family Snack's product line (see Figures 1 and 2).
- b. Use the requirement graph from Part a to complete Steps 1–5.
 1. Describe the Snack Pack in terms of its simple components.
 2. Describe the Gift Pack in terms of its simple components and composite products.
 3. Describe the Gift Pack in terms of its simple components only.
 4. Describe the Family Pack in terms of its simple components and composite products.
 5. Describe the Family Pack in terms of its simple components only.

Mathematics Note

A **matrix** (plural **matrices** or **matrixes**) is a rectangular arrangement of rows and columns used to organize information. A matrix of i rows and j columns has **dimensions** $i \times j$ (read “ i by j ”). Matrices are named using bold, uppercase letters or descriptive words. Each item in a matrix is an **element**.

A company can use a requirement graph to create a **total requirement matrix**. In a total requirement matrix, each item sold is represented in a column heading, while each simple component or composite product is represented in a row heading. Each element indicates the number of that simple component or composite product required to produce the corresponding item sold.

For example, the matrix **T** shown below is a total requirement matrix for the product line described by the requirement graph in Figure 3. In this matrix, G represents graph paper, R represents ruler, C represents compass, P represents protractor, E represents Euclid Set, and S represents Complete Set. In matrix **T**, the values 10, 20, 10 in the S column represent the total number of simple components required for a Complete Set. This total includes the simple components for the two Euclid sets that are part of a Complete Set.

$$\mathbf{T} = \begin{array}{c} \begin{array}{cccccc} & \mathbf{G} & \mathbf{R} & \mathbf{C} & \mathbf{P} & \mathbf{E} & \mathbf{S} \\ \mathbf{G} & [1 & 0 & 0 & 0 & 0 & 10] \\ \mathbf{R} & [0 & 1 & 0 & 0 & 10 & 20] \\ \mathbf{C} & [0 & 0 & 1 & 0 & 5 & 10] \\ \mathbf{P} & [0 & 0 & 0 & 1 & 0 & 10] \\ \mathbf{E} & [0 & 0 & 0 & 0 & 1 & 2] \\ \mathbf{S} & [0 & 0 & 0 & 0 & 0 & 1] \end{array} \end{array}$$

Since matrix **T** has 6 rows and 6 columns, its dimensions are 6×6 . The element in the third row, fifth column indicates that there are 5 compasses in each Euclid Set.

- c. Use the requirement graph from Part a to create a total requirement matrix **R** for Family Snack’s product line.

Discussion

- a. What does the element 5 represent in the requirement graph in Figure 3?
- b. 1. In the total requirement matrix you created for Family Snack’s product line, why is each element along the diagonal the number 1?
2. What does the element 0 represent in this matrix?
- c. Describe the relationship between a requirement graph and its corresponding requirement matrix.

- d. As mentioned earlier, Family Snack has received an order for 8 boxes of nuts, 20 pieces of jerky, 25 jars of jam, 10 Snack Packs, 5 Gift Packs, and 40 Family Packs. How can you use a requirement matrix to help determine the number of boxes of nuts, pieces of jerky, and jars of jam needed to fill this order?

Assignment

- 1.1 Family Snack has received an order for 20 boxes of nuts, 60 pieces of jerky, 48 jars of jam, 24 Snack Packs, 12 Gift Packs, and 2 Family Packs. How many boxes of nuts, pieces of jerky, and jars of jam are needed to fill the entire order?
- 1.2 Family Snack has decided to introduce three more products—individual jars of honey, the Honey Pack, and the Hive Pack. The Honey Pack contains 3 jars of honey and 2 Snack Packs, while the Hive Pack contains 3 jars of honey and 3 Gift Packs.
- Create a requirement graph for Family Snack’s product line that includes these new products.
 - Create a total requirement matrix for this product line that includes the new products.
 - Determine the number of jars of jam in one Hive Pack.
- 1.3 The owners of Sammie’s Spreadables sell a variety of cracker spreads. The requirement matrix for the company’s product line is shown below.

$$\mathbf{S} = \begin{matrix} & \begin{matrix} \text{shrimp} \\ \text{lobster} \\ \text{spinach} \\ \text{Six-Pack} \\ \text{Ten-Pack} \end{matrix} \\ \begin{matrix} \text{shrimp} \\ \text{lobster} \\ \text{spinach} \\ \text{Six-Pack} \\ \text{Ten-Pack} \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

- Create a requirement graph that corresponds with matrix \mathbf{S} .
- How can you identify the simple components in matrix \mathbf{S} ?
- Sammie’s Spreadables has received an order for 8 Ten-Packs. Describe how you could use matrix \mathbf{S} to determine the number of each simple component needed to fill the order.

* * * * *

- 1.4** The Holiday Hands Company sells packages of cheese and meat products. There are six items in their product line: cheddar cheese logs, cheese-and-bacon logs, summer sausage logs, Snack Packs, Hand-Out Packs, and Pig-Out Packs. The Snack Pack contains two cheddar cheese logs and three summer sausage logs. The Hand-Out Pack contains three cheddar cheese logs and two cheese-and-bacon logs. The Pig-Out Pack contains one Snack Pack and two Hand-Out Packs.
- Create a requirement graph for Holiday Hands' product line.
 - Create a total requirement matrix for Holiday Hands' product line.
 - Determine the number of summer sausage logs in one Pig-Out Pack.
 - Determine the number of cheddar cheese logs in one Pig-Out Pack.

- 1.5** Use the requirement matrix shown below to complete Parts **a–c**.

$$\mathbf{T} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}$$

- Create a requirement graph that corresponds with this requirement matrix.
 - Identify the simple components in matrix **T** and describe how you recognized them.
 - Describe how you could use the total requirement matrix to determine the amount of each simple component needed to fill an order for 10 pieces of item F.
- 1.6** In addition to those described in Problem **1.3**, Sammie's Spreadables has added four more items to its product line. These include a pepperoni spread, a jalapeño spread, a Twelve-Pack, and a Combo Pack. The Twelve-Pack contains a Six-Pack, 3 pepperoni spreads, and 3 jalapeño spreads. The Combo Pack contains a Ten-Pack, 4 pepperoni spreads, and 3 jalapeño spreads.
- Create a total requirement graph for Sammie's expanded product line.
 - Create a total requirement matrix for Sammie's expanded product line.
 - Determine the number of spinach spreads in one Combo Pack.

Activity 2

In Activity 1, you used requirement graphs and requirement matrices to represent Family Snack's product line. In this activity, you use matrices to organize and analyze other types of data.

Exploration

Family Snack employs three salespeople: Keyes, Zhang, and Troy. In September, Keyes sold 8 packages of nuts, 12 packages of jerky, 0 jars of jam, 16 Snack Packs, 28 Gift Packs, and 8 Family Packs.

In the same month, Zhang sold 12 packages of nuts, 4 packages of jerky, 24 jars of jam, 8 Snack Packs, 24 Gift Packs, and 12 Family Packs.

Troy's September sales included 12 packages of nuts, 0 packages of jerky, 12 jars of jam, 12 Snack Packs, 36 Gift Packs, and 4 Family Packs.

- a. Use a matrix to organize this information in each of the following ways. Record the dimensions of each matrix.
 1. Designate each row by a salesperson's name and each column by a product name.
 2. Designate each row by a product name and each column by a salesperson's name.

Mathematics Note

Two matrices are **equal** if they have the same dimensions and if corresponding elements are equal.

For example, consider matrices **A** and **B** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 7 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 7 & 6 \end{bmatrix}$$

These matrices are equal because the dimensions of both are 3×2 and because the element in row 1, column 1 of matrix **A** is the same as the element in row 1, column 1 of matrix **B**; the element in row 1, column 2 of matrix **A** is the same as the element in row 1, column 2 of matrix **B**; the element in row 2, column 1 of matrix **A** is the same as the element in row 2, column 1 of matrix **B**; and so on.

- b. Are the matrices created in Part a equal? Explain your response.

- c. In October, Keyes sold 8 packages of nuts, 4 packages of jerky, 24 jars of jam, 12 Snack Packs, 8 Gift Packs, and 36 Family Packs.
- In the same month, Zhang sold 16 packages of nuts, 8 packages of jerky, 28 jars of jam, 16 Snack Packs, 20 Gift Packs, and 32 Family Packs.
- Troy's October sales included 8 packages of nuts, 12 packages of jerky, 4 jars of jam, 16 Snack Packs, 8 Gift Packs, and 0 Family Packs.
- Select one of the matrices you created in Part **a** for September's sales. Using the dimensions of this matrix, create the corresponding matrix for October's sales.
- d. 1. Create a matrix that displays the total number of each product sold by each salesperson during September and October.
2. Create a matrix that shows the change in the sales of each product by each salesperson from September to October.
- e. As the holiday season approaches, the company expects November sales to be twice that of the previous month. Create a matrix that shows the predicted sales of each product for each salesperson.

Mathematics Note

Matrix addition can be performed on two matrices with the same dimensions by adding the corresponding elements of each matrix. For example, the addition of two 2×3 matrices is shown below.

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -8 & 9 \\ 10 & 11 & -2 \end{bmatrix} = \begin{bmatrix} (1+7) & (2-8) & (3+9) \\ (-3+10) & (4+11) & (5-2) \end{bmatrix} = \begin{bmatrix} 8 & -6 & 12 \\ 7 & 15 & 3 \end{bmatrix}$$

Matrix subtraction can be performed in a similar manner. For example, the following equation shows the subtraction of one 2×2 from another.

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} (1-0) & (-2-1) \\ (-3-2) & (4-(-5)) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -5 & 9 \end{bmatrix}$$

In **scalar multiplication**, each element of a matrix is multiplied by a constant, or **scalar**. The multiplication of a matrix **M** by a scalar k is denoted by $k \cdot \mathbf{M}$.

For example, consider the matrix **M** below.

$$\mathbf{M} = \begin{bmatrix} 5 & 4 \\ -2 & 8 \\ 0 & -1 \end{bmatrix}$$

Multiplying **M** by the scalar 5 produces the following result.

$$5 \cdot \mathbf{M} = 5 \cdot \begin{bmatrix} 5 & 4 \\ -2 & 8 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5(5) & 5(4) \\ 5(-2) & 5(8) \\ 5(0) & 5(-1) \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ -10 & 40 \\ 0 & -5 \end{bmatrix}$$

- f. Use technology to complete Steps **1–3** below. Assume that November sales are represented by the matrix you created in Part e.
1. Determine a matrix that displays the total sales of each product by each salesperson during October and November.
 2. Determine a matrix that shows the change in the sales of each product by each salesperson between September and November.
 3. Determine a matrix that shows total sales of each product by each salesperson for September, October, and November.
- g. The mean of two numbers can be found by dividing their sum by 2 or multiplying their sum by $1/2$. Use scalar multiplication to complete Steps **1** and **2** below.
1. Determine a matrix that shows mean sales of each product by each salesperson during September and October.
 2. Determine a matrix that shows mean sales of each product by each salesperson during September, October, and November.

Discussion

- a. Describe how you determined the mean sales for all three months in Part g of the exploration.
- b. Do you think that any two matrices **A** and **B** can be added together? Explain your response.
- c. When matrices are added, subtracted, or multiplied by a scalar, what are the dimensions of the resulting matrix? Why does this occur?
- d. Do you think that matrix addition is commutative? In other words, does $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, where **A** and **B** are matrices? Justify your response.

Assignment

2.1 In Parts **a–e** below, perform the indicated operation, if possible. If the operation is not possible, explain why.

a. $\begin{bmatrix} 5 & 9 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 2 & 8 & -2 \end{bmatrix}$

c. $\pi \cdot \begin{bmatrix} 5 & 7 \\ 11 & -4 \\ 3 & 0 \end{bmatrix}$

d. $4 \cdot \begin{bmatrix} 2 & 6 & -9 \\ 3 & 7 & 11 \end{bmatrix} - \begin{bmatrix} 8 & 15 & 2 \\ -5 & 6 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 4 \\ 1 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.2 To promote the company's products, Family Snack's business manager plans to visit six U.S. cities. In the matrix below, each element represents the distance in kilometers between two of these cities.

	Miami	New York	Chicago	Kansas City	Seattle	Los Angeles
Miami	0	1747	1901	1986	4374	3742
New York	1747	0	1141	1755	3853	3922
Chicago	1901	1141	0	662	2779	2792
Kansas City	1986	1755	662	0	2410	2170
Seattle	4374	3853	2779	2410	0	1534
Los Angeles	3742	3922	2792	2170	1534	0

- a. What does the element in row 4, column 5 represent?
- b. The element 2779 appears twice in this matrix. Describe both of its positions and identify what each represents.
- c. What do the zeros indicate in this matrix?

2.3 Each of Family Snack’s four production employees is capable of performing the following daily tasks:

- Task 1: Making and packaging jam.
- Task 2: Roasting and packaging nuts.
- Task 3: Preparing and packaging jerky.
- Task 4: Filling and delivering orders.

Since their skill levels differ from task to task, the time necessary for each employee to complete each task varies as described below.

Employee 1 can do task 1 in 8 hr, task 2 in 7 hr, task 3 in 4 hr, and task 4 in 10 hr.

Employee 2 can do task 1 in 10 hr, task 2 in 4 hr, task 3 in 5 hr, and task 4 in 8.5 hr.

Employee 3 can do task 1 in 6 hr, task 2 in 3.5 hr, task 3 in 4 hr, and task 4 in 9 hr.

Employee 4 can do task 1 in 8 hr, task 2 in 6.5 hr, task 3 in 8 hr, and task 4 in 6 hr.

- a. Create a matrix to organize this data. Designate each employee by a row heading.
- b. Assign one task to each employee so that all the tasks are completed in the least amount of time. (Assume that all tasks can be performed simultaneously.)
- c. Is there more than one possible response to Part **b**? If so, describe another solution.

2.4 A Family Corporation sportswear outlet, Family Funwear, sells sweat pants in three colors—blue, red, and yellow—and four sizes: S, M, L, and XL. The number of each item in their current inventory is shown in matrix **I** below.

$$\mathbf{I} = \begin{array}{l} \text{blue} \\ \text{red} \\ \text{yellow} \end{array} \begin{array}{cccc} \text{S} & \text{M} & \text{L} & \text{XL} \\ \left[\begin{array}{cccc} 11 & 25 & 24 & 8 \\ 12 & 27 & 22 & 10 \\ 9 & 15 & 18 & 12 \end{array} \right] \end{array}$$

- a. What are the dimensions of matrix **I**?
- b. What does the element in row 2, column 3 represent?
- c. After receiving a new shipment of sweat pants, the store’s stock of each item increases by 20. Create a matrix to represent this new inventory.
- d. In preparation for an upcoming sale, the manager triples the store’s original inventory. Create a matrix to represent this new inventory.
- e. Create a matrix that, when added to matrix **I**, results in the matrix for Part **c**.

* * * * *

2.5 Family Bicycles, another division of the Family Corporation, builds and sells three types of bicycles: mountain bikes, touring bikes, and cross-country bikes. Each type is sold in three different frame sizes: 43 cm, 48 cm, and 54 cm.

In the 43-cm frame size, Family Bicycles has 112 mountain bikes, 117 touring bikes, and 111 cross-country bikes in stock. In the 48-cm frame size, they have 190 mountain bikes, 122 touring bikes, and 92 cross-country bikes. In the 54-cm frame size, they have 101 mountain bikes, 216 touring bikes, and 132 cross-country bikes.

- a. Create a matrix to represent the company's entire stock of bicycles. Designate each frame size by a row heading.
- b. The company's marketing department predicts that sales of each style and size of bicycle will increase by 20% next month.
 1. Describe how to create a matrix that displays the number of each bicycle that must be in stock to meet next month's sales.
 2. If the current inventory represents just enough stock to supply this month's orders, how many of each bicycle should be stocked next month? Explain your response.

2.6 One of the stores that sells Family Bicycles handles only mountain bikes and touring bikes. The inventory matrices \mathbf{I}_1 and \mathbf{I}_{31} show the number of bicycles in stock at the beginning of sales on December 1 and at the close of sales on December 31, respectively:

$$\mathbf{I}_1 = \begin{array}{l} \text{mountain} \\ \text{touring} \end{array} \begin{array}{ccc} 43 & 48 & 54 \\ \left[\begin{array}{ccc} 5 & 17 & 13 \\ 11 & 9 & 18 \end{array} \right] \end{array} \quad \mathbf{I}_{31} = \begin{array}{l} \text{mountain} \\ \text{touring} \end{array} \begin{array}{ccc} 43 & 48 & 54 \\ \left[\begin{array}{ccc} 3 & 11 & 5 \\ 6 & 0 & 10 \end{array} \right] \end{array}$$

Assuming the shop did not receive any new bicycles during December, complete Parts **a–g** below.

- a. How many mountain bikes did the store have in stock on December 1?
- b. How many 48-cm bikes were in stock at the start of the month?
- c. What was the total number of bicycles in stock on December 1?
- d. Write a 2×3 matrix that represents the number of bicycles sold during December. Explain how you determined the elements in this matrix.
- e. Which individual model and size sold best during the month?
- f. How many 54-cm mountain bikes were sold during December?
- g. At the end of the month, the store manager orders more of each model and size for which no more than one-third of the initial inventory remains. Which bicycle(s) should be reordered?

- 2.7** Family Bicycle offers four different frame colors: white, red, blue, and green. Its current inventory of mountain, touring, and cross-country bicycles is represented in matrix **C** below.

$$\begin{array}{r}
 \text{MT} \quad \text{TR} \quad \text{CC} \\
 \text{white} \left[\begin{array}{ccc} 100 & 100 & 67 \end{array} \right] \\
 \text{red} \left[\begin{array}{ccc} 121 & 135 & 89 \end{array} \right] \\
 \text{C} = \text{blue} \left[\begin{array}{ccc} 82 & 125 & 34 \end{array} \right] \\
 \text{green} \left[\begin{array}{ccc} 100 & 95 & 145 \end{array} \right]
 \end{array}$$

Next week, the company will add the bicycles represented in matrix **N** to its inventory.

$$\begin{array}{r}
 \text{MT} \quad \text{TR} \quad \text{CC} \\
 \text{white} \left[\begin{array}{ccc} 6 & 12 & 6 \end{array} \right] \\
 \text{red} \left[\begin{array}{ccc} 12 & 0 & 4 \end{array} \right] \\
 \text{N} = \text{blue} \left[\begin{array}{ccc} 0 & 12 & 6 \end{array} \right] \\
 \text{green} \left[\begin{array}{ccc} 16 & 6 & 4 \end{array} \right]
 \end{array}$$

- a. **1.** Describe how matrix addition can be used to determine a matrix that represents the company's total inventory after next week.
 - 2.** Write the matrix that represents the company's total inventory after next week.
- b.** The matrix **S** represents the inventory the company must have in stock before the summer sales begin. How many more bicycles should the company produce to reach this goal?

$$\begin{array}{r}
 \text{MT} \quad \text{TR} \quad \text{CC} \\
 \text{white} \left[\begin{array}{ccc} 116 & 132 & 87 \end{array} \right] \\
 \text{red} \left[\begin{array}{ccc} 145 & 147 & 103 \end{array} \right] \\
 \text{S} = \text{blue} \left[\begin{array}{ccc} 94 & 152 & 54 \end{array} \right] \\
 \text{green} \left[\begin{array}{ccc} 130 & 117 & 158 \end{array} \right]
 \end{array}$$

2.8 Family Funwear plans to add four different styles of knitted sweaters to its clothing line. To accomplish this goal, the company has purchased four knitting machines and hired four new workers. During some initial trials, the company gathered the following production data:

In one day, worker 1 can produce 3 of style A, 6 of style B, 7 of style C, or 4 of style D.

Worker 2 can produce 4 of style A, 5 of style B, 5 of style C, or 6 of style D.

Worker 3 can produce 6 of style A, 3 of style B, 4 of style C, or 4 of style D.

Worker 4 can produce 5 of style A, 5 of style B, 3 of style C, or 6 of style D.

- Create a matrix to organize this data. Designate each worker by a row heading.
- In order to produce the maximum number of sweaters each day, which worker should knit each style? Justify your response.
- What is the maximum number of sweaters that the four workers can produce in one day?
- Is there more than one possible response to Part **b**? If so, describe another solution.

* * * * *

Activity 3

All businesses record information on profit and loss. One way to organize and analyze this information is with matrices. In this activity, you explore how Family Snack uses matrices to keep track of its profits.

Exploration 1

Family Snack’s employees earn a commission on every sale. As part of the company’s bookkeeping procedures, the business manager monitors both the weekly sales and the weekly commissions of each salesperson. The matrix **W** below shows one week’s sales of cases of Snack Packs, Gift Packs, and Family Packs by Keyes and Zhang.

$$\begin{array}{rcc}
 & \text{S} & \text{G} & \text{F} \\
 \mathbf{W} = & \text{K} & \begin{bmatrix} 4 & 7 & 2 \end{bmatrix} \\
 & \text{Z} & \begin{bmatrix} 2 & 6 & 3 \end{bmatrix}
 \end{array}$$

The company sells Snack Packs for \$45.00 per case, Gift Packs for \$70.00 per case, and Family Packs for \$95.00 per case. This information can be represented in a **column matrix**, as shown below. Each element in the matrix \mathbf{M}_p represents the price per case of a particular item.

$$\mathbf{M}_p = \begin{matrix} \text{Price} \\ \text{S} [45.00] \\ \text{G} [70.00] \\ \text{F} [95.00] \end{matrix}$$

- a.
1. Determine the value of the sales made by each employee in matrix \mathbf{W} .
 2. Describe how you determined the values in Step 1.

Mathematics Note

When a matrix \mathbf{P} with dimensions $k \times m$ is multiplied by a matrix \mathbf{Q} with dimensions $m \times n$, the dimensions of the **product matrix** $\mathbf{P} \cdot \mathbf{Q}$ are $k \times n$. The number of columns in the first matrix must equal the number of rows in the second matrix.

To obtain the product matrix, row 1 of the first matrix is “multiplied” with column 1 of the second matrix, as shown in Figure 4. This process is repeated until each row of the first matrix has been multiplied with each column of the second matrix.

$$\begin{matrix} \left[\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \square & \square & \square \\ \hline \end{array} \right] \cdot \begin{matrix} \left[\begin{array}{|c|c|} \hline d & \square \\ \hline e & \square \\ \hline f & \square \\ \hline \end{array} \right] \\ \text{column 1} \end{matrix} = \begin{matrix} \left[\begin{array}{|c|c|} \hline ad + be + cf & \square \\ \hline \square & \square \\ \hline \end{array} \right] \\ \text{product matrix} \end{matrix} \end{matrix}$$

Figure 4: Multiplying row 1 by column 1

When row 1 of the first matrix is multiplied with column 1 of the second matrix, the sum of the products is placed in row 1, column 1 of the product matrix. In general, when row i of the first matrix is multiplied with column j of a second matrix, the sum of the products is placed in row i , column j of the product matrix.

For example, the equation below shows a 2×3 matrix \mathbf{P} multiplied with a 3×2 matrix \mathbf{Q} . The dimensions of the product matrix $\mathbf{P} \cdot \mathbf{Q}$ are 2×2 .

$$\begin{bmatrix} 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 7 \\ 5 & 8 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 6 \cdot 5 + 9 \cdot 9 & 2 \cdot 7 + 6 \cdot 8 + 9 \cdot 0 \\ 1 \cdot 2 + 3 \cdot 5 + 4 \cdot 9 & 1 \cdot 7 + 3 \cdot 8 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 115 & 62 \\ 53 & 31 \end{bmatrix}$$

- b.** The commission earned for selling a case of Snack Packs is \$2.25, for a case of Gift Packs is \$4.20, and for a case of Family Packs \$7.20.
1. Write a column matrix \mathbf{M}_C to represent the commission earned by selling a case of each product.
 2. Use matrix multiplication to determine the commission earned by each salesperson for the sales shown in matrix \mathbf{W} .
- c.** In Parts **a** and **b**, you used two different column matrices to represent Family Snack's selling prices and commissions. As shown below, these two matrices could also be combined into a single 3×2 matrix.

$$\mathbf{M} = \begin{array}{c} \text{Price} \quad \text{Com.} \\ \text{S} \begin{bmatrix} 45.00 & 2.25 \end{bmatrix} \\ \text{G} \begin{bmatrix} 70.00 & 4.20 \end{bmatrix} \\ \text{F} \begin{bmatrix} 95.00 & 7.20 \end{bmatrix} \end{array}$$

1. Use the process described in the mathematics note to find the product $\mathbf{W} \cdot \mathbf{M}$.
 2. Record the dimensions of the product matrix.
 3. Describe the information contained in the product matrix.
- d.**
1. Use the process described in the mathematics note to find the product $\mathbf{M} \cdot \mathbf{W}$.
 2. Record the dimensions of the product matrix.
 3. Describe the information contained in the product matrix.

Discussion 1

- a. Compare the methods you used to determine the values in Parts **a** and **b** of the exploration. Which appears to be the better method and why?
- b. If the product matrix $\mathbf{A} \cdot \mathbf{B}$ exists, what must be true about the number of columns in matrix \mathbf{A} and the number of rows in matrix \mathbf{B} ?
- c. What advantages are there to performing the operation $\mathbf{W} \cdot \mathbf{M}$ instead of the operations $\mathbf{W} \cdot \mathbf{M}_P$ and $\mathbf{W} \cdot \mathbf{M}_C$?
- d.
 1. Was the information displayed in the product matrix $\mathbf{W} \cdot \mathbf{M}$ meaningful? Explain your response.
 2. Was the information displayed in the product matrix $\mathbf{M} \cdot \mathbf{W}$ meaningful? Explain your response.
- e. Which employee earned more in commissions for the sales shown in matrix \mathbf{W} ? Justify your response.

Exploration 2

In Exploration 1, you used matrix multiplication to analyze the sales of two of Family Snack's employees. In this exploration, you use technology to help the company organize and manipulate some additional sales information.

- a. Use technology to find the product $\mathbf{W} \cdot \mathbf{M}$. Compare this product with your results in Part c of Exploration 1.
- b. Besides Keyes and Zhang, Family Snack employs another salesperson, Troy, as well as a new trainee, Lia. In the week represented in matrix \mathbf{W} , Troy sold 3 cases of Snack Packs, 9 cases of Gift Packs, and 1 case of Family Packs. Lia sold 6 cases of Snack Packs, 0 Gift Packs, and 0 Family Packs.

Combine this information with the information in matrix \mathbf{W} to create a new matrix \mathbf{W}_2 .

- c. Determine the product matrix $\mathbf{W}_2 \cdot \mathbf{M}$ and record its dimensions. Describe the information contained in this matrix.
- d. Evaluate the product matrix $\mathbf{M} \cdot \mathbf{W}_2$ and interpret the results.
- e. Matrix \mathbf{V} below shows the cases of nuts, jerky, and jam sold by each salesperson during the week represented in matrix \mathbf{W}_2 . Combine this information with that in matrix \mathbf{W}_2 to create a new matrix \mathbf{W}_3 .

$$\mathbf{V} = \begin{array}{c} \text{N} \quad \text{J} \quad \text{M} \\ \text{K} \left[\begin{array}{ccc} 2 & 3 & 0 \end{array} \right] \\ \text{Z} \left[\begin{array}{ccc} 3 & 1 & 6 \end{array} \right] \\ \text{T} \left[\begin{array}{ccc} 3 & 0 & 3 \end{array} \right] \\ \text{L} \left[\begin{array}{ccc} 2 & 4 & 1 \end{array} \right] \end{array}$$

- f. Family Snack sells nuts for \$6.00 per case, jerky for \$10.00 per case, and jam for \$8.00 per case.

The commission earned for selling a case of nuts is \$0.18, for a case of jerky is \$0.30, and for a case of jam is \$0.32.

Combine this information with that in matrix \mathbf{M} to create a new matrix \mathbf{S} .

- g. Evaluate the product matrix $\mathbf{W}_3 \cdot \mathbf{S}$ and interpret the results.
- h. Evaluate the product matrix $\mathbf{S} \cdot \mathbf{W}_3$ and interpret the results.

Discussion 2

- a. Describe the process of multiplying an $m \times n$ by a $p \times q$ matrix.
- b. Why is the matrix multiplication $\mathbf{S} \cdot \mathbf{W}_3$ not possible?
- c. When using real numbers, multiplication is commutative. For example, $6 \cdot 14 = 14 \cdot 6$. Matrix multiplication, however, is not commutative. Explain why this is the case.

Assignment

- 3.1 Consider the five matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 2 & 9 & 1 \\ 0 & 6 & 1 \\ 1 & 2 & 6 \\ 0 & 5 & 1 \\ 8 & 3 & 1 \end{bmatrix}$$

If possible, find each of the following product matrices. When matrix multiplication is possible, show how the elements in the product matrix are formed. If multiplication is not possible, explain why not.

- a. $\mathbf{A} \cdot \mathbf{B}$
 - b. $\mathbf{C} \cdot \mathbf{D}$
 - c. $\mathbf{E} \cdot \mathbf{B}$
- 3.2 Consider the following two matrices.

$$\mathbf{X} = \begin{bmatrix} 7 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 2 \end{bmatrix}$$

- a.
 1. Find the product matrix $\mathbf{X} \cdot \mathbf{Y}$ and describe its dimensions.
 2. How are the dimensions of the product matrix related to the dimensions of \mathbf{X} and \mathbf{Y} ?
- b.
 1. Find the product matrix $\mathbf{Y} \cdot \mathbf{X}$ and describe its dimensions.
 2. How are the dimensions of the product matrix related to the dimensions of \mathbf{X} and \mathbf{Y} ?
- c. Describe how your responses to Parts **a** and **b** show that matrix multiplication is not commutative.

- 3.3** A store has placed an order with Family Snack represented by the matrix **O** shown below. Each element in the matrix indicates the number of a specific item in the order.

$$\mathbf{O} = \begin{matrix} & \text{N} & \text{J} & \text{M} & \text{S} & \text{G} & \text{F} \\ \text{N} & 20 & & & & & \\ \text{J} & & 60 & & & & \\ \text{M} & & & 48 & & & \\ \text{S} & & & & 24 & & \\ \text{G} & & & & & 12 & \\ \text{F} & & & & & & 2 \end{matrix}$$

- a. The total requirement matrix for Family Snack's product line is matrix **R** shown below. Use matrix **R** to determine the number of each simple component needed to fill the order.

$$\mathbf{R} = \begin{matrix} & \text{N} & \text{J} & \text{M} & \text{S} & \text{G} & \text{F} \\ \text{N} & 1 & 0 & 0 & 2 & 4 & 12 \\ \text{J} & 0 & 1 & 0 & 6 & 12 & 36 \\ \text{M} & 0 & 0 & 1 & 0 & 3 & 12 \\ \text{S} & 0 & 0 & 0 & 1 & 2 & 6 \\ \text{G} & 0 & 0 & 0 & 0 & 1 & 3 \\ \text{F} & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

- b. Another customer has ordered 150 boxes of nuts, 600 pieces of jerky, 48 jars of jam, 60 Snack Packs, and 12 Gift Packs. Determine the number of each simple component needed to fill this order.

- 3.4** Last year, Family Snack offered an incentive program to its sales staff. Employees received 10 incentive points for each month in which they ranked first in sales, 6 points for each month in which they ranked second, 4 points for third, and 3 for fourth.

- a. Create a matrix **S** that displays the incentive points awarded for each ranking.
- b. Employee 1 ranked first during 4 months, third during 4 months, and fourth during 4 months. Employee 2 ranked first during 3 months, second during 4 months, and fourth during 5 months. Employee 3 ranked first 2 months, second 5 months, and third 5 months. Employee 4 ranked first, second, third, and fourth for 3 months each.

Create a matrix **R** that displays the number of months at each rank for each of the four employees.

- c. The employee with the most incentive points receives an all-expenses-paid vacation. Determine the order in which the four employees finished in the contest, along with their scores.

* * * * *

- 3.5** Matrix **R** is the total requirement matrix for Family Snack's product line. Family Snack has received orders $O_1, O_2, O_3, O_4,$ and O_5 from five different customers, as shown in matrix **O** below.

$$\mathbf{R} = \begin{array}{c} \begin{array}{c} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{N} \left[\begin{array}{c|c} 1 & 0 \quad 0 \quad 2 \quad 4 \quad 12 \\ \hline \text{J} \left[\begin{array}{c|c} 0 & 1 \quad 0 \quad 6 \quad 12 \quad 36 \\ \hline \text{M} \left[\begin{array}{c|c} 0 & 0 \quad 1 \quad 0 \quad 3 \quad 12 \\ \hline \text{S} \left[\begin{array}{c|c} 0 & 0 \quad 0 \quad 1 \quad 2 \quad 6 \\ \hline \text{G} \left[\begin{array}{c|c} 0 & 0 \quad 0 \quad 0 \quad 1 \quad 3 \\ \hline \text{F} \left[\begin{array}{c|c} 0 & 0 \quad 0 \quad 0 \quad 0 \quad 1 \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \end{array} \end{array} \quad \mathbf{O} = \begin{array}{c} \begin{array}{c} \text{O}_1 \quad \text{O}_2 \quad \text{O}_3 \quad \text{O}_4 \quad \text{O}_5 \\ \text{N} \left[\begin{array}{c|c} 12 & 20 \quad 60 \quad 20 \quad 8 \\ \hline \text{J} \left[\begin{array}{c|c} 10 & 18 \quad 45 \quad 12 \quad 2 \\ \hline \text{M} \left[\begin{array}{c|c} 16 & 12 \quad 14 \quad 22 \quad 8 \\ \hline \text{S} \left[\begin{array}{c|c} 8 & 10 \quad 18 \quad 8 \quad 8 \\ \hline \text{G} \left[\begin{array}{c|c} 4 & 4 \quad 8 \quad 4 \quad 2 \\ \hline \text{F} \left[\begin{array}{c|c} 2 & 4 \quad 12 \quad 8 \quad 0 \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \end{array} \end{array}$$

- How many boxes of nuts are needed to fill O_3 ?
 - How many boxes of nuts are needed to fill all five orders?
 - How many pieces of jerky are needed to fill O_2 ?
 - How many pieces of jerky are needed to fill all five orders?
 - How many jars of jam are needed to fill all five orders?
- 3.6** A bicycle shop use matrices to track its monthly sales. The number of bikes sold in July are displayed in matrix **J** below.

$$\mathbf{J} = \begin{array}{c} \begin{array}{c} 43 \text{ cm} \quad 48 \text{ cm} \quad 54 \text{ cm} \\ \text{mountain} \left[\begin{array}{c|c} 2 & 6 \quad 5 \\ \hline \text{touring} \left[\begin{array}{c|c} 5 & 9 \quad 8 \end{array} \right. \end{array} \right. \end{array} \end{array}$$

Matrix **S** shows the selling price in dollars for each model.

$$\mathbf{P} = \begin{array}{c} \begin{array}{c} \text{MT} \quad \text{TR} \\ 43 \text{ cm} \left[\begin{array}{c|c} 250 & 280 \\ \hline 48 \text{ cm} \left[\begin{array}{c|c} 270 & 300 \\ \hline 54 \text{ cm} \left[\begin{array}{c|c} 290 & 320 \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \end{array}$$

- What matrix operation would allow you to calculate the value of the July sales for each model? Perform this operation. Use the column and row headings for matrices **P** and **J** to determine appropriate headings for the product matrix.
- Describe what each element in the product matrix represents. If some elements have no clear interpretation, identify them and explain why this occurs.
- Is it possible to perform the matrix operation in Part **a** if the order of the matrices is reversed? If so, describe the elements that have clear interpretations. If the operation is not possible, explain why not.
- Are the total July sales for the bicycle shop more or less than \$15,000? Explain your response.

3.7 Consider the five matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 7 & 8 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 0 & 5 & -2 \\ 1 & 8 & -4 & 9 \\ 11 & 6 & -5 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 5 & -6 \\ 9 & -4 & 0 \\ 9 & 3 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 7 & -6 & 7 \\ 2 & -1 & 1 & 0 \\ 5 & -8 & 3 & -11 \\ 2 & 9 & 1 & 7 \end{bmatrix}$$

If possible, find each of the following product matrices using a matrix manipulator. When matrix multiplication is possible, identify the dimensions of the product matrix. If multiplication is not possible, explain why not.

- $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$
- $\mathbf{A} \cdot \mathbf{C}$ and $\mathbf{C} \cdot \mathbf{A}$
- $\mathbf{B} \cdot \mathbf{C}$ and $\mathbf{C} \cdot \mathbf{B}$
- $\mathbf{C} \cdot \mathbf{D}$ and $\mathbf{D} \cdot \mathbf{C}$
- $\mathbf{B} \cdot \mathbf{E}$ and $\mathbf{E} \cdot \mathbf{B}$

Research Project

In this module, you have explored matrix addition, subtraction, and multiplication. Use the general 2×2 matrices below to further investigate matrix arithmetic.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

- Is multiplication of 2×2 matrices associative? In other words, does $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$?
 - Is the distributive property of multiplication over addition valid for 2×2 matrices? In other words, does $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$?
 - The identity element for real-number multiplication is 1. For example, $25 \cdot 1 = 25$. Is there an identity for multiplication of 2×2 matrices?
 - For all real numbers, multiplication by 0 results in a product of 0 (the Zero Product Property). Is there a matrix which, when multiplied by \mathbf{A} , \mathbf{B} , or \mathbf{C} , results in a matrix whose elements are all zeros? If so, is there more than one such matrix?
-

Summary Assessment

- 1.** Family Snack has decided to add crackers to its product line. The new Snack Pack contains 1 box of nuts, 1 box of crackers, 10 pieces of jerky, and 1 jar of jam; the new Gift Pack contains 2 Snack Packs and 2 jars of jam; and the new Family Pack contains 3 Gift Packs and 4 boxes of crackers.
 - a.** Construct a requirement graph for Family Snack's new product line.
 - b.** Use your requirement graph to construct a total requirement matrix for the new product line.

- 2.** An apartment complex offers one-, two-, and three-bedroom apartments. The complex contains 10 furnished and 10 unfurnished one-bedroom apartments; 5 furnished and 10 unfurnished two-bedroom apartments; and 5 furnished and 5 unfurnished three-bedroom apartments.

A furnished one-bedroom apartment rents for \$375 a month; an unfurnished one-bedroom rents for \$345. A furnished two-bedroom rents for \$400 per month; an unfurnished one rents for \$370. A furnished three-bedroom rents for \$450 per month; an unfurnished one rents for \$420.

 - a.** Create a 2×3 matrix that displays the apartment inventory.
 - b.** Create a 3×2 matrix that displays the rent for each type of apartment.
 - c.** Find the product of the rent matrix from Part **b** and the inventory matrix from Part **a**.
 - d.** Describe the meaning of each element in the matrix from Part **c**. If some elements in the matrix have no clear interpretation, explain why this occurs.
 - e.** The managers of the apartment complex wish to raise the rents by 10%. Construct a matrix showing the new rents.
 - f.** The monthly electric bills average \$80 for a one-bedroom apartment, \$95 for a two-bedroom apartment, and \$105 for a three-bedroom apartment. Show how matrix multiplication can be used to determine the total electric bill for the entire apartment complex.

- g.** In addition to the electric bills from Part **f**, the apartment managers also must pay bills for other utilities, such as water and sewer. Monthly water bills average \$18 for one-bedroom apartments, \$26 for two-bedroom apartments, and \$42 for three-bedroom apartments. Monthly sewer bills average \$15 for one-bedroom apartments, \$20 for two-bedroom apartments, and \$30 for three-bedroom apartments.

Show how matrix multiplication can be used to find the monthly utility bills for the entire apartment complex. The utility bill should be broken into water, sewer, and electricity.

Module Summary

- One way to represent a company's product line is with a **requirement graph**. A requirement graph is a tree diagram that shows each product's components at each level of its assembly. A complete requirement graph has simple components at the end of each branch.
- A **matrix** (plural **matrices** or **matrixes**) is a rectangular arrangement of rows and columns used to organize information. A matrix of i rows and j columns has **dimensions** $i \times j$ (read " i by j "). Matrices are named using bold, uppercase letters or descriptive words. Each item in a matrix is an **element**.
- A company can use a requirement graph to create a **total requirement matrix**. In a total requirement matrix, each item sold is represented in a column heading, while each simple component or composite product is represented in a row heading. Each element indicates the number of that simple component or composite product required to produce the corresponding item sold.
- Two matrices are **equal** if they have the same dimensions and if corresponding elements are equal.
- **Matrix addition** can be performed on two matrices with the same dimensions by adding the corresponding elements of each matrix. **Matrix subtraction** can be performed in a similar manner.
- **Scalar multiplication** is the multiplication of each element of a matrix by a constant, or **scalar**. The multiplication of a matrix **M** by a scalar k is denoted by $k \cdot \mathbf{M}$.
- When a matrix with dimensions $k \times m$ is multiplied with a matrix with dimensions $m \times q$, the dimensions of the **product matrix** are $k \times q$. To obtain the product matrix, each row of the first matrix is "multiplied" with each column of the second matrix, as shown below. In general, when row i of the first matrix is multiplied with column j of a second matrix, the sum of the products is placed in row i , column j of the product matrix.

$$\begin{array}{c}
 \left[\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \square & \square & \square \\ \hline \end{array} \right] \cdot \left[\begin{array}{|c|c|} \hline d & \square \\ \hline e & \square \\ \hline f & \square \\ \hline \end{array} \right] = \left[\begin{array}{|c|c|} \hline ad + be + cf & \square \\ \hline \square & \square \\ \hline \end{array} \right] \\
 \text{row 1} \quad \bullet \quad \text{column 1} \qquad \qquad \qquad \text{product matrix}
 \end{array}$$

Selected References

DeLange, J. *Matrices*. Utrecht, The Netherlands: Research Group on Mathematics Education, 1990.

Harshbarger, R. J., and J. J. Reynolds. *Mathematical Applications for Management, Life, and Social Sciences*. Lexington, MA: D.C. Heath, 1992.

Vazsonyi, A. *Finite Mathematics, Quantitative Analysis for Management*. Santa Barbara, CA: Wiley/Hamilton, 1977.

A New Angle on an Old Pyramid



What mathematical skills did the ancient Egyptians use to build the pyramids? In this module, you explore some of the theories regarding the construction of these architectural wonders—and see how modern mathematical knowledge could have made part of the job much simpler.

Clay Burkett • Randy Carspecken



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

A New Angle on an Old Pyramid

Introduction

Tutankhamen and Khufu were kings of ancient Egypt. Khufu ruled around 2500 B.C., while Tutankhamen (King Tut) governed about 1200 years later. Although King Tut's artifacts are among the most popular items displayed in museums today, it was King Khufu who built the Great Pyramid at Giza, one of the "seven wonders" of the ancient world.

Still standing after 4500 years, the Great Pyramid is an amazing architectural monument. Designed as Khufu's burial tomb, the structure is a regular square pyramid which originally stood 147 m high. Each side of its square base measured 230 m, as shown in Figure 1.

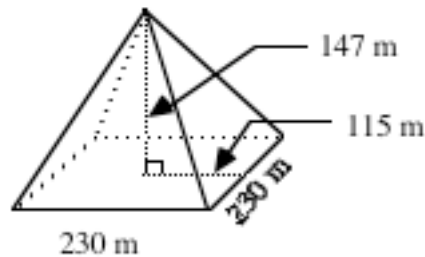


Figure 1: Dimensions of the Great Pyramid

The Great Pyramid consists of about 2.3 million stone blocks, each with a mass of approximately 1000 kg. As many as 100,000 laborers may have worked for 20 summers to complete this engineering marvel. Two basic kinds of stones were used: rectangular blocks and casing blocks. The casing blocks gave the faces of the pyramid a smooth, continuous slope.

The base of the Great Pyramid is nearly a perfect square. To build a structure of this size to such exacting specifications, the Egyptian engineers and surveyors must have applied some mathematics. Due to the lack of written records, however, historians are unsure about what mathematics the builders actually knew. In this module, you investigate some mathematical ideas that historians believe may have been used to build the pyramid.

Activity 1

Before beginning construction, modern engineers use precise measuring tools to stake out the corners of a building's foundation. How did the Egyptians locate the corners of the Great Pyramid without the benefit of 20th-century surveying tools? In this activity, you review some of the difficulties involved in performing this task.

Exploration

- a. On a large sheet of paper, construct one corner of a square by following Steps 1–3 below. When marking points and line segments on the paper, make them dark enough to show through another sheet of paper.
1. Mark a point C near the lower left-hand corner of the paper, 5 cm from each edge.
 2. Draw a line segment from point C to the lower right-hand corner of the paper. As shown in Figure 2, place point D on the segment 30 cm from point C .

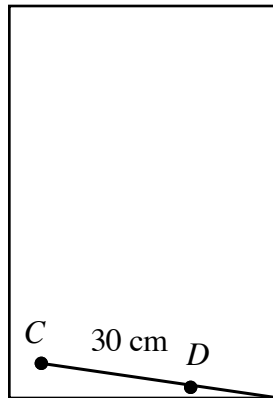


Figure 2: Segment from point C to point D

3. Use a protractor and a meterstick to draw a line segment that passes through point C , is perpendicular to \overline{CD} , and has length of 1 m. Label the point at the end of the segment as P .
- b. Compare the corner you created with those of others in the class by completing the following steps.
1. Select one sheet of paper from Part a on which to record the class results.
 2. Place the selected sheet on top of another sheet and align the two drawings of \overline{CD} .

3. Mark the location of point P on the lower sheet on the top sheet.
 4. Repeat Steps 2 and 3 until the information for the entire class has been collected on the selected sheet.
- c.
1. On the common sheet of paper, identify the two locations of P that are farthest apart. Measure the distance between these points to the nearest millimeter.
 2. Label these two points X and Y .
 3. Express the length of \overline{XY} in centimeters.
 4. Draw $\triangle CXY$. The common sheet should now resemble the diagram in Figure 3.

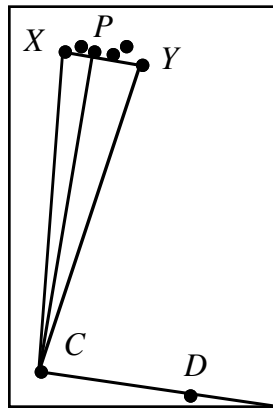


Figure 3: $\triangle CXY$ on the common sheet of paper

- d. Imagine R is a point on \overrightarrow{CX} 100 m from C and S is a point on \overrightarrow{CY} 100 m from C . Predict the distance between R and S .

Discussion

- a. Why was there a difference between the corner you drew and those drawn by others in your class?
- b. What type of triangle is $\triangle CXY$? What are its special features?
- c. Two figures are **similar** if they have the same shape and corresponding sides are proportional. The symbol used to indicate similarity is \sim . If $\triangle ABC$ is similar to $\triangle DEF$, explain why the notation $\triangle ABC \sim \triangle EDF$ is an incorrect way of expressing this similarity.

Mathematics Note

In plane geometry, if the angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar. This is referred to as the **Angle-Angle-Angle (AAA) Property**.

The symbol used to indicate congruence is \cong . For example, consider triangles ABC and DEF in Figure 4. As shown by the markings, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. Since their corresponding angles are congruent, $\triangle ABC \sim \triangle DEF$.

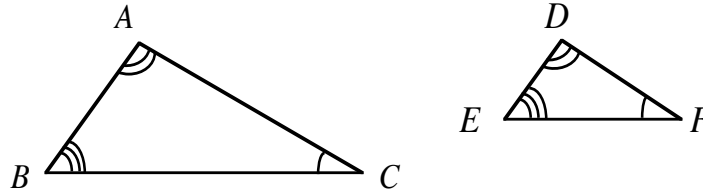


Figure 4: Two similar triangles

Since these triangles are similar, the measures of their corresponding sides are proportional, as shown below.

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

- d. In the exploration, which triangle is similar to $\triangle CXY$? Explain your response.
- e. How could you use the pair of similar triangles identified in Part d of the discussion to determine the distance between R and S ?

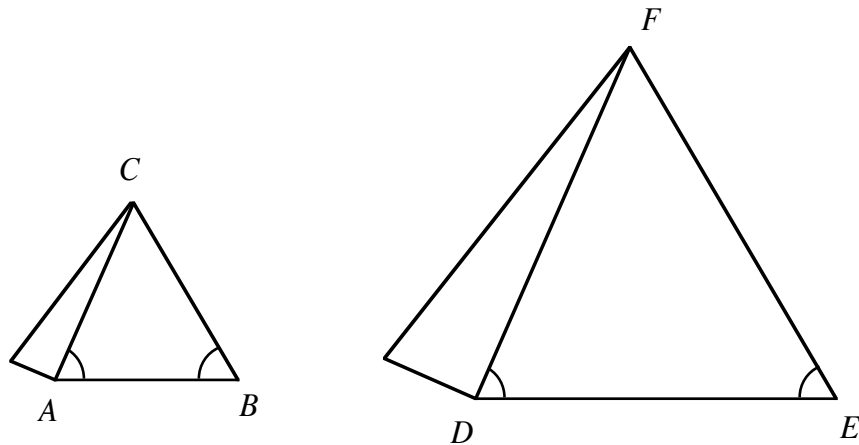
Assignment

- 1.1 Suppose that King Khufu ordered two different surveyors to stake the corners of the Great Pyramid. At 1 m from the first corner, the distance between the two surveyors' lines equaled the distance you measured in Part c of the exploration. How far apart will their two lines be after 230 m, the side length of the base of the Great Pyramid? Explain how you determined your response.

1.2 As mentioned in the introduction, the Great Pyramid is a regular square pyramid. This means that the base is square, the altitude passes through the center of the base, and the four lateral faces are congruent isosceles triangles. To investigate some of the properties of these lateral faces, complete Parts **a–e** below.

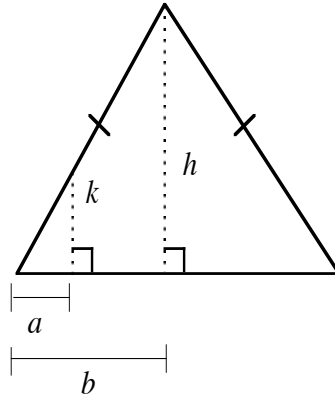
- Cut an isosceles triangle from the template supplied by your teacher. Fold the triangle along its line of symmetry.
- Use a protractor and a ruler to determine the relationship between the line of symmetry and the base of the triangle.
- What is the relationship between the triangle's altitude and its line of symmetry?
- When the isosceles triangle is folded along its line of symmetry, what type of triangle is formed?
- What is the relationship between the line of symmetry for the isosceles triangle and its vertex angle?

1.3 Two pyramids are similar when their bases are similar and their faces are similar. The diagram below shows two regular square pyramids. As shown by the corresponding markings, $\angle CAB$ and $\angle FDE$ are congruent. Are these pyramids similar? Explain your response.



1.4 What is the least amount of information necessary to determine if two triangles are similar? Explain your response.

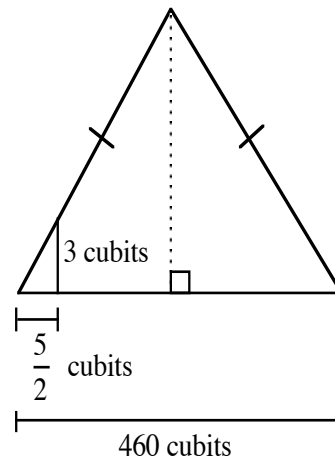
- 1.5 a. The diagram below shows a cross section of a regular square pyramid where h is the height of the pyramid. The ancient Egyptians were aware that the ratios h/k and b/a are equal. Explain why this relationship is true.



- b. Describe a simple rule for determining when two right triangles are similar.

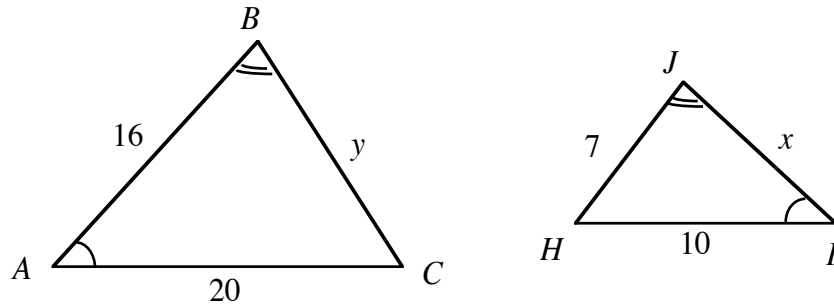
- 1.6 One source of information on ancient Egyptian mathematics is the Rhind Papyrus, a scroll about 5.5 m long and 33 cm wide. Written about 1650 B.C. by a scribe named Ahmes, the Rhind Papyrus is a practical handbook of mathematical problems.

As shown in the diagram below (not drawn to scale), one of these problems asks for the height of a square pyramid given certain measurements.

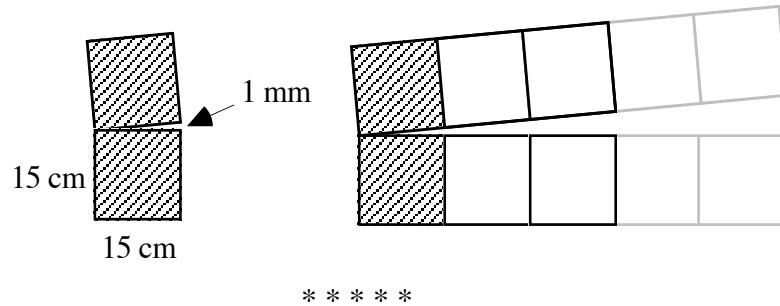


Find the height of the pyramid in cubits. **Note:** In ancient Egypt, the *cubit* was a unit of measurement based on the distance from the elbow to the tip of the fingers in an average person's forearm.

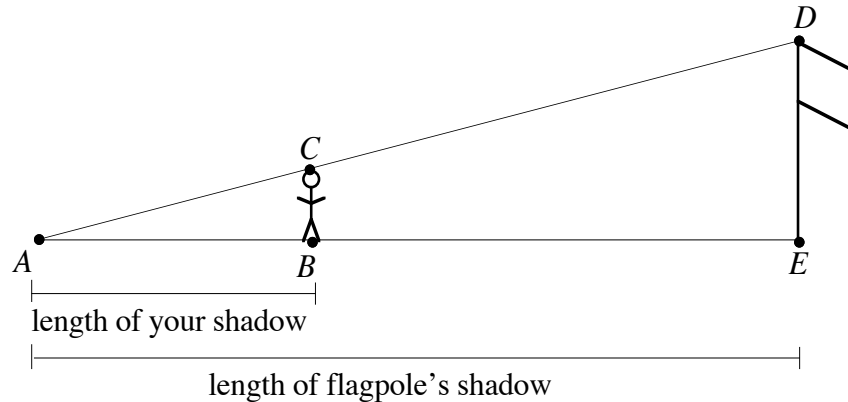
- 1.7 Use proportions to determine the values of x and y in the following triangles. (Angles which are marked in the same manner are congruent.)



- 1.8 When placing the giant stone blocks on some levels of the Great Pyramid, workers had to align the blocks edge to edge and corner to corner. The task was similar to installing a tile floor—except on a much larger scale. In the diagram below, two square tiles have been placed with one pair of corners touching. The other pair of corners, however, has a gap of 1 mm. If the next tiles are aligned exactly with these two (as shown in the diagram), then how wide will the gap be after 3 m of tile have been installed? Explain your answer using similar triangles.

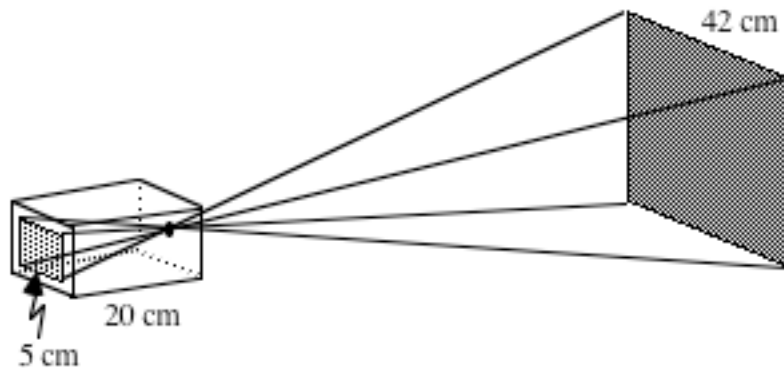


- 1.9** Your math class has been asked to determine the height of the school flagpole. Since you have no tools to actually measure the flagpole, the class devised a method which involves similar triangles. The diagram below shows a sketch of this situation.



- a. Explain why triangles ABC and AED are similar.
 - b. Describe how to use the properties of similar triangles to find the height of the flagpole.
- 1.10** A pinhole camera consists of a sealed box with a pinhole opening at one end and a piece of film at the opposite end. When light enters the box through the pinhole, an inverted image is projected on the film.

Consider a pinhole camera 20 cm long, as shown in the diagram below, containing a square piece of film, 5 cm on each side. The object to be photographed occupies a square 42 cm on each side.



If the photographer wants the image to exactly fill the film, how far away should the camera be placed? Explain your reasoning. (Hint: The lines in the diagram indicate the corresponding vertices of the object and the image. The two pyramids formed are similar.)

- 1.11** If two triangles are similar, are their corresponding altitudes proportional? Use diagrams to help support your response.

Activity 2

Since the ancient Egyptians did not measure angles in degrees, they must have used other methods than those in Activity 1 to make their corners “right.” One theory suggests that the Egyptians used a rope subdivided by knots into 12 congruent segments. To make a corner, the rope was formed into a triangle with sides of 3, 4, and 5 units. As shown in Figure 5, the special characteristics of this kind of triangle might have allowed the Egyptians to form a right angle. In this activity, you examine such triangles and develop a method for deciding when a corner is “right.”

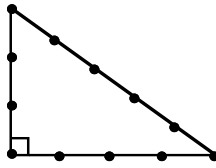


Figure 5: A 3-4-5 triangle

Mathematics Note

The **Pythagorean theorem** states that, in a right triangle, the square of the length of the longest side (the hypotenuse) equals the sum of the squares of the lengths of the other sides (the legs).

In Figure 6, for example, $\angle C$ is a right angle, \overline{AB} is the hypotenuse, and \overline{BC} and \overline{AC} are the legs. Since $\triangle ABC$ is a right triangle, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse.

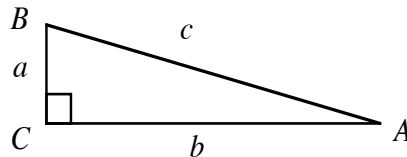


Figure 6: A right triangle

Notice that the vertices of the triangle in Figure 6 are labeled with capital letters, while the sides opposite these vertices are identified with the corresponding lowercase letters. This is a conventional way to label the parts of triangles.

For example, consider $\triangle RST$ in Figure 7. In this triangle, $s = 13$ m, $t = 5$ m, and $r = 12$ m. Because $\triangle RST$ is a right triangle, $s^2 = t^2 + r^2$ or $13^2 = 5^2 + 12^2$.

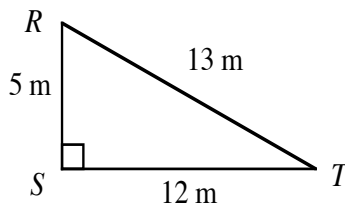


Figure 7: Right triangle RST

Exploration

Although the Pythagorean theorem is evident in the Egyptian rope triangle, it is the **converse** of this theorem that would have been more useful to the Egyptians.

Mathematics Note

The **converse** of a statement in the form “If A, then B” is the statement “If B, then A.” The converse of a true if-then statement may or may not be true.

For example, consider the following if-then statement:

“If the country is Egypt, then the Nile River passes through the country.”

The converse of this statement is:

“If the Nile River passes through a country, then the country is Egypt.”

Since the Nile flows through other countries besides Egypt, the first if-then statement in this example is true, while its converse is not.

- a.
 1. Write the converse of the Pythagorean theorem.
 2. Do you think that the converse is true?
- b. Use a geometry utility to test the truth of the converse of the Pythagorean theorem by completing Steps 1–5.
 1. Construct $\triangle ABC$ and two lines perpendicular to the base \overline{AB} , as shown in Figure 8.

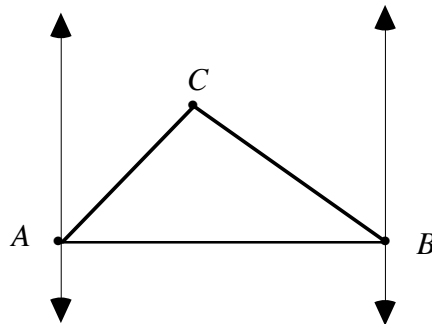


Figure 8: Construction of $\triangle ABC$

2. Create a table with headings like those in Table 1 below.

Table 1: Triangle data

$m\angle C$	AB	AC	BC	$(AB)^2 - [(AC)^2 + (BC)^2]$

3. Record the appropriate measurements for $\triangle ABC$ in Table 1.
4. Move point C to another location between the two lines (but without touching either line). Record the appropriate measurements in Table 1.
5. Repeat Step 4 for several other locations of C . Include some positions that make $m\angle C$ acute (less than 90°), some that make it a right angle (exactly 90°), and some that make it obtuse (greater than 90° but less than 180°).
- c.
1. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is positive, is $\angle C$ acute, right, or obtuse?
 2. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is negative, is $\angle C$ acute, right, or obtuse?
 3. When the value of $(AB)^2 - [(AC)^2 + (BC)^2]$ is 0, is $\angle C$ acute, right, or obtuse?

Discussion

- a. How is the expression $(AB)^2 - [(AC)^2 + (BC)^2]$ related to the equation for the Pythagorean theorem, $a^2 + b^2 = c^2$?
- b. In an obtuse triangle, two angles are acute and one is obtuse. The longest side is opposite the obtuse angle. Based on your data in Table 1, describe how the sides of an obtuse triangle are related.
- c. In an acute triangle, all three angles are acute. Based on your data in Table 1, describe how the sides of an acute triangle are related.
- d. Does the converse of the Pythagorean theorem appear to be true? Explain your response.
- e. How could you use the converse of the Pythagorean theorem to determine if a corner is “square”?

Assignment

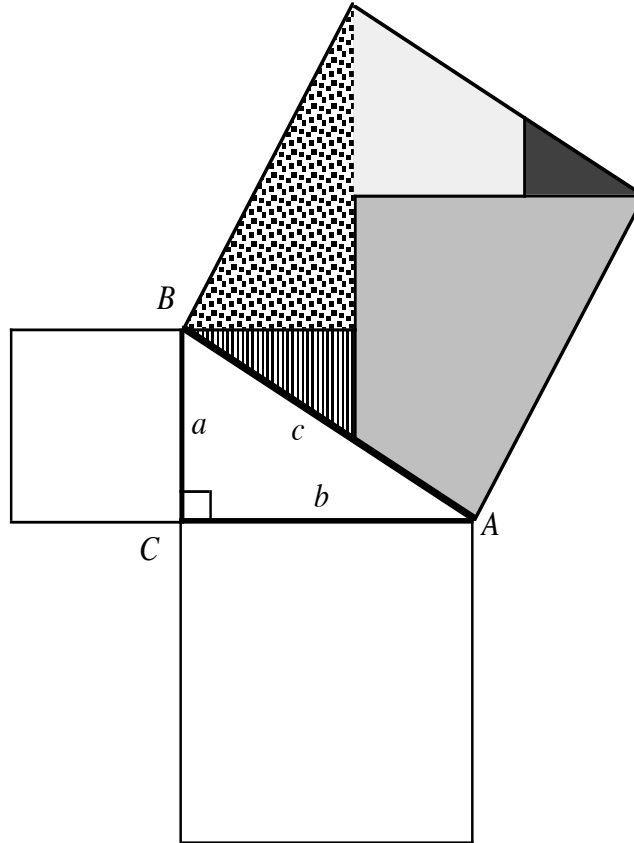
- 2.1** The scholars of Babylon recorded much of their mathematics on clay tablets. Many of these tablets still exist. One remarkable tablet, designated “Plimpton 322,” dates from sometime around 1900 B.C. This fragment of a once larger tablet contains several columns of numbers.

The table below shows a portion of the contents of Plimpton 322, as well as some numbers (in parentheses) that historians believe were in the original tablet. Each row of three numbers in the table is a *triple*.

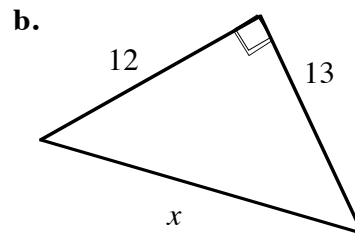
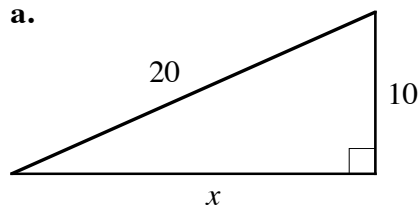
Side 1	Side 2	Side 3
(120)	119	169
(72)	65	97
(60)	45	75
(360)	319	481

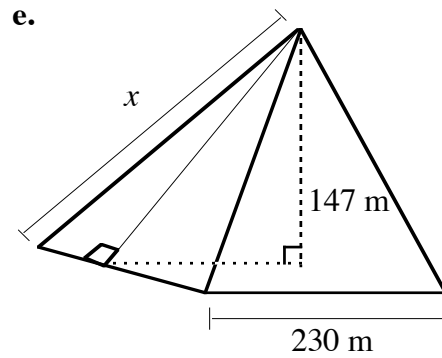
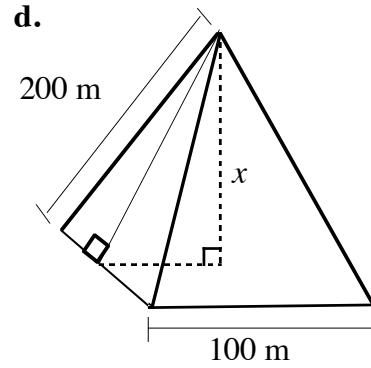
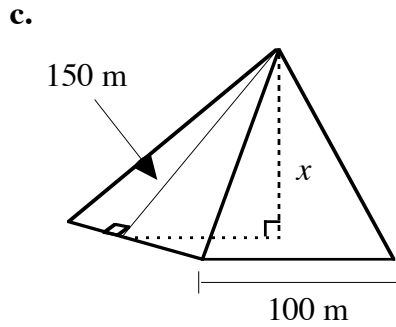
- a.** Use the converse of the Pythagorean theorem to determine what type of triangle—acute, obtuse, or right—is formed when each triple in the table represents the lengths of the sides of a triangle.
- b.** If you double each number in a triple from Plimpton 322, does the resulting triple still show the same relationship you described in Part **a**? Explain your response.
- 2.2** To create a surveying tool similar to the one the Egyptians might have used, mark 12 congruent segments on a string. Considering the length of each segment as 1 unit, form a square whose sides are exactly 3 units long. (Do not use protractors or other tools to help set your corners.) Verify that your result is a square.
- 2.3** Although the early Babylonians were aware of the relationship represented in the Pythagorean theorem, they had not yet developed the algebraic notation we use today. Instead, they used a sort of “geometric algebra.”

The following diagram shows a right triangle ABC with squares constructed on each side. Five shaded pieces subdivide the largest square. Obtain a template from your teacher and cut out the shaded pieces from the largest square. Arrange the pieces so that they exactly fill the two smaller squares. Explain how your arrangement geometrically demonstrates the algebraic expression $a^2 + b^2 = c^2$.



- 2.4** Given three segments, it is not always possible to use them to form a triangle.
- Identify at least three different sets of three segments which do not form triangles.
 - Explain why each set of segments does not form a triangle.
 - Describe the relationship that exists among the lengths of three segments when they cannot be used to form a triangle.
- 2.5** Determine the unknown length x in each of Parts **a–e** below. The pyramids in Parts **c–e** are regular square pyramids.

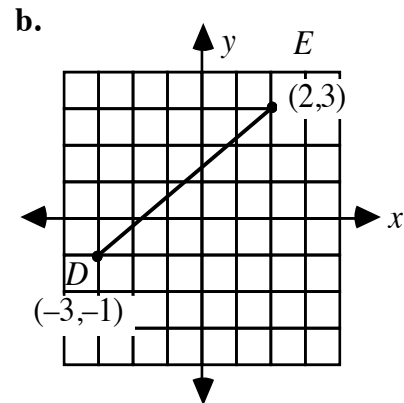
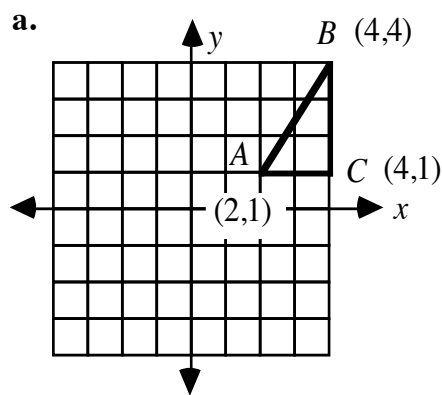


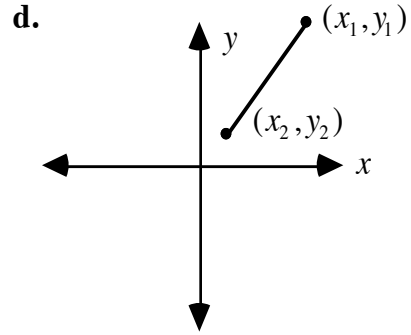
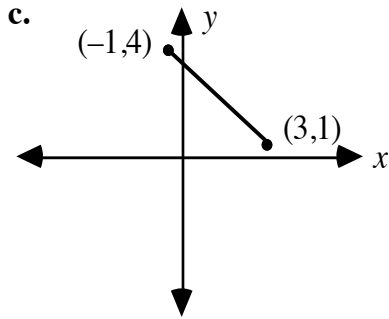


2.6 Without constructing the triangles, determine if the three lengths in each of the following would form an acute triangle, an obtuse triangle, a right triangle, or no triangle.

- a. 5, 12, 13
- b. 6, 8, 12
- c. 7, 10, 19
- d. 13, 8, 15

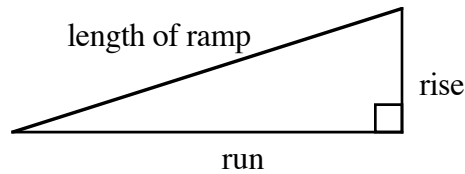
2.7 Use the properties of right triangles to find the distance between each pair of points in Parts **a–d**. (In Part **a**, an appropriate triangle has been drawn for you. In Part **d**, use an expression to represent the distance.)



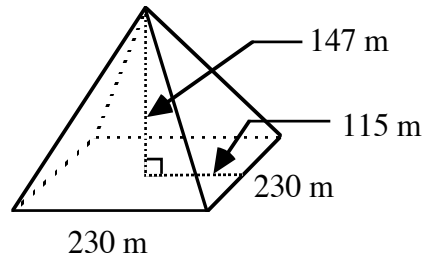


2.8 Some of the granite blocks used in the Great Pyramid are 8.2 m long and 1.2 m thick. Each block has a mass of 49 metric tons, and some of them are set 61 m above the base of the pyramid.

Egyptologists believe that the builders used inclined ramps to raise the stone blocks to these levels. As shown in the diagram below, some researchers have suggested that the incline ratio (slope or rise:run) of these ramps was 1:3.



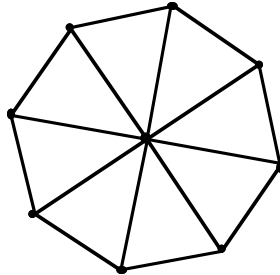
The dimensions of the Great Pyramid are shown in the diagram below. If the capstone (the stone on the top) is 1 m tall, how long was the ramp required to raise it?



* * * * *

2.9 Leah and Harlan are building a concrete foundation for their new garage. The foundation is a square 8 m on each side. After marking the boundaries, they decide to measure the diagonals to confirm that the corners are right angles. If the corners are right angles, how long should the diagonals be?

- 2.10** After finishing their garage, Leah and Harlan decide to build a gazebo in their backyard. (A gazebo is a small, roofed structure with open sides.) As shown in the following diagram, the floor of their gazebo is a regular octagon. The floor supports lie along the perimeter and the diagonals of the octagon.



- The diagonals of the octagon form eight congruent triangles. What type of triangles are these?
- An altitude drawn from the center of the octagon to the base of each triangle is 2 m long. Each diagonal is 5 m long. Determine the total length of the wood required for the floor supports.

* * * * *

Activity 3

Judging from the information recorded on Plimpton 322, the ancient Babylonians (and the ancient Egyptians too, perhaps) were aware of some uses for the ratios of the sides of right triangles. These ratios are important in many modern applications, from architecture to navigation. The study of these ratios and their properties is the focus of **trigonometry**. In this activity, you explore how the ancient Egyptians might have used some basic trigonometric ideas in the construction of the pyramids.

Exploration

The Great Pyramid of Giza has 203 distinct levels or steps, like those shown in Figure 9. These steps are not all the same height.

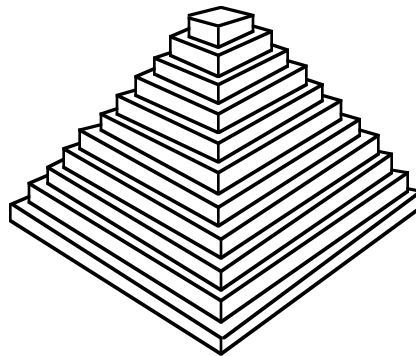


Figure 9: Pyramid steps

After the steps were built, casing blocks were used to give the pyramid's walls their smooth outer surface. To create a smooth slope, the outer faces of all the casing blocks—regardless of their height—had to be slanted at the same angle to their bases. Figure 10 shows a portion of a pyramid wall with three casing blocks.

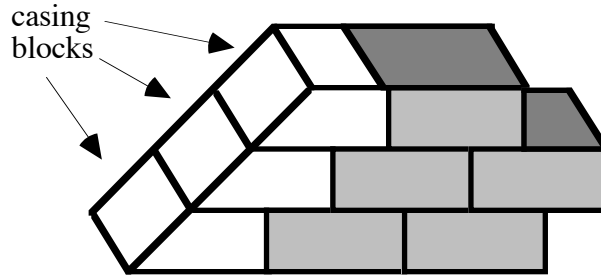


Figure 10: A portion of a pyramid wall with casing blocks

An angle formed by two intersecting planes is a **dihedral angle**. The measure of a dihedral angle is the measure of the angle whose sides are the two rays formed by the intersections of the faces and a plane perpendicular to the edge. As shown in Figure 11, the measure of the dihedral angle of the Great Pyramid is 52° .

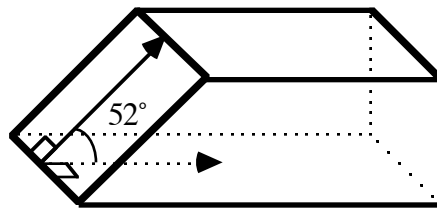


Figure 11: The dihedral angle of the Great Pyramid's casing blocks

Figure 12 shows cross-sectional drawings of the casing blocks for the bottom and top steps of the Great Pyramid. The bottom step is 141 cm thick, while the top step is 56 cm thick. The heights of the remaining 201 steps lie somewhere between these two values.

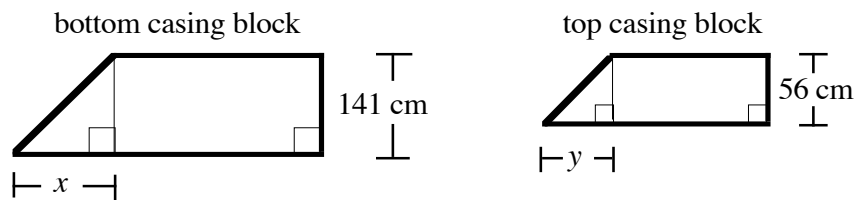


Figure 12: Cross-sections of two casing blocks

To maintain a constant dihedral angle measure of 52° , the stonecutters had to be very precise. However, the Egyptians did not measure blocks using angles or degrees. Instead, they used a relationship called the *seqt*, the ratio of the horizontal “run” of a slope to its vertical “rise.” The *seqt* of the bottom casing block in Figure 12, for example, is $x/141$. In this exploration, you experiment with a ratio from trigonometry related to the Egyptian *seqt*.

- a.
1. On a geometry utility, construct a horizontal line. Construct and label two points A and C on the line.
 2. Construct a line perpendicular to \overleftrightarrow{AC} through C . Label a point B on the perpendicular.
 3. Construct \overline{AB} . Your construction should now resemble the one shown in Figure 13.

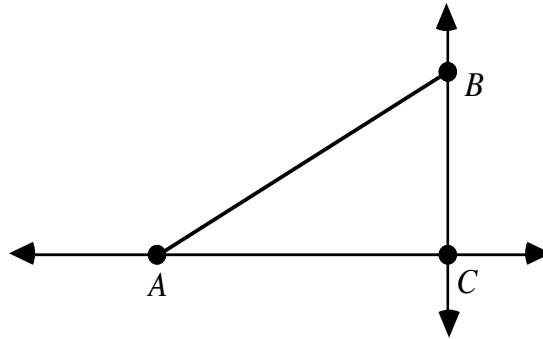


Figure 13: Construction of right triangle

4. Hide the lines and construct \overline{AC} and \overline{BC} , as shown in Figure 14.
Note: Save this construction for use in Activity 4.

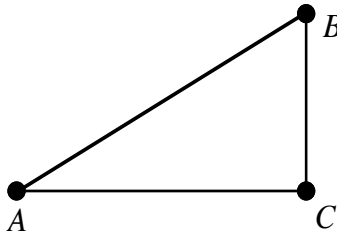


Figure 14: Right triangle

5. Drag point B until $m\angle A = 52^\circ$ (the measure of the dihedral angle of the Great Pyramid).
- b.
- Measure the legs of right triangle ABC and calculate the ratio below.
- $$\frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$
- Note:** For convenience, this ratio will be referred to in this module as the ratio “opposite/adjacent .”
- c.
- Drag point C along the horizontal line to form other right triangles.
1. As the lengths of the legs change, describe any patterns you observe in the ratios of the lengths.
 2. Explain why these right triangles are similar.

- d. Create a table with headings like those in Table 2. Drag point B to change the measure of $\angle A$ and record the ratio of the length of the leg opposite $\angle A$ to the length of the leg adjacent to $\angle A$.

Table 2: Ratios of length of opposite leg to adjacent leg

$m\angle A$	opposite/adjacent
52°	
15°	
30°	
45°	
60°	
75°	

Mathematics Note

In any right triangle, the **tangent** of the measure of an acute angle is the ratio of the length of the leg opposite the angle to the length of the leg adjacent to the angle.

For example, Figure 15 shows a right triangle ABC .

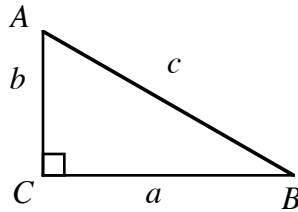


Figure 15: A right triangle ABC

In this case, the tangent of $\angle A$ is the ratio of a to b . This can be written as follows:

$$\tan \angle A = \frac{a}{b}$$

The tangent of $\angle B$ is the ratio of b to a . This can be written as $\tan \angle B = b/a$.

The tangent ratio is useful for determining unknown lengths in triangles. Consider the triangle in Figure 16, for example.

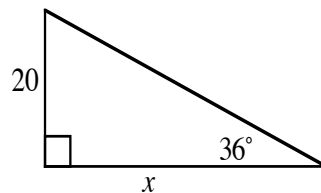


Figure 16: Triangle with the length of one leg unknown

Since $\tan 36^\circ$ equals the ratio of the length of the opposite leg to the length of the adjacent leg, the value of x can be found by solving the following equation.

$$\begin{aligned}\tan 36^\circ &= \frac{20}{x} \\ x(\tan 36^\circ) &= 20 \\ x &= \frac{20}{\tan 36^\circ} \approx 27.53\end{aligned}$$

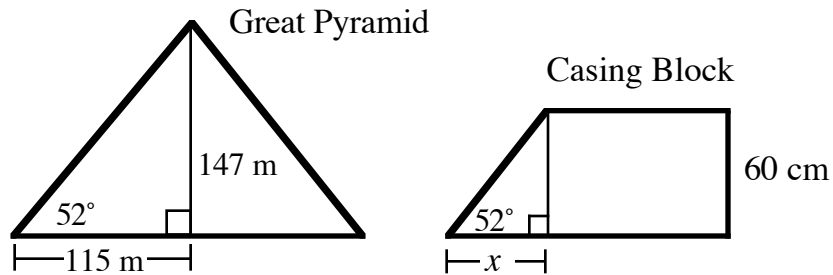
- e. Add a column to Table 2 and label it “ $\tan \angle A$.” Using technology, find and record the tangent of each angle measure in the left-hand column. Compare your results with the ratios in the column labeled “opposite/adjacent .”
- f. Many calculators feature a key labeled “ \tan^{-1} .” This represents the inverse tangent command. Use technology to determine and record the inverse tangent of each value you determined in Part e. Describe any patterns you observe.

Discussion

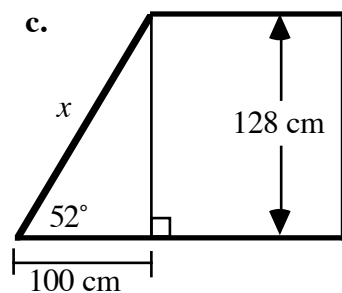
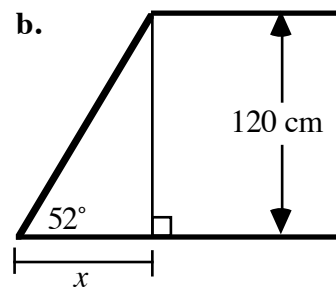
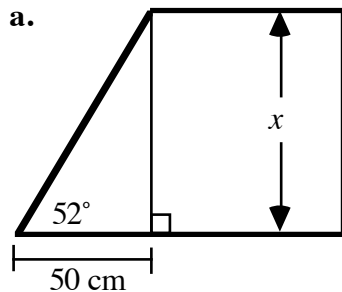
- a. In Part c of the exploration, what pattern did you observe in the ratio opposite/adjacent ? Explain why this pattern occurs.
- b. Compare the values you recorded in Table 2 with others in your class. Describe any similarities or differences you observe.
- c. What happens to the tangent as the measure of an acute angle changes?
- d. How is the tangent related to the Egyptian seqt?
- e.
 1. Given that the tangent of an angle is approximately 1.73, how could you use the values in Table 2 to determine the measure of the angle?
 2. Describe how this process is related to the \tan^{-1} key on a calculator.
- f. Why would you expect $\tan 45^\circ$ to equal 1?

Assignment

- 3.1 The figure below shows a cross-sectional drawing of the Great Pyramid and a cross-sectional drawing of a casing block.

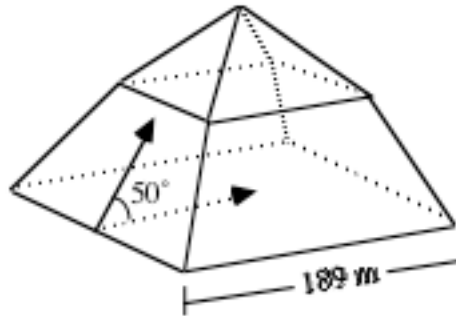


- Explain why the right triangle shown in the casing block is similar to the right triangle shown in the Great Pyramid.
 - Determine the value of x in the casing block using each of the following methods:
 - proportions and similar triangles
 - the tangent ratio.
- 3.2 The following diagrams show three casing blocks from the Great Pyramid. Determine the unknown length in each case.

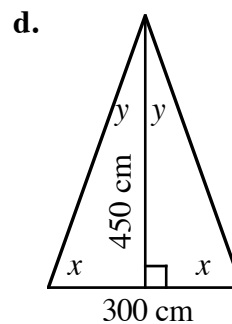
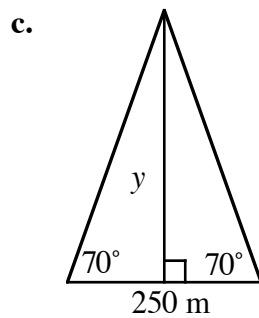
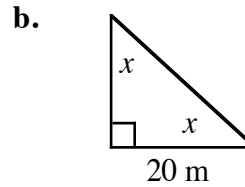
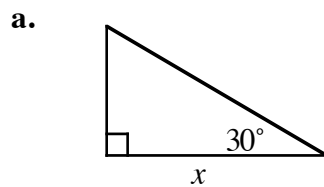


- 3.3 Consider a pyramid whose base has the same dimensions as the Great Pyramid's. If its casing blocks have dihedral angle measures of 71° instead of 52° , how tall is the pyramid? (Hint: Use the given information to sketch an appropriate triangle.)

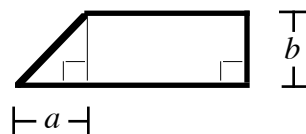
- 3.4** Describe what a calculator displays for $\tan 90^\circ$ and explain why this occurs.
- 3.5** King Khufu's father, Snefru, built a pyramid at Dashur known as the Bent Pyramid. As shown in the diagram below, the base of the Bent Pyramid is a square 189 m on each side. The 50° dihedral angle measure at its base was abandoned after construction reached a height of 73.5 m. A smaller angle measure, about 37° , was used to finish the pyramid.



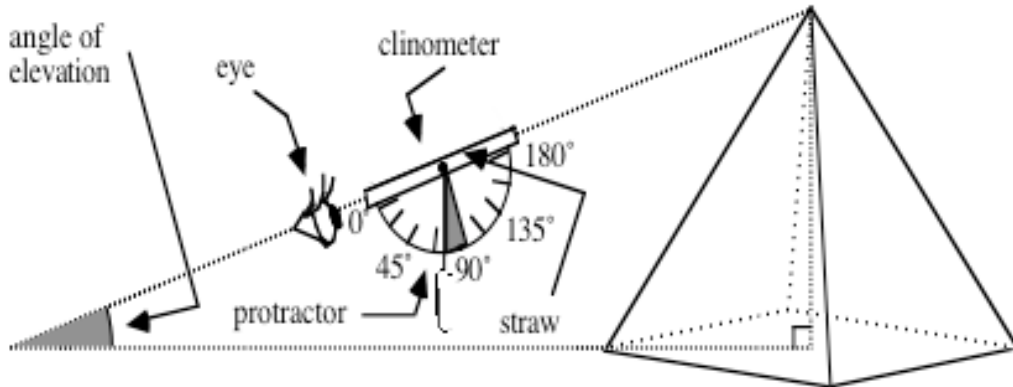
- a. How tall would the Bent Pyramid have been if completed using the 50° dihedral angle measure?
- b. How tall is the actual Bent Pyramid?
- 3.6** Find the unknown lengths and angle measures in each of the following triangles.



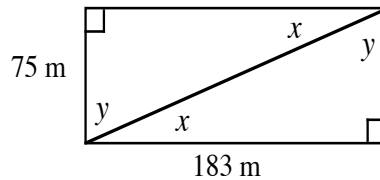
- 3.7** The ancient Egyptians used the ratio a/b to measure the slant of the casing block below. Express this ratio in terms of the tangent of the measure of the block's dihedral angle.



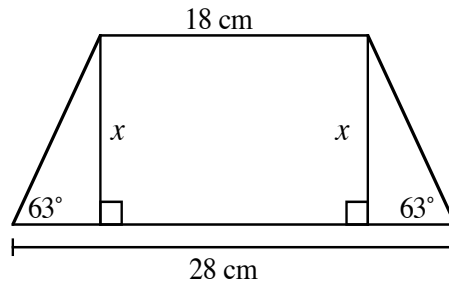
- 3.8** A clinometer measures angle of elevation. The simple clinometer shown in the diagram below consists of a drinking straw, a weighted string, and a protractor. Using this tool, the angle of elevation (the shaded angle in the diagram) is the difference between the angle indicated by the weighted string and 90° . Use this diagram to describe how a clinometer—along with some knowledge of trigonometry—could be used to calculate the height of a pyramid.



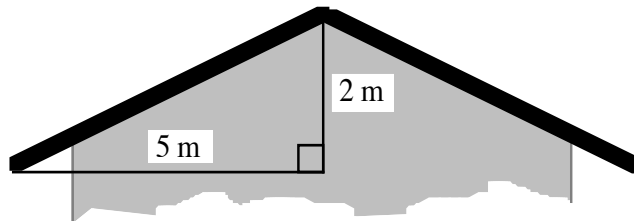
- 3.9** a. Find the unknown angle measures in the figure below.



- b. Find the unknown lengths in the following figure.



- 3.10** When describing the roof of a house, the ratio of the vertical distance to the horizontal distance is often referred to as *pitch*. The figure below shows a roof with a pitch of $2/5$. Determine the measure of the angle which this roof makes with the horizontal.



Research Project

Construct a clinometer like the one described in Problem 3.8. Use your clinometer to estimate the height of your school, a flagpole, or some other tall object. Describe the methods you used to make each estimate. If possible, find the actual height of the object and compare this value to your estimate. Discuss some possible explanations for any differences you observe.

Activity 4

The word *trigonometry* is derived from the Greek words for “three-angle measurement.” Right-triangle trigonometry involves the relationships among the sides and angles of right triangles. The tangent ratio is only one of these relationships.

As noted in Activity 3, the Egyptians used the *seqt*. In Plimpton 322, the Babylonians worked with yet another such ratio. In this activity, you explore some other trigonometric ratios and examine their usefulness as problem-solving tools.

Exploration

- a. Consider $\triangle ABC$ in Figure 17. In Activity 3, you found that the tangent ratio for a given measure of $\angle A$ remains the same for all right triangles that contain that angle. List all the other ratios that you think will remain constant for $\angle A$.

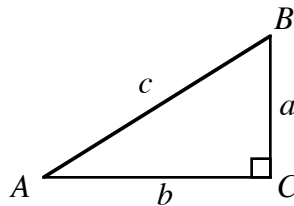


Figure 17: Right triangle ABC

- b. Using your construction of a right triangle from Activity 3, measure $\angle A$ and the sides of $\triangle ABC$. Calculate all the ratios that exist among the lengths of the sides.
- c. Drag point C to create other, similar right triangles. Verify that the ratios you identified in Part a remain constant.

- d. Create a table with headings like those in Table 3 below. Use your geometry utility to complete the table.

Table 3: Ratios of lengths in right triangles

$m\angle A$	opposite/hypotenuse	adjacent/hypotenuse
5°		
15°		
30°		
45°		
60°		
75°		

Mathematics Note

In any right triangle, the **sine** of the measure of an acute angle is the ratio of the length of the leg opposite that angle to the length of the hypotenuse.

For example, Figure 18 shows a right triangle ABC .

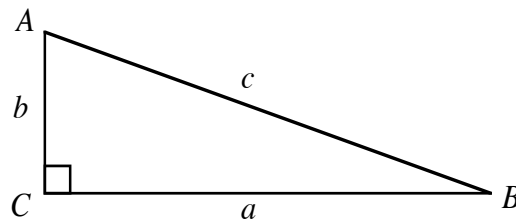


Figure 18: A right triangle ABC

In this right triangle, the sine of $\angle A$ is the ratio of a to c . This can be written as:

$$\sin \angle A = \frac{a}{c}$$

In any right triangle, the **cosine** of the measure of an acute angle is the ratio of the length of the leg adjacent to that angle to the length of the hypotenuse. In the right triangle in Figure 18, the cosine of $\angle A$ is the ratio of b to c . This can be written as:

$$\cos \angle A = \frac{b}{c}$$

For example, consider right triangle DEF in Figure 19. In this case, $\sin \angle E = 8/10 = 0.8$ and $\cos \angle E = 6/10 = 0.6$.

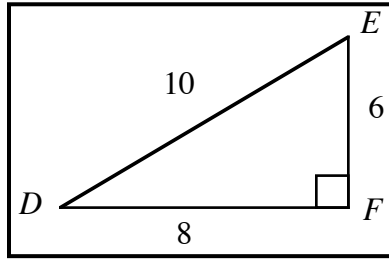


Figure 19: Right triangle DEF

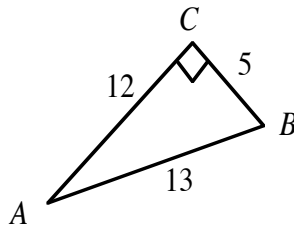
- e. Add two columns to Table 3 using the headings “ $\sin \angle A$ ” and “ $\cos \angle A$.” Use technology to determine the sine and cosine of each angle measure in the left-hand column. Compare your results with the ratios in the columns labeled “opposite/hypotenuse” and “adjacent /hypotenuse .”
- f. Use your geometry utility to determine the minimum and maximum values of $\sin \angle A$ and $\cos \angle A$ in right triangle ABC .
- g. Many calculators feature keys labeled “ \sin^{-1} ” and “ \cos^{-1} .” These represent the inverse sine and inverse cosine commands, respectively. Use the values you determined in Part e to experiment with these commands. Describe any patterns you observe.

Discussion

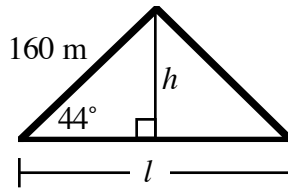
- a. Describe how the two ratios in Table 3 are related to $\sin \angle A$ and $\cos \angle A$.
- b. What trends did you observe in the values of sine and cosine as the measure of $\angle A$ increased?
- c. What is the measure of $\angle A$ when $\sin \angle A$ and $\cos \angle A$ are equal?
- d.
 1. Describe the information you would need to find the measure of an angle using the inverse sine command.
 2. Describe the information you would need to find the measure of an angle using the inverse cosine command.
- e. In a right triangle ABC where $\angle C$ is the right angle, $\sin \angle A$ equals $\cos \angle B$. Explain why this occurs.

Assignment

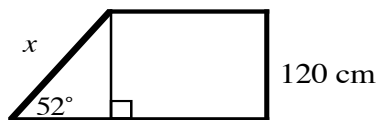
- 4.1 Use the right triangle below to answer Parts **a–d**.



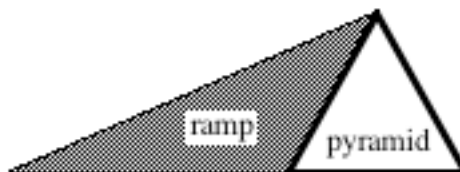
- a. For $\angle B$, which trigonometric ratio is represented by $5/13$?
 - b. For $\angle A$, which trigonometric ratio is represented by $5/13$?
 - c. For $\angle B$, which trigonometric ratio is represented by $12/5$?
 - d. For $\angle A$, which trigonometric ratio is represented by $12/13$?
- 4.2
- a. The figure below shows a cross-section of a pyramid. Use trigonometric ratios to determine the height h and the length l .



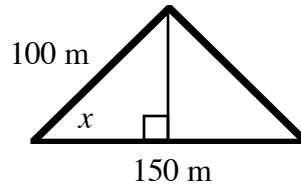
- b. The figure below shows a cross-section of a casing block. Determine the unknown length x .



- c. To move stones to the top of a pyramid, the Egyptians may have built a ramp of sand. If the pyramid in the diagram below is 100 m tall and the incline of the ramp is 25° , determine the length of the ramp.



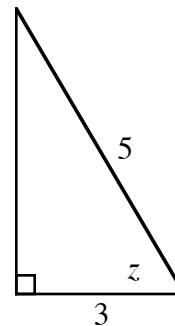
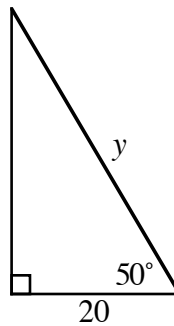
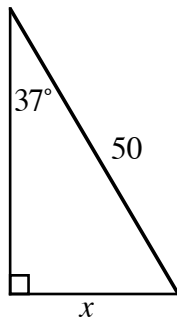
- 4.3 Use appropriate technology and the inverse cosine command to find the unknown angle measure x in the figure below.



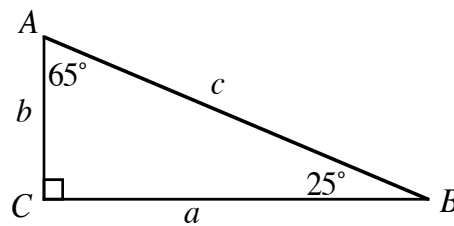
- 4.4 Use the definitions of the sine, cosine, and tangent ratios to prove that the following equation is true.

$$\tan \angle A = \frac{\sin \angle A}{\cos \angle A}$$

- 4.5 Determine the unknown length or angle measure in each of the three right triangles below.

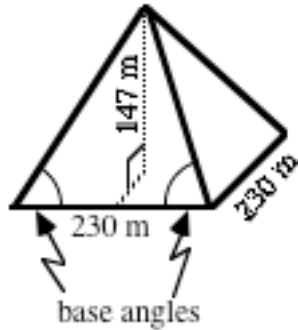


- 4.6 a. Using $\triangle ABC$ below, explain why $\sin 65^\circ = \cos 25^\circ$ and $\tan 65^\circ = 1/\tan 25^\circ$.

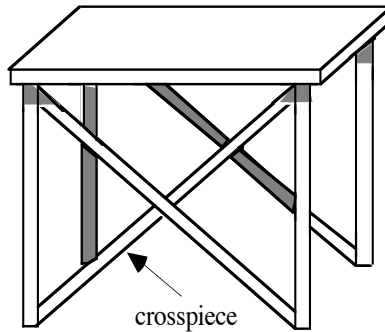


- b. Generalize your response to Part a for all pairs of acute angles whose measures add up to 90° . Use technology to verify this conjecture.

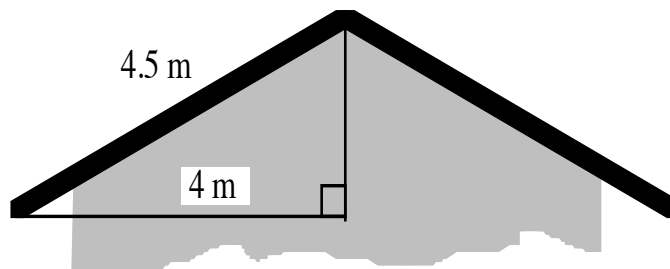
- 4.7 The diagram below shows the base angles on a face of the Great Pyramid. Determine the measure of these angles.



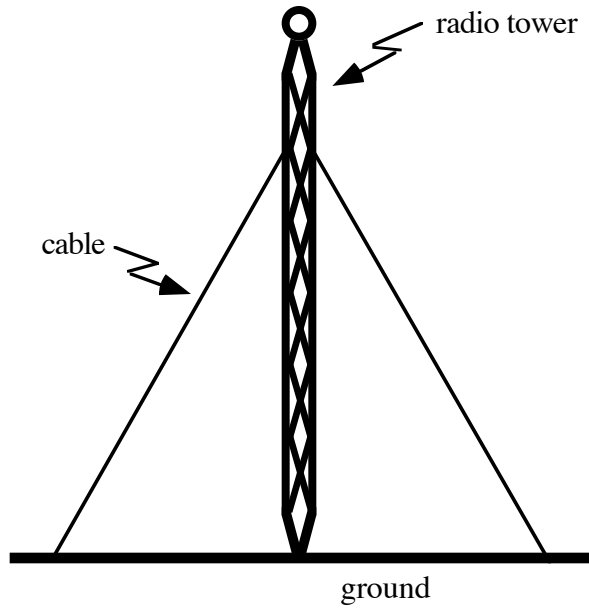
- 4.8 On a highway, a uniform grade of 4% means that there is a rise of 4 m for every 100 m of horizontal distance.
- If you are driving up a road with a 4% grade, what angle measure does the path of your car make with the horizontal?
 - What is your change in elevation after traveling 32 m on this road?
- 4.9 During construction of the Great Pyramid, workers used scaffolding to raise themselves to an appropriate height. The scaffold in the diagram below lifted stone masons to a height 4 m above the previous level. The measure of the angle between each crosspiece and vertical support is 60° . What is the length of the crosspiece? Describe how you determined your response.



- 4.10 Determine the measure of the angle which the roof in the following diagram makes with the horizontal.



- 4.11** A radio tower is anchored to the ground by four cables, two of which are shown in the figure below. Each cable is bolted to the tower 20 m above the ground. The angles formed by the cables with the ground measure 60° .



- How many meters of cable have been used to anchor the tower?
- How far is it from the bottom of the tower to the anchor point of each cable?

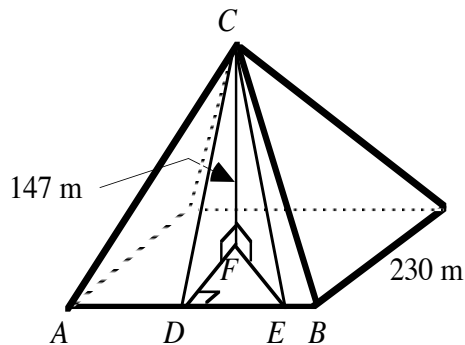
* * * * *

Summary Assessment

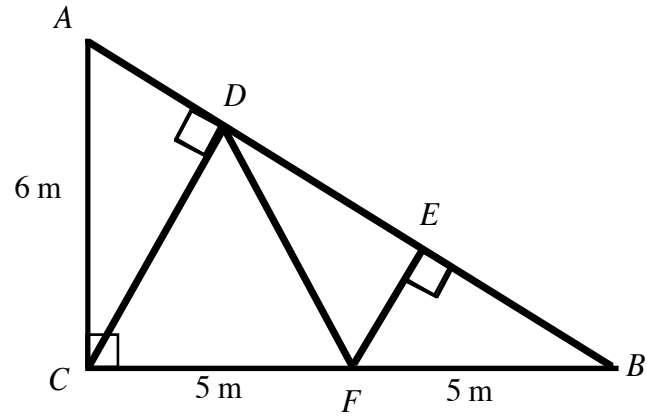
- The annual flooding of the Nile River was an important part of life for the ancient Egyptians. Imagine that you must determine the width of the Nile at its crest near the Great Pyramid. You know the distance from the edge of the pyramid to the near bank is 1 km, and that you are 3.5 km from the opposite bank. From where you are standing, the angle of elevation to the pyramid's peak is approximately 1.5° . Use the figure below to help calculate the width of the Nile.



- In the diagram of the Great Pyramid below, $\angle CDF$ and $\angle CEF$ are angles with one side in the plane of the pyramid's base and the other in the plane of one of its faces. The dihedral angle CDF measures 52° . The length of \overline{DE} is 77 m. Triangles CFD , CFE , and FDE are all right triangles. Use the Pythagorean theorem and trigonometric ratios to show that the measure of the dihedral angle is different from $m\angle CEF$.

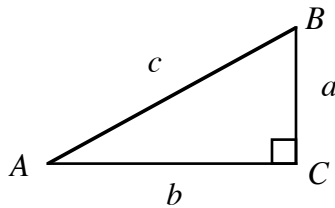


3. A truss is a combination of beams used to support the roof of a building. The diagram below shows some dimensions for a truss for a house. Use the measures indicated in the diagram to determine the lengths of segments EF , DF , and CD .



Module Summary

- Two figures are **similar** if they have the same shape and corresponding sides are proportional.
- In plane geometry, if the angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar. This is referred to as the **Angle-Angle-Angle (AAA) Property**.
- The symbol used to indicate congruence is \cong .
- The **Pythagorean theorem** states that, in a right triangle, the square of the length of the longest side (the hypotenuse) equals the sum of the squares of the lengths of the other sides (the legs).
- The **converse** of a statement in the form “If A, then B” is the statement “If B, then A.” The converse of a true if-then statement may or may not be true.
- An angle formed by two intersecting planes is a **dihedral angle**. The measure of a dihedral angle is the measure of the angle whose sides are the two rays formed by the intersections of the faces and a plane perpendicular to the edge.
- In any right triangle, the **tangent** of the measure of an acute angle is the ratio of the length of the leg opposite the angle to the length of the leg adjacent to the angle. For right triangle ABC in the figure below, $\tan \angle A = a/b$.



- In any right triangle, the **sine** of the measure of an acute angle is the ratio of the length of the leg opposite that angle to the length of the hypotenuse. For right triangle ABC in the figure above, $\sin \angle A = a/c$.
- In any right triangle, the **cosine** of the measure of an acute angle is the ratio of the length of the leg adjacent to that angle to the length of the hypotenuse. For right triangle ABC in the figure above, $\cos \angle A = b/c$.
- The sine, cosine, and tangent are **trigonometric** ratios.

Selected References

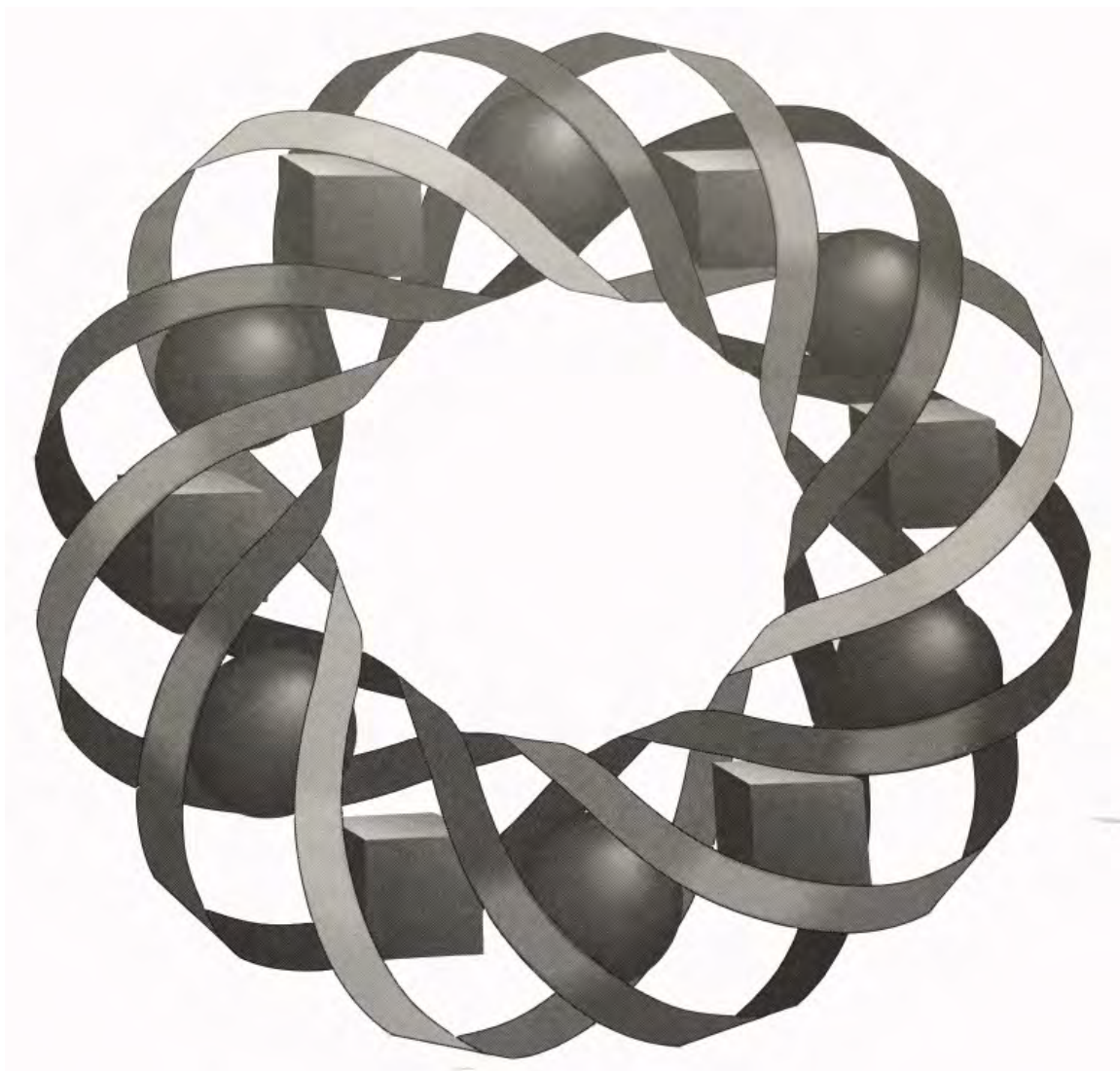
Badawy, A. *Architecture in Ancient Egypt and the Near East*. Cambridge, MA: M.I.T. Press, 1966.

Davidovits, J., and M. Morris. *The Pyramids*. New York: Hippocrene Books, 1988.

Eves, H., and H. Eves. *Introduction to the History of Mathematics with Cultural Connections*. Philadelphia, PA: Saunders College Publishing, 1990.

Macaulay, D. *Pyramid*. Boston, MA: Houghton Mifflin, 1975.

When to Deviate from A Mean Task



You can't argue with statistics. Or can you? In this module, you look at several different strategies for presenting—and interpreting—statistical information.

Randy Carspecken • Pete Stabio



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

When to Deviate from a Mean Task

Introduction

At Two Eagle High School, the senior class is responsible for writing and editing the annual yearbook. This year's seniors want to include stories about events such as the recycling drive and the Academic Bowl. Using statistics reported for each story, the seniors claim their class is the best in the school. The other classes, however, are unhappy with this assessment. They want to do their own analysis of the data. In this module, you develop some statistical methods for responding to the seniors' claim.

Activity 1

Statistics are used by all sorts of people and in all forms of communication. They can be used to explain points of view, promote special interests, and compare specific quantities. A statistical summary of data can be represented in many different ways, often in graphs or charts. Two of the most commonly used methods for displaying statistics are **histograms** (a type of bar graph) and **pie charts**.

Mathematics Note

Histograms display information using rectangles or bars. The bars must be of uniform width and the scales must have uniform intervals.

For example, Figure 1 shows a histogram of the number of movies attended by Two Eagle students in September. The horizontal axis shows the number of movies attended, while the vertical axis shows the number of students in each category.

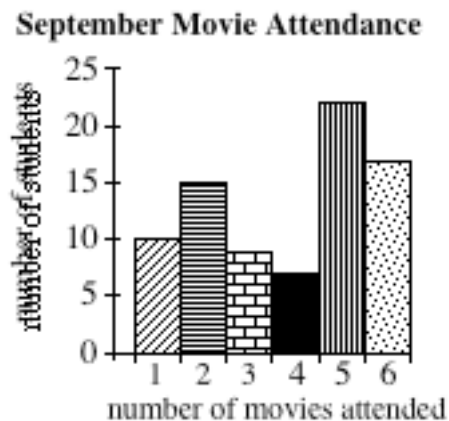


Figure 1: Histogram of September movie attendance

Exploration

The seniors on the Two Eagle yearbook staff want to include the first semester honor roll in their annual. They chose the graph shown in Figure 2 to display their statistics.

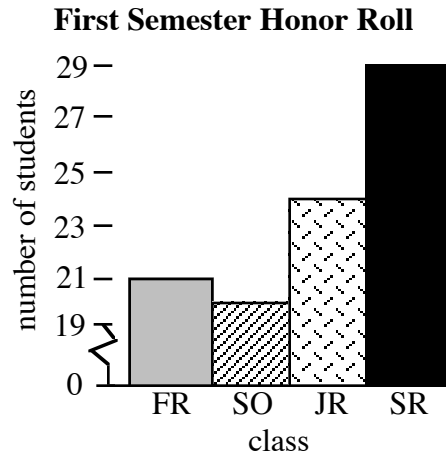


Figure 2: A histogram of the first semester honor roll

Notice the break in the vertical axis. In this case, the break indicates that the numbers from 0 to 19 are not shown on the graph.

Mathematics Note

A **frequency table** consists of two columns. One column contains a data item; the other displays the number of observed occurrences of that item. The number of occurrences of any particular data item is its **frequency**.

For example, the histogram in Figure 1 was generated from the frequency table shown in Figure 3. This table summarizes the results of a survey of 80 students.

Number of Movies	Frequency
1	10
2	15
3	9
4	7
5	22
6	17
Total	80

Figure 3: Frequency table for September movie attendance

- Use the data presented in Figure 2 to create the corresponding frequency table.
- Use your frequency table from Part a to create a histogram with a vertical axis that begins at 0 and ends at 30, without a break. Compare this new graph with the original in Figure 2. **Note:** Save your frequency table and histogram for use later in this module.
- Two Eagle High has a total of 104 freshmen, 81 sophomores, 110 juniors, and 128 seniors. Create another histogram to show the percentage of honor-roll students in each class. Compare this histogram with the one shown in Figure 2.

Discussion

- How accurately does the graph in Figure 2 portray the number of students from each class on the honor roll?
- Compare your graphs from Parts b and c of the exploration. Which graph do you prefer? Explain your response.
- How could the statistics encountered in the exploration be used to favor a particular class?

Mathematics Note

Pie charts (or circle graphs) consist of circular regions divided into sectors, each representing a percentage of the whole. For example, the pie chart in Figure 4 illustrates the same data as the histogram in Figure 1.

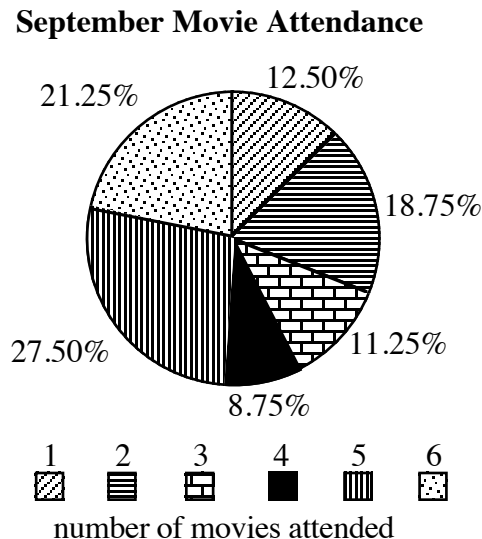


Figure 4: Pie chart of September movie attendance

Like the corresponding histogram, the pie chart in Figure 4 can be generated using a frequency table. The percentages are determined by dividing each frequency by the total. Then, to determine the measure of the central angle that should represent each frequency, these percentages are multiplied by 360° .

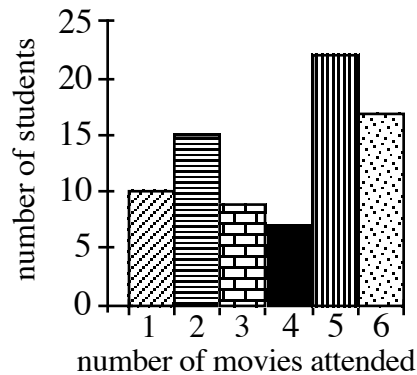
For example, 10 students attended 1 movie in September. This represents $10/80$ or 12.5% of the total. The measure of the central angle for the corresponding sector of the circle is $0.125 \cdot 360^\circ = 45^\circ$.

- d. The sum of the percentages in the pie chart in Figure 4 is 100%. How could you round each percentage to the nearest whole number, while maintaining a sum of 100%?

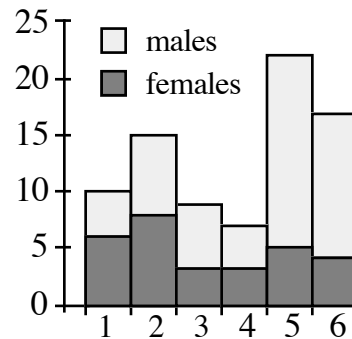
Assignment

- 1.1 a. Use the frequency table from Part a of the exploration to construct a pie chart.
- b. Compare the pie chart to the histogram in Figure 2. What advantages or disadvantages do you observe with the pie chart?
- c. If the number of students in the sophomore class doubled, while the number of sophomores on the honor roll remained the same, would you need to adjust your pie chart? Explain your response.
- d. Upon seeing your pie chart, one student commented, “The seniors had the largest percentage of their class on the honor roll.” Explain why this is a misinterpretation of the pie chart.
- 1.2 In order to display additional information on the same graph, a histogram can be “stacked.” Both histograms below were created using information from the same survey.

September Movie Attendance

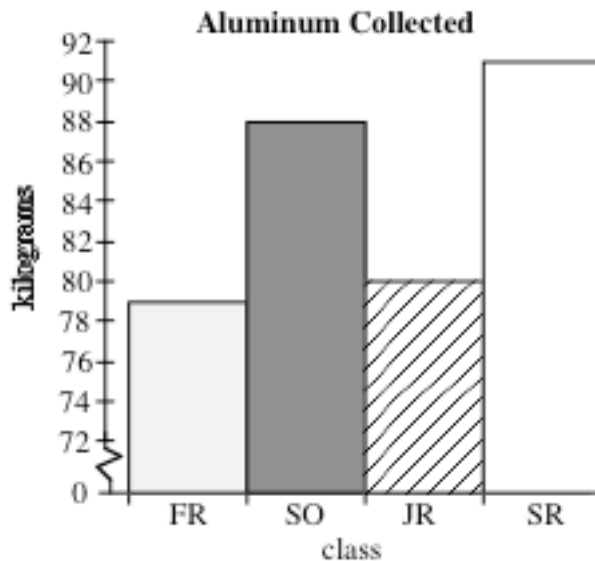


September Movie Attendance



- a. Use the stacked histogram on the right to make an interpretation of the data that was not possible with the histogram on the left.
- b. Explain why it might be difficult to use the stacked histogram to compare the numbers of male students in each attendance category.

- 1.3** Your frequency table from Part **a** of the exploration describes the number of students from each class on the first semester honor roll. Two Eagle High has a total of 104 freshmen, 81 sophomores, 110 juniors, and 128 seniors.
- Create a stacked histogram that displays both these sets of information.
 - Using your stacked histogram, can you identify the class with the largest number of students on the first semester honor roll? Explain your response.
 - Using your stacked histogram, can you identify the class with the greatest percentage of students on the first semester honor roll? Explain your response.
- 1.4** The following graph summarizes the results of Two Eagle High's recycling drive. The yearbook staff plans to use this graph to show that the seniors collected more aluminum than any other class.



- How could you change the histogram to represent the freshman class more positively?
- Two Eagle High has a total of 104 freshmen, 81 sophomores, 110 juniors, and 128 seniors. Using this information, create a graph which shows that a class other than the seniors was most effective in collecting aluminum for the recycling drive.

- 1.5** The numbers of males and females inducted in the Two Eagle High School Honor Society are displayed in the table below.

Class	Gender	No. of Applicants	No. Inducted
Junior	male	20	12
	female	10	5
Senior	male	20	6
	female	40	16

- a. Using the data in this table, create a graph to support each of the following statements.
1. The females at Two Eagle High are better students than the males.
 2. The males at Two Eagle High are better students than the females.
 3. The honor society's selection process favored seniors.
 4. The honor society's selection process favored juniors.
- b. Do you think it is possible to use accurate statistics to support an inaccurate or deceptive position? Explain your response.

* * * * *

- 1.6** The following table shows the land area and 1990 population for Alaska, Texas, California, and Montana.

State	Land Area (km²)	1990 Population
Alaska	1,477,267	550,043
California	403,970	29,760,021
Montana	376,991	799,065
Texas	678,358	16,986,510

- a. **1.** Create a pie chart of the information on land area.
- 2.** Use your chart to compare the areas of each of the following pairs of states: Alaska and Texas; Alaska and California; Alaska and Montana
- b. **1.** Create a pie chart of the information on population.
- 2.** Use this chart to compare the populations of each of the following pairs of states: California and Texas; California and Alaska; California and Montana.

- 1.7** The following table shows the land area and 1990 population of the five most densely populated states.

State	Land Area (km ²)	1990 Population
Connecticut	12,550	3,287,116
Maryland	25,316	4,781,468
Massachusetts	20,300	6,016,425
New Jersey	19,215	7,730,188
Rhode Island	2,706	1,003,464

- a. Jody claims that Rhode Island has the most people per square kilometer because it has the smallest area. Is her claim correct? Explain your response.
- b. Use a histogram to compare the population densities of the five states.
- 1.8** Use each of the following types of graphs to emphasize a characteristic of your state.
- a. a pie chart
- b. a histogram

* * * * *

Research Project

Design and write a survey on a topic of interest to your school. The survey should consist of no more than 10 carefully worded questions.

Give the survey to 30 people in your school.

After completing Activities **1–4** of this module, analyze the collected data using appropriate graphs and statistical measurements.

Prepare an oral presentation that describes the conclusions you drew from the results of the survey.

Activity 2

When reporting statistical information, the elements of an entire data set are often represented by a single number. For example, instead of listing the size of each class in your school, you might give the school's mean class size. In this case, the mean (or average) class size is a single statistic "representing" all class sizes. Statistics such as the mean, median, and mode are **measures of central tendency**.

Exploration

The Two Eagle High Physical Education Department administers a fitness test before allowing students to participate in sports. This test includes a check of the student's resting and active pulse rates. In this exploration, you collect and analyze data on pulse rates for three levels of activity.

- a. After sitting quietly for a period of five minutes, measure and record your pulse rate (beats per minute).
- b. Exercise for one minute. Immediately after exercising, measure and record your pulse rate again.
- c. Rest from your exercise for a period of two minutes. Measure and record your pulse rate one more time.
- d. Organize the class data in a table. **Note:** Save this table for use in the assignment.
- e. For each level of activity, determine a representative pulse rate for the class.

Discussion

- a. What were the representative pulse rates for your class? How did you choose these numbers?
- b. How did changing the level of activity affect the representative pulse rate?
- c. In discussing an individual's pulse rate, what additional information might be important?
- d. Do you think that your class statistics are representative of the entire school?

Assignment

- 2.1** The following table shows the number of siblings reported by nine students at Two Eagle High School.

Student	Number of Siblings
A	1
B	0
C	2
D	2
E	0
F	3
G	2
H	6
I	1

- Which measure of central tendency best represents the number of siblings per student? Justify your choice.
 - Create a frequency table of this data. For example, because students B and E each have 0 siblings, the frequency for 0 siblings is 2.
 - Use your frequency table to create the corresponding histogram. Represent the number of siblings on the horizontal axis and the frequency on the vertical axis.
- 2.2** Measures of central tendency are often used to describe an entire set of data. In situations involving small data sets, measures of central tendency can be misleading.
- Identify four people in your community with different occupations. Make a table that displays each person's occupation and estimated annual income. Determine the mean, median, and mode of the income data.
 - Add the occupation and estimated annual income of a professional athlete to your table. Determine the mean, median, and mode of the income data for this new group.
 - Does the mean income you calculated in Part **a** accurately represent the incomes of the original group? Explain your response.
 - Does the mean income you calculated in Part **b** accurately represent the incomes of the new group? Explain your response.
 - Which of the three measures of central tendency—mean, median, or mode—is most affected by data values very different from the majority of values? Which of the three measures of central tendency is affected the least?

- 2.3 Describe a set of data in which the mode does not exist.
- 2.4 Describe some possible pitfalls associated with using a measure of central tendency to describe a set of data.

Mathematics Note

A **stem-and-leaf plot** displays the values in a data set. To simplify interpretation, data is usually ordered and a legend is included.

For example, the stem-and-leaf plot in Figure 5 displays the number of hours worked by the Two Eagle yearbook staff during the month of September.

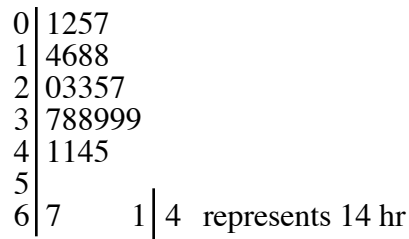


Figure 5: Hours worked by Two Eagle yearbook staff

In the row that reads 203357, the stem is 2. The numbers 0, 3, 3, 5, and 7 are leaves. This row represents the data points 20, 23, 23, 25, and 27 hr. The stem 5 has no leaves. This indicates that none of the staff worked from 50 hr to 59 hr.

A **back-to-back stem-and-leaf plot** can be used to compare two sets of data. The stems are placed in the center column of the plot. One set of leaves is placed to the left of the stems, the other set to the right.

For example, the back-to-back stem-and-leaf plot in Figure 6 displays the same information on work hours as Figure 5. It also shows how those hours were distributed between the boys and the girls.

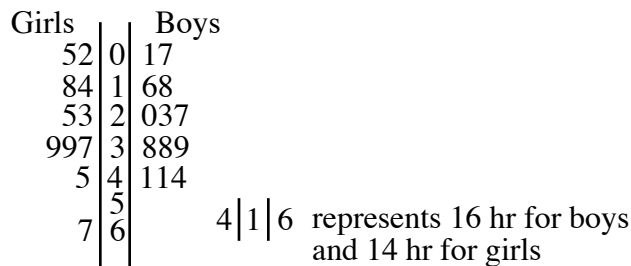


Figure 6: Hours worked by girls and boys on the yearbook staff

- 2.5** The **range** of a data set is the difference between the greatest data value and the least data value. The range provides one indication of how widely the data varies.
- Determine the range of the data in Figure 5.
 - Can the range of a set of data be negative? Explain your response.
- 2.6** The heights in centimeters of 10 students at Two Eagle High are listed below. The heights of females are underlined.
- 153, 157, 163, 165, 166, 169, 170, 173, 176, 185
- Determine the mean, median, and mode of the data.
 - Construct a stem-and-leaf plot of the data, using the first two digits of the heights as the stems and the last digits as the leaves.
 - Find the range of heights.
 - Create a histogram of the data using the following intervals for heights in centimeters: [150, 159], [160, 169], [170, 179], and [180, 189].
 - Separate the data into two categories: heights of female students and heights of male students. Use the data to make a back-to-back stem-and-leaf plot.
 - Use the back-to-back stem-and-leaf plot to create a stacked histogram.
- 2.7** Advertisers often omit important information in their sales pitches. One advertisement in the Two Eagle yearbook claims that “after more than a million miles of testing, our automobiles had a repair-free record.”
- Does the “repair-free” statistic alone convince you that the cars are reliable? Describe one way in which a “million miles of testing” could fail to prove reliability.
 - To convince you that the cars are reliable, which measure of central tendency should the advertisement mention?
- 2.8**
- Based on the pulse-rate data you collected in the exploration, create an advertisement about the physical fitness of America’s youth. Make any claim you can support.
 - What are some of the weaknesses of your claim?

* * * * *

- 2.9 a. Use measures of central tendency to compare the New England states with the Mountain states.

	State	Land Area (km ²)	1990 Population
New England States	Connecticut	12,550	3,287,116
	Maine	79,939	1,227,928
	Massachusetts	20,300	6,016,425
	New Hampshire	23,231	1,109,252
	Rhode Island	2,706	1,003,464
	Vermont	23,956	562,758
Mountain States	Arizona	296,400	3,665,228
	Colorado	268,660	3,294,394
	Idaho	214,235	1,006,749
	Montana	376,991	799,065
	Nevada	284,397	1,201,833
	New Mexico	314,334	1,515,069
	Utah	212,816	1,722,850
	Wyoming	251,501	453,588

- b. Would you rather live in the New England states or the Mountain states? Use the statistics from Part a support your choice.

- 2.10 a. Create a back-to-back stem-and-leaf plot to represent the land areas of the states in the following two regions. (You may want to round these values before creating your stem-and-leaf plot.)

	State	Land Area (km ²)
Region 1	Alabama	131,443
	Florida	139,852
	Georgia	150,010
	Mississippi	121,506
	North Carolina	126,180
	South Carolina	77,988
	Tennessee	106,759
Region 2	Illinois	143,987
	Indiana	92,904
	Iowa	144,716
	Michigan	147,136
	Ohio	106,067
	Wisconsin	140,673

- b. Compare the land areas for the two regions.

* * * * *

Activity 3

To help identify potential athletes, the Athletic Department would like a profile of all Two Eagle High students. Since larger hands give basketball or volleyball players an advantage, the coaches of those two teams are especially interested in student hand spans. A hand span is the distance from the end of the thumb to the end of the little finger when the two are stretched as far apart as possible.

Exploration

The hand spans, in centimeters, of 22 sophomores at Two Eagle High School are listed below (hand spans of females are underlined): 15.4, 23.4, 19.9, 14.8, 17.6, 19.3, 20.7, 22.3, 20.0, 16.4, 18.1, 19.6, 21.9, 24.1, 17.5, 23.9, 25.2, 18.5, 24.1, 17.6, 15.8, 23.4.

- a.
 1. Order the hand-span data from least to greatest.
 2. Determine the median of the entire set of data.
 3. Determine the median of the “upper half” of the data. This number is the **upper quartile**.
 4. Determine the median of the “lower half” of the data. This number is the **lower quartile**.
 5. Calculate the difference between the upper and lower quartiles. This difference is the **interquartile range**.
 6. Data values that are more than 1.5 times the interquartile range above the upper quartile or 1.5 times the interquartile range below the lower quartile are **outliers**. Identify the outliers, if any, in the hand-span data.

Mathematics Note

A **box-and-whisker plot** (or box plot) is a way to represent a set of data graphically. A box plot displays the median, lower quartile, and upper quartile of a data set, as well as any outliers. It may be constructed either horizontally or vertically.

A box-and-whisker plot can be created using the following steps.

- Order the data from least to greatest.
- Draw a scale that covers the range of data.
- Draw a rectangular box that stretches from the lower quartile to the upper quartile. (The width of the box is not significant.)
- Place a bar inside the box to mark the median.
- Find the interquartile range.
- Determine if there are any outliers.
- Mark any outliers with asterisks.
- Draw the **whiskers** (segments that connect each end of the box to the farthest data values which are *not* outliers).

For example, the following data set describes the number of minutes spent doing mathematics homework by 30 Two Eagle sophomores: 0, 4, 5, 9, 9, 9, 10, 16, 16, 18, 18, 18, 18, 19, 21, 21, 21, 21, 22, 22, 22, 23, 24, 26, 26, 27, 29, 29, 39, 45.

For this data set, the lower quartile (LQ) is 16, the median (M) is 21, the upper quartile (UQ) is 24, and the interquartile range is $24 - 16 = 8$. The outliers are values less than $16 - (1.5)8 = 4$, or greater than $24 + (1.5)8 = 36$. A box plot for this data is shown in Figure 7.

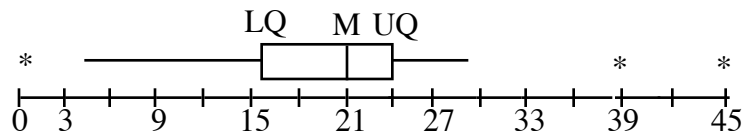


Figure 7: Box-and-whisker plot

- Create three different box-and-whisker plots of the hand-span data. One should display the data for all 22 students, one should display just the boys' hand spans, and one should display just the girls' hand spans. Compare the three plots. **Note:** Save these plots for use in the assignment.
- Measure and record the span of your dominant hand in centimeters.
- Add your hand-span measurement to the hand spans of all 22 students. Create a box-and-whisker plot for this new data set. Compare this plot to the plot for all 22 students from Part **b**.

- e. Add your hand-span measurement to either the boys' or the girls' data. Create a box-and-whisker plot for this new set of data. Compare this plot to the corresponding plot from Part **b**.

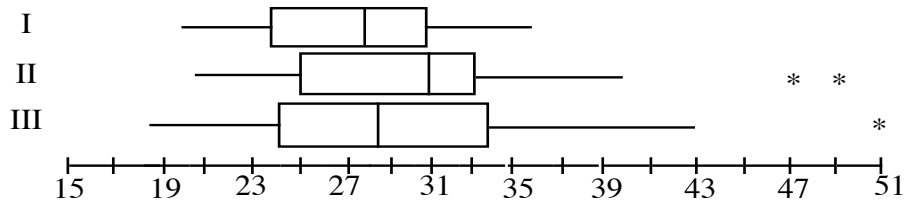
Discussion

- a. What would have more impact on a box-and-whisker plot—adding a data value within the interquartile range, or adding an outlier? Explain your response.
- b. What advantages are there to using the median instead of the mean when describing a data set?
- c. Is the mean of a data set readily identifiable in a box-and-whisker plot? Explain your response.
- d. What are some of the advantages and disadvantages of using a box-and-whisker plot to illustrate data?
- e. A box-and-whisker plot naturally leads to a so-called “five-point analysis” of a data set. What do you think these five points are?

Assignment

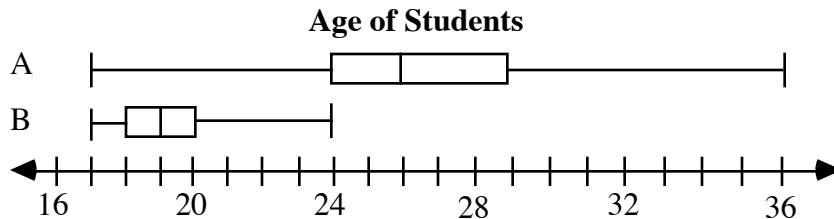
- 3.1 Use the box-and-whisker plots from Part **b** of the exploration to help answer the following questions about box-and-whisker plots in general. Explain your response for each one.
 - a. Are the lengths of the whiskers related to the interquartile range?
 - b. How would adding an outlier to the data affect the plot?
 - c. How would adding a data point within the interquartile range affect the plot?
- 3.2 Using as many statistical measures as you can, describe how your hand span compares to each of the following:
 - a. the hand spans of all 22 Two Eagle sophomores
 - b. the hand spans of the sophomore girls
 - c. the hand spans of the sophomore boys.
- 3.3 The world's tallest man, Robert Wadlow, had a hand span of about 37 cm. If this value were added to the data set in the exploration, would it be considered an outlier?

- 3.4** The following graph shows box plots for the number of sit-ups performed by three different classes at Two Eagle High.



- Which class had the person who did the most sit-ups?
 - For which class did the number of sit-ups vary the least?
 - Which class had the greatest upper quartile?
 - In which class did the bottom 25% perform the most sit-ups?
 - Which class has the greatest number of students?
- 3.5** Create a set of data with at least 16 items in which the mean and the median have the same value and there is at least one outlier. Explain the process you used to create the data set. Make a box plot of your data.

- 3.6** A survey conducted at two different universities, A and B, produced the following box plots.



- Use the information in the plots to compare the students at university A to the students at university B.
 - Describe some information which may have been determined from the data but is not shown in the box plots.
- 3.7**
- Create a box plot for each of the following sets of data.
 - {5,5,5,5,5,30,30,30,30,30,70,70,70,70,70,95,95,95,95,95}
 - {5,6,7,28,37,41,50,61,68,76,89,92,95}
 - Compare the two plots.
 - Explain why a five-point analysis using box plots sometimes can be misleading.
 - Identify a type of graph that would better show the differences in the two data sets. Create the corresponding graphs.

Activity 4

Measures of central tendency are not always useful when comparing data sets. If the mean, median, and a mode of a data set are the same, or relatively close, other statistical measures are needed to provide a better picture of the data.

Exploration 1

Every year, Two Eagle High sponsors a competition between classes called the Academic Bowl. The rules for the contest are listed below.

- Each class may enter only one team.
- Each team must consist of four students from one class.
- Each match is scored on a 100-point scale.
- Team scores are determined by averaging the scores of the four team members.
- Team members may work independently or cooperatively.
- Team members may change after any match.

The teams meet five times during the year, with the overall winner advancing to a state competition. Based on the information in Table 1, the seniors believe that they should be declared the winners.

Table 1: Results of the Academic Bowl

	Freshmen	Sophomores	Juniors	Seniors
Match 1	100	74	82	99
Match 2	60	90	72	66
Match 3	77	85	88	85
Match 4	94	82	89	100
Match 5	85	85	85	66

You have been selected as a judge for this year's contest. How will you choose a winner?

- Enter the data from Table 1 in a spreadsheet. Record the scores for each team in its own column. **Note:** Save this spreadsheet for use in Exploration 2.
- Create a histogram for each team's scores. To simplify comparisons, all four histograms should be the same size and have the same scales.
- Use your histograms along with the measures of central tendency (mean, median, and mode) to determine the winner of the Academic Bowl.
- Describe the process you used to choose a winner.

Discussion 1

- a. Which class did you select as the winner? What criteria did you use?
- b. Using the histograms you created in Part **b** of Exploration 1, compare the range in scores for each class.
- c. Would the difference in the range of scores for each class provide a reasonable criterion for selecting a winner? Explain your response.
- d. Which of the rules for the competition might affect the range of scores? Explain your response.

Exploration 2

In Exploration 1, you found that measures of central tendency alone may not provide enough information to select a winner for the Academic Bowl. Along with the mean, median, or mode, the variation among scores might be used to help determine a winner.

Mathematics Note

There are several methods for measuring variation in a set of data. One of these measures, **mean absolute deviation**, describes the average distance from the mean for the numbers in a data set. For a set of data with n members represented by x_1, x_2, \dots, x_n , the mean absolute deviation is given by:

$$\frac{|x_1 - \mu| + |x_2 - \mu| + \dots + |x_n - \mu|}{n}$$

where the notation $||$ represents absolute value and μ , the lowercase Greek letter *mu*, is the mean of the data.

For example, consider the five match scores for the seniors in the Academic Bowl. The mean (μ) of these scores is 83.2. The distance from the mean for each score—in other words, the absolute value of the difference between the score and the mean—is shown in Figure 8. The mean absolute deviation of the scores is the mean of these distances: $68.8/5 \approx 13.8$.

Score	Distance from Mean
99	15.8
66	17.2
85	1.8
100	16.8
66	17.2
Total	68.8

Figure 8: Distance from mean of the seniors' scores

- a. Use the data in Table 1 to determine the mean absolute deviation for each team's scores.
- b. Create a set of scores using any five values from 50 to 100. Experiment with these values to obtain a set of scores with a mean absolute deviation of 0. Record your data set.
- c. Create another set of scores with any five values from 50 to 100. Experiment with these to obtain a set of scores with a mean absolute deviation as great as possible. Record your data set.

Discussion 2

- a. Why is absolute value used in the calculation of mean absolute deviation?
- b. Mean absolute deviation is defined as the average distance from the mean for the numbers in a data set. How does the formula given in the mathematics note illustrate this definition?
- c. Figure 9 below shows a histogram of the number of parking tickets issued by a traffic officer in five days. The horizontal line indicates the mean number of tickets per day. How is the distance from the top of each bar to the line related to mean absolute deviation?

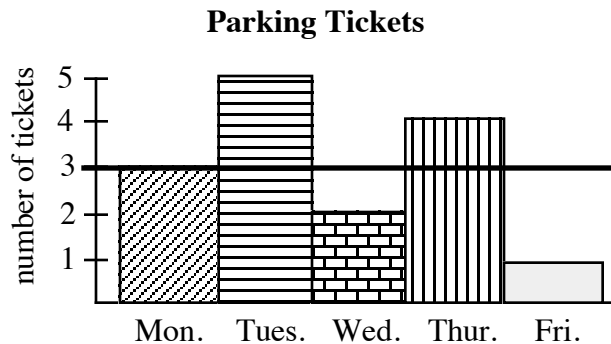


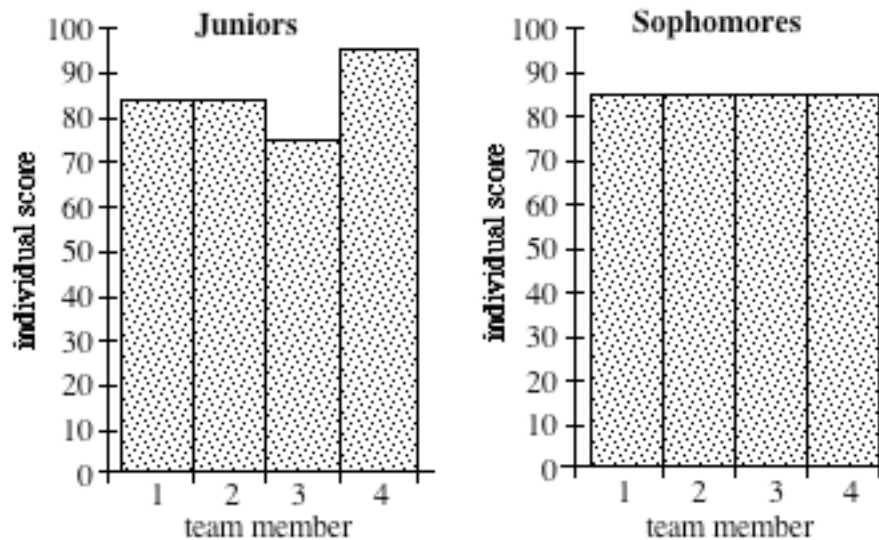
Figure 9: Histogram of parking tickets

- d. How do the histograms from Exploration 1 reflect the fact that the mean absolute deviations for the teams are different?
- e. How would considering the mean absolute deviation of each team's scores affect your choice for the winner of the Academic Bowl?
- f. How might mean absolute deviation and measures of central tendency be used to predict a team's score in a sixth match?
- g. Which team is least likely to score 95 in a sixth match? Explain your response.

Assignment

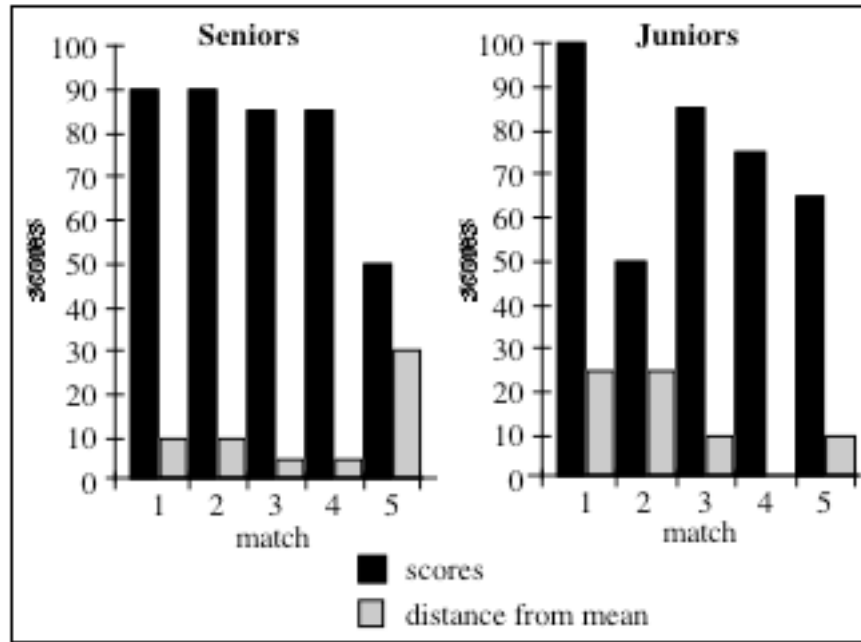
- 4.1** The following graphs display the scores for each member of the junior and sophomore teams in the fifth match of the Academic Bowl.

The members of the junior team scored 85, 85, 75, and 95, while all the members of the sophomore team scored 85.



- Can you differentiate between these two data sets using measures of central tendency alone? Explain your response.
 - Use the juniors' scores to describe how two unequal scores can have equivalent measures of deviation from the mean.
 - If team member 4 on the junior team scored 90 instead of 95, and team member 3 scored 80 instead of 75, how would this affect the mean absolute deviation? Explain why this occurs.
 - How would the mean absolute deviation be affected if a fifth member of the junior team scored 85? Explain your response.
- 4.2** The individual scores for the seniors in the fifth match of the Academic Bowl were 65, 99, 45, and 55. For the freshmen, the individual scores were 77, 93, 77, and 93.
- Use this information, along with that given in Problem 4.1, to determine the mean absolute deviation for the scores of each team in the fifth match.
 - Why can the mean absolute deviation for the sophomores' scores be determined without any calculations?
 - According to the rules for the Academic Bowl, team members may work independently or cooperatively. Describe the method which you think each team may have used. Explain your reasoning.

- 4.3 The two histograms below display the results of an academic bowl at a neighboring school. Each graph shows both the team scores and their distances from the mean for five matches.



- Identify the scores that deviate more than “an average amount” from the mean.
 - What statistical measure did you use to determine “an average amount” in Part a?
 - What is the significance of the missing bar in the fourth match for the juniors?
- 4.4 As described in the mathematics note, the mean of the Two Eagle seniors’ five match scores is 83.2, with a mean absolute deviation of 13.8. The scores within 1 mean absolute deviation of the mean are contained in the interval [69.4, 97].
- Describe how the interval [69.4, 97] was determined.
 - When is a data point 0 mean absolute deviations from the mean?
 - Determine the interval that contains scores within 2 mean absolute deviations of the mean. (Hint: Make sure that the extreme values in this interval make sense.)

- 4.5** The scores for the senior team in Problem 4.3 were 90, 89, 87, 86, and 51. The scores for the junior team were 100, 50, 85, 75, and 65.
- Calculate the mean absolute deviation for each team's scores.
 - How does the mean absolute deviation describe the variation in scores for each team?
 - For each team, identify all scores more than 1 mean absolute deviation from the mean.

- 4.6** The winning team in the Academic Bowl advances to the state competition. In the past, teams that score at least 80 points typically place among the top three at state. Select a winning team from Two Eagle High and justify your choice.

* * * * *

- 4.7** The table below shows some statistics for two basketball players at Two Eagle High. Rita is the leading scorer for the girls' team and Bernard is the leading scorer for the boys' team.

Player	Rita	Bernard
Average Points per Game	25	25
Mean Absolute Deviation (in Points per Game)	5	15
Points Scored in Last Game	35	35

- Which player is the more consistent scorer? Explain your response.
- For each player, identify the range of values that are within 1 mean absolute deviation of the mean.
- Which player's performance in the last game is more impressive? Explain your response.
- Both the boys' and girls' teams score an average of 60 points per game, with the same mean absolute deviation. Compare the range of the average scoring of Rita's teammates to that of Bernard's teammates.

- 4.8** At the end of the cross country season, the coach presents an award to the most improved runner. This award goes to the athlete whose time in the sixth race of the season is most extraordinary when compared with the mean time of the previous five races. The table below shows the times in each race for the top three candidates. Which runner should receive the award? Justify your choice.

	Runner 1	Runner 2	Runner 3
Race 1	25 min, 30 sec	20 min, 10 sec	22 min, 15 sec
Race 2	23 min, 45 sec	19 min, 50 sec	25 min, 20 sec
Race 3	24 min, 15 sec	21 min, 30 sec	20 min, 15 sec
Race 4	22 min, 5 sec	19 min, 45 sec	24 min, 25 sec
Race 5	21 min, 10 sec	18 min, 40 sec	20 min, 10 sec
Race 6	21 min, 5 sec	19 min, 20 sec	21 min, 45 sec

* * * * *

Activity 5

When considered along with measures of central tendency, mean absolute deviation can help you interpret a set of data. Another measure of the spread in a data set is **standard deviation**. Although mean absolute deviation and standard deviation are usually close in value, standard deviation is preferred by statisticians because of its mathematical properties.

Mathematics Note

Standard deviation is a common measure of the spread in a data set. It is calculated by first squaring the individual distance from the mean (the deviation) for each piece of data, then finding the mean of these squared deviations. The standard deviation is equal to the non-negative square root of the mean of the squared deviations.

Standard deviation is often represented by the symbol σ (the lowercase Greek letter *sigma*). For a population with n members represented by x_1, x_2, \dots, x_n , where μ is the mean, the standard deviation is given by the formula below:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \dots + (x_n - \mu)^2}{n}}$$

For example, consider the scores for the freshmen team in the fifth match of the Academic Bowl. The mean score (μ) is 85.25. The sum of the squared deviations may be calculated by using a spreadsheet, as shown in Figure 10.

Score (x_n)	$(x_n - \mu)$	$(x_n - \mu)^2$
85	-0.25	0.06
74	-11.25	126.56
92	6.75	44.56
90	4.75	22.56
	Total	194.74

Figure 10: Determining standard deviation

The mean of the sum of the squared deviations is $194.74/4 \approx 48.69$. The standard deviation is $\sqrt{48.69} \approx 6.98$.

Discussion

- How are the formulas for standard deviation and mean absolute deviation similar? How are they different?
- How is the difference between the mean and each data point kept positive in the formula for standard deviation?
- What is the purpose of taking the square root of the mean squared deviation?
- Why is it important to consider only the non-negative square root?

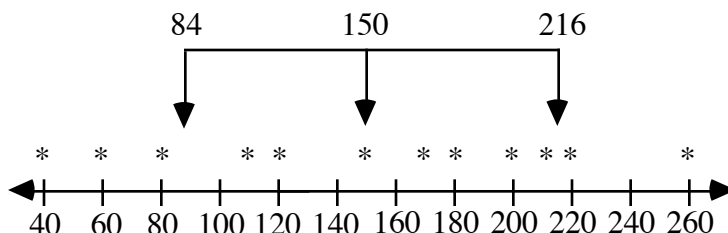
Assignment

- In the five matches of Two Eagle's Academic Bowl, the senior team scored 99, 66, 85, 100, and 66. Calculate the mean and the standard deviation for these scores.
- Many calculators have built-in functions to compute the mean and standard deviation for a set of data.

Enter the Academic Bowl scores for the juniors (82, 72, 88, 89, and 85), the sophomores (74, 90, 85, 82, and 85), and the freshmen (100, 60, 77, 94, and 85) into your calculator.

 - Use a calculator function to find the mean for each team's scores. One symbol commonly used to represent the mean is \bar{x} . What notation does your calculator use?
 - Use a calculator function to find the standard deviation for each team's scores. One common notation for standard deviation is σ . What notation does your calculator use?

- 5.3** The past 12 drama performances at Two Eagle High attracted the following audiences: 220, 210, 180, 170, 120, 60, 40, 80, 110, 150, 200, and 260. This data is displayed in the line plot below. The mean for the 12 performances is identified by the middle arrow. The other two arrows indicate values 1 standard deviation from the mean.



- How many performances were attended by an audience more than 1 standard deviation from the mean?
- In this data set, are there any audiences more than 2 standard deviations from the mean? Explain your response.

* * * * *

- 5.4** The following table shows the mass in kilograms for the players on a high school football team.

54	85	78	67	75
75	76	75	80	85
62	90	88	79	68
55	95	65	85	92
62	83	72	90	88

- Find the mean mass for the players on the team.
 - Calculate the standard deviation for the mass of the players.
 - What percentage of the players have a mass more than 1 standard deviation from the mean?
 - Are there any players whose masses are more than 2 standard deviations from the mean? If so, identify their masses.
- 5.5** The interval $[230, 450]$ represents values within 1 standard deviation of the mean for a data set.

- Determine the mean of the data set.
- Determine the interval that represents values within 2 standard deviations of the mean.
- Determine the interval that represents values within 3 standard deviations of the mean.

* * * * *

Summary Assessment

At Two Eagle High, the biology teacher has two classes: third period and fifth period. Both classes take the same tests. On previous tests, the average score in each class has been very close.

On the most recent test, the third-period class received the following scores: 52, 60, 66, 71, 72, 73, 75, 77, 78, 80, 82, 84, 85, 88, 89, and 90.

On the same test, the fifth-period class received these scores: 76, 77, 78, 79, 80, 80, 82, 83, 84, 85, 85, 88, 90, 90, 90, 92, 92, 96, 96, and 99.

1. Create a back-to-back stem-and-leaf plot of this data.
2. Determine the mean test score for each class.
3. Create a box plot of the data for each class and write a summary of your observations.
4.
 - a. Determine the standard deviation for the test scores from each class.
 - b. What percentage of the scores in each class are within 1 standard deviation of the mean?

Module Summary

- **Histograms** display information using rectangles or bars. The bars must be of uniform width and the scales must have uniform intervals.
- **Pie charts** (or circle graphs) consist of circular regions divided into sectors that each represent a percentage of the whole.
- A **frequency table** consists of two columns. One column contains a data item; the other displays the number of observed occurrences of that item. The number of occurrences of any particular data item is its **frequency**.
- The **mean**, **median**, and **mode** are **measures of central tendency**. They are often described as “representative” of the numbers in a data set.
- A **stem-and-leaf plot** displays the values in a data set. To simplify interpretation, data is usually ordered and a legend is included. A **back-to-back stem-and-leaf plot** may be used to compare two sets of data.
- The **range** is a measure of spread found by subtracting the least data value from the greatest data value.
- The **lower quartile** is the median of the lower half of the data points. The **upper quartile** is the median of the upper half of the data points.
- The **interquartile range** is the difference between the upper quartile and the lower quartile.
- **Outliers** are extreme data values more than 1.5 times the interquartile range above the upper quartile or below the lower quartile.
- A **box-and-whisker plot** (or box plot) displays the median, lower quartile, upper quartile, and outliers (if any) of a data set. The **whiskers** are segments that connect each end of the box to the farthest values which are not outliers.
- **Mean absolute deviation** is a measure of spread that describes the average distance from the mean for the numbers in a data set. For a set of data with n items represented by x_1, x_2, \dots, x_n , the mean absolute deviation is given by:

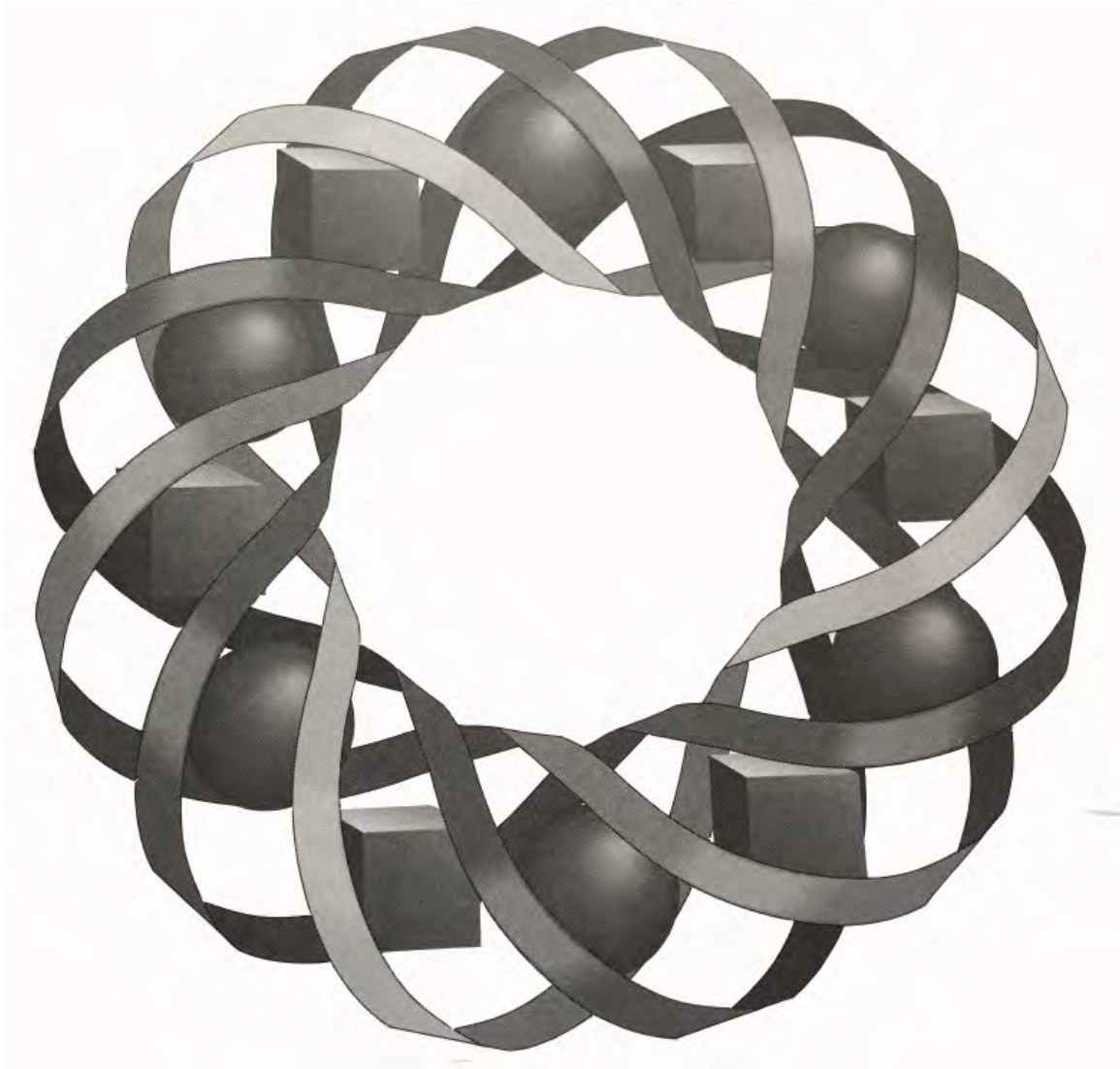
$$\frac{|x_1 - \mu| + |x_2 - \mu| + \dots + |x_n - \mu|}{n}$$

where the notation $||$ represents absolute value and μ is the mean of the data.

- **Standard deviation** is another measure of spread. For a population with n members represented by x_1, x_2, \dots, x_n , where μ represents the mean, the standard deviation is given by:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \dots + (x_n - \mu)^2}{n}}$$

Who Gets What and Why?



Why do some U.S. states have more senators than House representatives? In this module, you examine the process—and the politics—of apportionment.

Patricia Bean • Dean Preble • Teri Willard



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Who Gets What and Why?

Introduction

In the not-too-distant future, a world of unstable countries and changing borders is a real possibility. In the early 1990s, for example, the Soviet Union splintered into 15 different countries, each with its own government. As the next century approaches, maps will continue to be redrawn. The following scenario describes one possible version of the future.

The year is 2096. Antarctica, the continent that lies about the South Pole, is the last remaining land mass not yet exploited for its natural resources. First discovered by whalers in the early 1800s, it remained unexplored until the 20th century. However, repeated expeditions and the eventual establishment of several research stations led to conflicting territorial claims. To head off future disagreements, 12 nations signed an Antarctic treaty in 1959. The treaty went into effect in 1961. Although it has been modified over the years, three fundamental principles remain: the suspension of territorial claims, the freedom of scientific research, and a ban on all military activities on or around the Antarctic continent.

In order to maintain peace in the late 21st century and beyond, the countries of the seven major continents have agreed to form seven distinct global regions. To replace the United Nations, they have created the Global Congress for Peace (GCP). The most pressing issues facing the GCP are fair representation for the people of the new global regions, and the threat of war over rights to the resources of Antarctica.

The problem of ownership in Antarctica raises many of the same difficult issues at the core of other international disputes. Who owns Antarctica? Is there some reasonable method for dividing the region, or parts of it, among individual nations or groups of nations?

Discussion

- a. Name two countries whose governments are currently involved in a border dispute.
- b. Identify some of the causes for the dispute.
- c. What factors must be considered when determining boundaries between two neighboring countries?

Activity 1

Figure 1 shows a map of the continent of Antarctica. With more than 8.3 million square kilometers (km^2) of land and ice, Antarctica is a vast potential source of minerals and fresh water. The continent may contain coal, iron, copper, nickel, lead, silver, cobalt, manganese, and titanium. To whom do these resources belong? How should the Global Congress distribute the mining rights?

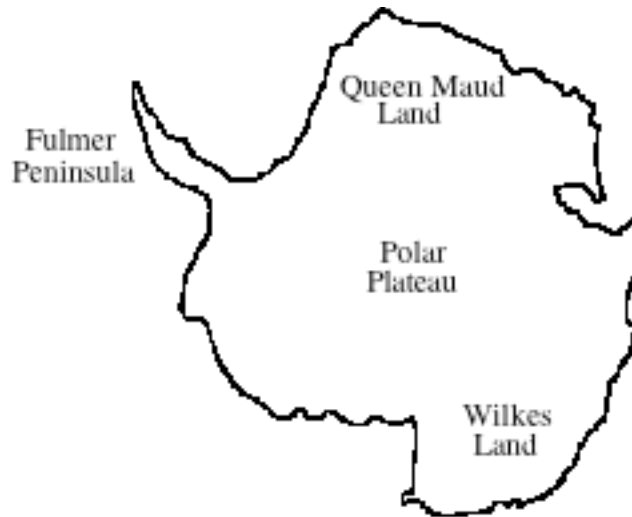


Figure 1: Antarctica

Exploration

In this exploration, you devise methods for distributing Antarctica's resources and assigning delegates to the Global Congress.

- Table 1 contains population estimates and land areas for the seven global regions in 2096. Use this information to find the population density (in people per square kilometer) for each region.

Table 1: Populations and land areas of global regions

Region	Population	Area (km^2)
Asia	$6.32 \cdot 10^9$	$1.77 \cdot 10^7$
Africa	$1.35 \cdot 10^9$	$1.95 \cdot 10^7$
North America	$8.60 \cdot 10^8$	$1.56 \cdot 10^7$
South America	$6.04 \cdot 10^8$	$1.15 \cdot 10^7$
Europe	$1.04 \cdot 10^9$	$3.18 \cdot 10^6$
Oceania	$5.40 \cdot 10^7$	$5.47 \cdot 10^6$
Sino-Soviet	$5.84 \cdot 10^8$	$1.44 \cdot 10^7$

- b. Assume that Antarctica's natural resources are distributed evenly about its land area. Using a map of the continent, divide it among the seven global regions listed in Table 1 so that each receives a fair share.
- c. In 2096, the Global Congress decides to limit itself to a total of 49 delegates. Determine a method for assigning a specific number of delegates to represent each region based on population.

Discussion

- a. How did you determine the amount of land to allot to each global region?
- b. What problems did you encounter when dividing 49 delegates among the seven regions?
- c. In the United States, the distribution of representatives is referred to as **apportionment**. How does your method of apportionment compare to those suggested by others in your class?

Assignment

- 1.1 In 1996, the United Nations allowed only one representative from each member country.
 - a. Do you agree with this form of representation? Explain your response.
 - b. Suggest an alternative form of representation and describe why you believe it might be more fair.
- 1.2 Since the establishment of the U.S. Constitution, the United States has had two chambers of Congress: the Senate and the House of Representatives. Each state receives two senators, regardless of land area or population. In 1996, the House of Representatives consisted of 435 members, divided according to population. Why do you think the country's founders included both forms of representation?

* * * * *
- 1.3
 - a. Express the values in Table 1 as values with one significant digit.
 - b. Use the values from Part a to find both the total population and the total area of the seven regions.
 - c. Express the population and area of each region as a fraction of the total. Record these fractions in a table similar to Table 1.
 - d. Find the sums of the population fractions and area fractions in Part c.
 - e.
 - 1. Using the information from Part c, divide the 49 representatives to the Global Congress among the seven regions.
 - 2. Compare this apportionment to the one you made in Part c of the exploration.

- 1.4 Juanita wants to give her collection of 957 rare books to four former students: Elizabeth, Bill, Isabella, and Kareem. She plans to divide the collection according to the number of years of education each student has completed after high school.

Elizabeth attended college and graduate school for a total of 10 years, Bill for 4 years, Isabella for 6 years, and Kareem for 7 years. Determine how many books each former student should receive and describe the method you used.

* * * * *

Activity 2

For advice on potential methods of apportionment, the Global Congress consulted an American historian. The historian reported that in 1787, the U.S. Constitution (Article 1, Section 2) set forth some basic rules for the apportionment of delegates to the House of Representatives. These rules included the following:

- representatives will be apportioned based on the population of the state
- populations will be determined by a census every 10 years
- the method to be used will be determined by congressional law every 10 years
- each state will receive at least one representative.

The historian also reported that several methods of apportionment were used during the nation's first 200 years. For example, a method devised by Thomas Jefferson was used from 1790 to 1830.

Exploration

In this exploration, you consider a hypothetical country with four regions. This country must fairly apportion a total of 10 representatives. **Note:** Throughout this module, assume that every region must receive at least one representative.

Historical Note

The **standard divisor** is the ratio of the total population to the total number of representatives.

For example, if the total population is P and the total number of representatives is d , the standard divisor is P/d . Notice that this value also describes the mean number of people per representative.

- a. Table 2 shows the population of each of the four regions. Use this information to find the standard divisor.

Table 2: Populations of four regions

Region	Population
A	813
B	652
C	1385
D	95
Total	2945

- b.
1. To determine the approximate number of representatives to assign to a region, divide its population by the standard divisor. This value is known as the **standard quota** for the region.
 2. Round each standard quota to the nearest hundredth and record it in a table with headings like those in Table 3.

Table 3: Populations of four regions

Region	Population	Standard Quota
A	813	2.76
B	652	
C	1385	
D	95	
Total	2945	

3. Find the sum of the standard quotas.
- c.
1. Use the standard quotas to assign an appropriate number of representatives to each region. (Remember that each region must receive at least one representative.)
 2. Determine the total number of representatives assigned. Did you allot all 10 representatives?
- d. Each of the historical methods of apportionment deals with the decimal portions of the standard quotas in a different way. The Jefferson method of apportionment rounds each standard quota down to the previous whole number.
1. Use the Jefferson method to round each standard quota in Table 3.
 2. Use the rounded quotas to assign an appropriate number of representatives to each region. (Remember that each region must receive at least one representative.)
 3. Determine the total number of representatives assigned. Did you allot all 10 representatives?

- e. When the total number of representatives assigned is different than the number available, the Jefferson Method states that the standard divisor must be changed.

Raise or lower the value of the standard divisor until the total number of representatives assigned is 10.

Discussion

- a. The standard quota is the basis for many methods of apportionment. Describe a mathematical expression for the standard quota in terms of the total population (P), the total number of representatives (d), and the population of the region (R).
- b. What problems arise when using the standard quota to assign a number of representatives to each region?
- c. Does the method you used in Parts **a–c** of the exploration assign representatives according to the guidelines set forth in the U.S. Constitution?
- d. Describe how technology could be used to round standard quotas according to the Jefferson method.
- e. Describe how technology could be used to modify the standard divisor according to the Jefferson method.

Assignment

- 2.1 Suppose that the country described in the exploration decides to add two more representatives to its governing body. This results in a total of 12 representatives. Use the Jefferson method to assign the appropriate number of representatives to each region described in Table 2.
- 2.2 In 1842, Congress replaced the Jefferson method of apportionment with one proposed by Daniel Webster. The Webster method was used for the remainder of the 1840s and again from 1900 to 1940. Although the two methods have some similarities, the Webster method rounds standard quotas to the nearest integer.
 - a. Round the standard quotas in the table below using the Webster method.

Region	Population	Standard Quota
A	813	2.76
B	652	2.21
C	1385	4.70
D	95	0.32

- b. Use the rounded quotas to assign a number of representatives to each region. (Remember that each region must receive at least one representative.)
- c. When the total number of representatives assigned does not equal the number available, the Webster method—like the Jefferson method—modifies the standard divisor. Adjust the standard divisor until the total number of representatives assigned is 10. Describe the process you used to find a new divisor.

2.3 After the 1850 census, Congress adopted a method of apportionment based on the work of Alexander Hamilton. The Hamilton method determines the standard divisor, calculates the standard quota for each state, then rounds each quota down to the previous whole number.

If the total number of representatives assigned is less than the number available, the decimal parts of the standard quotas are ranked from greatest to least. This ranking is used to assign the remaining seats in the House of Representatives. The state with the greatest decimal part receives the first extra seat, the state with the second greatest receives the second extra seat, and so on, until all extra seats are assigned.

- a. Use the Hamilton method to apportion 10 representatives among the four regions in Table 2.
- b. For each region, determine the number of people represented by each delegate.

2.4

- a. Write a paragraph comparing the Jefferson, Hamilton, and Webster methods of apportionment.
- b. Compare the number of representatives received by each region in Table 2 using the three methods.

2.5 As described in Activity 1, the Global Congress for Peace has a total of 49 delegates. Use each of the following methods of apportionment to assign delegates to the seven regions in Table 1.

- a. the Jefferson method
- b. the Webster method
- c. the Hamilton method

* * * * *

2.6 Suppose that the country described in the exploration adds 10 more representatives to its governing body, for a total of 20. Use each of the following methods of apportionment to assign representatives to the four regions in Table 2.

- a. the Jefferson method
- b. the Webster method
- c. the Hamilton method

- 2.7** In 2096, scientists estimate that the total amount of fresh water available from the Antarctic region is $1.58 \cdot 10^{10}$ kL (kiloliters). The Global Congress wants to apportion this resource among the seven regions in Table 1.
- The water will be allotted in units of 1 million kiloliters ($1 \cdot 10^6$ kL). Determine the number of units of this size in the total amount of water available.
 - The total population of the seven regions is approximately $1.08 \cdot 10^{10}$. To determine the standard divisor in this situation, divide the total population by the total number of $1 \cdot 10^6$ kL units. Round the divisor to the nearest whole number.
 - Using the result from Part **b**, find the standard quota for each region.
 - Determine the allotment of fresh water for each region using either the Jefferson or the Webster method of apportionment.

* * * * *

Research Project

To learn more about the history of apportionment in the United States, complete either Part **a** or Part **b** below.

- The following table shows the apportionment methods used in the United States from 1790 to 1990. (In 1941, the size of the U.S. House of Representatives was fixed at 435.) Select one of the entries in the table and write a report describing its historical circumstances.

Census Years	Apportionment Method Used
1790–1830	Jefferson
1840	Webster
1850	Hamilton
1860–1870	Hamilton (modified)
1880–1890	Hamilton
1900–1910	Webster
1920	No new apportionment
1930	Webster
1940–1990	Huntington

- Identify an event in U.S. history that was affected by apportionment. Describe the event and explain how apportionment affected its occurrence or outcome.

Activity 3

The method of apportionment for the U.S. House of Representatives has been the subject of frequent debate. In 1929, Congress mandated that this apportionment be done in one of three ways: by applying the method used for the previous apportionment, by applying the Webster method, or by applying a method of equal proportions.

The method of equal proportions was suggested by Joseph Hill in 1911 and presented to Congress by Edward Huntington. When Hill designed this method, he decided that the number of people represented by each delegate was important.

Discussion 1

- a. In the apportionment of 1990, 434 of the 435 representatives to the U.S. House were assigned with relative ease. The lone remaining seat, however, was coveted by several states. Massachusetts, with a population of 6,029,050, had already been assigned 10 representatives. The state of Washington, with a population of 4,887,945, had been assigned 8 representatives.

Given that M is the population of Massachusetts in 1990 and m its number of seats in the U.S. House, the number of people represented by each delegate can be described by the ratio M/m .

How could you use similar notation to describe the number of people represented by each delegate for the state of Washington?

- b. Using the ratios described in Part a, which state—Massachusetts or Washington—was better represented in the U.S. House in 1990? Justify your response.
- c. To determine which state should receive an unassigned delegate, Hill used what he referred to as **relative difference**. When Huntington presented Hill's method of apportionment to Congress, he defined relative difference using an expression like the one below, where W/w is greater than M/m :

$$\frac{\frac{W}{w} - \frac{M}{m}}{\frac{M}{m}}$$

Use this expression to find the relative difference between Massachusetts and Washington in 1990.

- d. What would happen to the relative difference if M/m were greater than W/w ?

Exploration

In the method of apportionment proposed by Huntington and Hill, relative difference is used to decide which state has priority for receiving another delegate. This priority is determined by minimizing the relative difference. In this exploration, you examine the relative differences among several states.

- a. In the 1990 apportionment, Massachusetts initially received 10 representatives, while Washington initially received 8 representatives. One unassigned seat remained.
 1. Determine the relative difference that occurs when Massachusetts receives the additional seat.
 2. Determine the relative difference that occurs when Washington receives the additional seat.
- b. In order to minimize the effect of adding another representative, would you give the additional seat to Massachusetts or to Washington?
- c. In 1988, New Jersey had 14 seats in the U.S. House. During the 1990 reapportionment, the state was initially assigned only 13 representatives. Naturally, many of New Jersey's 7,748,634 citizens felt that they should receive the remaining seat.
 1. Determine the relative difference between New Jersey and Washington if New Jersey receives the additional seat.
 2. Determine the relative difference between New Jersey and Washington if Washington receives the additional seat.
 3. Determine the relative difference between New Jersey and Massachusetts if New Jersey receives the additional seat.
 4. Determine the relative difference between New Jersey and Massachusetts if Massachusetts receives the additional seat.
- d. In order to minimize the effect of adding another representative, would you give the additional seat to Massachusetts, Washington, or New Jersey?
- e. When determining which of three states should receive an additional representative, you calculated a relative difference six times.

How many times would you have to calculate relative difference to determine priority among each of the following:

 1. four states?
 2. five states?

Discussion 2

- a. In the following expression, M represents the population of Massachusetts in 1990 and m its number of seats in the U.S. House, while W represents the population of Washington and w its number of seats in the house. What does the entire expression represent?

$$\frac{\frac{W}{w} - \frac{M}{m+1}}{\frac{M}{m+1}}$$

- b. Using M , m , W , and w , describe an expression for the relative difference when Washington receives an additional House seat.
- c. In Part a of the exploration, you determined two relative differences: one when Massachusetts received the additional seat, and one when Washington received it.
1. Use an inequality to describe the relationship between these two relative differences.
 2. How could you express this inequality using M , m , W , and w ?

Assignment

- 3.1 Expressions like the one described in Part a of Discussion 2 are sometimes called **complex fractions**. Simplify each of the following complex fractions to a simple fraction by performing the indicated operations.

a. $\frac{\frac{3}{5}}{\frac{4}{7}}$

b. $\frac{\frac{2}{5} + \frac{1}{4}}{\frac{4}{3}}$

c. $\frac{\frac{6}{5} + \frac{9}{4}}{\frac{2}{3}}$

d. $\frac{\frac{4}{3} - \frac{2}{x+1}}{\frac{2}{x+1}}$

- 3.2 In Part c of Discussion 2, you used the variables M , m , W , and w to write the following inequality.

$$\frac{\frac{W}{w} - \frac{M}{m+1}}{\frac{M}{m+1}} > \frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}}$$

- a. 1. Explain why the fraction on the left-hand side of the inequality can be rewritten as shown below.

$$\frac{\frac{W}{M}}{\frac{w}{m+1}} - \frac{\frac{M}{M}}{\frac{m+1}{m+1}}$$

2. Explain why this expression simplifies to the following:

$$\frac{\frac{W}{M}}{\frac{w}{m+1}} - 1$$

- b. Use the method described in Part a to simplify the right-hand side of the inequality.
- c. The entire inequality can be simplified further by adding 1 to both sides. Write the simplified inequality.
- d. The inequality can be simplified again by multiplying both sides by the common denominator of the remaining two complex fractions. However, if the common denominator is negative, this multiplication would change the direction of the inequality.
1. Is the common denominator shown below positive or negative? Explain your response.

$$\left(\frac{M}{m+1}\right)\left(\frac{W}{w+1}\right)$$

2. Multiply both sides of the inequality by this common denominator and write the simplified inequality.
- e. The simplified inequality you wrote in Part d should now contain two fractions with numerators W^2 and M^2 , respectively.
1. Determine whether each of these fractions is positive or negative.
2. Describe how the inequality can be rewritten so that the numerators are W and M .
3. Rewrite the inequality so that the numerators are W and M . The values on each side of this inequality are referred to as **priority numbers**.
4. Why was it important to determine whether the fractions were positive or negative before completing Step 3?

- f. In Parts a–e, you demonstrated that these inequalities are equivalent:

$$\frac{\frac{W}{w} - \frac{M}{m+1}}{\frac{M}{m+1}} > \frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}} \qquad \frac{W}{\sqrt{w(w+1)}} > \frac{M}{\sqrt{m(m+1)}}$$

1. Verify that each inequality is true for the 1990 values of M , m , W , and w . (Massachusetts had a population of 6,029,050 and 10 representatives. Washington had a population of 4,887,945 and 8 representatives.)
2. Which inequality do you think is easier to use? Explain your response.

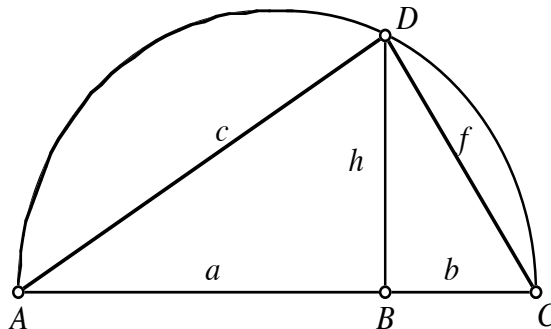
Mathematics Note

The **geometric mean** of two positive numbers a and b is \sqrt{ab} . For example, the geometric mean of 4 and 5 is $\sqrt{4 \cdot 5} = \sqrt{20}$ or about 4.47.

- 3.3
- a. Consider the arithmetic sequence 0, 2, 4, 6, 8. The arithmetic mean of the terms 0 and 8 is 4, while the arithmetic mean of 2 and 8 is 4. Describe how the arithmetic mean is calculated.
 - b. Consider the geometric sequence 2, 4, 8, 16, 32. The geometric mean of the terms 2 and 8 is 4, while the geometric mean of 2 and 16 is $\sqrt{32}$. Determine the geometric means of each of the following:
 1. 8 and 32
 2. 4 and 32
- 3.4
- a. Where did the geometric mean occur in your comparison of relative differences for Massachusetts and Washington?
 - b. How would you use priority numbers to determine which state should receive the additional delegate?

* * * * *

- 3.5 The geometric mean of a and b also can be described as the height h of the triangle circumscribed in the semicircle below.



- a. In the diagram, the measures of $\angle ADC$, $\angle ABD$, and $\angle CBD$ are all 90° . Use similar triangles to show that $h = \sqrt{ab}$.

- b. 1. If $a = 5.1$ cm and $b = 2.1$ cm, what is the value of h ?
2. If $a = 7.4$ cm and $h = 4.8$ cm, what is the value of b ?
3. If $b = 4.0$ cm and $h = 5.9$ cm, what is the value of a ?

* * * * *

Activity 4

As you observed in Activity 3, when the number of states vying for an additional representative increases, the number of comparisons required also increases. The method of equal proportions reduces the number of calculations necessary by assigning priority numbers based on relative differences.

Since 1929, the method of equal proportions has become known as the Huntington method. Using this method, each state receives one representative. Priority numbers are then used to apportion the remaining House seats.

Exploration 1

Table 4 shows the populations of six regions in a hypothetical country. These six regions share a total of 45 representatives to the country's governing body.

Table 4: Populations and representatives for six regions

Region	Population	Rounded Quota	(Rounded Quota) + 2
A	264		
B	481		
C	330		
D	1652		
E	1442		
F	477		
Total	4646		

In this exploration, you use the Huntington method to assign an appropriate number of representatives to each region.

- a. In order to determine how many priority numbers to calculate, you must estimate the number of representatives each region will receive. To make this estimate, complete Steps 1 and 2 below.
 1. Determine the standard quota for each region and round each one up to the next integer. Record the rounded quotas in Table 4.
 2. Since the rounded quota may be less than the number of representatives assigned, add 2 to each rounded quota and record these values in Table 4. This will ensure that enough priority numbers are calculated for each region.

Historical Note

The **method of equal proportions** is based on a comparison of each state or region's priority numbers. These **priority numbers** are determined according to the following formula:

$$\text{priority no.} = \text{population} \cdot \frac{1}{\sqrt{(\text{current no. of reps.})(\text{possible no. of reps.})}}$$

Using this method, states or regions may have several priority numbers, depending on whether one or more representatives could be added. For example, if a region can be considered for nine representatives, nine priority numbers must be calculated (one for each representative).

In the apportionment of 1990, the lone remaining seat in the U.S. House was to be assigned to either Massachusetts, with 10 representatives and a population of 6,029,050, or the state of Washington, with 8 representatives and a population of 4,887,945. Using the method of equal proportions, the priority number for Washington's 9th representative was found as follows:

$$\begin{aligned}\text{priority no.} &= 4,887,945 \cdot \frac{1}{\sqrt{8(9)}} \\ &= 4,887,945 \cdot 0.11785113 \\ &= 576,049.843\end{aligned}$$

In a similar manner, the priority number for Massachusetts' 11th representative was found as shown below:

$$\begin{aligned}\text{priority no.} &= 6,029,050 \cdot \frac{1}{\sqrt{10(11)}} \\ &= 6,029,050 \cdot 0.095346259 \\ &= 574,847.362\end{aligned}$$

Since the priority number for Washington was greater than the priority number for Massachusetts, Washington received the additional seat in the U.S. House.

- b.** In Activity 3, you assigned representatives by comparing relative differences. To assign 45 representatives among 6 regions by this method, you would have to make 410 comparisons.

How many priority numbers do you need to calculate to assign these 45 representatives using the method of equal proportions?

Discussion 1

- a. The partial spreadsheet in Table 5 below shows one way to calculate priority numbers for the six regions. Why are the values in each row of column C one more than the corresponding values in column B?

Table 5: Calculating priority numbers

	A	B	C	D	E
1	Region	Current No. of Reps.	Possible No. of Reps.	Pop.	Priority Number
2	A	1	2	264	186.68
3	A	2	3	264	107.78
4	A	3	4	264	76.21
5	A	4	5	264	59.03
6	B	1	2	481	340.12
7	B	2	3	481	196.37
8	B	3	4	481	138.85
9	B	4	5	481	107.55
10	B	5	6	481	87.82
11	B	6	7	481	74.22
12	C	1	2	330	233.35
13	C	2	3	330	134.72
⋮	⋮	⋮	⋮	⋮	⋮

- b. The estimated number of priority numbers needed for each region was based on the standard quota.
- Why have four priority numbers been calculated for region A?
 - How many priority numbers should be calculated for region C?
- c. Describe a spreadsheet formula that could be used to determine the priority numbers in column E.
- d. Which region shown in the partial spreadsheet has the highest priority for receiving the next representative? Do you think this will be the highest priority number among all six regions?

Exploration 2

- a. Complete the partial spreadsheet given in Table 5.
- b. According to the Huntington method, each of the six regions automatically receives 1 representative. Use priority numbers to assign the remaining 39 representatives.

Discussion 2

- a. Describe how the priority numbers for Massachusetts and Washington could be expressed using the variables M , m , W and w .
- b. In Problem 3.2, you used the following inequality to compare relative differences for Massachusetts and Washington:

$$\frac{W}{\sqrt{w(w+1)}} > \frac{M}{\sqrt{m(m+1)}}$$

Compare this inequality to the expressions you described in Part a.

- c. By comparing the results of an apportionment with the corresponding standard quotas, some critics have found fault with the Huntington method. Compare the number of representatives assigned by the Huntington method with the standard quota for each region in Table 6. What potential drawbacks do you observe?

Table 6: Standard quotas for six regions

Region	Population	Standard Quota
A	264	2.557
B	481	4.659
C	330	3.196
D	1652	16.001
E	1442	13.967
F	477	4.620
Total	4646	

Assignment

- 4.1 As described in Activity 2, you can assign representatives using the Hamilton method of apportionment by completing the following steps:
- finding the standard divisor
 - calculating the standard quota
 - rounding each quota down
 - assigning any remaining representatives by ranking the decimal parts of the standard quotas from greatest to least.
- a. Use the Hamilton method to apportion 45 representatives among the six regions in Table 4.
- b. Compare your results in Part a with the apportionment made using the Huntington method.

- 4.2** Table 4 in Exploration 1 shows the populations of six regions. Suppose that the legislature for these regions decides to increase the total number of representatives from 45 to 50. Use the Huntington method of apportionment to assign the additional seats.
- 4.3** As described in Activity 1, the Global Congress consists of 49 representatives from the seven regions described in the table below. Use the Huntington method to assign an appropriate number of representatives to each region.

Region	Population	Area (km²)
Asia	$6.32 \cdot 10^9$	$1.77 \cdot 10^7$
Africa	$1.35 \cdot 10^9$	$1.95 \cdot 10^7$
North America	$8.60 \cdot 10^8$	$1.56 \cdot 10^7$
South America	$6.04 \cdot 10^8$	$1.15 \cdot 10^7$
Europe	$1.04 \cdot 10^9$	$3.18 \cdot 10^6$
Oceania	$5.40 \cdot 10^7$	$5.47 \cdot 10^6$
Sino-Soviet	$5.84 \cdot 10^8$	$1.44 \cdot 10^7$

* * * * *

- 4.4** As shown in Table 4, the population of region D is 1652, while the population of region E is 1442. Suppose that one person from region E moves to region D. Using the Huntington method of apportionment, describe how this change will affect the number of representatives assigned to each region. (Assume that the total number of representatives for the six regions remains 45.)

* * * * *

Summary Assessment

Like the U.S. House of Representatives, the Canadian House of Commons is apportioned according to population. A total of 295 members of the House of Commons are divided among 10 provinces and 2 territories. The 1991 population of each of these regions is shown below.

Province	Population
Alberta	2,545,553
British Columbia	3,282,061
Manitoba	1,091,942
New Brunswick	723,900
Newfoundland	568,474
Nova Scotia	899,942
Ontario	10,084,885
Prince Edward Island	129,765
Quebec	6,895,963
Saskatchewan	988,928
Northwest Territories	57,649
Yukon Territory	27,797

Source: Statistics Canada.

1. Use each of the following methods of apportionment to assign representatives to the provinces and territories of Canada.
 - a. the Jefferson method
 - b. the Webster method
 - c. the Hamilton method
 - d. the Huntington method
2. In this situation, which of the four methods of apportionment do you prefer? Justify your selection.

Module Summary

- In the United States, the division and sharing of representatives is referred to as **apportionment**.
- The **standard divisor** is the ratio of the total population to the total number of representatives.
- The **standard quota** for each region is calculated by dividing its population by the standard divisor.
- The **Jefferson method** rounds each standard quota down to the previous integer. Using this method, the standard divisor is adjusted until the sum of the rounded quotas equals the number of available representatives.
- The **Webster method** rounds standard quotas to the nearest integer. Like the Jefferson method, it modifies the standard divisor until the sum of the rounded quotas equals the number of available representatives.
- The **Hamilton method** determines the standard divisor, calculates standard quotas, then rounds each quota down to the previous whole number. If the total number of representatives assigned is less than the number available, the decimal parts of the standard quotas are ranked from greatest to least. This ranking is used to assign any remaining seats. The region with the greatest decimal part receives the first extra seat, the region with the second greatest receives the second extra seat, and so on, until all remaining seats are assigned.
- The **geometric mean** of two positive numbers a and b is $\sqrt{a \cdot b}$.
- The **method of equal proportions**, or **Huntington method**, is based on a comparison of each state or region's priority numbers. These **priority numbers** are determined according to the following formula:

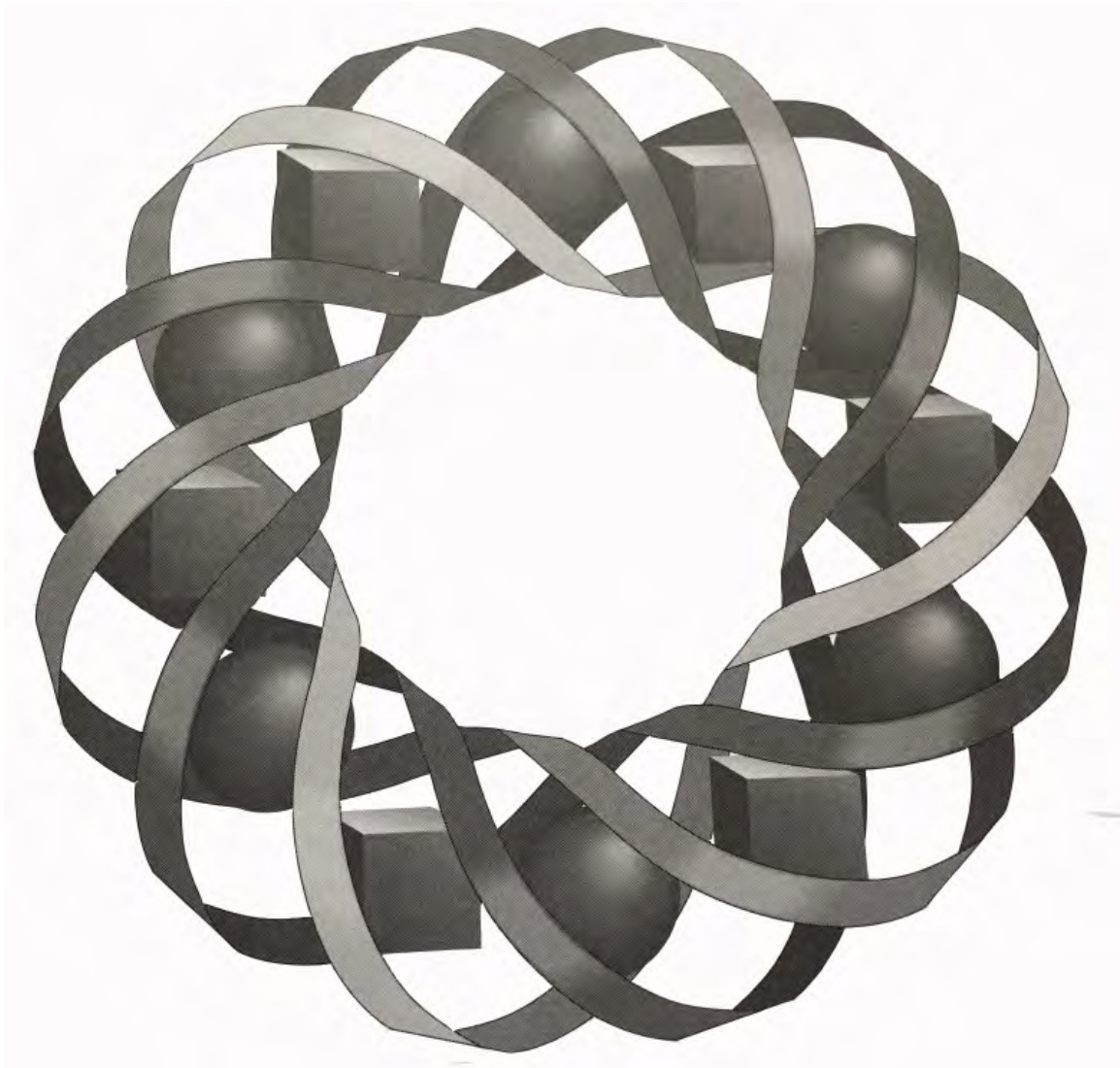
$$\text{priority no.} = \text{population} \cdot \frac{1}{\sqrt{(\text{current no. of reps.})(\text{possible no. of reps.})}}$$

In this formula, a region's population is multiplied by the reciprocal of the geometric mean of two consecutive numbers. Using this method, a region may have several priority numbers, depending on whether one or more representatives could be added.

Selected References

- Balinski, M. L., and H. P. Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. New Haven, CT: Yale University Press, 1982.
- Bennett, S., D. De Temple, M. Dirks, B. Newell, J. M. Robertson, and B. Tyus. "The Apportionment Problem: The Search for a Perfect Democracy." High School Mathematics and Its Applications (HiMAP) Project. Module 8. Arlington, MA: COMAP, 1986.
- Blay, S. K. N. "New Trends in the Protection of the Antarctic Environment: The 1991 Madrid Protocol." *American Journal of International Law* 86(April 1992): 377–399.
- Huntington, E. V. "The Apportionment of Representatives in Congress." *Transactions of the American Mathematical Society* 30 (January 1928): 85–110.
- Litwiller, B. H., and D. R. Duncan. "Apportionment Examples: An Application of Decimal Ordering." *The Mathematics Teacher* 76 (February 1983): 89–91.
- Statistics Canada. *The Canada Year Book 1994*. Ottawa: Statistics Canada, 1994.
- Sullivan, J. J. "Apportionment—A Decennial Problem." *The Mathematics Teacher* 75 (January 1982): 20–25.
- Tannenbaum, P., and R. Arnold. *Excursions in Mathematics*. Englewood Cliffs, NJ: Prentice-Hall, 1992.
- U.S. Congress. *House Reports (Public)*. Vol. 1 (62nd Congress, 1st Session). Washington, DC: U.S. Government Printing Office, 1911. pp. 43–65.
- U.S. Department of Commerce. "Strength in Numbers: Your Guide to 1990 Census Redistricting Data." Washington, DC: Bureau of the Census, 1990.
- U.S. Department of State. *U.S. Treaties and Other International Agreements*. Vol. 12, Part 1. Washington, DC: U.S. Government Printing Office, 1962.

What Are My Child's Chances?



Did you ever wonder why your eyes match your mother's? Or why you have your father's chin? In this module, you study the genetics of several inherited traits.

Patricia Bean • Bonnie Eichenberger



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

What Are My Child's Chances?

Introduction

Genetic diseases are passed on from generation to generation. Many of them can be crippling, even fatal. While effective therapies are available for some of these disorders, others are currently untreatable.

Scientists are developing tests to identify genetic diseases in unborn children and to detect adult carriers. Through prenatal screening, and by determining the probability that a particular disease will appear in certain families, many genetic diseases can be prevented. A genetics counselor is an important part of this effort.

The mathematical science of probability, combined with a knowledge of biology and medicine, can help a genetics counselor determine the chances that a child will inherit a genetic disease.

Activity 1

Genetics is the study of **heredity**, the process by which characteristics are passed from one generation to the next. The biological structures that control heredity are **genes**. Your particular combination of genes determines your eye color and hair color, for example, as well as many of the other characteristics that distinguish you from siblings, parents, and friends.

Most physical traits are determined by several sets of genes. Some, however, such as the presence of free earlobes, are controlled by a single set of genes. Single sets of genes also control blood type and determine the availability of certain hormones and enzymes in the body. How can you calculate a child's chances of having free earlobes or type A blood? In the following activities, you discover how mathematics can help you predict the likelihood of receiving a particular trait.

Exploration

Table 1 describes the characteristics associated with five physical traits. Read the descriptions and determine whether or not you have each trait.

Table 1: Five traits determined by single sets of genes

Trait	Description
Mid-digit hair	The middle section of each finger has hair on it.
Widow's peak	When the hair on the head is pulled back, a distinct point in the hairline can be seen in the center of the forehead.
Free earlobes	The bottom parts of the earlobes are not attached to the side of the head.
Rolled tongue	The tongue can be rolled to form a "U" shape.
Folded hands	When the hands are folded so that fingers interlace, the left thumb falls naturally on top.

To explore your own inherited characteristics, you can use a wheel of traits. Figure 1 shows a wheel for the five traits described above.

- a. Place your finger at the trait in the center of the wheel, "mid-digit hair." If you have this trait, move your finger to the "yes" portion. If you do not have it, move your finger to the "no" portion.

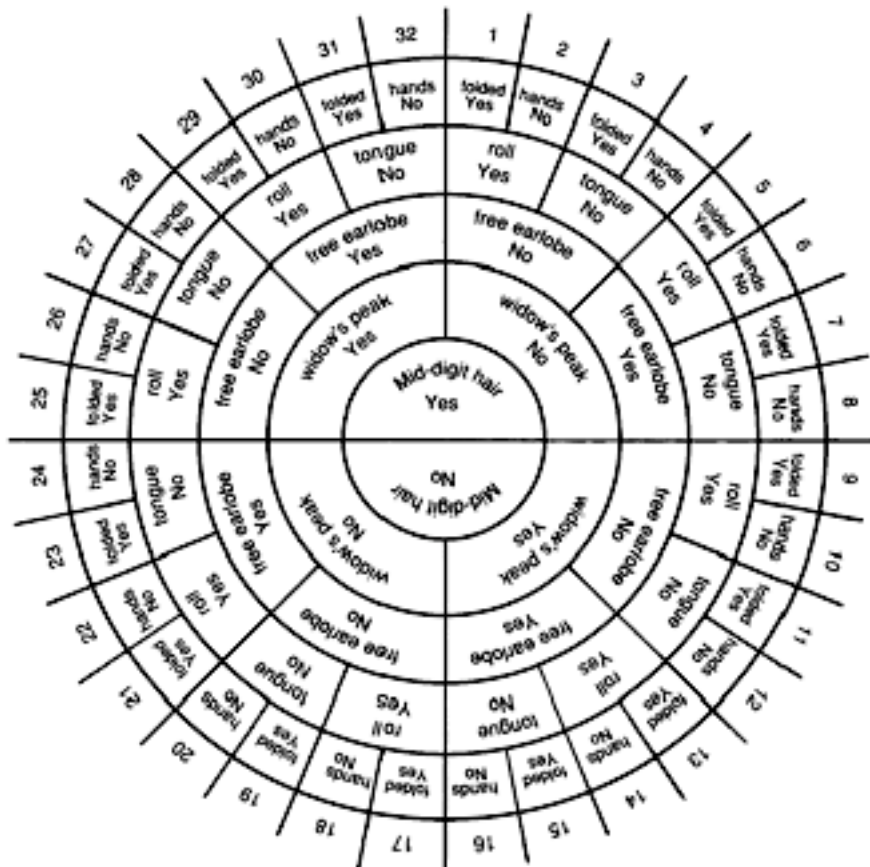


Figure 1: Wheel of traits

- b. From the portion of the ring where you placed your finger, move outward to the next ring and determine whether or not you have the trait “widow’s peak.” Again, move your finger to the appropriate portion of the ring.
- c. Continue this process for the next three traits, working your way outward through each ring of the wheel.
- d. The last portion of the ring where you placed your finger determines your personal number. Record this number.
- e. Organize the class results from Parts a–c in Table 2. **Note:** Save this table for use in Problem 1.1.

Table 2: Class results from wheel of traits

Trait	No. with Trait	No. without Trait
mid-digit hair		
widow’s peak		
free earlobes		
rolled tongue		
folded hands		

- f. The set of all possible outcomes for an experiment is the **sample space**. An **event** is a subset of the sample space.

One method of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event is the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

Using the class data, calculate the experimental probability of having each of the five traits on the wheel.

Discussion

- a. Is each trait on the wheel equally likely to occur in your class? Explain your response.
- b. Why does the wheel in Figure 1 have 32 different personal numbers?
- c. Given the five traits in the wheel in Figure 1, how many different pairs of traits exist?

Mathematics Note

Given any event E , the event that E does not occur is its **complement**. The complement of E can be represented by the symbol E' , read “ E prime” or “ E complement.” The sum of the probabilities of two complementary events is 1.

For example, consider an experiment in which a die is tossed 5 times and the number of sixes is counted. If S represents the event that a six occurs, the event that a six does not occur can be represented by S' . If 3 sixes are obtained, the experimental probability of S is $3/5$, while the experimental probability of S' is $2/5$. The sum of the probabilities for these two complementary events is 1.

- d. Describe how you could use a Venn diagram to show the relationship between two complementary events E and E' .
- e. If you know the experimental probability of having a specific trait, how can you determine the experimental probability of not having that trait?
- f. How could you use the experimental probabilities found in the exploration to estimate the number of students in your school with each trait?

Assignment

- 1.1 a. Use the class data to determine the experimental probability of not having each trait on the wheel. Organize these probabilities in a table like the one below.

Trait	Experimental Probability of Having Trait	Experimental Probability of Not Having Trait
mid-digit hair		
widow's peak		
free earlobes		
rolled tongue		
folded hands		

- b. Using the class data, predict how many students in your school have free earlobes.
 - c. What is the sum of the experimental probabilities of having and not having each trait? Why does this occur?
- 1.2 Suppose that you used the wheel of traits in Figure 1 to collect data from across the nation. Do you think the combination of five traits that occurred most often in your class would occur most often in the national data? Why or why not?

- 1.3 In one class of students, the experimental probability of having a widow’s peak was 0.7. What was the experimental probability of not having a widow’s peak in this class? Explain how you determined your response.
- 1.4 Describe the event that is complementary to having the trait “folded hands.”

* * * * *

- 1.5 In 1991, approximately 2,041,000 fires were reported in the United States. The table below shows how these fires were distributed by type.

Type of Fire	Number Reported
Structure—accidental	542,000
Structure—incendiary device used	62,000
Structure—suspicious origin	36,000
Outside of structure (includes timber, crops, and outside storage)	54,000
Brush and rubbish	806,000
Vehicle	428,000
Other	113,000

Source: U.S. Bureau of the Census.

- a. What is the probability that a fire reported in 1991 was a structure fire?
- b. What percentage of all reported fires in 1991 involved incendiary devices or suspicious causes?
- c. What is the probability that a fire reported in 1991 was not a vehicle fire?
- 1.6 In the United States, heart disease is the most common cause of death for adults. The table below shows the number of deaths due to heart disease for men of various age groups in 1990.

Age (years)	No. in Age Group	No. of Deaths
25–34	21,564,000	3278
35–44	18,009,000	18,585
45–54	12,232,000	46,078
55–64	9,955,000	948,276
65–74	7,907,000	1,716,056
75–84	3,745,000	169,828
over 85	841,000	70,864

Source: U.S. Bureau of the Census.

- a. For men aged between 45 and 54 in 1990, what was the probability of dying from heart disease?
- b. What percentage of the men who died from heart disease in 1990 were between 65 and 74 years old?

- c. For men aged 25 and over in 1990, what was the probability of dying from heart disease?
- d. For men aged between 25 and 34 in 1990, what was the probability of not dying from heart disease?

* * * * *

Activity 2

The genes you inherited from your parents are carried on 23 pairs of **chromosomes**. Figure 2 shows one pair of chromosomes, magnified many thousands of times.



Figure 2: A pair of human chromosomes

Like chromosomes, genes also come in pairs. Each member of a pair is called an **allele** (pronounced “uh leel”). During fertilization, two alleles combine—one from each parent. These two alleles may be alike or different. However, only one trait appears for each pair.

The pair of alleles that determines the presence or absence of a particular characteristic is the **genotype**. The trait that actually occurs is referred to as the **phenotype**. When the alleles in a pair are different, the trait that appears is the **dominant** trait. The other is referred to as the **recessive** trait.

Exploration

In the case of earlobes, there are two possible phenotypes: free earlobes or attached earlobes. Having free earlobes is the dominant trait, while having attached earlobes is the recessive trait.

In genetics, the dominant trait is typically represented by a capital letter, while the recessive trait is represented by the corresponding lowercase letter. If F represents free earlobes and f represents attached earlobes, then the four possible combinations of these two alleles can be written as follows: FF, Ff, fF, and ff.

However, the combinations Ff and fF, which each contain one dominant and one recessive allele, are genetically the same. This means there are only three possible genotypes: FF, Ff, and ff. When the two alleles in a pair are different, the dominant trait masks the recessive one. Since having free earlobes is the dominant trait, the genotypes FF and Ff both result in the phenotype of free earlobes. Only the genotype ff results in the phenotype of attached earlobes.

- a. A parent can have any one of the three possible genotypes: FF, Ff, or ff.
 1. Select two genotypes, one to represent each parent.
 2. List all the genotypes that are possible in their children.
- b. Describe the method you used in Part a to ensure that all possible combinations were identified.

Science Note

The act of combining genes from two parents is called a **cross**. In a cross, each allele in a parent's genotype is assumed to have an equal chance of being passed on to the offspring.

Given the genotypes of both parents for a particular trait, you can use a **Punnett square** to help identify the possible combinations from a cross. For example, Figure 3 shows a Punnett square for the cross of two parents with the same genotype, Ff. The possible genotypes from this cross are FF, Ff, and ff.

		Mother	
		F	f
Father	F	FF	Ff
	f	Ff	ff

Figure 3: Punnett square for the cross Ff × Ff

- c. Draw a Punnett square for the cross you chose in Part a. Identify all the possible genotypes and all the possible phenotypes.

Mathematics Note

If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

The theoretical probability of an event E can be written as $P(E)$.

For example, consider the cross of two parents with the same genotype, Ff. As shown in Figure 3, the sample space contains 4 equally likely outcomes: FF, Ff, Ff, and ff. Since the FF genotype occurs 1 time in the sample space, its theoretical probability can be expressed as $1/4$. In other words, $P(FF) = 1/4$. Similarly, $P(ff) = 1/4$. Since the genotype Ff occurs 2 times in the sample space, $P(Ff) = 2/4$, or $1/2$. The sum of the probabilities for all the possible outcomes in the sample space is 1.

Because the allele for free earlobes (F) is dominant, the genotypes FF and Ff both result in the phenotype of free earlobes. For the cross shown in Figure 3, the probability that a child has free earlobes is $3/4$, while the probability that a child has attached earlobes is $1/4$.

- d. For each genotype identified in Part c, determine the probability that a child of these parents will possess that genotype.
- e. For each possible phenotype identified in Part c, determine the probability that a child of these parents will have that phenotype.

Discussion

- a.
 - 1. In Part c of the exploration, how many different genotypes did you identify?
 - 2. How many different phenotypes did you identify?
- b. Why is it possible for your responses to Part a of the discussion to differ from those of your classmates?
- c. How do the parents' genotypes affect the probability of the child's phenotype?

Assignment

- 2.1** Consider the cross $FF \times ff$, where F represents free earlobes and f represents attached earlobes.
- Draw a Punnett square for this cross.
 - What is the probability of each possible genotype?
 - Describe the phenotype that results for each possible genotype.
 - What is the probability of each possible phenotype?
- 2.2** Repeat Problem **2.1** using the cross $Ff \times ff$.
- 2.3** Consider the cross $Rr \times RR$, where R represents the ability to roll the tongue, a dominant trait.
- Determine the probability of the genotype RR. Justify your response using a Punnett square.
 - Determine the probability of the genotype rr. Defend your response using a tree diagram.
 - What is the probability that a child of these parents has the ability to roll the tongue? Explain your response.
- 2.4** Having a widow's peak is a dominant trait, represented by the capital letter W. Consider the cross $Ww \times ww$, where the father has a widow's peak.
- Which pair of alleles belongs to the father? Explain your response.
 - Using a Punnett square, identify the genotypes that can result from this cross.
 - Determine the probability that a child of this cross does not have a widow's peak.
- 2.5** Consider the cross $Ww \times Ww$, where W represents the trait widow's peak.
- What is the probability that a child of this cross receives one dominant allele (W) and one recessive allele (w)?
 - Why do the two outcomes Ww and WW result in the same phenotype?
 - For a child of this cross, what is the ratio of the probability of having a widow's peak to the probability of not having one?

* * * * *

- 2.6 Both of Deirdre's parents have free earlobes, a dominant gene. Deirdre does not.
- Use Punnett squares to find all the possible genotypes for a child of Deirdre's parents.
 - Determine the probability that a child of Deirdre's parents does not have free earlobes.
- 2.7 Cystic fibrosis is a genetic disease. Its victims lack certain chemicals in the body and typically have severe respiratory and digestive problems. For children in the United States, it is the leading cause of death due to disease.

A baby can be born with cystic fibrosis if both its parents are **carriers** of the disease. Carriers are people who show no signs of the disease but can pass it on to their children. Carriers have the genotype Cc, while people with the disease have the genotype cc.

- Draw a Punnett square that represents a cross between two parents that are carriers of cystic fibrosis.
- What is the probability that a child with cystic fibrosis is born to parents who are both carriers of the disease?
- One out of every 25 people in the United States is a carrier of cystic fibrosis. What are the chances that a healthy person chosen at random in the United States is not a carrier of cystic fibrosis?

* * * * *

Activity 3

In the previous activities, you examined traits one at a time. When genetic counselors advise their clients, however, they must consider the parents' traits as groups. Some characteristics, for example, are associated only with males, while some are associated only with females. A few characteristics are linked to certain other traits regardless of sex. Grouping traits affects the probabilities in each situation. It also affects the way in which those probabilities are calculated.

Discussion 1

- a. Consider a father with the genotype Rr and a mother with the genotype RR, where R represents a dominant trait.
1. Assuming that each allele in a parent's genotype has an equal chance of being passed on to any offspring, what is the probability that a child of this couple will receive the dominant allele from the father?
 2. What is the probability that a child of this couple will receive the dominant allele from the mother?
 3. What is the probability that a child of this couple will have the genotype RR?
- b. If the mother's genotype were Rr, would this change your response to Part a.1?

Mathematics Note

Two events A and B are **independent** if $P(A \text{ and } B) = P(A) \cdot P(B)$. It follows that for independent events A, B, and C, $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$. This definition can be extended to any number of independent events.

Given two independent events, the occurrence of one has no effect on the likelihood of the occurrence of the other. Two events that are not independent are said to be **dependent**.

For example, consider tossing a coin heads up and rolling a 4 on a die. These two events are independent if $P(\text{head and } 4) = P(\text{head}) \cdot P(4)$. The sample space for tossing a coin and rolling a die consists of 12 equally likely outcomes: a head with each of the six faces on the die and a tail with each of the six faces. Only one of these outcomes includes a head and a 4. Therefore, $P(\text{head and } 4) = 1/12$.

The theoretical probability of tossing a head is $1/2$, while the theoretical probability of rolling a 4 is $1/6$. Therefore,

$$P(\text{head}) \cdot P(4) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Since $P(\text{head and } 4) = P(\text{head}) \cdot P(4)$, the two events are independent.

- c. In the cross $Rr \times Rr$, the probability of the genotypes RR and rr is $1/4$, while the probability of Rr is $1/2$. How could you find the probability of each genotype using the probabilities of receiving one allele from the father and one from the mother?

Exploration

- a. Select two of the following five traits: mid-digit hair, widow's peak, free earlobes, rolled tongue, and folded hands.
- b.
 1. Determine the number of students in your class who have only the first trait (A), only the second trait (B), and both the first and second traits (A and B). Record this data in a table with headings like those in Table 3 below.

Table 3: Experimental probabilities of two traits

Trait A:		
Trait B:		
Total Number in Class:		
Trait	Frequency	Experimental Probability
A		
B		
A and B		

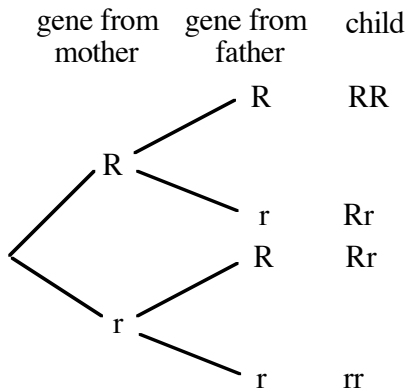
2. Find the experimental probabilities of having trait A, having trait B, and having both A and B. Record these probabilities in Table 3.
- c. Choose another pair of traits from the five traits listed in Part a. Repeat Part b using this pair of traits. Continue this process until you have completed a table for each of the 10 possible pairs of traits.
- d. Experimental probabilities are often used to approximate theoretical probabilities. Judging from the experimental probabilities in your tables from Parts b and c, which pairs of traits appear to represent independent events?

Discussion 2

- a.
 1. Using the information in your tables, do any of the pairs of traits appear to represent independent events? Explain your response.
 2. Do any of the pairs of traits appear to represent dependent events? Explain your response.
- b.
 1. Identify two events that appear to be independent. Describe how you could demonstrate that they are independent.
 2. Identify two events that appear to be dependent. Describe how you could demonstrate that they are dependent.
- c. Given the theoretical probability of having a particular trait, how would you predict the number of students in your class with that trait?

Assignment

- 3.1** Assume that each of the following five traits is independent of the others: mid-digit hair, widow's peak, free earlobes, rolled tongue, and folded hands. Using the class data from the exploration, estimate the probability of having free earlobes and the ability to roll your tongue and no widow's peak.
- 3.2** The tree diagram below shows the possible genotypes for the cross $Rr \times Rr$.



- a. Make a copy of this diagram. On each branch of the tree, write the probability of a child receiving that allele.
 - b. Use the tree diagram to find the probability of each genotype. Describe the method you used.
 - c. Create a Punnett square for the cross $Rr \times Rr$. Use it to calculate the probability of each genotype.
 - d. How do the probabilities you found using the tree diagram compare with the probabilities found using a Punnett square?
- 3.3** Two potential parents both have the genotype Mm for mid-digit hair. The father has the genotype WW for widow's peak, while the mother has the genotype Ww . Assuming that mid-digit hair and widow's peak are independent traits, determine the probability that a child of these parents has both mid-digit hair and a widow's peak.

3.4 **Antigens** are proteins that can activate the immune system. An individual's blood type is determined by the presence or absence of two antigens, A and B, on red blood cells. Type A and type B blood each have one antigen on red blood cells, type O blood contains neither antigen, while type AB contains both.

Blood type is also affected by another antigen, the Rh factor. Blood that contains this factor is referred to as Rh positive (Rh+), while blood that does not contain this factor is referred to as Rh negative (Rh-). The presence or absence of the Rh factor, along with the four basic blood types, results in eight different kinds of blood: A+, B+, AB+, O+, A-, B-, AB-, and O-.

A biology class has collected data on the blood of 26 students. The following table summarizes this data.

Blood Type	No. of Students	Rh Factor	No. of Students
A	10	Rh+	20
B	4	Rh-	6
AB	1		
O	11		

- Determine the experimental probability of each of the following blood types: A, B, AB, and O.
- Determine the experimental probability each of the following: Rh+ and Rh-.
- Assuming that blood type and Rh factor are independent traits, draw a tree diagram that shows the estimated probability of each of the eight possible kinds of blood.

* * * * *

3.5 The presence of curly hair (H) is dominant over straight hair (h). If the parents' genotypes are hh and Hh, determine the probability that their two children will both have curly hair. Assume that the traits of one child are independent of the traits of the other.

3.6 As mentioned in Problem 2.7, 1 out of every 25 people in the United States is a carrier of cystic fibrosis. What is the probability that two people chosen at random are both carriers?

- 3.7** A marketing firm surveyed 5000 people about their favorite sodas and snack foods. The results of the survey are shown in the table below. Each value in the “percentage” column represents the percentage of the survey group who preferred a particular soda or snack food.

Soda	Percentage	Snack Food	Percentage
cola A	30	corn chips	30
cola B	25	potato chips	40
cherry cola	10	pretzels	10
orange	5	variety mix	15
root beer	15	other	5
other	15		

- a. What is the probability that a person chosen at random from this group prefers cola?
 - b. What is the probability that a person chosen at random from this group prefers cola and pretzels?
 - c. What is the probability that a person chosen at random from this group does not prefer root beer?
- 3.8** Huntington’s disease is a rare and deadly genetic disorder. All carriers eventually develop the disease. Since Huntington’s disease may lie dormant until a person is between 35 and 45 years old, many carriers have children before they develop symptoms.
- a. Draw a Punnett square of a cross between a carrier of Huntington’s disease (Hh) and a person who is not a carrier (hh).
 - b. What is the probability that a child from this cross does not carry the disease?
 - c. If the parents from Part **a** have one child who is not a carrier, what is the probability that a second child will carry the disease?
 - d. What is the probability that the parents from Part **a** have two children who are not carriers?

* * * * *

Research Project

Research one of the following genetic diseases: cystic fibrosis, Tay-Sach’s disease, Huntington’s disease, sickle-cell anemia, or phenylketonuria (PKU). Determine its symptoms, the frequency of its occurrence, and the probability that parents pass the disease to their children.

Activity 4

Your individual combination of traits makes you different from every other person on the planet. Some of these inherited traits, however, can automatically prevent you from having some others. In this activity, you discover how the probability of having one trait or another can be affected by the relationship between them.

Exploration

- a. To complete Steps 1 and 2 below, let trait A represent a preference for writing with the left hand and trait B represent a preference for writing with the right hand.
 1. Determine the number of students in your class with trait A only, trait B only, both trait A and trait B, and either trait A or trait B. Record this data in a table with headings like those in Table 4.

Table 4: Experimental probabilities of two traits

Trait A:		
Trait B:		
Total Number in Class:		
Trait	Frequency	Experimental Probability
A		
B		
A and B		
A or B		

2. Find the experimental probabilities of having trait A only, having trait B only, having both trait A and trait B, and having either trait A or trait B. Record these probabilities in your table.
- b. Repeat Part a for each of the following pairs of traits.
 1. blue eyes and dark hair
 2. dark hair and folded hands

Mathematics Note

Given two events A and B, the theoretical probability of either A or B occurring can be found as follows:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For example, suppose event A is drawing a black card from a standard deck of 52 playing cards, and event B is drawing a king from a standard deck of playing cards. Since there are 26 black cards in a standard deck, $P(\text{black}) = 26/52$. Similarly, because there are 4 kings in the deck, $P(\text{king}) = 4/52$. The probability of either A or B occurring can be calculated as shown below:

$$\begin{aligned} P(\text{black or king}) &= P(\text{black}) + P(\text{king}) - P(\text{black and king}) \\ &= \frac{26}{52} + \frac{4}{52} - \left(\frac{26}{52} \cdot \frac{4}{52} \right) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned}$$

- c. Use the formula in the mathematics note to determine $P(A \text{ or } B)$ for each of the following pairs of traits.
1. left-handedness and right-handedness
 2. blue eyes and dark hair
 3. dark hair and folded hands
- d. Compare your results from Part c to the experimental probability of A or B for each pair of traits.

Discussion

- a. Did your results in Part c of the exploration support the formula for determining $P(A \text{ or } B)$? Explain your response.
- b. For which pair(s) of traits in the exploration was the experimental probability of A and B occurring at the same time 0?

Mathematics Note

Two events are **mutually exclusive** if they cannot occur at the same time in a single trial. For two mutually exclusive events A and B, $P(A \text{ and } B) = 0$.

For example, consider the toss of a single coin. In this case, heads and tails are mutually exclusive, since both cannot occur at the same time on a single toss. However, because one person could be both right-handed and have the ability to roll the tongue, these two events are not mutually exclusive.

- c.
 1. In your class data, which pairs of traits were mutually exclusive?
 2. Do you think that these traits are always mutually exclusive? Explain your response.
- d. For two mutually exclusive events A and B, the theoretical probability of A or B can be calculated using either the formula given in the mathematics note or the one shown below:
$$P(A \text{ or } B) = P(A) + P(B)$$
Why are these two formulas equivalent for mutually exclusive events?
- e. Are complementary events always mutually exclusive? Explain your response.
- f. Are mutually exclusive events always complementary? Use an example to support your response.
- g. Draw a Venn diagram that shows the relationship between two mutually exclusive events. How does this diagram show that $P(A \text{ and } B) = 0$?

Assignment

- 4.1 Decide whether or not each of the following pairs of traits represents two mutually exclusive events. Justify your response for each pair.
 - a. having the ability to roll the tongue and not having this ability
 - b. having a widow's peak and having curly hair
 - c. having two blue eyes and having two brown eyes
- 4.2 Familial hypercholesterolemia causes high levels of cholesterol in the blood and can lead to clogging of the arteries at a young age. This genetic disorder is caused by the dominant allele (D). People with the genotype DD are severely affected, those with the genotype Dd are mildly affected, and those with the genotype dd are not affected at all.

Hank and Erma both have the genotype Dd. Use two different methods to find the probability that a child of this couple is either mildly affected or not affected at all.

- 4.3** The Norgaard family has five children. Both parents have the genotype Mm for mid-digit hair. The father's genotype for widow's peak is WW, while the mother's is Ww.
- What is the probability that a child of these parents has either mid-digit hair or a widow's peak?
 - How many of the family's five children would you expect to have either mid-digit hair or a widow's peak? Justify your response.

* * * * *

- 4.4** A survey of 20 students reported the following hair colors: 10 brown, 6 black, 3 blond, and 1 red. If one student is randomly selected from this group, what is the probability that the student has blond hair or red hair? (Assume that each student has only one hair color.)
- 4.5** In 1990, the Internal Revenue Service audited 0.8% of all tax returns. In that same year, Carla's Accounting Service prepared tax returns for 250 clients. Assume that each taxpayer had the same chance of receiving an audit.
- What was the probability of receiving an audit in 1990?
 - Estimate the number of Carla's clients who received an audit.
 - Eli and Marc both used Carla's Accounting Service in 1990. Determine the probability of each of the following.
 - both Eli and Marc received audits
 - either Eli or Marc received an audit

- 4.6** Umberto plans to buy a chain with a tumbler lock to secure his bicycle. The first lock he examines has three tumblers, each of which can be set to any digit from 0 to 9. Only one combination of three numbers will open the lock.
- What is the probability of guessing the correct combination on the first attempt?
 - The second lock Umberto examines has five tumblers. Is a five-tumbler lock more secure than a three-tumbler lock? Explain your response.

* * * * *

Activity 5

Like hair color and free earlobes, your gender is an inherited trait. As shown in Figure 4, the sex chromosomes for a female are indicated by XX, while those for a male are indicated by XY. Each parent contributes one chromosome to the child's pair. The father can contribute either an X or a Y chromosome; the mother can contribute only an X chromosome.

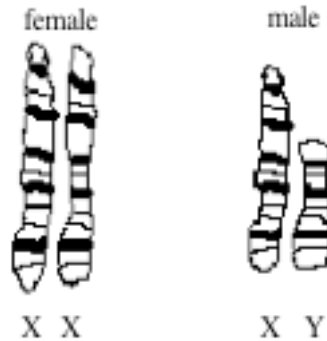


Figure 4: Human sex chromosomes

Exploration

In this exploration, you use a simulation to investigate the probability that a combination of sex chromosomes results in a female child. You then use another simulation to examine the probability that a family with two children will include one male child and one female child.

- a. Assume that the probability of receiving either allele from either parent is the same. Predict the probability that a combination of sex chromosomes results in a female child.
- b.
 1. To model the mother's chromosomes, label both sides of a coin "X."
 2. To model the father's chromosomes, label a second coin "X" on one side and "Y" on the other.
- c.
 1. Simulate the combination of male and female chromosomes by tossing the two coins. Determine and record the gender of the resulting combination.
 2. Repeat Step 1 19 more times.
 3. Determine the experimental probability that a combination of male and female chromosomes results in a female child.
- d. Use a Punnett square or tree diagram to determine the theoretical probability that a combination of male and female chromosomes results in a female child.
- e. Predict the probability that two independent combinations of male and female chromosomes result in one male and one female child.

- f.
 1. To simulate two independent combinations of sex chromosomes, toss the two coins twice. Determine and record the genders of the resulting pair of children.
 2. Repeat Step 1 19 more times.
 3. Determine the experimental probability that two independent combinations of male and female chromosomes result in one male and one female child.
- g. Determine the theoretical probability that two independent combinations of male and female chromosomes result in one male and one female child.
- h. Compile the class results from Parts c and f. Using the class results, determine the experimental probability of each of the following.
 1. A combination of male and female chromosomes results in a female child.
 2. Two independent combinations of male and female chromosomes results in one male and one female child.

Discussion

- a. When tossing two coins, why is the outcome for each coin independent of the other?
- b. In Part a of the exploration, did you predict that a male child and a female child are equally likely? Why or why not?
- c.
 1. How did the experimental probabilities you calculated using the results of 20 trials compare with the experimental probabilities calculated using the class results?
 2. Which probabilities were closer to the theoretical probabilities?
- d. In Part f of the exploration, you simulated two combinations of sex chromosomes by tossing the coins twice.
 1. How many outcomes are there in the sample space for this experiment?
 2. How would the size of the sample space change if the experiment simulated three combinations of sex chromosomes?
 3. How would the size of the sample space change if the experiment simulated n combinations of sex chromosomes?

Assignment

- 5.1**
- a. Consider a couple with three children. Draw a tree diagram that shows all the possible outcomes for the genders of the children.
 - b. Assume that having a boy and having a girl are equally likely outcomes. Determine the probability that the couple has a girl, a girl, and a boy, in that order.
 - c. Determine the probability that exactly two of the children are girls.
 - d. Are your responses to Parts **b** and **c** different? Why or why not?
- 5.2** A family has four boys. Assuming that the birth of each child is an independent event, is their fifth child more likely to be a girl or a boy? Explain your reasoning.

- 5.3** To find the probability that a family with five children will have five girls, a student used the following formula:

$$P(5 \text{ girls}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$$

Do you agree with this student's method? Why or why not?

* * * * *

- 5.4** After analyzing its customer data, a rental car agency found that approximately 2% of its customers were involved in accidents.
- a. What is the probability that a customer chosen at random is involved in an accident?
 - b. If 5 different customers are chosen at random, what is the probability that all 5 are involved in accidents? Explain your response.
 - c. If the agency rents 250 cars in one day, how many should they expect to be involved in accidents?
- 5.5** In one lottery game, players may choose any sequence of six digits. A computer generates the winning number by randomly selecting one digit at a time, in order from first to sixth.
- a. What is the probability that the fourth digit of the winning number is 7?
 - b. What is the probability that the winning number is 682399?
 - c. What is the probability that the winning number is 123456?
 - d. What is the probability that the winning number is 000000?
- 5.6** In the game of craps, a player rolls two dice. A roll of 7 or 11 results in a win.
- a. What is the probability of winning on one roll? Explain your response.
 - b. What is the probability of winning on three rolls in a row? Explain your response.

* * * * *

Summary Assessment

In order to ensure a safe and steady supply of blood for medical use, many hospitals maintain banks of donated blood. The human body normally contains about 5 L of blood. Each volunteer donates about 0.5 L. In healthy people, that 0.5 L is readily replenished.

Imagine that you work for a hospital blood bank. The blood bank likes to have at least 5 L of each blood type available—enough for one complete transfusion, if necessary. The hospital is planning to make a public service announcement requesting more donors. Your job is to determine which blood types are most needed.

1. An individual's blood type—O, A, B, or AB—is determined by a combination of the parents' genes, one allele from each parent. The allele for type O blood is always recessive, while the alleles for types A and B are always dominant. The combination of alleles A and B produces the blood type AB.

 - a. List all the possible genotypes for each blood type.
 - b. Assuming that each possible genotype is equally likely to occur in the population, determine the probability of having each blood type.
2. The blood bank currently has a total of 75 L of blood. How many liters of each blood type would you expect them to have?
3. The presence or absence of the Rh factor also affects blood type. In the United States, about 85% of the population is Rh positive (Rh+); the rest is Rh negative (Rh-).

The presence or absence of the Rh factor, along with the four basic blood types, results in eight different kinds of blood: A+, B+, AB+, O+, A-, B-, AB-, and O-. How many liters of each would you expect the blood bank to have?

4. In its public service announcement, which types of blood should the hospital say are most needed? Explain your response.

Module Summary

- The set of all possible outcomes for an experiment is the **sample space**. An **event** is a subset of the sample space.
- One method of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event is the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

- Given any event E, the event that E does not occur is its **complement**. The complement of E can be represented by the symbol E' , read “E prime” or “E complement.” The sum of the probabilities of two complementary events is 1.
- If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

The theoretical probability of an event E can be written as $P(E)$.

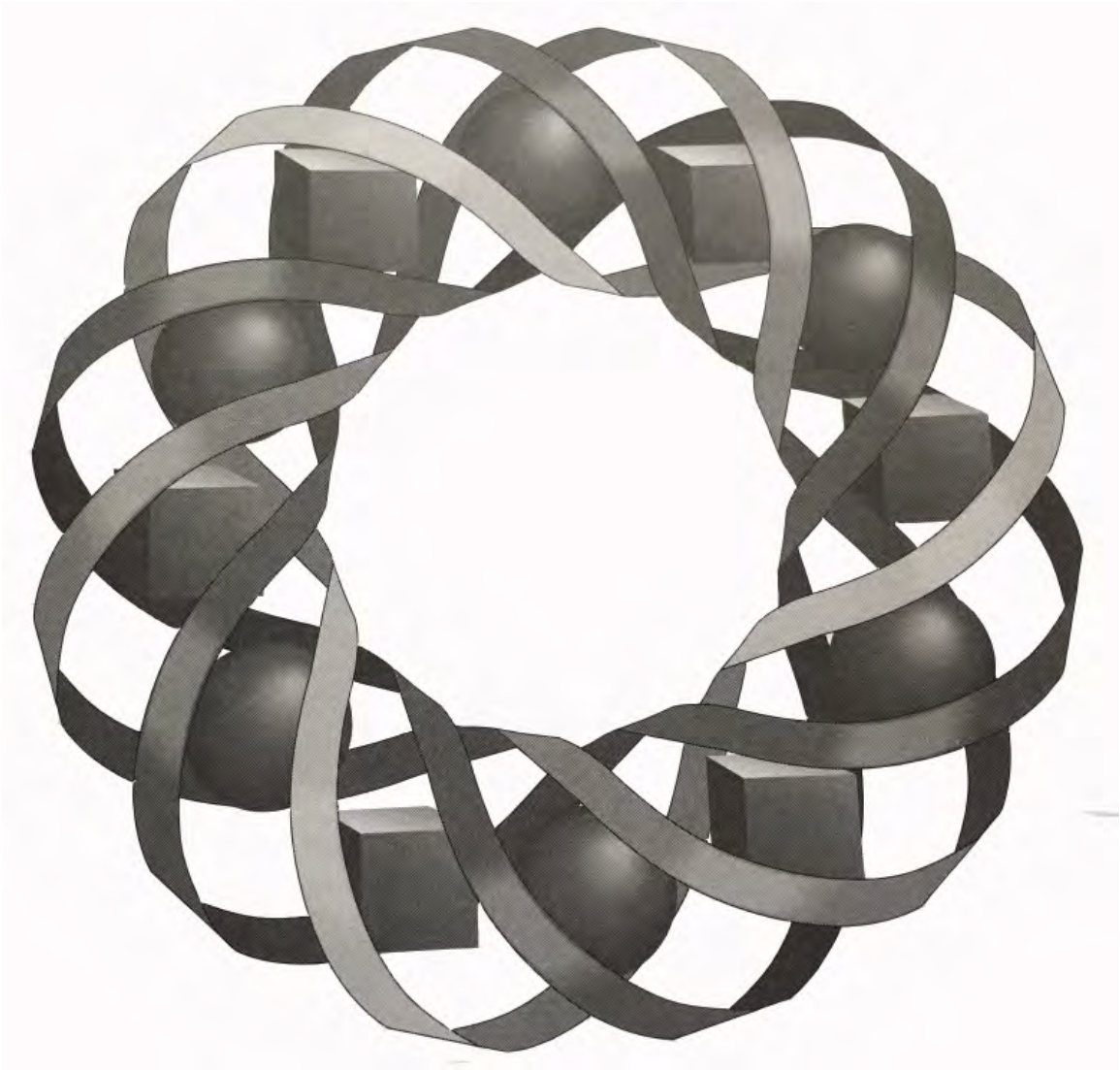
- Two events A and B are **independent** if $P(A \text{ and } B) = P(A) \cdot P(B)$.
It follows that for independent events A, B, and C,
 $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$. This definition can be extended to any number of independent events.
- Two events that are not independent are said to be **dependent**.
- Given two events A and B, the theoretical probability of either A or B occurring can be found as follows:
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
- Two events are **mutually exclusive** if they cannot occur at the same time in a single trial. For two mutually exclusive events A and B, $P(A \text{ and } B) = 0$.
- **Genetics** is the study of **heredity**, the process by which characteristics are passed from one generation to the next. The biological structures that control the heredity of traits are **genes**.
- Human genes are carried on 23 pairs of **chromosomes**. Genes also come in pairs; each member of a pair is an **allele**.

- When the alleles in a pair are different, the trait that appears is called the **dominant** trait. The other is called the **recessive** trait.
- The pair of alleles that determines the presence or absence of a particular characteristic is the **genotype**. The trait that actually occurs is referred to as the **phenotype**.
- The act of combining genes from two parents is called a **cross**.
- Given the genotypes of both parents for a particular trait, you can use a **Punnett square** to help identify the possible combinations from a cross.

Selected References

- Goodspeed, J. K. "But My Parents Both Have *Brown Eyes*." *Science and Children* 22 (January 1985): 9–12.
- Grollman, S. *The Human Body: Its Structure and Physiology*. London: The Macmillan Co., 1969.
- Ince, S. "Genetic Counseling." White Plains, NY: March of Dimes Birth Defects Foundation, 1987.
- Mange, A. P., and E. J. Mange. *Genetics: Human Aspects*. Sunderland, MA: Sinauer Associates, 1990.
- Rosenberg, N. S. "Genetic Counseling." Undergraduate Mathematics and Its Applications (UMAP) Project. Module 456. Arlington, MA: COMAP, 1987.
- Schraer, W. D., and H. J. Stoltze. *Biology: The Study of Life*. Needham, MA: Prentice-Hall, 1991.
- U.S. Bureau of the Census. *Statistical Abstract of the United States: 1993*. Washington, DC: U.S. Government Printing Office, 1993.
- Note:** To order copies of the March of Dimes pamphlet, call 1-800-367-6630, or write the March of Dimes, P.O. Box 1657, Wilkes-Barre, PA 18703.

There's No Place Like Home



What geometric shape does your home resemble? What do you think the houses of the future will look like? In this module, you explore some traditional American Indian housing designs—from the ground up.

Kyle Boyce • Clay Burkett



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

There's No Place Like Home

Introduction

Traditional American Indian dwellings vary from tribe to tribe and from region to region. Those who lived in what is now South Dakota used tipis, the tribes of the American Southwest chose pueblos, while the peoples of the Far North built igloos (also called *illuliaqs*). Differences in climate, terrain, building materials, and cultures were all important factors in the development of appropriate housing designs.

The design of a house affects such practical considerations as the amount of building material needed, the amount of space enclosed, and the relative heating or cooling efficiency of the dwelling. As the end 21st century begins, both building materials and heating fuels are growing scarce. Houses of the future must make efficient use of our resources.

Examining some traditional American Indian housing designs may help suggest some possible designs for efficient homes in the next century. In this module, you examine several types of structures, the geometric shapes that represent them, and the properties that help make buildings efficient.

Activity 1

The amount of floor space in a home is important to the comfort of its owners. Many colonial houses had rectangular floors. In this case, the area of the floor is relatively easy to determine. The floor of a large tipi, however, could be a regular polygon with as many as 18 sides.

Exploration

How could you find the area of a floor shaped like an 18-sided regular polygon? The area of a geometric figure refers to the region enclosed by it. The area of a polygon may be found by estimating, by using a formula, or by other means.

In this exploration, you calculate area by inscribing a regular polygon inside a circle. An **inscribed polygon** is one in which each vertex lies on a circle. The **radius** of an inscribed regular polygon is equal to the radius of the circle.

For example, Figure 1 shows a square inscribed in a circle, with the radius of the square drawn from the square's center to one of the vertices.

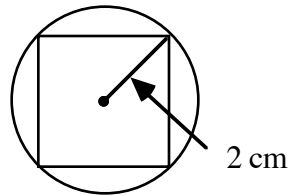


Figure 1: Inscribed square with a radius of 2 cm

- a.
 1. Construct a circle on a geometry utility. Record its radius, area, and perimeter.
 2. Select three points on the circle to represent the vertices of an inscribed triangle.
 3. Connect the center of the circle to each of the three points to form three central angles. A **central angle** is an angle with its vertex at the center of a circle.
 4. Move the points along the circle, one at a time, until the central angles all have equal measures.
 5. Connect the three points on the circle to form a regular triangle.
- b.
 1. Use the geometry utility to calculate the triangle's area and perimeter.
 2. Record this information in a table.
- c. Using a process similar to that described in Part **a**, inscribe a regular quadrilateral, pentagon, hexagon, octagon, and 18-sided polygon in circles of the same radius.
- d. In your table from Part **b**, record the area and perimeter of each regular polygon, as well as the area and perimeter of the circle in which they were inscribed. **Note:** Save your data for use in the discussion.

Discussion

- a. How did you determine the locations of the vertices for your inscribed polygons?
- b. In Parts **a** and **c** of the exploration, why does creating congruent central angles guarantee that a regular polygon will be formed?
- c.
 1. What methods do you know for determining the area of regular polygons?
 2. What are the advantages and disadvantages of each one?

- d. Consider a set of regular polygons inscribed in the same circle. As the number of sides in a polygon increases, describe what happens to each of the following characteristics:
1. the shape of the polygon
 2. the area of the polygon
 3. the perimeter of the polygon.
- e. A **tangent** to a circle is a line, segment, or ray that intersects a circle in one point and is perpendicular to a radius at that point. Consider a pentagon whose sides are each tangent to a circle, as shown in Figure 2 below. In this case, the polygon **circumscribes** the circle.

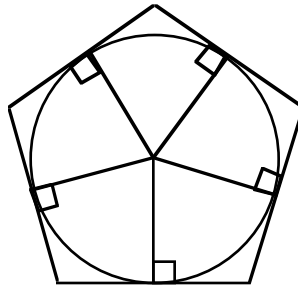


Figure 2: A circle circumscribed by a regular pentagon

Consider a set of regular polygons that circumscribe the same circle. As the number of sides in a polygon increases, describe what happens to each of the following characteristics:

1. the shape of the polygon
2. the area of the polygon
3. the perimeter of the polygon.

- f. Figure 3 below shows a regular pentagon inscribed in a circle. In this case, \overline{AG} is an **apothem**, a segment whose measure is the perpendicular distance from the center of a regular polygon to one of its sides.

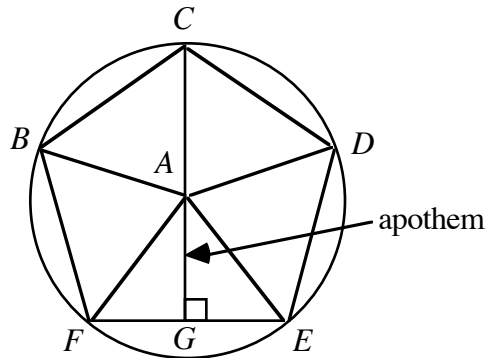


Figure 3: A regular pentagon

One formula for the area of a regular polygon is

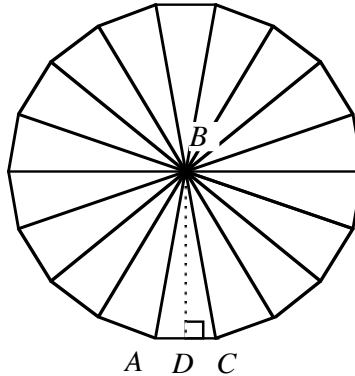
$$A = \frac{1}{2}ap$$

where a is the length of the apothem and p is the perimeter of the polygon. (See the Level 1 module, “A New Look at Boxing.”)

1. As the number of sides of an inscribed regular polygon increases, what do the values of a and p approach in relation to the circle?
2. How does this affect the formula for the area of a regular polygon given above?

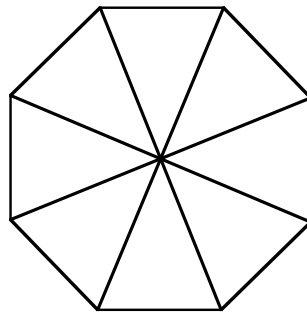
Assignment

- 1.1** The diagram below shows a view of a tipi from above. The floor of the tipi is an 18-sided regular polygon with a radius of 5 m.



Use this diagram to determine each of the following:

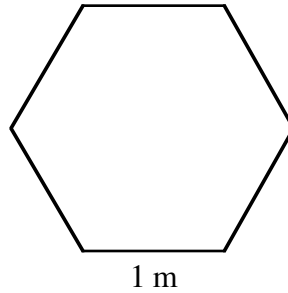
- a. the measures of $\angle ABC$ and $\angle ABD$
 - b. the length of \overline{AD} (using trigonometry)
 - c. the length of \overline{AC}
 - d. the length of \overline{BD}
 - e. the area of $\triangle ABC$
 - f. the area of the tipi's floor.
- 1.2** The Powder House in Williamsburg, Pennsylvania, was built in 1714 and used for storing gunpowder. As shown in the diagram below, the structure has a regular octagonal base. Given that the longest diagonals measure 16 m, determine the area of the floor.



- 1.3 Igloos are built from blocks of snow. The floor of an igloo is a circle. If an igloo has a diameter of 3.5 m, what is the area of its floor?

* * * * *

- 1.4 The diagram below shows the regular hexagonal floor of a modern domed camping tent. Given that each side is 1 m long, determine the area of the floor.



* * * * *

Research Project

Choose one American Indian tribe—or several tribes from a single region—and study their traditional buildings. Describe each structure’s practical, seasonal, and cultural uses. Then describe each structure from a mathematical point of view, including its shape, geometric properties, and typical dimensions.

Activity 2

When an architect designs a house, the desired amount of floor space is only one consideration. Available building materials also play a role in determining an appropriate design. In some situations, appearance is a primary consideration; in others, durability and cost are the main concerns. In any case, estimating the amount of materials required is an important part of the planning process. When determining the amount of material needed to build a house, it is helpful to calculate the area of the walls, floor, and roof.

American Indian families in the Yukon region of Alaska and Canada often built double lean-tos such as the one shown in Figure 4.

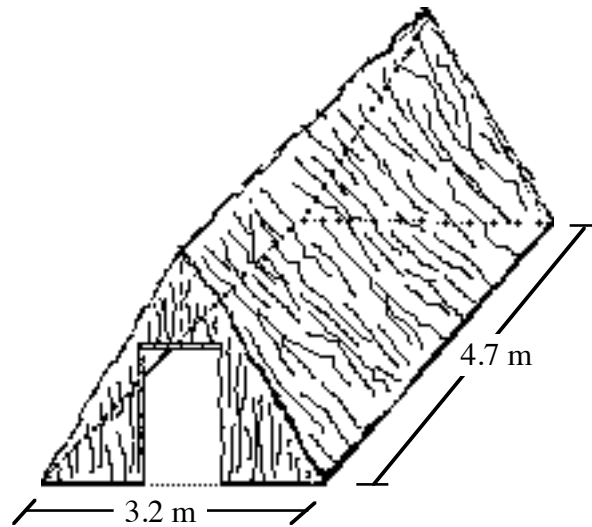


Figure 4: A double lean-to

A double lean-to can be described as a right triangular prism with bases that are equilateral triangles. The total surface area of a prism is the sum of the lateral surface area and the areas of the two bases. For example, Figure 5 shows a net for a right triangular prism. In this case, the three rectangles in the net are the lateral faces of the prism. The sum of their areas is the prism's lateral surface area.

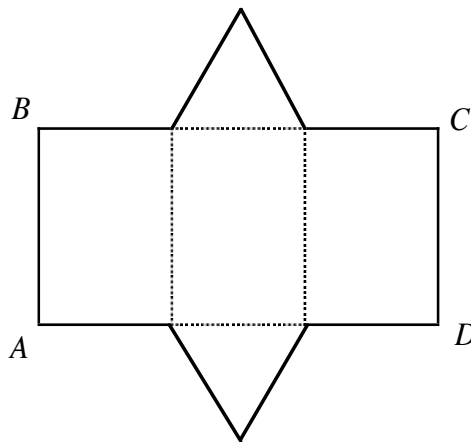


Figure 5: Net for a right triangular prism

Discussion

- a.
 1. Why is it incorrect to consider a rectangle as a base of the prism in Figure 5?
 2. Why is this prism called a “right” prism?

- b. How are the dimensions of rectangle $ABCD$ in Figure 5 related to the perimeter of the prism's base?
- c. Describe how to find the surface area of a right prism with a square base.
- d. Figure 6 shows a net for a right circular cylinder.

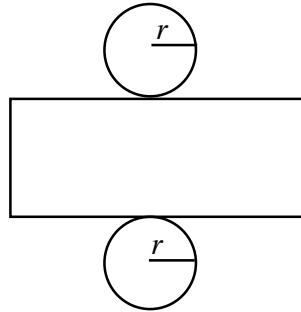
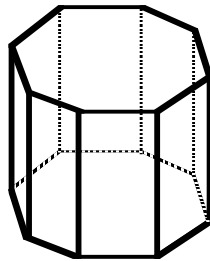


Figure 6: Net for a right circular cylinder

1. What does the length of the rectangle's longer side represent?
2. What part of the net represents the height of the cylinder?
3. What do the circles represent?
4. Express the length of the rectangle's longer side in terms of the radius of the circle.
5. Describe how to find the total surface area of a cylinder.

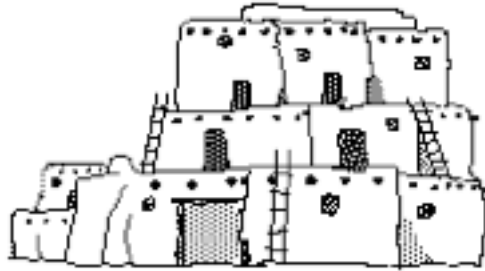
Assignment

- 2.1 The Powder House described in Problem 1.2 can be modeled by a right regular octagonal prism. A similar building, shown in the diagram below, is 11 m high. Each side of a base is 6 m long.

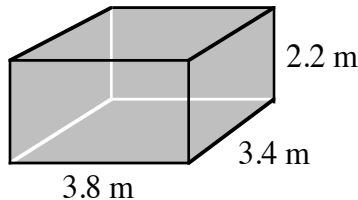


- a. Sketch a net for the building.
- b. Label the dimensions of one of the net's lateral faces.
- c. Determine the building's lateral surface area.
- d. Calculate the area of the bases.
- e. Determine the total surface area of the building.

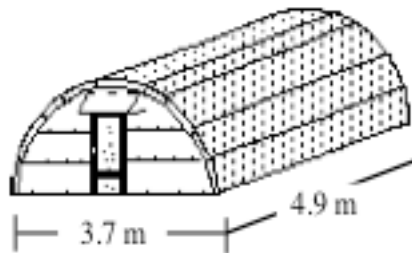
- 2.2 a. Sketch a net for a prism whose bases are regular heptagons.
 b. As the number of sides in the bases of a prism increase, what geometric shape does the prism approach?
- 2.3 To help estimate the amount of material needed to build the double lean-to shown in Figure 4, determine its total surface area.
- 2.4 The Pueblo Indians of the Rio Grande region lived in rectangular flat-roofed homes. As shown in the diagram below, these dwellings resemble an early version of the modern apartment building.



The right rectangular prism below shows some possible dimensions for a single-family unit in a Pueblo village. Use these measurements to determine the total surface area of one family's home.

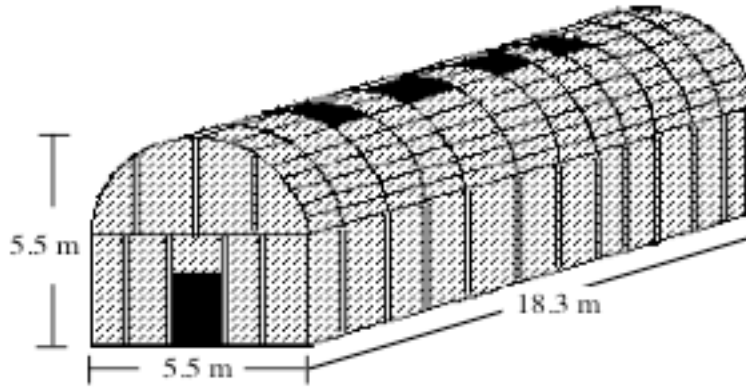


- 2.5 A traditional wigwam of the Chippewa tribe and a modern Quonset hut share the same basic shape: a half cylinder. The Chippewas used wigwams as sweat lodges. Quonset huts with translucent walls are commonly used as greenhouses. Complete Parts **a–d** for the Quonset hut shown below.



- a. What is the area of the floor?
 b. What is the area of the two walls formed by the semicircles?
 c. What is the lateral surface area?
 d. What is the total surface area, including the floor?

- 2.6 The Iroquois Indians of New York lived in rectangular barrel-roofed houses. The roof of each structure was made by bending poles into a semicircular shape (a half barrel). The entire building, except for the floor, was covered with bark. How much tree bark would be needed to cover the house shown in the diagram below?



* * * * *

Activity 3

Like today's families, early American Indian families lived in dwellings of various shapes and sizes. The ease of heating and cooling these structures was an important part of their design. Efficient heating and cooling are also major considerations for nearly all modern homes.

Since you heat the space inside a home (volume) and lose heat through the outside walls, roof, and floor (surface area), you must consider both surface area and volume when estimating efficiency. One way to do this involves calculating the ratio of a building's surface area to its volume. Although this rule of thumb is not always applicable, it does give one indication of a home's potential efficiency.

Exploration

- a.
 1. Arrange eight cubes so that each cube shares at least one entire face with another cube.
 2. Describe the shape you created in Step 1 and make a sketch of it. For example, Figure 7 shows a sketch of one possible arrangement.

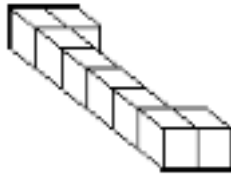


Figure 7: An arrangement of eight cubes

- b. Repeat Part a for several different arrangements of the cubes.

Mathematics Note

The **volume** (V) of a right prism can be found by multiplying the area of one of its bases (B) by its height (h). The height of a prism is the distance between the two bases. In general, the formula for the volume of a right prism is:

$$V = B \cdot h$$

For example, consider a right hexagonal prism with a height of 15 m. If the area of each base is 256 m^2 , the volume of the prism can be found as follows:

$$V = B \cdot h = 256 \text{ m}^2 \cdot 15 \text{ m} = 3840 \text{ m}^3$$

- c. For each shape you created in Parts a and b, determine the volume, the total surface area, and the ratio of surface area to volume. Record these values in a table. For example, the volume (V) of the shape in Figure 7 is 8 units^3 , its surface area (S) is 34 units^2 , and the ratio S/V is 4.25.

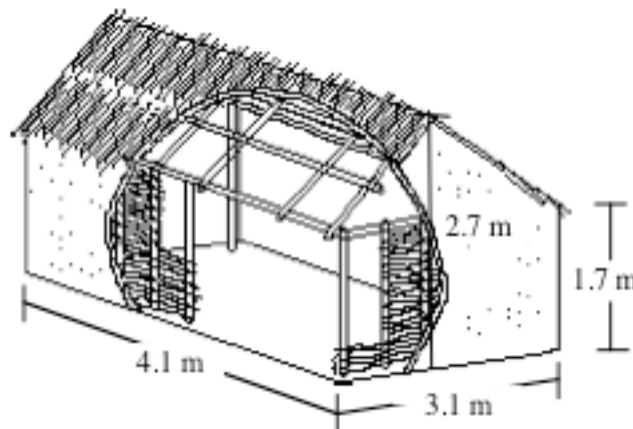
Discussion

- a.
 1. Which of the shapes you created in the exploration has the greatest surface area?
 2. Which has the least surface area?
 3. Which has the greatest volume?
- b. Why might the ratio of surface area to volume provide a reasonable way to evaluate the heating and cooling efficiency of a house? (Disregard the impacts of different building materials or insulation when considering your response.)

- c.
 1. Which of the shapes you created in the exploration has the greatest ratio of surface area to volume?
 2. Which has the smallest ratio of surface area to volume?
- d. If you were building a house in a cold climate and needed to conserve heating fuel, which shape would you choose? Explain your response.
- e. If you were building a house to collect and store solar energy, which shape would you choose? Explain your response.

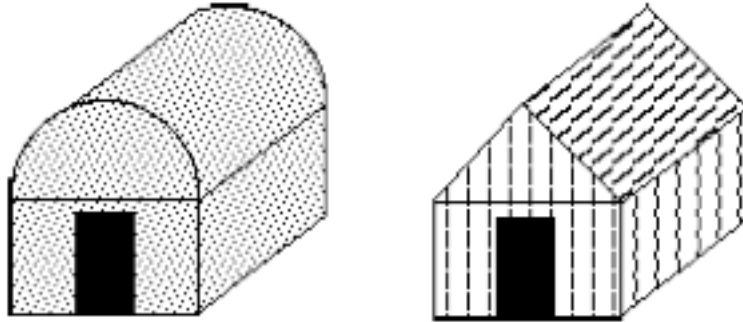
Assignment

- 3.1 The Yuma people of the Colorado River region lived in “Mohave-type” houses. Each one-story dwelling was a right rectangular prism with a square floor. These houses were about 2.2 m tall and had flat roofs. The square floors ranged from 6 m to 7.5 m on each side.
 - a. What is the range of the volumes for these dwellings?
 - b. What is the ratio of surface area to volume for a Yuma dwelling with a square floor 6 m on each side?
- 3.2 As mentioned in Problem 2.4, the traditional homes of the Pueblo Indians are rectangular flat-roofed structures that resemble apartment buildings. A typical family room was about 2.2 m high. The dimensions of the floor were approximately 4 m by 3.25 m.
 - a. What is the ratio of surface area to volume for this room?
 - b. How does the amount of space in a Pueblo dwelling compare with that of the smallest Yuma dwelling?
- 3.3 The summer houses of one Southeastern tribe were rectangular, gabled dwellings with thatched roofs and mud walls. The gabled roof was open, with smoke holes located along the ridge. One example of this type of house is shown in the diagram below.



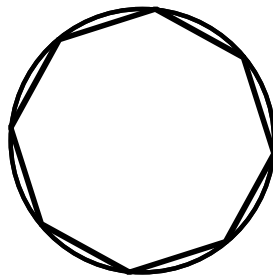
- a. What is the volume of this house?
- b. What is its ratio of surface area to volume?

- 3.4** The basic shapes of the two structures shown below were used by many different American Indian tribes.



Consider a structure of each type with identical, rectangular bases measuring 3.0 m by 3.8 m. Each structure has walls of the same height, as well as the same overall height, 3.4 m.

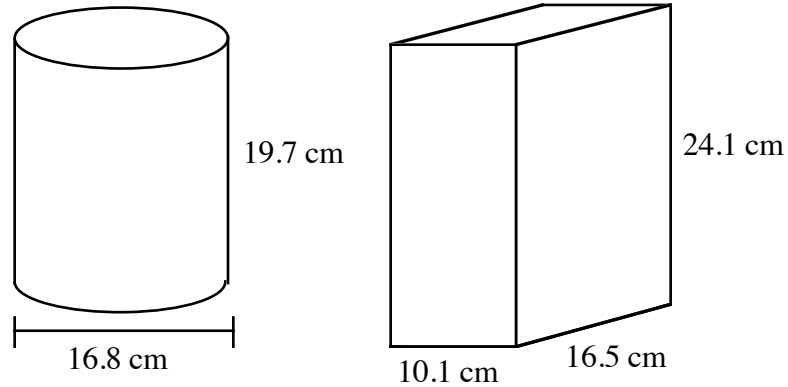
- Which structure has the greater surface area? Explain your response.
 - Which has the greater volume? Explain your response.
 - If the two structures were built of the same materials, which one would you expect to retain heat more efficiently? Explain your response.
- 3.5** The figure below shows the top view of a right prism inscribed in a cylinder. The bases of the prism are many-sided regular polygons.



- As the number of sides in a base of the prism increases, how does the volume of the prism compare to the volume of the cylinder?
- The formula for the volume of a right prism is $V = B \cdot h$, where B represents the area of a base and H represents the height.
 - Can this formula be used to find the volume of a right cylinder? Explain your response.
 - What formula would you use to determine the area of a base of the cylinder?

* * * * *

- 3.6** A manufacturer of painting supplies uses a cylindrical container for paint and a container shaped like a rectangular prism for paint thinner. The dimensions of these two containers are shown in the diagram below.



- Determine the lateral surface area, total surface area, and volume of the cylindrical container.
- Determine the lateral surface area, total surface area, and volume of the container shaped like a rectangular prism.
- If both containers are made from the same material, which one do you think would be cheaper to make? Explain your response.

* * * * *

Activity 4

For the American Indian tribes who followed the bison herds, a tipi was the ideal home. It was warm in winter, cool in summer, portable, and easily made from available materials. Figure 8 shows one example of a tipi.



Figure 8: A tipi

Some tipis approximated the shape of a regular pyramid. The base was a regular polygon, and the number of sides ranged from 6 in smaller tipis to 18 in larger ones. For example, the pyramid in Figure 9 represents a tipi with a hexagonal base 6 m on each side. The **height** of a pyramid is the distance from the vertex to the base. The **slant height** of a pyramid is the height of its lateral faces. Each lateral face is a triangle.

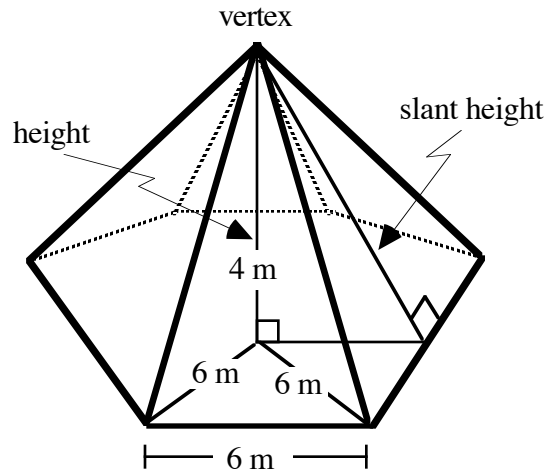


Figure 9: A pyramid with a regular hexagonal base

Discussion 1

- a. What type of triangle is each of the lateral faces of a regular pyramid?
- b. As the number of lateral faces in a pyramid increases, describe what happens to each of the following:
 1. the shape of the base
 2. the shape of the lateral faces
 3. the shape of the pyramid.
- c. Describe how to find the surface area of a pyramid.

Exploration

In this exploration, you construct a net for a model of a tipi.

- a.
 1. Construct a net, including the base, for a regular hexagonal pyramid.
 2. Verify that the net forms a pyramid.
- b.
 1. Cut out the lateral faces of the net. Arrange them so that the vertex angles of the triangles all meet at a single point and there are no gaps between the sides.
 2. Tape the triangles together to form a net of the **lateral surface** of the pyramid.

Discussion 2

- a.
 1. Describe the net of the lateral surface of the pyramid.
 2. As the number of lateral faces in a pyramid increases, what happens to the shape of this net?

Mathematics Note

An **arc** of a circle is a part of the circle whose endpoints are the intersections, with the circle, of the sides of a central angle.

The **measure of an arc** is the measure of its central angle. A **minor arc** has a measure less than 180° ; a **major arc** has a measure greater than 180° ; a **semicircle** has an arc measure of exactly 180° .

The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circle's circumference by the fractional part of the circle that the arc represents.

A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. The **area of a sector** can be found by multiplying the circle's area by the fractional part of the circle that the sector represents.

For example, Figure 10 shows a circle with center at O and a radius of 10 cm. The shaded sector is bounded by a central angle of 40° and the minor arc AB . The unshaded sector has a central angle of 320° and is bounded by the major arc ACB .

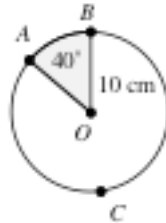


Figure 10: A sector of a circle

Since the measure of arc AB is 40° and the central angle of the entire circle is 360° , the length of arc AB can be found as follows:

$$\frac{40}{360} \cdot 2\pi(10) \approx 7 \text{ cm}$$

Similarly, the area of the shaded sector is:

$$\frac{40}{360} \cdot \pi(10)^2 \approx 35 \text{ cm}^2$$

- b. Consider the expression used to find the length of arc AB in the previous mathematics note. In this expression, what does each of the following quantities represent?
- $2\pi(10)$
 - $40/360$
- c. Consider the expression used to find the area of the shaded sector in the previous mathematics note. In this expression, what does the quantity $\pi(10)^2$ represent?
- d. As the number of lateral faces in a pyramid increases, how does the net of the lateral surface compare to the sector of a circle?
- e. The lateral surface of a cone can be represented by a sector of a circle. For example, Figure 11 shows a sector of a circle of radius r , along with the corresponding cone of slant height s .

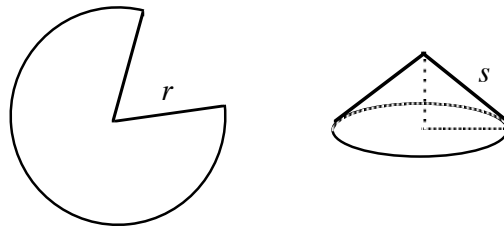
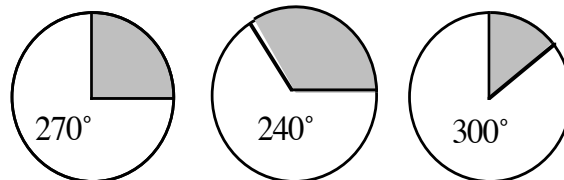


Figure 11: A sector formed into a cone

- What part of the sector determines the slant height of the cone?
- What part of the sector determines the circumference of the base of the cone?
- Describe how to find the radius of the cone's base given the radius of the sector and the measure of its central angle.

Assignment

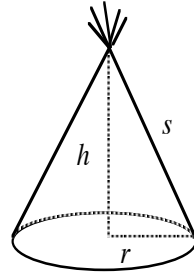
- 4.1 Determine the total surface area of the regular hexagonal pyramid in Figure 9.
- 4.2 The diagram below shows three circles, each with a radius of 5 cm, along with the measures of the central angles of the unshaded sectors.



- What is the area of each circle?
 - What is the circumference of each circle?

- b.
 1. What fraction of the circle is represented by each unshaded sector?
 2. What is the area of each unshaded sector?
 3. What is the length of the arc that bounds each unshaded sector?
- c. Consider a cone constructed from the sector with a central angle that measures 270° .
 1. What part of the cone is represented by the length of the arc that bounds the sector?
 2. What is the height of the cone?
 3. What is the area of the cone's base?
 4. What is the total surface area of the cone?

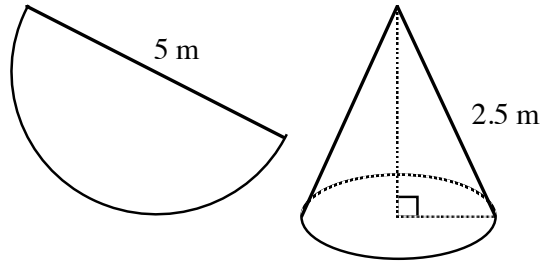
4.3 Although tipis are actually pyramids, their shapes can be modeled by cones. As shown in the diagram below, when the covering material for a tipi is cut from a circle and draped over the supporting poles, it resembles a cone without a base. In this case, the covering material represents the lateral surface of the cone.



- a. Sketch a net of a cone, including the base.
 1. Label the radius of the base r .
 2. Label the part of the net that represents the slant height of the cone s .
- b.
 1. Determine the length of the “curved” part of the lateral surface in terms of the radius of the cone's base.
 2. Find the circumference of a circle with radius s .
 3. What fraction of a circle with radius s is represented by the lateral surface of the cone? Express your response in terms of r (the radius of the cone's base) and s (the cone's slant height).
- c.
 1. Find the area of a circle with radius s .
 2. Determine the area of the lateral surface of the cone.

- d. Write a formula for the total surface area of a cone.
- e. Describe how to find the height h of a cone using r and s .

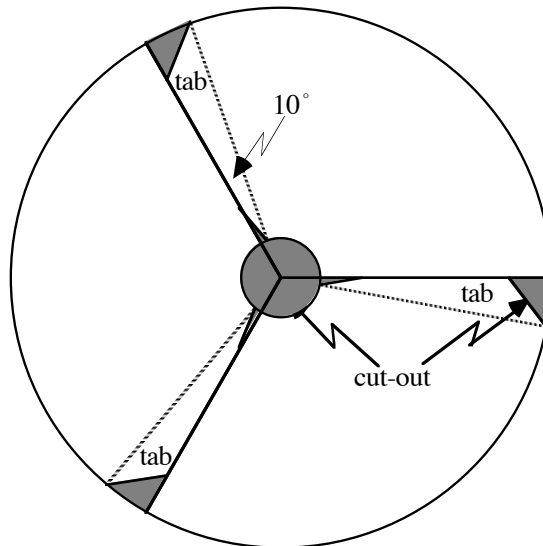
4.4 In some modern tipi designs, the covering material is cut in the shape of a semicircle, then draped over the poles. In the model shown below, the cover was cut from a circle of radius 2.5 m.



- a. What is the diameter of the tipi's floor?
- b. What is the height of the tipi?
- c. Do these seem like reasonable measurements for a tipi? Explain your response.

* * * * *

4.5 To help customers add engine oil, many service stations provide paper funnels. One manufacturer of such funnels uses the circular template shown in the diagram below. By cutting the template along the solid lines, three funnels (with tabs) can be produced. The dotted lines indicate the folds for the tabs. The circumference of the larger opening of each funnel is 32 cm, while the circumference of the smaller opening is 5 cm.



- a. Determine the radius of the circular template.
- b. Find the radius of the inner circle that is cut from the template.
- c. Determine the height of the paper funnel.

* * * * *

Activity 5

A structure's volume—and its ratio of surface area to volume—are important considerations for heating or cooling the space inside. In the previous activity, you modeled tipi coverings using sectors of a circle. In this activity, you experiment with the size of the sectors and observe the resulting effects on a tipi's volume.

Exploration 1

In this exploration, you investigate the relationship between the volumes of two figures with the same height and radius of the base: a right circular cylinder and a right circular cone.

- a. Construct a right circular cone by completing the following steps.
 1. On a sheet of paper, draw a circle with a radius of at least 4 cm. As shown in Figure 12, label center A and a point on the circle B .

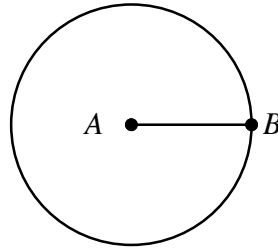


Figure 12: Circle and radius AB

2. Cut out the circle, then make a cut along \overline{AB} .
 3. Investigate the right circular cones you can form by moving point B to various points on the circumference of the circle.
 4. Select one cone. Tape the overlapping edge to the cone's surface.
- b. Measure the height of the cone and the radius of its base in centimeters.
- c.
 1. Create a net for the lateral surface of a right circular cylinder with the same height and radius as your cone. Include a tab on one edge, as illustrated in Figure 13.

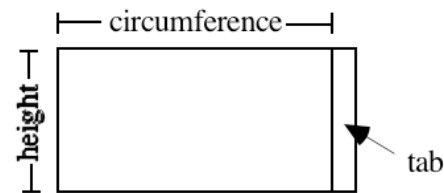


Figure 13: Lateral surface of right circular cylinder

2. Tape the edges of the net together to form the lateral surface of the cylinder.

- d.
 1. Estimate the ratio of the volume of the cylinder to the volume of the cone.
 2. Place the cylinder upright on a flat surface.
 3. Fill the cone with rice, then pour the rice into the cylinder. Repeat until the cylinder is full.
 4. Record the number of full cones required to fill the cylinder.
Note: Save the rice for use in Exploration 2.

Discussion 1

- a. Compare the volumes of cones and right circular cylinders with equal heights and radii.
- b. Use the formula for the volume of a right circular cylinder and your results in Exploration 1 to determine a formula for the volume of a right circular cone.

Exploration 2

In this exploration, you investigate the relationship between the volumes of two figures with the same height and the same square base: a right prism and a regular pyramid.

- a. Use a geometry utility to create a net for a regular pyramid with a square base. Print a copy of the net.
- b. Cut the triangular faces from the net and tape them together to form the lateral surface of the pyramid.
- c.
 1. Measure the height of the pyramid in centimeters.
 2. Measure the length of a side of the base in centimeters.
- d.
 1. Create a net for a right prism with the same height and base as the pyramid from Parts a–c.
 2. Cut out the net, then cut off one of the bases. Tape the edges together to create an open container.
- e.
 1. Estimate the ratio of the volume of the prism to the volume of the pyramid.
 2. Fill the pyramid with rice, then pour the rice into the prism. Repeat until the prism is full.
 3. Record the number of full pyramids required to fill the prism.

Discussion 2

- a. Compare the volumes of regular pyramids and right prisms with equal heights and congruent bases.

Mathematics Note

The volume of a pyramid or a cone can be found using the following formula:

$$V = \frac{1}{3}B \cdot h$$

where B is the area of the base and h is the height of the pyramid or cone.

- b.
1. Are your results in Exploration 1 consistent with the formula given in the mathematics note? Explain your response.
 2. Are your results in Exploration 2 consistent with the formula?
- c. Describe how to determine the volume of a regular pyramid with a height of h units and a square base in which each side is s units long.
- d. Figure 14 below shows a right circular cylinder, a hemisphere, and a right circular cone, with the same radii and “heights.” (For the purposes of this comparison, the radius of a hemisphere also can be thought of as its “height.”)

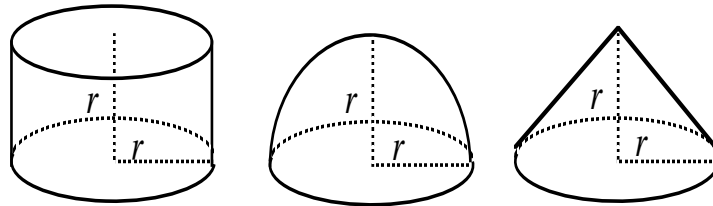


Figure 14: A cylinder, a hemisphere, and a cone

1. Rank the volumes of each figure from largest to smallest.
2. Express the volumes of the cylinder and the cone in terms of r .
3. Make a conjecture about the formula for the volume of a hemisphere with radius r .
4. Based on your conjecture above, what would be the formula for the volume of a sphere of radius r ?

Mathematics Note

A **sphere** is the set of all points in space that are the same distance from a given point, the **center** of the sphere. The common distance is the **radius** of the sphere. The **diameter** of a sphere is twice the radius.

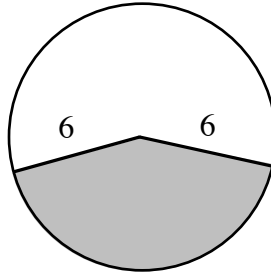
The volume of a sphere with radius r can be found using the following formula:

$$V = \frac{4}{3}\pi r^3$$

Assignment

- 5.1**
- Find the volume of a right circular cone with a radius of 10 cm and a height of 15 cm.
 - Determine the height of a right circular cylinder with the same radius and volume as the cone in Part **a**.
 - Determine the height of a right circular cone with a radius of 8 cm and the same volume as the cone in Part **a**.
- 5.2**
- Consider a tipi shaped like a pyramid with a regular dodecagon as its base. (A dodecagon has 12 sides). The tipi's height is 5 m and each side of the base is 2 m long. What is the volume of this tipi?
 - Consider a tipi that can be modeled by a right circular cone. The radius of its base is 3.75 m and its height is 5 m. What is the volume of this tipi?
 - Explain why your responses to Parts **a** and **b** are close in value.
- 5.3**
- A cone-shaped structure with a small ratio of surface area to volume might be more efficient to heat than one with a larger ratio. Consider a building shaped like a right circular cone with a slant height of 6 m, a height of h meters and a radius of r meters.
- Write an equation that expresses the height h in terms of the radius r .
 - Use your equation from Part **a** to write a formula for the volume of the cone in terms of r .

- c. The shaded sector in the following diagram represents a net for the lateral surface of a cone with a slant height of 6 m. The arc that bounds the sector has a length equal to the circumference of the base of the cone: $2\pi r$.



Write an expression for the lateral surface area of the cone in terms of r .

- d. Create a spreadsheet with the following headings. Use the spreadsheet to investigate the ratio of lateral surface area to volume for cones with a slant height of 6 m and radii from 0.1 m to 5.9 m, in increments of 0.1 m.

Radius (m)	Volume (m³)	Lateral Surface Area (m²)	<u>Surface Area</u> Volume
0.1			
0.2			
0.3			
⋮			
5.8			
5.9			

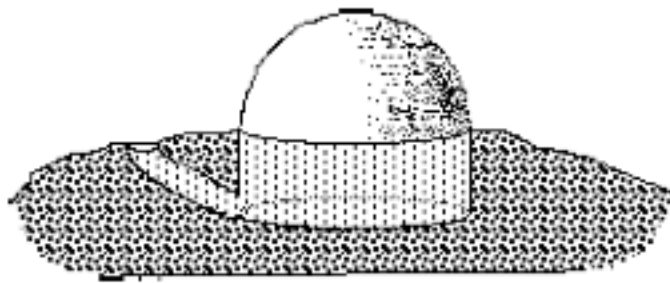
- e. Identify the height and radius of the cone in Part **d** with the smallest ratio of surface area to volume.

Mathematics Note

The surface area of a sphere with radius r can be found using the formula below:

$$S = 4\pi r^2$$

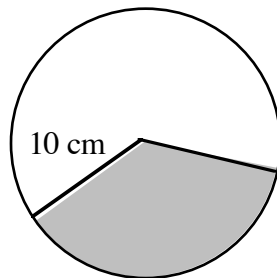
- 5.4** The traditional house of the Maidu and Miwok tribes of central California was a cylindrical pit covered by a hemispherical dome. These pits ranged from 1.5 m to 3.5 m deep, and 3.5 m to 18 m in diameter. The diagram below shows an example of this type of dwelling.



- The pit in the diagram above is 1.5 m deep and has a diameter of 3.5 m. What is the total volume of this house?
- What is the total surface area of this house?
- As shown in the diagram, part of the house lies below ground level. How do you think this will affect the relative efficiency of heating or cooling?

* * * * *

- 5.5** The diagram below shows a circle with a radius of 10 cm. The shaded sector has a central angle that measures 120° .



Imagine that the two sectors are cut apart, then folded into cones. In each cone, the cut edges touch, but do not overlap.

- Find the total surface area of each cone, including the base.
- Find the volume of each cone.
- Find the ratio of surface area to volume for each cone.

- 5.6** The diagram below shows one example of an igloo. The inside diameter of the igloo is 6.0 m, while its walls are made of packed snow 0.1 m thick.

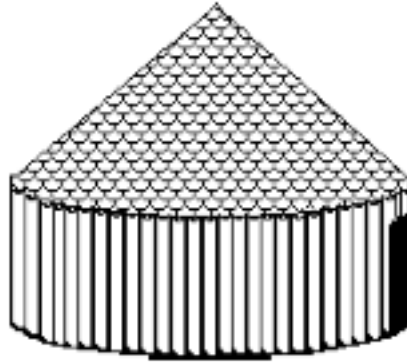


- a. The living space inside this igloo is a hemisphere. What is the volume of the living space?
 - b. What is the volume of the packed snow in the igloo's walls (ignoring the entryway)?
 - c. What is the ratio of lateral surface area to volume for the inside of the igloo? What might this ratio tell you about the relative heating efficiency of an igloo?
 - d. Consider a house built in the shape of a half cylinder (such as a Quonset hut) with the same volume as the igloo in Part a and a height of 3 m. Disregarding the floor, what is this home's ratio of surface area to volume?
 - e. Which shape—a hemisphere or a half cylinder—do you think would be more practical for housing in Arctic regions? Explain your response.
- 5.7** The bathysphere is a spherical steel vessel designed for undersea research. Designed by American naturalist and explorer William Beebe, it made its first dive in 1930.
- a. The outside radius of the bathysphere is approximately 75 cm. Determine the volume of water it displaces.
 - b. Find the volume of the interior of the bathysphere given that its steel walls are 3.8 cm thick.
 - c. When the first bathysphere was designed, weight was an important concern. Assuming that weight is proportional to surface area, compare the weight of the bathysphere with that of a cube of equal volume.

* * * * *

Summary Assessment

1. The Miwok people of central California were one of several tribes who built structures with cylindrical walls and conical roofs known as roundhouses. Roundhouses were used to host ceremonial dances and rituals. The diagram below shows one example of a roundhouse.



The floor of the roundhouse in the diagram has a diameter of 10.6 m. The height of the walls is 2.1 m; the total height of the building is 4.6 m.

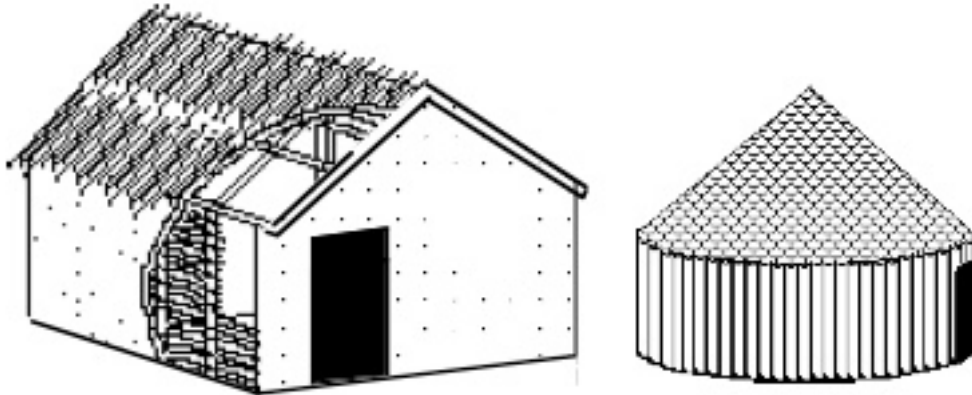
- a. What is the total surface area of this roundhouse?
 - b. What is its volume?
2. As shown in the following diagram, the living space in an igloo is shaped like a hemisphere, while its entryway is shaped like a half-cylinder.



The inside radius of the igloo in the diagram is 2.5 m. The entryway is 1 m long and has an inside radius of 0.5 m.

Determine the inside surface area and volume of the igloo, including the entryway. Describe any assumptions you make in solving this problem.

3. Oklahoma's Delaware tribe did not build roundhouses. Their ceremonial lodges were shaped like square prisms with gabled roofs. In the diagram below, a Delaware lodge is shown on the left, while a Miwok roundhouse appears on the right. For both of these structures, the area of the floor is 49 m^2 , the height of the walls is 3 m, and the total height of the building is 4.7 m.



- Compare the ratios of total surface area to volume (including the floor) for the two buildings.
- If you lived in a region where building materials were scarce, which type of structure would you choose: the roundhouse or the gabled lodge? Explain your response.
- Which design might be easier to heat or cool? Explain your response.

Module Summary

- An **inscribed polygon** is one in which each vertex of the polygon lies on a circle. The **radius** of a regular polygon is the radius of the circle in which the polygon can be inscribed.
- A **tangent** to a circle is a line, segment, or ray that intersects a circle in one point and is perpendicular to a radius at that point.
- The **volume** (V) of a right prism can be found by multiplying the area of one of its bases (B) by its height (h). The height of a prism is the distance between the two bases. In general, the formula for the volume of a right prism is:

$$V = B \cdot h$$

- The volume of a right cylinder with radius r and height h can be found using the following formula:

$$V = (\pi r^2)h$$

- An **arc** of a circle is a part of the circle whose endpoints are the intersections, with the circle, of the sides of a central angle.
- The **measure of an arc** is the measure of its central angle. A **minor arc** has a measure less than 180° ; a **major arc** has a measure greater than 180° ; a **semicircle** has an arc measure of exactly 180° .
- The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circle's circumference by the fractional part of the circle that the arc represents.
- A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. The **area of a sector** can be found by multiplying the circle's area by the fractional part of the circle that the sector represents.
- The lateral surface of a cone is a sector of a circle.
- The volume of a pyramid or a cone can be found using the following formula:

$$V = \frac{1}{3}B \cdot h$$

where B is the area of the base and h is the height of the pyramid or cone.

- A **sphere** is the set of all points in space that are the same distance from a given point, the **center** of the sphere. The common distance is the **radius** of the sphere. The **diameter** of a sphere is twice the radius.
- The surface area of a sphere with radius r can be found using the formula:

$$S = 4\pi r^2$$

- The volume of a sphere with radius r can be found using the following formula:

$$V = \frac{4}{3}\pi r^3$$

Selected References

- Blackman, W. *Geometry in Architecture*. Berkeley, CA: Key Curriculum Press, 1984.
- Draiver, H. E. *Indians of North America*. Chicago: University of Chicago Press, 1961.
- Hall, S. *The Fourth World: The Heritage of the Arctic*. New York: Alfred A. Knopf, Inc., 1987.
- Haskin, F. J. *10,000 Answers to Questions*. Detroit, MI: Gale Research Co., 1970.
- The 1993 Information Please Almanac*. Boston, MA: Houghton Mifflin, 1993.
- Laubin, R., and G. Laubin. *The Indian Tipi: Its History, Construction, and Use*. New York: Ballantine Books, 1985.
- Maletsky, E. M. "Visualization, Estimation, Computation." *Mathematics Teacher* 75 (December 1982): 759–64.
- Maxwell, J., ed. *America's Fascinating Indian Heritage*. Pleasantville, NY: Readers Digest Association, 1978.
- Nabokov, P., and R. Easton. *Native American Architecture*. New York: Oxford University Press, 1989.

Making Concessions



If there's going to be a prom this spring, the class needs to raise some cash. What to sell? How much to buy? How much to charge? Unless the class makes good choices, the big spring dance could be doomed.

Bonnie Eichenberger • Paul Swenson



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Making Concessions

Introduction

Even before the class treasurer had made her report, everyone knew that cash was in short supply. Without more money, the spring dance would have to be canceled. The class discussed several fund-raising suggestions before deciding to sell concessions at the remaining home sporting events.

Since other school organizations already controlled the candy sales, the class would have to sell something else. Before they could start this project, however, they needed to answer some key questions. What should they sell? How much should they sell it for? How much could they afford to buy? And how could they make the most profit?

Anyone who opens a store or restaurant must consider these same questions. In the following activities, you examine some methods which could help the class make sound business decisions.

Activity 1

To increase their chances for success, businesses gather as much information as possible before making important decisions. Before entering a new market or introducing a new product, businesses often conduct surveys or **feasibility studies**. These studies may identify both potential customers and appropriate prices.

Exploration

When the class conducted a feasibility study, they found that students at their school were most interested in buying pizza, canned soda pop, and nachos. The class has asked you to make some recommendations on the remaining questions.

They have about \$100 to spend on pizzas, pop, and nachos, and want to earn as much money as possible. Each pizza contains eight slices. Soda pop comes in six-packs. Nachos must be purchased in kits consisting of four bags of tortilla chips and one container of cheese. Each kit makes 16 servings.

- a. Estimate the cost to the class of each pizza, six-pack, and nacho kit.
- b. Describe how the class should sell each item—by weight, volume, piece, or some other method.
- c. Determine how much they should charge for each item sold.
- d. Keeping in mind the \$100 limit, how many of each item—pizza, pop, and nachos—should the class purchase for resale?

- e. Record your decisions from Parts **a–d** in a table with headings like those in Table 1 below. Use this information to estimate the potential profit for the class.

Table 1: Estimated profit for concessions

Item	How Sold	Selling Price	Cost per Item	Quantity Purchased	Amount Spent	Potential Profit
pizza						
pop						
nachos						
Total						

- f. Summarize your recommendations in a report. **Note:** Save your report for use in the assignment.

Discussion

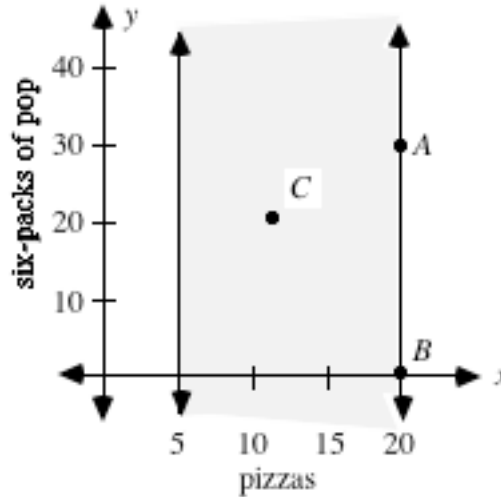
- a. What difficulties did you encounter in reaching your decisions?
- b. What additional information would have been helpful?
- c. What other questions might be important to ask in a feasibility study?
- d. What assumptions did you make when estimating potential profit?
- e. Compare your recommendations with those of others in the class.
1. For each item, what is the least quantity recommended for purchase?
 2. For each item, what is the greatest quantity recommended for purchase?
 3. What is the range of estimated profit?
- f. What would be the least possible profit in this situation? Explain your response.

Assignment

- 1.1 a. Assuming it is possible for the class to sell none of the items they purchase, what is the range of their estimated profit?
- b. Represent the range of estimated profit on a number line.
- c. Write an inequality to represent the range for estimated profit.
- d. In your response to Part c, did you use the symbols $<$ and $>$ or the symbols \leq and \geq ? Explain how you made your choice.

- 1.2** Suppose the class recommended that 5–20 pizzas should be purchased for resale. The illustration below shows a graphical representation of this information on an xy -coordinate system.

The recommended number of pizzas is represented on the x -axis. The recommended number of six-packs—which has not yet been determined—is represented on the y -axis.



- Interpret the meaning of the coordinates for each of the following:
 - point A
 - point B .
 - If point C represents any point in the shaded region, describe its possible range of coordinates.
 - Is there any portion of the shaded region in which the coordinates of the points are not relevant to this setting? Explain your response.
 - Suppose the class recommended that 10–40 six-packs of pop be purchased. On a copy of the coordinate system shown above, represent the range of six-packs that might be ordered.
 - Graph the intersection of the region that describes the recommended range of 5–20 pizzas with the region that describes the recommended range of six-packs.
- 1.3** In Part **e** of the discussion, your class described its recommended range for each item.
- On a coordinate system like the one in Problem **1.2**, graph the recommended ranges for pizzas and six-packs. Represent the number of pizzas on the x -axis and the number of six-packs on the y -axis.
 - Describe what the graph represents in terms of items to be purchased.

- 1.4** Repeat the process described in Problem **1.3** for each of the following.
- Graph the recommended ranges for six-packs and nacho kits. Represent the number of six-packs on the x -axis and the number of nacho kits on the y -axis.
 - Graph the recommended ranges for pizzas and nacho kits. Represent the number of pizzas on the x -axis and the number of nacho kits on the y -axis.

Mathematics Note

A **feasible region**, or set of **feasible solutions**, consists of all the points that satisfy the limitations, or linear **constraints** of a problem. The vertices of the feasible region are **corner points**.

For example, Figure 1 shows the feasible region for the constraints $x \geq 1$, $x \leq 4$, $y \geq 0$, and $y < 3$.

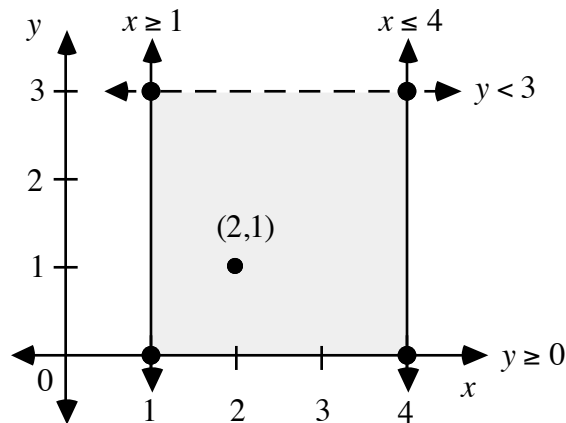


Figure 1: A feasible region

The x -coordinates of points in the shaded region are greater than or equal to 1 and less than or equal to 4. The y -coordinates of these points are greater than or equal to 0 and less than 3. The corner point with coordinates $(4,0)$ is in the feasible region. The corner point with coordinates $(4,3)$ is not in the feasible region.

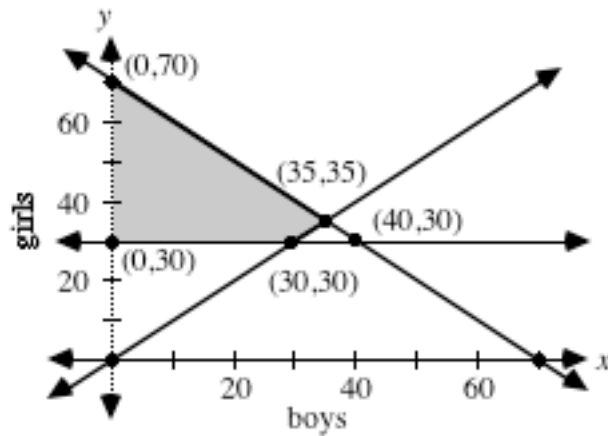
- 1.5** Use your graph from Problem **1.2e** to complete Parts **a–c** below.
- Write inequalities to describe the intersection of the two graphs (the feasible region).
 - In your inequalities from Part **a**, did you use the symbols $<$ and $>$ or the symbols \leq and \geq ? Explain your choice.
 - Do all the points in the feasible region indicate reasonable values considering the problem setting? Explain your response.

- 1.6** The class has decided to start concession sales using only pizza and pop.
- a.** Estimate a reasonable cost for each pizza and six-pack. You may wish to consult local merchants on prices.
 - b.**
 - 1.** Write inequalities to represent the number of pizzas that can be purchased for \$100 or less, where x represents number of pizzas. (Your inequalities should indicate that the class cannot purchase a negative number of pizzas.)
 - 2.** Write inequalities to represent the number of six-packs that can be purchased for \$100 or less, where y represents number of six-packs. (Your inequalities should indicate that the class cannot purchase a negative number of six-packs.)
 - 3.** Graph these inequalities on a two-dimensional coordinate system where y represents number of six-packs and x represents number of pizzas.
 - c.**
 - 1.** In Part **b**, you assumed that the entire \$100 could be spent on either pizza or six-packs. Write an inequality in which \$100 is the limit for the total cost of both items.
 - 2.** Graph this constraint on the same coordinate system as in Part **b**. You may need to solve the inequality for y before graphing.
 - d.** Label the feasible region on your graph and explain what it represents in this situation.
 - e.**
 - 1.** Determine the coordinates of the corner points of the feasible region and describe the method you used to identify them.
 - 2.** Describe at least one other method you could use to determine the coordinates of the corner points.

* * * * *

- 1.7** A student newspaper plans to conduct a survey on dating. The editors would like to poll at least 30 boys and 30 girls, but no more than a total of 120 students. To reflect their ratio in the school population, the survey group should consist of at least 5 girls for every 7 boys.
- a.** Write an inequality to represent each of the following:
 - 1.** the number of boys to be surveyed
 - 2.** the number of girls to be surveyed
 - 3.** the total number of students to be surveyed
 - 4.** the ratio of the number of girls to be surveyed to the number of boys to be surveyed.
 - b.** Graph the inequalities from Part **a** on a single coordinate system. Identify the feasible region.
 - c.** Determine the coordinates of the corner points and identify those that are included in the feasible region.

- 1.8 The feasible region shown in the graph below is defined by four constraints.



- Write each constraint as an inequality.
- Identify the corner points that are included in the feasible region.
- What might these constraints represent in terms of the number and gender of students in a newspaper survey?

Activity 2

Business analysts often use feasible regions to examine all the possible solutions to a problem—and eliminate those that do not satisfy one or more constraints. These constraints might include limitations on raw materials, on production costs, or on manufacturing output. The relationships among these constraints affect a company’s potential profit.

One method for analyzing potential profit is **linear programming**. To use this method, the constraints on a situation must all be linear. After determining a feasible region, an **objective function**—in this case, an equation representing profit—can be evaluated for points in this region. Using this process, it is possible to identify the conditions which will result in the maximum potential profit.

Linear programming also can be used to identify the conditions which will result in minimum values, such as production costs or manufacturing time. In this activity, you use an objective function to determine the maximum profit the class can make from their concession stand.

Exploration 1

Suppose that the class makes a profit of \$3.00 on each pizza and \$2.00 on each six-pack. If x represents number of pizzas, y represents number of six-packs, and P represents profit, the objective function in this situation is $P = 3x + 2y$.

- a. Substitute four different values for P in the objective function $P = 3x + 2y$. Use technology to graph these four equations.
- b.
 1. What similarities or differences do you observe among the graphs?
 2. How is the value of P related to the y -intercept of $P = 3x + 2y$?

Discussion 1

- a. What similarities did you observe in the graphs of the four equations?
- b. Why do you think these similarities occur?
- c. As the values of x increase in any given profit equation, does the profit increase or decrease? Explain your response.
- d. In relation to the other graphs, where is the graph that indicates the greatest profit? Why do you think this occurs?
- e. In relation to the other graphs, where is the graph that indicates the least profit?

Exploration 2

One method for determining the maximum profit involves substituting the coordinates of a point in the feasible region in the objective function, then calculating the resulting profit. This process is continued for each point until the maximum profit is found. Since it requires checking every point in the feasible region, this method can require an unreasonable amount of time. In this exploration, you develop an alternative method for finding maximum profit.

- a. Suppose that the class can buy pizzas for \$5.00 each and soda pop for \$2.00 per six-pack. Given that they can spend no more than \$100.00, the constraints on this situation are $0 \leq x \leq 20$, $0 \leq y \leq 50$, and $5x + 2y \leq 100$, where x represents number of pizzas and y represents number of six-packs.

Use technology to graph the feasible region described by these constraints. **Note:** The technology you choose may not be able to graph the boundary of a constraint that is not a function. In this case, it may be possible to “draw” a line to represent that boundary.

- b.
 1. Select a point in the feasible region. Use the objective equation $P = 3x + 2y$ to calculate the profit at that point.
 2. Substitute the profit calculated in Step 1 for P in $P = 3x + 2y$. Graph this equation on the coordinate system from Part a.
- c.
 1. As you observed in Exploration 1, the graphs of $P = 3x + 2y$ for all values of P are parallel.

Use this fact to help you find the location, with respect to the feasible region, of the line that indicates the maximum profit.
 2. Determine the maximum profit in this situation.
- d. Compare the value you found for the maximum profit with those of others in the class.

Discussion 2

- a.
 1. What is the minimum number of boundary lines required to determine a vertex of a feasible region?
 2. What is the minimum number of equations required to identify the coordinates of a vertex?
- b. Depending on the technology you used, it may not have been possible to graph all of the boundaries of the feasible region. Describe any problems you encountered when graphing the feasible region and explain why they occurred.
- c. Where did the line that indicates maximum profit intersect the feasible region?
- d. In the module “Under the Big Top but Above the Floor,” you used the substitution method to solve systems of equations. Describe how you could use this method to find the coordinates of the corner points of a feasible region.
- e. How could you determine the maximum profit without substituting the coordinates of every point in the feasible region in the objective equation?
- f. Suppose that the graph of the objective function was parallel to the boundary that contained the point where the maximum profit occurred. How would this affect your ability to determine the maximum value?
- g. Given the objective function and the coordinates of the point at which the maximum profit occurs, how could you determine the maximum profit?

Mathematics Note

When the corner points are part of the feasible region, the **corner principle** provides a method of finding the maximum or minimum values of an objective function. According to this principle, the maximum and minimum values of an objective function occur at the corner points of the feasible region.

For example, Figure 2 shows a feasible region with five corner points. According to the corner principle, the maximum value of an objective function is located at one of the corner points.

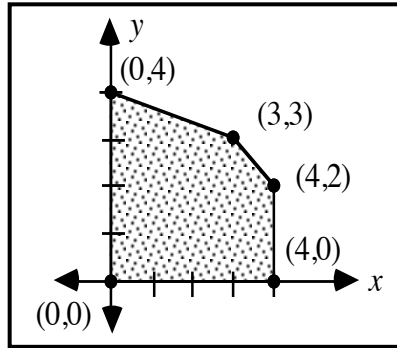


Figure 2: Feasible region with five corner points

Given the objective function $P = 2x + y$, the maximum value can be found by substituting the coordinates of each of the five corner points in this equation, then comparing the resulting values for P .

As shown in Figure 3 below, the maximum value of 10 occurs at the corner point with coordinates (4,2).

Coordinates (x, y)	Objective Function ($P = 2x + y$)
(0,0)	$2(0) + 0 = 0$
(0,4)	$2(0) + 4 = 4$
(3,3)	$2(3) + 3 = 9$
(4,2)	$2(4) + 2 = 10$
(4,0)	$2(4) + 0 = 8$

Figure 3: Substituting the coordinates of corner points in the objective function

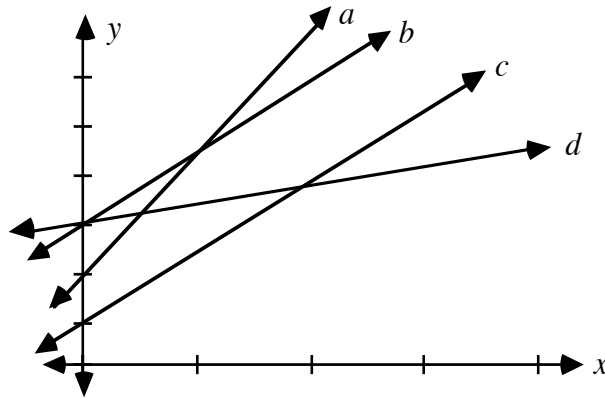
The minimum value also can be found using the corner principle. For the objective function $P = 2x + y$, the minimum value of 0 occurs at the corner point with coordinates (0,0).

- h.** What if, by applying the corner principle, you obtained a solution which indicated that the class should buy $5/8$ of a six-pack of pop. Is this solution reasonable? Why or why not?

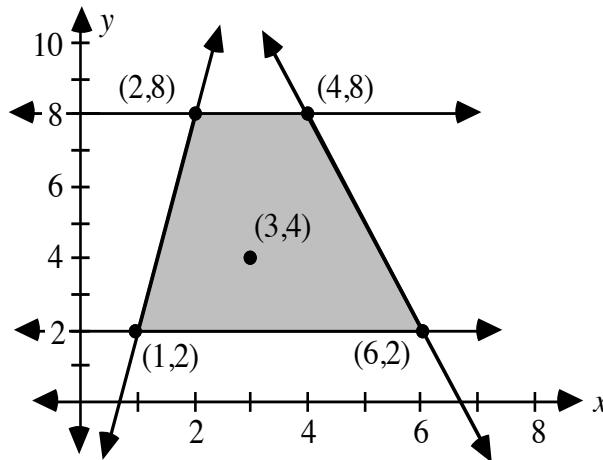
- i. Suppose that the objective function for the feasible region in Figure 2 is $P = x + 3y$. In this case, the maximum value of P is 12. The graph of $12 = x + 3y$ contains the segment joining the vertices $(0,4)$ and $(3,3)$. What does this indicate about where the maximum value occurs?

Assignment

- 2.1 Which of the lines in the graph below, if any, appear to represent the same profit function for different profit values? Explain your response.



- 2.2 The figure below shows a feasible region with four corner points.



- a. Use the information in the graph to represent each of the following with a sketch and with an equation.
1. the maximum value of the objective function $P = 3x + 2y$
 2. the maximum value of the objective function $P = 15x + y$
 3. the minimum value of the objective function $P = 3x + 2y$
 4. the minimum value of the objective function $P = 15x + y$
- b. Describe how the slope of the objective function affects your use of the corner principle.

- 2.3** According to the results of their feasibility study, the class should purchase no more than 16 eight-slice pizzas. The study also indicated that the class should buy no more than 22 six-packs of pop. Each pizza costs \$4.96, while each six-pack costs \$1.84. The class has \$100.00 to spend on these items.
- Write inequalities to represent the constraints in this situation.
 - Graph the feasible region defined by these constraints.
 - The class plans to charge \$2.00 for a slice of pizza and \$1.00 for a can of pop. How much profit will the class earn by selling one pizza? How much profit will they earn by selling one six-pack?
 - Use your responses to Part **c** to write an objective function for profit from selling pizzas and pop.
 - Find the maximum profit the class can earn if they sell all the pizza and pop they purchase. Graph the equation that represents the maximum profit on the coordinate system from Part **b**.
 - To realize this maximum profit, how many pizzas and six-packs should the class buy? Does this solution seem reasonable? Explain your response.

Mathematics Note

The **substitution method** can be used to solve a system of linear equations. For example, Figure 4 shows a feasible region with five corner points.

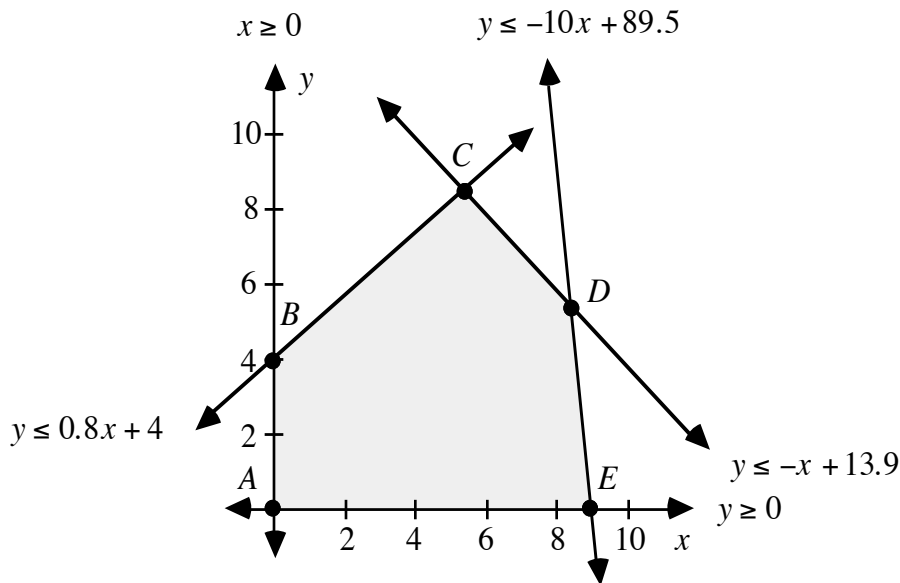


Figure 4: A feasible region with five corner points

In this case, the substitution method can be used to identify the coordinates of each corner point. For instance, point C is the intersection of the boundaries defined by the system of equations $y = -x + 13.9$ and $y = 0.8x + 4$. To find the coordinates of C , the system can be solved as shown in the steps below.

Substitute the value of y in one equation into the other equation.	$0.8x + 4 = -x + 13.9$
Solve for x .	$1.8x = 9.9$ $x = 5.5$
Substitute this value for x in one of the original equations.	$y = 0.8x + 4$ $= 0.8(5.5) + 4$ $= 8.4$

Since the solution to the system is $(5.5, 8.4)$, these are the coordinates of point C . To verify this solution, these coordinates should be substituted into each of the original equations.

- 2.4**
- Verify the coordinates for point C found in the mathematics note.
 - Determine the coordinates of the corner points A , B , D , and E in Figure 4. Verify your solutions by substituting these coordinates into the appropriate equations.

- 2.5**
- Graph the feasible region described by the following constraints.

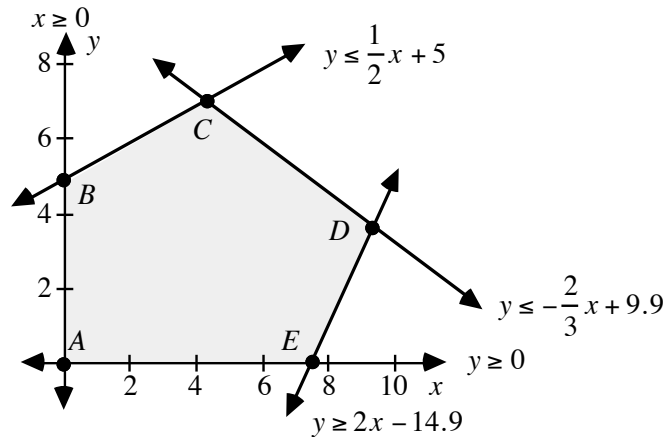
$$\begin{cases} 3x + y \leq 15 \\ x + 2y \leq 20 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

- Determine the coordinates of the corner points of the feasible region.
 - Given the constraints in Part **a**, find the minimum value of each of the following objective functions:
 - $P = 2y - x$
 - $P = 3y + 2x$
- 2.6**
- Repeat Problem **2.3** if each pizza costs the class \$6.00 and each six-pack costs \$2.00.
 - How would you try to find the maximum profit if the corner principle resulted in a solution that requires the purchase of fractional amounts of pizza or six-packs?
 - What do you think a businessperson might do if faced with the situation in Part **b**?

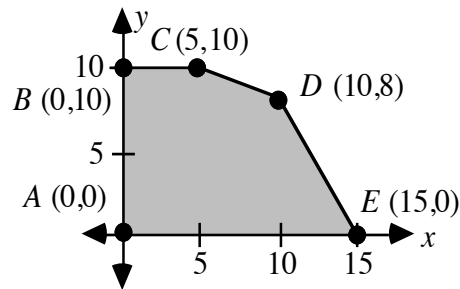
- 2.7 Imagine that your class decides to sell nachos and canned soft drinks as a fund-raising project. You have \$100.00 available to purchase these items. Determine the maximum potential profit for the class. Describe any assumptions you make in analyzing this situation.

* * * * *

- 2.8 The feasible region shaded in the following graph is defined by five constraints. Determine the coordinates of the corner points of this region.



- 2.9 The following graph shows a feasible region with five corner points.



- a.
 1. Which point in the feasible region results in the maximum value of the objective function $P = 5x + 5y$?
 2. Which point in the feasible region results in the minimum value of the objective function $P = 5x + 5y$?
- b.
 1. Determine the slope of the line that passes through points D and E .
 2. Determine the value of the objective function $P = 8x + 5y$ at D and E .
 3. Write $P = 8x + 5y$ in slope-intercept form and identify the slope.
 4. Describe how your responses to Steps 1 and 2 are related.
 5. A graph of the objective function $P = 8x + 5y$ for three values of P would consist of three lines. Describe how these lines would be related to \overline{DE} .
- c. Find an example of an objective function that has its maximum value at each of the following:

1. both point D and point C
2. point C only
3. point E only.

* * * * *

Research Project

Plan a fund-raiser for your mathematics class that involves selling two items at a dance, game, or other school-related activity. Design a survey to determine what two items to sell, how much to charge for each item, and how much of each item to purchase. Administer the survey. Assume that you can interest enough investors in your plan to raise \$50.00. Use this constraint and the results from the survey to determine the constraints for the fund-raiser.

Use linear programming to determine the maximum profit that each investor could expect to earn. Prepare a report for the class to persuade each student to become an investor. You may wish to include the survey questionnaire, survey results, initial costs, expected profit, graphs, and sample advertising posters.

Activity 3

In Activity 2, you used the corner principle to determine the maximum or minimum value of an objective function. Identifying the coordinates of corner points often involves solving a system of equations, either by substitution or by some other method. In this activity, you use matrices to solve systems of linear equations.

Exploration

Using matrices to solve a system of equations involves two steps: writing a matrix equation to represent the system, then solving the resulting matrix equation.

- a. Consider the following system of equations:

$$\begin{cases} -8x + 10y = 40 \\ 10x + 10y = 139 \end{cases}$$

The **matrix equation** for this system is shown below. Simplify this equation by completing the matrix multiplication on the left-hand side of the equal sign.

$$\begin{bmatrix} -8 & 10 \\ 10 & 10 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 139 \end{bmatrix}$$

- b. Compare the simplified matrix equation with the original system of equations. Describe any similarities you observe.

Mathematics Note

Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

may be written as the matrix equation:

$$\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, matrix **X** is the **variable matrix**, since it represents the variables of the system, while matrix **C** is the **constant matrix**, since it represents the constants of the system.

For example, the system

$$\begin{cases} 3x + 4y = 22 \\ -5x + 2y = 1 \end{cases}$$

may be written as the following matrix equation:

$$\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} 3 & 4 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 1 \end{bmatrix}$$

The **multiplicative inverse** of matrix **M** is written \mathbf{M}^{-1} and has the property that $\mathbf{M} \cdot \mathbf{M}^{-1} = \mathbf{M}^{-1} \cdot \mathbf{M} = \mathbf{I}$, where **I** is the identity matrix for matrix multiplication. The identity matrix for 2×2 matrices is shown below.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For example, consider the following two matrices:

$$\mathbf{M} = \begin{bmatrix} -5 & 8 \\ -2 & 3 \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 3 & -8 \\ 2 & -5 \end{bmatrix}$$

Since \mathbf{M}^{-1} is the multiplicative inverse of \mathbf{M} , their product is the identity matrix \mathbf{I} :

$$\begin{bmatrix} 3 & -8 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 & 8 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- c. Use technology to verify that the value of \mathbf{M}^{-1} given in the mathematics note is correct.
- d. Use technology to verify that $\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$ for two different 2×2 matrices \mathbf{A} .
- e. A linear equation of the form $a \cdot x = b$ can be solved for x by multiplying both sides of the equation by the multiplicative inverse of a .

Similarly, a matrix equation of the form $\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$ can be solved for matrix \mathbf{X} by multiplying both sides of the equation by the multiplicative inverse of \mathbf{M} , as shown below, and using the associative property for matrix multiplication:

$$\mathbf{M}^{-1} \cdot (\mathbf{M} \cdot \mathbf{X}) = \mathbf{M}^{-1} \cdot \mathbf{C}$$

Note that each side of the equation must be multiplied on the left by \mathbf{M}^{-1} and recall that $\mathbf{M}^{-1} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{M}^{-1} = \mathbf{I}$.

1. Write the following system of linear equations as a matrix equation of the form $\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$:

$$\begin{cases} 5x + 2y = 22 \\ 2x + y = 9 \end{cases}$$

2. Use technology to find the inverse of the coefficient matrix \mathbf{M} .
3. Solve the matrix equation for the variable matrix \mathbf{X} by multiplying each side by \mathbf{M}^{-1} . Write the resulting equation in the form below:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$$

4. Verify by substitution that $\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$.
5. Write the solution of the system as an ordered pair.
6. Verify your solution by substituting these coordinates in the original equations.
7. Check the solution by graphing the two equations on the same coordinate system.

- f. When a system of equations has no solution, it is said to be **inconsistent**.
1. Verify that the system shown below is inconsistent by graphing the two equations on the same coordinate system.

$$\begin{cases} x + 2y = 4 \\ 2x + 4y = 13 \end{cases}$$

2. When an attempt is made to solve an inconsistent system using matrices, one step of the process cannot be completed. Repeat Part e using the system of equations above until you discover where the process fails.

Discussion

- a. In Part c of the exploration, you used technology to verify that the value of \mathbf{M}^{-1} given in the mathematics note was correct. Describe two different ways that you can use technology to make this verification.
- b. If a system is inconsistent, what can you conclude about the coefficient matrix \mathbf{M} ?
- c. Describe the coefficient matrix for the following system of equations:

$$\begin{cases} 3x = 7 \\ 5x + 2y = 11 \end{cases}$$

- d. When multiplying both sides of a matrix equation $\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$ by the multiplicative inverse of \mathbf{M} , each side of the equation must be multiplied on the left by \mathbf{M}^{-1} . Why is this necessary?
- e. Matrices also can be used to solve systems of equations involving more than two equations and two unknowns. Consider the following system of three equations and three unknowns:

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 15 \\ x + 3y - 2z = -7 \end{cases}$$

Describe how you could rewrite this system as a matrix equation including a coefficient matrix, a variable matrix, and a constant matrix.

Assignment

- 3.1 a. Find the multiplicative inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$$

- b. Verify that your response to Part a is correct.
c. Create a 2×2 matrix that does not have an inverse. Defend your response.

- 3.2 Solve each of the following matrix equations and verify your solutions.

a. $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 5 & 2 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

- 3.3 Use matrices to solve the following system of equations:

$$\begin{cases} 80x + 33y = 466 \\ -20x + 73y = 46 \end{cases}$$

- 3.4 a. Graph the feasible region described by the following constraints.

$$\begin{cases} 3x + 8y \leq 48 \\ 3x + 2y \leq 24 \\ -x + y \leq 0 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

- b. Determine the coordinates of the corner points of the feasible region.
c. Given these constraints, find the maximum value of the objective function $P = 2y + 3x$.

- 3.5** Your class has \$100.00 to buy pizza and canned soda pop for a concession booth. Each pizza costs \$5.00 and each six-pack costs \$2.00. The constraints on your purchases are listed below:
- no more than \$100.00 can be spent on pizza and pop
 - at least \$10.00 must be spent on pop
 - no more than \$80.00 can be spent on pizza
 - the amount spent on pizza cannot exceed 4 times that spent on pop.
- a. Express these constraints as inequalities.
 - b. Graph the feasible region described by these constraints.
 - c. The class expects a profit of \$11.00 on each pizza and \$2.00 on each six-pack. Determine the maximum potential profit in this situation.

* * * * *

- 3.6** The manager of a municipal water system must decide how to provide the town with at least $20 \cdot 10^6$ L per day. Some of this the water can be drawn from a nearby reservoir. The rest must be purchased from a company that pumps water from a large underground aquifer.

In order to maintain the level of the reservoir, no more than $12 \cdot 10^6$ L per day can be drawn from this source. The town's contract with the water company states that the city must purchase at least $14 \cdot 10^6$ L per day, but no more than $20 \cdot 10^6$ L per day.

- a. Express these constraints as inequalities.
- b. Graph the feasible region described by these constraints.
- c. Water from the reservoir costs the town \$80.00 for $1 \cdot 10^6$ L, while water from the water company costs \$100.00 for $1 \cdot 10^6$ L.
 1. Determine the town's smallest possible daily water bill.
 2. Determine the town's largest possible daily water bill.

* * * * *

Activity 4

In Activity 1, you explored the potential sale of three items at a concession stand: pizza, pop, and nachos. Do you think it is possible to use linear programming and the corner principle to determine the maximum profit in this situation?

Discussion 1

- When considering the sale of two items, the feasible solutions form a region on a coordinate plane. The boundaries of the region are formed by lines whose equations are determined by constraints.

Suppose that the class wanted to decide which two of the three items—pizza, pop, or nachos—to sell. How many coordinate planes would be needed to graph the feasible regions for all the possible pairs?
- How many variables are necessary to model a situation in which three items are sold?
- Is it possible to graph the constraints for all three items on a single coordinate plane? Explain your response.
- Is it possible to represent the constraints on all three items on another type of coordinate system? If not, why not? If so, what geometric figure do you think could be used to represent the boundaries?

Exploration

- Cut the top, front, and one side from a cardboard box.
- Use the remaining inside corner to create a three-dimensional coordinate system. As shown in Figure 5, label the two axes that lie in the bottom of the box x and y , respectively. Label the vertical axis z . Divide each axis into intervals of at least 1 cm and label the intervals accordingly.

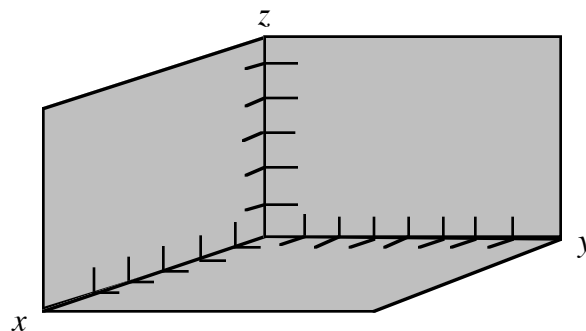


Figure 5: A three-dimensional coordinate system

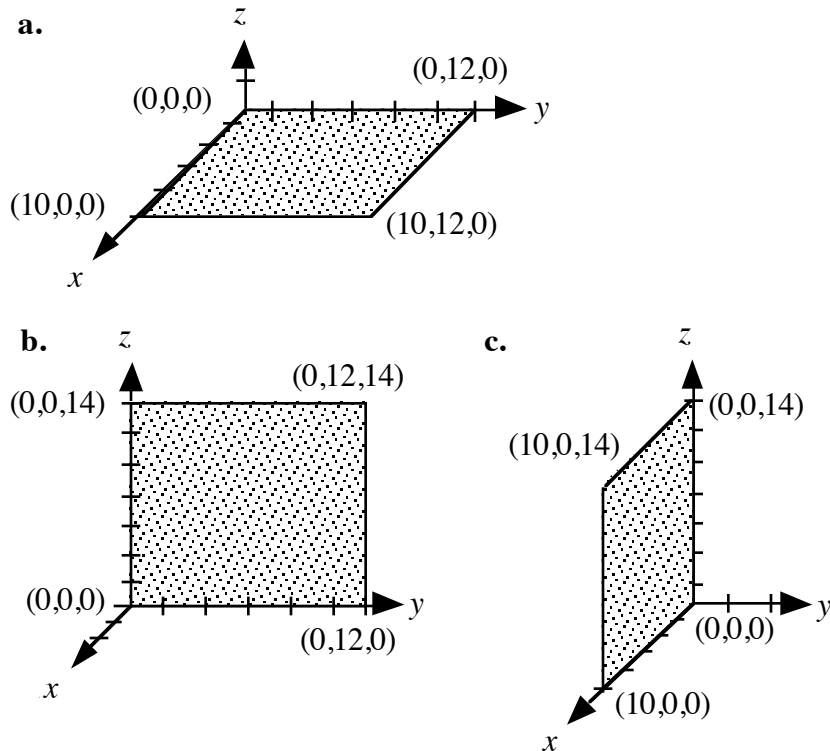
- c. The model shown in Figure 5 represents the first octant of a three-dimensional coordinate system. On the planes of the first octant, shade and label the regions defined by the following sets of constraints.
- Region A : $0 \leq x \leq 6, 0 \leq y \leq 4, z = 0$
- Region B: $0 \leq y \leq 4, 0 \leq z \leq 5, x = 0$
- Region C: $0 \leq x \leq 6, 0 \leq z \leq 5, y = 0$
- d. Cut shapes that approximate these regions from leftover cardboard.
- e. 1. Position the cutout of region A on your coordinate system to represent the constraints $0 \leq x \leq 6, 0 \leq y \leq 4$, and $z = 1$. Note its position.
2. Move the same cutout along the z -axis to the position that represents the constraints $0 \leq x \leq 6, 0 \leq y \leq 4$, and $z = 2$. Compare this position with the position in Step 1.
3. Repeat the process described in Step 2, increasing the value of z by 1 unit each time, until the cutout is positioned at the edge of the shaded region on the yz -plane.
- f. Tape the cutouts of regions A, B, and C in the first octant so that they form the boundaries of a shape that includes the set of feasible solutions described by $0 \leq x \leq 6, 0 \leq y \leq 4$, and $0 \leq z \leq 5$.
- g. Identify the corner points of the shape formed in Part f.

Discussion 2

- a. Describe the set of points obtained as you moved the cutout of region A in Part e of the exploration.
- b. 1. What is the least number of planes that can intersect in exactly one point? Explain your response.
2. Imagine that the constraints of a problem indicate that four planes define the set of feasible solutions. Why is 4 the maximum number of corner points in this situation?
3. What is the maximum number of corner points if five planes define the feasible solutions? Explain your response.
- c. 1. What shape is formed in Part f of the exploration? How many corner points does this shape have?
2. Would you expect this shape to be the same for every set of feasible solutions involving six planes? Why or why not?
- d. 1. Where would you expect the maximum value of an objective function to occur in terms of the set of feasible solutions from Part f?
2. How could you verify the location of the maximum value?
- e. Do you think it is possible to graphically represent a situation in which you consider more than three items for sale? Explain your response.

Assignment

- 4.1** In the three graphs below, x represents the number of pizzas, y the number of six-packs, and z the number of nacho kits. For each graph, identify the items whose constraints are represented, and write inequalities to describe the shaded region.



- 4.2** Describe what the inequalities in Problem **4.1** mean in terms of the number of pizzas, six-packs, and nacho kits the class should purchase.
- 4.3** In Problem **4.1a**, the boundaries of the feasible region are line segments. The equations of the lines that form these boundaries are $x = 10$, $x = 0$, $y = 12$, and $y = 0$. You can use these equations to determine the intersection of this plane with other planes.
- Write the equations that represent the boundary lines for the region in Problem **4.1b**.
 - Write the equations that represent the boundary lines for the region in Problem **4.1c**.
 - Describe how you could find the points of intersection of the boundary lines given only their equations.
- 4.4** Sketch the shape that defines the feasible solutions when the constraints on all three items in Problem **4.1** are considered at once. List the coordinates of the corner points of this shape.

4.5 Use the corner points from Problem **4.4** to find the maximum value for the objective function $P = x + y + z$. Describe the method you used to find the maximum value.

4.6 Use matrices to find the intersection of the three planes represented by each of the systems of equations below. Identify any system that is inconsistent.

a. $3x - 2y + 3z = 0, -x + 4y + z = 8, x - 2y + z = 4$

b. $2x + y + 2z = 4, 3x - 6y + 2z = 12, 4x + 2y + 4z = 12$

c. $x + y = 3, 2x + z = 9, 3x - 4y + z = 20$

4.7 Suppose that the class decides to sell pizza, pop, and nachos. Each pizza costs the class \$5.00 and each six-pack costs \$2.00. Nacho kits contain four bags of tortilla chips and one container of cheese. Each bag of tortilla chips costs \$2.00; each container of cheese also costs \$2.00.

There are eight slices in each pizza and the class plans to sell them for \$2.00 a slice. A can of pop sells for \$1.00. Each nacho kit contains enough chips and cheese to make 16 orders of nachos. One order of nachos sells for \$1.50.

a. Determine the profit on each pizza, six-pack, and nacho kit sold.

b. Let x represent the number of pizzas, y the number of six-packs, and z the number of nacho kits purchased by the class. Write an objective function to calculate the maximum profit the class can expect to earn.

4.8 a. The class in Problem **4.7** plans to spend no more than \$50.00 on any one of the three items to be sold: pizza, pop, and nachos. Write inequalities to describe these constraints.

b. How many planes enclose the feasible solutions for these constraints?

c. How many points of intersection must be examined to determine the maximum profit? Explain your response.

4.9 Using the information determined in Problems **4.7** and **4.8**, find the maximum profit the class can expect to earn.

* * * * *

4.10 Using the cost of pizza, pop, and nacho kits from a local store and a \$50 spending limit for each item, find the maximum profit that your class could earn selling concessions. Record all assumptions you make in analyzing this situation.

4.11 Use matrices to determine the solution to the system of equations below.

$$\begin{cases} x + y + z \leq 15 \\ 2x + y + 2z \leq 26 \\ 5x + 2y + 3z \leq 43 \end{cases}$$

* * * * *

Summary Assessment

1. As shift manager of a fast-food restaurant, you must make sure that enough meals are prepared to serve customers quickly—but not so many that food is wasted.

During a typical lunch-time rush, you sell no more than 100 single burgers and no more than 90 double burgers. A single burger requires 100 g of hamburger, while a double burger requires 200 g. To help avoid spoilage, you plan to use no more than 20 kg of hamburger during the rush.

Before being served to a customer, each single burger requires 6 seconds in a microwave oven, while each double burgers requires 8 seconds. During the lunch-time rush, there are no more than 16 minutes of available time in the microwave.

The restaurant earns a profit of \$0.70 on each single burger and \$0.90 on each double burger. Assuming that customers purchase every burger prepared, determine how many of each type should be made in order to maximize profits.

2. As class president, you are analyzing the best way for your class to earn money with its concession stand. The class officers have decided to sell hot dogs, potato chips, and soda pop. The table below shows the cost and selling price for each of these items.

Item	hot dog	bag of chips	can of pop
Cost	\$0.15	\$0.20	\$0.35
Selling Price	\$1.00	\$1.00	\$1.00

The class has \$108.00 in its fund-raising account. The total cost for hot dogs, chips, and pop cannot exceed this amount. Judging from last year's sales, the class will sell at least twice as many hot dogs as bags of chips, and at least 3 times as many cans of pop as the combined numbers of chips and hot dogs. Determine the profit at the corner point defined by these three constraints.

Module Summary

- A **feasible region**, or set of **feasible solutions**, consists of all the points that satisfy the limitations, or linear **constraints** of a problem. The vertices of the feasible region are **corner points**.
- **Linear programming** is a method used to maximize or minimize an objective (such as profit), subject to linear constraints. The function that describes the objective is called the **objective function**.
- When the corner points are part of the feasible region, the **corner principle** provides a method of finding the maximum or minimum values of an objective function. According to this principle, the maximum and minimum values of an objective function occur at the corner points of the feasible region.
- The **substitution method** for solving a system of linear equations can be used to solve a system of equations involving any number of variables.
- Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

may be written as the matrix equation:

$$\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, matrix **X** is the **variable matrix**, since it represents the variables of the system, while matrix **C** is the **constant matrix**, since it represents the constants of the system.

- The **multiplicative inverse** of matrix **M** is written \mathbf{M}^{-1} and has the property that $\mathbf{M} \cdot \mathbf{M}^{-1} = \mathbf{M}^{-1} \cdot \mathbf{M} = \mathbf{I}$, where **I** is the identity matrix for matrix multiplication.

Selected References

Tan, S. T. *Applied Finite Mathematics*. Boston, MA: PWS Publishing, 1994.

Crazy Cartoons



How are cartoon characters made? How do they change shape? How can you make an inanimate object appear to move? A cartoonist—or a math student—can help answer these questions.

Monty Brekke • John Carter • Janet Kuchenbrod



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Crazy Cartoons

Introduction

Think about your favorite animated cartoon characters. You've probably seen them stretched, flipped, smashed, and racing away in a blur of speed. In order to create these sequences, cartoonists once drew hundreds of individual frames.

In each frame, the size, shape, or position of the character was slightly altered. When the frames were transferred to film and run through a projector, the character appeared to move.

Although many of today's animated cartoons are created with computers, the process still follows much of the same logic. In this module, you develop the geometry skills that can help you create your own cartoon.

Discussion

- a. Figure 1 below shows several different images of a cartoon bug. Describe how the bug in frame A could be changed to create the image in each of the other frames.

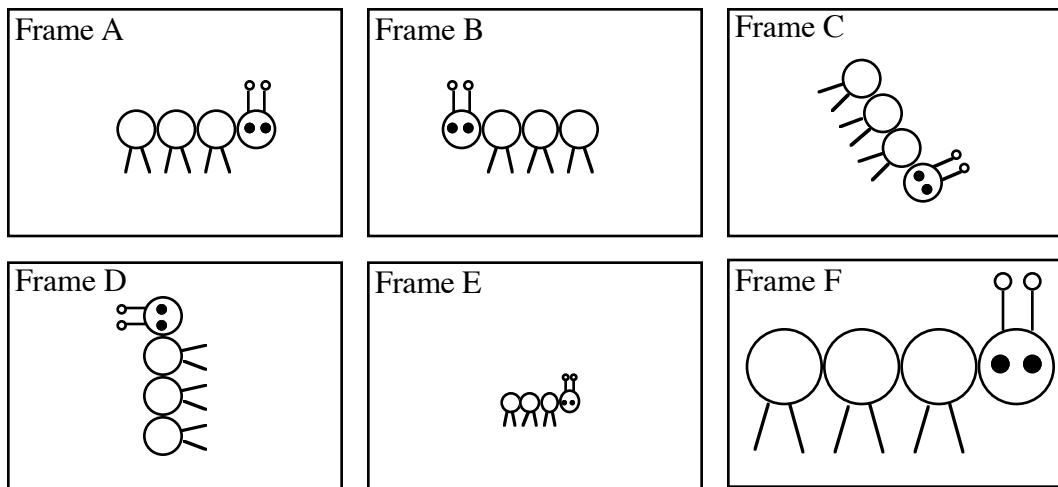


Figure 1: Transformations of a bug

- b. Describe how the bug in frame A could be changed to the bug in frame B through two or more modifications.

Activity 1

Each change of the bug in Figure 1 is an example of a **transformation**. In this activity, you investigate a type of transformation that alters the size of a figure proportionally.

Mathematics Note

A **one-to-one correspondence** is a function between two sets: the domain and the range. It pairs each element in the domain with exactly one element in the range, and each element in the range with exactly one element in the domain.

For example, Figure 2 shows a one-to-one correspondence between the fingers of the left hand and the fingers of the right hand.

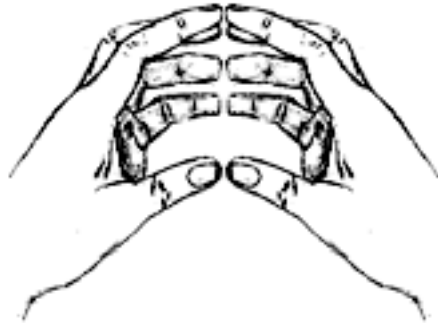


Figure 2: A one-to-one correspondence between two hands

A **transformation** in a plane is a one-to-one correspondence of the plane onto itself. For example, Figure 3 shows a reflection in the y -axis. In this transformation, point $A(2,5)$ is paired with $A'(-2,5)$. This type of pairing occurs for every point in the plane.

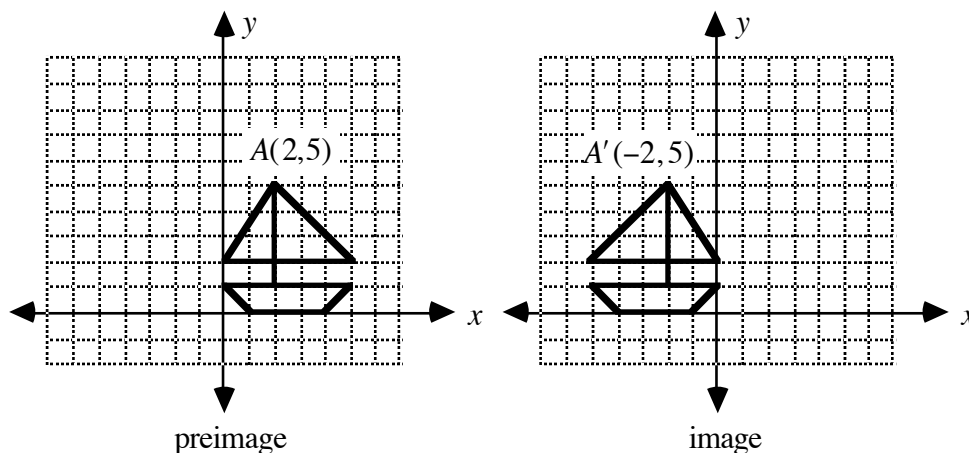


Figure 3: A transformation of a sailboat

Exploration

Before drawing a cartoon of your own, some experience with simple figures may be helpful. In this exploration, you use graph paper to construct a **point-perspective drawing**. Artists use point-perspective drawings to represent three-dimensional objects in two dimensions. For example, Figure 4 shows two rectangular prisms drawn using this method, with point P as the **point of perspective**.

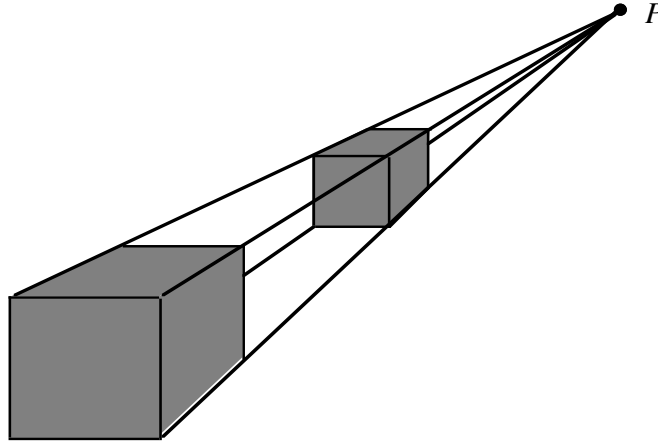


Figure 4: Point-perspective drawing of two rectangular prisms

- On a sheet of graph paper, plot the ordered pairs $A(1,1)$, $B(2,1)$, and $C(1,3)$. Draw triangle ABC .
- On the same sheet of graph paper, plot the following ordered pairs: $A'(3,3)$, $B'(6,3)$, and $C'(3,9)$. Draw triangle $A'B'C'$.
- Use your drawing, the Pythagorean theorem, and the inverse trigonometric functions to complete Table 1 below.

Table 1: Measures of corresponding parts of ΔABC and $\Delta A'B'C'$

Preimage		Image	
AB		$A'B'$	
BC		$B'C'$	
CA		$C'A'$	
$m\angle ABC$		$m\angle A'B'C'$	
$m\angle BCA$		$m\angle B'C'A'$	
$m\angle CAB$		$m\angle C'A'B'$	

- Determine each of the following ratios:
 - $A'B'/AB$
 - $B'C'/BC$
 - $C'A'/CA$

- e. Draw $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. Extend these segments until they intersect at a common point, the point of perspective. Label this point P and identify its coordinates.

Mathematics Note

As shown in Figure 5, the Pythagorean theorem can be used to derive a formula for the distance between any two points, (x_1, y_1) and (x_2, y_2) , on a coordinate plane.

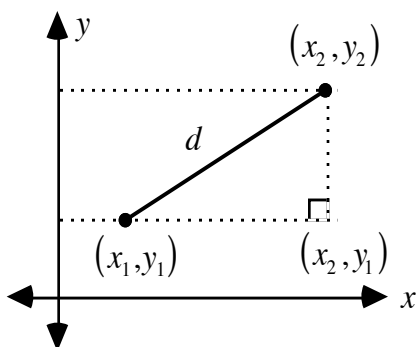


Figure 5: Distance between two points on a coordinate plane

According to the **distance formula**, the distance d between (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, consider the points $(-2, 5)$ and $(-4, -6)$. Using the distance formula, the distance between these two points can be found as follows:

$$\begin{aligned} d &= \sqrt{(-4 - (-2))^2 + (-6 - 5)^2} \\ &= \sqrt{(-2)^2 + (-11)^2} \\ &= \sqrt{125} \approx 11.18 \end{aligned}$$

- f. Use the distance formula to find the lengths of each of the following segments: \overline{PA} , \overline{PB} , \overline{PC} , $\overline{PA'}$, $\overline{PB'}$, and $\overline{PC'}$.
- g. Determine each of the following ratios:
1. PA'/PA
 2. PB'/PB
 3. PC'/PC
- h. Compare the ratios you determined in Part g with those found in Part d.

Discussion

- a. How is the Pythagorean theorem used to derive the distance formula?
- b. What did you observe about the ratios of corresponding lengths in Parts d and g of the exploration?

Mathematics Note

A **dilation** is a transformation that pairs a point P , the **center**, with itself and any other point X with a point X' on ray PX so that $PX'/PX = r$, where r is the **scale factor**.

A dilation with its center at the origin is a transformation such that every point Q with coordinates (x,y) has an image Q' with coordinates (rx,ry) , where r is the scale factor, and $r \neq 0$. The scale factor also may be written as a ratio of the corresponding dimensions of the image to the preimage.

For example, Figure 6 shows $\triangle DEF$ with vertices at the points with coordinates $(1,1)$, $(2,1)$ and $(1,2)$ and its dilation with center at the origin O and scale factor 2. The vertices of the image, $\triangle D'E'F'$, are the points with coordinates $(2,2)$, $(4,2)$ and $(2,4)$, respectively.

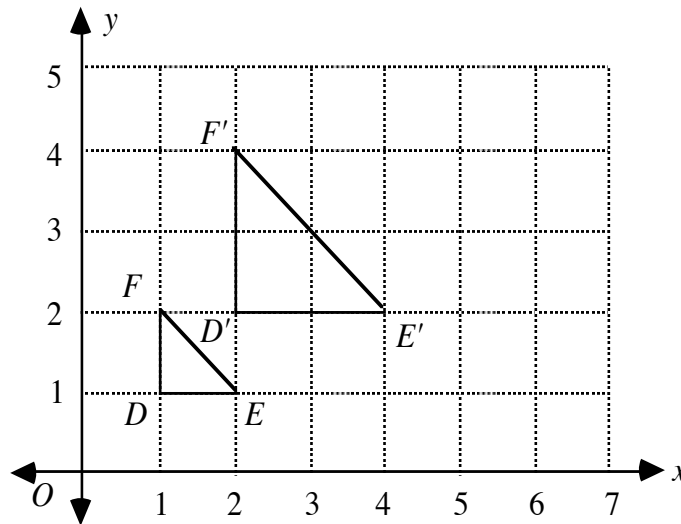


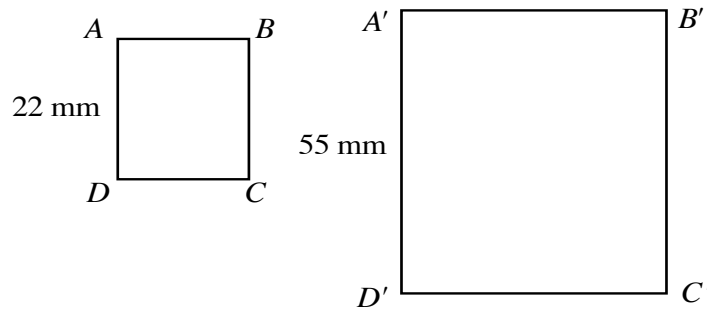
Figure 6: Preimage and image of a triangle under a dilation

- c. The transformation of $\triangle ABC$ to $\triangle A'B'C'$ is a dilation. Describe the similarities between a dilation and a point-perspective drawing.
- d. In Part c of the exploration, what did you observe about the measures of corresponding angles of $\triangle ABC$ and $\triangle A'B'C'$?
- e. In a dilation, what geometric properties appear to remain the same in both the image and the preimage?
- f.
 1. Describe the geometric relationship between $\triangle ABC$ and $\triangle A'B'C'$.
 2. Identify any other triangles on your graph that share this same relationship.

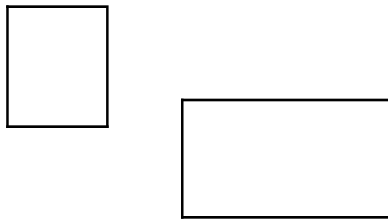
- g.**
1. How does the scale factor for the dilation compare to the ratios of corresponding lengths for $\triangle ABC$ and $\triangle A'B'C'$?
 2. How does the scale factor compare to the ratio of the perimeters of the two triangles?
 3. How does the scale factor compare to the ratio of the areas of the two triangles?
- h.** If you moved the center of the dilation, do you think that the relationships identified in Parts **f** and **g** would still be true?
- i.** How could you change the scale factor to produce an image that is smaller than the preimage?
- j.** In general, what is the ratio of the distance from the center of dilation to a point on the preimage to the distance from the center of dilation to the corresponding point on the image?

Assignment

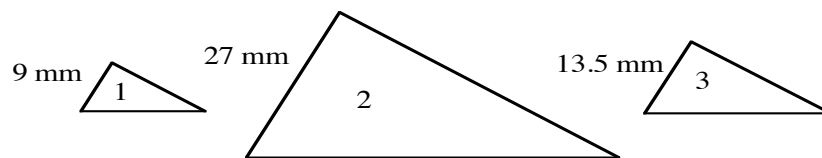
- 1.1 a.** The two figures in the following diagram are similar. Describe how to find the center of the dilation that transforms $ABCD$ to $A'B'C'D'$.



- b.** Find the scale factor of the dilation.
- 1.2** Is it possible to find a point of perspective in the drawing below, where either figure can be the preimage and the other its image? Explain your response.



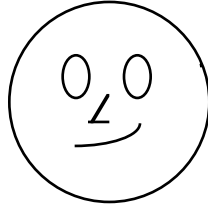
- 1.3**
- Draw the triangle formed by the points $A(5,10)$, $B(-5,5)$, and $C(10,-15)$.
 - Find the image of the triangle under a dilation with center at the origin O and a scale factor of $2/5$.
 - Label the vertices of the image and identify their coordinates.
 - Determine the scale factor of the dilation of $\triangle ABC$ for each of the following:
 - $A'(10,20)$, $B'(-10,10)$, and $C'(20,-30)$
 - $A'(-1,-2)$, $B'(1,-1)$, and $C'(-2,3)$
 - $A'(12.5,25)$, $B'(-12.5,12.5)$, and $C'(25,-37.5)$
- 1.4**
- On the same sheet of graph paper, draw the images of the triangle from Problem **1.3a** in a series of four dilations with center at the origin O and scale factors of $5/4$, $3/2$, $7/4$, and 2 .
 - Suppose that each image was taped to a separate file card and the cards arranged in the order given in Part **a**. If you held the cards on one side and flipped through them—as if flipping the edge of a telephone book—what would appear to happen to the original triangle?
 - For any pair of images in Part **a**, describe how to find a dilation that transforms one image to the other.
- 1.5** The similar triangles below represent different images of a cartoon character's eye.



- Determine the scale factor of a dilation that transforms triangle 1 to triangle 2.
- Determine the scale factor of a dilation that transforms triangle 2 to triangle 3.
- Determine the scale factor of a dilation that transforms triangle 1 to triangle 3.
- Considering the arrangement of the triangles in the diagram above, could the dilations in Parts **a** and **b** have the same center? Explain your response.

* * * * *

- 1.6** The coordinates of the vertices of quadrilateral $ABCD$ are $A(0,0)$, $B(0,3)$, $C(5,3)$, and $D(5,0)$. The coordinates of the vertices of its image are $A'(0,0)$, $B'(0,6)$, $C'(10,6)$, and $D'(10,0)$. Find the center of dilation and the scale factor for this transformation.
- 1.7** Sketch a dilation of the face shown below using any point on a coordinate grid as the point of perspective. On your sketch, label the point of perspective, one point in the preimage, and the corresponding point in the image. Describe how to determine the scale factor of the dilation.



- 1.8** The coordinates of the vertices of quadrilateral $ABCD$ are $A(1.9,1.2)$, $B(4.9,1.2)$, $C(4.9,3.2)$ and $D(1.9,3.2)$. The coordinates of the vertices of its image are $A'(5.3,6.7)$, $B'(11.3,6.7)$, $C'(11.3,10.7)$ and $D'(5.3,10.7)$.
- Graph the two quadrilaterals on the same coordinate plane.
 - Show that the quadrilaterals are similar.
 - What is the scale factor of the dilation that transforms $ABCD$ to $A'B'C'D'$?
 - On your graph from Part **a**, draw the lines of perspective.
 - Estimate the coordinates of the point of perspective.
 - Determine the exact coordinates of the point of perspective. (Hint: Find the equations of two lines of perspective.)

* * * * *

Activity 2

Welcome to Skip's world! Although Skip is relatively easy to draw, his face still requires 19 ordered pairs to define. In this activity, you use matrices to store the ordered pairs that define Skip and his images.

Exploration

- a. Figure 7 shows a picture of Skip. Graph Skip on a coordinate grid with his chin on the x -axis and the highest point on his cap on the y -axis.

Label your axes so that the coordinates for Skip's cap can be represented by the following points: $(9,6)$, $(0,6)$, $(0,8)$, $(7,7)$, and $(7,6)$.

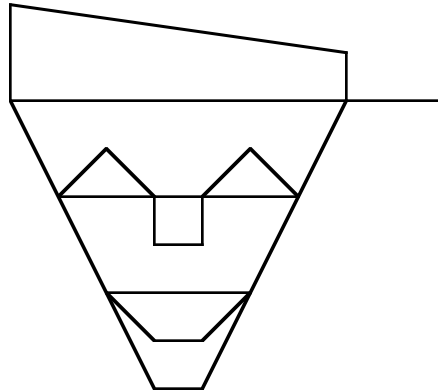


Figure 7: Skip, a cartoon character

Mathematics Note

A point P with coordinates (x,y) can be represented in a matrix as shown below.

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Using matrices to represent several points in this way can be helpful in many applications. For example, one of Skip's eyes is a triangle with vertices at $(1,4)$, $(2,5)$, and $(3,4)$. These points can be mathematically described by the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \end{bmatrix}$$

- b. List the coordinates for Skip's mouth in a matrix \mathbf{M} .
- c. Using matrix \mathbf{M} , draw Skip's mouth on another sheet of graph paper. Draw a dilation of Skip's mouth with center at the origin and a scale factor of 3.
- d. Write a matrix \mathbf{M}' for the ordered pairs that define the image of Skip's mouth from Part c.
- e. Dilations, along with some other types of transformation which you will examine later, can be accomplished using matrix multiplication. For example, Skip's mouth can be dilated by multiplying \mathbf{M} on the left by a 2×2 matrix. Determine the dilation matrix \mathbf{D} necessary to transform \mathbf{M} to \mathbf{M}' , where \mathbf{D} is of the following form:

$$\mathbf{D} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(Hint: Find the values for a , b , c , and d so that $\mathbf{D} \cdot \mathbf{M} = \mathbf{M}'$.)

- f. List the coordinates for Skip's entire face in a matrix \mathbf{S} .

Note: When using some forms of technology to draw closed figures, you should enter the coordinates of each vertex in the order in which you would like them to be connected.

To determine this order, trace your graph of Skip without lifting your pencil from the paper or retracing any segments. Mark both your starting and stopping points. You may have to list both these points in the matrix—even if they have the same coordinates.

Enter the coordinates of your starting point in the first column, the coordinates of the second point in the second column, those of the third point in the third column, and so on. List the coordinates of the stopping point in the last column of the matrix.

- g. Multiply \mathbf{D} , the dilation matrix found in Part e, by \mathbf{S} . Compare the elements in \mathbf{S} to those in \mathbf{S}' .

Discussion

- a. Describe how a dilation of Skip's mouth by a scale factor of 3, with center at the origin, relates to scalar multiplication.
- b. Why must matrix \mathbf{S} be multiplied on the left by the 2×2 dilation matrix \mathbf{D} ?
 1. By what 2×2 matrix could you multiply matrix \mathbf{M} to dilate Skip's mouth by a scale factor of 3, with center at the origin?
 2. By what 2×2 matrix could you multiply matrix \mathbf{M} to dilate Skip's mouth by a scale factor of $1/4$, with center at the origin?
 3. By what 2×2 matrix could you multiply matrix \mathbf{M} to dilate Skip's mouth by a scale factor of 1, with center at the origin?

- d. In the module “Marvelous Matrices,” you learned that the 2×2 identity matrix for matrix multiplication is:

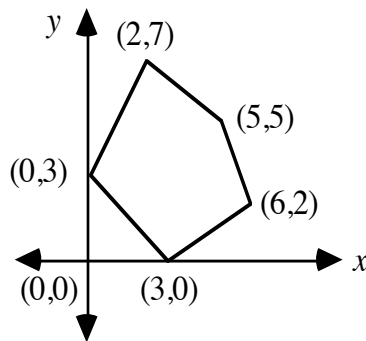
$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Describe how the product of a scalar and the 2×2 identity matrix relates to a matrix that produces a dilation with center at the origin.

- e. Suppose Skip is dilated by a scale factor of n , with center at the origin. Describe how to determine the matrix that represents Skip’s image.
- f. In a dilation, what is the geometric relationship between an image and its preimage?

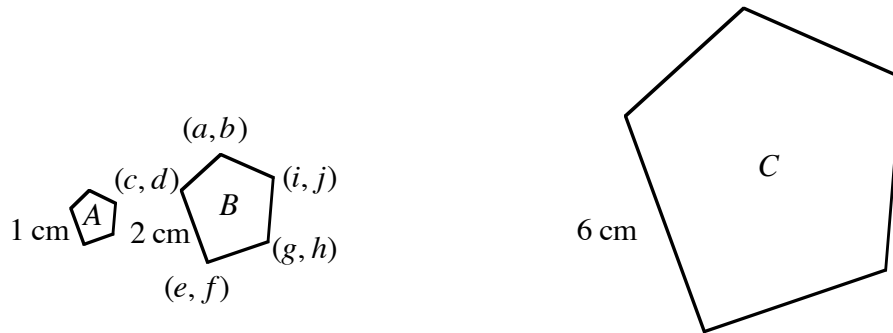
Assignment

- 2.1 a. Write a matrix to represent the vertices of the polygon below so that connecting the points in order results in a closed figure.



- b. 1. Write a matrix expression that produces a dilation of the polygon in Part a by a scale factor of 3.5, with center at the origin.
2. Write a different matrix expression from the one in Part b1 that produces the same dilation of the polygon.
- c. Use technology to find the coordinates of the image polygon. Write the results in a matrix.
- 2.2 a. Write a matrix expression to show the dilation of Skip by a scale factor of 5, with center at the origin.
- b. Write a matrix expression to show the dilation of Skip’s cap by a scale factor of $3/4$, with center at the origin.
- c. Use technology to find the coordinates of the image of Skip’s mouth under a dilation with center at the origin and a scale factor of $3/4$. Write the coordinates of the image in a matrix.

2.3 The diagram below shows three similar polygons.



- a. Write a matrix for polygon B .
 - b. If polygon B is the preimage, what scale factors are needed to create polygons A and C , respectively?
 - c. Consider a dilation, with center at the origin, in which polygon B is the preimage and polygon A is the image. Write a matrix equation to describe this transformation.
 - d. Consider a dilation, with center at the origin, in which polygon A is the preimage and polygon C is the image. Write a matrix equation to describe this relationship.
- 2.4 Use technology to determine the image matrix that represents a dilation of Skip by a scale factor of 3, with center at the origin. Graph both the preimage and the image.

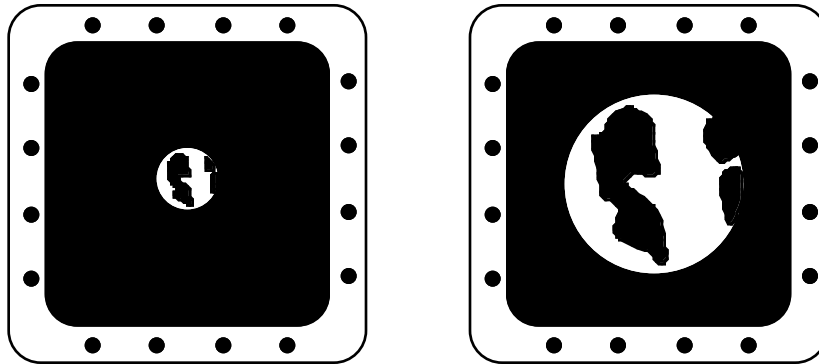
* * * * *

2.5 The following matrix represents a company's daily production of sweatshirts, in two colors and four sizes.

	small	medium	large	x-large
blue	$\begin{bmatrix} 30 & 70 & 30 & 40 \end{bmatrix}$			
red	$\begin{bmatrix} 50 & 80 & 20 & 10 \end{bmatrix}$			

The company plans to quadruple its daily production. Write a matrix to represent this information.

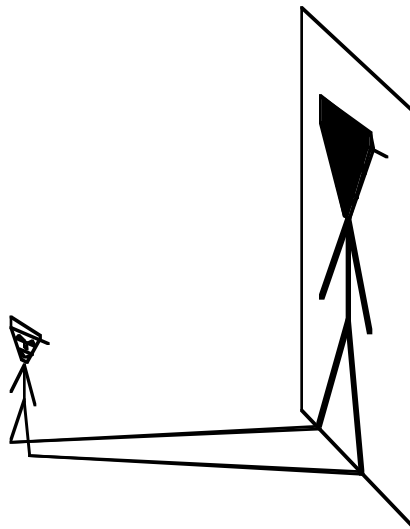
- 2.6 The following figure shows two views from the window of a spacecraft traveling at a constant velocity. As the spacecraft approaches earth, the planet's image appears to dilate.



6:00 P.M. Tuesday

1:00 A.M. Wednesday

- Use a ruler to determine the difference in scale between the two images of earth.
 - If the spacecraft continues to approach earth at the same velocity, what will the diameter of the earth's image be after another 7 hr?
- 2.7 Skip's friend Chip used a flashlight to produce a giant shadow of Skip, as shown in the illustration below.



- Use a copy of this illustration to determine where Chip held the flashlight.
- If Chip wants Skip's shadow to be five times as tall as Skip, describe where he should hold the flashlight.
- If Chip keeps moving the flashlight toward Skip, what will appear to happen to Skip's shadow? Explain your response.

Activity 3

Imagine Skip going down a slide. When he reaches the bottom, only his position in the frame has changed. This type of transformation is a **translation**. As shown in Figure 8, Skip's change in position can be represented by a **translation vector**. The length of the translation vector represents the distance Skip has moved; the arrow indicates the direction of travel. Any translation vector may be represented as the sum of its horizontal and vertical components.

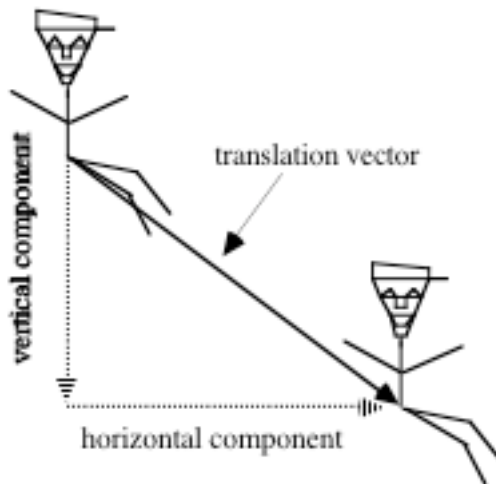


Figure 8: Skip on the slide

Exploration

- a.
 1. Trace a picture of Skip from Figure 8. Align your tracing with Skip's position at the top of the slide, then move Skip along the translation vector. Record your observations.
 2. Beginning again at the top of the slide, move Skip along the vertical component of the translation vector shown in Figure 8, then along the horizontal component. Record your observations.

- b. Matrix C below represents Skip's cap at the top of the slide, while matrix C' represents his cap at the bottom of the slide.

$$C = \begin{bmatrix} 9 & 0 & 0 & 7 & 7 \\ 6 & 6 & 8 & 7 & 6 \end{bmatrix} \quad C' = \begin{bmatrix} 13 & 4 & 4 & 11 & 11 \\ 3 & 3 & 5 & 4 & 3 \end{bmatrix}$$

Graph the positions of Skip's cap as defined by these matrices.

- c. Determine the changes that have occurred in the coordinates of each point of Skip's cap.

- d. A matrix equation of the following form, where \mathbf{T} is a translation matrix, can be used to represent the movement of Skip's cap from the top of the slide to the bottom: $\mathbf{C} + \mathbf{T} = \mathbf{C}'$.

Determine the matrix \mathbf{T} that makes this equation true.

- e. Use technology to verify your equation from Part d.

Discussion

- a. In Part a of the exploration, how did Skip's image at the bottom of the slide after moving him along the vertical and horizontal components compare with the image after moving him along the translation vector?
- b. What is the geometric relationship between the image at the bottom of the slide and its preimage?
- c. Describe the change in position of Skip's cap from the top of the slide to the bottom.

Mathematics Note

A **translation** is a transformation that pairs every point $P(x,y)$ with an image point $P'(x+h, y+k)$. A translation of this type moves every point h units horizontally and k units vertically.

A translation can be described using matrix addition. In this case, the number of columns in the translation matrix must match the number of points used to defined the figure to be translated. A translation matrix is represented in the following form:

$$\mathbf{T}_{P,P'} = \begin{bmatrix} h & h & h & \dots \\ k & k & k & \dots \end{bmatrix}$$

For example, consider the triangle whose vertices are represented in matrix \mathbf{A} .

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 5 \\ 3 & 8 & -1 \end{bmatrix}$$

To find the vertices of the image of this triangle under a translation -7 units horizontally and 3 units vertically, you can perform the following matrix addition:

$$\mathbf{A} + \mathbf{T} = \mathbf{A}'$$

$$\begin{bmatrix} 2 & 7 & 5 \\ 3 & 8 & -1 \end{bmatrix} + \begin{bmatrix} -7 & -7 & -7 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -2 \\ 6 & 11 & 2 \end{bmatrix}$$

- d. What translation matrix would you use to translate Skip's cap 4 units to the left and 5 units down?
- e. The lengths of the vertical and horizontal components represent vertical and horizontal distances. How could you use these distances to find the length of the translation vector?

- f. How could you find the measure of the angle formed by the translation vector and its vertical component?
- g. Describe how to find the distance from any point in Skip's preimage to the corresponding point in the image.
- h. Figure 9 shows the translation of point $P(x,y)$ to $P'(x+h,y+k)$.

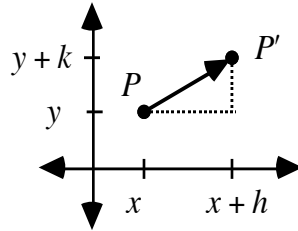
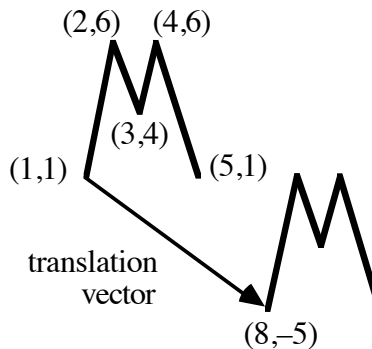


Figure 9: A translation of point P

1. What is the length of the translation vector?
2. How could you determine the direction of the translation vector?

Assignment

- 3.1 Use your results from Part **d** of the exploration to describe the length of the translation vector in Figure 8, along with the measure of the angle it forms with its vertical component.
- 3.2 Consider a polygon with vertices at $(1,1)$, $(5,1)$, $(4,4)$, and $(0,4)$.
 - a. Write a matrix equation that describes a translation of this polygon 4 units to the right and 2 units down.
 - b. Show this translation on a sheet of graph paper.
- 3.3 The following diagram shows the translation of a letter M.



- a. Describe the matrix that accomplishes the translation.
- b. Write a matrix equation to summarize the translation.
- c. What is the measure of the angle formed by the translation vector and its vertical component?
- d. What is the length of the translation vector?

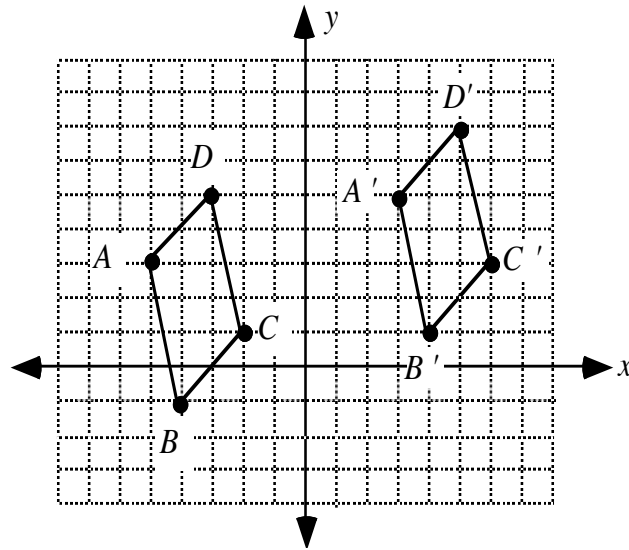
- 3.4**
- Use technology to draw the preimage and image of the letter M in Problem 3.3.
 - Suppose that you drew five more images of the letter M, all evenly spaced between the preimage and image in Part a. If you taped each image to a separate card and flipped the cards like the pages of a book, what would you see?
 - Describe how to find a translation matrix that transforms each image in Part b to the next, given only the coordinates of the original preimage and final image.
- 3.5**
- Describe a translation matrix that could be used to slide Skip's face to a region below and to the left of its original position on a grid.
 - Describe the matrix that contains the coordinates of the image points.
- 3.6** Consider the hexagon described by matrix **P** below.

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & -1 & -2 & -1 & 2 \\ 1 & 7 & 7 & 1 & -4 & -4 \end{bmatrix}$$

- Translate this figure 3 units horizontally and -8 units vertically, then dilate it by a scale factor of 2, with center at the origin. Write a separate matrix equation to describe each transformation.
 - Create a graph of the preimage and the final image.
 - If possible, write a single matrix equation that describes the combination of transformations. If a single matrix equation is not possible, explain why not.
- Reverse the order of the transformations, then repeat Part a.
- Compare your results from Parts a and b. Did you think they would be the same? Were they? Why or why not?

* * * * *

- 3.7 a. Describe the translation shown in the diagram below.



- b. Write a matrix that can be used to translate $ABCD$ to $A'B'C'D'$.
- c. Determine the distance between each point in the preimage and its corresponding point in the image.
- 3.8 Consider $\triangle ABC$ with vertices $A(2,4)$, $B(-4,6)$ and $C(1,-2)$. A translation using the following matrix results in the image $\triangle A'B'C'$.

$$\begin{bmatrix} 5 & 5 & 5 \\ -4 & -4 & -4 \end{bmatrix}$$

A translation of $\triangle A'B'C'$ by the matrix below produces the image $\triangle A''B''C''$.

$$\begin{bmatrix} -7 & -7 & -7 \\ 2 & 2 & 2 \end{bmatrix}$$

- a. Find the coordinates of $\triangle A''B''C''$.
- b. 1. Describe the translation of $\triangle ABC$ to $\triangle A''B''C''$.
2. Write a translation matrix that describes this transformation.
- c. Determine the distance between each point in $\triangle ABC$ and its corresponding point in $\triangle A''B''C''$.

Activity 4

Skip's friend Chip is washing dishes when the phone rings. While talking on the phone Chip's head is tilted as shown in Figure 10. The **rotation** of his head represents another type of transformation. In this activity, you explore some special cases of rotations.

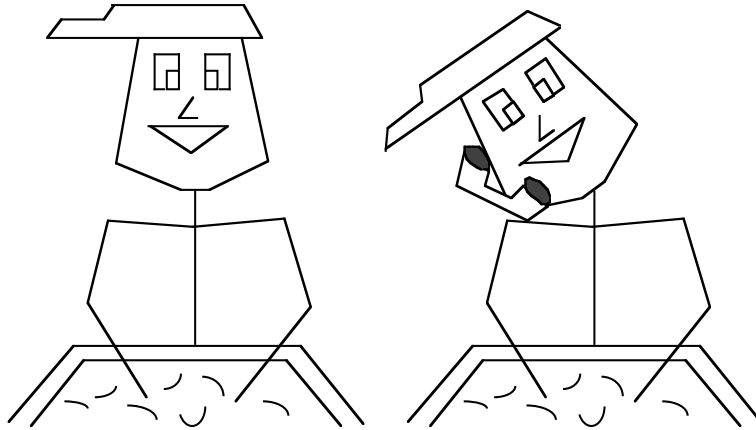


Figure 10: Chip washing dishes and talking on the phone

Exploration

- a. Chip's mouth can be represented by matrix \mathbf{P} below.

$$\mathbf{P} = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \end{bmatrix}$$

1. Plot Chip's mouth on a coordinate grid.
 2. On a sheet of tracing paper, trace Chip's mouth, the x - and y -axes, and the origin O .
- b. Position the tracing over the original so that it aligns perfectly. Place the tip of your pencil on the origin. While holding this point with the pencil gently turn the tracing so that the x -axis rotates 180° counterclockwise. Write the matrix \mathbf{P}' for the image of Chip's mouth.
- c. Make a conjecture about the relationship between the matrix of the preimage and the matrix of its image under a counterclockwise rotation about the origin of 180° .
- d. Find a 2×2 matrix \mathbf{R} so that $\mathbf{R} \cdot \mathbf{P}$ produces a counterclockwise rotation about the origin of 180° .
- e. Repeat Parts **b–d** using a counterclockwise rotation about the origin of each of the following measures:
1. 360°
 2. 90° .

Discussion

- Describe the image that results from a rotation of 0° about the origin.
- What 2×2 matrix, when multiplied on the left of a preimage matrix, would produce a rotation of 0° about the origin?

Mathematics Note

A **rotation** is a transformation that pairs one point C , the center, with itself and every other point P with a point P' that lies on a circle with center C such that $m\angle PCP'$ is the magnitude of the rotation.

Counterclockwise rotations about a point are denoted by positive degree measures. Clockwise rotations are represented by negative degree measures.

In Figure 11, for example, $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a 90° rotation about point C .

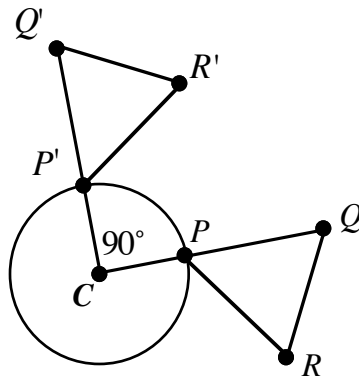


Figure 11: Triangle rotated 90° about point C

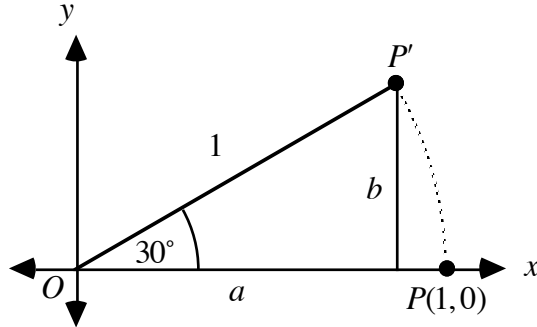
- Name at least two rotations that yield the same result as a counterclockwise rotation about the origin of 360° .
- What clockwise rotation would produce the same result as a counterclockwise rotation about the origin of 90° ? How can you generalize the relationship between equivalent clockwise and counterclockwise rotations?
- Do you think that two counterclockwise rotations of 90° with the same center are equivalent to one counterclockwise rotation of 180° ? Explain your response.
- In a rotation, what is the geometric relationship between an image and its preimage?

Assignment

- 4.1 Skip's mouth can be described by matrix \mathbf{M} below.

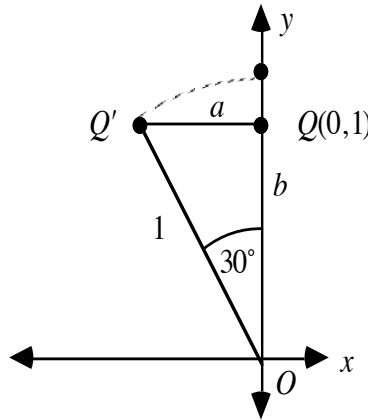
$$\mathbf{M} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

- a. Use matrix multiplication to determine the coordinates of the image that results from each of the following:
1. a 360° rotation about the origin
 2. a 180° rotation about the origin
 3. a 90° rotation about the origin.
- b. Graph the image of Skip's mouth produced by each rotation in Part a.
- 4.2 a. What matrix can be used to describe the image of Skip's mouth after a 270° rotation about the origin?
- b. Find a 2×2 matrix \mathbf{R} so that $\mathbf{R} \cdot \mathbf{M}$ produces a counterclockwise rotation about the origin of 270° .
- 4.3 As shown in the diagram below, the image of a point $P(1,0)$ under a 30° rotation with center at the origin is P' .



- a. 1. Use the cosine ratio to determine the distance a .
2. Use the sine ratio to determine the distance b .
3. Write the coordinates of point P' in a matrix.
- b. Determine the coordinates of P' under a 30° rotation, with center at the origin, for each of the following:
1. $P(2,0)$
 2. $P(c,0)$.

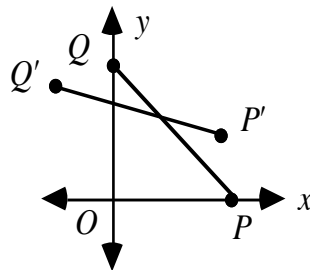
- 4.4 As shown in the following diagram, the image of a point $Q(0,1)$ under a 30° rotation with center at the origin is Q' .



- a.
 1. Use the sine ratio to determine the distance a .
 2. Use the cosine ratio to determine the distance b .
 3. Write the coordinates of point Q' in a matrix.
 - b. Determine the coordinates of Q' under a 30° rotation, with center at the origin, for each of the following:
 1. $Q(0,2)$
 2. $Q(0,d)$.
- 4.5 Matrix S below represents the line segment connecting $P(1,0)$ and $Q(0,1)$.

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- a. If \overline{PQ} is rotated 30° about the origin, as shown in the following graph, what matrix could be used to represent $\overline{P'Q'}$?



- b. If \overline{PQ} is rotated 20° about the origin, what matrix could be used to represent $\overline{P'Q'}$?
- c. If \overline{PQ} is rotated n° about the origin, where $0 \leq n < 90$, what matrix could be used to represent $\overline{P'Q'}$?

- 4.6** a. The following matrix **S** represents the line segment connecting $P(2,0)$ and $Q(0,2)$.

$$\mathbf{S} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

If \overline{PQ} is rotated 30° about the origin, what matrix could be used to represent $\overline{P'Q'}$?

- b. The matrix **S** below represents the line segment connecting $P(c,0)$ and $Q(0,c)$.

$$\mathbf{S} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

1. If \overline{PQ} is rotated 30° about the origin, what matrix could be used to represent $\overline{P'Q'}$??
2. If \overline{PQ} is rotated n° about the origin, where $0 \leq n < 90$, what matrix could be used to represent $\overline{P'Q'}$?

- 4.7** Use matrix multiplication to perform a rotation of Skip's cap. Make a sketch of the preimage and image and list the appropriate matrix equation.

- 4.8** A triangle can be represented by matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 2 \end{bmatrix}$$

Determine a matrix **A'** that represents the image of this triangle under each of the following transformations:

- a. a 180° rotation about the origin
- b. a 90° rotation about the origin.

Mathematics Note

An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.

For example, Figure 12 shows an equilateral triangle ABC and its center O . This object has rotational symmetry about O . When $\triangle ABC$ is rotated 120° about O , each point in the image coincides with a point in the preimage.

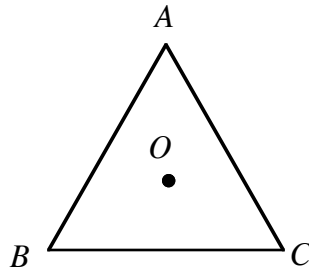
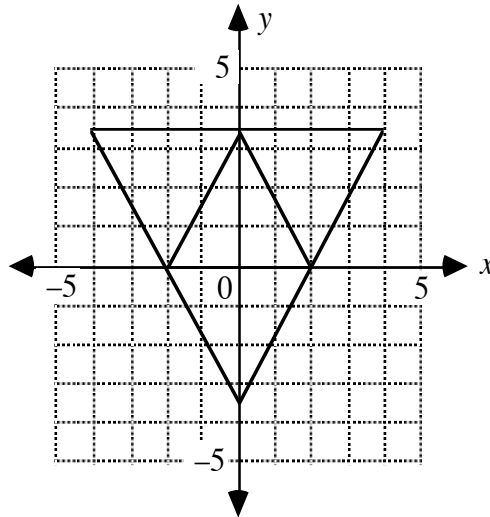


Figure 12: An equilateral triangle

- 4.9** When $\triangle ABC$ in Figure 12 is rotated 120° about O , each point in the image coincides with a point in the preimage. Identify at least two other degree measures for which this is true.
- 4.10** Determine which cards in an ordinary deck of playing cards have rotational symmetry.
- 4.11** Does Skip's face have rotational symmetry? Explain your response.
- 4.12**
- Draw a polygon that has rotational symmetry when rotated 90° about its center.
 - What regular polygon has rotational symmetry when rotated 45° about its center?
 - Determine the smallest positive degree measure which shows rotational symmetry for each of the following:
 - a regular hexagon
 - a regular pentagon
 - a regular decagon
 - a regular polygon with n sides.

* * * * *

4.13 The following graph shows a net for a triangular pyramid.



This net can be represented by matrix \mathbf{S} below.

$$S = \begin{bmatrix} -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \\ 0 & 3.4 & 3.4 & 0 & -3.4 & 3.4 & 3.4 & 0 \end{bmatrix}$$

- a. Determine a matrix \mathbf{S}' that represents the image of this net under each of the following rotations with center at the origin:
1. 180°
 2. 270°
 3. 360°
 4. 0°
 5. 90° .
- b. Identify the degree measures, where $0 < n < 360$, for which the net shows rotational symmetry about its center.

4.14 Consider a figure contained entirely in the first quadrant. What angle of rotation, with center at the origin, is required to produce an image of this figure that is contained entirely in the second quadrant?

Activity 5

In previous activities, you have seen transformations of Skip and Chip using dilations, translations, and rotations. Another common type of transformation is a **reflection**, as shown in Figure 13.

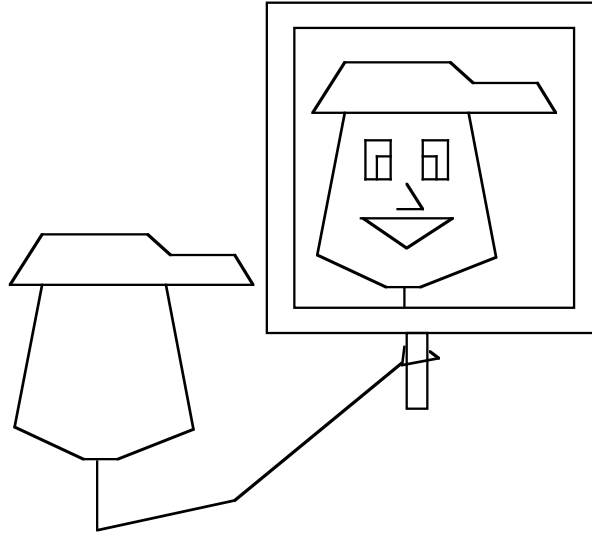


Figure 13: Chip looking in the mirror

Exploration

- a. Matrix \mathbf{P} below describes the vertices of a triangle.

$$\mathbf{P} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

1. Graph this triangle on a coordinate plane.
2. Consider the x -axis as a mirror. Graph the reflection of the triangle in the x -axis.
3. Write the coordinates of the image in matrix form.
4. Draw line segments connecting the corresponding vertices of the preimage and the image.
5. Describe the relationship between the line of reflection and each segment you drew in Step 4.

- b. Matrix **H** below describes the vertices of a quadrilateral.

$$\mathbf{H} = \begin{bmatrix} -1 & 2 & 6 & 8 \\ 0 & 3 & 4 & -2 \end{bmatrix}$$

1. Graph this quadrilateral on a coordinate plane.
 2. Graph the reflection of the quadrilateral in the x -axis.
 3. Write the coordinates of the image in matrix form.
- c. Repeat Parts **a** and **b** using the y -axis as a mirror.
- d.
 1. On a new coordinate system, graph the line $y = x$.
 2. Reflect the triangle described by matrix **P** in this line and write the coordinates of the image in matrix form.
 3. Repeat Steps **1** and **2** using the quadrilateral defined by matrix **H**.
- e. Reflect the polygons described by matrices **P** and **H** in the line $y = -x$.

Mathematics Note

A **reflection** in a line is a transformation that pairs each point on the line with itself and each point in the preimage with a corresponding point in the image so that the line of reflection is the perpendicular bisector of the segment connecting the point in the preimage with its image.

For example, Figure 14 shows the reflection of a polygon in line m .

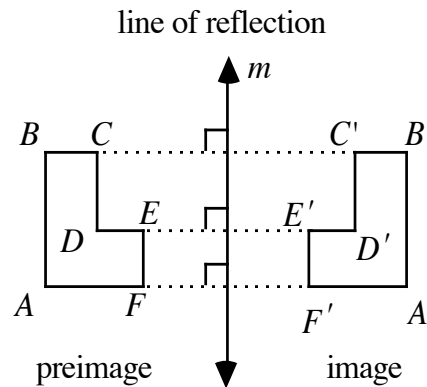


Figure 14: Reflection of a polygon in a line

A reflection in line m is denoted by r_m . For example, a reflection in the x -axis may be written as r_x .

- f. When a matrix of ordered pairs is multiplied on the left by one of the four matrices below, the product is one of the reflections explored in Parts **a–e**. Identify the line of reflection associated with each matrix below.

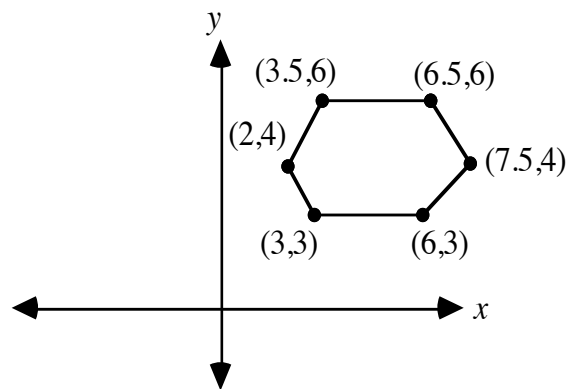
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Discussion

- a. 1. What patterns did you observe in the matrices when reflecting an object in the x -axis?
 2. What patterns did you observe when reflecting in the y -axis?
- b. Using your results from Parts **d** and **e** of the exploration, write a general statement that describes a reflection in each of the following lines:
 1. $y = x$
 2. $y = -x$.
- c. What similarities or differences are there in the matrix operations for the four types of transformations you have explored so far?
- d. How would you summarize the geometric relationships between the preimage and the image in these four types of transformations?

Assignment

- 5.1 a. The following graph shows the coordinates of the vertices of a hexagon. Write a matrix expression that describes the reflection of this hexagon in the y -axis.



- b. List the coordinates of the image in a matrix.
- c. Graph the preimage and the image of the hexagon in Part **a**. Reflect each of these figures in the x -axis.
- d. Select any one of the four figures from Part **c**. Using this figure as the preimage, can each of the other figures be generated by a single reflection in a line? Explain your response.

5.2 Use matrices to produce a reflection of Skip in the line $y = x$ or the line $y = -x$, followed by a reflection in the x -axis or the y -axis. Show the resulting image on a sheet of graph paper.

5.3 When more than one transformation is performed on a figure, the result is a **composite transformation**.

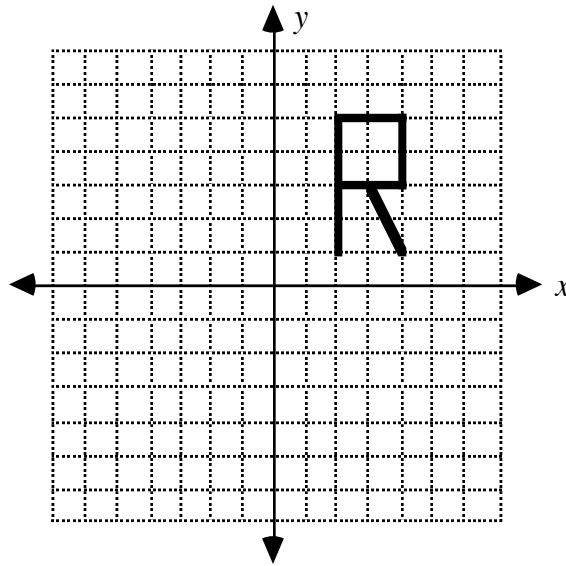
a. Reflect the quadrilateral described by matrix \mathbf{M} below in the x -axis.

$$\mathbf{M} = \begin{bmatrix} -1 & 2 & 8 & 2 \\ 5 & 8 & 5 & 2 \end{bmatrix}$$

b. Dilate the image from Part **a** by a scale factor of $1/2$, with center at the origin. Show your results graphically and identify the coordinates of the vertices.

c. Write a matrix representation of the composite transformation.

5.4 The figure below shows a letter R graphed on a coordinate plane.



a. Reflect the letter R in the x -axis, then reflect the resulting image in the y -axis.

b. Rotate the letter R 180° about the origin.

c. Compare the images you obtained in Parts **a** and **b**. Use matrix multiplication to confirm your observations.

Mathematics Note

An object has a **line of symmetry** if, when reflected in that line, each point in the image coincides with a point in the preimage.

The equilateral triangle in Figure 15, for example, has three lines of symmetry. When $\triangle ABC$ is reflected in line l , m , or n , each point in $\triangle A'B'C'$ coincides with a point in $\triangle ABC$.

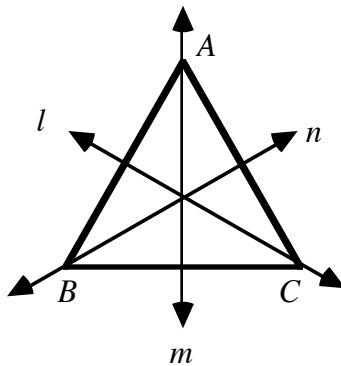
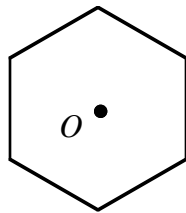


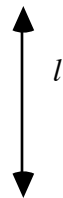
Figure 15: Lines of symmetry of an equilateral triangle

5.5 Identify the line(s) of symmetry for each of the following.

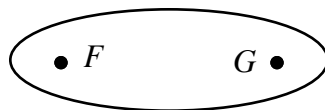
a.



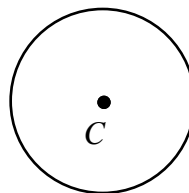
b.



c.



d.



5.6 Does Skip's face have a line of symmetry? Explain your response.

- 5.7** If the quadrilateral described by matrix **D** is reflected in a line, the resulting image is described by matrix **Z**.

$$\mathbf{D} = \begin{bmatrix} -3 & -6 & -4 & -2 \\ 1 & 2 & 4 & 3 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} -1 & -2 & -4 & -3 \\ 3 & 6 & 4 & 2 \end{bmatrix}$$

- a. What is the equation of the line of reflection?
 - b. Determine a matrix that may have been used to perform the reflection.
 - c. Verify your response to Part **b** using a matrix equation.
- 5.8** Parts **a–c** below each describe two composite transformations. For each part, use matrix operations to determine if the two composite transformations produce the same final image.
- a.
 1. a dilation with center at the origin and a scale factor of 4, followed by a translation 2 units to the left and 5 units up
 2. a translation 2 units to the left and 5 units up, followed by a dilation with center at the origin and a scale factor of 4
 - b.
 1. a reflection in the line $y = x$, followed by a dilation with center at the origin and a scale factor of 7
 2. a dilation with center at the origin and a scale factor of 7, followed by a reflection in the line $y = x$
 - c.
 1. a reflection in the y -axis, followed by a reflection in the line $y = -x$
 2. a 90° rotation about the origin
- 5.9** Make a sketch of an object that has at least three different lines of symmetry.

* * * * *

Research Project

Create your own animated cartoon using the transformations examined in this module. (Your collection of Skip's transformations should help you consider the possibilities.) Your cartoon should be at least 20 frames long. So that others may view your cartoon, tape each image onto a 3×5 card. Include a written summary with a mathematical description of the transformations from each frame to the next. You may also create a cartoon on a programmable calculator. See your teacher for details.

Summary Assessment

1. When composite transformations are repeated over and over again, some interesting patterns may develop. **Note:** Read Parts **a–c** below before beginning work on this problem.

- a. Graph the triangle described by matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 0 & 16 & 16 \\ 0 & 0 & 16 \end{bmatrix}$$

- b. Perform each of the following transformations in the order given below. Graph the image that results from this composite transformation on the same grid as the triangle in Part **a**.

1. a translation of -2 units horizontally and 6 units vertically
2. a rotation of 90° about the origin
3. a dilation with center at the origin and a scale factor of $3/4$

- c. Using the image that resulted from the composite transformation in Part **b** as the preimage, repeat Steps **1–3** above. Graph the resulting image on the same grid as the triangle in Part **a**.

Continue this process until you have graphed five triangles. Describe the resulting figure.

2. Create your own picture using repeated composite transformations. Use a geometry utility to help explore some possibilities. Summarize the transformations using matrix equations.

Module Summary

- A **one-to-one correspondence** is a function between two sets: the domain and the range. It pairs each element in the domain with exactly one element in the range, and each element in the range with exactly one element in the domain.
- A **transformation** in a plane is a one-to-one correspondence of the plane onto itself.
- The Pythagorean theorem can be used to derive a formula for the distance between any two points, (x_1, y_1) and (x_2, y_2) , on a coordinate plane. According to the **distance formula**, the distance d between (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- A **dilation** is a transformation that pairs a point P , the **center**, with itself and any other point X with a point X' on ray PX so that $PX'/PX = r$, where r is the **scale factor**.
- A dilation with its center at the origin is a transformation such that every point Q with coordinates (x, y) has an image Q' with coordinates (rx, ry) , where r is the scale factor, and $r \neq 0$. The scale factor also may be written as a ratio of the corresponding dimensions of the image to the preimage.
- A point P with coordinates (x, y) can be represented in a matrix as shown below.

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- An object's change in position can be represented by a **translation vector**. The length of the translation vector represents the distance the object has moved; the arrow indicates the direction of travel. Any translation vector may be represented as the sum of its horizontal and vertical components.
- A **translation** is a transformation that pairs every point $P(x, y)$ with an image point $P'(x + h, y + k)$. A translation of this type moves every point h units horizontally and k units vertically.
- A translation can be described using matrix addition. In this case, the number of columns in the translation matrix must match the number of points used to defined the figure to be translated. A translation matrix is represented in the following form:

$$\mathbf{T}_{P, P'} = \begin{bmatrix} h & h & h & \cdots \\ k & k & k & \cdots \end{bmatrix}$$

- A **rotation** is a transformation that pairs one point C , the center, with itself and every other point P with a point P' that lies on a circle with center C such that $m\angle PCP'$ is the magnitude of the rotation.

Counterclockwise rotations about a point are denoted by positive degree measures. Clockwise rotations are represented by negative degree measures.

- An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.
- A **reflection** in a line is a transformation that pairs each point on the line with itself and each point in the preimage with a corresponding point in the image so that the line of reflection is the perpendicular bisector of the segment connecting the point in the preimage with its image.
- When more than one transformation is performed on a figure, the result is a **composite transformation**.
- An object has a **line of symmetry** if, when reflected in that line, each point in the image coincides with a point in the preimage.

Selected References

Barasch, M. *Theories of Art: From Plato to Winkelman*. New York: New York University Press, 1985.

Hulse, C. *The Rule of Art: Literature and Painting in the Renaissance*. Chicago: University of Chicago Press, 1990.

Papy, G. *Modern Mathematics*. London: Collier-Macmillan, 1968.

Hurry! Hurry! Hurry!

Step Right Up!



Have you ever played a carnival game? Better yet, have you ever won a big prize? In this module, you use probability to explore the carnival midway.

Bill Chalgren • Mike Trudnowski • Anne Watkins



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Hurry! Hurry! Hurry! Step Right Up!

Introduction

A carnival barker beckons to you from the midway: “Step right up! You look like a winner!” The barker’s booth is decorated with enticing prizes. The barker tells you that you can choose from one of two games. One requires you to throw a dart at a board while blindfolded, the other involves twirling a spinner on an identical board. To win, either the dart or the arrow must land in the unshaded portion of the board. The two game boards are shown in Figure 1 below.

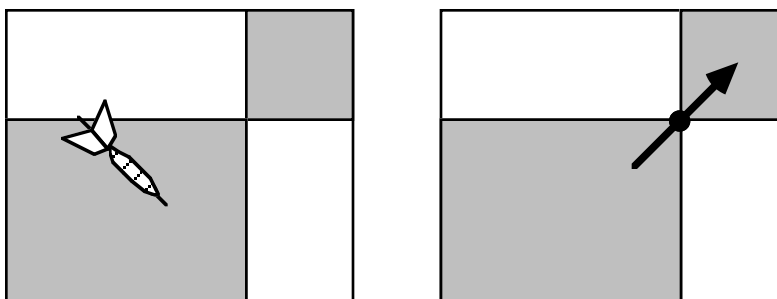


Figure 1: Two carnival game boards

“Put your ticket down,” says the barker, “choose a game, and take a chance.” When you hesitate, the barker assures you that you continue throwing darts until one hits the board and that the spinner is equally likely to stop at any point on its arc.

Discussion

- a. Which has a better chance of landing in the unshaded portion of the board: a dart thrown while blindfolded, or the arrow on the spinner? Explain your response.
- b. When determining whether or not the arrow landed in the unshaded portion, why should you consider only the tip of the arrow?
- c.
 1. What kinds of carnival games have you played or seen others play?
 2. Were those games based on skill or on chance? Explain your reasoning.
- d. Which of the games shown in Figure 1 would you rather play? Why?
- e. Do you think that carnival games are fair? Explain your response.

Activity 1

As you observed in the introduction, the chances of a dart landing in the unshaded portion of the game board in Figure 1 are not the same as the chances of the spinner landing there. In this activity, you create **geometric models** that will allow you to determine why these probabilities are not the same.

Exploration

In the coin-drop game, each player tries to drop a dime so that it lands entirely inside a $3\text{ cm} \times 3\text{ cm}$ square on a grid. If the dime lands completely within a square, the player wins. If any part of the dime touches or crosses a line on the grid, the player loses. Figure 2 shows some possible game situations:

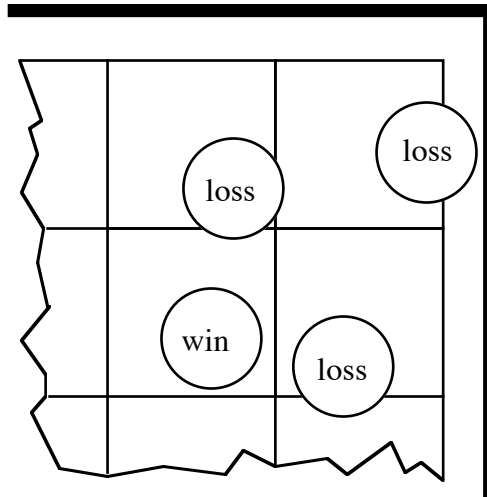


Figure 2: Sample coin-drop outcomes

- a.
 1. Mark a full sheet of graph paper into $3\text{ cm} \times 3\text{ cm}$ squares.
 2. Use folded cardboard to form walls around the sheet of graph paper, as shown in Figure 3. Each wall should be placed about 1 cm from the edge of the graph paper. Tape the walls together.

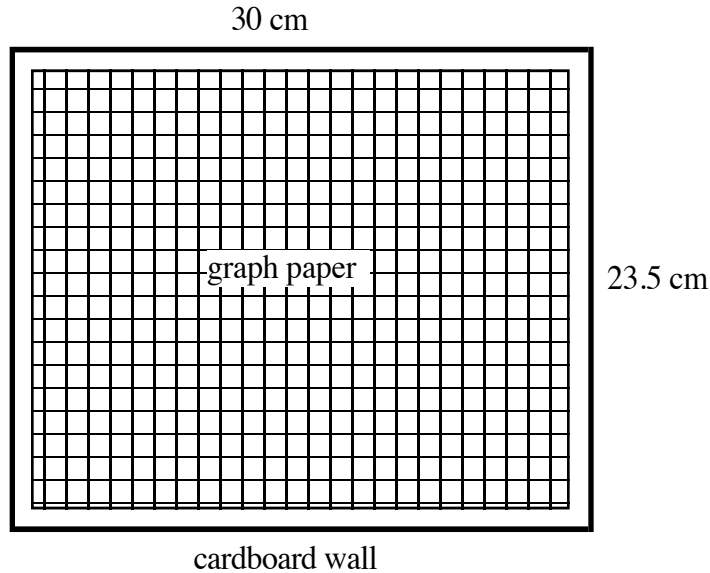


Figure 3: Graph paper with cardboard walls

- b.** Drop a dime onto the grid from the height of a chair seat (about 45 cm). Record the result of the drop as a win or loss.
- c.** Repeat Part **b** nine more times.
- d.**
 1. Compile the total number of wins and losses for the entire class.
 2. Use the class data to determine the experimental probability of winning the game.

Mathematics Note

If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event E is determined using the following ratio:

$$P(E) = \frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

When the outcomes in an event can be represented as points in a **geometric model**, the theoretical probability of the event may be found using the ratio below:

$$P(E) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

Some examples of geometric measures are lengths, angle measures, areas, and volumes.

For example, consider a carnival game that involves throwing a dart at balloons. There are 16 balloons with a mean diameter of 10 cm attached to a square board 60 cm on each side. To win a prize, players must break a balloon. As shown in Figure 4, the balloons can be represented as circles, while the board can be represented as a square.

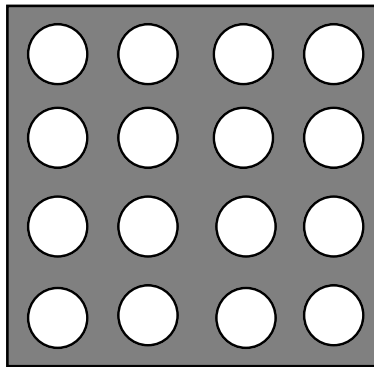


Figure 4: Balloons on a board

Assuming that the dart is equally likely to hit any point on the board, the probability of breaking a balloon (and winning a prize) can be found as follows:

$$P(\text{win}) = \frac{\text{area of 16 circles}}{\text{area of board}} = \frac{16 \cdot (\pi \cdot 5^2)}{60 \cdot 60} \approx 0.35 = 35\%$$

- e. Consider one 3 cm \times 3 cm square on the grid. To determine the theoretical probability of winning using a geometric model, you must first identify the area that models all winning outcomes.
 1. How close can a dime's center come to the side of a 3 cm \times 3 cm square and still represent a win?
 2. Using the center of the dime as a reference point, determine the area of the region that models all winning outcomes.
 3. Use the area determined in Step 2 and the area representing all possible outcomes to find the theoretical probability of winning.
- f. Compare the experimental probability of winning you determined in Part d with the theoretical probability from Part e.

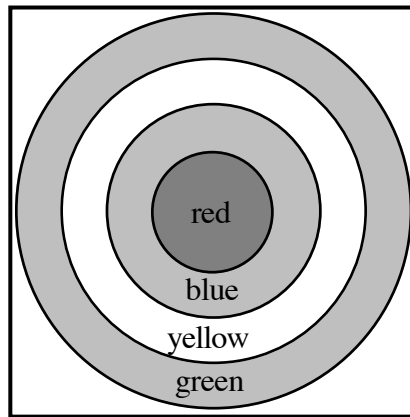
Discussion

- a. In a spinner game, the tip of the arrow serves as the reference point for determining the outcome of each spin. In the coin-drop game, you used the center of the dime. Why is this an appropriate reference point?
- b. Does the number of 3 cm \times 3 cm squares on the grid affect the probability of winning the coin-drop game? Explain your response.

- c. In Part **f** of the exploration, how did the experimental probability of winning the coin-drop game compare to its theoretical probability?
- d. Describe how you could make the coin-drop game impossible to win.

Assignment

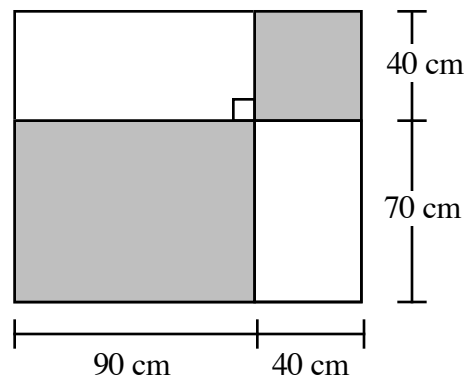
- 1.1 In another version of the coin-drop game, players drop quarters onto the grid instead of dimes.
 - a. Describe the area within a $3\text{ cm} \times 3\text{ cm}$ square that represents a win.
 - b. Determine the theoretical probability of winning the coin-drop game using a quarter.
- 1.2 To attract more players, the carnival decides to allow players to drop dimes onto a grid of $4\text{ cm} \times 4\text{ cm}$ squares.
 - a. Describe the area within a $4\text{ cm} \times 4\text{ cm}$ square that represents a win.
 - b. Determine the theoretical probability of winning this version of the coin-drop game.
- 1.3 In another kind of carnival game, players throw darts at a circular board. As shown in the diagram below, the board is divided into four concentric regions, each painted a different color.



The radius of the red region in the center of the board is 4 cm, while the width of each of the remaining regions is 3 cm.

- a. Assuming that a dart has an equal chance of landing at any point on the board, what is the theoretical probability that it lands in the yellow region? Explain your response.
- b. Is throwing a dart at a target typically a random event? Explain your response?

- 1.4** In Bowl-o-rama, a bowling pin with a diameter of 8 cm is placed somewhere near the center of a lane 100 cm wide. To keep the ball in play, the lane has rails on each side. To win the game, a blindfolded player must knock down the pin using a bowling ball 20 cm in diameter.
- Assuming that the pin will fall if it is touched by the ball, what is the probability of winning the game? Justify your response.
 - Would placing the bowling pin against one of the rails affect the probability of winning the game? Explain your response.
- 1.5** In the introduction to this module, a carnival barker offered you a chance to play either a dart game or a spinner game. (See Figure 1.) To win, either the dart or the arrow must land in the unshaded portion of the board. The measurements of the game board are shown below.



Based on the theoretical probabilities of winning, which game would you rather play? Explain your response.

* * * * *

- 1.6** Consider two real numbers, x and y , where $0 \leq x \leq 5$ and $0 \leq y \leq 5$.
- Graph all the possible ordered pairs (x,y) on a two-dimensional coordinate system.
 - Determine the area of the region that represents all the possible ordered pairs (x,y) .
 - Graph all the possible ordered pairs (x,y) where $x + y \leq 3$, $0 \leq x \leq 5$, and $0 \leq y \leq 5$.
 - Given a point (x,y) selected at random from Part **a**, determine the probability that $x + y \leq 3$. Justify your response.
 - Given a point selected at random from Part **a**, determine the probability that $x + y \geq 6$. Justify your response.

- 1.7** After jumping from an airplane, a parachutist must land safely in a square parking lot 75 m on each side. The parking lot is free of hazards, except for at the corners. In each corner of the lot, there is a utility pole with a street light. If the parachutist lands within 10 m of a utility pole, the parachute's lines will get caught on the light.
- Draw a sketch of the parking lot and the utility poles. Shade the areas in which the parachutist should not land.
 - Determine the area in which it is safe to land.
 - Assuming that the parachutist is equally likely to land anywhere in the parking lot, determine each of the following probabilities:
 - the parachutist lands safely
 - the parachute's lines get caught on a security light.
 - What is the relationship between the two probabilities in Part c?
- 1.8** Consider two real numbers, x and y , where $1 \leq x \leq 3$ and $1 \leq y \leq 3$.
- Graph all the possible ordered pairs (x,y) on a two-dimensional coordinate system.
 - Determine the area of the region that represents all the possible ordered pairs (x,y) .
 - Graph all the possible ordered pairs (x,y) where $y/x \leq 1$, $1 \leq x \leq 3$, and $1 \leq y \leq 3$.
 - Given a point (x,y) selected at random from Part a, determine the probability that $y/x \leq 1$. Justify your response.
 - Given a point selected at random from Part a, determine the probability that $y/x \geq 1$. Justify your response.

* * * * *

Activity 2

In another carnival game, players roll a six-sided die. The game costs \$1.00 to play. The values of the prizes that correspond with the six possible rolls are shown in Table 1 below.

Table 1: Value of prizes in die game

Roll of Die	Value of Prize
1	\$1.50
2	\$1.00
3	\$0.50
4	\$0.00
5	\$0.00
6	\$0.00

Judging from the values of the prizes, do you think this is a fair game? In the following exploration, you use mathematics to help answer this question.

Exploration

- a. Play the die game 10 times. Record the number of wins for each prize.
- b.
 1. Compile the data from Part a for the entire class.
 2. Use the class data to determine the experimental probability of winning each prize.
- c. Determine the theoretical probability of winning each prize.
- d. Compare the experimental probability of winning each prize to the theoretical probability.

Mathematics Note

The mean value of an experiment is the **expected value**. Expected value is determined by adding the products of the value of each event and its corresponding theoretical probability.

For example, suppose a carnival game uses the spinner shown in Figure 5. The measures of the three central angles are 45° , 135° , and 180° , respectively. Using a geometric model, the probability of winning \$25 is $45/360 = 1/8$, the probability of winning \$15 is $135/360 = 3/8$, and the probability of winning \$0 is $180/360 = 1/2$.

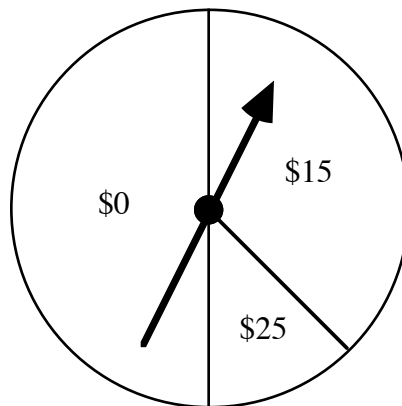


Figure 5: Spinner with three central angles

The expected value for this game is the sum of the products of each prize value and its corresponding probability. This can be calculated as follows:

$$\frac{1}{8} \cdot \$25 + \frac{3}{8} \cdot \$15 + \frac{1}{2} \cdot \$0 = \$8.75$$

This indicates that, for a large number of plays, the mean amount won per play is likely to be close to \$8.75.

- e. Determine the expected value of the die game.

Discussion

- When playing the die game, are you equally likely to win each prize? Explain your response.
- A **fair game** is one in which the expected value equals the cost to play. Is the die game mathematically fair? If so, justify your response. If not, suggest at least one way to make it fair.
- Is it reasonable to expect a carnival game to be mathematically fair? Explain your response.

Assignment

- 2.1** In another version of the die game, prizes are awarded as shown in the following table.

Roll of Die	Value of Prize
1	\$2.00
2	\$0.00
3	\$0.50
4	\$0.00
5	\$0.50
6	\$0.00

Determine the cost to play that makes this a fair game.

- 2.2** In one version of the coin-drop game described in Activity 1, the theoretical probability of winning is $4/25$. The game costs \$1.00 to play. Winning players receive a prize worth \$5.00. Losing players receive no prize.
- Determine the expected value for this game.
 - Given that the prize value remains \$5.00, change the cost of playing to make this a fair game.
 - Given that the cost of playing remains \$1.00, change the value of the prize to make this a fair game.
- 2.3** In another version of the coin-drop game, the theoretical probability of winning is $1/25$. This game also costs \$1.00 to play. However, winning players receive a prize worth \$15.00.
- Determine the expected value for this game.
 - Given that the prize value remains \$15.00, change the cost of playing to make this a fair game.
 - Given that the cost of playing remains \$1.00, change the value of the prize to make this a fair game.
- 2.4** Imagine that you are the manager of a carnival. One of the game operators has designed a new game. In this game, players pick 1 card from an ordinary deck of 52 cards. An ace wins \$10.00, a face card (king, queen, or jack) wins \$1.00, and all other cards win nothing. Determine the cost to play to make this a fair game.

* * * * *

- 2.5** Joan and Juan are playing a game that involves randomly selecting two real numbers in the interval $[0, 5]$. Joan offers to pay Juan \$0.50 every time the sum of the two numbers is less than or equal to 3, if he will pay her \$0.20 every time the sum is greater than 3.
- Before deciding whether to accept these rules, Juan wants to know his chances of winning. Use a geometric model to determine the probability that Juan will win.
 - Determine the expected value of the game for Juan.
 - Do you think Juan should accept Joan's rules? Explain your response.
 - Modify the prize amounts so that the expected values for both players are equal.

* * * * *

Research Project

Design your own carnival game. Calculate the expected value and determine the cost to play required to make it a fair game. Exchange your game with a classmate. Check each other's designs to verify that the games are fair. Present your game to the class.

Activity 3

In the fishing derby, players hook three rings, one at a time, from a tank of swirling water. The tank contains 10 rings. Four of the rings are marked with a small red dot and six are marked with a small blue dot.

Players receive 1 point for each blue ring and 2 points for each red ring. After each ring is caught, it is placed back in the tank before the player's next try. Table 2 below shows the total number of points required to win each prize.

Table 2: Prizes in the fishing derby

Prize	Number of Points Needed
rabbit's foot key chain	3
celebrity poster	4
black hat with a pink feather	5
large teddy bear	6

To win a particular prize, a player must have the exact number of points for that prize after hooking three rings. To win the celebrity poster, for example, a player must have exactly 4 points. Players with 5 or 6 points cannot win the poster, even though they may have had exactly 4 points at some time before hooking the last ring. Points earned in one game cannot be added to points earned in another game.

Exploration 1

- a. To simulate the fishing derby, mark four balls with a red dot and six balls with a blue dot. Place the balls in a container.
 1. Without looking, draw one ball from the container. Record the color of its dot, then return it to the container.
 2. Repeat Step 1 two more times, mixing the balls before each draw.
 3. Determine the total number of points for the three draws. As in the actual fishing derby, each blue dot is worth 1 point and each red dot is worth 2 points.
- b.
 1. Play the game 10 times. Record the points earned in each game.
 2. Compile the experimental results for the entire class.
 3. Using the compiled results, determine the experimental probability of winning each prize.
- c. Create a tree diagram to show the possible outcomes in the fishing derby after a player's first catch. Label each branch with a theoretical probability.
- d. Create a tree diagram to show the possible outcomes after all three catches. Label each branch with a theoretical probability.

Mathematics Note

Two events A and B are **independent** if $P(A \text{ and } B) = P(A) \cdot P(B)$. Three events A, B, and C are independent if each pair of events is independent and $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$. This definition can be extended to any number of independent events.

Given two independent events, the occurrence of one has no effect on the likelihood of the occurrence of the other. Two events that are not independent are said to be **dependent**.

For example, consider tossing two coins—a penny and a nickel—one at a time. Since the outcome of tossing the penny does not affect the outcome of tossing the nickel, the two tosses are independent events.

The probability of obtaining heads on either coin, or $P(H)$, is $1/2$. The probability of obtaining heads on the penny and the nickel, or $P(H \text{ and } H)$, is:

$$P(H \text{ and } H) = P(H) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

- e. In the fishing derby, hooking the first ring, hooking the second ring, and hooking the third ring are independent events. Use your tree diagram from Part **d** and the definition given in the mathematics note to determine the theoretical probability of each outcome.
- f.
 - 1. Determine the theoretical probability of winning each prize in the fishing derby.
 - 2. Compare these probabilities with the experimental probabilities you found in Part **b**.

Discussion 1

- a. From your tree diagram in Part **d** of Exploration **1**, how many possible outcomes are there in the fishing derby?
- b. Describe the different sets of three rings that result in each of following numbers of points:
 - 1. 3
 - 2. 4
 - 3. 5
 - 4. 6
 - 5. 7
- c. Why do you think the rings with red dots are worth more points than the ones with blue dots?
- d.
 - 1. Does hooking a red ring on the first try affect the probability of hooking a blue one on the second try? Explain your response.
 - 2. Does it affect the probability of hooking a red ring on the second try? Why or why not?
- e. Which prize do you think is awarded most often? Why?
- f.
 - 1. Describe how you could use technology to model the fishing derby.
 - 2. Would you expect this simulation to produce the same results as the simulation in Exploration **1**? Explain your response.
- g. Which information is likely to better represent the long-term results of playing the fishing derby: the data from 10 simulations, or the compiled class data? Explain your reasoning.

Exploration 2

In the next town on the carnival circuit, the operator of the fishing derby changes the rules of the game. The change requires players to keep each ring hooked instead of returning it to the tank. This new version of the derby is supposed to give players a better chance of winning a hat or a teddy bear. The rest of the game remains the same.

- Predict how the rule change will affect the probability of winning each prize.
- Create a tree diagram to show the possible outcomes in the new fishing derby. Label each branch with a theoretical probability.

Mathematics Note

Conditional probability is the probability of an event occurring given that an initial event has already occurred. The probability that event B occurs, given that event A has already occurred, is denoted by $P(B|A)$.

In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

For example, consider a game in which players draw two chips from a box containing three chips: two red and one blue. Each player draws a chip then, without replacing the first chip, draws another. Since the first chip drawn affects the probabilities in the second draw, they are not independent events. Figure 6 shows a tree diagram for this game. The probability of drawing a red on the second draw, given a red on the first draw is $1/2$. The probability of drawing a red on the second draw, given a blue on the first draw, or $P(B|R) = 2/2$.

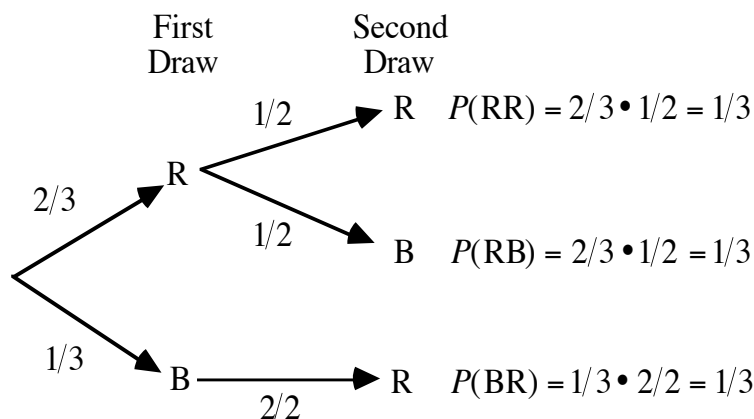


Figure 6: Conditional probabilities

- c. Use your tree diagram to determine the theoretical probability of winning each prize.
- d. Create a simulation for this version of the game.
- e.
 1. Conduct your simulation a total of 10 times. Record the points earned and the prize won each time.
 2. Compile the results from the entire class.
 3. Using the compiled results, determine the experimental probability of winning each prize.
 4. Compare these experimental probabilities with the theoretical probabilities you determined in Part c.

Discussion 2

- a. Which part of your simulation models the rule that each ring caught must be removed from the tank?
- b. Are the chances of getting 6 points in the new fishing derby better or worse than in the original version of the game?
- c. Does the rule change appear to have improved the chances of winning a hat or a teddy bear?
- d.
 1. In the game modeled in Figure 6, why are the probabilities of obtaining a red chip on the second draw different from the probability of obtaining a red chip on the first draw?
 2. What is the probability of drawing two blue chips in this game?

Assignment

- 3.1
 - a. What is the theoretical probability of winning a prize in the original fishing derby? Explain your response.
 - b. What is the theoretical probability of earning 7 points in the original fishing derby? Explain your response.
 - c. What is the expected value, in points, of the original fishing derby?
- 3.2
 - a. What is the expected value, in points, of the new fishing derby?
 - b. Which version of the fishing derby would you rather play? Explain your response.

- 3.3** Both versions of the fishing derby cost \$1.00 to play. The values of the prizes are shown in the following table.

Prize	Value
rabbit's foot key chain	\$0.25
celebrity poster	\$0.60
black hat with pink feather	\$0.80
large teddy bear	\$2.00

- a. Is the original fishing derby a fair game? If so, justify your response. If not, describe how you could modify the game to make it fair.
- b. Is the new fishing derby a fair game? If so, justify your response. If not, describe how you could modify the game to make it fair.
- 3.4** Guinn and Ebdul are playing the stair-step game. This game starts on the middle step of a long staircase. Each player's turn consists of flipping two coins and moving up or down according to the following rules: if no heads appear on a player's two coins, the player does not move; if one head appears, the player moves up one stair; if two heads appear, the player moves down one stair.

The staircase has 25 steps. The game is over when one player wins by reaching the top step, or loses by reaching the bottom step.

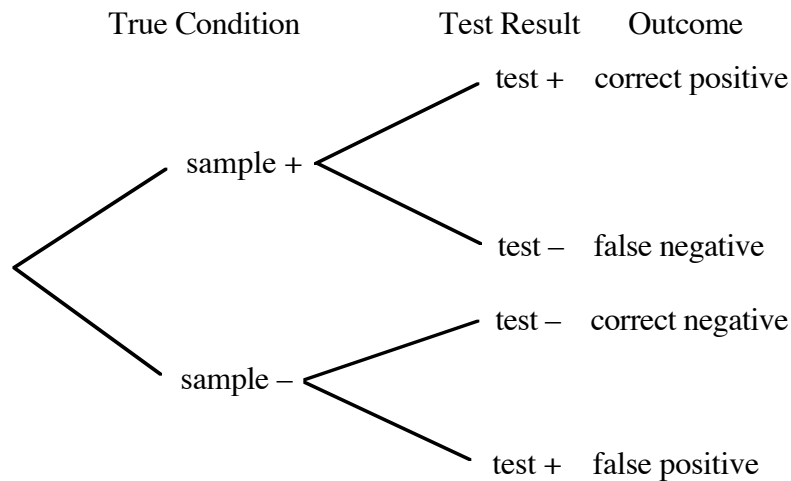
- a. If movement is described in terms of the number of stairs moved upwards, then each player's turn in the game has three possible results: 1 step, 0 steps, and -1 step.
- What is the probability of each of these results?
- b. What is the expected value, in number of stairs moved upwards, for each turn in the stair-step game?
- c. Where on the staircase do you think Guinn will be after 10 turns?
- d. How many turns do you think it will take for a player to reach the top step? Explain your response.

* * * * *

3.5 In 1992, the Centers for Disease Control (CDC) reported that 1 out of every 250 people in the United States had the HIV virus. For people in either group (HIV positive and HIV negative), the HIV test result is correct 99.8% of the time and false 0.2% of the time.

The tree diagram below shows the possible outcomes of the test for a person selected at random from the U.S. population. “Sample +” indicates that the person who gave the sample has the HIV virus. “Test +” indicates that the sample tested positive for HIV.

A test that incorrectly characterizes the sample is called a false test. A “false negative” means that the sample tested negative for HIV, but actually contains the virus. A “false positive” means that the sample tested positive, but does not contain the HIV virus.

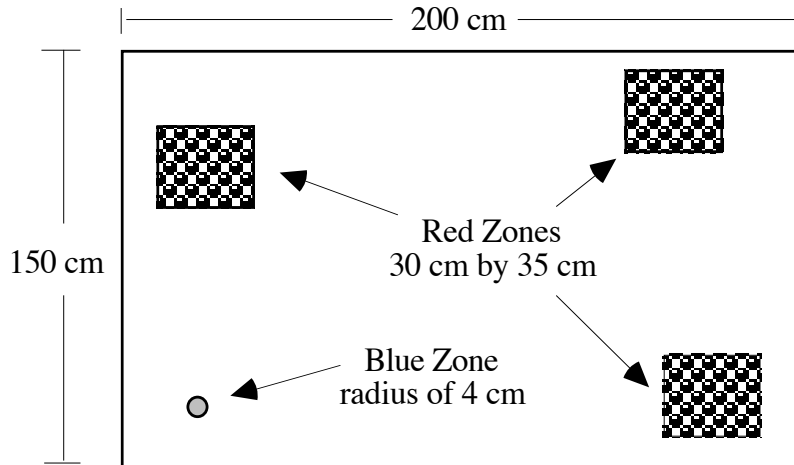


- a. Using a tree diagram like the one above, calculate the probability of each outcome in the sample space.
- b. Use your results from Part **a** to answer the following questions. If a random sample of 1,000,000 people were tested for HIV, how many of them do you think would test:
 1. positive and actually have HIV (correct positive)?
 2. negative and actually have HIV (false negative)?
 3. negative and not have HIV (correct negative)?
 4. positive and not have HIV (false positive)?
- c. Use your results in Part **b** to estimate the probability that a sample contains the HIV virus, given that the test result is positive.

* * * * *

Summary Assessment

1. As part of its annual fund-raiser, a local charity has designed a game called Auto Dart. Each player throws one dart at a computerized board. After the dart lands, the board lights up, displaying three red zones and one blue zone. The dimensions of the board, along with one possible arrangement of the zones, is shown in the diagram below.



If the dart landed in the blue zone, the player wins a car worth \$10,000. If the dart landed in any of the three red zones, the player wins a miniature replica of the car valued at \$25. If the dart landed anywhere else on the board, the player wins nothing. After each throw, the computer randomly relocates the winning zones. These zones are always the same size and never overlap.

Imagine that you are the charity's financial advisor. Analyze the Auto Dart game and decide if the charity should use it in the fund-raiser. Write a letter to the charity explaining your recommendation. Your letter should include the following:

- a. the probability of winning each prize in Auto Dart
- b. the expected value for the game
- c. the lowest price, rounded to the nearest dollar, that the charity could charge for the game and still expect to make a profit
- d. the number of miniature replicas and full-sized cars the charity can expect to award to the first 1000 players
- e. the probability of two players in a row winning full-sized cars.

2. A carnival manager has created a new game called Blind Draw. In this game, players draw three small balls from a container. The container holds 20 balls: 10 green balls worth 1 point each, 8 blue balls worth 2 points each, and 2 red balls worth 5 points each.

Players cannot see the contents of the container and remove each ball after it is drawn. The point total for three balls determines the prize won. A total of 6 through 8 points wins a pennant, 9 through 11 points wins a small stuffed animal, and 12 or more points wins a large stuffed animal. Points from one game cannot be carried over to another game.

The pennants cost the carnival \$0.40 each, the small stuffed animals cost \$1.75 each, and the large stuffed animals cost \$3.00 each. The manager plans to charge \$1.00 to play the game.

As the carnival's financial advisor, write a letter to the manager describing your recommendations on this game. Use mathematics to support your position.

Module Summary

- If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event E is determined using the following ratio:

$$P(E) = \frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

- When the outcomes in an event can be represented as points in a **geometric model**, the theoretical probability of the event may be found using the ratio below:

$$P(E) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

Some examples of geometric measures are lengths, angle measures, areas, and volumes.

- The mean value of an experiment is the **expected value**. Expected value is determined by adding the products of the value of each event and its corresponding theoretical probability.
- A **fair game** is one in which the expected value equals the cost to play.
- Two events A and B are **independent** if $P(A \text{ and } B) = P(A) \cdot P(B)$. Three events A, B, and C are independent if each pair of events is independent and $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$. This definition can be extended to any number of independent events.

Given two independent events, the occurrence of one has no effect on the likelihood of the occurrence of the other. Two events that are not independent are said to be **dependent**.

- **Conditional probability** is the probability of an event occurring given that an initial event has already occurred. The probability that event B occurs, given that event A has already occurred, is denoted by $P(B|A)$.
- In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Selected References

Harshbarger, R. J., and J. J. Reynolds. *Finite Mathematics for Management, Life, and Social Sciences*. Lexington, MA: D. C. Heath and Co., 1992.

Heintz, R. E. "It's in the Bag." *Mathematics Teacher* 70 (February 1977): 132–36.

North Carolina School of Science and Mathematics, Department of Mathematics and Computer Science. *Geometric Probability*. Reston, VA: National Council of Teachers of Mathematics, 1988.

U.S. Bureau of the Census. *Statistical Abstract of the United States: 1993*. Washington, DC: U.S. Government Printing Office, 1993.

Zuwaylif, F. H. *General Applied Sciences*. Reading, MA: Addison-Wesley, 1970.

Atomic Clocks Are Ticking



You've found the remains of an old cooking fire. How do you tell if that lump of charcoal is 10 years old, or 10,000? In this module, you explore one method that archaeologists use to tell time.

John Freal • Sandy Johnson • Anne Merrifield



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Atomic Clocks Are Ticking

Introduction

Andrea shone her flashlight at the walls of a cavern. One wall was covered with drawings of animals and people, just like in a museum. Jared guessed that the pictures could be hundreds—or even thousands—of years old. Excited about their discovery, the two friends decided to tell Professor Cordova about the cave.

Three years after Professor Cordova's first visit, there have been many changes. A sign warns visitors not to touch the ancient drawings. The floor at the cave's mouth has been carefully excavated to reveal layers of charcoal and ash. Professor Cordova explains to visitors that people have used this cave for thousands of years. Samples of the charcoal from old cooking fires are being used to determine the approximate age of the site.

One visitor asks how anyone can possibly determine the age of a lump of charcoal. Professor Cordova then describes the **carbon dating** process. Anything that has once been alive contains carbon atoms, including a charred piece of wood. A very small fraction of those atoms are radioactive. They emit tiny particles that scientists can measure. As time passes, the number of radioactive atoms decreases in a predictable way, acting as an atomic clock. Because scientists know the original level of radioactive carbon in living things, they can read the clock by determining the amount that remains.

Activity 1

The process by which the number of radioactive atoms decreases is **radioactive decay**. In this activity, you learn more about the pattern of decay described by Professor Cordova.

Exploration

In this exploration, you use a model to simulate the process of radioactive decay. You need a container with a lid, 32 chips, and graph paper or a spreadsheet for recording data.

Read Parts **a–g** before beginning the exploration. The chips represent radioactive atoms. Predict how the number of chips will change during the simulation. Record your prediction, then begin.

- a. Put a mark on one side of all 32 chips and place them in the container.
- b. Close the lid, then shake the container.

- c. Open the container and remove each chip with a mark showing.
- d. Count the number of chips remaining.
- e. Record both the number of chips remaining and the shake number. (After shake 0, there are 32 chips.)
- f. Repeat Parts **b–e** until no chips remain in the container.
- g. Graph the data. Plot the shake number along the x -axis and the number of chips remaining along the y -axis. **Note:** Save your data for use in the assignment.

Discussion

- a. Describe any patterns you observed in your data or graph.
- b. What similarities or differences are there among the patterns found by the class?
- c. What fraction of the previous number of chips would you expect to remain after each shake?
- d. What happens to any number representing a population when it is multiplied by a factor between 0 and 1?

Mathematics Note

An equation of the form $y = a \cdot b^x$ is an exponential function. The function can be used to describe a pattern of **exponential growth** or **exponential decay**.

When this equation describes the growth or decay in a population, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the rate of growth or decay). The independent variable x represents number of time periods, while the dependent variable y represents the total population after x time periods. When used to model growth or decay, the exponential function will be of the form $y = a(1 \pm r)^t$.

In exponential growth, r is positive and represents the **growth rate**. In exponential decay, r is negative and represents the **decay rate**.

For example, the value of business equipment often depreciates exponentially. If the value of a machine originally worth \$1000 decreases by 19% each year, the equation that models its depreciation is $y = 1000(1 - 0.19)^x$ or $y = 1000(0.81)^x$. In this case, x represents the number of years since the machine was purchased and y represents the value of the machine after x years. The graph in Figure 1 shows the depreciation of this machine over a 10-year period.



Figure 1: Depreciation of equipment over 10 years

- e. Given the example outlined in the **Mathematics Note**, how would you interpret the value of y in each of the following equations?
- $y = 1000(0.81)^{1/2}$
 - $y = 1000(0.81)^{1/3}$
- f. Describe how you could use a calculator to find the value of each of the following expressions.
- $(0.81)^{1/2}$
 - $(0.81)^{1/3}$
- g. What is the mathematical meaning of $(0.81)^{1/2}$?
- h. From your response to Part g, what do you think is the mathematical meaning of $(0.81)^{1/3}$? Verify your response.

Mathematics Note

The roots of a number can be represented using exponents of the form $1/n$, where n is a natural number. If the n th root of a exists, it can be represented as follows:

$$\sqrt[n]{a} = a^{1/n}$$

For example, $\sqrt[4]{64} = 64^{1/4}$, $\sqrt[5]{32} = 32^{1/5}$, and $\sqrt{16} = 16^{1/2}$.

- i. In the equation $y = a \cdot b^x$, $b = 1 + r$.
1. What are the possible values of b when the equation models exponential growth?
 2. What are the possible values of b when the equation models exponential decay?

Assignment

- 1.1 Evaluate each of the following expressions:
- a. $125^{1/3}$
 - b. $81^{1/4}$
 - c. $343^{1/3}$
 - d. $\sqrt[10]{1024}$
- 1.2 Use the data you collected in the exploration to complete Parts **a–e**.
- a. Find the percent decrease in the population after each shake. For example, if there are 17 chips remaining after shake 1, the percent decrease can be found using the following ratio:

$$\frac{32 - 17}{32} = \frac{15}{32} \approx 0.47 = 47\%$$

- b. 1. Determine the mean of the percents of decrease for all shakes.
2. Considering the theoretical probability of a chip landing with the marked side up, what would you expect this mean to be? Explain your response.
 - c. Describe the relationship between the rate of decay and the mean of the percents of decrease.
 - d. Use the mean from Part **b** to determine the value of b , where $b = 1 + r$.
 - e. Write an equation of the form $y = a \cdot b^x$ that models the exponential decay of the population of chips.
- 1.3 Complete the following table.

	Initial Population	Rate of Growth or Decay	Model Equation of the Form $y = a \cdot b^x$
a.	500	8%	
b.			$y = 20,000 \cdot (0.89)^x$
c.	100	-32%	

- 1.4 A 1-g sample of carbon contains $5.02 \cdot 10^{22}$ atoms, of which 1 out of every 1 trillion ($1 \cdot 10^{12}$) is radioactive. How many radioactive atoms are there in this sample?

- 1.5** Suppose that you began the exploration in Activity 1 with $4 \cdot 10^{14}$ chips. Predict the number of chips remaining after the third shake. Explain your reasoning.
- 1.6** An object contains $3.8 \cdot 10^{13}$ radioactive atoms. The annual rate of decay is 0.5. Estimate the number of radioactive atoms the object contained in the previous year. Describe how you found your solution.
- 1.7** A sample contains $1 \cdot 10^{12}$ radioactive atoms which decay at the rate of 50% every decade. How many decades will it take for the number of radioactive atoms to be less than 1 billion? Explain your reasoning.
- 1.8** Each of the equations below models a pattern of either exponential decay or exponential growth. For each equation, complete the following steps.
- Identify whether the equation models growth or decay.
 - Determine the value of r .
 - Use appropriate technology to graph the function and describe the basic shape of the graph.
- a. $y = 350 \cdot (1.055)^x$
- b. $y = 100 \cdot (0.50)^x$
- c. $y = 5000 \cdot (0.75)^x$
- * * * * *
- 1.9** The balance of an interest-bearing account also can be modeled by an exponential equation. For example, consider an initial deposit of \$1000 at an annual interest rate of 8%. The interest earned each year is deposited in the account at the end of the year.
- a. Assuming that no withdrawals or deposits are made, write an equation that models the account balance after x years.
 - b. What is the account balance at the end of 5 years?
 - c.
 1. Create a graph that shows the increase in the account balance over the next 20 years.
 2. Using the graph, estimate how many years it will take for the initial deposit to double.
 - d. If you deposited \$1000 in an account of this type, what would the account balance be when you are 65 years old?

- 1.10** The following graph shows the change in a town's population over a 20-year period.



- What was the town's initial population?
- About how many years did it take for the population to decrease by half?
- Judging from the graph, what was the population at year 15?
- Using the method described in Problem 1.2, determine the mean of the percents of decrease in the population over the 20-year period.
- Find an equation of the form $y = a \cdot b^x$ that models this data, where y represents population and x represents years.
 - Use this equation to verify your response to Part c.

Activity 2

In living plants and animals, only about 1 in every $1 \cdot 10^{12}$ carbon atoms is radioactive. When a plant or animal dies and new carbon is no longer absorbed or metabolized, these radioactive carbon atoms decay. When determining the age of an artifact using carbon dating, scientists can estimate the number of radioactive carbons remaining. This number, along with carbon's known rate of decay, allows scientists to determine the age of the sample.

Given carbon's rate of decay and the number of radioactive atoms present at a particular time, it is also possible to calculate the number of radioactive atoms at any time in the past. Such calculations may require the use of negative exponents.

In this activity, you examine other ways to represent the model of exponential decay from the exploration in Activity 1.

Exploration 1

- a. Create a spreadsheet with the headings shown in Table 1 below. Complete the table for values of x from 4 to -4 , where x is an integer.

Table 1: Spreadsheet for Exploration 1

x	2^x	$(1/2)^x$	$1/2^x$	2^{-x}
4	16	0.0625		
3	8			
2				
\vdots				
-2				
-3				
-4				

- b. Describe any patterns you observe in the spreadsheet.
- c. 1. Create a scatterplot of the values of 2^{-x} versus the values of x .
2. On the same coordinate system from Step 1, create a scatterplot of the values for 2^x versus the values for x .

Discussion 1

- a. Based on your results in Table 1, what is the value of 2^0 ?
- b. Describe the relationship among the following expressions: $(1/2)^x$, $1/2^x$, and 2^{-x} .
- c. What is the relationship between 2^x and each of the other expressions in Table 1?
- d. In the Level 1 module “Are You Just a Small Giant,” you learned that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

where n is a non-negative integer and $b \neq 0$. From your spreadsheet, it appears that

$$\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$$

Are these two relationships in conflict? Explain your response.

- e. Describe some other ways to represent the quantity 2^{-3} .
- f. How do the graphs from Part c of the exploration compare?

Mathematics Note

If a is a nonzero real number, then $a^0 = 1$.

If a is a nonzero real number and n is an integer, then

$$a^{-n} = \frac{1}{a^n}$$

For example, $2^{-3} = 1/2^3$ and $3^4 = 1/3^{-4}$. This relationship is also true when $a \neq 0$ and n is a real number.

- g.** Describe how to express each of the following using a negative exponent:
- 2^3
 - $a \cdot b^2$
- h.** Describe how to express each of the following without using a negative exponent:
- 2^{-3}
 - $a \cdot b^{-2}$

Exploration 2

In this exploration, you examine some properties of integer exponents that can help you determine rates of decay.

- a.** Create a spreadsheet with headings like those in Table 2 below. The spreadsheet should be created so that the values you select for b , x , and y in the first three cells of each row are used to complete the rest of the entries in that row.

Table 2: Spreadsheet for Exploration 2

b	x	y	b^{x+y}	$b^{x \cdot y}$	b^{x^y}	$(b^x)^y$	$b^x \cdot b^y$	b^x / b^y

- b.** Select values for b , x , and y and enter them in the appropriate columns. Note the values generated in the remaining columns. Describe any relationships you observe.
- c.** Change any combination of the values of b , x , and y . Determine if your observations from Part **b** hold true for these new values.

Discussion 2

- a. Compare your observations with those of others in the class. Do the generalizations you made seem to hold true for all examples?

Mathematics Note

When two exponential expressions containing the same base are multiplied, their product is the base raised to a power equal to the sum of the exponents. For natural numbers m and n ,

$$\begin{aligned} a^m \cdot a^n &= \overbrace{a \cdot a \cdot \dots \cdot a}^{m \text{ terms}} \cdot \overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ terms}} \\ &= \overbrace{a \cdot a \cdot \dots \cdot a}^{m+n \text{ terms}} = a^{m+n} \end{aligned}$$

This result also is true for any real numbers m and n when $a > 0$.

For example, $3^{-2} \cdot 3^4 = 3^{-2+4} = 3^2 = 9$.

When an exponential expression is raised to a power, the result equals the base in the original expression raised to the product of the powers. For natural numbers m and n ,

$$(a^m)^n = \overbrace{a^m \cdot a^m \cdot \dots \cdot a^m}^{n \text{ terms}} = a^{m \cdot n}$$

This result is also true for any real numbers m and n when $a > 0$.

For example, $(3^{-2})^3 = 3^{-2 \cdot 3} = 3^{-6} = 1/729$.

When two exponential expressions containing the same base are divided, their quotient is the base raised to a power equal to the exponent of the dividend minus the exponent of the divisor. For natural numbers m and n ,

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} = a^{m+(-n)} = a^{m-n}$$

This result also is true for any real numbers m and n when $a > 0$.

For example, $3^7/3^4 = 3^{7-4} = 3^3 = 27$.

- b. Explain how you could use the relationships described in the mathematics note above to find the value of x in each of the following:
- $4^{-5} \cdot 4^x = 4^{10}$
 - $(4^{-5})^x = 4^{10}$
- c. What value of x satisfies the equation $(5^3)^x = 5^1$? Describe how you identified this value.

- d. What number do you think is represented by each of the following expressions?

1. $16^{0.25}$

2. $16^{3/4}$

Mathematics Note

When $a \geq 0$, a rational exponent in the form m/n can be represented as follows, where m/n is in lowest terms:

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

For example, $8^{0.4} = 8^{4/10} = 8^{2/5} = (8^{1/5})^2 = (\sqrt[5]{8})^2 \approx 2.30$.

- e. The number of radioactive atoms in a sample at a given time can be modeled by the equation $y = 64 \cdot 0.67^{4.3}$. What does 4.3 represent in this situation?
- f. Describe another way to represent the expression $b^{4.3}$.
- g. Because $2/6 = 1/3$, one might incorrectly believe that $(-8)^{2/6} = (-8)^{1/3}$.
1. Express $(-8)^{2/6}$ and $(-8)^{1/3}$ in as many different forms as you can. Evaluate each expression and compare the results.
 2. According to the mathematics note above, the relationship below is true only when the exponent m/n is in lowest terms.

$$a^{m/n} = \sqrt[n]{a^m}$$

Using $(-8)^{2/6}$ and $(-8)^{1/3}$ as examples, explain why you think this condition is necessary.

Assignment

- 2.1 The number of radioactive atoms in a sample decays at a rate of 50% per year and can be modeled by an equation of the form $y = a \cdot b^x$.
- a. Determine the proportion of the original number of radioactive atoms that remain for each of the following values of x :
 1. 2
 2. 0
 3. -2
 - b. In this situation, what does $x = -2$ represent?

- 2.2** The equation $y = 32 \cdot 2^x$ models the number of bacteria present in a population after x minutes.
- In this situation, what does the expression $32 \cdot 2^4$ represent?
 - Given this same population of bacteria, what does the expression $32 \cdot 2^{-4}$ represent?
 - Write the expression in Part **b** without using a negative exponent.
- 2.3**
- In Activity **1**, you modeled the number of chips remaining after x shakes using the equation $y = 32 \cdot (1/2)^x$. Write an equivalent equation using a negative exponent.
 - Use the equation $y = 32 \cdot (1/2)^x$ to predict the shake number at which each of the following populations of chips would remain:
 - 8 chips
 - 2 chips
 - 64 chips
 - 128 chips
 - 32 chips.

- 2.4** The two relationships below were described in Part **d** of Discussion **1** and in the previous **Mathematics Note** on page 278, respectively:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \text{and} \quad a^{-m} = \frac{1}{a^m}$$

Use these relationships to show that the following relationship is true, where $a \neq 0$ and $b \neq 0$.

$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$$

- 2.5**
- Rewrite the expression $(1/3)^3$ using an integer as the base.
 - Rewrite the expression $(1/a)^b$ using a as the base, where $a \neq 0$.
 - Rewrite the expression $(3/5)^2$ without using parentheses.
 - Rewrite $(3/5)^2$ using a negative exponent.
 - Rewrite $(4/3)^{-2}$ using a positive exponent.
- 2.6** A fragment of animal bone originally contained $5.12 \cdot 10^{14}$ radioactive carbon atoms. Now it contains only $3.2 \cdot 10^{13}$ radioactive carbons.
- Given a rate of decay of 50% for each interval of time, represent this situation in an equation of the form $y = a \cdot b^x$.
 - Determine the age of the artifact.

- 2.7**
- How does a change in the initial population affect the graph of the equation $y = a \cdot b^x$?
 - How does a change in the rate of decay affect the graph of the equation $y = a \cdot b^x$?

2.8 The equation $y = 4000 \cdot (17/20)^x$ models the decay of radioactive atoms in an object. In a paragraph, describe the situation represented by this equation.

2.9 To solve the equation $64 = x^3$ for x , both sides of the equation can be raised to the exponent $1/3$, as follows:

$$64 = x^3$$

$$(64)^{1/3} = (x^3)^{1/3}$$

$$64^{1/3} = x$$

- Explain why $(x^3)^{1/3}$ can be replaced with x .
 - Find the value of x in the equation $64^{1/3} = x$.
- 2.10** Solve each of the following equations for x and describe how you reached your solutions.

a. $x^6 = 729$

b. $x^{0.125} = 2$

c. $10^5 \cdot x = 10^{16}$

2.11 Each of the following equations is a model of exponential growth or decay in the form $y = a \cdot b^x$. For each one, determine the value of b , explain whether it models growth or decay, and identify the rate.

a. $1024 = 64 \cdot b^4$

b. $13 = 10 \cdot b^3$

c. $1.6 \cdot 10^{13} = 5.0 \cdot 10^{14}(b^5)$

d. $107 = 1000 \cdot b^{10}$

* * * * *

- 2.12** The table below shows some data collected while a container of boiling water cooled to room temperature.

Number of 2-min Intervals Elapsed	Degrees above Room Temperature (°C)
1	74
2	70
3	67
4	65
5	63
6	60
7	58
8	56

- Create a scatterplot of the data.
 - Find the mean of the percents of decrease in the temperature for these 2-min intervals.
 - Using the mean from Part **b** as a rate, write an equation of the form $y = a \cdot b^x$ that models the data.
 - When the water was first poured, its temperature was 100° C. Use this fact and your model to estimate the temperature of the room.
 - Should your equation from Part **c** be used to make predictions about the temperature of the water several minutes before the time frame given in the table? Explain your response.
- 2.13** Identify the equation in column B that is equivalent to each equation in column A.

Column A	Column B
$y = 5 \cdot (1 - 0.5)^3$	$y = 5 \cdot (1/2)^{-3}$
$y = 5 \cdot (2)^3$	$y = 5 \cdot (0.5)^{-1/3}$
$y = 5 \cdot (2)^{1/3}$	$y = 5 \cdot (2)^{-3}$

Activity 3

All living organisms contain carbon atoms. A relatively small proportion of these carbon atoms are unstable. Over time, these unstable atoms, known as carbon-14 atoms, decay. As they decay, they emit energy. Scientists can estimate the age of a sample by measuring the energy emitted from these radioactive atoms.

The time required for one-half of the carbon-14 atoms in a given sample to decay is the **half-life**. Carbon-14 has a half-life of 5730 years. Other unstable atoms have half-lives that range from fractions of seconds to billions of years. For example, the half-life of carbon-11 is about 21 min, while the half-life of uranium-238 is 4.5 billion years. These unstable atoms rarely have half-lives that are exactly 1 unit of time. In the following exploration, you modify models of exponential decay to include half-lives that are not whole numbers.

Exploration

Although a typical artifact may contain trillions of radioactive atoms, in this exploration, you again consider a sample that has only 32 unstable atoms.

- a. Create a table with headings like those in Table 3. Round 0 represents the time when the number of radioactive atoms in the sample is 32.

Table 3: Radioactive atom simulation

Round	Number of Radioactive Atoms
0	32
1	
2	
⋮	
9	
10	

- b. Suppose the rate of decay is $1/6$. This means each radioactive atom has 1 chance in 6 of decaying.

1. To model a rate of decay of $1/6$, randomly generate an integer from 1 to 6 for each radioactive atom remaining in the sample. The last entry in the second column of Table 3 represents the number of radioactive atoms remaining in the sample.

For example: In Round 1, generate a list of 32 random integers since 32 is the last entry in the second column of Table 3.

2. Count the number of times that the integer 1 appears in the list. Reduce the quantity of radioactive atoms in the sample by this number.

3. Record the remaining number of radioactive atoms in the second column of Table 3. This ends the round. The number recorded represents the number of radioactive atoms that remain in the sample.
- c. Repeat Part b for 10 rounds, or until no radioactive atoms remain.
 - d. Create a scatterplot of the data. Represent the round number on the x -axis and the number of radioactive atoms on the y -axis.
 - e.
 1. What equation of the form $y = a \cdot b^x$ would you expect to model this simulation?
 2. Graph this equation on the coordinate system from Part d.
 - f.
 1. Use the data to estimate the half-life, in number of rounds, of these radioactive atoms.
 2. Use your equation from Part e to determine the half-life of the atoms.

Discussion

- a. What fraction of the number of atoms did you expect to remain after each round? Explain your response.
- b. How does the half-life of the atoms in this simulation compare with the half-life of the atoms in the simulation in Activity 1?
- c.
 1. Describe how you determined a model equation of the form $y = a \cdot b^x$ in Part e of the exploration.
 2. How does this equation compare with the one that modeled the simulation in Activity 1?

Assignment

- 3.1 A sample originally contained $1.05 \cdot 10^{11}$ radioactive atoms, which have been decaying at a rate of 0.00003 atoms/year .
 - a. Write an equation of the form $y = a \cdot b^x$ to model this situation.
 - b. The number of radioactive atoms currently in the sample is $2.3 \cdot 10^{10}$. Use your model to estimate the age of the sample.
- 3.2
 - a. Find a value for x when:
 1. $3^x = 9$
 2. $3^x = 27$
 - b. If the equation $3^x = 16$ has a solution, between what two whole numbers must x lie?
 - c. Use a guess-and-check method to find x to the nearest hundredth.
 - d. Find x to the nearest hundredth when $2.5^x = 20$.

- 3.3** In the exploration in this activity, you conducted a simulation with 32 radioactive atoms and a rate of decay of $1/6$. This situation can be modeled by the exponential equation $y = 32 \cdot (5/6)^x$.

The half-life is the time required for the number of radioactive atoms to decrease to 16. Substituting this value into the model equation yields $16 = 32 \cdot (5/6)^x$. Use this equation to estimate the half-life.

- 3.4** **a.** Each of the following equations models the radioactive decay of a substance over time, where x represents years. Determine the half-life for each substance and describe the methods you used.

1. $y = 4000(0.61)^x$

2. $y = 5.3 \cdot 10^7 (0.61)^x$

3. $y = 0.61^x$

- b.** Considering your responses to Part **a**, how does the initial number of radioactive atoms appear to affect the half-lives of substances with the same rate of decay?

- 3.5** Kim is a doctor in the Department of Nuclear Medicine at the county hospital. She uses radioactive tracers to detect disease or injury in her patients. When comparing levels of radioactivity in the tracers, Kim designates the initial level as 100%. Later measurements are then recorded as percentages of that level. Using this method, radioactive decay can be modeled by equations of the form $y = 100 \cdot b^x$. When $y = 50$, the value of x equals the half-life.

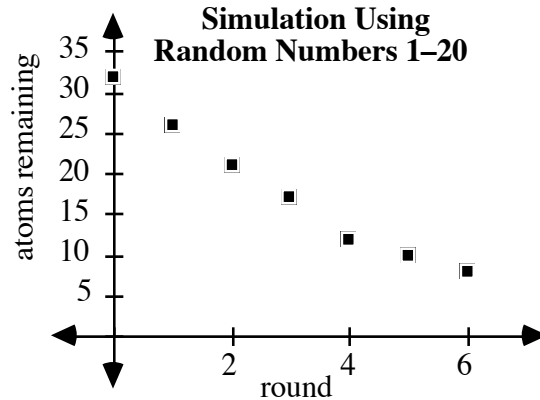
- a.** In one case, Kim has monitored the radioactivity of a tracer at the same time each day for two consecutive days. The second day's radioactivity is 80% of the first day's. Write an equation of the form $y = 100 \cdot b^x$ to model this situation, where x represents time in days.
- b.** Create a graph of the model for the next 10 days.
- c.** Estimate the half-life of the tracer.
- d.** Kim received the radioactive tracer 2 days after it was manufactured. If the rate of decay is constant, what was the radioactivity of the tracer when it was first produced? Explain your response.

- 3.6** The comparative equation $50 = 100 \cdot b^x$ can be used to calculate the value of b for a given half-life. Determine the value of b and identify the rate of decay for each of the following half-lives:

- a.** 12.5 hours
- b.** 57.3 centuries

- 3.7** Imagine that you are examining a sample that contains radioactive atoms with a half-life of 5.0 years.
- The radioactive decay of these atoms can be modeled by the equation $y = 100 \cdot b^x$ where x represents time in years. What is the value of b in this equation?
 - What is the radioactive material's rate of decay?
 - If the level of radioactivity remaining is 93% of the original level, how old is the sample you are examining? Explain your response.
- 3.8** Adrian's foot has bothered him since Wednesday. The X-rays taken on Friday looked normal, but his doctor suspects a possible stress fracture. On Monday, the doctor will inject a radioactive tracer to help detect the site of the fracture. The tracer has a half-life of 6.0 hr.
- What percentage of the tracer's radioactivity remains after each hour?
 - Write an equation of the form $y = 100 \cdot b^x$, where x represents hours, to model the decay of this tracer.
 - After the tracer has been in Adrian's body for 45 min, the doctor will scan Adrian's foot for signs of radioactivity. What percentage of the tracer is left in his body at this time?
 - How many hours will pass before only 10% of the tracer remains?
 - The radioactive tracer was manufactured 3 hr before it was injected into Adrian. How much more radioactive was the tracer at the time of its manufacture?
- 3.9** As mentioned in the introduction to Activity 3, the half-life of carbon-14 is 5730 years. Use an equation of the form $y = a \cdot b^x$ to determine the rate of decay for carbon-14 per century. Hint: If the initial amount of carbon-14 is 100%, the amount remaining after 57.3 centuries is 50%.
- * * * * *
- 3.10** Use technology to find a value of x so that $256^x = 4$. Express this value as a decimal and as a fraction.
- 3.11** One of your classmates has conducted a simulation like the one described in Activity 3. Instead of randomly generating integers from 1 to 6 however, she generated integers from 1 to 20, inclusive, and removed an atom for each multiple of 5 that appeared in the list.
- What fraction of the atoms would you expect to remove after each round? Explain your response.
 - What fraction of the atoms would you expect to remain in the container after each round?

- c. A scatterplot of the data from your classmate's simulation is shown below. Use it to estimate the half-life of the substance.



- d. Find an equation of the form $y = a \cdot b^x$ that models the data.

3.12 The Trojan Nuclear Power Plant near Portland, Oregon, is shutting down. One of the waste products of its operation is radioactive strontium-90. Strontium-90 is extremely hazardous to people and other living things. Its half-life is 28.0 years. As an environmental engineer, you must design leak-proof containers to hold strontium-90 until less than 1% of its radioactivity remains.

- What percentage of a sample of strontium-90's initial radioactivity remains after 1 year?
- How many years will it take for a sample of strontium-90 to have less than 1% of its present radioactivity?
- Most of the plant's strontium-90 was buried in temporary storage containers 5 years ago. How much more radioactive was this material at the time of its burial?
- To inform the public of the radioactivity present in the new, leak-proof containers, your firm has decided to attach a warning label to each one. Design a label that includes a graph showing the percentage of radioactivity remaining as a function of the year.

3.13 In 1972, a candy bar cost \$0.25. In 1996, the same candy bar sold for \$0.65.

- Assuming that the cost of the candy bar grew exponentially, determine the annual rate of growth.
- Write an equation that could be used to model the cost of the candy bar in a given year.
- Use your equation to estimate the year in which the candy bar cost only \$0.10.
- Predict the year in which the candy bar will cost \$1.00.

* * * * *

Summary Assessment

Professor Cordova found many layers of charcoal in the fire pit near the mouth of Andrea and Jared's cave. Each layer contained a different percentage of the carbon-14 activity of living trees. Her data is shown in the table below. Layer 1 was taken from the top of the pit; layer 11 was taken from the bottom.

Layer	Percentage of Carbon-14 Radioactivity
1	91.6
2	83.3
3	81.6
4	78.5
5	75.0
6	71.2
7	65.4
8	47.4
9	42.9
10	39.9
11	32.1

1. Professor Cordova believes humans have occupied this site for no more than 12,000 years. Given that the half-life of carbon-14 is about 57.3 centuries, the rate of decay is approximately 0.012. Find the approximate age of each layer of charcoal.
2. Near the cave, Professor Cordova found what appears to be a mammoth bone. Mammoths became extinct about 11,500 years ago. What percentage of carbon-14 radioactivity would you expect to find in this artifact? Explain your reasoning.
3. As trees grow, new carbon-14 is absorbed only by the outermost ring of the trunk. The amount of carbon-14 radioactivity in the inner rings decays as if the wood were actually dead. Because of this phenomenon, the age of trees can be determined by carbon dating.
 - a. To verify her carbon-dating procedure, Professor Cordova decides to test a newly fallen tree near the cave. The outermost ring of the tree has 112% of the carbon-14 radioactivity of the innermost ring. Use this information to estimate the age of the tree.
 - b. The age of a tree also can be found by counting the growth rings in the trunk, one ring for each year of life. The rings of the tree indicate that it was about 940 years old. How does your estimate in Part a compare with this value?

Module Summary

- The process by which the number of radioactive atoms decreases is **radioactive decay**.
- An equation of the form $y = a \cdot b^x$ is an exponential function. The function can be used to describe a pattern of **exponential growth** or **exponential decay**.

When this equation describes the growth or decay in a population, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the rate of growth or decay). The independent variable x represents number of time periods, while the dependent variable y represents the total population after x time periods. When used to model growth or decay, the exponential function will be of the form $y = a(1 \pm r)^t$.

- In exponential growth, r is positive and represents the **growth rate**. In exponential decay, r is negative and represents the **decay rate**.
- The roots of a number can be represented using exponents of the form $1/n$, where n is a natural number. If the n th root of a exists, it can be represented as follows:

$$\sqrt[n]{a} = a^{1/n}$$

- If a is a nonzero real number, then $a^0 = 1$.
- If a is a nonzero real number and n is an integer, then

$$a^{-n} = \frac{1}{a^n}$$

- When two exponential expressions containing the same base are multiplied, their product is the base raised to a power equal to the sum of the exponents. For natural numbers m and n ,

$$\begin{aligned} a^m \cdot a^n &= \overbrace{a \cdot a \cdot \dots \cdot a}^{m \text{ terms}} \cdot \overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ terms}} \\ &= \overbrace{a \cdot a \cdot \dots \cdot a}^{m+n \text{ terms}} = a^{m+n} \end{aligned}$$

This result also is true for any real numbers m and n when $a > 0$.

- When an exponential expression is raised to a power, the result equals the base in the original expression raised to the product of the powers. For natural numbers m and n ,

$$(a^m)^n = \overbrace{a^m \cdot a^m \cdot \dots \cdot a^m}^{n \text{ terms}} = a^{m \cdot n}$$

This result is also true for any real numbers m and n when $a > 0$.

- When two exponential expressions containing the same base are divided, their quotient is the base raised to a power equal to the exponent of the dividend minus the exponent of the divisor. For natural numbers m and n ,

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} = a^{m+(-n)} = a^{m-n}$$

This result also is true for any real numbers m and n when $a > 0$.

- When $a \geq 0$, a rational exponent in the form m/n can be represented as follows, where m/n is in lowest terms:

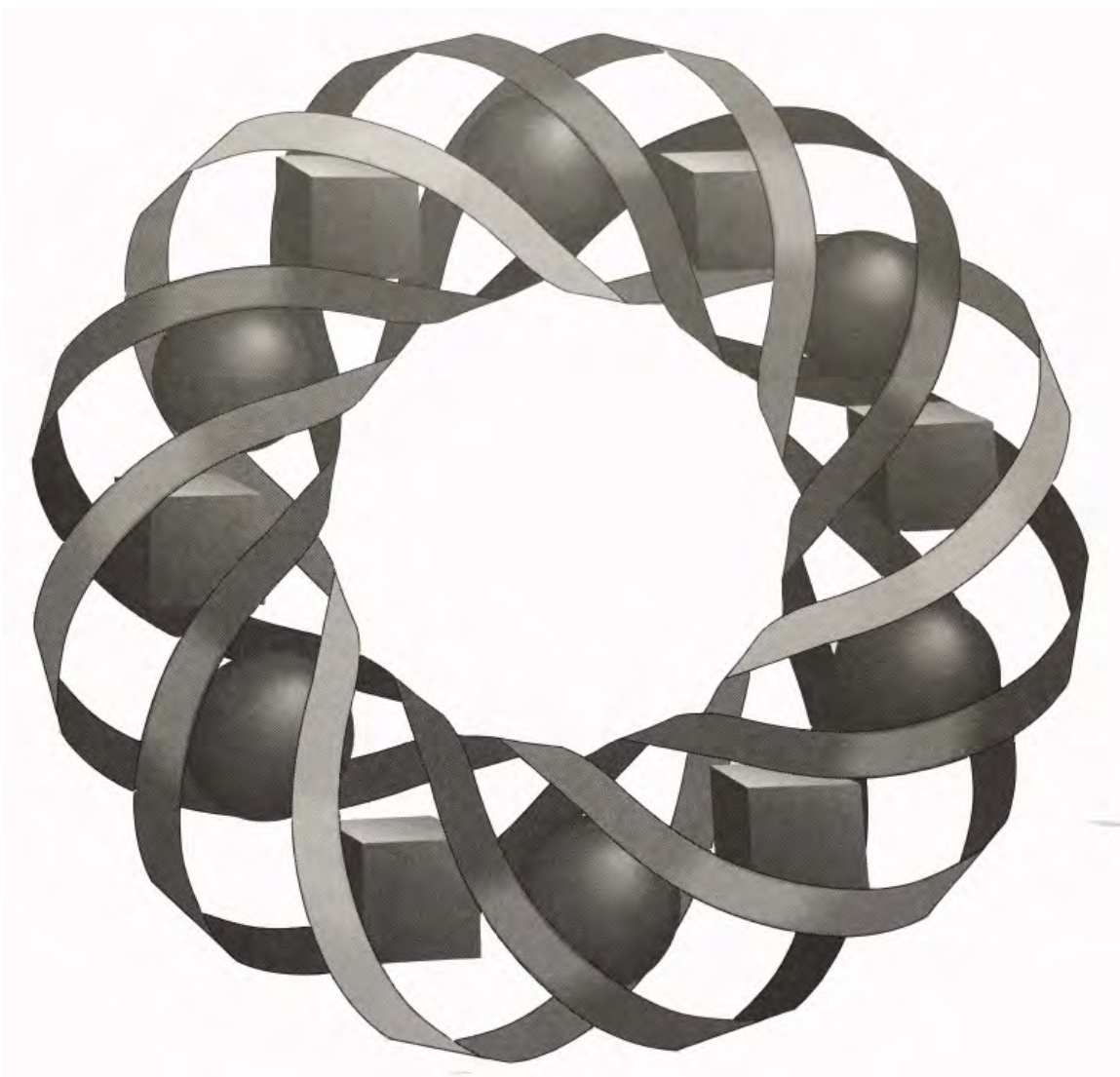
$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Selected References

Libby, W. F. *Radiocarbon Dating*. Chicago, IL: University of Chicago Press, 1952.

Taylor, R. E. *Radiocarbon Dating: An Archaeological Perspective*. Orlando, FL: Academic Press, 1987.

And the Survey Says . . .



How can television newscasters predict the next president of the United States long before the votes are counted? In this module, you explore some of the basics of sampling and surveys.

Staci Auck • Kyle Boyce • Tom Teegarden



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

And the Survey Says . . .

Introduction

Opinion polls have become a familiar feature of American life. You read about them in magazines and newspapers, and hear them discussed on radio and television. Such surveys are administered by a number of different polling organizations. Some—like Gallup, Roper, and Harris—are independent companies. Others, like the New York Times and CBS, are members of the news media themselves.

The pollsters question you about everything from religion to politics, from your views on health care to your preferred brand of toothpaste. The information they obtain has many applications. A candidate for the U.S. Senate might use polls to plan a campaign strategy. A Hollywood producer might use audience surveys to edit an upcoming movie. A cookie manufacturer might use consumer taste preferences to target a profitable new market.

How do polling organizations discover the likes and dislikes of a nation of more than 250 million? Fortunately for the pollsters, reliable results can be obtained by surveying only about 1500 people—a small percentage of the entire population. Why is this sample big enough? This module will help you answer that question.

Activity 1

Whether you watch television, listen to the radio, or read newspapers, what you get from the news often comes in the form of statistics. Statistics arise from the collection of data. In the case of opinion polls, data is collected by surveying small samples of the entire population. In this activity, you investigate some methods pollsters use to determine whom to survey and examine some of the factors that might prevent a sample from representing the entire population.

Mathematics Note

A **population** is a group of all objects, individuals, or observations about which information is to be gathered. A quantity that describes a population is a **parameter**.

A **sample** is a portion of the population. A quantity that describes a sample is a **statistic**.

A **bias** is a factor that prevents a sample from representing the entire population.

A **census** is the collection of data about an entire population.

For example, a pollster's study of voter preferences for an upcoming presidential election might describe the responses from all registered voters as a *population*. The percentage of voters who intend to vote for a particular candidate is a *parameter*.

A *sample* of the population might include 1500 registered voters who answered their home telephones on a Monday morning. The percentage of those surveyed who intend to vote for a particular candidate is a *statistic*.

The fact that most working people are not home on weekday mornings would be a *bias* of this sample, since these people would not have a chance of being selected. To perform a *census* of the population, the polling organization would have to question every registered voter in the country—a nearly impossible task.

Exploration

In this exploration, you describe the characteristics of a population of beans based on the data obtained from samples. **Note:** Do not examine the contents of your container before beginning the exploration.

- a.
 1. Shake the container containing the population of beans.
 2. Take a small handful of beans from the container. This is one sample of the population.
 3. Record the percentage of each kind of bean in the sample.
 4. Return the sample to the container and mix it thoroughly with the other beans.
- b. Repeat Part a four more times.
- c. Based on the data collected in your five samples, predict the percentage of each kind of bean in the entire population.
- d. Take a census of the beans. Determine the percentage of each kind of bean in the population. Compare the actual percentages to your predictions in Part c.

Discussion

- a.
 1. Which values in the exploration are parameters?
 2. Which values are statistics?
- b. How did the results of the census compare with your predictions based on five samples?
- c. Describe any biases in the selection process that might have prevented samples from accurately representing the entire population.
- d. How could you eliminate bias in this sampling method?

- e. What kinds of bias could affect a real-life survey?
- f. Consider the following survey question, “Would you prefer the powerful and graceful eagle or the clumsy bison as the school mascot?” How could you reword this question to minimize possible bias?

Assignment

- 1.1 Some television programs offer their audiences a chance to express opinions on everything from favorite rock videos to favorite presidential candidates. Viewers typically dial a phone number to register their votes. Is such a survey biased? If so, describe the biases that might exist. If not, explain why not.
- 1.2 To help determine consumer preferences, an electronics company conducts a poll in a shopping mall. The pollsters stand outside a department store and ask all the people who pass by if they own a videocassette recorder (VCR). Describe any biases this sample might contain.
- 1.3 Washington High School has 2380 students: 500 seniors, 597 juniors, 626 sophomores, and 657 freshmen. Elections for student government are coming up next month. In order to predict the next student body president, the school newspaper has asked you to take a preliminary poll.
 - a. What is the population for this poll?
 - b. Describe how to conduct a census of the population.
 - c. Describe one method for sampling the population.
 - d. What advantages or disadvantages are there to the sampling method you described in Part c?
- 1.4 In 1936, incumbent Franklin Delano Roosevelt was campaigning against Alf Landon for the presidency of the United States. Shortly before the election, pollsters for the *Literary Digest* conducted a survey. They sent postcard ballots to people selected at random from a list of automobile registrations and from a telephone directory. Their results indicated that Landon would win the election. That prediction was incorrect. In a landslide victory, Franklin Roosevelt won all but two states—Maine and Vermont.
 - a. What biases in the poll may have caused the incorrect prediction?
 - b. How might the pollsters have obtained a more accurate picture of the nation’s preferences?
- 1.5 According to pollster George Gallup, “Nothing is so difficult, nor so important, as the selection and wording of questions.” In fact, poor word choice and misleading questions have biased many surveys. Write a survey question that introduces bias due to its own wording and explain the cause of that bias.

* * * * *

- 1.6** Jordan is conducting a survey of music preferences at his school using the following questionnaire:

Which of the following music types do you prefer?

- (1) rock and roll (2) alternative
(3) country western (4) classical
(5) heavy metal (6) other

To obtain a sample of 50 people, he polls 20 students in his mathematics class, 20 friends and acquaintances sitting nearby at lunch, and 10 fellow band members at an after-school practice session.

- a. Is Jordan’s sample representative of the entire school population?
 - b. Describe any biases that might exist in his survey question or sampling techniques.
 - c. Suggest how Jordan could change the sampling method to minimize bias.
- 1.7** For an article in the school paper, Susanne wants to survey students on a proposal to open the campus for lunch. This has been a hot topic for weeks. If the school board approves the proposal, the lunch period will be lengthened from 30 minutes to 45 minutes, which in turn will add 15 minutes to the end of the day. She plans to ask the following question:

“Do you want to have an open campus for the lunch period in order to go anywhere you want to eat instead of being confined to the school cafeteria?”

- a. What do you think is Susanne’s opinion on this proposal?
- b. Write a question that shows a bias for the opposing view.
- c. Write a question on this proposal that minimizes bias.

* * * * *

Research Project

Select and complete one of the projects described below.

- a. Find a report of an interesting survey in a magazine or newspaper. Write a summary of its results, including a description of the population, the sample size, and the sampling method. Identify potential sources of bias in the survey and suggest how these biases might have been avoided.
 - b. Identify a poll or survey (other than the one mentioned in Problem **1.4**) that produced an incorrect prediction. Describe some of the biases that may have caused the mistake. Present your findings to the class.
-
-

Activity 2

In order to design reliable surveys, many polling organizations work hard to remove biases from both their samples and their survey questions. At the same time, however, they must also try to control the costs of administering the poll. Selecting an appropriate sampling method can help achieve all of these goals. In this activity, you examine several sampling methods that can reduce polling costs, simplify administration, and minimize bias.

Exploration

The map in Figure 1 shows a small town located at the intersection of two highways. The town is divided into 40 different districts, each identified by a number. On this map, the area of each district indicates its population (1 square unit represents 100 people).

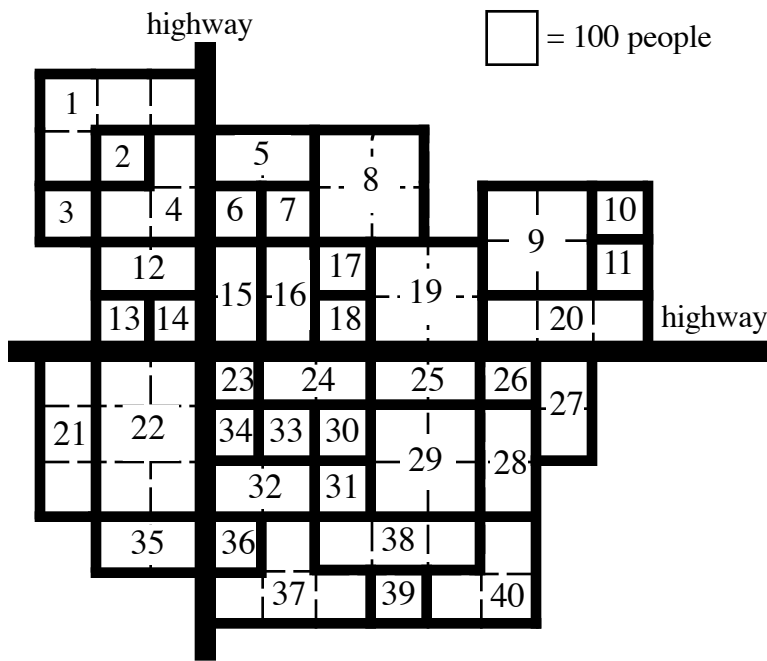


Figure 1: Population of a town, by district

- Select a sample of 4 of the 40 districts from the map in Figure 1.
 - Determine the population of each district in your sample.
 - Using only your sample, predict the mean number of people per district for the entire town.
 - Using only your sample, estimate the population of the entire town.
- Determine the population of the entire town. Compare the actual value with your prediction in Part a.

Mathematics Note

A **simple random sample** is selected so that each member of the population has the same chance of being included in the sample.

For example, consider a population of students in a classroom. To obtain a simple random sample of this population, each student's name could be written on an identical slip of paper. The slips of paper could then be placed into a bowl, mixed thoroughly, and drawn one at a time.

- c.
 1. Randomly generate four different integers in the interval $[1, 40]$. Use these integers to identify four districts on the map in Figure 1.
 2. Use this sample to predict the mean number of people per district.
 3. Use this sample to estimate the population of the entire town.

Mathematics Note

Stratified sampling requires that a population be divided into parts. Each part is a **stratum** (plural, **strata**). To produce a stratified sample, simple random samples are taken from each stratum. These samples do not necessarily have to be the same size.

For example, differences in age could determine the strata of a voting population. To select a stratified sample of this population, a polling organization might survey groups of people with ages 18–28 years, 29–45 years, 46–60 years, and 61 years or older.

Systematic sampling is accomplished by collecting data from every n th unit of a population after randomly choosing a starting point.

For example, a systematic sample of the customers in a shopping mall could be obtained by selecting every fifth person, beginning with the first person who enters after 10:00 A.M.

- d.
 1. Devise a stratified sampling method to obtain a sample of four districts from the map in Figure 1.
 2. Use your stratified sample to predict the mean number of people per district and to estimate the population of the entire town.
- e.
 1. Devise a systematic sampling method to obtain a sample of four districts from the map in Figure 1.
 2. Use your systematic sample to predict the mean number of people per district and to estimate the population of the entire town.

Discussion

- a. Compare your results in the exploration with those of others in your class.

Is it reasonable to expect population estimates made using the same sampling technique to be similar? Why or why not?
- b. You used four different sampling methods in the exploration: sampling based on intuition, simple random sampling, stratified sampling, and systematic sampling. Where might bias occur in each of these methods?
- c. Which method has the most potential for bias? Which has the least potential for bias? Explain your responses.

Assignment

- 2.1 Parts **a–f** each describe a sampling method. Identify those that produce a simple random sample. If a method does not produce a simple random sample, explain why this occurs.
 - a. Select the first 20 students who enter the cafeteria for lunch.
 - b. Assign a number to each student in the school, then use a random number generator to select 20 students.
 - c. Assign a number to each student in the school, then use a random number generator to select 10 boys and 10 girls.
 - d. Write all student identification numbers on identical slips of paper, place the slips in a bin, mix them up, and select 20 from the bin.
 - e. Select the first person who walks into a pep assembly and every 10th person thereafter.
 - f. Select only those students whose school identification numbers end with a 0.
- 2.2 Which of the sampling methods described in Problem **2.1** represent stratified sampling? Which represent systematic sampling?
- 2.3
 - a. The editors of a school paper want to survey student preferences for an upcoming election. The school has 400 students. Suggest one way to obtain a sample of 20 students from this population using each of the following methods:
 1. simple random sampling
 2. stratified sampling
 3. systematic sampling
 - b. Identify potential sources of bias in each sampling technique you suggested in Part **a**.

- 2.4** Identify an issue of concern at your school. Devise a method of surveying student opinion on this issue.
- Describe your poll in a paragraph, including a list of survey questions and an explanation of your sampling technique.
 - Describe both some advantages and some disadvantages of the method you devised.
- 2.5** Write a paragraph describing a situation in which a biased sample could be used to influence people. Identify the source of bias and suggest why this influence might be considered misleading.
- * * * * *
- 2.6** Consider a district that contains 1400 registered voters, 748 of which are women. Describe how you could obtain a sample of 100 registered voters from this population using each of the following methods:
- systematic sampling
 - simple random sampling
 - stratified sampling
- 2.7** A car manufacturer is conducting a marketing study. The proposed sampling method consists of interviewing customers who request test drives at a dealership.
- Will this sampling method produce a simple random sample of all potential car buyers?
 - Describe the advantages and disadvantages of the proposed method.

* * * * *

Activity 3

Amelia and Bernadette are both running for president of Washington High School. Two weeks before election time, the school newspaper surveyed the entire school population of 2380 students. According to the results of this census, 40% of the students favored Amelia, 30% preferred Bernadette, 20% were undecided, and 10% did not plan to vote.

With the election approaching quickly, Amelia felt that she had a comfortable lead. A week after the census, however, she took a strong stand in favor of an unpopular dress code. Concerned that Amelia might have lost some support, her campaign staff surveyed a random sample of 25 students. Of those polled, only 8 students planned to vote for Amelia.

Now Amelia is worried. Do the results of this most recent poll mean her support has slipped to $\frac{8}{25}$ or 32%? Her campaign manager points out that these are the findings of only one sample. Another sample might show completely different results. “But if that’s true,” says Amelia, “what’s the use of taking any samples at all?” Before drawing any conclusions, Amelia and her friends decide to investigate the sampling process.

Exploration

If 40% of Washington High students plan to vote for Amelia, how likely is it for a random sample of 25 students to show only 32% support? In this exploration, you use simulations to model the results of such samples.

- a. The percentage of a population with a given characteristic is a **population proportion**. Create a population of beans in which 40% have a distinguishing characteristic other than size or shape.

In this model, the marked beans represents students who planned to vote for Amelia. The remaining 60% represents students who did not indicate that they planned to vote for Amelia.
- b. In a simple random sample of 25 beans from this population, about how many would you expect to be marked?
- c. The percentage of a sample with a given characteristic is a **sample proportion**. Take a random sample of 25 beans from the container. Record the percentage of marked beans in the sample, then return the sample to the container and mix the beans thoroughly.
- d. Repeat Part c nine more times, for a total of 10 samples.
- e.
 1. Combine the results of your 10 samples with the results of the rest of your class and sort the data from least to greatest.
 2. Determine the mean of the sample proportions.
 3. Display the class data in a histogram, using intervals with a width of 4%. Represent the percentage of marked beans on the x -axis and the frequency on the y -axis.
 4. Find the percentage of sample proportions that are less than or equal to 32%.
- f.
 1. Use technology to simulate Parts a–d of the exploration. Generate as many samples as the technology allows, not to exceed the number of samples in the class data from Part e.
 2. Determine the mean of the sample proportions.
 3. Create a histogram of the data using the same intervals as the histogram in Part e.

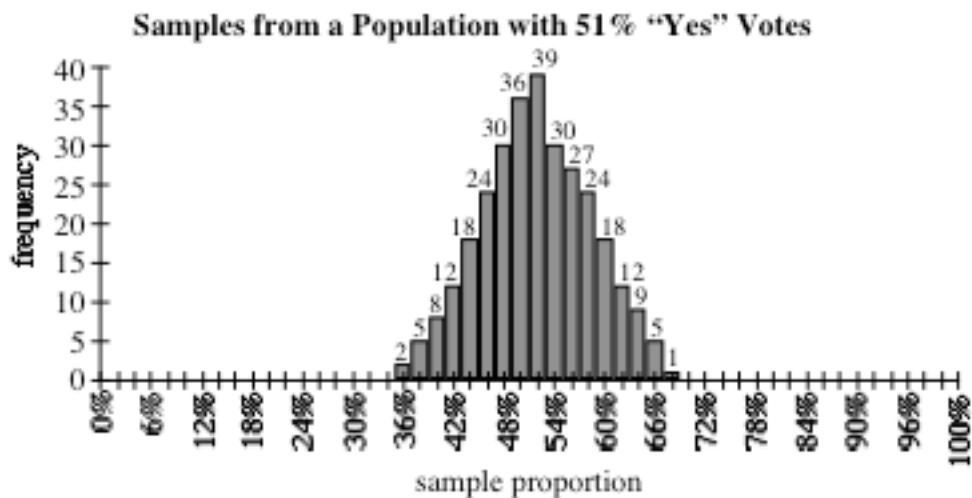
Discussion

- a. What patterns did you observe in the two histograms created in the exploration?
- b. How does the mean of the sample proportions compare with the population proportion?
- c.
 1. If the population proportion had been 52% instead of 40%, how would the histograms have been affected?
 2. What would the histograms have looked like if twice as many samples were taken?
- d.
 1. In the class data, what percentage of the sample proportions were less than or equal to 32%?
 2. Given this percentage, should Amelia be concerned about a loss of student support? Explain your response.
- e. How might you change the sampling method described in Part c of the exploration to obtain a more accurate characterization of the population?

Assignment

- 3.1** In order for your community to build a new swimming pool, 51% of the voting population must approve its construction. Before hiring an architect, the local council decided to assess community support by surveying a random sample of 25 voters. Of those surveyed, 9 people indicated that they would vote for the new pool.

To determine what conclusions they could draw from this sample, the council used a computer simulation to collect numerous samples of size 25 from a population in which 51% favored a proposal. The following histogram shows the results of this simulation.



- a. What percentage of the people surveyed indicated that they would vote for the pool?
- b. What is the total number of samples in the simulation?
- c. What is the mean of the sample proportions in the simulation?
- d. Should the town council hire an architect? Write a recommendation to the council that supports your position.

3.2 In the election for president of Washington High School, 10% of the students do not intend to vote.

- a. What percentage of the school population would have to vote for Amelia in order for her to win? Explain your response.
- b. **1.** Use a simulation to obtain 90 samples of size 25 from a population that would narrowly elect Amelia.
2. Create a histogram of this data.
- c. When Amelia’s campaign crew polled a sample of 25 students, they found that only 8 planned to vote for Amelia. What is the estimated probability that, in a sample of 25 students, 8 or fewer would support Amelia?
- d. Do you think Amelia has a good chance of winning the election? Justify your response.

* * * * *

3.3 A group of friends is playing a board game. In this game, players use a six-sided die to determine the number of spaces they may move around the board. Because 4 of the first 5 players have each rolled a six, the group suspects that the die may not be fair.

- a. Create a simulation to model the roll of a fair die. Use the simulation to obtain 300 samples of size 5 and determine the number of sixes in each sample.
- b. Create a histogram to display the results of the simulation.
- c. Use the histogram to estimate the probability of obtaining a six in 4 out of 5 rolls with a fair die.
- d. Do you think that the die in the board game is fair? Explain your response.

* * * * *

Activity 4

Amelia was relieved to know that her popularity might not have slipped. With 20% of voters still undecided and the election just a few days off, she felt that she still had a chance to win.

Her campaign staff, however, was beginning to have some doubts. In their previous survey of 25 students, only 32% indicated that they would vote for Amelia. What if they had surveyed a sample of 100 students? What would a 32% rating have meant then?

As the staff discussed these questions, they realized that selecting a random sample of size 25 from the Washington High population could result in many different combinations of students. In fact, for a population of 2380, there are about 10^{60} different samples of size 25. Some of these samples would reflect an accurate picture of Amelia's support, while others might give misleading information. How could they possibly be confident in their results?

The campaign staff paid a visit to their mathematics teacher. She told them that even though they could never survey all the possible samples, they could still obtain a reasonable estimate of Amelia's chances in the election using the mean and standard deviation of all possible sample proportions.

Mathematics Note

The mean of all possible sample proportions for samples of size n equals the population proportion p .

Standard deviation is a measure of the spread in a data set. For a population consisting of all possible sample proportions for a given sample size, the standard deviation can be calculated using the following formula:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

where p is the population proportion and n is the sample size.

For example, if 40% of Washington High students support Amelia, the population proportion p is 0.4. For samples of size 20, the standard deviation of all possible sample proportions is:

$$\sigma = \sqrt{\frac{0.4(1-0.4)}{20}} \approx 0.11 = 11\%$$

Exploration 1

In this exploration, you investigate the effects of sample size on the reliability of the resulting data.

- a. Assume that 40% of the Washington High student population still supports Amelia.
 1. Use a simulation to obtain 90 samples of size 25 from this population. Record the proportion of favorable votes in each sample.
 2. Sort the data from least to greatest.
 3. Create a histogram of the sample proportions.
- b. Using the formula given in the mathematics note, calculate the standard deviation of all possible sample proportions for the simulation in Part a.
 1. Determine the percentage of sample proportions from Part a that lie within 1 standard deviation of the population proportion.
 2. Determine the percentage of sample proportions that lie within 2 standard deviations of the population proportion.
- c. Repeat Parts a and b using a sample size of 50.
- d. Repeat Parts a and b using a sample size of 100. **Note:** Save this data for use in Exploration 2.

Discussion 1

- a. Compare your histogram from Part a of Exploration 1 with the class histogram from Activity 3.
- b. Describe how changing the sample size affected each of the following:
 1. the shape of the histograms
 2. the standard deviation of all possible sample proportions
 3. the interval that represents values within 1 standard deviation of the population proportion.
- c. Statisticians have found that, for reasonably large sample sizes, approximately 68% of sample proportions fall within 1 standard deviation of the population proportion, while approximately 95% fall within 2 standard deviations of the population proportion.
 1. How do these values compare with the percentages you found in Exploration 1?
 2. What do these percentages indicate about the results of a single sample?
- d. If the campaign staff found that 32% of a sample of 100 students planned to vote for Amelia, what should they conclude? Explain your response.

Exploration 2

As election day approaches, the number of undecided voters grows smaller and smaller. Some choose Amelia, while others throw their support to Bernadette. Amelia's campaign staff decides to take one last poll, this time with a larger sample size.

One thing worries them, however. In the two weeks that have passed since the school-wide census, the proportion of students who favor Amelia almost certainly has changed. The campaign staff knows that no matter what the level of her support, the sample proportion is likely to be within 2 standard deviations of the population proportion. The hard part involves calculating this standard deviation, since its formula depends on knowing the population proportion.

Without knowing the actual population proportion, how could they draw a reliable conclusion from their sample? In this exploration, you determine how confident you can be in the results of a single sample.

- a. The formula for the standard deviation of all possible sample proportions, where p is the population proportion and n is the sample size, is shown below:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

- Graph this equation for a sample size of 100 ($n = 100$).
 - Use the graph to determine the largest possible standard deviation for a sample size of 100.
 - What population proportion corresponds with this maximum standard deviation?
- b. Repeat Part a for some other sample sizes. Use your results to make a general statement about the maximum standard deviation for any sample size.
- c. In Part d of Exploration 1, you simulated 90 samples of size 100 from a population with a known population proportion of 40%.
Add and subtract 1 maximum standard deviation (MSD) from each of these sample proportions to obtain 90 intervals. For example, given a sample proportion of 34%, the corresponding interval would be:
[34 – MSD, 34 + MSD]
- Determine the number of intervals from Part c that contain the population proportion (40%).
 - Combine your results from Part d with those of the rest of the class. Determine the percentage of intervals that contain the population proportion.
 - Repeat Parts c–e using twice the maximum standard deviation ($2 \cdot \text{MSD}$).

Discussion 2

- a. Describe the domain and range of the following function, when $n = 100$:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

- b. Would you be more confident in predicting that an unknown population proportion is within 1 MSD of a sample proportion, or within 2 MSDs of a sample proportion? Explain your response.
- c. In Part c of Exploration 2, why do you think it was important to use the maximum standard deviation to determine each interval?
- d. How do you think that changing the sample size would affect these intervals?
- e. Would you be more confident in predicting an unknown population proportion using a sample size of 100 or a sample size of 400? Explain your response.

Mathematics Note

The **maximum standard deviation** of all possible sample proportions occurs when the population proportion $p = 0.5$. The value of 2 maximum standard deviations for all possible sample proportions equals:

$$2 \cdot \sigma = 2 \cdot \sqrt{\frac{0.5(1-0.5)}{n}} = \frac{2 \cdot 0.5}{\sqrt{n}} = \frac{1}{\sqrt{n}}$$

A **confidence statement** declares that a population parameter lies within a specific range of values. Given the value of 2 maximum standard deviations calculated above, the following confidence statement can be made, where n is the sample size: “It is highly likely that the population proportion is within $1/\sqrt{n}$ of the sample proportion.”

For example, consider a random sample of 1500 registered voters. Of those surveyed, 700 plan to vote for candidate A. In this case, the sample proportion is $700/1500$ or 47%. The value of 2 maximum standard deviations is $1/\sqrt{1500}$ or 2.6%. Therefore, the following confidence statement can be made: “It is highly likely that the proportion of the population who plan to vote for candidate A is within the interval [44.4%, 49.6%].”

The interval within which a parameter is likely to fall is often reported using a **margin of error**. Identifying a margin of error that is twice the maximum standard deviation provides an interval that is highly likely to contain the population proportion.

For example, the results of the survey described above might be reported as follows: “At the time of our poll, it was estimated that 47% of registered voters preferred candidate A, with a margin of error of 2.6%.”

- f.
 1. When using the maximum standard deviation to identify a margin of error, how is the margin of error affected by the size of the sample?
 2. How is it affected by the size of the population?
- g. Why do you think a pollster would be willing to predict the outcome of a national election based on a random sample of only 1500 registered voters?

Assignment

- 4.1 On the day before the election, Amelia's campaign staff surveyed a random sample of 400 students. Of those polled, 49% favored Amelia.
 - a. Write a confidence statement describing the true proportion of Amelia's supporters.
 - b. If you were Amelia's campaign manager, what would you say about her chances of winning the election? Explain your response.
- 4.2 A television commercial made the following claim: "Our survey shows that 4 out of 5 dentists prefer sugarless gum for their patients who chew gum." Can you make a confidence statement about this proportion? If so, make one. If not, explain why not.
- 4.3 A poll taken the day before a congressional election reported the following results: "Senator Rodriguez currently has 54% of the vote. The poll's margin of error is $\pm 3\%$." Is Senator Rodriguez assured of re-election? Explain your response.
- 4.4 For its annual election survey, one polling organization selects a random sample of 1000 registered voters. Using the maximum standard deviation of all possible sample proportions, will the survey's margin of error be more or less than three percentage points? Explain your response.
- 4.5 The editors of the Washington High newspaper surveyed a random sample of 100 students on the morning of the election. Of those surveyed, 56 planned to vote for Amelia. What predictions can you make about the election? Justify your response.

* * * * *

- 4.6** To assess customer satisfaction with its new checkout system, a supermarket surveyed a random sample of 40 shoppers. Of those surveyed, 83% were happy with the new system. In what interval is the actual proportion of satisfied customers likely to fall? Explain your response.
- 4.7** As part of its quality control process, a car dealership surveyed a random sample of 25 customers. Of those surveyed, 73% expressed satisfaction with the service they received. Using the maximum standard deviation of all possible sample proportions, what is the margin of error for this survey? Explain your response.

* * * * *

Summary Assessment

Imagine that you are a quality control specialist for a light bulb manufacturer. Your job involves testing bulbs to make sure that the production line is working properly. When the equipment is functioning as designed, the proportion of defective bulbs is less than 5%. Since the factory produces many thousands of bulbs each day, you must use a sampling technique to estimate the actual proportion.

1. Design a practical method of sampling light bulbs so that the sample proportions provide a reliable estimate of the population proportion.
2. Use technology to create a simulation of the sampling process. Display the results in a histogram.
3. Determine the margin of error for a sample proportion obtained using your method.
4. Describe how you would respond to a sample proportion of 6.2% defective bulbs obtained using your method.

Module Summary

- A **population** is a group of all objects, individuals, or observations about which information is to be gathered. A quantity that describes a population is a **parameter**.
- A **sample** is a portion of the population. A quantity that describes a sample is a **statistic**.
- A **bias** is a factor that prevents a sample from representing the entire population.
- A **census** is the collection of data about an entire population.
- A **simple random sample** is selected so that each member of the population has the same chance of being included in the sample.
- **Stratified sampling** requires that a population be divided into parts. Each part is a **stratum** (plural, **strata**). To produce a stratified sample, simple random samples are taken from each stratum. These samples do not necessarily have to be the same size.
- **Systematic sampling** is accomplished by collecting data from every n th unit of a population after randomly choosing a starting point.
- The percentage of a population with a given characteristic is a **population proportion**.
- The percentage of a sample with a given characteristic is a **sample proportion**.
- The mean of all possible sample proportions for samples of size n equals the population proportion p .
- **Standard deviation** is a measure of the spread in a data set. For a population consisting of all possible sample proportions for a given sample size, the standard deviation can be calculated using the following formula:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

where p is the population proportion and n is the sample size.

- The **maximum standard deviation** of all possible sample proportions occurs when the population proportion $p = 0.5$. The value of 2 maximum standard deviations for all possible sample proportions equals:

$$2 \cdot \sigma = 2 \cdot \sqrt{\frac{0.5(1-0.5)}{n}} = \frac{2 \cdot 0.5}{\sqrt{n}} = \frac{1}{\sqrt{n}}$$

- A **confidence statement** declares that a population parameter lies within a specific range of values.
- The interval within which a parameter is likely to fall is often reported using a **margin of error**.

Selected References

- Gallup, G. *The Sophisticated Poll Watcher's Guide*. Ephrata, PA: Princeton Opinion Press, 1972.
- Landwehr, J. M., J. Swift, and A. E. Watkins. *Exploring Surveys and Information from Samples*. Palo Alto, CA: Dale Seymour Publications, 1988.
- Mizrahi, A., and M. Sullivan. *Finite Mathematics with Applications for Business and Social Sciences*. New York: John Wiley & Sons, 1973.
- Moore, D. S. *Statistics: Concepts and Controversies*. New York: W. H. Freeman and Co., 1991.
- Sincich, T. *Statistics by Example*. San Francisco, CA: Dellen Publishing Company, 1982.

Traditional Design



In this module, you examine the geometric properties of some traditional American Indian art forms.

Todd Fife • Anne Merrifield



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Traditional Design

Introduction

The traditional artwork of many American Indian peoples is prized for its beauty, meaning, and symmetry. Its forms are as diverse as the tribes themselves—varying by region, by culture, and by individual artist. In this module, you examine the star quilts of the Assiniboine and Sioux, the medicine wheels of the Plains Indians (particularly the Northern Cheyenne), and the sandpaintings of the Navajo.

Exploration

American Indian art often emphasizes nature and its forces through pictures and symbols. The symbol shown in Figure 1, for example, represents both a mountain and abundance. It may have been painted on a prayer bowl or woven into a blanket.

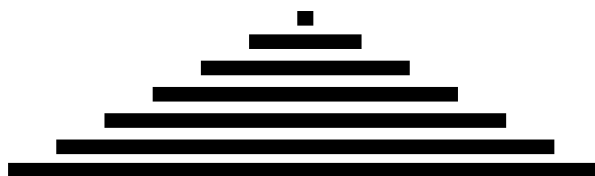


Figure 1: Representation of a mountain (abundance)

The geometric properties of designs like this one can be examined through paper folding. In this exploration, you will practice some basic paper-folding techniques. One such technique allows you to investigate the line symmetry of symbols like the one in Figure 1.

Mathematics Note

An object has **line symmetry** if the object is its own image in a reflection in a line. This line is the **line of symmetry**.

For example, Figure 2 shows three lines that each divide a rectangle into two congruent parts. This rectangle has line symmetry with respect to lines l and m . The reflection of the rectangle in line n , however, is not itself. Therefore, line n is not a line of symmetry for this figure.

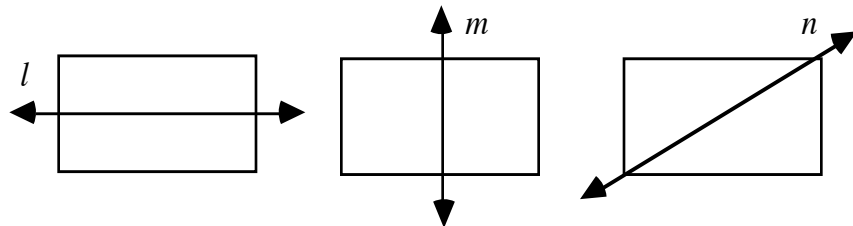


Figure 2: Lines dividing a rectangle into congruent parts

Complete Parts **a–c** using a reproduction of the symbol in Figure 1.

- a.
 1. Fold your paper along the symbol's line of symmetry. Unfold the paper and label the resulting crease line a .
 2. Measure the length of the bars on each side of crease a .
 3. Measure the angles formed by line a and the bars in the symbol.
 4. Make a conjecture about the relationship of the line of symmetry to the bars in the symbol.
- b.
 1. Fold the paper to form a line parallel to the bars in the symbol. Unfold the paper and label this crease line b .
 2. Identify an angle formed by crease a and crease b . As shown in Figure 3 below, fold the paper so that one side of the angle coincides with its other side. Unfold the paper and label this crease line c .

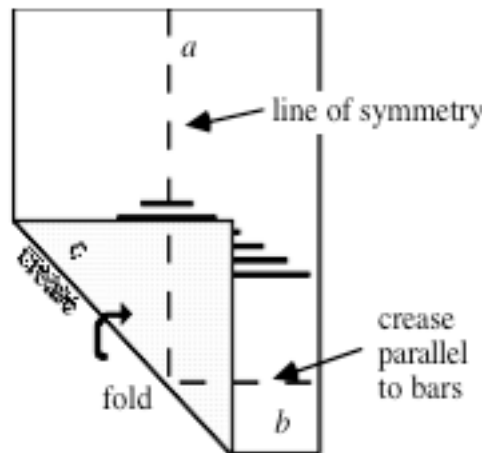


Figure 3: Folding one side of an angle onto the other

Mathematics Note

Perpendicular distance is the distance from a point to a line or a plane. It is equal to the length of the segment, perpendicular to the line or plane, that connects the point to that line or plane.

For example, Figure 4 shows a point A and a line l . Since \overline{AX} is perpendicular to line l , the perpendicular distance from A to l is 5 units.

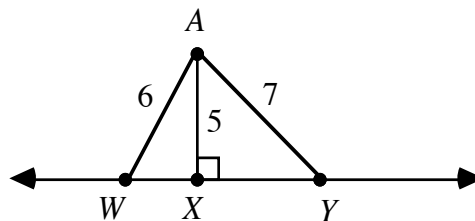


Figure 4: Perpendicular distance

- c.
 1. Select a point on crease c and measure the perpendicular distance from this point to each side of the angle formed by creases a and b .
 2. Measure the angles formed by creases c and b and the angles formed by creases c and a .
 3. Make a conjecture about the relationship of crease c to the angle formed by creases a and b .

Discussion

- a. Describe a paper-folding method that could be used to create each of the following:
 1. an object's reflection in a line
 2. a perpendicular bisector of a line segment
 3. a bisector of an angle
 4. a midpoint of a line segment.
- b. Describe a paper-folding method that could be used to form a square from a rectangular sheet of paper.

Activity 1

For hundreds of years, some American Indian families marked important events in the lives of their children by giving gifts. These gifts were not intended for the individual child, however, but for some other admired person. In this tradition, gift giving demonstrated the deep pride a family felt for its children.

Today, the Assiniboine and Sioux tribes of northeastern Montana preserve this tradition through the star quilt ceremony. Each star quilt represents many hours of painstaking labor. It is a great honor both to give away a star quilt and to receive one. Star quilt ceremonies are usually held at community gatherings, particularly athletic events. For example, the family of a basketball player or cheerleader might make a quilt to present between games at a state tournament. The person to receive the quilt might be a coach, an admired member of the community, or a player from another team.

The family often wraps the quilt around the shoulders of the chosen recipient while introductions are made and the audience stands respectfully. This ceremony is a powerful medium for reflecting the values, attitudes, pride, and identity of the presenting family.

Although there are many kinds of star quilts, all of them have the lone star as their primary focus. Figure 5 shows one example of a lone star pattern.

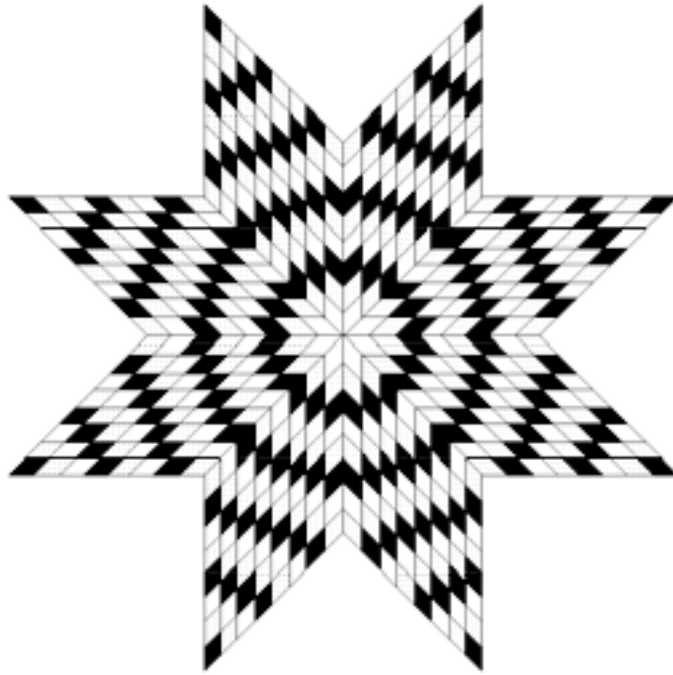


Figure 5: Lone star quilt pattern

Discussion 1

- a. Identify some of the geometric shapes you recognize in the lone star pattern in Figure 5.
- b.
 1. What basic shape do you think was used to construct the star?
 2. Describe the properties of this shape.

Exploration

- a. Fold and cut a standard sheet of paper to form a square. If no crease has been formed along a diagonal of the square, use another fold to create one.
- b. As shown in Figure 6a, use paper folding to bisect an angle formed by a side of the square and the creased diagonal. Using the same vertex—and without unfolding the paper—repeat this step for the adjacent angle, as shown in Figure 6b. Do not unfold this shape.

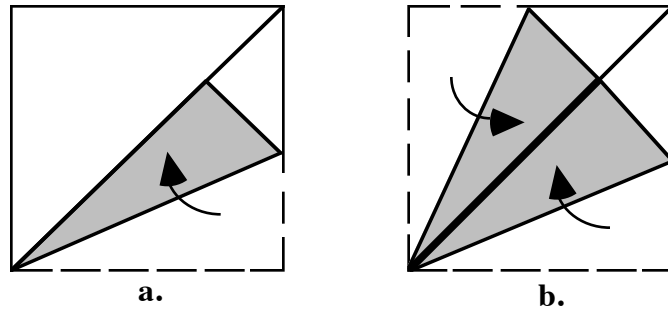


Figure 6: Folding in the sides of a square

- c. Repeat Parts **a** and **b** using the opposite vertex of the original square, as shown in Figure 7. The resulting shape is a **rhombus**.

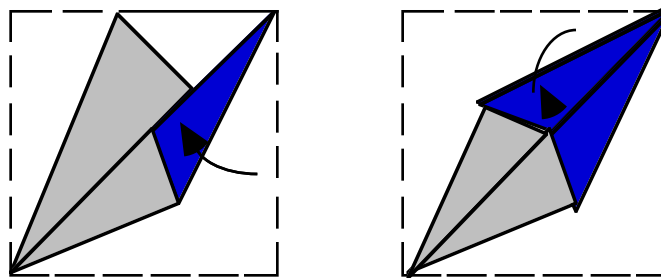


Figure 7: Folding at the opposite vertex

- d. **1.** By carefully examining the folds made in creating this rhombus, determine and record the measures of its interior angles.
2. Predict the relationship that occurs between opposite angles of a rhombus.
- e. Turn the rhombus over so that the previous folds are not visible. Fold it along the shorter diagonal. Unfold it, fold along the longer diagonal, and then unfold again. Make a conjecture concerning the diagonals of this rhombus.
- f. **1.** Find the midpoint of each side of the rhombus.
2. Using a straightedge, draw the line that contains the midpoints of two opposite sides. Identify the shapes formed by these lines and the sides of the rhombus.
3. Draw the line that contains the midpoints of the other two sides.
- g. Identify the four shapes formed by the lines in Part **f** and the sides of the rhombus.
- h. Record the measure of each angle of the shapes you identified in Part **g**. **Note:** Save the folded rhombus for use in Discussion **2** and in the assignment.

Mathematics Note

Two polygons are **similar** if there is a one-to-one correspondence between their vertices so that corresponding sides are proportional and corresponding angles are congruent.

For example, Figure 8 shows two triangles, the approximate lengths of their sides, and the approximate measures of their angles.

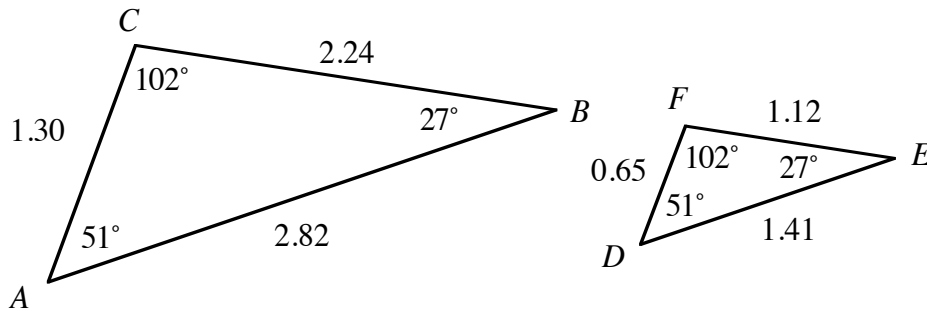


Figure 8: Two similar triangles

In this example, point A corresponds to point D , B to E , and C to F so that:

$$\frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF} = 2$$

and $\angle F \cong \angle C$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$. Therefore, triangle ABC is similar to triangle DEF . This can be denoted as follows: $\triangle ABC \sim \triangle DEF$.

Discussion 2

- Are the new shapes you identified in Part **g** of the exploration similar to the original rhombus created in Part **c**?
- Describe the relationship between the two parallelograms formed in Part **f** of the exploration.
- Compare the conjecture you made in Part **e** of the exploration concerning the diagonals of the rhombus with others in your class.
- Describe the relationships that exist among the interior angles of a rhombus.
 - What methods could you use to demonstrate these relationships?

- e. After connecting the midpoints of opposite sides in Part f of the exploration, your paper rhombus should have resembled the diagram in Figure 9.

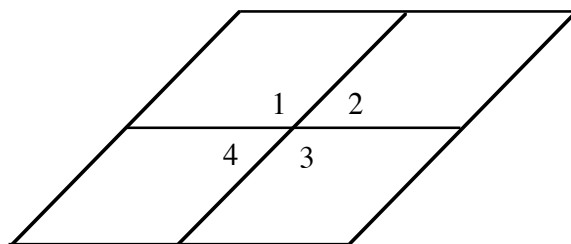


Figure 9: A rhombus

1. Angles 1 and 3 are **vertical angles**. What is the relationship between the measures of these angles?
 2. Does another pair of angles in Figure 9 have this same relationship?
 3. What characteristics do you think might define a pair of vertical angles?
 4. Using the rhombus you created in the exploration, identify several pairs of vertical angles formed by crease lines.
- f. Connecting the midpoints of opposite sides in Part f of the exploration created new, smaller shapes similar to the original rhombus. If you repeat this process on the smaller rhombi, what is the result?

Mathematics Note

A **rep tile** is a shape that can be partitioned into congruent shapes, each one similar to the original. For example, $\triangle ABC$ in Figure 10 is a rep tile.

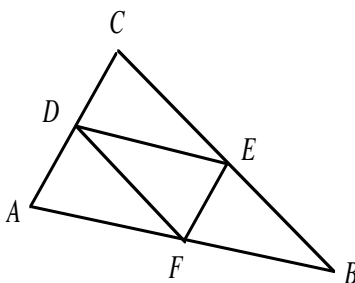


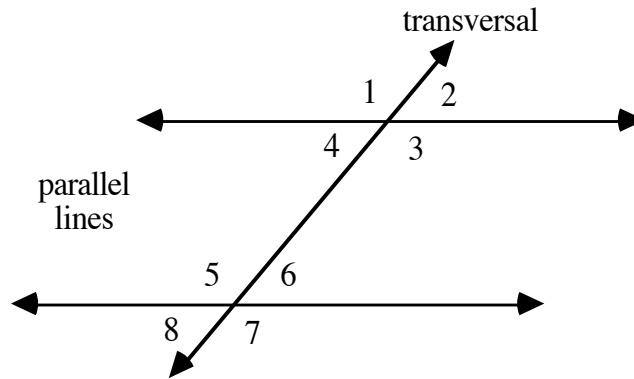
Figure 10: A rep tile

A rep tile can also be used to create a larger similar figure. For example, $\triangle DEF$ is a rep tile creating $\triangle ABC$. **Note:** The name *rep tile* is derived from the phrase “repetition of a tile.”

- g. Based on your response to Part f above, is a rhombus a rep tile?

Assignment

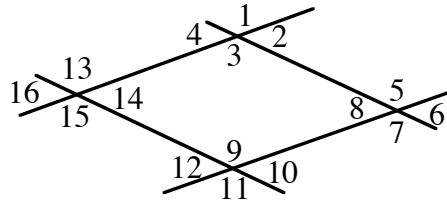
- 1.1 a. On a sheet of paper, extend the pattern shown in Figure 9 to show that it could tessellate the plane.
- b. 1. On your paper, find two parallel lines and a transversal similar to the ones shown in the following diagram.



Describe the relationship between the measures of $\angle 2$ and $\angle 6$, and between the measures of $\angle 4$ and $\angle 8$. These are pairs of **corresponding angles**.

2. Using the diagram above, identify as many sets of corresponding angles as possible.
3. Suggest a general definition for a pair of corresponding angles.
- c. Identify all pairs of vertical angles in the diagram in Part b.
- d. In the diagram in Part b, $\angle 4$ and $\angle 6$ are **alternate interior angles**, while $\angle 2$ and $\angle 8$ are **alternate exterior angles**.
1. Describe the relationship between the measures of $\angle 4$ and $\angle 6$.
2. Describe the relationship between the measures of $\angle 2$ and $\angle 8$.
3. Using the diagram in Part b, identify as many pairs of alternate interior and alternate exterior angles as possible.
4. Suggest how these pairs of angles might have received their names.

- 1.2 Consider the following rhombus with its sides extended.



- Determine the relationship that exists between $\angle 1$ and each of the remaining angles at that vertex.
- Explain why the measure of $\angle 1$ is equal to the measure of $\angle 5$, $\angle 13$, and $\angle 9$.
- What is the minimum number of angles you must measure in order to determine the measures of all 16 angles in the diagram?
- Suppose none of the sides of the figure were parallel. How would this change your response to Part **c**?

Mathematics Note

An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.

For example, Figure 11 shows a regular pentagon $ABCDE$ and its center O . This object has rotational symmetry about O . When pentagon $ABCDE$ is rotated 72° about O , each point in the image coincides with a point in the preimage.

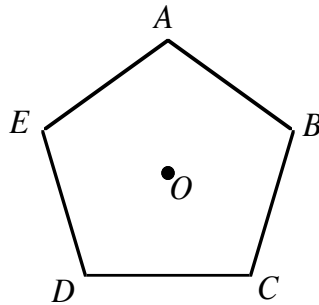
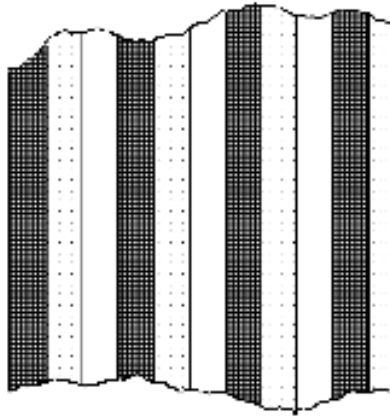


Figure 11: A regular pentagon and its center

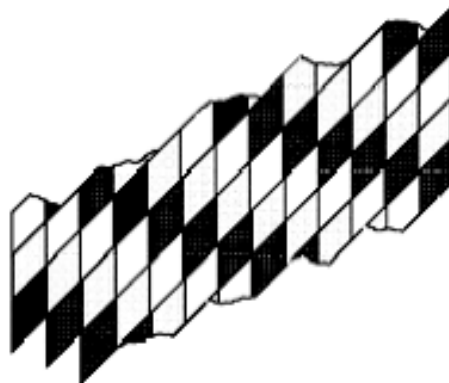
- 1.3 Obtain a copy of the lone star design from your teacher.
- Draw a line that passes through the center of the star. If you cut the star along this line, how would the two pieces be related?
 - Describe any symmetries that would exist between the two pieces.
 - Would your observations in Parts **a** and **b** hold true for any line that passes through the center of the star? Explain your response.
 - Summarize your responses to Parts **a–c** in a paragraph.

- 1.4** **a.** Because making a star quilt requires many hours of work, quilters often use certain cutting and sewing techniques to simplify the task. To make the lone star in Figure 5, for example, quilters generally start by sewing long strips of material together as shown below.

Suppose that you drew a set of parallel lines on this material so that each line was perpendicular to the strips. Describe the geometric figures that would be formed by the lines and strips.



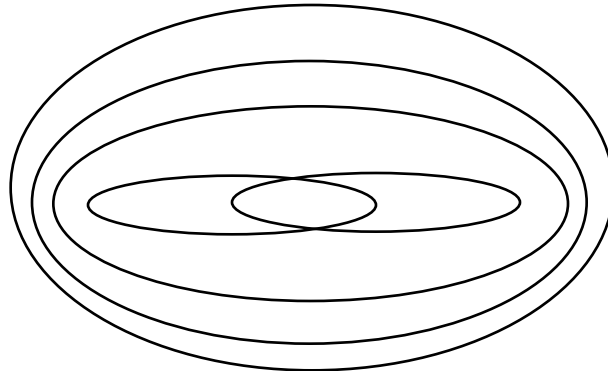
- b.** On a copy of the diagram above, draw a set of parallel lines that are not perpendicular to the strips of material. Describe what kind of geometric figures are obtained.
- c.** The lone star pattern in Figure 5 has eight evenly spaced “points.” What is the measure of the central angle for each of these points?
- d.** By cutting the quilt material along a set of parallel lines such as those in Part **b**, then shifting each new strip by one color, a quilter can create the following arrangement of rhombi. Describe how to draw this set of parallel lines. Verify your method using a copy of the diagram in Part **b**.



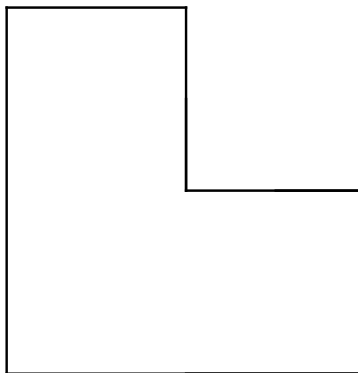
- e.** Imagine that you are stitching the strips created in Part **d** to make a lone star quilt. As you sew, the quilt starts to curl and buckle. Describe what might be causing this problem, and what you might do to correct it.

- 1.5** One way of classifying quadrilaterals is by the number of parallel sides. When characterized in this manner, a trapezoid can be defined as a quadrilateral with at least one pair of parallel sides. In the same fashion, a parallelogram can be defined as a trapezoid with two pairs of parallel sides.

Create a copy of the following Venn diagram. Using the definitions described above, label each region in the diagram with one of the following classifications: quadrilaterals, parallelograms, trapezoids, rectangles, squares, rhombi.



- 1.6** Use a geometry utility to examine the properties of the diagonals of each of the following polygons. Describe your observations.
- a. a trapezoid with exactly one pair of parallel sides
 - b. a parallelogram that is neither a rhombus nor a rectangle
 - c. a rectangle that is not a square
 - d. a rhombus that is not a square
 - e. a square.
- 1.7** a. Make a copy of the figure below. Draw line segments on the figure to create a rep tile. Hint: Begin by examining the midpoints of the edges.



- b. Design a rep tile that could be used in a quilt pattern. Describe the roles of angles and parallel lines, if any, in your design.

* * * * *

- 1.8 Describe four ways in which you could determine if a quadrilateral is a parallelogram.
- 1.9 Many buildings have rectangular floors. Describe how a carpenter can verify that a floor is a rectangle using only a tape measure.
- 1.10 List the quadrilaterals (trapezoids, parallelograms, rhombi, rectangles, or squares) that have each of the following properties:
 - a. a diagonal forms two congruent triangles
 - b. a diagonal bisects opposite angles
 - c. the two diagonals form four congruent triangles.

* * * * *

Activity 2

The medicine wheels of the Northern Cheyenne and other Plains peoples are both beautiful and symmetrical. Almost every aspect of each wheel—its symbols, color, and design—holds a special significance for the person who creates it.

In the past, simple wheels were built by placing small stones or pebbles on the ground. One pattern for such a wheel is shown in Figure 12. Each stone represented one of the many things in the universe. For example, an individual stone might have represented an animal, an individual person, or a nation.

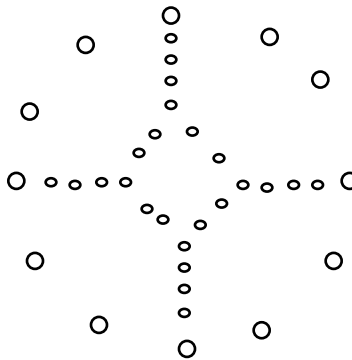


Figure 12: Medicine wheel formed with pebbles

Medicine wheels were also painted on personal shields. These shields were made from a variety of animal hides—including buffalo, bear, deer, coyote, weasel, and mouse—and often decorated with eagle plumes or tassels of animal fur. The designs and symbols on each wheel held deep personal meaning for the bearer.

Figure 13 shows a medicine wheel with two **concentric circles** (circles that have the same center), a set of evenly spaced quadrilaterals, and a Cheyenne design in the middle.

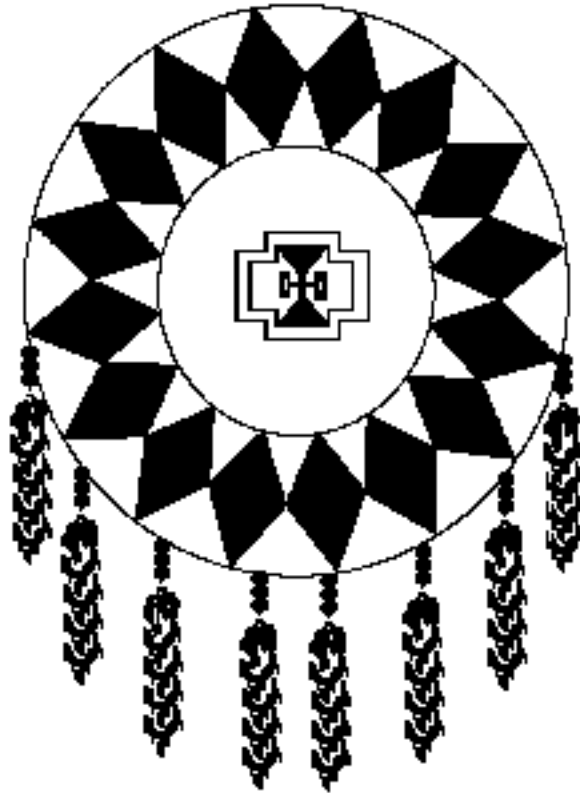


Figure 13: A Cheyenne medicine wheel

In this activity, you use some geometric properties of a circle to create your own medicine wheel.

Discussion 1

- a. Disregarding the tassels and beads, what types of symmetry do you observe in the medicine wheel in Figure 13?
- b. In the Level 2 module “Crazy Cartoons,” you explored four types of transformations: reflections, rotations, dilations, and translations. Explain how you could use transformations to reconstruct this medicine wheel—including tassels and beads—on a geometry utility.

Exploration

- a. Using a geometry utility, draw a circle whose diameter is approximately two-thirds the width of the screen.
- b.
 1. A **chord** is a line segment joining any two points on a circle. Draw two chords of your circle.
 2. Predict where the perpendicular bisectors of the two chords will intersect. Draw a point at that location.
 3. Construct the perpendicular bisectors of the chords. Mark the intersection point (if different from the location predicted in Step 2).
 4. Drag the endpoints of the chords to change their sizes and locations. Record your observations.
 5. Change the size of the circle. Record your observations.
- c.
 1. Draw a new circle and construct a diameter.
 2. Identify the intersection points of the diameter and the circle.
 3. Construct a line through one of the points of intersection.
 4. Measure an angle at the intersection of the diameter and the line constructed in Step 3.
 5. Adjust the angle constructed in Step 3 until its measure is 90° .
 6. Record your observations.

Mathematics Note

A **secant** of a circle is a line that intersects a circle in two points.

A **tangent** of a circle is a line, segment, or ray in the plane of the circle that intersects the circle in exactly one point and is perpendicular to a radius at that point. This intersection is the **point of tangency**.

For example, Figure 14 shows circle O with secant \overleftrightarrow{AB} , tangent \overleftrightarrow{BC} , and B , a point of tangency.

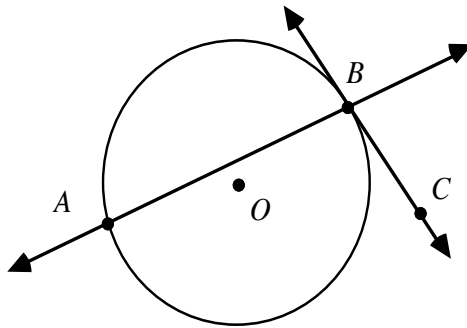


Figure 14: Circle with secant and tangent lines

Discussion 2

- a.
1. What is the measure of the angle between a diameter of a circle and a tangent to that circle whose point of tangency is the endpoint of the diameter?
 2. Suppose that a secant does not contain the center of a circle. If a tangent to the circle is then drawn as in Figure 14, what can be said about the measure of an angle formed by this secant and tangent?
- b. Describe how to use paper folding to find a line tangent to a circle.
- c. What is true about two tangents whose points of tangency are opposite endpoints of a diameter? Explain your response.
- d. Figure 15 shows a fragment of American Indian pottery. Before beginning reconstruction of the circular plate, a museum curator might first make a sketch of the original artifact, including its center. Using your observations from the exploration, describe how to find the center of this circular plate.



Figure 15: Pottery fragment

- e. Describe how to use paper folding to find the center of a circle.
- f. The perpendicular bisector of a chord is the set of points in the plane equidistant from the ends of the chord. This means that for any point C in Figure 16 below, $AC = BC$.

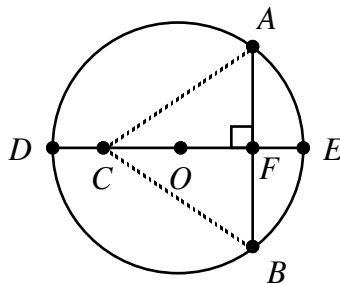


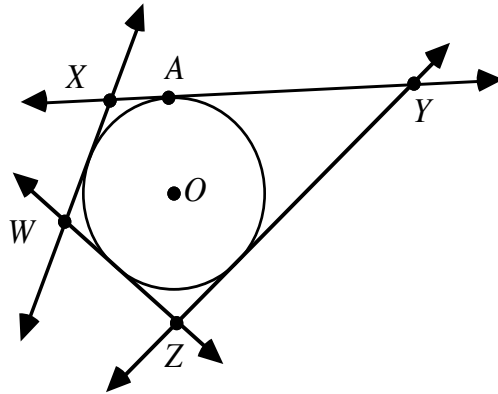
Figure 16: Chord AB and its perpendicular bisector

How does this verify that the intersection of the perpendicular bisectors of two chords is the center of the circle?

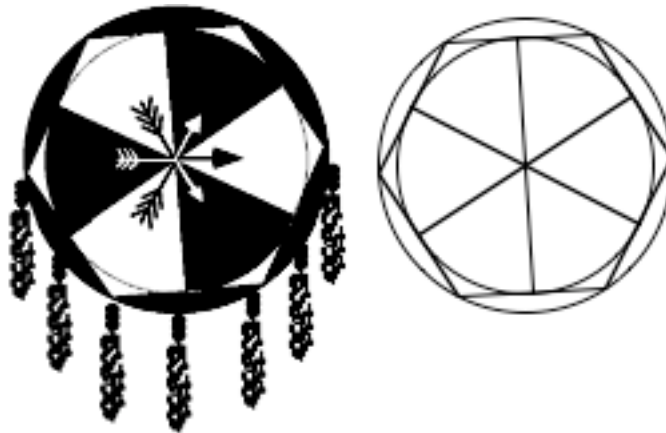
- g. In Figure 16, $\triangle ABC$ is isosceles and \overline{CF} is an altitude. What is the relationship between the altitude of an isosceles triangle and its base?

Assignment

- 2.1
- Using appropriate technology, construct a point A on a circle and a tangent to the circle at that point.
 - Continue the construction from Part **a** to create a circle inscribed in quadrilateral $XYZW$, as shown in the diagram below. Describe the process you used to complete the drawing.



- 2.2
- The diagram below shows two drawings of a medicine wheel. The drawing on the right represents the design of the wheel without the feathers, arrows, and shading of the drawing on the left.



- Recreate the drawing on the right using a geometry utility. Describe the techniques you used.
- Does the drawing you created in Part **a** have line symmetry? Explain your response.
 - Does it have rotational symmetry? Explain your response.

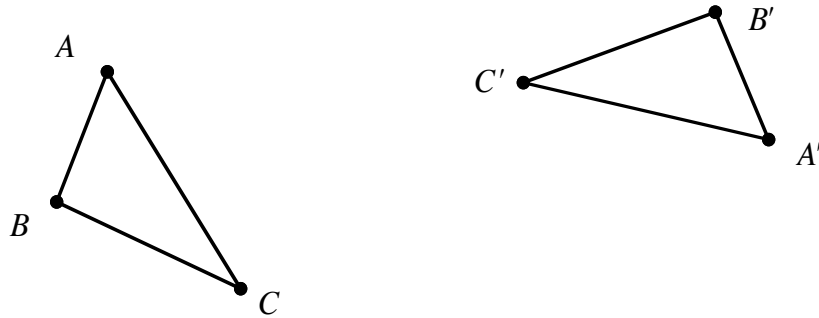
- 2.3 a. The diagram below shows the outline of a medicine wheel.



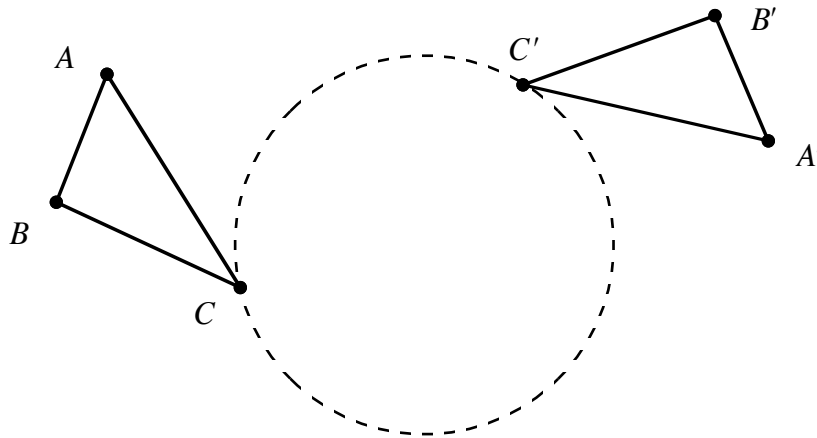
Using a copy of this diagram, find the center of the circle. Use the center to construct two smaller, concentric circles on the wheel.

- b. Describe how you found the center of the circle.
- c. Personal medicine wheels often display the symbols of an individual or clan. What symbols could you use to represent yourself, your class, or your school?
- d. Using the three circles drawn in Part a, create a medicine wheel that combines chords, tangents, and at least one traditional American Indian symbol in a design that radiates from the center.
- e. Describe how you used each required component in your design.

- 2.4 The following illustration shows $\triangle ABC$ and its image under a rotation, $\triangle A'B'C'$.



Any two corresponding points on the image and the preimage, such as C' and C , are contained on a circle whose center is also the center of rotation. The center of the circle is not shown in the illustration below.

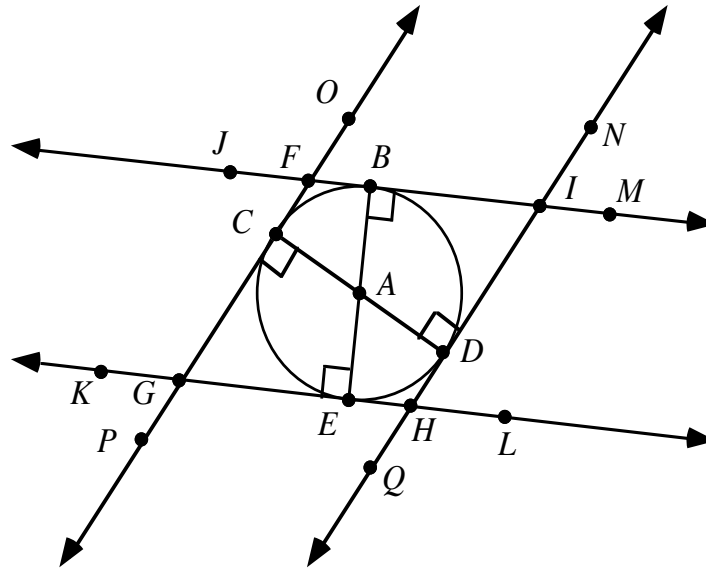


- Explain why the center of the circle must be contained on the perpendicular bisector of $\overline{CC'}$.
- Using a copy of the diagram, determine the center of rotation. Describe the method you used.

* * * * *

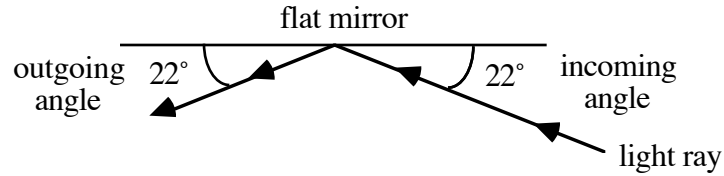
2.5

- a. In the following diagram, \overline{CD} and \overline{BE} both contain A , the center of the circle. What other geometric term describes these chords?

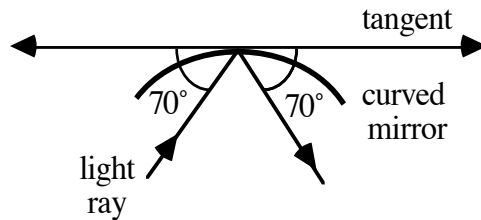


- b. In the diagram above, \overleftrightarrow{FI} and \overleftrightarrow{GH} are tangent to the circle at B and E , respectively, while \overleftrightarrow{FG} and \overleftrightarrow{IH} are tangent to the circle at C and D , respectively.
1. Explain why \overleftrightarrow{FI} and \overleftrightarrow{GH} are parallel.
 2. Is there another set of parallel lines in the diagram? Explain your response.
- c. 1. Recreate the diagram on a geometric utility.
 2. Measure $\angle KGP$, $\angle CGE$, $\angle KGC$ and $\angle CFB$.
 3. Any two of the angles listed in Step 2 can be used to illustrate a special angle relationship. Identify several of these relationships.
- d. Identify all pairs of congruent angles formed by \overleftrightarrow{IH} , \overleftrightarrow{FI} and \overleftrightarrow{GH} . Describe the relationship that exists between the angles in each pair.
- e. Make a conjecture about the lengths of \overline{FI} , \overline{IH} , \overline{HG} , and \overline{GF} . Confirm your conjecture by measuring these lengths.
- f. Do you believe $GFIH$ is a rhombus? Explain your response.

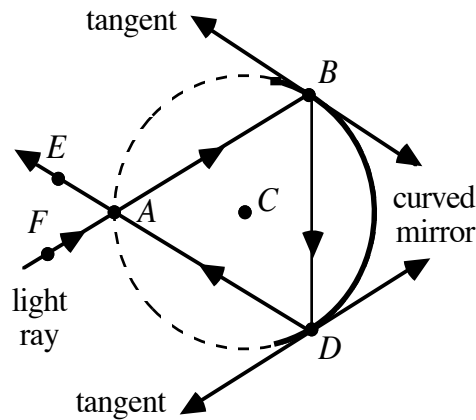
- 2.6 As you learned in the Level 1 module “Reflect on This,” when a light ray reflects off a flat mirror, the incoming angle is congruent to the outgoing angle. For example, the diagram below shows a light ray reflecting off a flat mirror at an angle of 22° .



When light reflects off a curved mirror, the incoming angle also is congruent to the outgoing angle. In this case, the angles are measured with respect to the tangent at the point of reflection. For example, the following diagram shows a light ray reflecting off a curved mirror at an angle of 70° .



Consider a mirror whose cross section is a portion of a circle with center at point C . As shown in the following illustration, points A , B , and D are on circle C . A light ray travels from A to B with an incoming angle of 60° , reflects off the mirror at B , reflects again off the mirror at D , then passes back through A .



- a. If segments are drawn from C to each vertex of $\triangle ABD$ in the diagram above, three triangles are formed: $\triangle ACB$, $\triangle ACD$, and $\triangle BCD$. What type of triangles are these?

- b. 1. Each segment described in Part a cuts an interior angle of $\triangle ABD$ into two smaller angles. What is the relationship between these two smaller angles? Explain your response.
2. What term can be used to identify each of these segments?
- c. Why must the outgoing light ray at point D be parallel to the tangent to the circle at point B ?

* * * * *

Activity 3

The art of the Navajo people of the southwestern United States is rich in symbolism and meaning. Navajo sandpaintings, for example, are an important component of traditional healing rituals. The sandpaintings themselves are done with five sacred, symbolic colors and usually represent a particular character in tribal legend.

Figure 17 shows one sandpainting design recreated on a geometry utility.

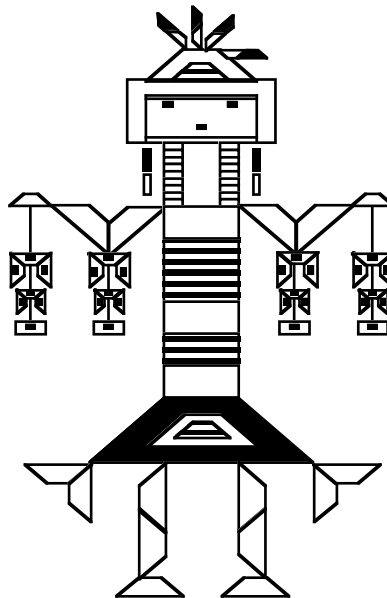


Figure 17: Design resembling a Navajo sandpainting

Discussion 1

- a. Where does symmetry appear to play a part in the design in Figure 17?
- b. Describe the two basic geometric figures used in the design.
- c. Where does similarity appear to play a part in the design?

Exploration

In this exploration, you examine one shape from the design in Figure 17 and investigate how that shape can be transformed to create other parts of the design.

- a.
 1. On a geometry utility, construct a rectangle to represent the head of the figure in the design. Label the vertices of the rectangle $HEAD$, as shown in Figure 18 below.
 2. Create a point P outside the rectangle to serve as a center of dilation.

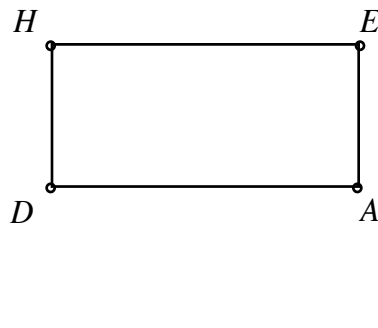


Figure 18: A rectangle with center of dilation

3. Construct $\triangle PEA$.
 4. Find the midpoints of \overline{PE} and \overline{PA} . Label these points E' and A' .
 5. Construct $\overline{E'A'}$.
- b. Verify that $\triangle PE'A'$ is similar to $\triangle PEA$.
 - c. Using the same techniques as in Part a, construct $\overline{D'A'}$, $\overline{H'D'}$, and $\overline{E'H'}$ to create rectangle $H'E'A'D'$.
 - d. Rectangle $H'E'A'D'$ is a **dilation** of rectangle $HEAD$ with center at point P and a scale factor of $H'E'/HE$. Describe the properties of the preimage that are preserved in a dilation.
 - e. Move point P to several other locations on your screen. Record your observations.
 - f. In Parts a and c, you used the midpoints of segments to construct rectangle $H'E'A'D'$.

Predict what would happen if you repeated this process using points $1/4$ the distance from P to each vertex of rectangle $HEAD$.

Discussion 2

- a.
 1. What is the scale factor for the dilation in Part **d** of the exploration?
 2. What would be the scale factor for the dilation suggested in Part **f** of the exploration?
- b. Describe how changing the location of the center of dilation in Part **e** affected the resulting image.
- c. Describe how you could use a dilation to create an image larger than the preimage.
- d. A transformation that produces an image congruent to the preimage is an **isometry**. Which of the following transformations—reflections, rotations, dilations, and translations—can be classified as isometries? Explain your responses.

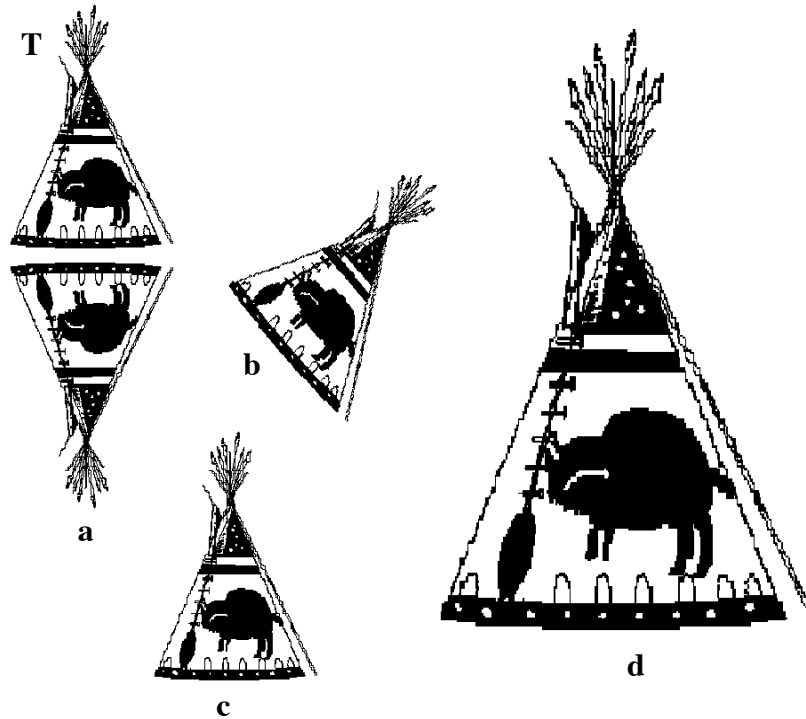
Assignment

- 3.1 a. Using the template supplied by your teacher, find the images of several points of the following figure under the transformations described below.



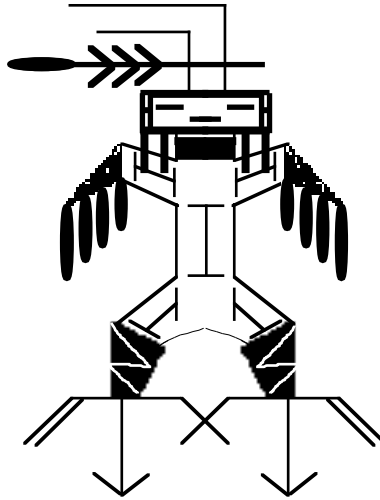
1. a dilation with center at point P and a scale factor of $1/2$
 2. a translation by the vector $\overrightarrow{PP'}$ of the image in Step 1
 3. a reflection in $\overleftrightarrow{PP'}$ of the image found in Step 2
 4. a 90° clockwise rotation, with center at point P , of the image found in Step 3
- b. Sketch the remainder of each image in Part **a**.

- 3.2** The Blackfeet tribe of the American Northwest sometimes painted symbols and designs on their tipis. In the diagram below, the tipi in the upper left-hand corner, labeled **T**, is the preimage. Use a copy of the diagram to complete Parts **a–e**.



- Image **a** is a reflection of **T**. Draw the line of reflection.
- Image **b** is a transformation of **T** using a rotation. Find the center of rotation.
- Image **c** is a translation of **T**. Draw a vector to represent the translation.
- Image **d** is a dilation of **T**. Find the center of dilation and identify the scale factor.
- The preimage **T** can also be transformed to the image **c** by a composite transformation involving two reflections. Draw two lines of reflection that could be used to perform this transformation.

- 3.3 Navajo sandpaintings frequently represent legendary characters. The following design is associated with Big Thunder.



- Create your own design for a sandpainting using transformational geometry. (You may want to use a geometry utility.)
- Write a paragraph describing how you used transformations to create your design.

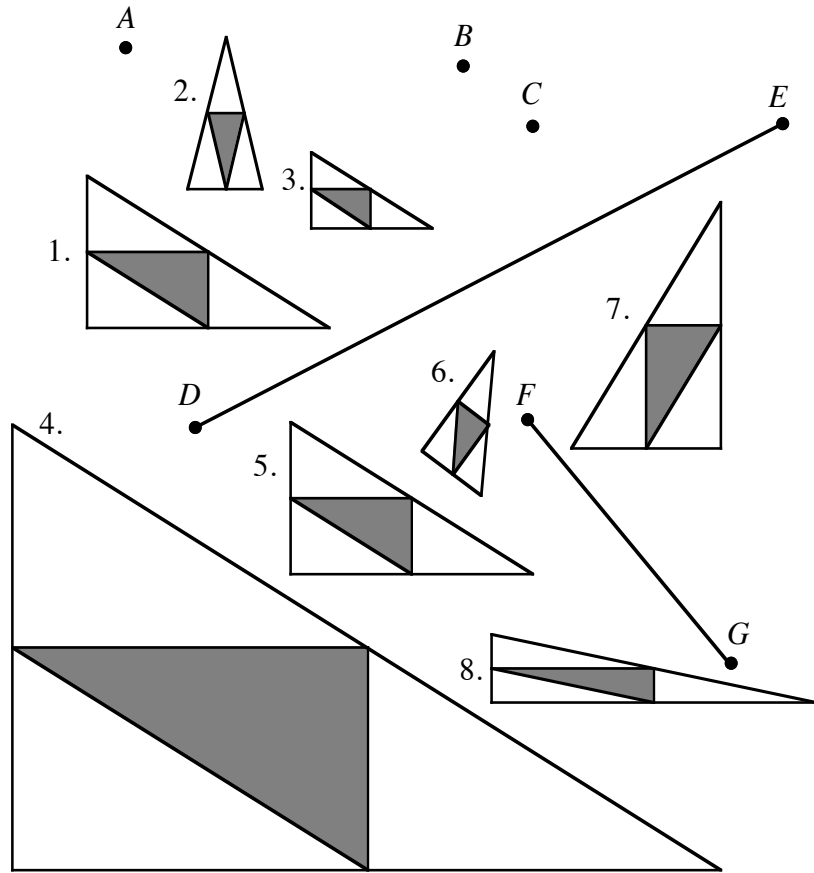
* * * * *

Research Project

Geometric designs appear in the art of many cultures and historical periods. Byzantine mosaics, for example, contain many polygonal designs and symmetries. Celtic knot designs are two-dimensional representations of three-dimensional patterns. And many well-known Renaissance painters used similarity and line perspective to portray landscapes and the human form.

Choose a culture and historical period. Research the use of geometric shape and design in the art of that time and place. Write a paper discussing your findings.

3.4 The following diagram shows eight numbered figures. Some of these figures are the images of figure 1 under transformations involving the labeled points and segments.

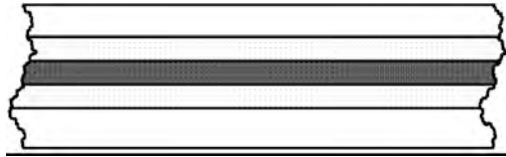


If possible, describe the transformation used to produce each numbered image, with figure 1 as its preimage. Each description should respond to the questions listed below:

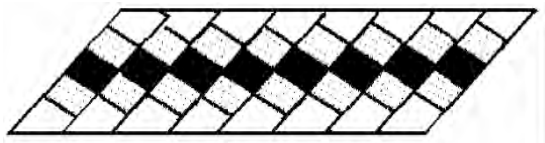
- a. Is the transformation an isometry?
- b. Is the image similar to the preimage?
- c. Which labeled points or lines were used in the transformation?
- d. If the transformation includes a dilation, what is the approximate scale factor?
- e. If the transformation includes a rotation, what is the approximate angle of rotation?
- f. If the transformation includes a translation, what is the approximate magnitude and direction of the translation vector?

Summary Assessment

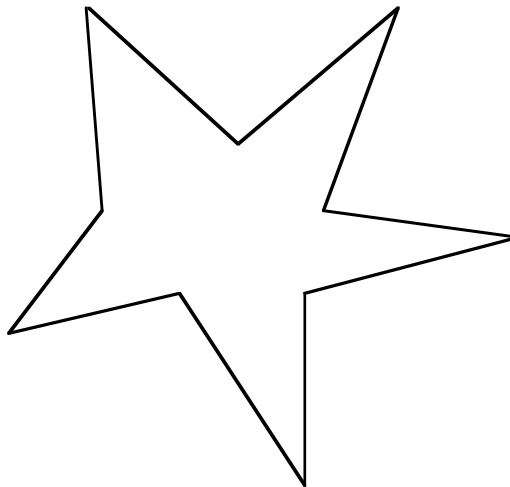
1. The people of Florida's Seminole tribe are known for their patchwork designs. As early as the 1880s, they used hand-cranked sewing machines to stitch their patterns. Although there are many variations in Seminole patchwork, each one is started by sewing strips of fabric together as shown in the diagram below.



- a. How could these strips of fabric be cut and sewn to create the following pattern?

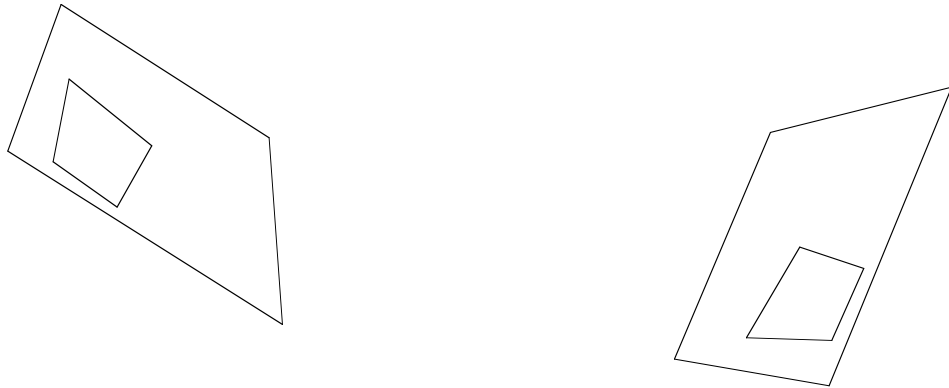


- b. Describe the method you developed in Part **a** in a paragraph. Include specific directions to complete the task and use geometric terms to explain why your method works.
2. A symbol often used by the Hopi people resembles the five-pointed star (or pentagram) shown below.



- a. Use appropriate technology to create a pentagram like the one above.
- b. Dilate the pentagram by a scale factor of your choice.
- c. Explain why the corresponding sides of the image and preimage are parallel.

3. The Navajo symbol for a medicine man's eye represents wisdom. The eye symbol on the right-hand side of the diagram below is the image of the one on the left under a rotation.



- a. Using a copy of this diagram, find the center of rotation. Describe the procedure you used to locate the center.
- b. Using the center found in Part a, dilate the preimage by a scale factor of $1/2$.

Module Summary

- An object has **line symmetry** if the object is its own image in a reflection in a line. This line is the **line of symmetry**.
- **Perpendicular distance** is the distance from a point to a line or a plane. It is equal to the length of the segment, perpendicular to the line or plane, that connects the point to that line or plane.
- Two polygons are **similar** if there is a one-to-one correspondence between their vertices so that corresponding sides are proportional and corresponding angles are congruent.
- **Vertical angles** are two non-straight, non-adjacent angles formed by two intersecting lines. Vertical angles are congruent.
- A **rep tile** is a shape that can be partitioned into congruent shapes, each one similar to the original.
- If two parallel lines are cut by a transversal, then the **corresponding angles** formed are congruent.
- If two parallel lines are cut by a transversal, then the **alternate interior angles** formed are congruent.
- If two parallel lines are cut by a transversal, the **alternate exterior angles** formed are congruent.
- A **rhombus** is a parallelogram with two consecutive congruent sides.
- An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.
- A **chord** is a line segment joining any two points on a circle.
- A **secant** of a circle is a line that intersects the circle in two points.
- A **tangent** of a circle is a line, segment, or ray in the plane of the circle that intersects the circle in exactly one point and is perpendicular to a radius at that point. This intersection is the **point of tangency**.
- The perpendicular bisector of a chord passes through the center of the circle.
- **Isometries** are transformations in which each preimage is congruent to its image.

Selected References

Better Homes and Gardens. *American Patchwork and Quilting*. Des Moines, IA: Meredith Corporation, 1985.

Krause, M. C. *Multicultural Mathematics Materials*. Reston, VA: National Council of Teachers of Mathematics, 1983.

Lafarge, O. *A Pictorial History of the American Indian*. New York: Crown Publishers, 1956.

Nabakov, P., and R. Easten. *Native American Architecture*. New York: Oxford University Press, 1989.

Storm, Hyemeyohsts. *Seven Arrows*. New York: Ballantine Books, 1972.

If the Shoe Fits . . .



How can you tell when two quantities—like age and heart rate—are related? In this module, you explore methods for finding and evaluating mathematical models of some important relationships.

Byron Anderson • Ruth Brocklebank • Pete Stabio



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

If the Shoe Fits . . .

Introduction

In one sense, moviegoers, doctors, economists, and sports fans all study relationships. You can tell when two characters on the screen might be more than friends from their first conversation. And you can guess from the crack of the bat when a fly ball might become a home run.

In many situations, however, it's not enough just to know that two things are related. It also may be important to know how they are related. For example, physiologists have found that as the amount of alcohol consumed increases, human reaction time also increases. In other words, the more people drink, the more time it takes for them to react to a sharp curve in the road—or a child's ball rolling in front of the car.

Discussion

Describe any relationship that may exist between the following quantities:

- a. the number of hours worked by an employee and the employee's gross pay
- b. the heights of basketball players and the numbers of points they score
- c. the distance a student lives from school and the student's arrival time at school in the morning
- d. the ages of drivers and the numbers of automobile accidents they cause
- e. the number of hours spent studying and a student's grade-point average

Activity 1

Medical scientists have studied the relationship between exercise and the human body for decades. Much of that research has focused on the body's need for additional oxygen during physical activity.

One way to determine if there is a relationship between two quantities is to gather and analyze data. After the data has been collected, it can be organized into ordered pairs and graphed. If there appears to be a trend or pattern, an equation can often be found to summarize the relationship mathematically. If the equation is a good model, it can then be used to make predictions for one quantity based on specific values of the other quantity.

Data that appears to have a linear pattern can be modeled with an equation of the form $y = mx + b$. Finding an equation that models a set of data is not an exact science. In most cases, the model will not fit every data point. In many cases, however, it is possible to find a line that models the data reasonably well. In this activity, you examine some data for oxygen consumption and heart rate, determine a linear model for the data, then decide how well the line models the data by calculating residuals.

Exploration

During a treadmill test, physiologists monitor heart rate and rate of oxygen consumption while the subject walks on a treadmill. Table 1 shows the results of a hypothetical treadmill test in which the treadmill's elevation was increased 2.5% every 3 min.

Table 1: Heart rate and oxygen consumption during a treadmill test

Heart Rate (beats/min)	Oxygen Consumption (L/min)
60	0.41
80	0.71
100	0.90
120	1.27
140	1.62
150	1.76
160	1.92
180	2.24
200	2.42

- a. Make a scatterplot of the data in Table 1. Let x represent heart rate and y represent oxygen consumption.
- b.
 1. Draw a line that you think fits the data reasonably well.
 2. Determine the equation of your line in the form $y = mx + b$. **Note:** Round the values of m and b to the nearest thousandth.

Mathematics Note

In real-world applications, a **linear model** seldom yields an equation that fits all the data—even when a line is a good model for the situation. The difference between the y -coordinate of a data point and the corresponding y -value of the model is a **residual**. Since data points may be located above or below the model, the values of the residuals may be positive or negative.

The **absolute value of a residual** is a measure of the distance from the data point to the model. The sum of the absolute values of the residuals provides information about how well a model fits the data. In general, the smaller the sum of the absolute values of the residuals, the more closely a model approximates the data.

For example, consider the coordinates of the data points shown in the two left-hand columns of Table 2 below. This data can be modeled by the linear equation $y = -1.46x + 230$. The table also lists the corresponding values of $y = -1.46x + 230$ for each x -value in the data, the absolute values of the residuals, and the sum of the absolute values of the residuals for this model.

Table 2: Data points and residuals

x -value	y -value	predicted y -value	residual
20	200	200.8	0.8
25	193	193.5	0.5
30	188	186.2	1.8
35	180	178.9	1.1
40	170	171.6	1.6
		Sum	5.8

- c.
 1. Determine the absolute value of the residual for each data point using your equation from Part **b**.
 2. Determine the sum of the absolute values of the residuals.
- d. Try to reduce the sum of the absolute values of the residuals by adjusting the slope and the y -intercept of your equation from Part **b**. Record the equation that results in the lowest sum.

Discussion

- a. What does your scatterplot indicate about the relationship between heart rate and oxygen consumption?
- b. What kinds of information could indicate that a line is a good model of a data set?

- c. Is the line you found in Part **b** of the exploration a good model of the data? Explain your response.
- d.
 1. In Part **d** of the exploration, what was the equation of the linear model that generated the smallest sum of the absolute values of the residuals?
 2. Describe how you adjusted the equation of your line from Part **b** to find this model.
- e. Describe how you would use a linear model to predict each of the following:
 1. the rate of oxygen consumption when the heart rate is 168 beats/min
 2. the heart rate when oxygen consumption is 1.5 L/min
- f. What are some of the dangers of using models to make predictions?

Assignment

- 1.1 The following table shows the suggested maximum heart rate (by age) for persons entering aerobic training programs.

Age (years)	Heart Rate (beats/min)
20	200
25	195
30	190
35	185
40	180
45	173
50	166
55	160
60	155
65	150
70	145

Source: Adapted from McArdle, Katch, and Katch, *Exercise Physiology*.

- a. Draw a scatterplot of the data.
- b. Find a linear equation that models the data. Graph this equation on your scatterplot.
- c. Determine the sum of the absolute values of the residuals for the equation from Part **b**.
- d. Use the equation from Part **b** to predict the suggested maximum heart rate for a person of age 15.
- e. Many physiologists consider a rate of about 220 beats/min to be the upper limit for the human heart. Given this information, does the equation from Part **b** appear to be valid for all ages? Explain your response.

- 1.2** In recent years, physical fitness has focused on aerobic conditioning. Regular aerobic exercise increases your ability to supply oxygen to muscles. When your muscles receive more oxygen, they work more efficiently.

The table below shows data for heart rate and oxygen consumption for a 20-year-old female, both before and after a 10-week aerobic conditioning program. Since the volume of oxygen consumed depends on a person's mass, oxygen consumption is measured here in milliliters per kilogram per minute.

	Before Aerobic Program	After Aerobic Program
Heart Rate (beats/min)	Oxygen Consumption (mL/kg/min)	Oxygen Consumption (mL/kg/min)
100	21.3	22.5
110	22.6	24.4
120	23.8	26.6
130	25.0	28.4
140	25.9	30.3
150	27.4	32.1
160	28.6	34.0
170	29.7	35.8
180	30.9	37.2

Source: Adapted from McArdle, Katch, and Katch, *Exercise Physiology*.

- On a single set of axes, create scatterplots of the data for both before and after the aerobic conditioning program. Let x represent heart rate and y represent oxygen consumption.
- Find a linear equation to model each set of data. Graph both equations on the scatterplot from Part **a**.
- Which line is the better model of its data set: the line that models the “before” data or the line that models the “after” data? Defend your choice using the sum of the absolute values of the residuals.
- Use your models to predict the heart rate—both before and after the conditioning program—when the rate of oxygen consumption is 40 mL/kg/min.

* * * * *

- 1.3** The table below shows the mean amount spent on health care in the United States, per person, for each year from 1984 to 1991.

Year	Amount Spent (\$) per Person
1984	1049
1985	1108
1986	1135
1987	1135
1988	1298
1989	1407
1990	1480
1991	1554

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of the data.
 - Find the equation of a line that models the data. Graph this equation on the scatterplot from Part a.
 - Determine the sum of the absolute values of the residuals for this model.
 - Using your model, predict the mean amount per person that Americans will spend on health care in the year 2000.
- 1.4** The table below shows the percentage of the U.S. population who participated in the work force for selected years from 1970 to 1992.

Year	Percentage in Work Force
1970	60.4
1980	63.8
1985	64.8
1990	66.4
1992	66.3

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of this data.
- Find the equation of a line that models the data.
- Use your model to estimate the percentage of the population who will participate in the work force during the year 2000.
- The U.S. Bureau of the Census predicts that 68.7% of the population will participate in the work force in the year 2000. How does this prediction compare with the one you made in Part c?

* * * * *

Activity 2

Scientists and engineers often make decisions or predictions based on observed data. These decisions can have significant, even life-saving, consequences. For example, the amount of radiation administered to cancer patients is determined using experimental models. Airplane designs depend on data collected using wind tunnels. And the decision to place a new drug on the market is made only after analyzing the results of clinical trials in both humans and animals. In all of these applications, models that accurately represent observed data are of vital importance.

When a set of data appears to have a linear relationship, there are several ways to obtain lines that model the data reasonably well. In the previous activity, you used trial and error to adjust the equation of the linear model until the sum of the absolute values of the residuals was reasonably small.

Another method for determining a linear model is based on a measure of central tendency: the median. The line generated by this method is the **median-median line**. In this activity, you examine the procedure for finding a median-median line.

Mathematics Note

When a set of data appears to have a linear relationship, one reasonable linear model is the **median-median line**.

To find this model, the data points are sorted according to the values of their x -coordinates. They are then divided into three groups of equal size, if possible. If not possible and there is a single extra data point, the middle group is increased by one. If there are two extra data points, the end groups are each increased by one. The coordinates of three **summary points** are found using the median x -coordinate and median y -coordinate of each group.

The median-median line is located parallel to the line through the outer summary points and $1/3$ the vertical distance from that line to the middle summary point. For example, Figure 1 shows a scatterplot of the data points $(1,2)$, $(2,2)$, $(3,4)$, $(4,5)$, $(5,4)$, $(6,5)$, and $(7,7)$.

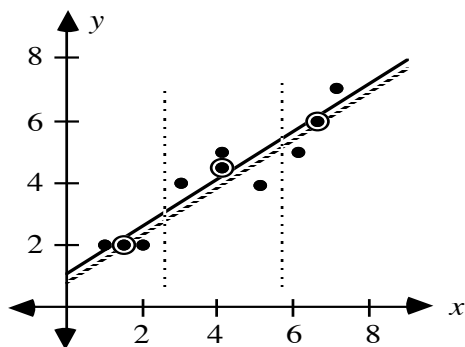


Figure 1: Median-median line through sample data

The dotted vertical lines divide the data into two groups with two points each and a middle group with three data points each. The circled points—(1.5,2), (4,4.5), and (6.5,6), respectively—are the summary points. The solid line is the median-median line. It is located parallel to the line that connects the first and last summary points (the dotted line), and $\frac{1}{3}$ of the vertical distance from the dotted line to the middle summary point. In this case, the equation of the median-median line is $y = 0.8x + 0.97$.

Exploration

In this exploration, you collect data on arm span and shoe size, then use a median-median line to model the relationship between these measurements.

- a.
 1. Measure and record your arm span in centimeters. (Arm span is the distance across the back from fingertip to fingertip, with arms extended away from the body at shoulder level.)
 2. Record your shoe size to the nearest half-size.
 3. Compile the class data for arm span and shoe size.
- b. Make a scatterplot of the data. Let x represent shoe size and y represent arm span.
- c. Use the following steps to divide the data points on your graph into three groups.
 1. Create three groups of equal size, if possible. Any data points with the same x -coordinate should be placed in the same group.

The first group should contain the points that have the smallest x -coordinates; while the third group should contain the points that have the largest x -coordinates. The middle group should contain the remaining data points.

If the data cannot be divided into three equal groups, divide them as described in Step 2 or Step 3.
 2. If there is one extra point, place it in the middle group.
 3. If there are two extra points, place one in the first group and one in the third group.
- d. Use the following process to graph the summary point for each group.
 1. Find the median of the x -coordinates of the group.
 2. Find the median of the y -coordinates of the group.
 3. Use the medians of the x -coordinates and the y -coordinates to form an ordered pair.
 4. Plot and circle this ordered pair on your scatterplot from Part b.

- e. Find the median-median line and determine its equation by following the steps below.
1. Draw a line that passes through the summary points of the first and third groups and find the equation of this line.
 2. Find the point on the line drawn in Step 1 that has the same x -coordinate as the summary point of the middle group. Label this point on the graph.
 3. Find the coordinates of the point that is $1/3$ of the vertical distance between the point labeled in Step 2 and the summary point of the middle group. Graph this point.
 4. Draw the line that is parallel to the line in Step 1 and contains the point graphed in Step 3. This is the median-median line.
 5. Find the equation of the median-median line. **Note:** Save this equation, along with the data from this exploration, for use in the assignment and in the next activity.

Discussion

- a. Why do you think medians are used to determine summary points?
- b. Does the median-median line appear to be a good model of the class data? Defend your response.
- c. How does the y -intercept of the line connecting the first and third summary points compare with the y -intercept of the median-median line?
- d.
 1. Use the median-median line to predict the arm span of a person with a shoe size of 15.
 2. Based on your classroom data, does this prediction seem reasonable? Explain your response.

Assignment

- 2.1
 - a. According to the 1990 *Guinness Book of World Records*, the arm span of the tallest human on record, Robert Wadlow, was 289 cm. Use the median-median line from Part e of the exploration to estimate Wadlow's shoe size.
 - b. Robert Wadlow's actual shoe size was 37. How does your estimate compare with this value?
- 2.2
 - a. Use technology to find the median-median line of the shoe-size data from the exploration.
 - b. How does this line compare to the median-median line you calculated by hand in the exploration?

- 2.3 Since 1985, hospitals have recorded the number of HIV-positive patients discharged after an illness. The table below shows the number of discharged patients who were HIV-positive from 1985 to 1991.

Year	HIV-Positive Patients (in thousands)
1985	23
1986	44
1987	67
1988	95
1989	140
1990	146
1991	165

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of this data.
- Find the median-median line that models the data.
- Does the number of discharged patients who are HIV-positive appear to be increasing linearly? Defend your answer.

* * * * *

- 2.4 The table below shows the number of self-employed workers in the United States for selected years from 1970 to 1992.

Year	Self-Employed Workers (in thousands)
1970	7031
1975	7427
1980	8642
1985	9269
1989	10,008
1990	10,160
1991	10,341
1992	10,017

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of the data.
- Find the median-median line that models the data.
- Use the median-median line to estimate the number of people who were self-employed in 1988.
 - According to the U.S. Bureau of the Census, the actual number of self-employed workers in 1988 was 9917. How does your prediction compare with this value?

- 2.5 The table below shows the number of births performed by cesarean section in the United States for selected years from 1970 to 1991.

Year	Number of Cesarean Sections (in thousands)
1970	195
1980	619
1983	808
1984	813
1985	877
1986	906
1987	953
1988	933
1989	938
1990	945
1991	933

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of this data.
- Find the median-median line that models the data.
- Does the median-median line appear to be a reasonable model?
- Do you think that the number of cesarean births will increase linearly as time passes? Explain your response.

* * * * *

Activity 3

As you saw in Activity 2, the median-median line provides one possible linear model for a set of data. However, the method used to determine the line tends to disregard data points that have large residuals. In this activity, you use residuals to investigate another method for finding the equation of a linear model.

Exploration 1

- Create a scatterplot of the data for arm span and shoe size from Activity 2.
- Draw a line that you think fits the data reasonably well.
 - Determine the equation of your line in the form $y = mx + b$.

Note: In this exploration, round all values to the nearest thousandth.

Mathematics Note

A **linear regression equation** for a set of data is the equation of a linear model that minimizes the sum of the squares of the residuals.

For example, Table 3 lists the coordinates of some data points, the corresponding y -values of a linear model, the squares of the residuals, and the sum of the squares of the residuals.

Table 3: Data points and squares of residuals

x-value	y-value	predicted y-value	(residuals)²
20	200	200.8	0.64
25	193	193.5	0.25
30	188	186.2	3.24
35	180	178.9	1.21
40	170	171.6	2.56
		Sum	7.90

Adjusting the equation of a linear model to obtain the smallest sum possible results in the linear regression equation $y = -1.46x + 230$. A method for determining such an equation is called the **least-squares method**. Most forms of technology use the least-squares method to determine linear regression equations.

- c.
 1. Determine the square of the residual for each data point using your equation from Part **b**.
 2. Determine the sum of the squares of the residuals.
- d. Try to reduce the sum of the squares of the residuals by adjusting the slope and the y -intercept of your equation from Part **b**. Record the equation that results in the lowest sum.
- e. Use technology to find the linear regression equation for the data.

Discussion 1

- a. How did the model you found in Part **d** of the exploration compare with each of the following:
 1. the median-median line found in Activity 2?
 2. the linear regression equation found using technology?
- b.
 1. When would you expect the regression line for a set of data to differ greatly from the median-median line?
 2. When would you expect the regression line to be approximately the same as the median-median line?
- c. In what types of situations would a regression line provide a better model than a median-median line?

Exploration 2

One way to analyze a linear model is to create a **residual plot**. In the following exploration, you use residual plots to evaluate models of two different data sets.

Note: In this exploration, round all values to the nearest thousandth.

- a. Table 4 shows some data on oxygen consumption and heart rate during a treadmill test.

Table 4: Heart rate, oxygen consumption, and residuals

Heart Rate (beats/min)	Oxygen Consumption (L/min)	Predicted Oxygen Consumption	Residual
60	0.41		
80	0.71		
100	0.90		
120	1.27		
140	1.62		
150	1.76		
160	1.92		
180	2.24		
200	2.42		

One possible model for this data is the linear regression equation $y = 0.015x - 0.509$, where y represents oxygen consumption and x represents heart rate. Create a scatterplot of the data and graph the equation $y = 0.015x - 0.509$ on the same coordinate system.

- b.
1. Use the equation $y = 0.015x - 0.509$ to determine the predicted oxygen consumption for each heart rate. Record these values in Table 4.
 2. Determine the value of the residual for each data point. Record these values in Table 4.
- c. Does the line $y = 0.015x - 0.509$ appear to fit the data well? Explain your response.

Mathematics Note

In a **residual plot**, the x -values of the data are represented on the x -axis and the residuals are represented on the y -axis. A scatterplot is created using the ordered pairs (x -value, residual). The plot is then examined for possible patterns. A graph in which the points are randomly scattered above and below the x -axis typically indicates that a reasonable model has been selected.

For example, consider the coordinates of the data points shown in the two left-hand columns of Table 5 below. This data can be modeled by the linear equation $y = 0.75x + 1.14$. The table also lists the corresponding values of $y = 0.75x + 1.14$ for each x -value in the data and the value of the residual for each data point.

Table 5: Data points, predicted y -values, and residuals

x -value	y -value	predicted y -value	residual
1	2	1.89	0.11
2	2	2.64	-0.64
3	4	3.39	0.61
4	5	4.14	0.86
5	4	4.89	-0.89
6	5	5.64	-0.64
7	7	6.39	0.61

Figure 2 shows the corresponding residual plot. Since the points appear to be randomly scattered above and below the x -axis, the equation appears to be a reasonable model for the data.

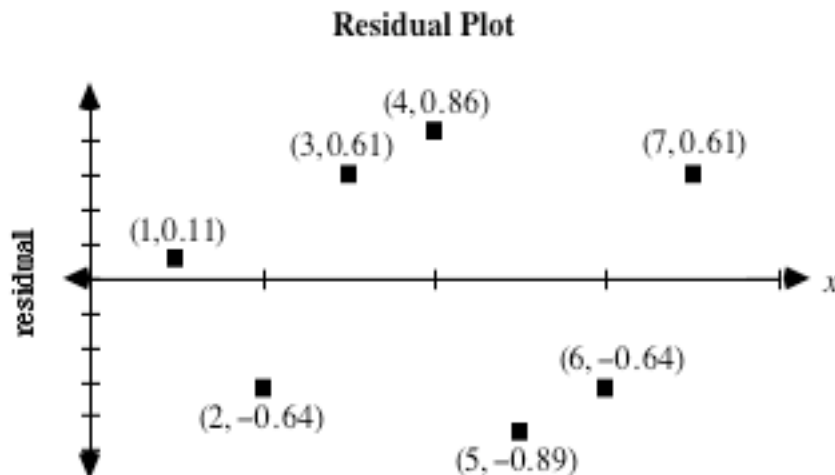
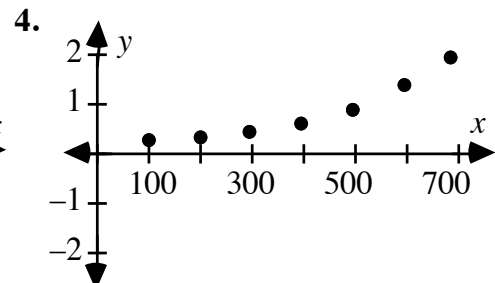
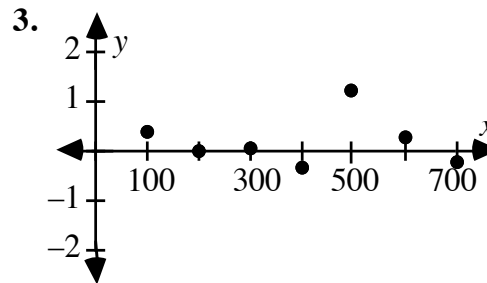
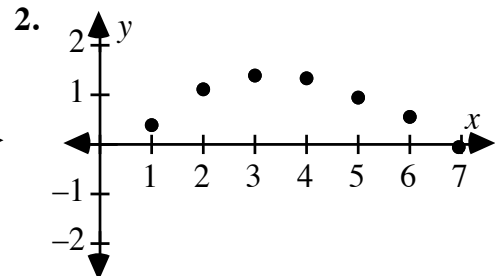
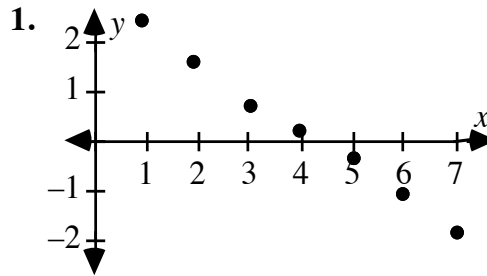


Figure 2: A residual plot

- d. Create a residual plot for the equation $y = 0.015x - 0.509$ and the data in Table 4. Record any patterns you observe.
- e. Repeat Parts a–d using the equation $y = 0.02x - 1$ as a model of the data.
- f. Compare the two residual plots from Parts d and e.

Discussion 2

- a. Which equation in the exploration is the better model of the data in Table 4? Justify your response.
- b. Describe how you would use a residual plot to determine whether a linear model is appropriate for a set of data.
- c. Each of the four graphs below is a residual plot for a linear model of a different data set. Which plots indicate that the linear model is appropriate for the data? Justify your responses.



Assignment

- 3.1** The table below shows the life expectancy for people born in selected years from 1970 to 1991.

Birth Year	Life Expectancy (years)	Birth Year	Life Expectancy (years)
1970	70.8	1983	74.6
1975	72.6	1984	74.7
1976	72.9	1985	74.7
1977	73.3	1986	74.7
1978	73.5	1987	74.9
1979	73.9	1988	74.9
1980	73.7	1989	75.1
1981	74.1	1990	75.4
1982	74.5	1991	75.7

Source: U.S. Bureau of the Census, 1995.

- Make a scatterplot of the data.
 - Find the equation of the regression line for the data. Graph this equation on the scatterplot from Part **a**.
 - Using the equation from Part **b**, predict the life expectancy of people born in 1973 and people born in 2010.
 - The U.S. Bureau of the Census predicts that people born in 2010 will have a life expectancy of 77.6 years. How does this prediction compare with the one you made in Part **c**?
- 3.2** The table below shows the energy used, in kilocalories per minute, by a person walking at speeds from 1.45 km/hr to 9.20 km/hr.

Walking Speed (km/hr)	Energy Used (kcal/min)
1.45	2.15
2.75	2.75
3.90	3.40
4.60	4.00
5.20	4.40
6.30	5.60
7.20	6.80
7.70	8.00
8.30	9.25
9.20	11.00

Source: Adapted from McArdle, Katch, and Katch, *Exercise Physiology*.

- Find a linear regression equation that models this data.
- Make a residual plot for the equation from Part **a**.

- c. What does the residual plot indicate about an appropriate model for the data? Explain your response.

3.3 The table below shows the U.S. birth rate, per 1000 women, for selected years.

Year	Birth Rate per 1000
1970	87.9
1980	68.4
1985	66.3
1989	69.2
1990	70.9

Source: U.S. Bureau of the Census, 1995.

- a. Find the linear regression equation that models the data.
 b. Make a residual plot for the model from Part a.
 c. Does the model appear to be appropriate for the data? Explain your response.

* * * * *

3.4 The table below shows the average daily cost of a stay in a U.S. community hospital, for the years from 1984 to 1991.

Year	Daily Cost (\$)
1984	411
1985	460
1986	501
1987	539
1988	586
1989	637
1990	687
1991	752

Source: U.S. Bureau of the Census, 1995.

- a. Find the linear regression equation for this data.
 b. Using the equation from Part a, predict the average daily cost of a hospital stay in the year 2000.
 c. Make a residual plot for the model from Part a.
 d. What does the residual plot indicate about an appropriate model for the data? Explain your response.

- 3.5** The table below shows the number of men and women employed in the United States for the years from 1986 to 1992.

Year	Males (in thousands)	Females (in thousands)
1986	60,892	48,706
1987	62,107	50,334
1988	63,273	51,696
1989	64,315	53,027
1990	64,435	53,479
1991	63,593	53,284
1992	63,805	53,793

Source: U.S. Bureau of the Census, 1995.

- a.** Find the linear regression equation that models the total employment of men and women for the years shown in the table.
- b.** According to this model, what is the average yearly increase in the number of American workers?
- c.** Make a residual plot for the model from Part **a**.
- d.** What does the residual plot indicate about an appropriate model for the data? Explain your response.
- e.** Repeat Parts **a–d** using the data for female workers only.
- f.** Repeat Parts **a–d** using the data for male workers only.
- g.** Compare the average yearly increases for number of female workers, number of male workers, and total number of workers.

* * * * *

Summary Assessment

1. The table below shows some data on body mass and oxygen consumption collected during treadmill tests.

Body Mass (kg)	Oxygen Consumption (L/min)
60	1.50
65	1.66
70	1.78
75	1.93
80	2.07
85	2.17
90	2.30
95	2.46
100	2.56

Source: Adapted from McArdle, Katch, and Katch, *Exercise Physiology*.

- a. Make a scatterplot of the data.
 - b.
 1. Find a linear regression equation that models the data.
 2. Find a median-median line that models the data.
 - c. Find the sum of the squares of the residuals for each model in Part b.
 - d. Make a residual plot for each model in Part b.
 - e. Which model better represents the data? Justify your response.
2. The amount of energy necessary to maintain normal body functions—such as heart beat and respiration—is the **basal metabolic rate**. Basal metabolic rate is typically measured in energy output per unit surface area per hour. It is relatively constant for people of the same age and sex.

The table below shows some data for males from 5 to 70 years of age. Determine whether a linear model is appropriate for this set of data. Justify your answer using graphs, equations, and residuals.

Age (years)	Metabolic Rate (kcal / m ² / hr)
5	50
10	44
20	40
30	38
35	37
55	34
70	33

Source: Adapted from McArdle, Katch, and Katch, *Exercise Physiology*.

Module Summary

- The difference between the y -coordinate of a data point and the corresponding y -value of the model is a **residual**. Since data points may be located above or below the model, the values of the residuals may be positive or negative.
- The **absolute value of a residual** is a measure of the distance from the data point to the model. The sum of the absolute values of the residuals provides information about how well a model fits the data. In general, the smaller the sum of the absolute values of the residuals, the more closely a model approximates the data.
- When a set of data appears to have a linear relationship, one reasonable linear model is the **median-median line**.

To find this model, the data points are sorted according to the values of their x -coordinates. They are then divided into three groups of equal size, if possible. If not possible and there is a single extra data point, the middle group is increased by one. If there are two extra data points, the end groups are each increased by one. The coordinates of three **summary points** are found using the median x -coordinate and median y -coordinate of each group.

The median-median line is located parallel to the line through the outer summary points and $\frac{1}{3}$ the vertical distance from that line to the middle summary point.

- A **linear regression equation** for a set of data is the equation of a linear model that minimizes the sum of the squares of the residuals. A method for determining such an equation is called the **least-squares method**. Most forms of technology use the least-squares method to determine linear regression equations.
- In a **residual plot**, the x -values of the data are represented on the x -axis and the residuals are represented on the y -axis. A scatterplot is created using the ordered pairs (x -value, residual). The plot is then examined for possible patterns. A graph in which the points are randomly scattered above and below the x -axis typically indicates that a reasonable model has been selected.

Selected References

- McArdle, W. D., F. I. Katch, and V. L. Katch. *Exercise Physiology*. Philadelphia: Lea & Febiger, 1991.
- North Carolina School of Science and Mathematics, Department of Mathematics and Computer Science. *Contemporary Precalculus through Applications*. Dedham, MA: Janson Publications, 1992.
- Rodahl, K. *The Physiology of Work*. Bristol, PA: Taylor & Francis, 1989.
- Thomas, T. R., and B. R. Londeree. "Calories Expended, Walking vs. Jogging." *The Physician and Sports Medicine* 17 (May 1989): 98.
- U.S. Bureau of the Census. *Statistical Abstract of the United States: 1995*. Washington, DC: U.S. Government Printing Office, 1995.

Take It to the Limit



What do hungry chickens, chain letters, and the perimeters of nested triangles have in common? This module introduces you to infinite sequences—and their limits.

Todd Fife • Mark Lutz • Lisa Wood



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Take It to the Limit

Introduction

Many advertisements use the power of mathematics to attract consumers. An ad for the latest weight-control plan, for example, might promise to help dieters “lose 10 pounds a week.” An ad for a new car might guarantee “payments of only \$199 a month for 24 months.” And an ad for chicken feed might lure poultry farmers by crowing “the mass of your birds will increase 10% a day.”

The numbers implied by these ads can be represented as different types of **sequences**, or ordered lists. Whether you find their claims believable or not depends on how you interpret them.

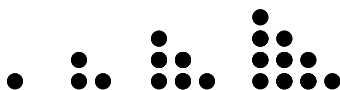
Discussion

- a. Determine a rule or pattern that allows you to identify a next possible term or object for each of the following:

1. 7, 13, 19, 25

2. 162, 54, 18, 6

3.



4. $-7, 49, -343, 2401$

5. $[2, 3], [2, 2.5], [2, 2.\bar{3}], [2, 2.25]$

6. 4, 5, 9, 14, 23

Mathematics Note

Rules and patterns often can be described using formulas.

A **recursive formula** for a sequence identifies the initial term (or terms), then defines all other terms using the previous term(s).

An **explicit formula** for a sequence defines any term based on its term number.

For example, consider the sequence 2, 5, 8, 11, 14, A recursive formula for this sequence is shown below:

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 3, \quad n > 1 \end{cases}$$

The same sequence can be described by the explicit formula $a_n = 2 + 3(n - 1)$.

- b. If possible, determine a recursive formula for each sequence in Part a.
- c. If possible, determine an explicit formula for each sequence in Part a.

Mathematics Note

In an **arithmetic sequence**, every term after the first is formed by adding a constant value, the **common difference**, to the preceding term.

For example, the sequence 2, 7, 12, 17, 22 is a finite arithmetic sequence with a common difference of 5.

In a **geometric sequence**, every term after the first is formed by multiplying the preceding term by a constant value, the **common ratio**.

For example, the sequence 2, 10, 50, 250, 1250 is a finite geometric sequence with a common ratio of 5.

- d.
 1. Which of the ordered lists in Part a can be represented by an arithmetic sequence?
 2. Which can be represented by a geometric sequence?
- e. Describe how to find the common difference for an arithmetic sequence.
- f. Describe how to determine the common ratio for a geometric sequence.

Activity 1

In this activity, you use formulas for arithmetic and geometric sequences to create and interpret sequential patterns.

Mathematics Note

The recursive formula for an arithmetic sequence, where d is the common difference, n is the term number, and a_n is the n th term, is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, n > 1 \end{cases}$$

The explicit formula for such a sequence is $a_n = a_1 + d(n - 1)$.

The recursive formula for a geometric sequence, where r is the common ratio ($r \neq 0$), n is the term number, and g_n is the n th term, is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = rg_{n-1}, n > 1 \end{cases}$$

The explicit formula for such a sequence is $g_n = g_1 r^{n-1}$.

Exploration

In this exploration, you use your knowledge of sequences to play a game. Read Parts **a–e** before beginning play.

- a.** To play this game, each person creates a sequence based on a specific rule or pattern. The first five terms of the sequence are written near the top of a sheet of paper. The paper is then passed to a second person.
- b.** After examining the first five terms, the second person identifies the sequence as *arithmetic*, *geometric*, or *neither*. The second person writes this identification on the sheet of paper below the sequence.

Using complete sentences, the second person then writes a rule for the sequence, folds the paper so that only the rule shows, and passes it to a third person.

- c.** Using only the rule written by the second person, the third person writes both a recursive formula and an explicit formula (if possible) for the sequence.

The third person then folds the paper so that only the two formulas are visible, and passes it to a fourth person.

- d.** Using only the formulas written by the third person, the fourth person writes the first five terms of the sequence on the paper, then passes it back to the sequence's original creator.
- e.** The creator unfolds the paper and checks to see that the new sequence matches the original five terms. If the two sequences do not match, any discrepancies are identified and recorded.

Discussion

- a.** Did the sequence found in Part **d** of the exploration match the original five terms of your sequence? If not, explain why the two patterns differed.
- b.** Describe how to determine whether a sequence is arithmetic, geometric, or neither.
- c.** Consider a sequence whose first three terms are 1, 2, 3. Describe a rule which allows the next three terms to be:
 - 1.** 4, 5, 6
 - 2.** 5, 8, 13
 - 3.** 6, 11, 20
- d.** Given the first three terms of an arithmetic sequence, must the next three terms follow the same pattern? Explain your response.

Assignment

- 1.1** Write the first five terms of the sequence defined by each of the following explicit formulas:
- $k_n = 9 - 3(n - 1)$
 - $k_n = -5(1.3)^{n-1}$
- 1.2** Determine the number of terms in each of the following sequences. Hint: Find an explicit formula for the sequence, then solve the formula for the term number n .
- 11, 13, 15, ..., 99
 - 20, 16, 12.8, ..., 6.5536
 - 7, 10.5, 14, ..., 357
 - 3, 6, 12, ..., 3072
- 1.3** Complete Steps 1–3 for each sequence in Parts a–e below.
- Identify the next two terms in the sequence.
 - Write an explicit or recursive formula, if one exists, that describes each pattern. If such a formula does not exist, write a description of the pattern you used to identify the next two terms.
 - Explain whether the sequence is arithmetic, geometric, or neither.
- 1, 3, 7, 15, 31, ...
 - $-50, 10, -2, 2/5, \dots$
 - 1, 4, 7, 10, ...
 - $2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, 2\frac{1}{5}, \dots$
 - 6, $-6, 6, -6, 6, -6, \dots$
- 1.4**
- Write the first five terms of a sequence that is both arithmetic and geometric. Explain why your sequence fits in both categories.
 - Write an explicit formula for your sequence when it is considered an arithmetic sequence.
 - Write an explicit formula for your sequence when it is considered a geometric sequence.

- 1.5**
- a. If the first term of an arithmetic sequence is 13 and the 50th term is 93, what is the common difference? Describe the process you used to solve this problem.
 - b. If the first term of a geometric sequence is 12 and the fifth term is 192, what is the common ratio? Describe the process you used to solve this problem.
 - c. If the first term of an arithmetic sequence is 11, the second term is 14, and the last term is 62, what is the common difference? How many terms are there in this sequence?
 - d. If the first term of a geometric sequence is 4, the second term is 20, and the last term is 7,812,500, what is the common ratio? How many terms are there in this sequence?
- 1.6** Obesity in pets can cause joint problems and is related to arthritis and other diseases. In some cases, veterinarians may recommend specially formulated pet foods to help control this condition. One such product claims that it can “help your pet lose 0.9 kg a month.” Imagine that a veterinarian has prescribed this diet for a dog with a mass of 54 kg. What would you expect the mass of the dog to be at the end of 12 months? Explain your reasoning.
- 1.7** In its ad for chicken feed, a supplier promises a 10% daily gain in mass. A newly hatched chick has an approximate mass of 36 g. When the chicken reaches 8 weeks of age, it is considered a marketable broiler.
- a. Consider the chicken’s mass as a geometric sequence. If the ad’s claim is true, what will the chicken’s mass be after 8 weeks?
 - b. Do you think your answer to Part a is reasonable? Explain your response.
 - c. After 56 days on the advertised feed, a broiler chicken has a mass of 1487 g. What percentage daily gain in mass should the advertiser claim? Explain your response.
- 1.8** As the term number n becomes increasingly large, predict what will happen to a geometric sequence with a common ratio:
- a. between -1 and 0
 - b. between 0 and 1
 - c. less than -1
 - d. greater than 1 .

* * * * *

- 1.9** Cut out a square of paper that measures 16 cm on each side.
1. Fold the paper in half.
 2. Record the fold number, the number of layers created by the fold, and the area of the top surface.
 - Repeat Part **a** as many times as you can.
 - Write both explicit and recursive formulas for the area of the top surface after n folds.
 - Write both explicit and recursive formulas for the number of layers after n folds.
 - What value does the area of the top surface appear to approach as the number of folds increases?
 - What value does the number of layers appear to approach as the number of folds increases?

1.10 In Problem **1.9**, you folded the paper in half at each stage. After seven or eight folds, you probably noticed that continued folding became nearly impossible. Instead of folding the paper at each stage, suppose you cut it in half and stacked the cut sheets.

- How big would the original sheet of paper have to be in order for the area of the top surface of the stack to be 1 cm^2 after 50 cuts?
- After 50 cuts, how tall would the stack of paper be? Give your estimate in kilometers and describe how you arrived at your response.

1.11 Describe the pattern for each of the following sequences. If possible, identify the sequence as arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference. If it is geometric, find the common ratio.

a. $1, 1, 2, 3, 5, 8, \dots, k_n$

b.
$$\begin{cases} k_1 = 2 \\ k_n = k_{n-1}(1.05), \quad n > 1 \end{cases}$$

c. $k_n = 2000 - (1000)(n - 1)$

d.
$$\begin{cases} k_1 = 0 \\ k_n = k_{n-1} + (n - 1), \quad n > 1 \end{cases}$$

e. $3, 3/2, 1, 3/4, 3/5, \dots, k_n$

1.12 For each sequence in Problem **1.11**, describe what happens to the terms as n becomes increasingly large.

* * * * *

Activity 2

Carl Friedrich Gauss (1777–1855) was one of the most productive and influential mathematicians of the 19th century. He made many important mathematical discoveries while still a student at the University of Göttingen.

Even as a child, Gauss showed a remarkable skill with numbers—in particular with the set of natural numbers $\{1, 2, 3, 4, \dots\}$. According to mathematical lore, one day his teacher asked the class to add all the natural numbers from 1 to 100. Students were instructed to place their slates on the table when finished. To the surprise of the teacher, young Gauss placed his slate on the table after only a few moments.

Mathematics Note

The indicated sum of the terms of a finite sequence is a **finite series**. A finite series of n terms, denoted S_n , may be written in expanded form as follows:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

where a_n represents the n th term of the sequence.

The sum of the terms of an arithmetic sequence is an **arithmetic series**. For example, the finite arithmetic sequence shown below has seven terms:

$$2, 4, 6, 8, 10, 12, 14$$

The sum of those terms, in expanded form, is the following arithmetic series:

$$S_7 = 2 + 4 + 6 + 8 + 10 + 12 + 14 = 56$$

To find the sum of the first 100 natural numbers, Gauss used a method involving a finite series. For example, the sum of the first 100 numbers can be written as the arithmetic series S_{100} :

$$S_{100} = 1 + 2 + \cdots + 99 + 100$$

This series can also be written in reverse order, as shown below.

$$S_{100} = 100 + 99 + \cdots + 2 + 1$$

These two series can then be added as follows:

$$\begin{array}{r} S_{100} = 1 + 2 + \cdots + 99 + 100 \\ S_{100} = 100 + 99 + \cdots + 2 + 1 \\ \hline 2S_{100} = 101 + 101 + \cdots + 101 + 101 \end{array}$$

Since the resulting equation contains 100 terms of 101, the sum of the two equations can be written as:

$$2S_{100} = 100(101)$$

Solving this equation for S_{100} , the sum of the first 100 natural numbers can be found as follows:

$$S_{100} = \frac{100(101)}{2} = 5050$$

Exploration

- a.
 1. Use a method like Gauss's to find the sum of the first 1000 natural numbers.
 2. Generalize this method to find the sum of the first n natural numbers.
- b. Use a similar method to find the sum of the first 50 even natural numbers.
- c. Consider an arithmetic sequence with a first term of 2 and a common difference of 4. Find the sum of the first 75 terms of this sequence. Hint: Determine the 75th term of the sequence before trying to find the sum.
- d. Consider an arithmetic sequence of n terms with a first term a_1 and a common difference d .
 1. Using a_1 and d , write expressions for the second, third, and fourth terms of the sequence.
 2. Using a_1 and d , write expressions for the n th term of the sequence and the three terms preceding the n th term.
 3. Find the sum of the first n terms of this arithmetic sequence.

Mathematics Note

The sum of the terms of a finite arithmetic sequence with n terms and a common difference d can be found using the following formula:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the sequence and a_n is the n th term.

For example, consider the sequence 7, 11, 15, ..., 59. The number of terms in this arithmetic sequence can be found using the general form of the explicit formula, $a_n = a_1 + (n - 1)d$. In this case, $a_1 = 7$, $a_n = 59$, and $d = 4$. By substituting as shown below, the equation can be solved for n , the number of terms.

$$59 = 7 + (n - 1)4$$

$$59 = 7 + 4n - 4$$

$$56 = 4n$$

$$14 = n$$

Since $n = 14$, the sum of those terms, S_{14} , can be found as follows:

$$S_{14} = \frac{14}{2}(7 + 59) = 462$$

Discussion

- a.** Verify that the following two expressions are equivalent:

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \quad S_n = \frac{n}{2}(a_1 + a_n)$$

- b.** Consider the sequences whose terms you added in Parts **a** and **b** of the exploration. Are these sequences arithmetic, geometric, or neither?
- c.**
- 1.** Multiplying the first 100 natural numbers by 2 produces the first 100 even natural numbers. Describe how to find the sum of this set of numbers.
 - 2.** Multiplying the first 100 natural numbers by k produces the first 100 multiples of k . Describe how to find the sum of this set of numbers.
- d.** Suppose that the following arithmetic series continued indefinitely:

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

What do you think the sum of the series would be?

Assignment

- 2.1** Write a formula to find the sum of the first n even natural numbers.
- 2.2** Find the sum of the first 2000 natural numbers.
- 2.3**
- a.** Generalize your response to Part **c** of the discussion to find a formula for the sum of the first n multiples of k .
 - b.** Use your formula from Part **a** to find the sum of the first 600 multiples of 3.

- 2.4 Consider the arithmetic sequence 3, 7, 11, ..., 451.
- What is the first term of the sequence?
 - What is the common difference?
 - How many terms are there in the sequence?
 - What is the sum of the terms of the sequence?
- 2.5 Can Gauss's method be used to find the sum of a geometric sequence? Explain your response.
- 2.6 To attract new customers, a car dealership displays the following advertisement.

New Car for Only \$8,888.00	
Starting at \$106.26 a Month!*	
*36-month lease	
Refundable Security Deposit	\$ 150.00
Down Payment	\$ 1,600.00
First Month's Payment	\$ 106.26
Cash Due at Lease Inception	\$ 1,856.26

- Do the monthly payments described in this ad form an arithmetic sequence? Explain your response.
 - Does the total amount paid over the term of the lease form an arithmetic series? Explain your response.
 - According to this ad, how much money will the lessee pay over the entire 36-month period?
 - Explain any discrepancies you observe between the amount determined in Part c and the advertised price of \$8,888.00.
- * * * * *
- 2.7 Consider a weekly newspaper with 15,000 subscribers. During the next year, the paper plans to increase its number of subscribers by 50 every week. If this plan succeeds, what will be the total number of newspapers delivered for the entire year?
- 2.8 Consider a sequence described by the explicit formula $a_n = -7 + 5n$, for $n \geq 1$.
- Write the first three terms of the sequence.
 - Find a_{150} .
 - Find S_{150} .

- 2.9 Consider an arithmetic sequence with a first term of 5 and a 41st term of 120.
- Find the sum of the first 41 terms of this sequence.
 - Identify the common difference of the sequence.
 - Write the second and third terms of the sequence.

* * * * *

Activity 3

Have you ever received an e-mail from a friend warning you about a computer virus? Such warnings usually urge you to send the warning to all the names in your e-mail address book.

What happens if the warning is a hoax? What are the consequences for all the servers that route the e-mail messages? Knowledge about geometric series helps answer such questions.

Exploration

You just received an e-mail warning your about Red Ant, particularly destructive computer virus that destroys all data stored on a computer. Concerned that this virus could cause major damage in your computer and those of your friends, you immediately e-mail a copy of the warning to two friends.

- Your two friends each e-mail a copy of the warnings to three of their friends. What is the total number of e-mail warnings your two friends send?
- Suppose each of the people to whom your friends send e-mail warnings in Part a also send e-mail warnings to three of their friends. How many total e-mail warnings will they send?
- Assuming every recipient continues to e-mail a copy of the warning to three of their friends, continue this pattern for three more stages. Record the total number of e-mail warning sent at each stage.
- Write the geometric sequence formed by the number of e-mail warnings sent at each stage.

Mathematics Note

The indicated sum of the terms of a geometric sequence is a **geometric series**. For example, the finite geometric sequence below has five terms.

2, 6, 18, 54, 162

The sum of those terms, in expanded form, is the following finite geometric series:

$$S_5 = 2 + 6 + 18 + 54 + 162 = 242$$

- e. Write the expanded form of the geometric series that represents the total number of e-mail warnings sent. List at least three terms at the beginning of the series and three terms at the end of the series. Use an ellipsis to indicate the middle terms.

Set the series equal to S_{200} and call this “equation 1.”

- f. If a geometric series has many terms, determining its sum by simple addition can be time consuming. Use the following steps to determine the total number of e-mail warnings sent.
1. Multiply both sides of equation 1 by the common ratio of the sequence. Call the result “equation 2.”
 2. Subtract equation 1 from equation 2.
 3. Solve the resulting equation for S_{200} .

- g. A geometric series with n terms, where g_1 is the first term and r is the common ratio, also can be represented in the following expanded form:

$$S_n = g_1 r^0 + g_1 r^1 + g_1 r^2 + \cdots + g_1 r^{n-3} + g_1 r^{n-2} + g_1 r^{n-1}$$

Use the process described in Part f to simplify this representation of the series.

Discussion

- a. Assuming every recipient continues to e-mail warnings to three of their friends, how many total e-mail warnings would you be sent? Is this a reasonable expectation? Why or why not?
- b. Suppose the virus warning is a hoax and people do e-mail warnings to everyone in their address list. What are some possible consequences?

Mathematics Note

The sum of the terms of a finite geometric sequence with n terms and a common ratio r can be found using the following formula:

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

where g_1 is the first term of the sequence and $r \neq 1$.

For example, consider the geometric sequence 2, 6, 18, 54, 162. In this case, $n = 5$, $r = 3$, and $g_1 = 2$. Using the formula given above, the sum of the sequence can be found as follows:

$$S_5 = \frac{2(3)^5 - 2}{3 - 1} = 242$$

- c. How does the formula given in the mathematics note above compare to the one you found in Part g of the exploration?
- d. Why must r not equal 1 in the formula given in the mathematics note?
- e. 1. Write a geometric series S_n where $r = 1$.
2. Find a formula for the geometric series S_n when $r = 1$.
- f. Describe how to demonstrate that the formula for a geometric series

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

is equivalent to each of the following expressions. Hint: $g_1 r^{n-1} = g_n$.

1. $S_n = \frac{g_1(r^n - 1)}{r - 1}$
2. $S_n = \frac{g_n r - g_1}{r - 1}$
- g. Given the first three terms and the last term of a geometric series, how could you determine the sum?

Assignment

- 3.1 a. Determine the sum of the terms of a geometric sequence in which $g_1 = 3.5$, $r = 0.6$, and $n = 17$.
b. Determine the sum of the terms of a geometric sequence in which $g_1 = 10$, $r = 10$, and $n = 10$.
- 3.2 Find each of the sums below.
a. $20 + 16 + 12.8 + \cdots + 6.5536$
b. $3 + 6 + 12 + \cdots + 3072$
- 3.3 Identify each of the following series as arithmetic, geometric, or neither and find the sum.
a. $11 + 13 + 15 + 17 + 19 + 21 + 23 + \cdots + 99$
b. $2 + (-3) + (-8) + \cdots + (-28)$
c. $2\frac{1}{2} + 2\frac{1}{3} + 2\frac{1}{4} + \cdots + 2\frac{1}{7}$
d. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

- 3.4** The Greek philosopher, Zeno of Elea, used paradoxes to demonstrate the difficulties of dividing time and space into an infinite number of parts. The following situation is similar to one of Zeno's paradoxes.

Imagine a race between the legendary Greek hero, Achilles, and a tortoise. Since Achilles can run 10 times faster than the tortoise, he gives the tortoise a 10-m head start. During the time it takes Achilles to run the first 10 m, the tortoise travels a distance of 1 m. Because of its head start, the tortoise still leads, but now by only 1 m.

During the time it takes Achilles to make up that 1 m, the tortoise travels 0.1 m. While Achilles runs that 0.1 m, the tortoise travels 0.01 m. As Achilles runs 0.01 m, the tortoise moves another 0.001 m. Imagine that this pattern continues for each successive interval of time. **Note:** As the race continues, the intervals of time used to analyze the situation become smaller and smaller.

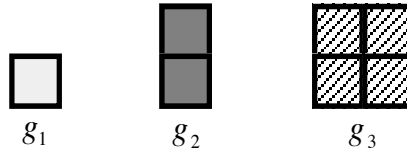
- a. If the tortoise has traveled a distance of 1.11111 m, how many time intervals have elapsed since the start of the race?
 - b. Find the distance covered by Achilles when the tortoise has traveled a distance of 1.11111 m. Describe the process you used.
 - c. Compare the distance traveled by Achilles in Part **a** with the total distance traveled by the tortoise, including its 10-m head start. Did Achilles catch the tortoise? Explain your response.
 - d. If the race continues in this fashion, do you think Achilles will ever catch the tortoise? Explain your response.
- 3.5** In one ancient legend, the king of Persia offered a reward to the inventor of the game of chess. The inventor suggested using the chessboard itself to determine the size of the reward: for the first square on the board, the reward would be 1 kernel of wheat. For the second square, the reward would be twice that, or 2 kernels. The inventor would receive twice as many again for the third square (4 kernels), and so on, until all 64 squares on the board had been filled.

The king agreed to this plan and sent a servant to bring the appropriate amount of grain.

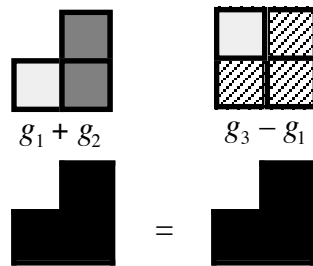
- a. How many kernels of wheat were in the reward?
- b. When kernels of wheat are spread one layer thick, $7.5 \cdot 10^{10}$ kernels will cover 1 km^2 . If the reward were spread one layer thick, how many square kilometers would it cover?
- c. The area of the United States is approximately $1.0 \cdot 10^7 \text{ km}^2$. If the reward were spread over the entire country, how many layers thick would it be?

3.6 Imagine that the inventor of chess asked the king to triple the amount of grain on each successive square of the chessboard. How many wheat kernels would there be in this reward? How does this amount compare to the reward in Problem 3.5?

3.7 a. Consider the sequence described by the explicit formula $g_n = 2^{n-1}$. On a sheet of graph paper, shade and label a rectangular region for each of the first six terms. The area of each shaded rectangle should correspond to the value of the term. A sample graph of the first three terms is shown below:



b. The figure below shows the sum of the first two terms of the sequence. Since $g_1 + g_2$ equals $g_3 - g_1$, the sum can be expressed in two different, but equivalent, ways.

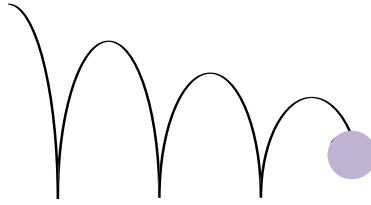


Demonstrate a similar property for each of the following:

1. the sum of the first three terms
 2. the sum of the first four terms
 3. the sum of the first five terms.
- c.** Generalize the pattern for the sum of the first n terms of the sequence.
- d.** Show how the formula you obtained in Part **c** is related to the general formula for a finite geometric series.

* * * * *

3.8 Consider a ball bouncing as shown in the figure below.



On the first bounce, the ball reaches a height of 6.00 m. On the second bounce, the ball reaches a height of 4.80 m. On the third bounce, the ball reaches a height of 3.84 m.

- a. Assuming that this pattern continues, determine a geometric sequence which models the height of the ball on each bounce.
- b. Write the expanded form of the geometric series for the first n terms of the sequence.
- c. Determine the value of the geometric series for each of the following:
 1. the first 10 terms
 2. the first 20 terms
 3. the first 30 terms
 4. the first 40 terms.
- d.
 1. What value does the geometric series appear to approach?
 2. What does this value represent?

3.9 A tree in a tropical rain forest grows 9 m in one year, 6 m in the next year, and 4 m in the following year.

- a. Assuming that this pattern continues, write a geometric series that models the tree's growth for 5 years.
- b. What is the common ratio?
- c. Given that the tree's initial height was 4 m, use a geometric series to determine the height of the tree after:
 1. 10 years
 2. 20 years
 2. 30 years
- d. What value does the height of the tree appear to approach?

* * * * *

Activity 4

Paradoxes are statements that seem contrary to common sense, yet somehow may be true. Zeno of Elea used the race between Achilles and the tortoise, for example, to support his argument that time could not be split into an infinite number of parts.

Because it is difficult to describe an infinite number, problems involving infinity have intrigued mathematicians for centuries. In this activity, you begin an informal investigation of some problems involving infinite processes.

Exploration

- a.
 1. Using a geometry utility, create a triangle large enough to fill the screen. Label the vertices A , B , and C .
 2. Locate the midpoint of each side of the triangle and label them D , E , and F , respectively. Connect these points to form $\triangle DEF$.
 3. Connect the midpoints of the sides of $\triangle DEF$ to form $\triangle GHJ$.
 4. Repeat this process to form $\triangle KLM$. Your drawing should now resemble the one shown in Figure 2.

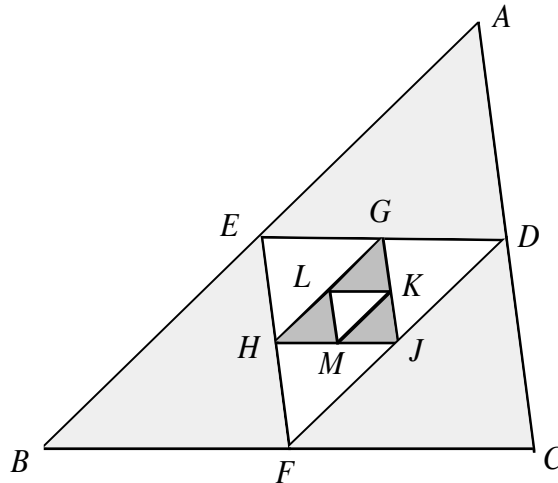


Figure 2: Triangles formed by connecting midpoints of sides

- b. Create a spreadsheet with headings like those in Table 1 and complete the first four rows.

Table 1: Data for sequence of triangles

Term No.	Name of Triangle	Area	Perimeter	$\frac{\text{Area of } \Delta}{\text{Area of Previous } \Delta}$	$\frac{\text{Perimeter of } \Delta}{\text{Perimeter of Previous } \Delta}$
1	ΔABC			none	none
2	ΔDEF				
3	ΔGHJ				
4	ΔKLM				
\vdots					
15					

- c. 1. Use the ratios of areas developed in Part b to extend the spreadsheet for 15 triangles.
 2. Create a scatterplot of area versus term number.
- d. 1. Use the ratios of perimeters developed in Part b to extend the spreadsheet for 15 triangles.
 2. Create a scatterplot of perimeter versus term number.
- e. 1. Use the data for the areas of the triangles to find each of the following sums: $S_1, S_2, S_3, \dots, S_{10}$.

For example, if the area of ΔABC is 100 cm^2 , the area of ΔDEF is 25 cm^2 , and the area of ΔGHI is 6.5 cm^2 , then $S_1 = 100$, $S_2 = 100 + 25 = 125$, and $S_3 = 100 + 25 + 6.5 = 131.5$.

2. Create a scatterplot of the sum versus the number of terms.
- f. Find the sums S_{20}, S_{30}, S_{50} , and S_{100} using the formula below:

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

Discussion

- a. 1. Considering the four triangles from Part a of the exploration, what do you notice about the areas of consecutive triangles?
 2. What do you notice about the perimeters of consecutive triangles?
- b. 1. If your spreadsheet columns containing areas and perimeters could be extended indefinitely, what value would the areas appear to approach?
 2. What value would the perimeters appear to approach?

Mathematics Note

In an **infinite sequence**, every term in the sequence has a successor. Infinite sequences do not have a finite number of terms, but continue indefinitely.

The **limit** of a sequence, $k_1, k_2, k_3, \dots, k_n, \dots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L .

For example, consider the infinite sequence below.

$$\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots$$

Figure 3 shows a graph of the first 32 terms of this sequence.

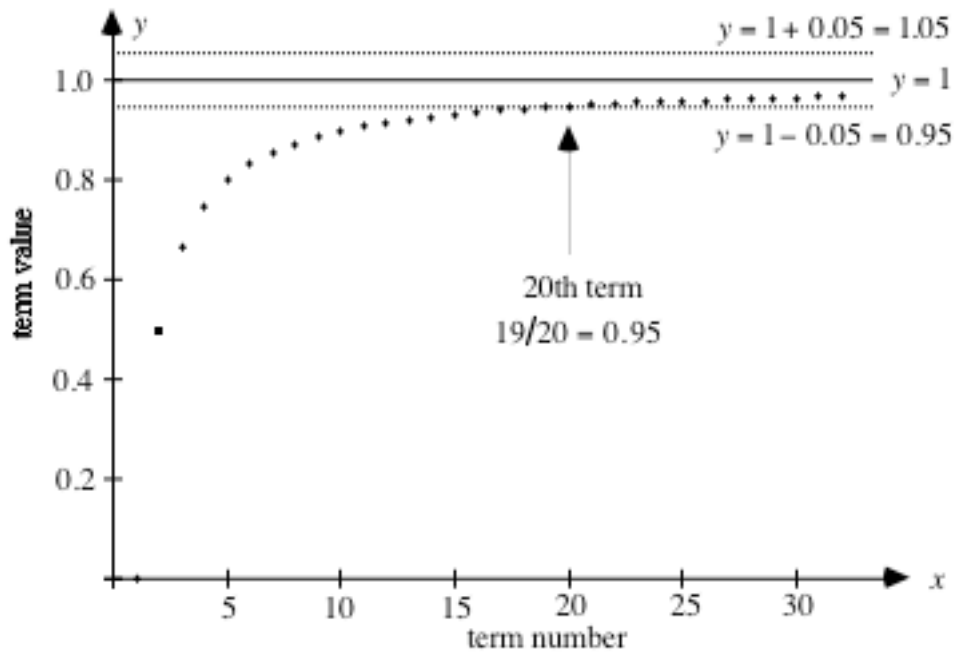


Figure 3: Graph of a sequence whose terms approach a limit

The limit of this sequence appears to be 1 because, for any prescribed level of accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of 1. For example, given a level of accuracy of 0.05, all of the terms of the sequence after k_{20} are within 0.05 of 1.

- c. What is meant by “any prescribed accuracy” as mentioned in the mathematics note?
- d. How do your responses to Part **b** of the discussion compare with the trends you observe in the scatterplots of the perimeter and area data?

- e. 1. How many triangles would you have to draw for the value of the perimeter to fall within 0.01 units of the limit?
 2. How many triangles would you have to draw for the value of the area to fall within 0.01 square units of the limit?
- f. In Part f of the exploration, you used the following formula to find sums for larger values of n , where $r = 1/2$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of $(1/2)^n$ as n gets very large.
 2. Describe what happens to the value of the formula below as n gets very large.

$$S_n = \frac{g_1 \cdot ((1/2)^n - 1)}{(1/2) - 1}$$

- g. Consider the following formula for a geometric series where $-1 < r < 1$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of r^n as n increases indefinitely.
 2. Describe what happens to the formula for S_n as n increases indefinitely.

- h. Consider the following formula for a geometric series where $r \leq -1$ or $r > 1$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of r^n as n increases indefinitely.
 2. Describe what happens to the formula for S_n as n increases indefinitely.

Mathematics Note

The sum of the terms of an infinite geometric sequence in which $-1 < r < 1$ can be found using the following formula:

$$S = \frac{g_1(-1)}{r - 1} = \frac{g_1}{1 - r}$$

For example, consider the infinite geometric series below:

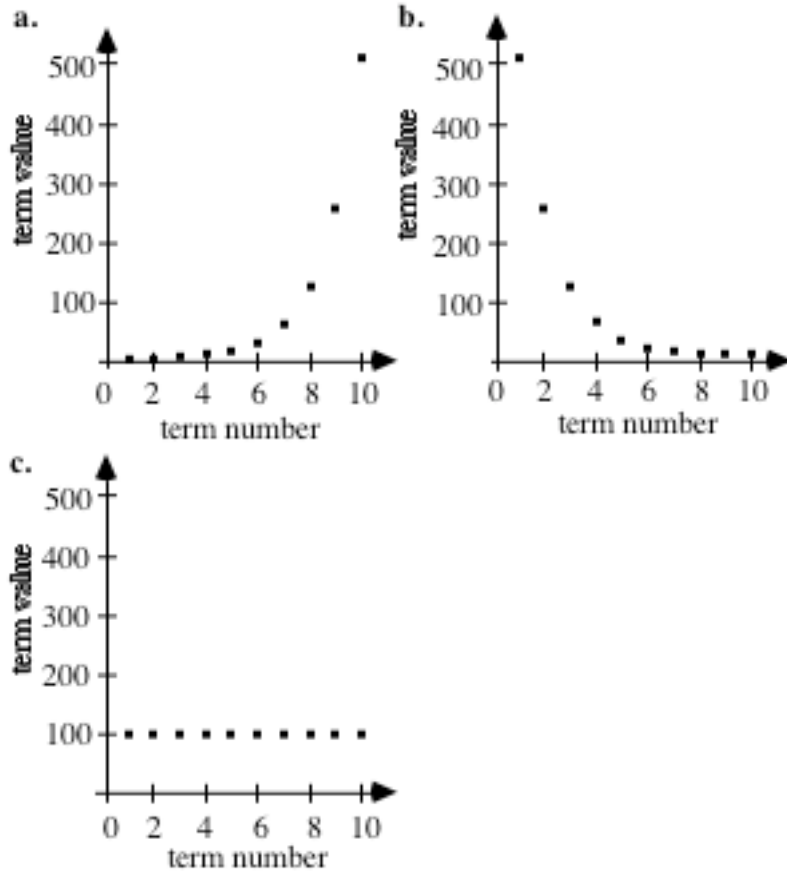
$$27 + 9 + 3 + 1 + \dots$$

In this case, $g_1 = 27$ and $r = 1/3$. The sum of the terms can be found as follows:

$$S = \frac{27}{1 - (1/3)} = \frac{27}{2/3} = 40.5$$

Assignment

- 4.1. Each graph in Parts a–c below shows a scatterplot of the first 10 terms of a sequence. As the number of terms increases indefinitely, which of the sequences appears to have a limit? Explain your responses.



- 4.2 Using a spreadsheet, enter the sequence 1, 2, 3, 4, ..., 50 in column A. In column B, enter the geometric sequence with a first term of 1 and a common ratio of 10.
- Suppose that the values in column A represent the term numbers of a sequence while the values in column B represent the term values. As the term number increases indefinitely, do the values in column B appear to approach a limit?
 - Determine the ratio of each term in column B to the preceding term. Enter these values in column C.
 - Suppose that the values in column A represent the term numbers of a sequence while the values in column C represent the term values. As the term number increases indefinitely, do the values in column C appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.01 of the limit.

d. Determine the reciprocal of each term in column B and enter these values in column D.

e. Suppose that the values in column A represent the term numbers of a sequence while the values in column D represent the term values. As the term number increases indefinitely, do the values in column D appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.005 of the limit.

f. Determine the ratio of each term in column D to the preceding term. Enter these values in column E.

g. Suppose that the values in column A represent the term numbers of a sequence while the values in column E represent the term values. As the term number increases indefinitely, do the values in column E appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.001 of the limit.

h. 1. Create a scatterplot of the term value versus the term number for the first six terms of the sequence in column B.

2. Create a scatterplot of the term value versus the term number for the first six terms of the sequence in column D.

3. Describe the differences you observe in the two scatterplots.

4.3 Repeat Problem 4.2 using a geometric sequence with a first term of -1 and a common ratio of 10 in column B.

4.4 Use a geometry utility to construct a square. Connect the midpoints of the sides to form a quadrilateral. Repeat the process two more times.

a. Justify that your original figure is a square.

b. What type of quadrilateral is each of the other three figures in your drawing? Explain your response.

c. Find the areas and perimeters of the four quadrilaterals.

d. 1. Calculate the ratio of the area of each quadrilateral to the area of the next larger one. What pattern do you observe in these ratios?

2. In the exploration in this activity, the ratio of the areas of consecutive triangles was 1:4. How does this ratio compare with the ratio of the areas of consecutive squares?

e. The ratio of the perimeters of consecutive triangles in the exploration was 1:2. How does this ratio compare with the ratio of the perimeters of consecutive squares?

f. 1. If you continued to construct quadrilaterals as described above, what would happen to the sequence of areas?

2. What would happen to the sequence of perimeters?

- 4.5** Determine the sum of the terms of the infinite geometric sequence below:

$$98, 19.6, 3.92, \dots$$

- 4.6** Consider the intervals described by the following expression, where n is the term number:

$$\left[3, 3 + \frac{1}{n}\right]$$

- Use this expression to write the first four terms of the sequence.
- As n increases indefinitely, does the sequence appear to approach a limit? Explain your response.

* * * * *

- 4.7** Consider the following infinite geometric sequence:

$$\frac{6}{10}, \frac{6}{100}, \frac{6}{1000}, \frac{6}{10,000}, \dots$$

- What is the common ratio?
- What does the limit of the sequence appear to be?
- After how many terms of the sequence are all successive terms within 0.001 of the limit?
- Find the sums $S_1, S_2, S_3, \dots, S_6$ and write them as a sequence.
Hint: $S_1 = 6/10$ and

$$S_2 = \frac{6}{10} + \frac{6}{100} = \frac{66}{100}$$

- If the sequence of sums continues indefinitely, what limit does it appear to approach?
- 4.8** While recovering from surgery, a patient takes a 250-mg pain relief tablet every 12 hours for 2 weeks. At the end of 12 hours, 10% of the medication remains in the body.
- Write a sequence that describes the amount of medication in the body right after taking the first tablet, the second tablet, the third tablet, and the fourth tablet.
 - How much medication is there in the body on the morning of the second day (after the patient takes the third tablet)?
 - What appears to be the limit of the amount of medication in the body?
 - On what day does the amount of medication become close to this limit?

- 4.9** Express $5/11$ as an infinite series.
- 4.10**
- a.** Write the first four terms of the infinite geometric sequence in which $g_1 = 0.27$ and $r = 0.01$.
 - b.** What is the sum of the infinite geometric series formed by the sequence in Part **a**?

* * * * *

Research Project

The idea of figures inside figures has intrigued many artists and mathematicians. To explore this idea in more detail, choose either Part **a** or **b** below.

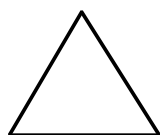
- a.** Waclaw Sierpinski, a Polish mathematician, experimented with triangles like those described in the exploration in Activity **4**. Starting with a large triangle, he created a new triangle by connecting the midpoints of the sides of the original triangle. Sierpinski then removed the interior triangle and created new triangles using the midpoints of the sides of the three remaining triangles. He repeated this process over and over, removing the interior triangle each time, and examining the areas and the perimeters of all the remaining triangles.

Explore Sierpinski's triangle for yourself using a geometry utility, a graphing calculator, or graph paper.
 - b.** Create a rectangle that is not a square. Construct consecutive quadrilaterals within the rectangle by connecting the midpoints of the sides. Explore the patterns created by the ratios of consecutive areas and consecutive perimeters of the quadrilaterals.
-

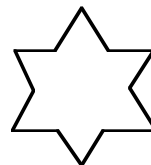
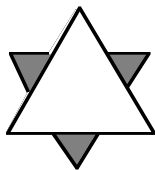
Summary Assessment

In 1906, Swedish mathematician Helge von Koch introduced a geometric figure that became known as the Koch curve. The Koch curve was later manipulated to create the Koch snowflake. The snowflake can be created with the following steps.

- Begin with an equilateral triangle. This is stage 1.
- Divide each edge of the figure into three equal segments.
- Use the middle segment of each edge to create an equilateral triangle, as shown in the diagram below. The resulting figure is stage 2.

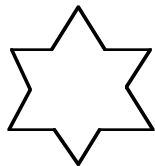


stage 1

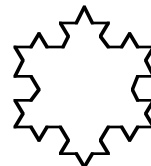
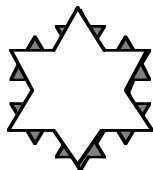


stage 2

- Continue dividing each edge of the figure into three equal segments and using the middle segment of each edge to create an equilateral triangle, as shown in the diagram below. The resulting figure is stage 3.



stage 2



stage 3

- To complete another stage, this process is repeated. When the process continues for an infinite number of stages, the figure becomes the Koch snowflake.
1. Assuming that the area of the original equilateral triangle is 1 unit^2 , determine the area of the snowflake at each of the following stages:
 - a. stage 1
 - b. stage 2
 - c. stage 15
 2. If the process described above is continued, what does the area of the snowflake appear to approach? Explain your response.

Module Summary

- A **sequence** is an ordered list. Each item in the list is a **term** of the sequence. A **finite sequence** has a specific number of terms.
- A **recursive formula** for a sequence identifies the initial term (or terms), then defines all other terms using the previous term(s).
- An **explicit formula** for a sequence defines any term based on its term number.
- In an **arithmetic sequence**, every term after the first is formed by adding a constant value, the **common difference**, to the preceding term. The recursive formula for an arithmetic sequence, where d is the common difference, n is the term number, and a_n is the n th term, is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, n > 1 \end{cases}$$

The explicit formula for such a sequence is $a_n = a_1 + d(n - 1)$.

- In a **geometric sequence**, every term after the first is formed by multiplying the preceding term by a constant value, the **common ratio**. The recursive formula for a geometric sequence, where r is the common ratio ($r \neq 0$), n is the term number, and g_n is the n th term, is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = rg_{n-1}, n > 1 \end{cases}$$

The explicit formula for such a sequence is $g_n = g_1 r^{n-1}$.

- The indicated sum of the terms of a finite sequence is a **finite series**. A finite series of n terms, denoted S_n , may be written in expanded form as follows:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

where a_n represents the n th term of the sequence.

- The sum of the terms of an arithmetic sequence is an **arithmetic series**.
- The sum of the terms of a finite arithmetic sequence with n terms and a common difference d can be found using the following formula:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the sequence and a_n is the n th term.

- The indicated sum of the terms of a geometric sequence is a **geometric series**.
- The sum of the terms of a finite geometric sequence with n terms and a common ratio r can be found using the following formula:

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

where g_1 is the first term of the sequence and $r \neq 1$.

- In an **infinite sequence**, every term in the sequence has a successor. Infinite sequences do not have a finite number of terms, but continue indefinitely.
- The **limit** of a sequence, $k_1, k_2, k_3, \dots, k_n, \dots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L .
- The sum of the terms of an infinite geometric sequence in which $-1 < r < 1$ can be found using the following formula:

$$S = \frac{g_1(-1)}{r - 1} = \frac{g_1}{1 - r}$$

Selected References

- Bennett, Jr., A. B. "Visualizing the Geometric Series." *Mathematics Teacher* 82 (February 1989): 130–36.
- Buchanan, O. L. *Limits: A Transition to Calculus*. Boston: Houghton Mifflin, 1985.
- Garland, T. H. *Fascinating Fibonacci: Mystery and Magic in Numbers*. Palo Alto: Dale Seymour Publications, 1987.
- Huntley, H. E. *The Divine Proportion: A Study in Mathematical Beauty*. New York: Dover Publications, 1970.
- Peitgen, H.-O., H. Jürgens, D. Saupe, E. Maletsky, T. Perciante, and L. Yunker. *Fractals for the Classroom*. New York: Springer-Verlag, 1991.

Algorithmic Thinking



What do baking a cake, programming a VCR, and adding a pair of two-digit numbers have in common? In this module, you explore how algorithms affect everything from cooking to computers.

Wendy Driscoll • Darlene Pugh • Todd Robins



© 1996-2019 by Montana Council of Teachers of Mathematics. Available under the terms and conditions of the Creative Commons Attribution NonCommercial-ShareAlike (CC BY-NC-SA) 4.0 License (<https://creativecommons.org/licenses/by-nc-sa/4.0/>)

Algorithmic Thinking

Introduction

Computers come in many shapes and in many forms, from high-tech factory robots to children's toys. In the 21st century, computers will have new applications and new abilities beyond the comprehension of today's users.

The only sure thing in this rapidly changing environment is the need for clever and competent programmers. Before a computer can perform any task, it must be programmed. Skilled programmers dedicate many working hours to crafting clear directions in a coded language that machines can understand. Each set of directions is written in the form of an **algorithm**, a step-by-step process for completing a task.

In this module, you will plan and write algorithms, represent them graphically using flowcharts, and explore some of their applications. During this process, you will also examine some other aspects of mathematics that have been important since ancient times.

Historical Note

The word *algorithm* is derived from the name of Mohammed ibn-Musa al-Khwarizmi, an Arabian scholar and mathematician who died around 850 A.D. When his influential book on the use of Hindu numerals (based on the work of an earlier Indian mathematician, Brahmagupta) appeared in Latin translation, many readers mistakenly attributed the number system to an Arabic source. (As a result, the numerals we use are commonly referred to as Hindu-Arabic numerals.)

The notation described in the translation came to be known as that of al-Khwarizmi, or *algorismi*. This latter word later evolved to *algorism* and, eventually, to *algorithm*.

Activity 1

The ability to write clear, concise directions is an important skill in industries ranging from food products to fiber optics. Consumers often form their most lasting impressions of a product not through the endless hours of advertising, but through those first 15 minutes with an instruction manual. In this activity, you take the initial steps in developing your algorithmic thinking skills by investigating some familiar types of algorithms.

Exploration

- a. Draw a simple picture that includes at least three geometric shapes. Do not allow anyone else to see the picture.
- b. Write a set of instructions that a person could use to duplicate the picture.
- c. Trade instructions with a partner.
- d. Perform the instructions as written and record how long it takes you to complete them.
- e. Compare your original picture with your partner's drawing. Make any revisions necessary to your set of instructions.
- f. Exchange instructions with another partner and repeat Part d. Note any difference in the time required to recreate the drawing.

Discussion

- a. How did the picture your partner created using your instructions compare with the original picture?
- b. Which picture took the least amount of time to reproduce?
- c. Which set of instructions resulted in the most accurate reproduction?
- d. Which set of instructions were the most efficient? Why? Explain your definition of *efficient*.
- e. Is it possible for more than one set of instructions to generate the same picture?
- f. How could your instructions have been made more clear?
- g. How would you change your instructions if the reader was an elementary student?

Mathematics Note

An **algorithm** is a step-by-step process for completing a task. It typically ends in a finite number of steps. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.

For example, the following algorithm could be used by bus drivers who must cross railroad tracks.

1. Stop between 50 and 100 feet from the railroad tracks.
2. Open the bus door.
3. Look and listen for a train.
4. If there is no train approaching, close the door and proceed.
5. If there is a train approaching, close the door and wait for the train to pass.
6. After the train passes, return to step 2.

Assignment

- 1.1 Consider the two sets of instructions below.

Washing Clothes	Preparing a Bath
Put clothes in washing machine.	Turn on the water in the tub.
Add soap.	Make sure the temperature is warm, but not too hot.
Turn on the machine.	Lay out clean towels.

Complete Parts **a–c** for each set of instructions.

- Write a short story about someone who followed the instructions explicitly.
 - Explain whether or not the instructions are successful algorithms.
 - Rewrite the set of instructions for washing clothes so that they are appropriate for elementary students.
- 1.2 Most recipes provide a list of ingredients and a set of directions for preparing a particular dish or meal. The following recipe is from a collection gathered by the Indian Studies Program in Harlem, Montana.

Raised Fry Bread
1 to $1\frac{1}{4}$ cup warm milk
1 package dry yeast
2 tablespoons soft shortening
2 teaspoons salt
2 tablespoons sugar
3 cups sifted flour
Dissolve yeast in milk. Add shortening, salt, sugar, and one-fourth of the flour. Beat until smooth, scraping sides and bottom of the bowl frequently. Add remaining flour and blend until smooth. Let rise in warm place about 30 minutes. Punch down and knead. Let rise again. Break into doughnut-size pieces and shape flat, cutting a hole in the center of the dough. Deep fry in hot grease, turning the bread until both sides are golden brown. Drain on a paper towel and serve while warm.

- Most recipes are designed so that any two people who follow it will obtain comparable results. Do you think this recipe will always produce the same results? Explain your response.
- In an algorithm, a **decision point** is a place where a judgment must be made. Are there decision points in the fry-bread recipe that influence later instructions or results? If so, identify them and describe some possible results of these decisions.

1.3 A **prime number** is a natural number greater than 1 with exactly two divisors. The ancient Greeks apparently knew about prime numbers since the time of the Pythagoreans—and specifically since the time of Philolaus (died ca. 390 B.C.).

- a. List the first 20 prime numbers.
- b. Write a set of instructions for determining if a natural number is prime.

1.4 a. Consider the purchase of an item which costs less than \$1.00 with a dollar bill. Beginning with the two steps described below, create an algorithm that will determine the appropriate number and type of coins to be given in change. The total number of coins should be kept to a minimum and the types of coins should include pennies, nickels, dimes, and quarters.

Step 1: Subtract the cost of the item from 100 cents.

Step 2: Divide the difference from step 1 by 25.

- b. Use an example to show that the algorithm works.
- c. Describe what basic knowledge might be required in order to understand the algorithm in Part a.

* * * * *

1.5 Some steps in an algorithm may include algebraic notation. For example, the following algorithm converts temperature in degrees Fahrenheit to temperature in degrees Celsius.

- Let F represent the temperature in degrees Fahrenheit and C the temperature in degrees Celsius.
- Substitute the value of F into the following equation:

$$C = \frac{5}{9}(F - 32)$$

- Solve the equation for C .
- a. Use this algorithm to determine the temperature in degrees Celsius that corresponds with a temperature of 99° F.
 - b. Rewrite the algorithm so that it converts temperature in degrees Celsius to temperature in degrees Fahrenheit.

1.6 A small programmable robot can travel a specific path if given the correct algorithm. For example, the algorithm “Triangle(x)” shown below instructs the robot to trace an equilateral triangle whose sides are x meters long.

- Forward(x)
- Right(120°)
- Forward(x)
- Right(120°)
- Forward(x)
- Right(120°)

- a. What path would the robot trace if given the algorithm “Triangle(10)”?
- b. Write an algorithm that will allow the robot to trace a square.
- c. Write an algorithm that allows the robot to trace a path of your choice. Have another person complete the algorithm by drawing the path the robot will follow.

* * * * *

Activity 2

In Activity 1, you examined sets of instructions that form algorithms. Algorithms can also be represented graphically as **flowcharts**.

Mathematics Note

A **flowchart** can be used to represent the steps of an algorithm. As seen in Figure 1, it may consist of input and output boxes, data processing boxes, and decision boxes. Ovals are used for “start” and “stop” commands.

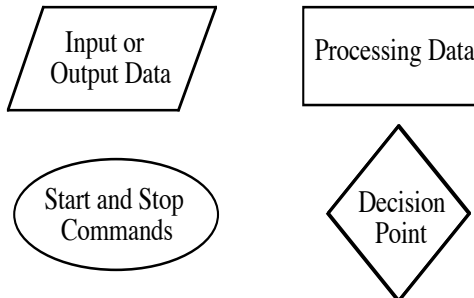


Figure 1: Flowchart symbols

To illustrate a step-by-step sequence of instructions, flowchart symbols are usually connected by arrows. The flowchart in Figure 2, for example, shows an algorithm for finding two matching socks in a drawer that contains several pairs of unmatched socks.

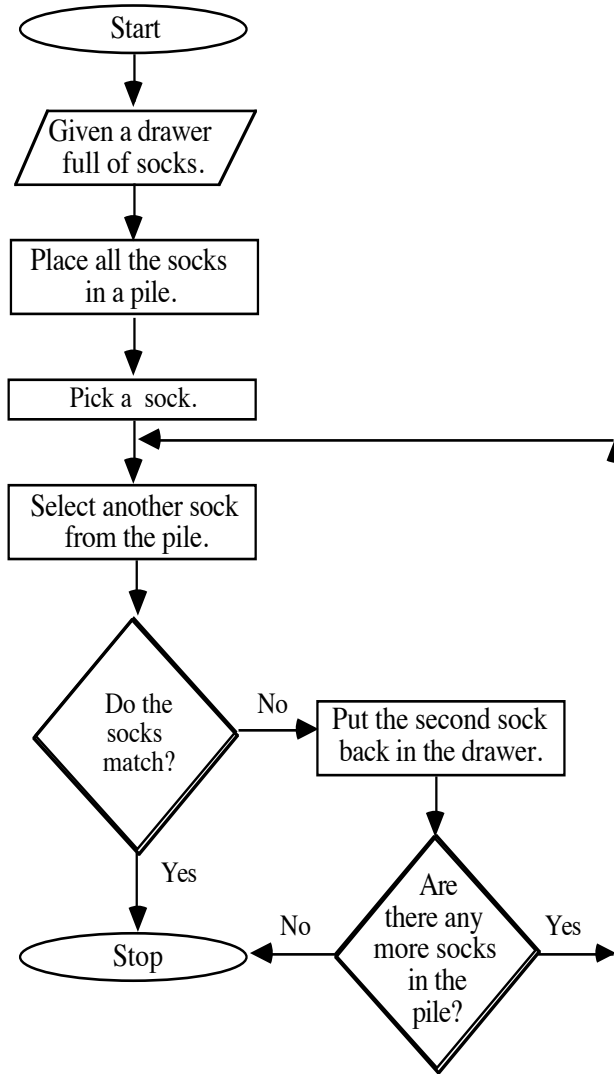


Figure 2: Flowchart for finding a pair of matching socks

Exploration 1

A flowchart must follow the same logical process as the algorithm it represents. In this exploration, you organize the parts of a flowchart into a logical order.

- a. Obtain a copy of the flowchart template for subtracting two natural numbers from your teacher. Cut out each of the command boxes.
- b. Organize the command boxes so that a person could use the algorithm to subtract two natural numbers. Draw arrows to show the appropriate directions and include “yes” and “no” answers for decision boxes.
- c. Exchange your flowchart with another student. Follow the commands as indicated to verify that the flowchart is properly organized.

Discussion 1

- a. Compare your flowchart with those of others in the class. Are all the flowcharts that correctly model the subtraction algorithm identical?
- b. Describe the method you used to organize the flowchart.
- c. What characteristics make one flowchart easier to read than another?

Exploration 2

In the following exploration, you create a flowchart for addition.

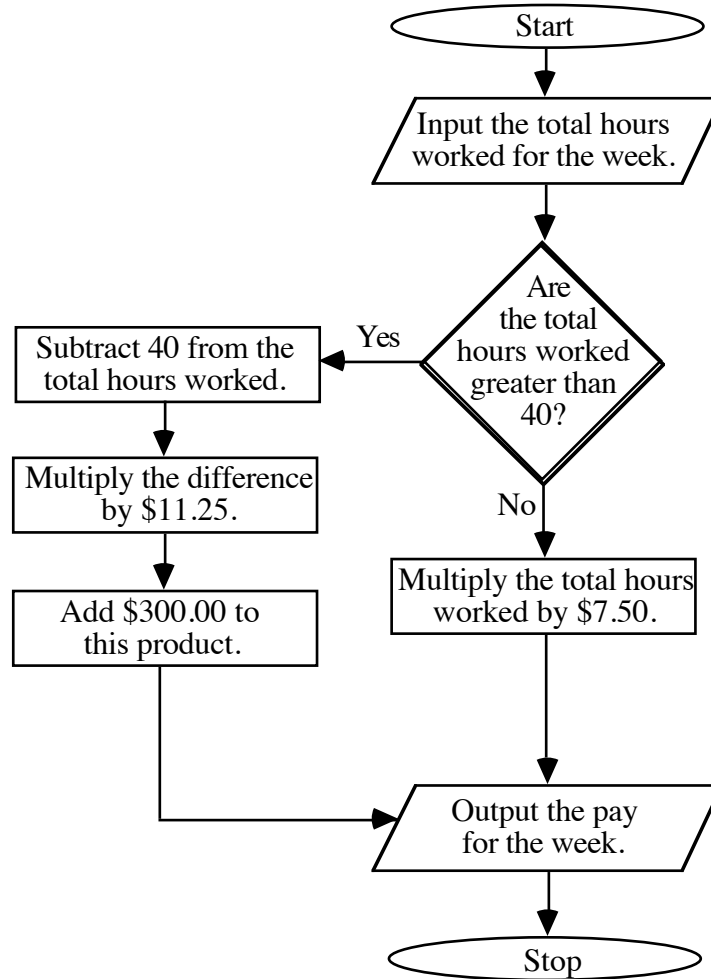
- a. Select a pair of two-digit numbers, such as 84 and 39. Working with pencil and paper, find the sum of these numbers using the algorithm you learned in elementary school.
- b. Write a set of instructions that someone else could use to add these two numbers.
- c. Draw a flowchart to illustrate the algorithm you developed in Part **b**.
- d. Exchange flowcharts with a partner. Follow your partner’s flowchart exactly as written to find the sum of a pair of two-digit numbers. When you have finished, return the flowchart and sum to the original writer.
- e. Revise your flowchart so that it may be used to add any pair of two-digit numbers.

Discussion 2

- a. How does your flowchart compare with those of others in the class?
- b.
 1. Which flowchart from Exploration 2 seems to be the best?
 2. What criteria did you use to make this decision?
- c. Describe how your flowchart could be adapted for finding the sum of a pair of three-digit numbers.

Assignment

- 2.1 To calculate an employee's weekly wages, Company A uses the number of hours worked during the week, along with the algorithm shown in the flowchart below:



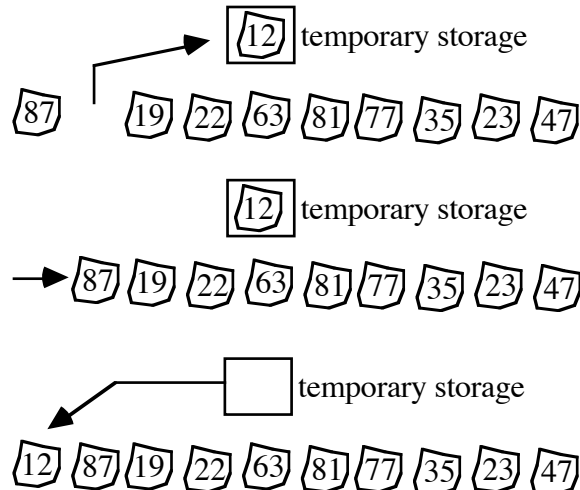
- a. Use the flowchart to calculate the weekly pay for an employee who works each of the following numbers of hours:
- 46 hours
 - 36 hours
- b. Company B calculates its employee's wages on a daily rate. The company pays \$7.50 per hour for 8 or fewer hours, and an additional \$3.75 per hour for any time over 8 hours.
- Create a flowchart that could be used to determine an employee's daily wage for Company B.
- c. For which company would you prefer to work? To support your response, use an example of the wages paid by each company for a specific number of hours worked.

- 2.2 Design a flowchart to determine if any natural number is prime.
- 2.3 In one version of the game of Nim, two players take turns adding either 1 or 2 to a sum until a total of 21 is reached. The rules for this game are listed below:
- The first player starts with either the number 1 or the number 2.
 - The second player adds either 1 or 2 to the first player's total.
 - Players continue to take turns adding 1 or 2 to the previous player's total until one player reaches exactly 21. That player wins the game.
- a. Play the game several times.
 - b. Describe an algorithm for winning at this game.
 - c. Do you think a computer could be programmed to win at this game? Explain your response.
- 2.4 Sorting algorithms are used to place items such as addresses, numbers, and names in order. Computer scientists have developed many algorithms that can sort quickly.

For example, suppose that you want to sort the 10 numbers in the following set from least to greatest: {87, 12, 19, 22, 63, 81, 77, 35, 23, 47}. Write each number on a slip of paper and arrange them as shown in the following diagram:



The **insertion sort algorithm** starts by removing the second number from the list, leaving an empty position. This number is placed in a temporary storage location and compared with the first number. If the first number is greater, it is moved to the right, into the empty position. The number in temporary storage is then inserted back into the list wherever the open position is located, as shown in the diagram below.



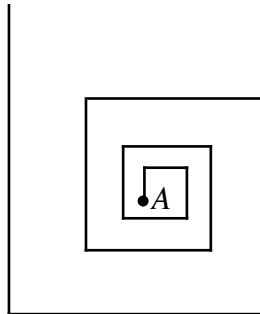
At this point in the sort, one comparison has been made. Next, the third number is moved into the temporary storage location and compared with the second number. If the second number is greater, it is moved to the right. If the first number is greater, it is also moved to the right. The number in storage is then moved into the empty position. Two more comparisons have been made at this point in the sort, for a total of three comparisons.

- a. Complete the insert sort algorithm for all 10 numbers, keeping track of the number of comparisons made at each step in the sort. How many comparisons were needed to completely sort the list?
- b. Draw a flowchart to model the insertion sort algorithm.
- c. Two other sorting algorithms used by computer programmers are the bubble sort and the merge sort. The bubble sort algorithm requires 45 comparisons to sort this same list of numbers, while the merge sort algorithm takes 24 comparisons. Which algorithm appears to be the most efficient: the insertion, bubble, or merge sort?

2.5 Draw a flowchart for finding the sum of any pair of three-digit numbers.

* * * * *

2.6 In Problem 1.6, you wrote an algorithm that instructed a programmable robot to trace a path. Create a flowchart that will allow this robot to trace a path similar to the one shown in the diagram below, beginning at point A.



2.7 The *Elements* of the Greek mathematician Euclid (ca. 300 B.C.) consist of 13 books on topics from elementary geometry to number theory. Book VII contains what has come to be known as the “Euclidean algorithm”—a scheme for finding the greatest common divisor of two natural numbers.

In the Euclidean algorithm, the smaller number s is subtracted from the greater number g . If the difference d is greater than s , then s is subtracted from d , creating a new d . This process is repeated (subtracting s from d) until d is less than s .

If d is a divisor of the original two numbers, then the process is complete and the greatest divisor is found. If d is not a divisor, then g is replaced with s , s is replaced with d , and the entire process is repeated until a divisor is found.

For example, the Euclidean algorithm can be used to find the greatest common divisor of 15 and 6 as shown below.

$$\begin{array}{r} 15 \quad 9 \\ -6 \quad -6 \\ \hline 9 \quad 3 \end{array}$$

Since 3 is a divisor of 15 and 6, it is the greatest common divisor of the two numbers.

Similarly, the Euclidean algorithm can be used to find the greatest common divisor of 24 and 9 as shown below:

$$\begin{array}{r} 24 \quad 15 \\ -9 \quad -9 \\ \hline 15 \quad 6 \end{array}$$

The difference found after the second subtraction is less than 9. However, since 6 is not a divisor of 24 and 9, the algorithm is continued using 6 and 9.

$$\begin{array}{r} 9 \\ -6 \\ \hline 3 \end{array}$$

Since 3 is a divisor of 24 and 9, it is the greatest common divisor of the two numbers.

- a. Use the Euclidean algorithm to find the greatest common divisor of each of the following pairs of numbers:
 1. 54 and 24
 2. 51 and 9
 3. 12 and 5
- b. Create a flowchart to model the Euclidean algorithm.
- c. Do you think the Euclidean algorithm could be modified to find the greatest common divisor of any two integers? Explain your response.

- 2.8 a. The process of division can be defined as “repeated subtraction.” Rewrite the Euclidean algorithm described in Problem 2.7 so that it uses division, rather than repeated subtraction, to find the greatest common divisor.
- b. Create a flowchart to model this algorithm.

* * * * *

Research Project

Working with a partner, develop a generalized version of the game of Nim described in Problem 2.3. For example, you might consider a game in which players add the digits 1 or 3 to reach a total of 36. In your generalized version of the game, let T be the target number and the digits 1 and n the numbers to be added. Your report should include precise directions for playing the game, rules of play, and a description of any algorithms for winning the game.

Activity 3

The algorithms you explored in previous activities used concepts from number theory. In the Level 1 module “Going in Circuits,” you investigated some algorithms derived from graph theory. In this activity, you examine an algorithm that involves geometry.

Exploration

You have investigated several methods of writing algorithms: step-by-step instructions, flowcharts, and formulas. In this exploration, the algorithms consist of a set of information and a procedure to perform on the given information.

- a. Use the algorithm in Table 1 to create a geometric shape.

Table 1: Algorithm for creating a geometric shape

Given:	A triangle.
Procedure:	<ol style="list-style-type: none"> 1. Locate the midpoints of each side. 2. Connect the midpoints consecutively to create a new inscribed polygon. 3. Repeat the procedure on the new inscribed polygon.

Mathematics Note

A **recursive definition** consists of some specific information, a procedure that is performed on the given information, and continuing steps that generate new information using the same procedure on previously generated information. The process of using a recursive definition is known as **recursion**.

For example, the following formula uses recursion to describe the arithmetic sequence: 1, 8, 15, 22, 29,

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 7, n > 1 \end{cases}$$

In this case, the first term, $a_1 = 1$, represents the given information. The process of repeatedly adding 7 to the previous term to obtain a new term is the procedure.

- b. Write an algorithm that uses recursion to create a sequence of inscribed squares like the one shown in Figure 3. Each square is a dilation by a scale factor of 0.8 of the next larger square. The center of dilation is the center of the largest square.

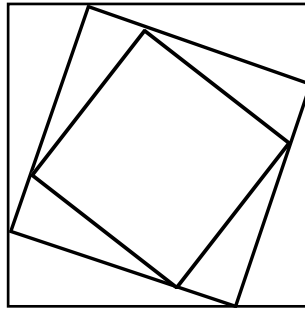


Figure 3: Inscribed squares

- c. Use your algorithm to continue the pattern until your drawing includes eight inscribed squares.

Discussion

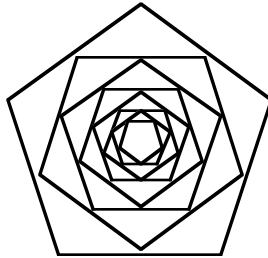
- a. Which step in the formula for an arithmetic sequence given in the mathematics note creates recursion?
- b. Which step in the algorithm in Table 1 creates recursion?
- c. Since the algorithm described in Table 1 has no terminating step, the recursion is indefinite. If the algorithm continues indefinitely, what is the limit of the perimeters of the inscribed polygons? Explain your response.
- d. 1. How could you change the algorithm in Table 1 so that it creates exactly 10 inscribed triangles?
2. How could you change the algorithm to create a geometric figure with inscribed quadrilaterals?

Assignment

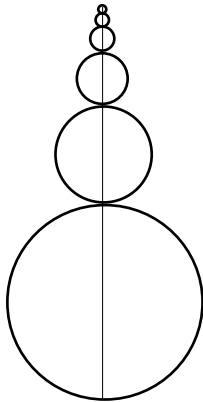
- 3.1** The following algorithm uses recursion to create a geometric sequence.

$$\begin{cases} g_1 = 3 \\ g_n = 4g_{n-1}, n > 1 \end{cases}$$

- Use the algorithm to write the first five terms of the sequence.
 - What part of the algorithm defines the given information?
 - What part of the algorithm defines the procedure?
- 3.2** Design an algorithm that could be used to recreate the diagram below. Each pentagon is a dilation by a scale factor of 0.8 of the next larger pentagon. The center of dilation is the center of the largest pentagon.



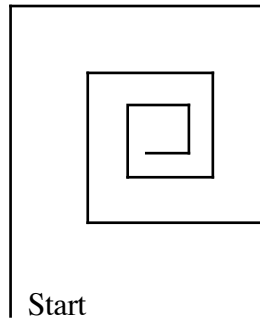
- 3.3** Design an algorithm that could be used to recreate the diagram below.



- 3.4** Write an algorithm that creates a geometric sequence.

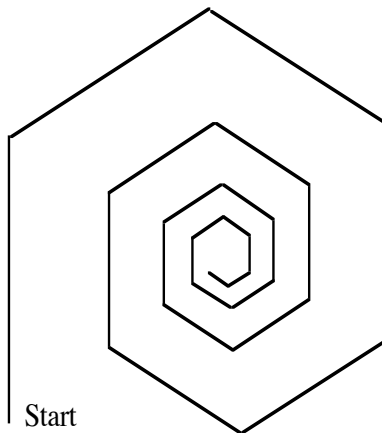
* * * * *

3.5 Consider the geometric figure shown in the following diagram.

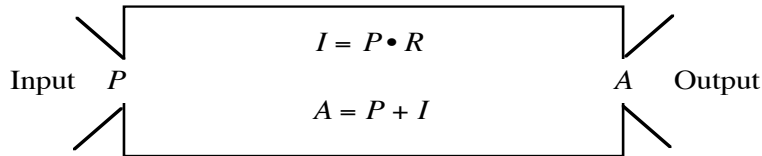


Using the algorithm described below, “Squiral(x),” a programmable robot can trace a similar figure.

- Squiral(x)
 - Forward(x)
 - Right(90°)
 - Squiral($0.8x$)
- a. If x is 5 m, how long is the first side of the figure? How long is the second side?
 - b. According to the algorithm, how many times will the robot go forward and turn?
 - c. Write a recursive algorithm that will allow the robot to trace the following figure.



3.6 Interest is the amount earned on invested money. A bank offers an investment account that pays interest at an annual rate of 8%. To demonstrate the potential earnings of this account, a bank officer uses the “interest machine” shown below. In this case, I represents interest, P represents the initial investment, R represents the annual interest rate, and A represents the account balance after 1 year. The output at the end of each year becomes the input for the following year.

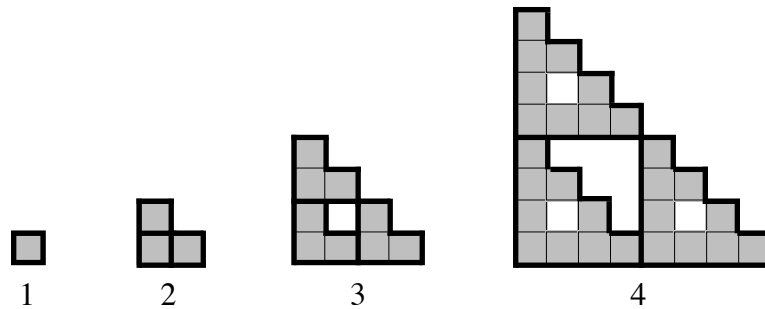


- a. Given an initial investment of \$500 in this account, determine the account balance at the end of each of the next three years.
- b. Is the algorithm used by the “interest machine” recursive? Explain your response.
- c.
 1. If no withdrawals are made from the account, what will it be worth at the end of the 20th year?
 2. What is the total amount of interest earned during this period?

* * * * *

Summary Assessment

1. a. The diagram below shows a sequence of figures. Term 2 can be created with three copies of term 1, term 3 can be created with three copies of term 2, and so on.

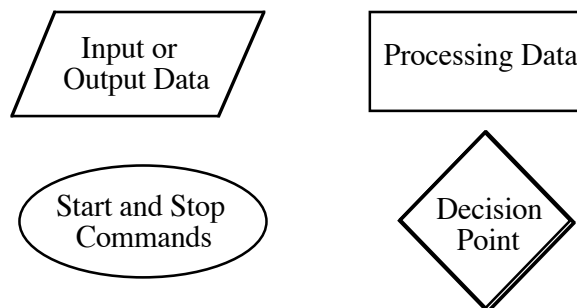


Write an algorithm that allows you to create each figure using three copies of the previous figure.

- b. Create a flowchart to illustrate the algorithm.
- c. Is the algorithm recursive? Explain your response.
- d. Use your algorithm to create the next figure in the sequence.
2. a. Create your own sequence of geometric figures.
- b. Write an algorithm that another person could use to recreate the same sequence.

Module Summary

- An **algorithm** is a step-by-step process for completing a task. It typically ends in a finite number of steps. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.
- A **decision point** in an algorithm is a place where a judgment must be made.
- A **prime number** is a natural number greater than 1 with exactly two divisors.
- A **flowchart** can be used to represent the steps of an algorithm. As seen in the diagram below, flowchart symbols may consist of input and output boxes, data processing boxes, and decision boxes. Ovals are used for “start” and “stop” commands. To illustrate a step-by-step sequence of instructions, these symbols are usually connected by arrows.



- A **recursive definition** consists of some specific information, a procedure that is performed on the given information, and continuing steps that generate new information using the same procedure on previously generated information. The process of using a recursive definition is known as **recursion**.

Selected References

- Coes, L., III. "Building Fractal Models with Manipulatives." *Mathematics Teacher* 86 (November 1993): 646–51.
- Guy, R. *Fair Game*. Arlington, MA: Consortium for Mathematics and Its Applications (COMAP), 1989.
- Graham, N. *Introduction to Computer Science*. St. Paul, MN: West Publishing Co., 1985.
- Harel, D. *Algorithmics*. Workingham, England: Addison-Wesley, 1987.
- Harel, D. *The Science of Computing*. Reading, MA: Addison-Wesley, 1989.
- Kenney, M. J., and C. R. Hirsch, eds. *Discrete Mathematics Across the Curriculum K-12*. Reston, VA: National Council of Teachers of Mathematics (NCTM), 1991.
- Kolman, B., and R. C. Busby. *Discrete Mathematical Structures for Computer Science*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
- Kraus, W. H. "Don't Give Up!" *Mathematics Teacher* 86 (February 1993): 111–12.
- Merris, R. *Introduction to Computer Mathematics*. Rockville, MD: Computer Science Press, 1985.
- Spencer, D. D. *Computers in Number Theory*. Rockville, MD: Computer Science Press, 1982.
- Stiffarm, N. Unpublished set of recipes. Indian Studies Program. Harlem, MT. 1993.

Level 2 Index

- Absolute value of residuals, 353, 370
- Addition of matrices, 10, 26
- Algebra, geometric, 42
- Algorithm, 405, 406, 422
- al-Khwarizmi, Mohammed ibn-Musa, 405
- Allele, 126, 144
- Alternate exterior angles, 326, 347
- Alternate interior angles, 326, 347
- Angle
 - alternate exterior, 326, 347
 - alternate interior, 326, 347
 - central, 150, 177
 - corresponding, 326, 347
 - dihedral, 47, 56
 - vertical, 325, 347
- Angle-Angle-Angle (AAA) Property, 34, 63
- Antigen, 134
- Apothem, 152
- Apportionment, 99, 116
 - by equal proportions, 111, 116
 - Hamilton method, 103, 116
 - Huntington method, 116
 - Huntington-Hill method, 105
 - Jefferson method, 101, 116
 - Webster method, 103, 116
- Arc, 164, 177
 - length of, 164, 177
 - major, 164, 177
 - measure, 164, 177
 - minor, 164, 177
- Arithmetic sequence, 376, 400
- Area
 - lateral surface, 155
 - sector, 164, 177
 - sphere, 177
 - total surface, 155
- Associative property of multiplication of matrices, 23

- Basal metabolic rate, 369
- Bias, 295, 313
- Boundary lines, 187, 188
- Box-and-whisker plot, 80, 93

- Carbon dating, 271

- Census, 295, 313
- Center of
 - dilation, 213, 241
 - rotation, 228, 242
 - sphere, 171, 177
- Central angle, 150, 177
- Chord, 332, 347
- Chromosome, 126, 144
- Circle(s), concentric, 331
- Circumscribe, 151
- Clinometer, 53
- Coefficient matrix, 195, 205
- Column matrix, 17
- Complement, 124, 144
- Components
 - simple, 3
 - vector, 241
- Composite
 - product, 4
 - transformation, 237, 242
- Concentric circles, 331
- Conditional probability, 260, 266
- Confidence statement, 309, 314
- Constant matrix, 195, 205
- Constraint, 184, 205
- Converse, 40, 63
- Corner
 - point, 184, 205
 - principle, 189, 205
- Corresponding angles, 326, 347
- Cosine, 55, 63
- Cos^{-1} , 56
- Cross, 127, 145
- Cubit, 36

- Decay
 - radioactive, 271, 290
 - exponential, 272, 290
 - rate of, 272, 290
- Decision point, 407, 422
- Dependent events, 131, 144, 258, 266
- Deviation
 - mean absolute, 84
 - standard, 89
- Difference
 - common, 376, 400
 - relative, 105

Dihedral angle, 47, 63
 Dilation, 213, 241, 340
 center of, 213, 241
 scale factor, 213, 241
 Dimension, 6, 26
 Distance
 formula, 212, 241
 perpendicular, 320, 347
 Divisor, greatest common, 415

Element, 6, 26
 Empirical probability, 123, 144
 Equal matrices, 9, 26
 Equation, linear regression, 362, 370
 Error, margin of, 309, 314
 Euclid, 415
 Euclidean algorithm, 415
 Event(s), 123, 144
 complement of, 124, 144
 dependent, 131, 144, 258, 266
 independent, 131, 144, 258, 266
 mutually exclusive, 138, 144
 Expected value, 255, 266
 Experimental probability, 123, 144
 Explicit formula, 375, 400
 arithmetic sequence, 376, 400
 geometric sequence, 376, 400
 Exponent(s)
 negative, 278
 properties of, 277–91
 Exponential
 decay, 272, 290
 growth, 272, 290

Fair game, 255, 266
 Feasible
 regions, 184, 205
 solutions, 184, 205
 Feasibility studies, 181
 Five-point analysis, 81
 Flowchart, 409, 422
 Formula
 explicit, 375, 400
 recursive, 375, 400
 Fractions, complex, 107
 Frequency, 68, 93
 table, 68, 93

Gauss, C. F., 381
 Genes, 121, 144
 Genetics, 121, 144

Genotype, 126, 145
 Geometric
 algebra, 42
 mean, 109, 116
 models of probability, 248, 249, 266
 sequence, 376, 400
 Graph(s)
 back-to-back stem-and-leaf plot, 76, 93
 box-and-whisker plot, 80, 93
 histogram, 67, 93
 pie chart, 67, 69, 93
 requirement, 5
 stem-and-leaf plot, 76, 93
 Greatest common divisor, 415

Half-life, 284
 Height
 pyramid, 163
 slant, 163
 Heredity, 121, 144
 Histogram(s), 67, 93
 stacked, 70
 Hypotenuse, 39

Identity for matrix multiplication, 23, 195, 205
 Image, 210
 Independent events, 131, 144, 258, 266
 Interquartile range, 79, 93
 Inverse
 cosine, 56
 multiplicative, of matrix, 195, 205
 sine, 56
 tangent, 50
 Isometry, 341, 347

Koch, H. von, 399

Lateral surface
 cone, 165
 cylinder, 155, 168
 pyramid, 163
 Least-squares method, 362, 370
 Limit of sequence, 393, 401
 Line(s)
 boundary, 187, 188
 median-median, 357, 370
 of reflection, 235, 242
 of symmetry, 238, 242, 319, 347

- Linear
 - model, 353
 - programming, 186, 205
 - regression equation, 362, 370
- Margin of error, 310, 314
- Matrix, 3, 6, 26
 - addition of, 10, 26
 - coefficient, 195, 205
 - column, 17
 - constant, 195, 205
 - dilation, 218
 - equal, 9, 26
 - identity for multiplication, 195, 205
 - inverse for multiplication, 195, 205
 - multiplication, 10, 17, 26
 - by scalar, 10, 26
 - reflection, 236
 - rotation, 227
 - subtraction, 10, 26
 - total requirement, 6, 26
 - translation, 223, 241
 - variable, 195
- Mean, 74, 93
 - absolute deviation, 84, 93
 - geometric, 109, 116
- Measure(s) of central tendency, 74, 93
 - mean, 74, 93
 - median, 74, 93
 - mode, 74, 93
- Median, 74, 93
 - median line, 357, 370
- Method of equal proportions, 111, 116
- Mode, 74, 93
- Model, linear, 353
- Multiplication,
 - associative property, 23
 - commutative property, 20
 - matrix, 17, 26
 - by scalar, 7, 26
 - identity, 23
 - Zero-product property, 23
- Mutually exclusive events, 138, 144
- Net, 155, 156
- Nim, 413
- Number
 - prime, 408, 422
 - priority, 108, 111, 116
 - root of, 273, 290
- Objective function, 186, 205
- One-to-one correspondence, 210, 241
- Outlier(s), 79, 93
- Paradox, 388, 391
- Parallelogram, 329
- Parameter, 295, 313
- Perpendicular
 - distance, 320, 347
 - bisector, 321, 333
- Phenotype, 126, 145
- Pie chart, 67, 69, 93
- Pitch, 53
- Plimpton, 322
- Point(s)
 - decision, 407, 422
 - of perspective, 211
 - of tangency, 332, 347
 - summary, 357, 370
- Point-perspective drawing, 211
- Polygon(s)
 - inscribed, 149, 177
 - regular, 149
 - similar, 324
- Population, 295, 313
 - proportion, 303, 313
- Preimage, 210
- Prime number, 408, 422
- Priority numbers, 108, 111, 116
- Prism
 - lateral surface, 155
 - right, 155
 - volume, 159, 177
- Probability
 - conditional, 260, 266
 - empirical, 123, 144
 - experimental, 123, 144
 - geometric model of, 248, 249, 266
 - theoretical, 128, 144, 249, 266
- Product matrix, 17, 26
- Proportion
 - population, 303, 313
 - sample, 303, 313
- Punnett square, 127, 145
- Pythagorean
 - theorem, 39, 63
 - triple, 42
- Pyramid
 - lateral surface, 163
 - regular square, 35
 - volume, 170, 177

Quartile
 lower, 79, 93
 upper, 79, 93

Radius
 circle, 149, 177
 polygon, 149, 177

Range
 interquartile, 79, 93
 of data, 77, 93

Rate
 basal metabolic, 369
 decay, 272, 290
 growth, 272, 290

Ratio, common, 376, 400

Rectangle, 329

Recursion, 417, 422

Recursive
 definition, 417, 422
 formula, 375, 400
 arithmetic sequence, 376, 400
 geometric sequence, 376, 400

Reflection, 234, 235, 242
 matrix of, 236

Regular square pyramid, 35

Relative difference, 105

Rep tile, 325, 347

Requirement graph, 5

Residual, 353
 plot, 363, 364, 370

Rhombus, 323, 329

Root of number, 273, 290

Rotation, 227, 228, 242
 matrix, 227

Rotational symmetry, 232, 242, 327, 347

Sample, 295, 313
 proportion, 303, 313
 simple random, 300, 313
 space, 123, 144
 stratified, 300, 313
 systematic, 300, 313

Sampling
 simple random, 300, 313
 stratified, 300, 313
 systematic, 300, 313

Scalar multiplication, 10, 26

Scale factor, 213, 241

Secant, 332, 347

Sector, 164, 177
 area, 164, 177

Seqt, 47

Sequence, 375, 400
 arithmetic, 376, 400
 geometric, 376, 401
 infinite, 393, 401
 limit of, 393, 401

Series
 arithmetic, 381, 400
 finite, 381, 400
 geometric, 386, 401
 sum, 381, 382, 387, 394, 400, 401

Similar
 figures, 33, 63
 polygons, 324, 347
 triangles, 34, 63

Sine, 55, 63

Sin^{-1} , 56

Sphere, 171, 177
 center, 171, 177
 diameter, 171, 177
 radius, 171, 177
 surface area, 173, 177
 volume, 171, 177

Square, 329
 Punnett, 127, 145

Standard
 deviation, 89, 93, 306, 313
 maximum, 308, 309, 313
 population, 89, 93, 306
 divisor, 100, 116
 quota, 101, 116

Statement, confidence, 309, 314

Statistic, 295, 313

Stem-and-leaf plot, 76, 93
 back-to-back, 76, 93

Stratum, 300, 313

Substitution method, 191, 205

Subtraction
 matrix, 10, 26
 repeated, 416

Summary points, 357, 370

Symmetry
 line of, 235, 242, 319, 347
 rotational, 232, 242, 327, 347

Tangent, 49, 63
 of circle, 151, 177, 332, 347

Tan^{-1} , 50

Theorem, Pythagorean, 39
Theoretical probability, 128, 144
Total requirement matrix, 6, 26
Trait, 121
 dominant, 126, 145
 recessive, 126, 145
Transformation, 210, 241, 341, 347
 composite, 237, 242
Translation, 222, 241
 matrix, 223, 241
 of a point, 224, 241
 vector, 222, 241
Transversal, 326, 347
Trapezoid, 329
Triangle
 obtuse, 41
 right, 39
Trigonometry, 46, 54
 ratios, 63

Variable matrix, 195, 205
Vertical angles, 325, 347
Volume
 cone, 170, 177
 cylinder, 161, 177
 prism, 159, 177
 pyramid, 170, 177
 sphere, 171, 177

Whisker, 80, 93

Zero-product property, 23

