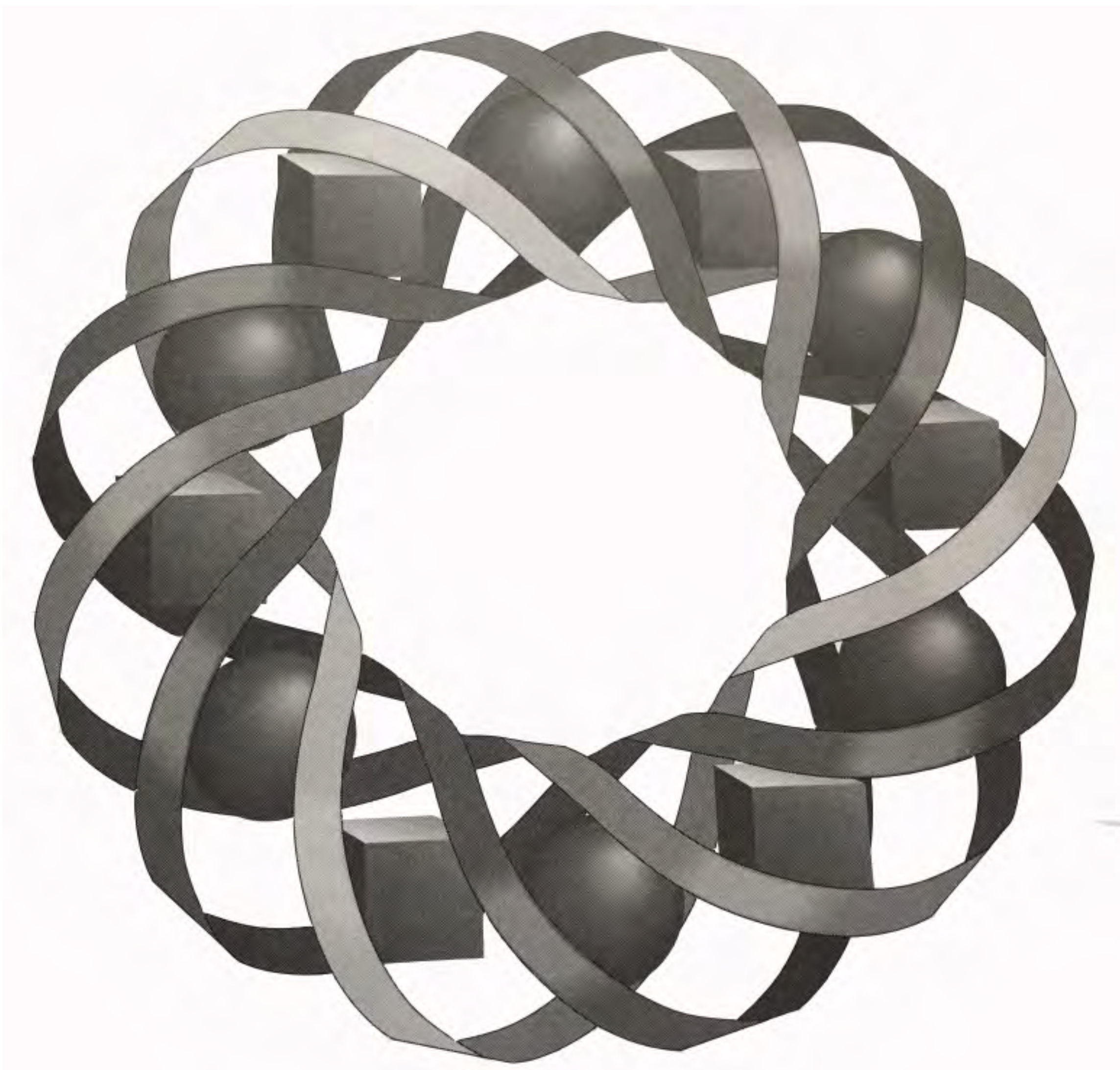
SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 3 Volumes 1-3

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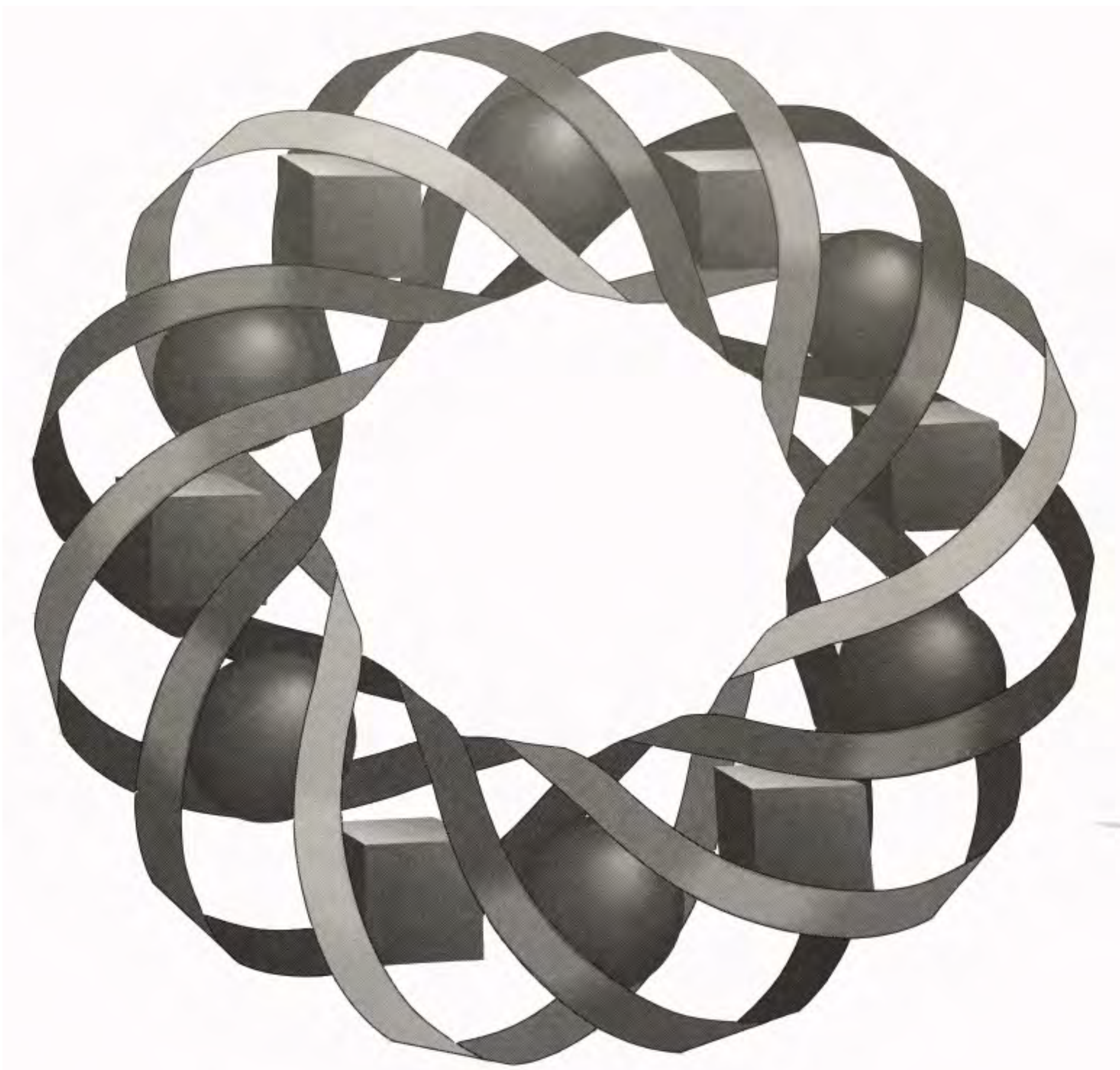


If you've ever hiked over unfamiliar terrain, chances are you consulted a topographic map to help find your way. (If you didn't, you may have wished that you had.) In this module, you use topography to investigate three-dimensional coordinate systems and the distances between points.

Pete Stabio • Randy Carspecken

From Here to There

Introduction

Topographic maps are two-dimensional representations of regions of the earth's surface. Besides identifying roads, trails, rivers, and other landmarks, topographic maps also contain information about the elevation of the terrain.

By knowing how to read a topographic map, you can determine the height of a mountain—and make a good guess about the difficulty of the climb. For example, Figure **1** shows a picture of the south side of a mountain. The horizontal lines are **level curves**. Each level curve represents points of equal elevation above sea level.

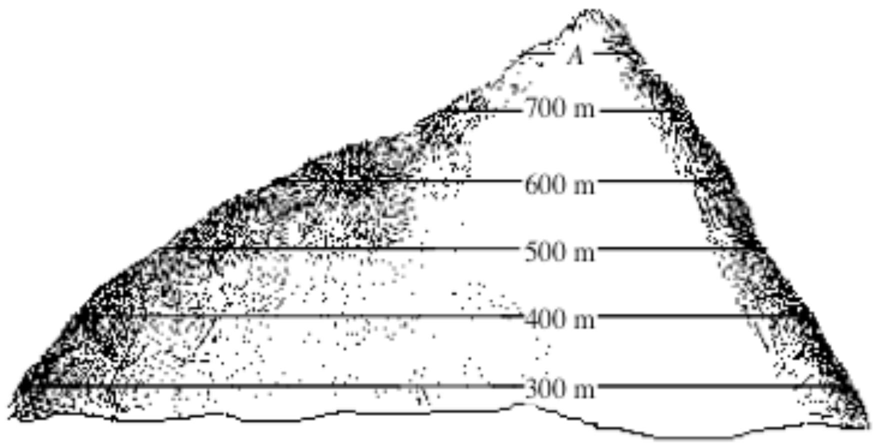


Figure 1: Level curves on the south side of a mountain

Although the view in Figure **1** provides some information about the landscape, a topographic representation of the mountain, as shown in Figure **2**, can be much more helpful. On a topographic map, the curved lines indicating points of equal elevation are **contour lines**.

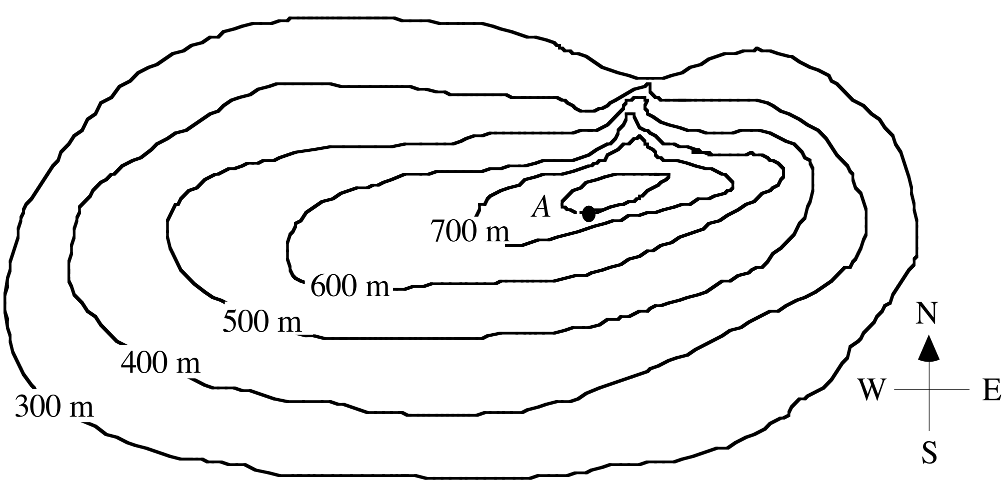


Figure 2: Topographic map of the mountain in Figure 1

Geography Note

When these terms are used to describe features of mountainous terrain, a **summit** is the highest point of the surrounding elevations and a **saddle** is a pass between two higher elevations. Some examples of these features are shown in Figure **3**.

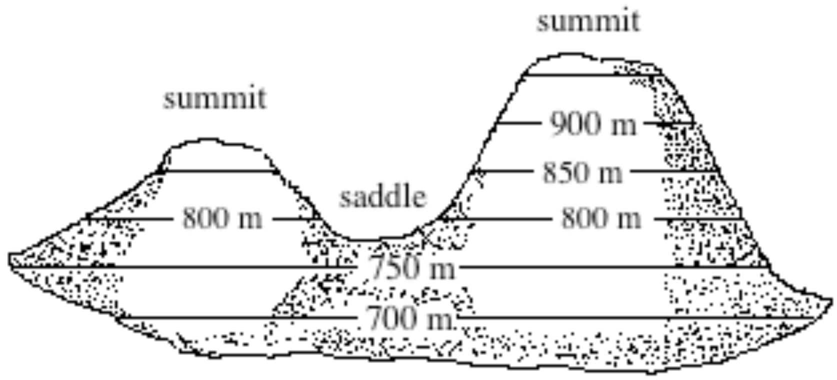


Figure 3: Some geographic features

Discussion

a. From what perspective does a topographic map show the earth's surface?

b. In Figure **1**, what do the quantities expressed in meters represent?

c. In Figure **2**, what is the elevation at contour line *A*?

d. On a topographic map, moving from one contour line to the next consecutive line indicates a constant change in altitude. However, the contour lines themselves may not be evenly spaced.

What is indicated by the space between two consecutive contour lines? Explain your response.

e.
1. How would you estimate the elevation of a point between two contour lines?
2. What physical characteristics of the terrain could make this estimate inaccurate?

f. How does the south side of the mountain in Figure **1** compare with its north side?

Activity 1

In this activity, you create a topographic map for a three-dimensional surface of your own design. To investigate some of the features of your map, you use a three-dimensional coordinate system.

Exploration

When trying to visualize a mountain from a topographic map, it may help to build a three-dimensional model of the terrain. In this exploration, you build a model that contains a saddle, a summit, a cliff, and a lake. At the same time, you create a topographic map of your model on a coordinate plane. **Note:** Read the instructions in Parts **a–g** before beginning your model.

a. Draw a pair of coordinate axes on a large sheet of paper (at least 60 cm by 28 cm).

b. **1.** On a sheet of cardboard slightly smaller than the paper in Part **a**, sketch the outline of the base of a mountain, such as the example shown in Figure **4**. Cut out this base.

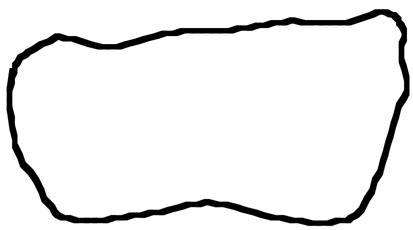


Figure 4: Outline of the base of a mountain

2. Trace the outline of the cardboard base on the coordinate plane, as shown in Figure **5**. This is the first contour line of your topographic map.

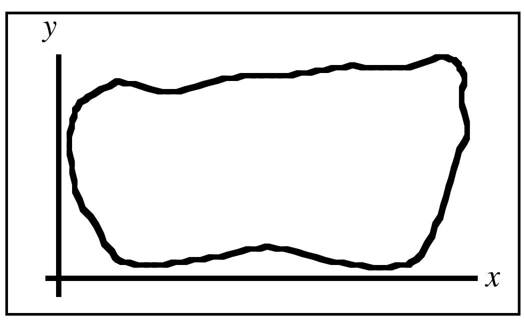


Figure 5: Coordinate plane with first contour

c. **1.** Trace the outline of the base on another sheet of cardboard. Using the tracing as a guide, draw the next contour of the mountain. As shown in Figure **6**, make this contour slightly smaller than the base.

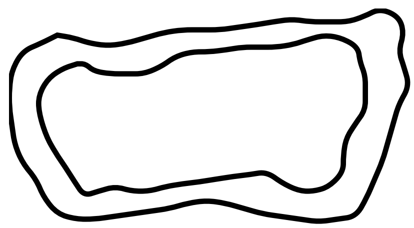


Figure 6: Cardboard with first and second levels

2. Cut out this second level and place it on the coordinate plane inside the outline of the base. Trace the outline of the second contour on the plane, as shown in Figure **7**.

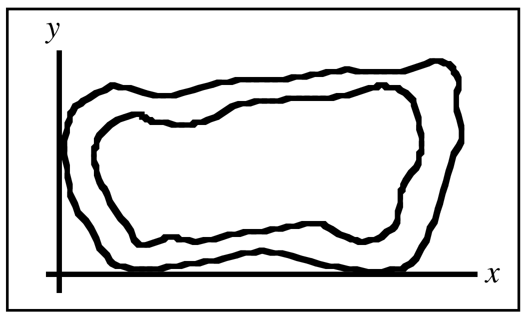


Figure 7: Topographic map of first two levels

d. Trace the outline of the second level on another sheet of cardboard to create a guide for the third contour. Then tape or glue the second level to the base in the same relative position as the contour lines on the coordinate plane.

e. Cut out the third level and place it on the coordinate plane inside the second contour. Trace the outline of the third contour on the plane.

f. Repeat the process described in Parts **d** and **e** until you have created a complete model of a mountain along with its topographic map. Your model should consist of 10 layers of cardboard; the map should have 10 contour lines. Both should include the following geographic features:

1. a lake

2. a summit

3. a saddle

4. a cliff at least half as high as the summit.

g. Measure the thickness of one layer of cardboard. Let this distance represent 100 m in elevation as well as 100 m along the *x*- and *y*-axes on your topographic map.

h. Use your map to identify an ordered pair (x,y) that represents the location of the summit.

i. Identify an ordered triple (x,y,z), where z represents elevation, that designates the location of the summit.

Discussion

a. Describe how the cliff, lake, saddle, and summit are represented on your topographic map.

b. Why does an ordered triple provide more information about the summit than an ordered pair?

c. Compare the method you used to find the thickness of one layer of cardboard with the methods used by others in your class.

d. Describe the z-coordinate of all the points that represent the surface of your lake.

e. Describe how to draw a path that provides a gradual ascent from the lake to the summit on your topographic map.

Assignment

1.1 Using your cardboard model from the exploration, locate a point on the top of the cliff. Locate another point at the bottom of the cliff, directly below the first one. Label the two points C_1 and C_2.

a. Identify an ordered triple for each point.

b. How do the coordinates of the ordered triples $C_1(x_1,y_1,z_1)$ and $C_2(x_2,y_2,z_2)$ compare?

c. What geometric figure is formed by the set of all points with coordinates (x,y,z) where the x- and y-coordinates are held constant? Explain your response.

d. Where are the points that correspond to ordered triples such as C_1 and C_2 located on a topographic map?

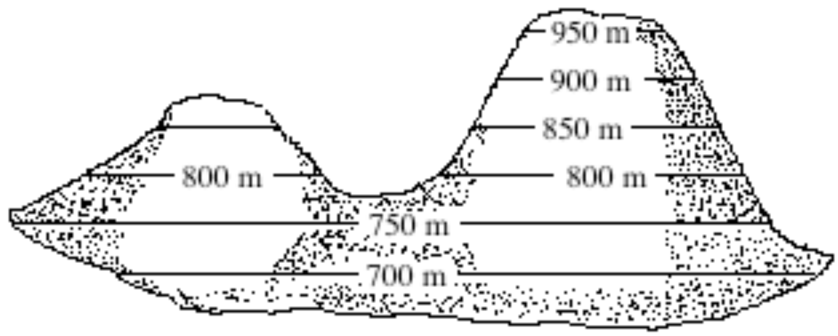
1.2 Using your cardboard model from the exploration, label two points L_1 and L_2 on opposite shores of the lake.

a. Identify an ordered triple for each point.

b. How do the coordinates of the ordered triples $L_1(x_1,y_1,z_1)$ and $L_2(x_2,y_2,z_2)$ compare?

c. What geometric figure is formed by the set of all points with coordinates (x,y,z) where the z-coordinate is held constant? Explain your response.

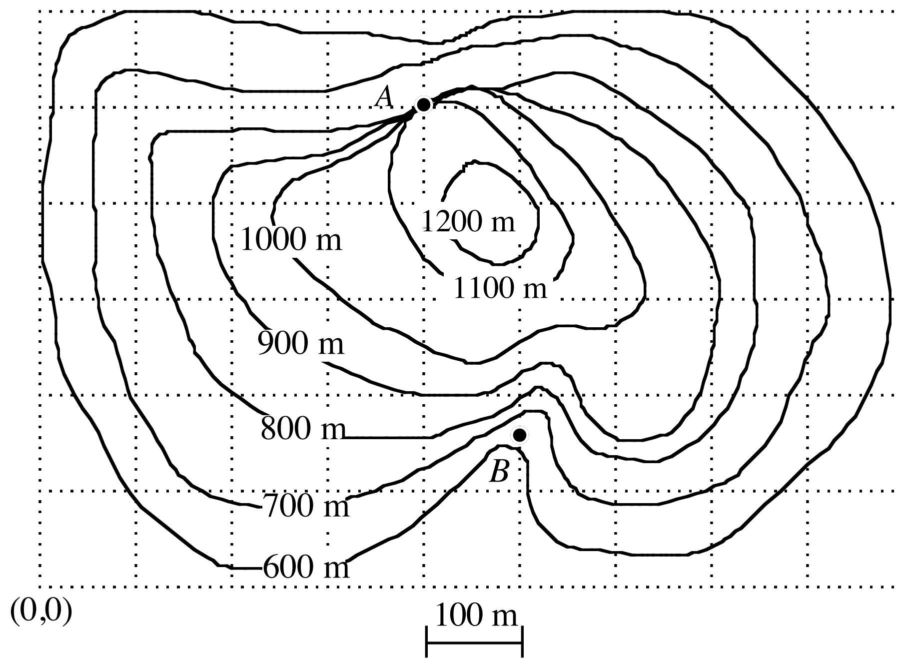
1.3 The diagram below shows the north side of a mountain. Draw a topographic map that might represent this terrain.



1.4 Use the topographic map below to complete Parts **a** and **b**.

a. Describe the features of the terrain at points *A* and *B*.

b. Identify the locations of points *A* and *B* using ordered triples.



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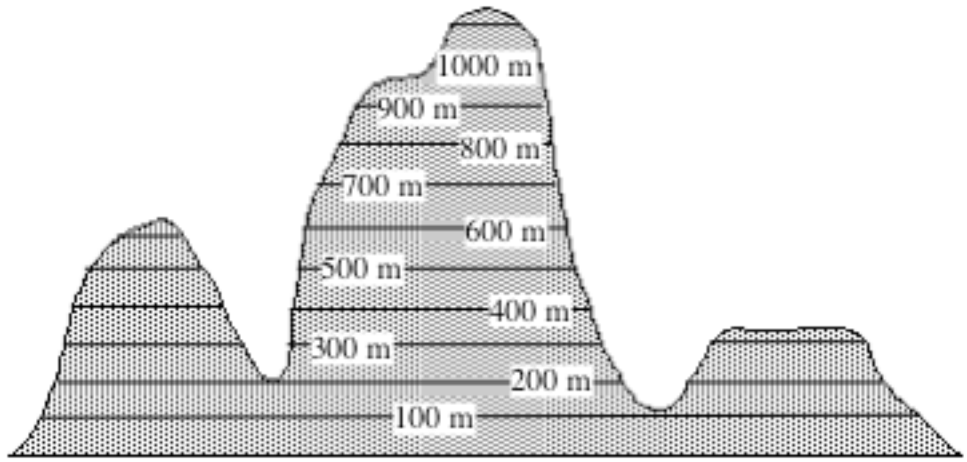
1.5 Create a topographic map for a right circular cone using at least five contour lines and your own grid system.

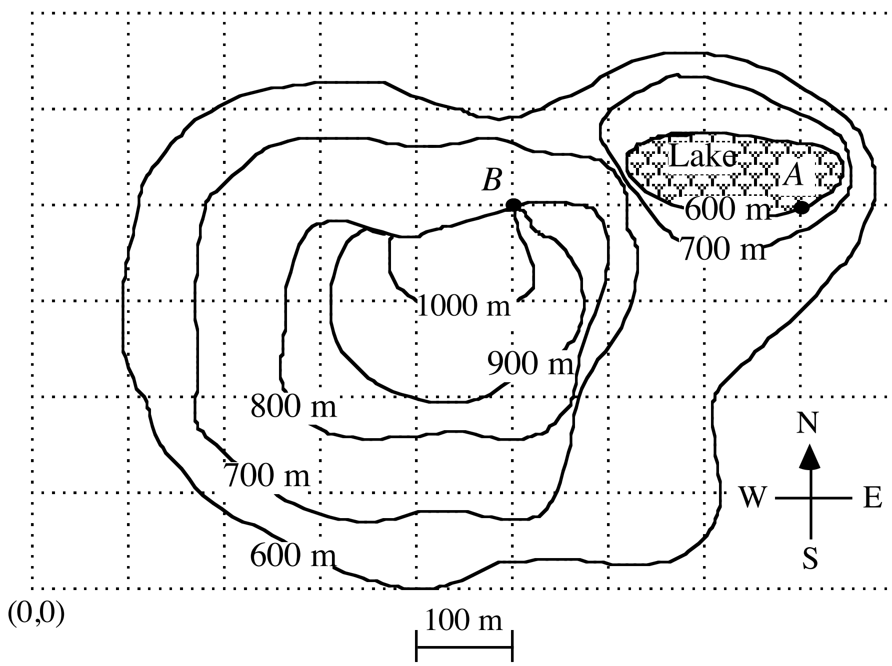
1.6 The diagram below shows the south side of a mountain range. Draw a topographic map that might represent this terrain.



1.7 Use the topographic map below to complete Parts **a** and **b**.

a. Describe the features of the terrain around points *A* and *B*.

b. Identify the locations of points *A* and *B* using ordered triples.



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Activity 2

In previous modules, you calculated distances in one or two dimensions. Finding the distance from the base of a mountain to its summit, however, could require the use of three-dimensional coordinates.

Exploration

In this exploration, you develop a method for finding the distance between any two points in a three-dimensional coordinate system.

Figure **8** shows a three-dimensional coordinate system and three points, P_1, P_2 and P_3. Each unit on the coordinate system represents 1 m. Each edge of the rectangular prism in Figure **8** either coincides with or is parallel to one of the axes.

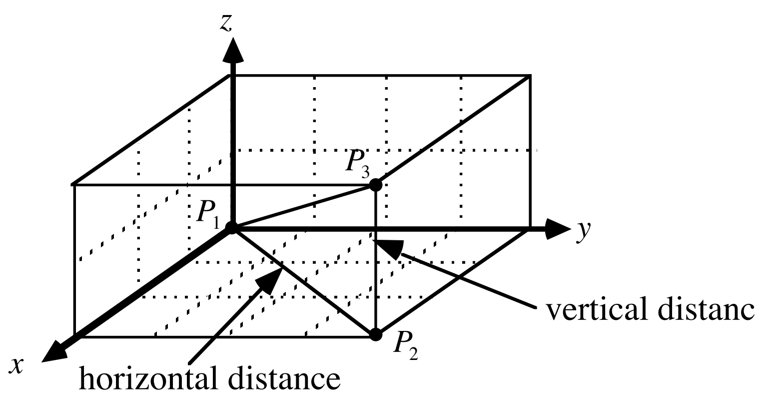


Figure 8: A three-dimensional coordinate system

a. One way to find the length of the segment with endpoints P_1 and P_3 is through the following steps.

1. Identify the coordinates of P_1, P_2, and P_3.
2. Find the distance between P_1 and P_2. This is the horizontal distance between P_1 and P_3.
3. Find the distance between P_2 and P_3. This is the vertical distance between P_1 and P_3.
4. Use the fact that triangle $P_1P_2P_3$ is a right triangle to find the distance between P_1 and P_3.

b. The three-dimensional coordinate system in Figure **9** shows two right triangles $P_1P_2P_3$ and $P_1P_3P_4$, with right angles at P_2 and P_4. The coordinates of P_3 are $(6,3,4)$, $\overline{P_2P_3}$ is parallel to the z-axis, and $\overline{P_3P_4}$ is parallel to the y-axis.

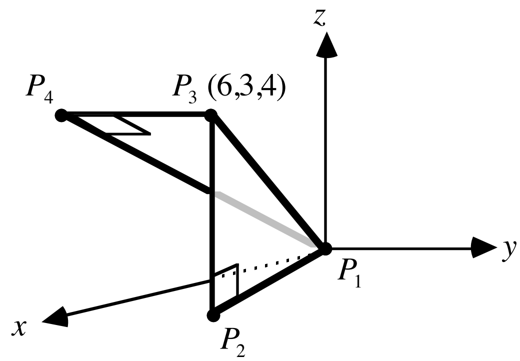


Figure 9: Two right triangles

1. Find the coordinates of P_2, a point in the xy-plane. Use these coordinates to find the distance from P_1 to P_3.
2. Find the coordinates of P_4, a point in the xz-plane. Use these coordinates to find the distance from P_1 to P_3.
3. Compare the distance found in Step **2** with the distance in Step **1**.

c. Draw and label a three-dimensional coordinate system.

1. Plot the two points with coordinates $(4,3,5)$ and $(0,2,8)$.
2. Find the distance between these two points. Hint: Draw right triangles.

d. The three-dimensional coordinate system in Figure **10** shows the locations of two points with coordinates (x_1,y_1,z_1) and (x_2,y_2,z_2). Find the distance between the two points.

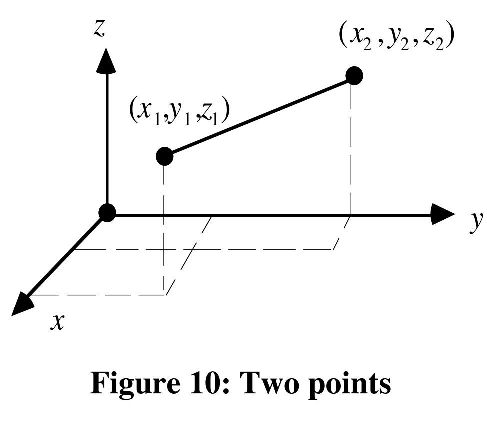


Figure 10: Two points

Discussion

a. How many times did you use the Pythagorean theorem to find the distance between P_1 and P_3 in Figure **8**?

b. How would you find the distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2)? Express your method using a mathematical formula.

c. Describe how you could find the distance between points *A* and *B* on the topographic map in Figure **11**.

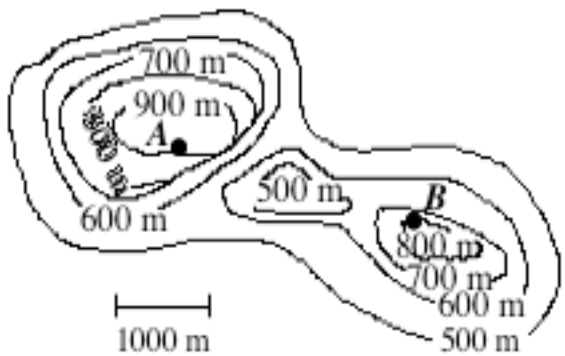


Figure 11: A topographic map

Assignment

2.1 Draw and label a three-dimensional coordinate system.

a. Plot the two points with coordinates $(5, 0, 2)$ and $(2, 2, 4)$.

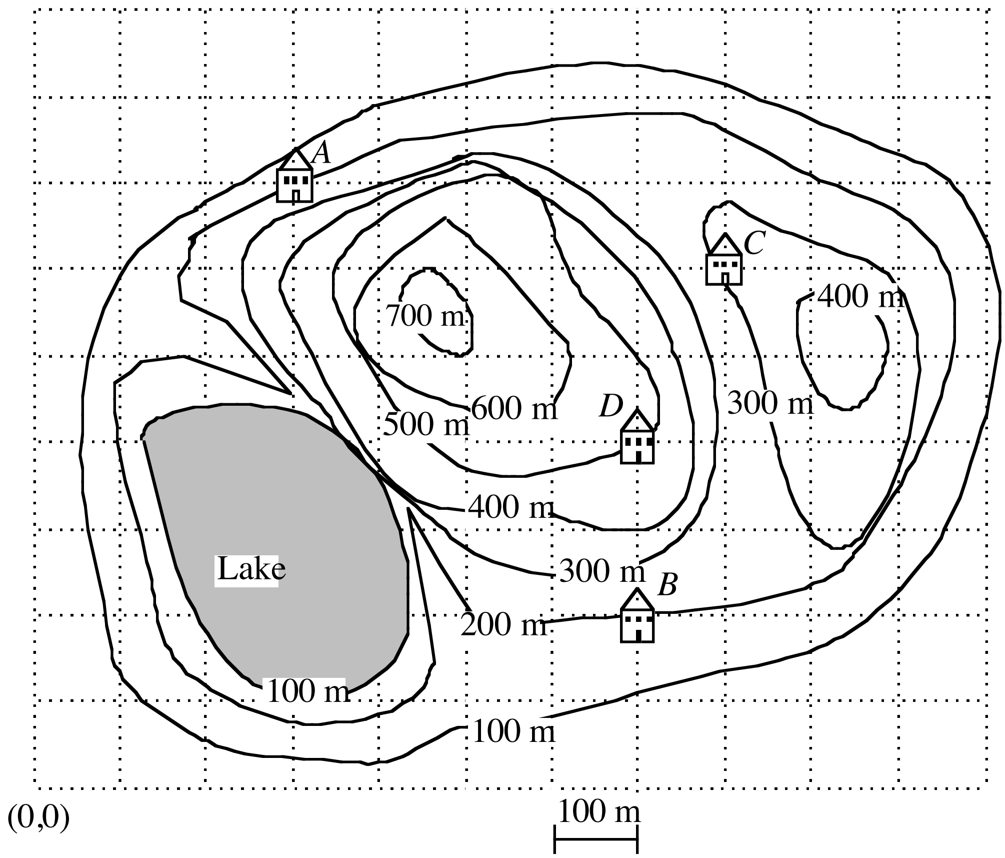
b. Find the distance between these points.

2.2 Draw and label a three-dimensional coordinate system.

a. Plot the two points with coordinates $(3, 4, 2)$ and $(5, 4, 0)$.

b. Find the distance between these points.

2.3 The figure below shows a topographic map of the terrain surrounding a lake, including four houses designated A, B, C, and D.



Use the map to find the distance between each of the following pairs of houses:

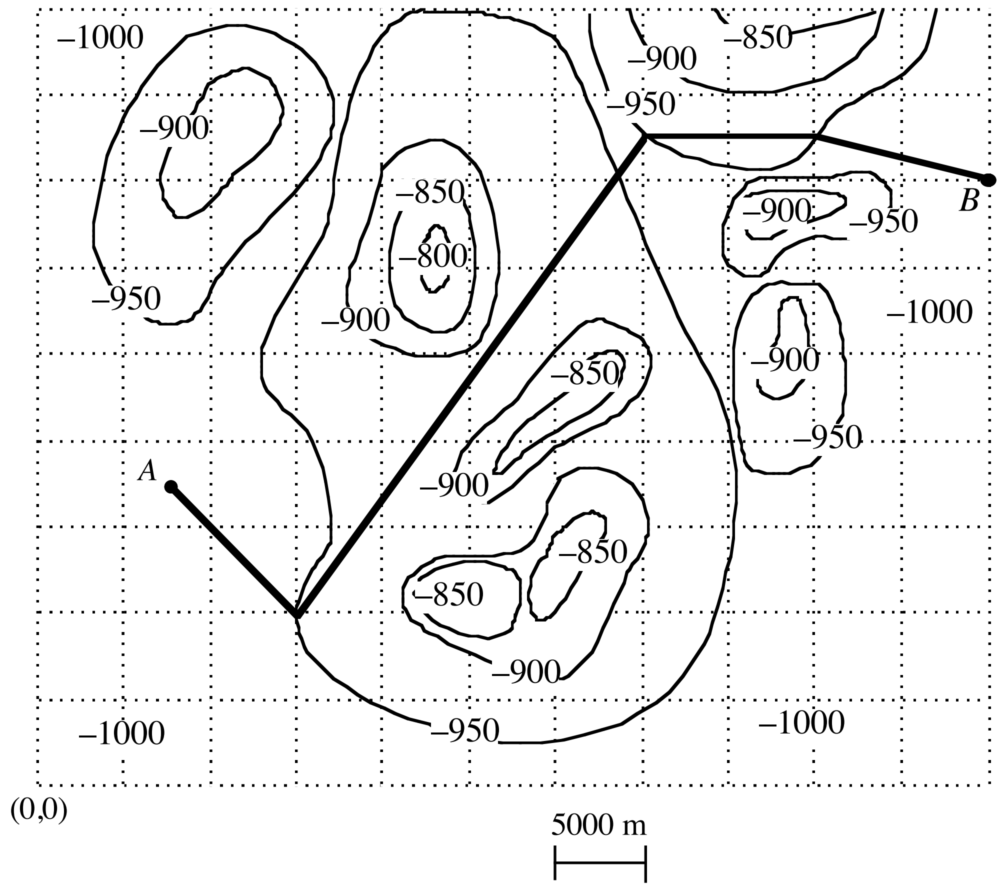
a. A and B

b. B and C

c. C and D

d. A and D.

2.4 The following diagram shows a topographic map of a region of the ocean floor. In this case, the contour lines represent distances below the surface of the ocean. For example, –850 indicates 850 m below sea level. The ocean floor for this region is 1000 m deep.



a. The captain of a submarine must navigate from point *A* to point *B* along the path shown. Assume the submarine is 10 m above the floor at each point along the path. What are the coordinates of point *A*, point *B*, and each point where the path turns?

b. The submarine must travel no closer than 10 m but no more than 25 m above the ocean terrain. What is the total distance the sub will travel from point *A* to point *B* along the given path?

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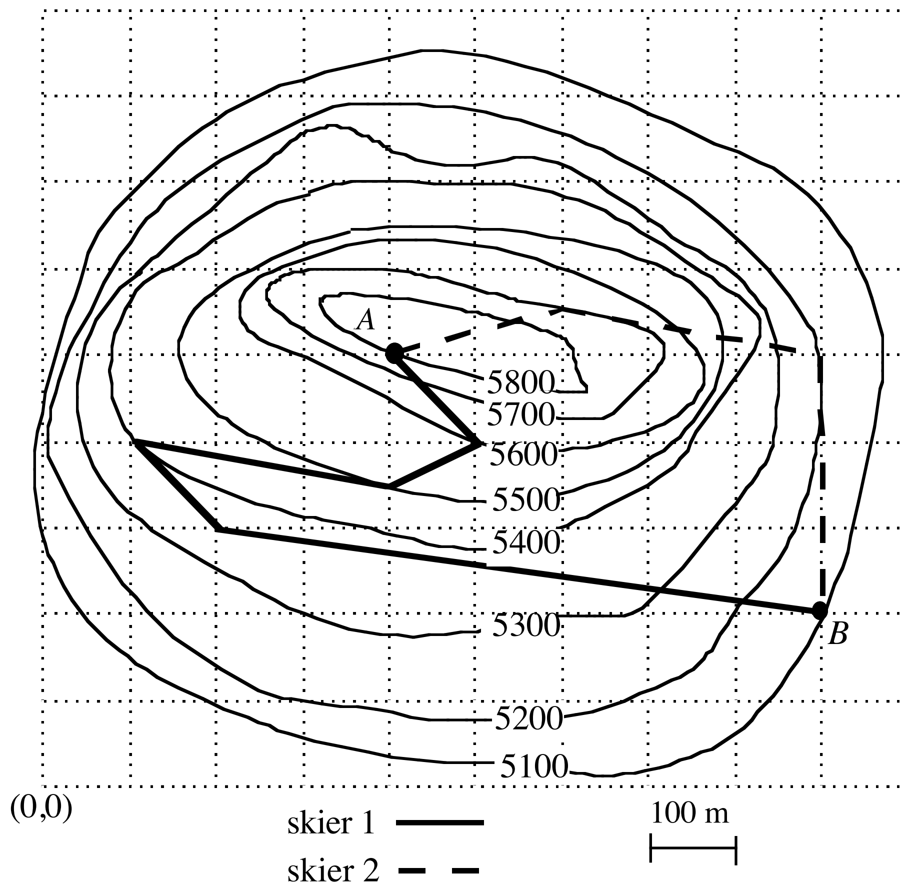
2.5 The topographic map below shows the routes taken by two skiers traveling down a mountain from point *A* to point *B*. Create a story for each skier's descent. In each story, include a description of the terrain and the approximate distance traveled. Assume that the terrain between contour lines is smooth, with no drops or rises other than those indicated on the map.



2.6 Use the map in Problem **2.5** to complete Parts **a–c** below.

a. **1.** Find the vertical distance between points *A* and *B*.

2. Find the horizontal distance between points *A* and *B*

3. Find the distance between *A* and *B* using the Pythagorean theorem.

b. **1.** Determine the coordinates of points *A* and *B*.

2. Find the distance between *A* and *B* using the coordinates of the points.

c. Compare your results in Parts **a** and **b**.

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Activity 3

Microwave signals are used in a variety of applications, including telephone transmissions. Since microwaves travel in a straight line and weaken considerably over distances greater than 5 km, engineers must place microwave repeaters along the transmission path to receive and relay signals.

In this activity, you use a topographic map to create a **profile** of mountainous terrain. Profiles can help provide a visual image of the landscape between two points on a topographic map.

Geography Note

A **profile** shows a vertical cross section or "side view" of the terrain. For example, Figure **12** shows a profile of a mountain valley.

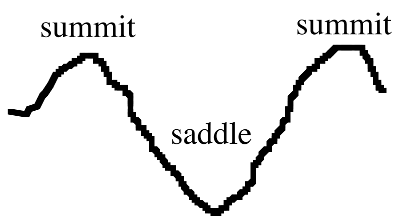


Figure 12: A profile of a mountain valley

Exploration

a.

1. On a topographic map provided by your teacher, label two points at least 10 km apart and located on different contour lines.
2. As shown in Figure **13**, connect the points with a **profile segment**.

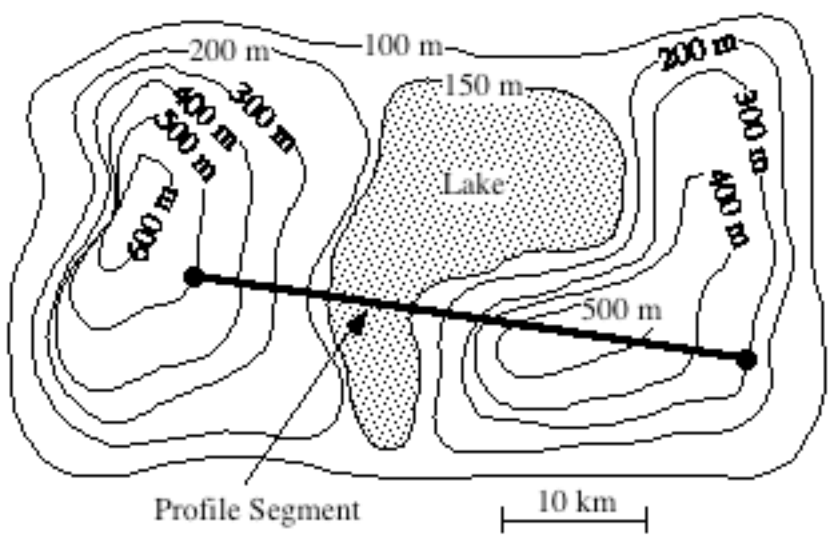


Figure 13: Profile segment drawn on a topographic map

b. **1.** Label the top line of a sheet of notebook paper with the highest elevation crossed by the profile segment.

2. Determine the change in elevation between consecutive contour lines on the map. This change in elevation is the **contour interval** (*I*). In Figure **13**, for example, the contour interval is 100 m.

3. Label each successive line on the notebook paper with an elevation *I* units less than the line above it. Continue this process until a line has been labeled with the lowest elevation crossed by the profile segment.

c. **1.** Place the top edge of your notebook paper along the profile segment, as shown in Figure **14**.

2. At each point where a contour line crosses the profile segment, draw a segment perpendicular to the profile segment. Extend each perpendicular segment to the line on the notebook paper that represents the same elevation as the corresponding contour line.

3. Mark the intersections of the perpendicular segments and their corresponding lines on the notebook paper.

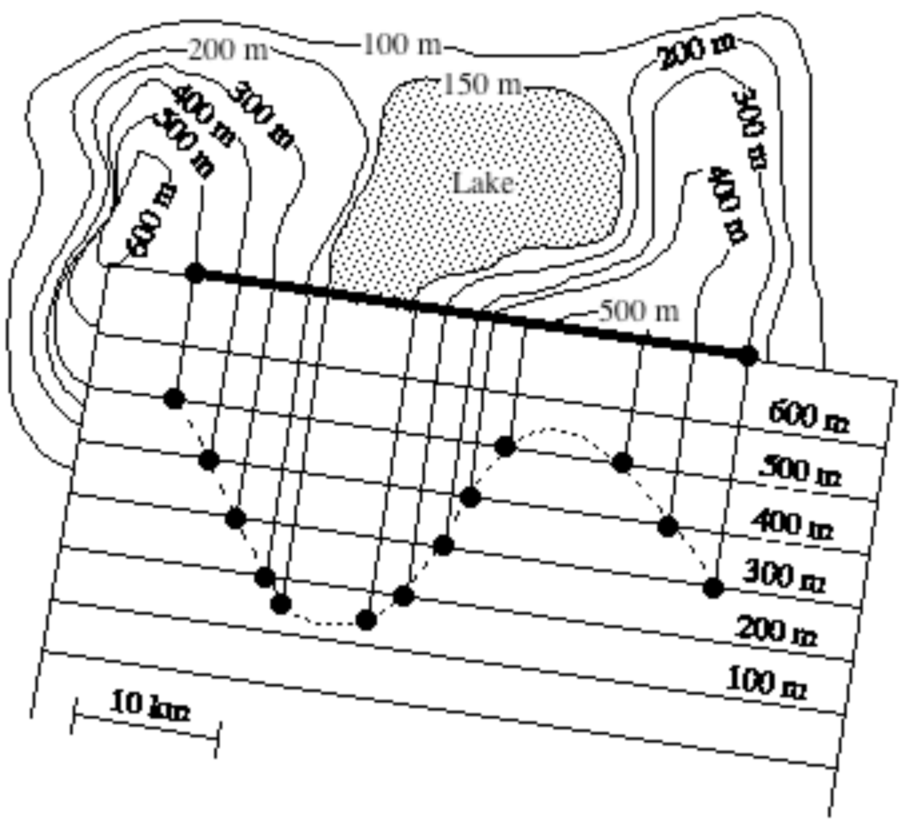


Figure 14: Creating a profile

d. Connect the points of intersection with a smooth curve. This curve represents a profile of the terrain between the two points you located on the map.

e. Compare the vertical scale on the profile with the horizontal scale. If they are different, make a sketch of the profile using the same vertical and horizontal scales.

f. Describe any similarities or differences you observe in the two profiles in Parts **d** and **e**.

Discussion

a. What information does a profile provide about a landscape?

b. Given a profile between two points, could you recreate the topographic map from which it was derived? Why or why not?

c. Microwaves travel in a straight line and the distance between repeaters must be less than 5 km along an unobstructed path.

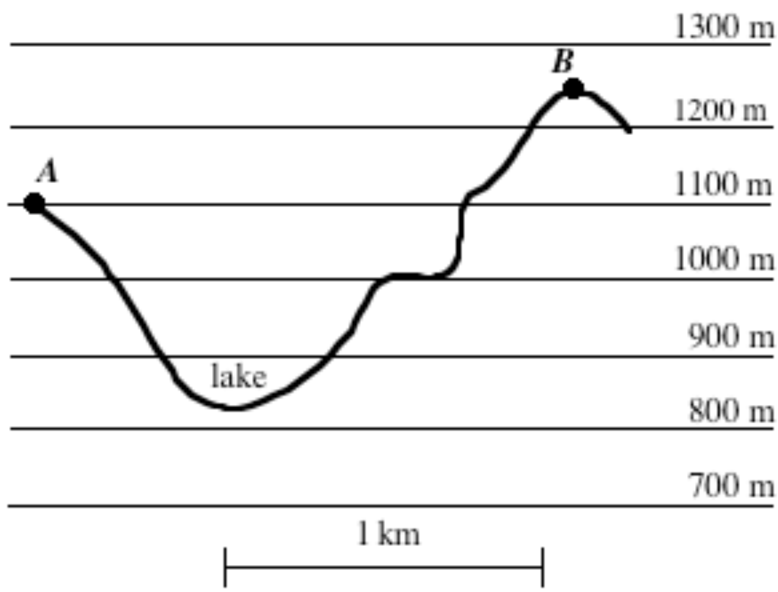
Imagine that the two points you located in the exploration are the sites of a microwave transmitter and receiver, respectively. Describe how you could use the profile to help locate appropriate sites for microwave repeaters.

d. Describe how the distance between two points on a topographic map can be approximated using a profile of the terrain.

e. Could you use a profile to determine the angle of elevation between two points (the angle formed by the segment joining the two points and a horizontal line)? Explain your response.

Assignment

3.1 The figure below shows a profile of a mountain landscape.



a. What is the elevation of the summit?

b. Approximately how tall is the cliff?

c. At what elevation is the lake?

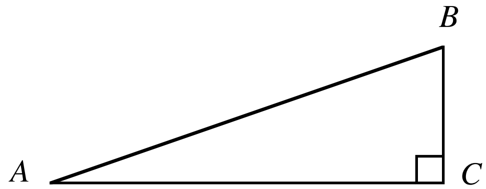
d. What is the distance from point *A* to point *B*?

e. **1.** Sketch a profile of this mountain on a coordinate grid, using the same scale for both the horizontal and vertical axes.

2. Use the ordered pairs that describe the locations of points *A* and *B* to verify the distance found in Part **d**.

3.2 Imagine that a microwave transmitter is located at point *A* on the profile in Problem **3.1**. A receiver is located at point *B*.

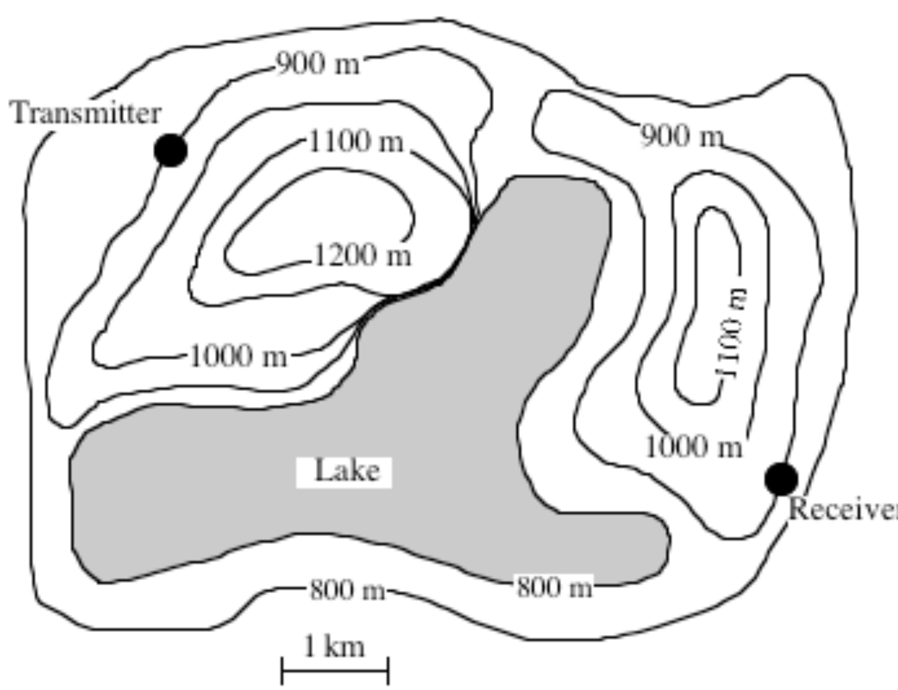
a. Using the distances found in Problem **3.1**, sketch a right triangle like the one in the diagram below to model this situation.



b. In the diagram, what trigonometric ratio is defined by BC/AB? Explain your response.

c. Using the trigonometric ratio you identified in Part **b**, determine the angle necessary to transmit a microwave signal directly from *A* to *B*.

3.3 The topographic map below shows the locations of a microwave transmitter and receiver. Both are built on towers 20 m above the ground.



a. In order to relay signals from the transmitter to the receiver, two microwave repeaters must be built between them. Identify possible locations for the repeaters. (Recall that microwaves travel in straight lines and require unobstructed paths no greater than 5 km long.)

b. Demonstrate that your proposed locations are adequate by:

1. drawing profiles of the terrain
2. finding the appropriate distances using ordered triples.

3.4 Microwave signals travel at the speed of light, about $3 \cdot 10^8$ m/sec. Determine the time required for a signal to travel from the transmitter to the receiver in Problem **3.3**.

3.5 The terrain represented by the topographic map in Problem **3.3** features two summits. Write a detailed paragraph describing a journey from one summit to the other. Include an estimate of the distance traveled and describe how you determined this distance.

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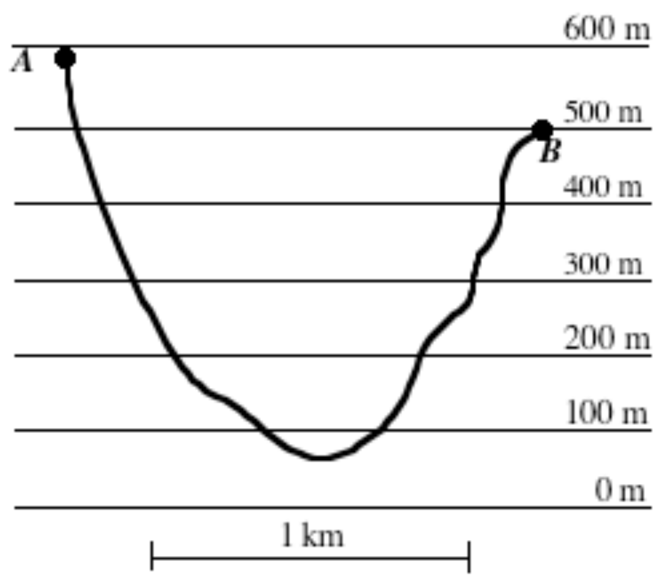
3.6 Create a topographic map that corresponds with the profile given in Problem **3.1**.

3.7 The figure below shows a profile of a valley.



a. What is the elevation of the lowest part of the valley shown in the profile?

b. What is the distance from point *A* to point *B*?

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Research Project

The U.S. Geological Survey (USGS) has published topographic maps of nearly the entire United States. Obtain a topographic map of a region near your school.

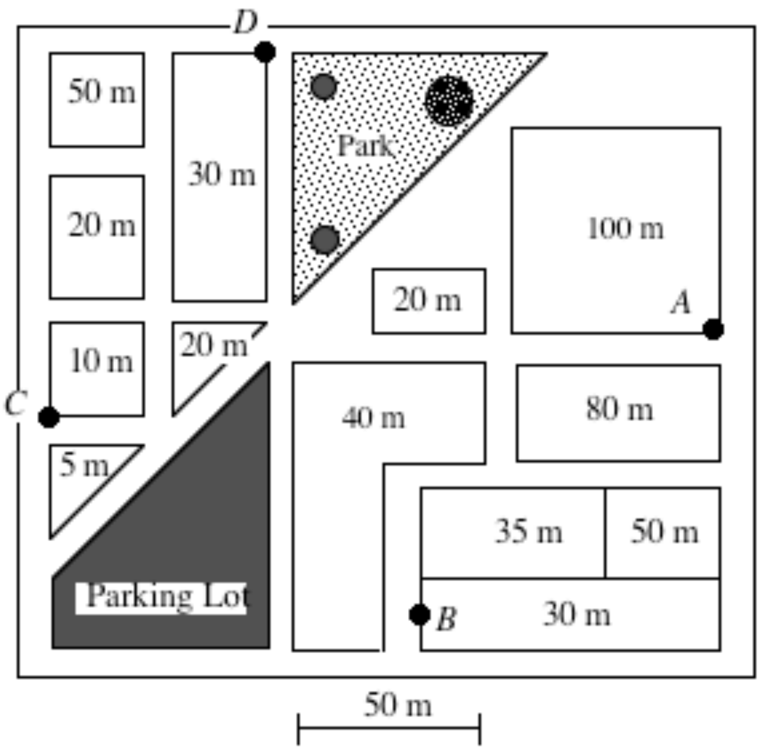
a. Identify several key features of the terrain. In a paragraph, describe how contour lines help characterize these features on the map.

b. Create a profile of the terrain.

c. Determine the straight-line distance and angle of elevation between the two highest points on the map.

Summary Assessment

The diagram below shows a map of a business district in a large city. Each polygon not otherwise labeled represents a building in the district. Each measure indicates the height above street level of the corresponding building. Use this map to complete Problems **1** and **2** below.



1. The local telephone company plans to build a microwave transmitter on the top of the building at point *A*. The transmitter will send signals to a receiver located on the top of a building at point *C*.

a. Determine whether or not there is an unobstructed line of sight between points *A* and *C* by creating a profile of the terrain.

b. Determine the distance between points *A* and *C* using the coordinates of ordered triples.

c. Determine the angle of elevation (measured from the horizontal) necessary to transmit the signal directly from point *A* to point *B*.

2. A cable television company is trying to determine the most cost-efficient way to relay signals from its headquarters at point *B* to a substation at point *D*. The company has two choices: burying a cable underground, or installing a system on the roofs of buildings in order to transmit signals through the air.

   a. Burying a cable underground will cost $450 per meter. To minimize the disruption of traffic, the cable may not pass under more than three streets and must run alongside buildings.

      1. Determine a possible route between points *B* and *D*.

      2. Calculate the cost of this plan.

   b. The transmitters required to send the signal through the air cost $35,000 each. The repeaters cost $15,000 each. The path between a transmitter and a repeaters must be unobstructed by other buildings. If a transmitter or repeaters must be located on the roof of a building not already owned by the company, the space must be purchased for $3000.

      1. Use a profile to determine the number and location of the transmitters and repeaters necessary to relay signals between points *B* and *D*.

      2. Calculate the cost of this plan.

   c. Write a letter to the president of the company describing both options. Include appropriate maps and profiles, a summary of the costs of each plan, and a report of your recommendations.

Module Summary

- The lines that indicate elevation on a topographic map are **contour lines**.
- A **summit** is the highest point of the surrounding elevations.
- A **saddle** is a pass between two higher elevations.
- The coordinates of a point in a three-dimensional rectangular coordinate system are written as an **ordered triple**.
- The **origin** of a three-dimensional rectangular coordinate system is the point where the three axes intersect. The origin has coordinates $(0,0,0)$.
- The distance *d* between two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) can be found using the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- A **profile** shows a vertical cross section or "side view" of terrain.

Selected References

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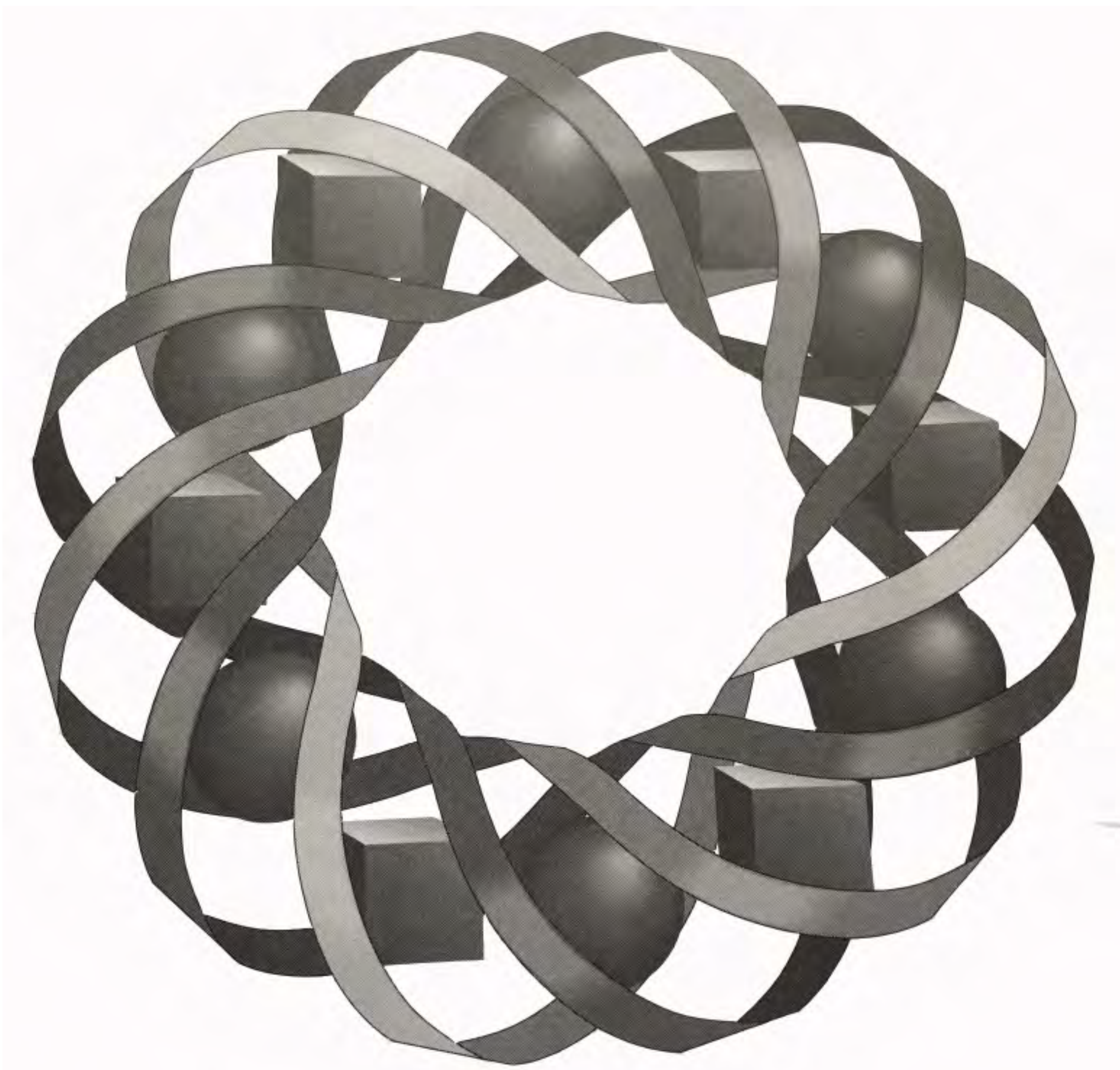
Van Heuvelen, A. *Physics*. Boston: Little Brown, 1982.

What Are You Eating?



The labels on packaged food contain a wealth of nutritional information. In this module, you use these labels and linear programming to help select foods that satisfy nutritional requirements.

Masha Albrecht • Darlene Pugh

What Are You Eating?

Introduction

Reduced fat. High in fiber. Low calorie. Light. Lite? To satisfy health-conscious consumers, many food manufacturers added health claims to their labels and packaging. Some of these claims were confusing; others were downright misleading.

In order to regulate this situation, the U.S. Congress passed the Nutrition Labeling and Education Act in 1990. By May 1994, nutrition labeling became mandatory for almost all processed foods. Regulations for labeling of meat and poultry became effective later that year.

The Food and Drug Administration (FDA) has found sufficient scientific evidence to support the following health claims:

- higher calcium intake reduces the risk of osteoporosis
- lower sodium use reduces high blood pressure
- high amounts of dietary saturated fat and cholesterol can lead to coronary heart disease
- high dietary fat intake increases the risk of cancer
- a diet high in fiber, fruits, and vegetables helps prevent cancer
- dietary fiber reduces the risk of coronary heart disease.

As a result of this evidence, the FDA recommends that no more than 30% of a day's calories be obtained from fat—with no more than 10% of each day's calories consisting of saturated fat. Carbohydrates should provide approximately 60% of each day's calories, while proteins should account for about 10%.

The FDA's recommendations, along with accurate food labels like the sample shown in Figure **1**, can help people plan diets and make informed nutritional choices. The upper part of the label provides some specific information about the nutritional content of the food, while the lower portion describes Percent Daily Values of fat, cholesterol, sodium, carbohydrates, and dietary fiber for both 2000-calorie and 2500-calorie per day reference diets.

Nutrition Facts

Serving Size 1 cup (228 g)
Servings Per Container 2

Amount Per Serving	
Calories 260	Calories from Fat 120

	% Daily Value*
Total Fat 13 g	20%
Saturated Fat 5 g	25%
Cholesterol 30 mg	10%
Sodium 660 mg	28%
Total Carbohydrate 31 g	10%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	

Vitamin A 4%	•	Vitamin C 2%
Calcium 15%	•	Iron 4%

* Percent Daily Values are based on a 2,000 calorie diet. Your daily values may be higher or lower depending on your calorie needs:

	Calories:	2,000	2,500
Total Fat	Less than	65 g	80 g
Sat Fat	Less than	20 g	25 g
Cholesterol	Less than	300 mg	300 mg
Sodium	Less than	2,400 mg	2,400 mg
Total Carbohydrate		300 g	375 g
Dietary Fiber		25 g	30 g

Calories per gram:
Fat 9 • Carbohydrate 4 • Protein 4

Figure 1: Sample Food Label

Source: *FDA Consumer Special Issue on Food Labeling*, May 1993.

The information on food labels is based on a reference diet of 2000 calories per day. This 2000-calorie diet is not recommended for all people. Some might need more calories; others might require less. For example, very active people, children under the age of four, and pregnant women have nutritional needs that vary from the 2000-calorie reference diet. On average, a moderately active teenage male needs at least 2800 calories per day, while a moderately active teenage female needs at least 2200 calories per day. A healthy diet involves more than just counting calories; it requires eating a variety of foods in moderation.

Discussion

a. Use the health claims cited by the FDA to describe examples of healthy and unhealthy meals.

b.
1. How did the FDA calculate the number of calories from fat shown on the sample label in Figure **1**?
2. What percentage of the calories in this food come from fat?
3. How is the Percent Daily Value (% DV) of total fat calculated from the information on the label?

c. How might you use the information on a typical food label to make your own nutritional decisions?

d.
1. Consider a person who requires 2500 calories per day and eats one serving of the food whose label is shown in Figure **1**. Describe the Percent Daily Values of total fat, total carbohydrate, and fiber received by this person.
2. How do these Percent Daily Values compare with those for a 2000-calorie diet?

e. One serving of the food on the label in Figure **1** has a mass of 228 g. However, the mass of the fat, cholesterol, sodium, carbohydrates, and protein in that serving totals only about 50 g. If the remainder of the serving is water, then what percentage of this food product is water?

f. One serving of a certain food provides 25% of the Daily Value for both carbohydrates and sodium for a person on a 2000-calorie diet.

1. How many servings of this food would provide 100% of the Daily Value of carbohydrates for a person on a 2000-calorie diet?
2. How many servings would provide 100% of the Daily Value of carbohydrates for a person on a 2500-calorie diet?
3. Use the nutrient information for 2000- and 2500-calorie diets listed on the label in Figure **1** to explain why more than four servings of this food would not be wise for a person who must limit sodium intake.
4. How might a person obtain 100% of their Daily Value for carbohydrates without consuming too much sodium?

Research Project

Develop a chart to keep track of the food you consume each day for a five-day period. Make an entry on your chart for every meal or snack. For each item, record the food type and amount eaten, as well as a nutritional summary. The summary should include calories, total fat, saturated fat, cholesterol, sodium, carbohydrates, and protein. Calculate totals for each day. Write a summary of your diet that includes:

- a comparison with the reference diet listed on food labels
- a consideration of your own caloric needs (this may vary from the reference diet)
- a comparison with the FDA recommendations for percentages of total fat, saturated fat, carbohydrates, and proteins.

Activity 1

The nutrients in foods include vitamins, minerals, fats, protein, and carbohydrates. When planning healthy meals, nutritionists sometimes evaluate each nutrient separately. Many people, for example, need to minimize their intakes of fat and sodium while increasing their consumption of dietary fiber. One method of analyzing the nutrient content of a meal involves **linear programming**.

Mathematics Note

Linear programming can be used to solve problems involving variables that are subject to linear **constraints**. The system of linear inequalities determined by these constraints defines a **feasible region**, a set of points that satisfies the system.

Typically, such problems require the identification of an optimal value for an **objective function**, either a maximum or a minimum. According to the **corner principle**, if the objective function has a minimum or maximum value, it will occur at a corner point (or vertex) of the feasible region.

For example, suppose that you wanted to find the minimum value of the function $f = 4x + 3y$ given the following system of constraints:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \geq 12 \\ 4x + 2y \geq 16 \\ x + y \leq 9 \end{cases}$$

Figure **2** below shows a graph of a feasible region that satisfies these constraints.

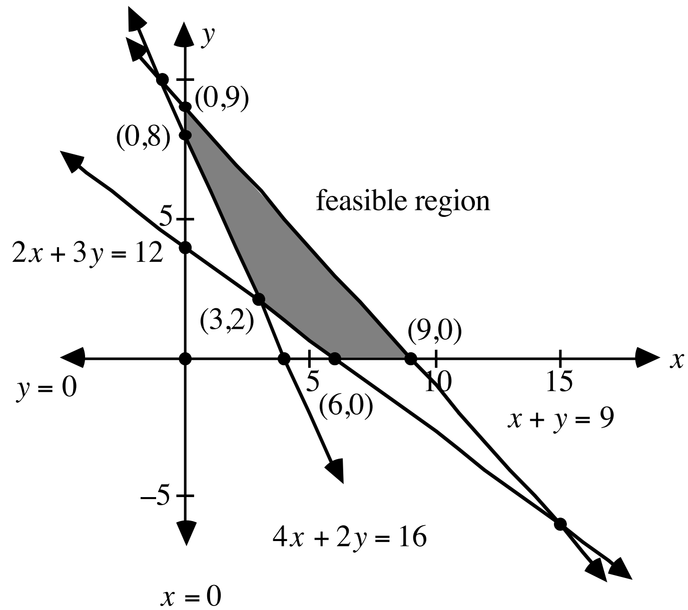


Figure 2: Feasible region defined by five constraints

In this case, the feasible region has five corner points—(0,9), (0,8), (3,2), (6,0), and (9,0)—all of which are contained in the feasible region. Using the corner principle, the minimum value of the objective function $f = 4x + 3y$ must occur at one of these points. By evaluating the function at each corner point, you can determine that the minimum value of 18 occurs at the point with coordinates (3,2).

Discussion 1

a. The five constraints described in the previous mathematics note correspond with the five lines graphed in Figure **2**. How many points of intersection are there for these lines?

b. The intersection of the lines $x = 0$ and $2x + 3y = 12$ in Figure **2** is not a corner point. Explain why this point is not included in the feasible region.

c. Identify the other pairs of lines in Figure **2** whose points of intersection are not corner points.

d.
1. Describe how you could find the *y*-intercept of the line $4x + 2y = 16$ without using a graph.
2. Describe how you could find the *x*-intercept without using a graph.

e.
1. Which corner points in Figure **2** are located on a coordinate axis?
2. How could you determine the coordinates of each of these points using algebra?
3. How could you determine the coordinates of the remaining corner point using algebra?

Exploration

Imagine that you are a nutritionist who wants to plan a lunch of tomato soup and rolls. The meal must contain at least 20% of the Daily Value of iron, at least 40% of the Daily Value of calcium, no more than 700 calories, and a minimum amount of fat. The percentages given in Table **1** are based on a 2000-calorie reference diet.

Table 1: Nutrients in one serving of tomato soup and rolls

Food	% DV Iron	% DV Calcium	Calories	Fat
tomato soup	4	17	140	4 g
roll	6	2	120	3 g

a. Using t to represent the number of servings of tomato soup and r to represent the number of servings of rolls, write inequalities for each of the following constraints.

1. The number of servings of tomato soup in the meal must be at least 0.
2. The number of rolls in the meal must be at least 0.
3. The total number of calories from a meal of soup and rolls can be no more than 700.
4. The meal of soup and rolls should provide at least 20% of the Daily Value of iron.
5. The meal of soup and rolls should provide at least 40% of the Daily Value of calcium.

b. Using the inequalities from Part **a**, graph the feasible region for the number of servings of soup and rolls in this meal, with values for t represented on the horizontal axis. Your graph should include the following:

1. the equations that determine the boundaries of the feasible region
2. the coordinates of the vertices of the polygon that defines the feasible region.

c. The two inequalities you determined in Steps **4** and **5** of Part **a** define the constraints placed on iron and calcium consumption. List the coordinates of a point in the first quadrant that:

1. does not satisfy either of these constraints
2. satisfies exactly one of these constraints
3. satisfies both of these constraints.

d.

1. Let f represent the grams of fat consumed in the meal. Write an equation that relates f, t, and r for a meal of tomato soup and rolls.
2. Suppose that the meal must contain 4 g of fat. Substitute $f = 4$ into your equation from Step **1**, and sketch a graph of the result on the same coordinate system as in Part **b**.
3. Repeat Step **2** for $f = 8$, $f = 12$, and $f = 16$.
4. For various values of f, what patterns do you notice in the resulting lines?

e.

1. Determine the approximate number of servings of rolls and soup that will minimize the amount of fat consumed while providing at least 20% of the Daily Value of iron and at least 40% of the Daily Value of calcium.
2. One serving of soup has a volume of 240 mL, while one serving of rolls has a mass of 120 g. Express your solution in Step **1** in terms of milliliters of soup and grams of rolls.
3. Calculate the number of calories in this meal.

Discussion 2

a. What is the significance of each point in a feasible region?

b. Why is it appropriate that the feasible region in the exploration is located in the first quadrant?

c. Describe how you graphed the inequalities and combined their graphs to find the feasible region.

d. The feasible region in the exploration appears to be bounded by three segments. Which constraint determines each of these segments?

e.

1. In Part **d** of the exploration, you wrote an equation that relates grams of fat to servings of soup and rolls. Consider the graph of this equation for $f = 8$ and explain the meaning of several points along the line.
2. The line determined when $f = 8$ does not pass through the feasible region. What does this indicate in terms of the meal of tomato soup and rolls?

f.

1. In Part **e** of the exploration, you were asked to minimize fat consumption while providing at least 20% of the Daily Value of iron and at least 40% of the Daily Value of calcium. Where in the feasible region did this occur?
2. Describe a simple way of finding the minimum or maximum values of an objective function over a feasible region.

g. If you were asked to specify the combination of soup and rolls in terms of whole numbers of servings, how would you report the combination that minimizes the fat content?

Assignment

1.1 A farm family would like to plant up to 100 acres of their land in corn and wheat. They can afford a maximum of $26,500 on crop expenses. Corn costs about $300 per acre to plant, grow, and harvest. Wheat costs about $250 per acre to plant, grow, and harvest.

a. Write a system of inequalities that describes the constraints in this situation.

b. Graph the feasible region.

c. Identify the pairs of equations whose intersections are corner points and find the coordinates of these points.

d. Write an objective function that describes the farm's profit if the profit per acre for corn is $100 and the profit per acre for wheat is $90.

e. Determine the maximum profit the family can make by planting corn and wheat. Justify your response.

1.2 A nutritionist would like a meal of cheese pizza and fruit salad to provide at least 40% of the Daily Value of vitamin C and at least 20% of the Daily Value of calcium, but no more than 500 calories. The table below shows the Percent Daily Values for these nutrients in one serving of each food, based on a 2000-calorie reference diet:

Food	% DV Vitamin C	% DV Calcium	Calories	Fat
cheese pizza	14	18	290	8 g
fruit salad	14	4	124	0 g

a. Using p for the number of servings of pizza and s for the number of servings of salad, express all the constraints as inequalities.

b. Graph the feasible region.

c. The intersection of the constraints $p \geq 0$ and $s \geq 0$ does not define a corner point. Describe the region defined by these constraints and explain how it is related to the feasible region.

d. Write an objective function based on the desire to minimize calories.

e. One serving of pizza has a mass of 210 g, while one serving of fruit salad has a mass of 248 g. Determine how many grams of pizza and fruit salad a person should eat to minimize the intake of calories while providing at least 20% of the Daily Value of calcium and at least 40% of the Daily Value of vitamin C.

f. Find the total calories and total fat for your response to Part **e**.

1.3 **a**. Choose two healthy foods. Use their labels to determine how you can receive at least 100% of the Daily Values of any two nutrients while minimizing your intake of sodium.

b. In order to confirm your response to Part **a**, use both an algebraic method and a graphical method to identify the vertices of the feasible region.

1.4 One average-size plum provides approximately 3% of the Daily Value of vitamin A and 10% of the Daily Value of vitamin C. One large slice (500 g) of watermelon provides approximately 24% of the Daily Value of vitamin A and 80% of the Daily Value of vitamin C.

Explain why it is not possible to obtain at least 80% of the Daily Value of vitamin C, but no more than 10% of the Daily Value of vitamin A, by eating a combination of plums and watermelon.

Note: Vitamin C is water soluble, but vitamin A is not. Unused amounts of vitamin A accumulate in the body.

* * * * *

1.5 A landscaping firm uses two brands of fertilizer, Green Grass and Grow More. Green Grass fertilizer contains 4 kg of phosphates and 2 kg of nitrates per bag, while Grow More fertilizer contains 6 kg of phosphates and 5 kg of nitrates per bag. One customer's lawn needs a mixture of at least 24 kg of phosphates and at least 16 kg of nitrates.

a. Write a system of inequalities that describes these constraints.

b. Graph the feasible region and identify its corner points.

c. Write the objective function that describes the total cost of fertilizer for this customer if Green Grass costs \$6.99 per bag and Grow More costs \$17.99 per bag.

d. Determine the minimum cost for fertilizer for this customer's lawn. Justify your response.

1.6 Two carpentry classes at school make and sell desks and chairs. One class assembles the furniture, while a second class varnishes each piece. It takes an average of 1 hr for the first class to assemble a chair and 4 hr to assemble a desk. It takes the second class an average of 2 hr to varnish each chair and 1 hr to varnish each desk. Each class has a maximum of 200 student hours to assemble or varnish furniture each week.

a. Write a system of inequalities that describes these constraints.

b. Graph the feasible region and list the coordinates of the corner points.

c. Write the objective function that describes the total profit if the two classes make \$15 per chair and \$25 per desk.

d. Determine the maximum profit the two carpentry classes can make in one week. Justify your response.

* * * * * * * * * *

Activity 2

When using linear programming to analyze meals, the feasible region is defined by a system of inequalities. In order to maximize your consumption of fiber or minimize your intake of fat, you may determine an objective function, then apply the corner principle and find a solution. In these situations, finding the coordinates of the vertices of the feasible region is an important task.

You already are familiar with at least two different methods for finding the intersection of two lines: graphing and substitution. In the following activities, you examine a third useful method for solving systems of linear equations.

Exploration

In this exploration, you investigate the use of matrices to represent systems of linear equations.

a. Solve each of the following systems of equations.

1. $\begin{cases} 2s + 3t = 2 \\ -5s + 0.5t = -21 \end{cases}$

2. $\begin{cases} 12x - y = 10 \\ 4x + 2y = 1 \end{cases}$

b. Check your solutions by substitution.

c. Simplify the following matrix equations, then solve them:

1. $\begin{bmatrix} 2 & 3 \\ -5 & 0.5 \end{bmatrix} \bullet \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -21 \end{bmatrix}$

2. $\begin{bmatrix} 12 & -1 \\ 4 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

d. Compare the systems of equations and solutions from Part **a** with the simplified matrix equations and solutions from Part **c**.

e. Based on your observations in Part **d**, express each of the following systems of equations as a matrix equation.

1. $\begin{cases} 5t + 7r = 30 \\ 2t + 8r = 25 \end{cases}$

2. $\begin{cases} x - 3y = -2 \\ 4x + 2y = 7 \end{cases}$

3. $\begin{cases} y = 3 - 2x \\ 3x + y = 4 \end{cases}$

Mathematics Note

Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

may be written as a matrix equation:

$$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \bullet & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} e \\ f \end{bmatrix} \end{array}$$

The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, the matrix **X** is the **variable matrix**, since it represents the variables of the system, while the matrix **C** is the **constant matrix**, since it represents the constants of the system.

For example, the system

$$\begin{cases} 2s + t = 10 \\ 3s + 2t = 5 \end{cases}$$

may be written as the following matrix equation:

$$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \\ \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} & \bullet & \begin{bmatrix} s \\ t \end{bmatrix} & = & \begin{bmatrix} 10 \\ 5 \end{bmatrix} \end{array}$$

Discussion

a. Identify the coefficient matrix, the variable matrix, and the constant matrix in each of the matrix equations in Part **c** of the exploration.

b. How many solutions are possible for a system of two linear equations with two variables?

c. How does solving a system of equations help you analyze a linear programming problem?

Assignment

2.1 **a.** Write the following system of equations as a matrix equation and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} -2x + 3y = -19 \\ 5x - 2y = 31 \end{cases}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.2 **a.** Write the following system of equations as a matrix equation and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} 2d = 14 + c \\ -5c + 10d = 70 \end{cases}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.3 **a.** Write the system of equations represented by the matrix equation below:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.4 A nutritionist is planning a meal of pasta and vegetables for a client. The client should receive no more than 40% of the Daily Value of cholesterol and at least 700 calories from the meal. The following table shows some of the nutritional information for single servings of pasta and mixed vegetables.

Food	**% DV of Cholesterol**	**% DV of Sodium**	**Calories**
pasta	35%	25%	450
vegetables	1%	6%	150

a. Write a system of inequalities that represents all of the combinations of pasta and vegetables that meet the client's requirements.

b. Create a graph of the feasible region.

c. Identify the corner point of the feasible region that does not lie on a coordinate axis. Write a system of equations whose solution defines this corner point.

d. Express the system of equations in Part **c** as a matrix equation.

e. One serving of pasta has a mass of 210 g, while one serving of vegetables has a mass of 200 g. Determine the number of grams of pasta and vegetables the client should eat to receive 40% of the Daily Value of cholesterol and 700 calories from the meal.

2.5 The following table lists the Percent Daily Values of several nutrients for one serving of crackers (30 g) and one serving of apples (100 g).

Food	**Vitamin A**	**Iron**	**Calcium**	**Fat**	**Calories**
crackers	2	6	3.3	3	120
apples	1.8	0.6	4.7	0.9	56

a. Write a system of inequalities that represents all of the combinations of crackers and apples which provide at least 20% of the Daily Value of vitamin A but no more than 25% of the Daily Value of fat.

b. Create a graph of the feasible region.

c. Identify the corner point of the feasible region that does not lie on a coordinate axis. Write a system of equations whose solution defines this corner point.

d. Express the system of equations in Part **c** as a matrix equation.

e. Determine the numbers of grams of crackers and apples that provide 20% of the Daily Value of vitamin A and 25% of the Daily Value of fat.

* * * * *

2.6 **a.** Write the system of equations represented by the matrix equation below:

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \end{bmatrix}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.7 The Jewelry Emporium makes rings and necklaces. Each ring requires 5 g of metal and 1.5 hr of labor, while each necklace requires 20 g of metal and 1 hr of labor. Each week, the staff uses 500 g of metal and works a total of 80 hours.

a. Write both a system of equations and a matrix equation that describe this situation.

b. Determine the number of rings and necklaces that the Jewelry Emporium makes each week.

* * * * * * * * * *

Activity 3

In Activity **2**, you expressed systems of equations in matrix form. You also reviewed some methods of solving systems of equations. In this activity, you discover how to solve a matrix equation using matrix operations.

Exploration

This exploration introduces the use of matrices to solve a matrix equation.

a. Represent the following system as a matrix equation in the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} -8s - 3t = 10 \\ 4s + 6t = 5 \end{cases}$$

Mathematics Note

If $a \bullet b = b \bullet a = 1$, where 1 is the **identity** for multiplication, then a and b are **multiplicative inverses**. The product of a and the identity 1 is a. In other words, $a \bullet 1 = 1 \bullet a = a$. In the real number system, for example, the multiplicative inverse of $3/5$ is $5/3$ since

$$\frac{5}{3} \bullet \frac{3}{5} = \frac{3}{5} \bullet \frac{5}{3} = 1$$

The number 0 has no multiplicative inverse.

The multiplicative inverse of a, where $a \neq 0$, may be written as a^{-1}. The product of a and a^{-1} is always the identity, 1. In other words, $a \bullet a^{-1} = a^{-1} \bullet a = 1$. The multiplicative inverse of $3/5$ can be written as $(3/5)^{-1}$, where $(3/5)^{-1} = 5/3$.

A **multiplicative identity matrix** (**I**) is always a square matrix with entries of 1 along the diagonal that passes from the upper left to the lower right. All the other elements in an identity matrix are 0.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The product of a square matrix **M** and the identity **I** is **M**. In other words, $\mathbf{M} \bullet \mathbf{I} = \mathbf{I} \bullet \mathbf{M} = \mathbf{M}$. If it exists, the multiplicative inverse of the matrix **M** may be written as $\mathbf{M}^{-1}$ and $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$.

Given a matrix **A** of the form shown below, its **determinant** is $ad - bc$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrix **A** has an **inverse matrix** $\mathbf{A}^{-1}$ if and only if $ad - bc \neq 0$. Although only square matrices have inverses, not every square matrix has an inverse.

b. Using appropriate technology, determine the multiplicative inverse of the coefficient matrix you identified in Part **a**.

c.
1. Determine the multiplicative identity **I** for any 2×2 matrix.
2. The product of any matrix **A** and the identity **I** is **A**. Prove that the matrix you found in Step **1** is the 2×2 identity by multiplying it by matrix **A** below:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

d. Solve the matrix equation you wrote in Part **a** by multiplying both sides of the equation by the multiplicative inverse of **M**. **Note:** Each side of the matrix equation must be multiplied on the *left* by $\mathbf{M}^{-1}$.

e. Check your solution in Part **d** using each of the following methods.

1. Substitute your values for *s* and *t* back into the original equations.
2. Graph the original equations and check their point of intersection.
3. Substitute your values for *s* and *t* into the variable matrix **X** and verify that $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ using matrix multiplication.

Discussion

a. Compare the process for solving the linear equation $m \bullet x = c$ with the process for solving the matrix equation $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$.

b. Given a matrix equation $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$, which of the following equations could represent a first step in solving for the variable matrix **X**? Justify your response.

1. $\mathbf{M}^{-1} \bullet (\mathbf{M} \bullet \mathbf{X}) = \mathbf{M}^{-1} \bullet \mathbf{C}$
2. $\mathbf{M}^{-1} \bullet (\mathbf{M} \bullet \mathbf{X}) = \mathbf{C} \bullet \mathbf{M}^{-1}$
3. $(\mathbf{M} \bullet \mathbf{M}^{-1}) \bullet \mathbf{X} = \mathbf{C} \bullet \mathbf{M}^{-1}$
4. $(\mathbf{M} \bullet \mathbf{X}) \bullet \mathbf{M}^{-1} = \mathbf{C} \bullet \mathbf{M}^{-1}$

c. Describe how each of the following systems can be represented as a matrix equation in the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$.

1. $\begin{cases} 3y - 2z = 3 \\ 2y + 5z = 21 \end{cases}$

2. $\begin{cases} s + 2t - 1v = 7 \\ 5s + t + 5v = 5 \\ s + t - v = 4 \end{cases}$

d. Suggest a general rule for representing a system of three equations in three unknowns in matrix form.

e.
1. Describe how to solve each system of equations in Part **c** of the discussion using matrix multiplication and multiplicative inverses.
2. Identify at least two different ways that you could check your solutions.

f.
1. Describe the $n \times n$ identity for matrix multiplication.
2. What is the product of this identity and an $n \times n$ matrix **A**?

g. **1.** How can you determine whether or not a 2×2 matrix has an inverse?

2. What can you conclude about a system if its coefficient matrix does not have an inverse?

h. **1.** If a system of two equations in two unknowns has no solution, how can you tell this from looking at a graph of the equations?

2. Consider the system of two linear equations in two unknowns shown below.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

This system can be represented by the following matrix equation:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

Using the slopes of the two lines, determine what the relationship between a/d and b/e indicates about the number of solutions to the system.

3. If the coefficient matrix in Step **2** has no inverse, write a true equation expressing the relationship among a, b, d, and e.

i. How can solving a matrix equation help you analyze a linear programming problem?

Assignment

3.1 **a**. Find the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

b. Verify that your response to Part **a** is correct.

c. Write a 2×2 matrix that does not have an inverse. Defend your response.

3.2 Solve each of the following matrix equations and justify your solutions.

a. $\begin{bmatrix} 7 & 5 \\ 4 & 10 \end{bmatrix} \bullet \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$

b. $\begin{bmatrix} 20 & 12 \\ 5 & 3 \end{bmatrix} \bullet \begin{bmatrix} s \\ m \end{bmatrix} = \begin{bmatrix} 25 \\ 10 \end{bmatrix}$

c. $\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \bullet \begin{bmatrix} f \\ r \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \end{bmatrix}$

3.3 A nutritionist is planning a meal of pasta and vegetables for a client. The client should receive at least 300 calories from the meal but no more than 15% of the Daily Value of sodium. The following table shows some of the nutritional information for single servings of pasta and mixed vegetables.

Food	Serving Size	% DV of Cholesterol	% DV of Sodium	Calories
pasta	210 g	35%	25%	450
vegetables	200 g	1%	6%	150

a. Write a system of four inequalities using two variables that describes the constraints in this situation.

b. Graph the feasible region.

c. Each corner of the feasible region is determined by two intersecting lines. Write the equations for the lines related to the 15% limit on the Daily Value of sodium and the minimum of 300 calories.

d. Write a matrix equation for the pair of equations from Part **c**.

e. Solve the matrix equation you wrote in Part **d** and determine whether or not your solution lies in the feasible region.

f. Find the remaining vertices of the feasible region and determine which one minimizes the cholesterol in the meal.

3.4 The following table lists the Percent Daily Values of several nutrients for one serving of crackers (30 g) and one serving of apples (100 g).

Food	Vitamin A	Iron	Calcium	Fat	Calories
crackers	2	6	3.3	3	120
apples	1.8	0.6	4.7	0.9	56

a. Imagine that you would like to design a snack of crackers and apples that provides at least 20% of the Daily Value of calcium and at least 10% of the Daily Values of vitamin A and iron, while providing no more than 15% of the Daily Value of fat. Write six inequalities using two variables that describe these constraints.

b. Graph the feasible region.

c. Determine the coordinates of the vertices of the feasible region.

d. Determine the combination of crackers and apples (in grams) that minimizes the number of calories.

e. Does your solution provide a reasonable snack of crackers and apples? If not, how do you think the snack should be modified?

* * * * *

3.5 As described in Problem **2.7**, the Jewelry Emporium manufactures rings and necklaces. Each week, the staff uses a maximum of 500 g of metal and works a maximum of 80 hours. Each ring requires 5 g of metal and 1.5 hr of labor, while each necklace requires 20 g of metal and 1 hr of labor. The profit on each ring is $90; the profit on each necklace is $40.

a. Write a system of inequalities that describes all the constraints in this situation.

b. Graph the system and shade the feasible region.

c. Determine the vertices of the feasible region. When a vertex does not lie on a coordinate axis, use matrices to find it.

d. Determine the maximum amount of profit that the Jewelry Emporium can earn each week. Justify your response.

3.6 The table below shows some of the nutrients in one piece of fried chicken and one ear of corn.

	Vitamin A	**Potassium**	**Iron**	**Calories**
fried chicken	100 units	0 mg	1.2 mg	122
ear of corn	310 units	151 mg	1.0 mg	70

a. Imagine that you want to plan a meal of chicken and corn that contains a minimum of 1000 units of Vitamin A, 200 mg of potassium, 6 mg of iron, and 600 calories. Write a system of inequalities that describes these constraints.

b. Determine the coordinates of the vertices of the feasible region.

c. Determine the minimum cost of a meal that meets the constraints in Part **a** if one piece of fried chicken costs $0.90 and one ear of corn costs $0.75. Justify your response.

* * * * * * * * * *

Activity 4

Healthy meals typically contain more than two kinds of food. As the number of variables and constraints increases, using linear programming to analyze nutrition can become more complicated. The usual graphical representation of a feasible region is difficult to extend to three variables, and impossible for four or more. Fortunately, the corner principle is valid for any number of variables and constraints. To identify these corner points, however, you must solve systems of equations algebraically.

Exploration 1

Nutritionists recommend a diet that includes fruit. Table **2** lists the Percent Daily Values of some nutrients for several kinds of fruit:

Table 2: Percent Daily Values for Nutrients in Several Fruits

One Serving (100 g)	Vit. A	Vit. C	Iron	Calcium	Fat	Carbo-hydrates	Fiber	Calories
apples	1.8	11.7	0.6	0.3	0.9	4.7	4.0	56
bananas	0.0	23.3	2.2	1.0	1.8	8.5	1.6	97
oranges	4.0	83.3	2.2	4.1	0.3	4.1	2.0	49
pears	0.4	6.7	1.7	0.8	1.1	5.1	5.6	61
peaches	26.6	11.7	2.8	0.9	0.2	3.2	2.4	38
peaches (in syrup)	7.1	4.0	2.0	0.3	0.0	4.8	1.2	56
raisins	0.4	1.5	11.7	8.6	0.3	25.7	3.2	78

Imagine that you must create a recipe for fruit salad using apples, bananas, and oranges. The salad should consist of at least one serving (100 g) of each fruit and provide at least 20% of the Daily Values of vitamin A, iron, and calcium. These constraints are described by the following system of inequalities, where a, b, and r represent servings of apples, bananas, and oranges, respectively:

$$\begin{cases} a \geq 1 \\ b \geq 1 \\ r \geq 1 \\ 1.8a + 0.0b + 4.0r \geq 20 \ (\text{vitamin A}) \\ 0.6a + 2.2b + 2.2r \geq 20 \ (\text{iron}) \\ 0.3a + 1.0b + 4.1r \geq 20 \ (\text{calcium}) \end{cases}$$

a. Determine at least three ordered triples (a, b, r) that are solutions to the equation $1.8a + 0b + 4.0r = 20$.

b. Describe the graph of all the possible solutions in Part **a**.

c. Determine at least three ordered triples (a, b, r) that are solutions to the equation $0.6a + 2.2b + 2.2r = 20$.

d. Describe the graph of all the possible solutions in Part **c**.

Mathematics Note

The solution set of an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all 0, is a plane.

The graph of the equation $x + 2y + z = 10$, for example, is a plane. A portion of this plane is shown in Figure **3**. The triangular region represents the portion of the plane that falls in the first octant.

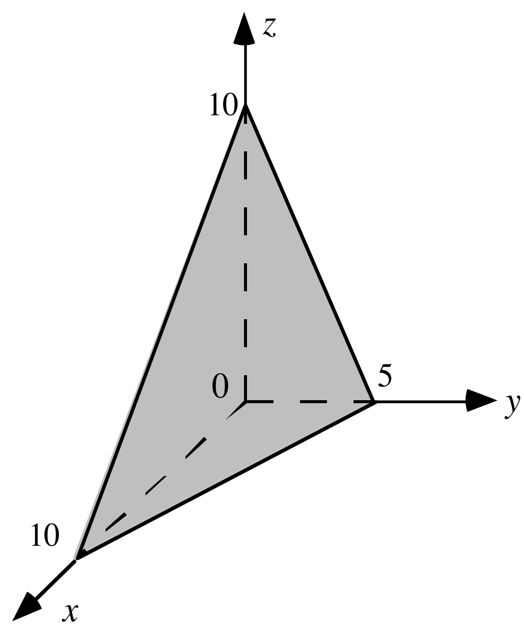


Figure 3: A portion of the graph of the equation $x + 2y + z = 10$

e. Use three index cards to represent three planes in space. Assuming that parallel planes do not intersect, make a model of each of the following situations:

1. three planes that are parallel to each other

2. two of three planes that are parallel to each other

3. three planes that intersect in a line

4. three planes that intersect in a point

5. three planes that are not parallel and have no points common to all of them.

Discussion 1

a. Why do you need three dimensions to graph the equations in Parts **a** and **c** of Exploration **1**?

b. If your models in Part **e** of Exploration **1** represent graphs of constraints, which, if any, includes a potential corner point for a set of feasible solutions? Explain your response.

c. What is the least number of planes that can intersect in a single point?

d. In a two-variable system, the graphs of inequalities are half-planes. Describe the graph of an inequality in a three-variable system.

e. Imagine that you are working with a problem that involves three variables.

1. Determine a system of inequalities for which the feasible region is the entire first octant.
2. Determine a system of inequalities for which the feasible region is a rectangular prism.

Exploration 2

In problems involving two variables, you used the corner principle to find the values of variables that minimized or maximized some objective function. The corner principle works equally well for problems with three or more variables. In this exploration, you solve some selected systems of three equations in three unknowns, without considering all the intersections necessary to analyze the situation completely. To ensure that a particular intersection yields a corner point, you should verify that its coordinates satisfy all the given constraints.

The corner points of the feasible region described by the six inequalities in Exploration **1** are located at the intersections of the following sets of planes:

$$\text{I: } \begin{cases} a = 1 \\ b = 1 \\ 0.6a + 2.2b + 2.2r = 20 \end{cases} \qquad \text{IV: } \begin{cases} a = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.6a + 2.2b + 2.2r = 20 \end{cases}$$

$$\text{II: } \begin{cases} b = 1 \\ r = 1 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases} \qquad \text{V: } \begin{cases} b = 1 \\ 0.6a + 2.2b + 2.2r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

$$\text{III: } \begin{cases} r = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases} \qquad \text{VI: } \begin{cases} 1.8a + 0.0b + 4.0r = 20 \\ 0.6a + 2.2b + 2.2r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

a. Write each of these systems of equations as a matrix equation.

b. Determine the corner point described by each set of planes, either by solving the system of equations using substitution, or by solving the corresponding matrix equation. Check each solution.

c. Write an equation for the total number of calories in a fruit salad containing *a* servings of apples, *b* servings of bananas, and *r* servings of oranges.

d. Using the corner points you found in Part **b** and the equation you wrote in Part **c**, determine a fruit-salad recipe that minimizes calories. Write the recipe in terms of the number of grams of each fruit in the salad.

e. Demonstrate that your solution in Part **d** appears to minimize calories by testing at least three other points in the feasible region.

Discussion 2

a. What do the three equations in the following system represent in terms of the situation described in Explorations **1** and **2**?

$$\begin{cases} r = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

b. In a two-variable system, an objective function defines a family of lines, some of which pass through the feasible region. Describe the graph of an objective function in a three-variable system.

c. Use geometry to explain why the corner principle works in three-variable systems.

Assignment

4.1 Defend or refute each of the following statements.

- **a.** The solution to a system of equations can be a single ordered pair of numbers.
- **b.** The solution to a system of equations can be a single ordered triple of numbers.
- **c.** A system of parallel lines represents a system of linear equations with at least one solution.
- **d.** A system of parallel planes represents a system of equations with no solutions.
- **e.** If two of three planes are parallel, the system of equations represented by the planes has no solutions.
- **f.** If three planes intersect in such a way that no two of them are parallel, then the system of equations represented by the three planes has at least one solution.

4.2 Consider the following system of three equations with the three variables x, y, and z:

$$\begin{cases} 0x + 0y + z = 1 \\ 0x + 0y + z = 2 \\ 0x + 0y + z = 3 \end{cases}$$

a. Describe the graph of this system.

b. Rewrite the system in matrix form.

c. Describe what happens when you solve this system using matrices.

4.3 Solve each of the following systems of equations and use geometric terms to describe the intersections of planes.

a. $$\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{cases}$$

b. $$\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 5 \\ 9x + 10y + 11z = 12 \end{cases}$$

c. $$\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 5 \\ 3x + 6y + 9z = 7 \end{cases}$$

d. $$\begin{cases} 2x - y + 3z = -9 \\ x + 3y - z = 10 \\ 3x + y - z = 8 \end{cases}$$

4.4 Imagine that you want to design a snack of bananas, peaches, and raisins that provides at least 10% of the Daily Values for vitamin A, iron, calcium, and fiber. Use the variables b, p, and r to represent the bananas, peaches, and raisins, respectively.

a. Use the information in Table **2** to write the four inequalities that represent these constraints.

b. The intersection of these four inequalities forms a three-dimensional region with four corners. Each of these corners is the intersection of the edges defined by three inequalities. Identify these four corners.

c. Explain why only one of these corners is a possible solution to the problem.

* * * * *

4.5 In Exploration **2**, you designed a recipe for fruit salad that provides 20% of the Daily Values of vitamin A, iron, and calcium while minimizing calories. Using the corner points determined in Exploration **2**, design another recipe for fruit salad that provides 20% of the Daily Values of the same three nutrients, while maximizing fiber.

* * * * * * * * * *

Summary Assessment

Since May 1994, the U.S. Food and Drug Administration (FDA) has required food manufacturers to show the Percent Daily Values of several nutrients on food labels. Find labels for two foods appropriate for a meal. The two labels should report more than 0% of the Daily Values for at least three of the same nutrients. Assuming that you are restricted to a 2000-calorie diet, determine how many servings of each food you must eat to obtain at least 50% of your Daily Values for the three nutrients, while minimizing your consumption of fat. Explain how you determined each amount and describe the nutritional benefits or drawbacks of the meal.

Module Summary

- **Linear programming** can be used to solve problems involving variables that are subject to linear **constraints**. The system of linear inequalities determined by these constraints defines a **feasible region**, a set of points that satisfies the system. Typically, such problems require the identification of an optimal value for an **objective function**, either a maximum or a minimum.
- According to the **corner principle**, if the objective function has a minimum or maximum value, it will occur at a corner point (or vertex) of the feasible region.
- Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

  may be written as the matrix equation:

$$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \bullet & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} e \\ f \end{bmatrix} \end{array}$$

  where matrix **M** is the **coefficient matrix**, matrix **X** is the **variable matrix**, and matrix **C** is the **constant matrix**.
- If $a \bullet b = b \bullet a = 1$, where 1 is the **identity** for multiplication, then a and b are **multiplicative inverses**. The product of a and the identity 1 is a. In other words, $a \bullet 1 = 1 \bullet a = a$. The number 0 has no multiplicative inverse.
- The multiplicative inverse of a, where $a \neq 0$, may be written as a^{-1}. The product of a and a^{-1} is always the identity, 1. In other words, $a \bullet a^{-1} = a^{-1} \bullet a = 1$.
- A **multiplicative identity matrix** (**I**) is always a square matrix with entries of 1 along the diagonal that passes from the upper left to the lower right. All the other elements in an identity matrix are 0.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- The product of a square matrix **M** and the corresponding identity **I** is **M**. In other words, $\mathbf{M} \bullet \mathbf{I} = \mathbf{I} \bullet \mathbf{M} = \mathbf{M}$.

- If it exists, the multiplicative inverse of the matrix **M** may be written as $\mathbf{M}^{-1}$ and $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$, where **I** is the identity for matrix multiplication.
- Given a matrix **A** of the form shown below, its **determinant** is $ad - bc$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  Matrix **A** has an **inverse matrix** $\mathbf{A}^{-1}$ if and only if $ad - bc \neq 0$. Although only square matrices have inverses, not every square matrix has an inverse.
- The solution set of an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all 0, is a plane.

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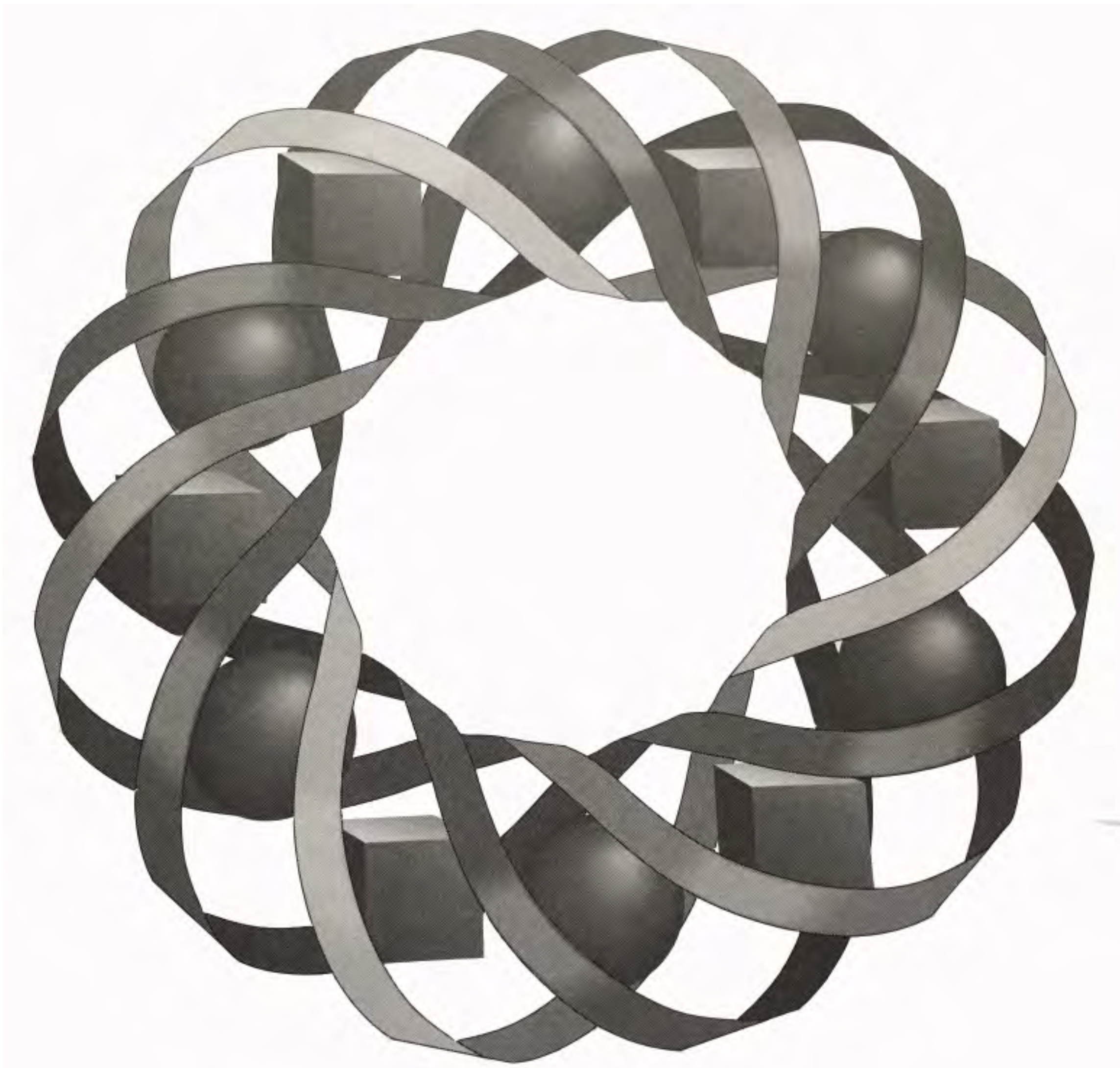
What’s Your Bearing?



In this module, you’ll explore how surveyors locate property lines, find the lengths of boundaries, and calculate the area of land parcels.

Todd Fife • Anne Merrifield

What’s Your Bearing?

Introduction

For anyone hiking in unfamiliar territory, a good compass can be just as essential as a good pair of shoes. Combined with a map, a compass allows you to determine not only your direction of travel, but also your location. A compass is an important tool in many professions—from forestry to ocean navigation.

Science Note

An **azimuth** can be described as the angle formed by rotating a ray clockwise from the ray representing north. In Figure **1**, for example, the ray from point *A* to point *B* has an azimuth of 60°. The ray from point *B* to point *C* has an azimuth of 120°.

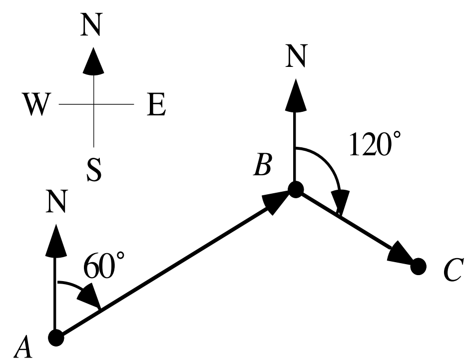


Figure 1: Azimuths of 60° and 120°

Exploration

Figure **2** shows the parts of a typical orienteering compass. The freely suspended magnetic needle indicates the direction of magnetic north. **Note:** To simplify the calculations in this module, *north* always refers to magnetic north.

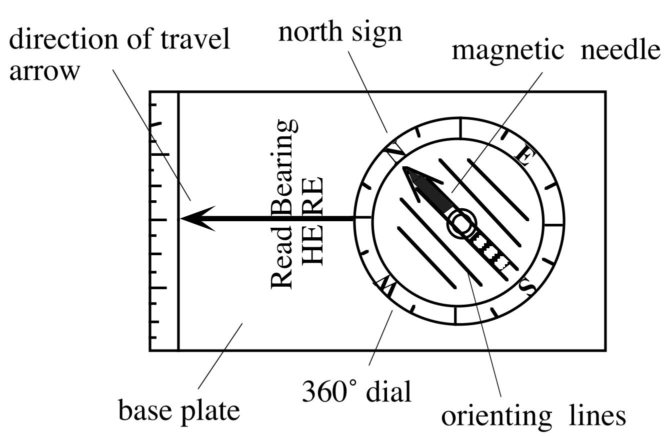


Figure 2: Parts of an orienteering compass

a. In this part of the exploration, you practice following a designated azimuth.

1. Select an azimuth. Turn the 360° dial on your compass until the desired degree measure aligns with the arrow indicating direction of travel.
2. Hold the compass level in your hand to permit the magnetic needle to swing freely. Point the arrow indicating direction of travel directly away from you.
3. While holding the compass steady, turn yourself until the north end of the magnetic needle points to the letter N on the dial. You are now facing your chosen azimuth.
4. Lift the compass close to eye level. Using the arrow indicating direction of travel as a line of sight, choose a distinct landmark in the direction of your azimuth. Lower the compass and walk the number of steps specified by your teacher toward the landmark.

b. Now that you have practiced with your compass, you can use it to follow a prescribed path.

1. Select a starting point and place a marker on the ground between your feet.
2. Select and record an initial azimuth. Walk along this initial azimuth for 30 steps, then stop.
3. Turn 120° clockwise from the direction in which you just walked. Record the new azimuth. Walk 30 steps along this azimuth and stop.
4. Turn another 120° clockwise from the last azimuth. Record the new azimuth. Walk 30 more steps in this direction, then stop.
5. You should now be back at your original starting point—indicated by the marker. If you did not return to the marker, try again.

c. Draw a detailed map of the path you completed, including the azimuth and length of each segment walked.

Discussion

a. In Part **b** of the exploration, what types of errors might have caused you to miss the marker?

b.
1. What type of geometric figure should be created by the path described in Part **b** of the Exploration? Explain your response in terms of the increase in azimuth at each turn.
2. How does the length of your steps affect this figure?

c. At the point where you stopped, how many more degrees should you turn in order to be facing in the same direction in which you started? Explain your response.

d. Would you expect to end up in the same starting place and with the same azimuth every time you followed a path that made three turns of 120˚? Explain your response.

e. How could you use a process like the one described in Part **b** of the exploration to make a rectangular path?

Activity 1

The directions indicated by azimuths also can be used to describe property boundaries. When a conflict arises over a property line, land surveys are often used to settle the dispute. A land survey is a map, similar to the one in Figure **3**, that shows the lengths and **surveying bearings** of the property lines as well as the area of the property.

Science Note

A **surveying bearing** is given as the number of degrees measured either to the east or west from a north or south reference. Since surveying bearings always measure acute angles, they are always greater than 0˚ and less than or equal to 90˚. In this module, *bearing* always refers to a surveying bearing.

Figure **3** shows two examples of bearings. The ray from point *A* to point *B* follows an azimuth of 52˚ and a bearing of N 52˚ E. This bearing means the ray from *A* to *B* is at an angle 52˚ east of north. The ray from point *B* to point *C* follows an azimuth of 111˚. Since bearings are always less than or equal to 90˚, the bearing for this ray is measured from south instead of north. The bearing of this ray is S 69˚ E. This means the ray from *B* to *C* is at an angle 69˚ east of south.

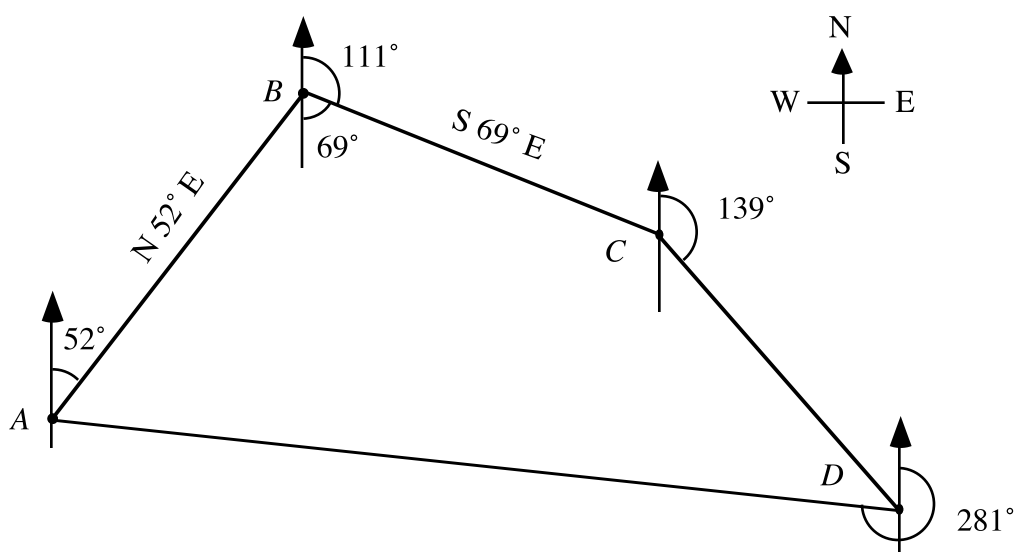


Figure 3: The relationship between azimuth and bearing

Exploration 1

In this exploration, you find the bearings for different azimuths.

a. Using the information in Figure **3**, find the bearings for:

1. the ray from point *C* to point *D*
2. the ray from point *D* to point *A*.

b. Figure **3** shows the bearings for the rays from *A* to *B* and from *B* to *C*. Determine the bearings for each of the following:

1. the ray from point *B* to point *A*
2. the ray from point *C* to point *B*.

Discussion 1

a. Describe how you determined whether a bearing should be measured from north or south.

b. Describe how to determine whether a ray is pointing east or west.

c. Explain why the bearing of the ray from point *A* to point *B* is different than the bearing of the ray from point *B* to point *A*.

d. The ray *MN* has a bearing of N 35° E. What is the bearing of ray *NM*? Describe how you determined your response.

e. Give two bearings for a ray pointing due east. Explain why both bearings are correct.

Exploration 2

In this exploration, you describe the boundaries of a plot of land.

a. One way to estimate distance is by paces. A pace is the distance covered with two normal walking steps. The length of a pace varies from person to person. Determine a method to estimate the length of your pace in meters. Record your estimate.

b. Using visible landmarks such as trees, corners of buildings, or flagpoles, identify a plot of land to be surveyed. Your plot should consist of at least five straight boundary lines.

c. Use a compass to determine the azimuth of each boundary, then estimate the length of each boundary in paces. Record these measurements.

d. Create a scale drawing of your plot. Identify the surveying bearing and length in meters of each boundary line.

e. Exchange maps with another student or group of students. To check the accuracy of the map, use the recorded bearings and lengths to walk the described boundaries. Determine whether or not the measurements match the actual boundaries of the property.

Discussion 2

a. What method did you use for estimating the length of your pace? How could your method be improved?

b. What problems did you encounter when trying to walk the boundaries described by the map in Part **e** of the exploration? What caused these problems?

c. Describe the process you used to calculate a surveying bearing from an azimuth.

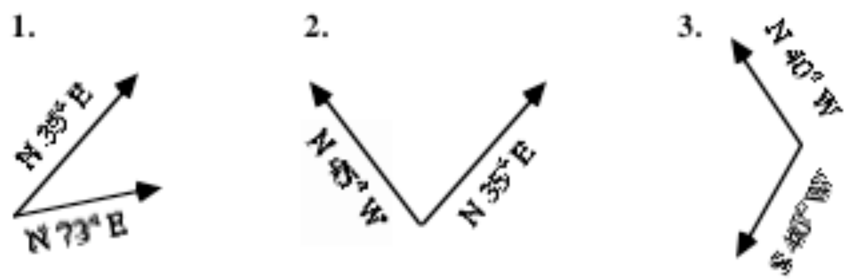
d. The method of surveying used in this exploration could introduce large amounts of error. Where and how might such errors occur?

e. How could you improve the accuracy of your land survey?

f. What tools do professional surveyors use to minimize the amount of error in their land surveys?

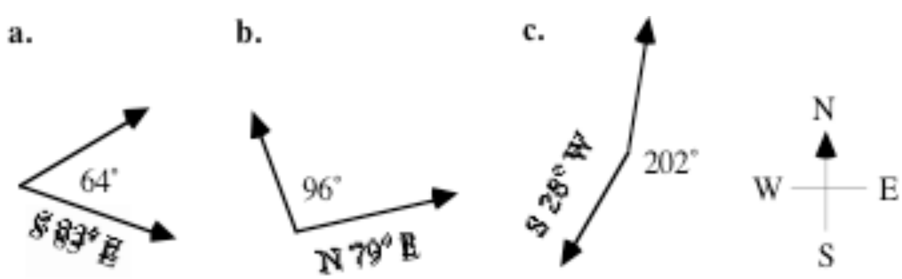
Assignment

1.1 **a.** Without using a protractor, find the measure of the angle between each of the following sets of bearings:

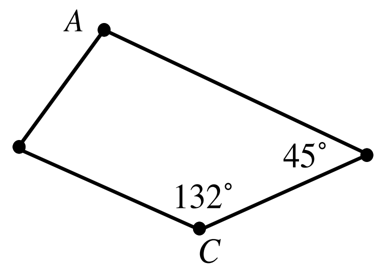


b. Develop a method for calculating the measure of an angle formed by two intersecting lines with known bearings.

1.2 Find the bearing of each unlabeled ray in Parts **a–c** below.



1.3 Use the following figure to find each bearing in Parts **a–c** below.

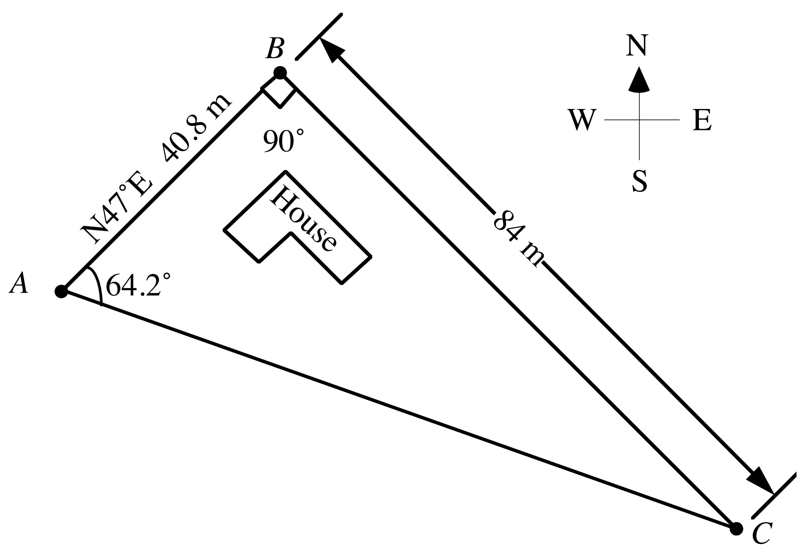


a. ray *BA*

b. ray *CB*

c. ray *CD*

1.4 A map of the Martin family's property shows some of the measurements taken during a land survey.



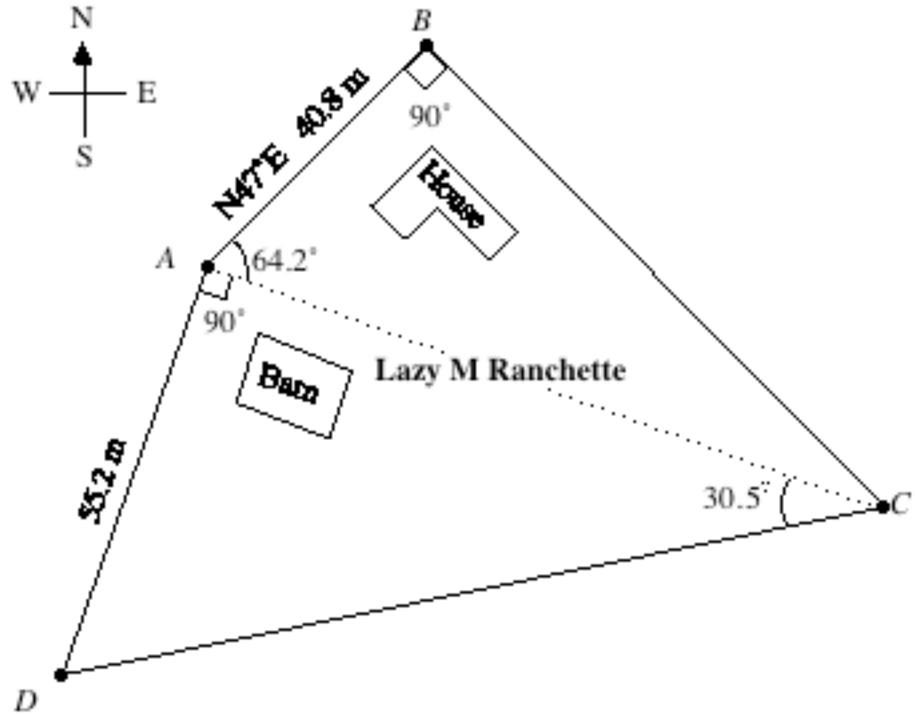
a. **1.** Determine the length of side *CA*.

2. Determine the measure of angle *C*.

b. **1.** Determine the bearing of ray *CA*.

2. Determine the bearing of ray *CB*.

1.5 The Martin family has purchased the lot adjoining their original property. As shown on the map below, they have named their new spread the Lazy M Ranchette. Use the information given on the map and the measurements you determined in Problem **1.4** to complete Parts **a–c**.



a. Calculate the length of side CD.

b. **1.** Determine the bearing of ray CD.

2. Determine the bearing of ray DA.

c. Determine the total area of the Lazy M Ranchette.

Science Note

Dividing a parcel of land into a system of connected triangles is known as **triangulation**. This technique is often used in surveying. The hexagonal plot of land in Figure **4**, for example, can be divided into four triangles so that each vertex of a triangle is also a vertex of the hexagon, and so that each triangle contains at least one side of the hexagon.

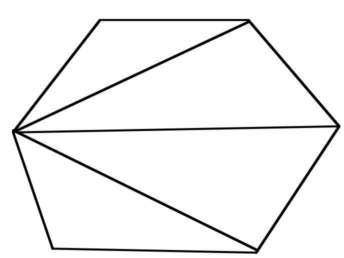


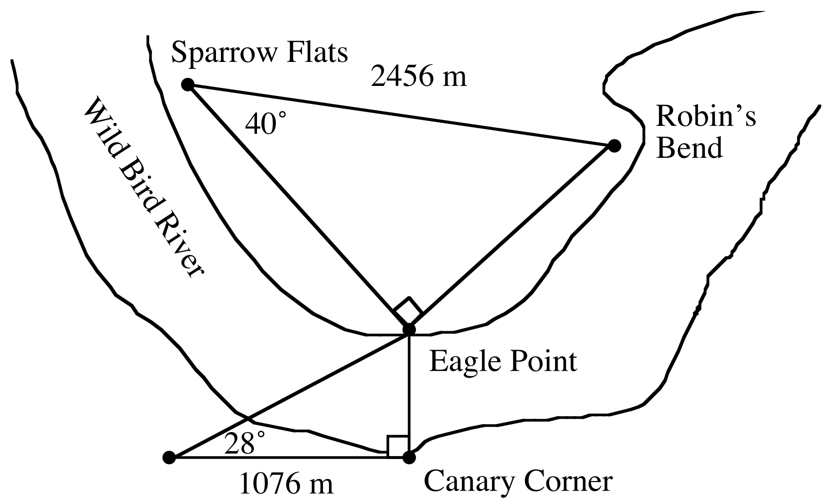
Figure 4: Hexagon divided into triangles

1.6 Describe the difficulties you might have encountered in completing a land survey of the Lazy M Ranchette if it had not been divided into triangles.

1.7 Draw a map of the completed land survey for the Lazy M Ranchette, including the lengths and bearings of all boundary lines, as well as the area of the property.

* * * * *

1.8 Imagine that you work for a construction company. Your firm has been hired to help develop the Wild Bird River Estates. A map of the property is shown below.



a. Your company must build a bridge from Canary Corner to Eagle Point. Use your knowledge of right-triangle trigonometry to determine the length of the bridge.

b. After the bridge is completed, the company must install sewer pipe from the pump stations at Sparrow Flats and Robin's Bend to the sewage treatment plant at Eagle Point.

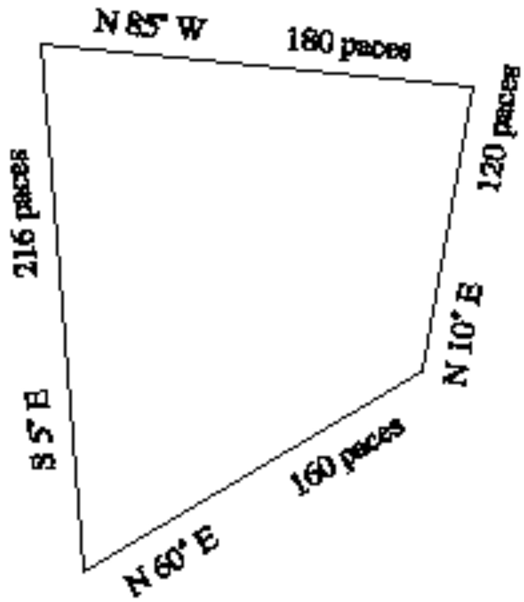
1. Find the distance from Sparrow Flats to Eagle Point and describe the method you used.
2. Find the distance from Robin's Bend to Eagle Point. Describe at least two different ways to determine this distance.

1.9 Measuring all the sides and angles on a piece of property can sometimes be expensive or impractical. In such cases, surveyors often make just a few measurements, then use trigonometry to determine the remaining distances and bearings.

a. Using triangulation, what is the minimum number of triangles into which any polygon can be divided?

b. How does trigonometry allow surveyors to take fewer measurements to complete a survey?

1.10 As part of a class project, Rocky made the following map of a park near his school. By mistake, however, he left the distance measurements in paces.



a. Using this map, Rosa tries to follow the boundaries of the park exactly as Rocky has directed. The size of her pace, however, is three-fourths that of Rocky's. Create a map that shows both the boundaries which Rocky surveyed and the path which Rosa took.

b. **1.** Does Rosa's path end at the same point where she started?

2. How does Rosa's path compare to the park's actual boundaries?

3. How does the area bounded by Rosa's path compare to the area of the park?

* * * * * * * * * *

Activity 2

The Westwolff family is considering the purchase of one of two triangular plots of land. To avoid potential boundary disputes, they have hired your company to survey both plots. A surveying team is sent out to take the necessary measurements. When the team returns to the office, the surveyors hand you the map shown in Figure **5** below.

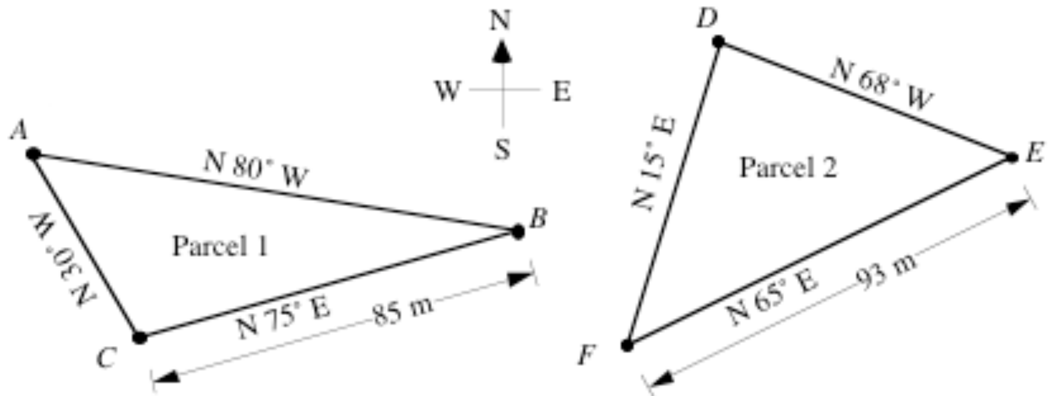


Figure 5: Field data for two land surveys

It is your job to determine the unknown lengths, calculate the areas of the plots, and complete the two surveys. What methods could you use to find this information? In the following explorations, you develop some methods for completing this job.

Exploration 1

As shown in Figure **5**, Parcel 1 appears to be shaped like an obtuse triangle. In this exploration, you use your knowledge of right-triangle trigonometry to interpret sine and cosine values for angle measures greater than 90˚.

a. Figure **6** below shows a triangle *ABC* with one vertex located at the origin of a two-dimensional coordinate system.

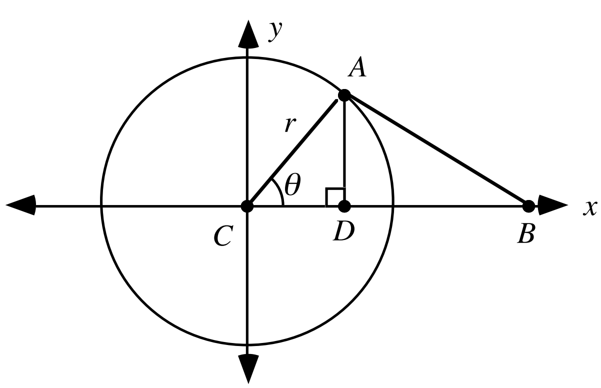


Figure 6: Triangle drawn on a set of coordinate axes

Using a geometry utility, create a drawing similar to the one in Figure **6**. In your construction, point C should be the center of the circle and the origin of a two-dimensional coordinate system. (If your geometry utility reports coordinates, make sure that the measure of r equals the x-coordinate of the point where the circle and the x-axis intersect.) Point A should be a freely moving point on the circle. Segment AD is the altitude of ΔABC through point A.

Note: Save this construction for use in Activity **3.**

b. The lengths CD and AD represent the absolute values of the x- and y-coordinates, respectively, of a point on the circle.

Use your construction in Part **a**, along with right-triangle trigonometry, to complete Table **1** for several angles with measures between 0 and 90°.

Table 1: Coordinates of point *A* in terms of *r*

$m\angle ACB$	r	$\cos\angle ACB$	$\sin\angle ACB$	Coordinates of A

c.
1. Describe the relationship between the cosine of θ, the radius r, and the x-coordinate of point A.
2. Describe the relationship between the sine of θ, the radius r, and the y-coordinate of point A.

d. Use the relationships you determined in Part **c** to extend Table **1** to include angle measures between 90° and 180°.

e. Use the data in Table **1** to create a scatterplot of $\sin\angle ACB$ versus $m\angle ACB$.

f. Use the data in Table **1** to create a scatterplot of $\cos\angle ACB$ versus $m\angle ACB$.

Discussion 1

a.
1. In Figure **6**, why does the length of $\overline{CD}$ represent the absolute value of the x-coordinate of point A?
2. Why does the length of $\overline{AD}$ represent the absolute value of the y-coordinate of A?

b.
1. If $r = 1$ in Figure **6** and $0^\circ \le \theta < 180^\circ$, what does the sine of θ represent?
2. What does the cosine of θ represent?

c. When the measure of $\angle ACB$ in Exploration **1** was between 90° and 180°, point A was in the second quadrant. Figure **7** shows an example of ΔACB with point A in the second quadrant and altitude AD.

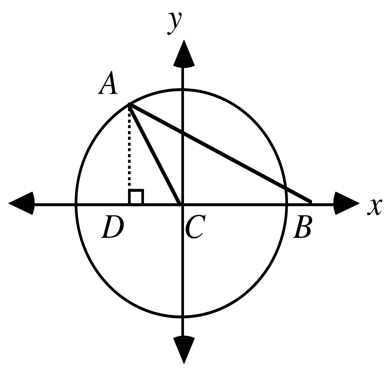


Figure 7: ΔABC with altitude AD

1. What are the signs of the x- and y-coordinates of point A?
2. Write an equation using $m\angle ACB$ and $m\angle ACD$.

d.

1. Considering that $r \sin \angle ACB$ is the y-coordinate of A, how do $\sin\angle ACB$ and $\sin\angle ACD$ compare?
2. Based on your response above, what appears to be the relationship between $\sin\theta$ and $\sin(180 - \theta)$? How does the scatterplot you created in Part **e** of Exploration **1** support this conjecture?

e.

1. Considering that $r \cos\angle ACB$ is the x-coordinate of A, how do $\cos\angle ACB$ and $\cos\angle ACD$ compare?
2. Based on your response above, what appears to be the relationship between $\cos\theta$ and $\cos(180 - \theta)$? How does the scatterplot you created in Part **f** of Exploration **1** support this conjecture?

f. Describe a method for finding the sine and cosine of an angle with a measure of 130°, given the sine and cosine of an angle with a measure of 50°.

g. In general, how could you find the sine and cosine of an angle with a measure of n° given the sine and cosine of an angle with a measure of $(180 - n)^\circ$?

Exploration 2

In this exploration, you apply what you know about right-triangle trigonometry to develop a method for determining the unknown sides and angles of non-right triangles.

a. Using a geometry utility, draw an acute triangle like the one shown in Figure **8**.

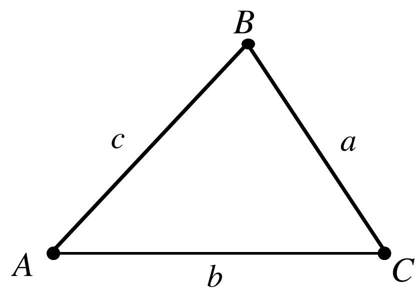


Figure 8: An acute triangle

b. Find the measures of $\angle BAC$, $\angle ABC$, and $\angle ACB$.

c. Find the lengths a, b, and c.

d. Construct an altitude from vertex B to the line containing the opposite side $\overline{AC}$. As shown in Figure **9**, label the altitude h and its intersection with line AC point D. The altitude divides ΔABC into two other triangles. Describe these triangles.

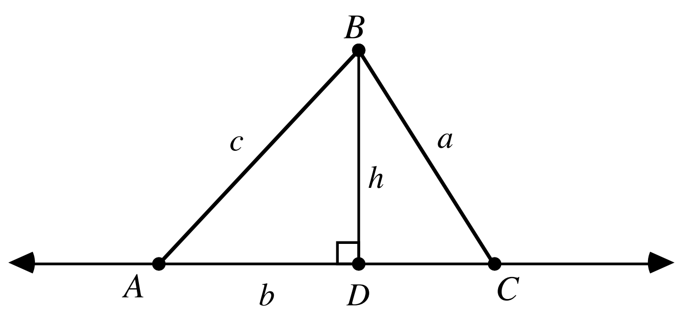


Figure 9: ΔABC with altitude BD

e. Write equations for $\sin\angle BAC$ and $\sin\angle ACB$ in terms of h, c, and a.

f. Solve for h in each equation in Part **e**.

g.
1. Use your equations from Part **f** and the geometry utility to calculate a value for the altitude.
2. Measure the altitude.
3. Observe how the values for h in Step **1** compare to the measured altitude as you drag any vertex of the triangle.

h. Since the two expressions from Part **f** are both equal to h, they are equal to each other. Set these two equations equal to each other and solve for the following ratio:

$$\frac{\sin\angle ACB}{c}$$

Discussion 2

a. What does the relationship you found in Part **h** of Exploration **2** indicate about the ratio of the sine of each angle in a triangle to the length of its opposite side?

Mathematics Note

The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles.

For example, Figure **10** shows a triangle ABC.

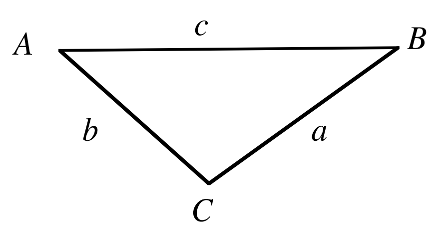


Figure 10: Triangle *ABC*

According to the law of sines,

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

b. Describe how the law of sines can be used to find the unknown values in Figure **11**.

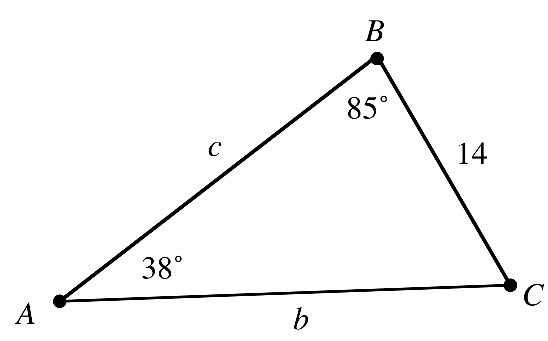


Figure 11: A triangle with some unknown measures

c. In one of the triangles in Figure **12**, the law of sines cannot be used to find the unknown sides and angles. Identify this triangle and explain why the law of sines cannot be used to determine these measurements.

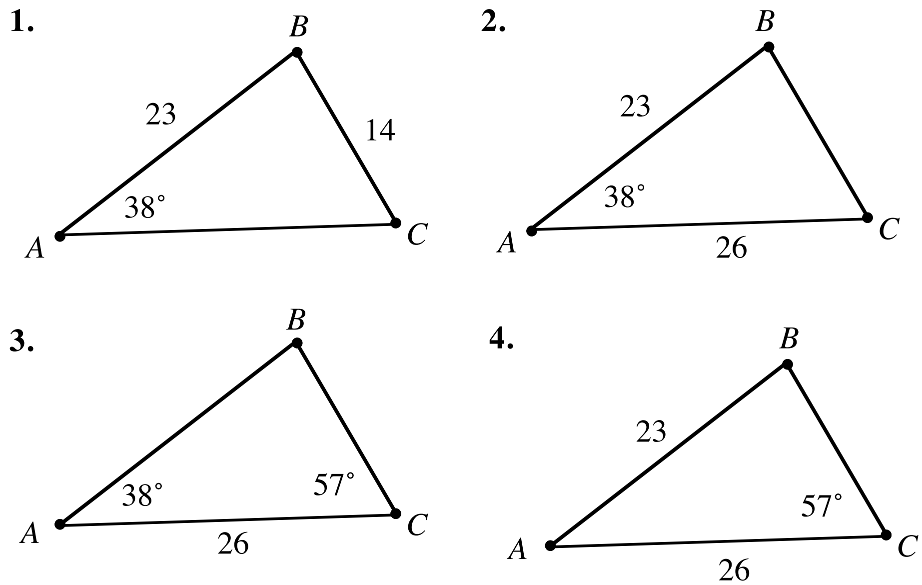


Figure 12: Triangles with unknown measures

d. Describe the minimum information needed to use the law of sines to determine the measures of all angles and sides of a triangle.

e. In Exploration **2**, you discovered a relationship between the sine of an angle and the altitude of a triangle. Describe how you could use this relationship to determine the area of the triangle in Figure **13**.

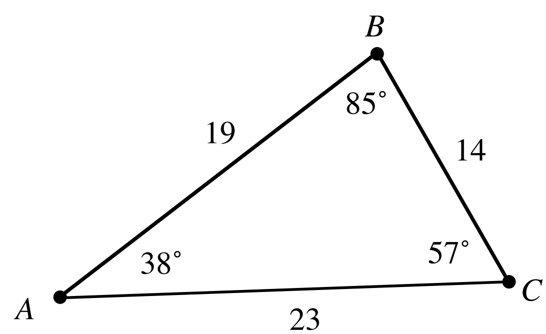
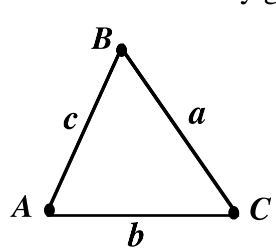


Figure 13: A triangle

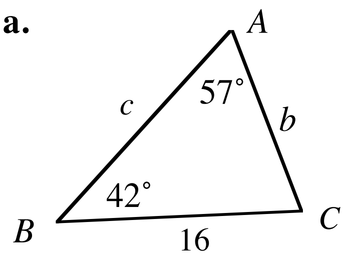
f. Write a formula to find the area of any general triangle ABC.



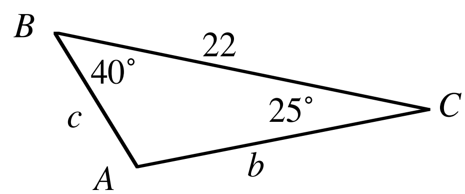
Assignment

2.1. Determine the measures of the unknown sides and angles in each of the following triangles. If any of the missing values cannot be determined using the law of sines, explain why not.

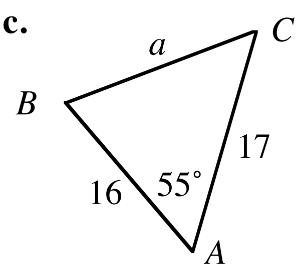
a.



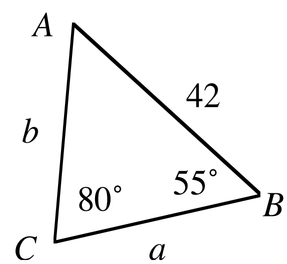
b.



c.

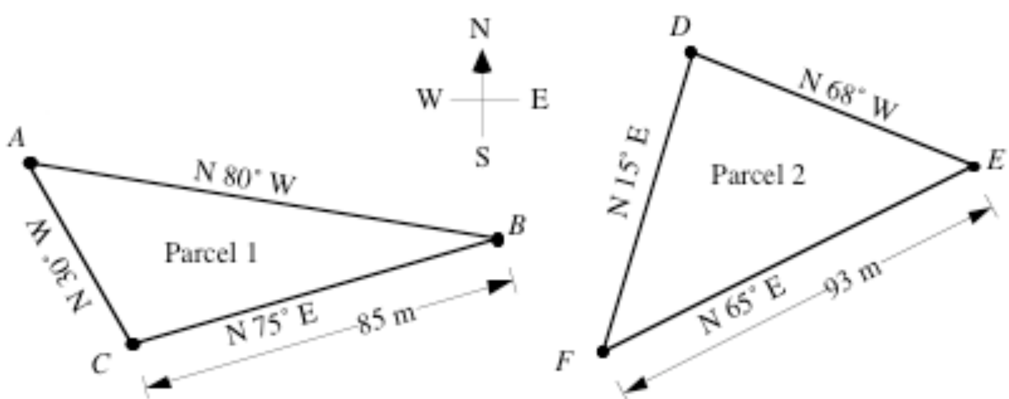


d.



2.2 Find the areas of the four triangles in Problem **2.1**.

2.3 As mentioned in the introduction to this activity, the Westwolff family plans to buy one of two triangular plots of land. The field data from the surveys of the two properties are shown in the figure below.



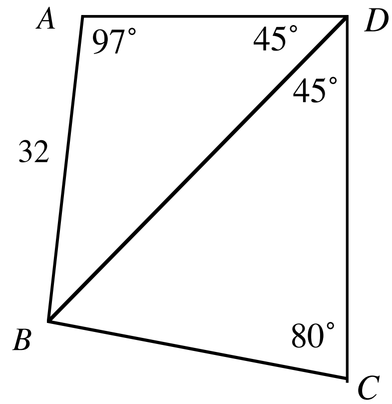
a. Determine the measure of each interior angle and the length of each side of Parcel 1.

b. Determine the measure of each interior angle and the length of each side of Parcel 2.

c. Calculate the areas of Parcels 1 and 2.

d. If Parcel 1 costs $21,000 and Parcel 2 costs $26,000, which one do you think is the better buy based on the cost per unit of area? Justify your response.

2.4 The diagram below shows a polygon *ABCD*.

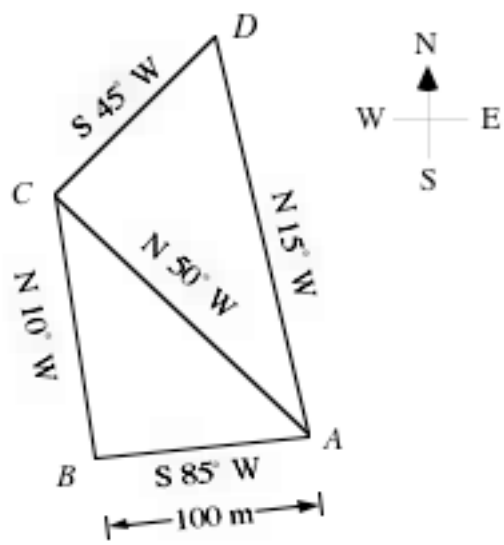


a. Use the information given in the diagram to calculate each of the following lengths:

1. *AD*
2. *BD*
3. *BC*
4. *DC*

b. Determine the area of polygon *ABCD*.

2.5 A surveying crew has collected the data shown on the following map.



a. Determine the measures of all interior angles on the map.

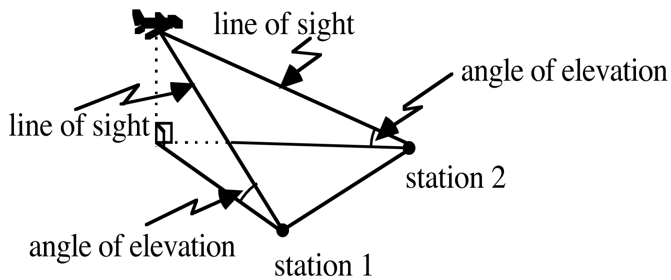
b. Determine the measures of the sides of all triangles on the map.

c. Calculate the area of each triangle.

d. Calculate the area of the plot of land described by polygon *ABCD*.

* * * * *

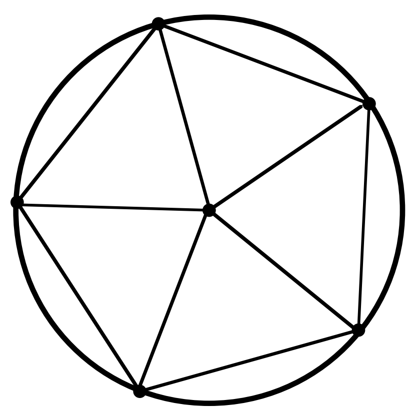
2.6 The process of triangulation also can be used to determine the location and altitude of an airplane from the ground. For example, the diagram below shows two tracking stations monitoring the location of a plane. Observers at each station measure the angle of elevation (the angle measured from horizontal) between themselves and the plane, as well as the angle formed by the line of sight to the plane and the line connecting the stations. Given these angles and the distance between the stations, they can calculate the location of the plane.



Suppose that the distance between the two stations is 2 km, the plane's angle of elevation at station 1 measures 39˚, and the plane's angle of elevation at station 2 measures 60˚. At station 1, the angle formed by the line of sight to the plane and the line between the stations reasures 43˚; at station 2, this angle measures 65˚.

What is the plane's altitude at the time these observations were recorded? Explain your response.

2.7 As part of her design for a new park, a city planner has decided to build a circular garden. The garden will feature five walkways radiating from its center like the spokes of a wheel, as shown in the diagram below. Five more walkways will connect the spokes to form a regular pentagon. If the radius of the circle is 40 m, find the sum of the lengths of all the walkways.



2.8 Imagine that you and an associate are on opposite sides of a river, surveying land for a new city bridge. There is a large tree on your side of the river. From the point where you are standing, the angle between your associate and the tree measures 48°. From the point where your partner is standing, the angle between you and the tree measures 62°. The distance from you to the tree is 160 m.

a. Make a sketch of this situation.

b. What is the distance between you and your associate? Explain your response.

* * * * * * * * * *

Activity 3

In Activity **2**, you used the law of sines to determine unknown measurements in triangles. However, you also discovered that the law of sines has some limitations. As a surveyor, you have been asked to finish a survey for the plot of land shown in Figure **14**. The results of the field measurements are indicated on the map. In this case, the lengths of two sides are known, as well as the measure of the angle between those sides. Is this enough information to proceed without taking more measurements?

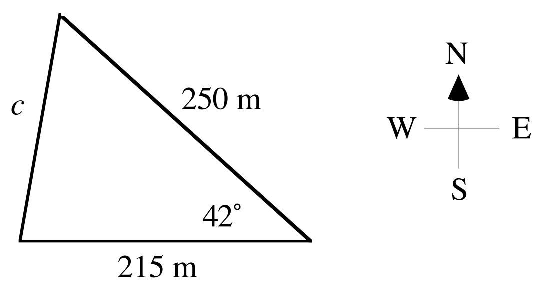


Figure 14: A partially completed land survey

Exploration 1

The Pythagorean theorem states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. As you have seen in other modules, this relationship can be represented by constructing squares on each side of a right triangle and comparing the areas of the squares. In this exploration, you use a similar model to explore the relationships among the sides of acute and obtuse triangles.

a. Using a geometry utility and your construction from Activity **2**, create a diagram similar to the one shown in Figure **15**. In this construction, point C is the center of the circle, point A is a moveable point on the circle, and ΔABC is a right triangle with $\overline{AB}$ as its hypotenuse.

The quadrilaterals on each side of the triangle are squares. The sides of each square are congruent to the corresponding side of the triangle. These squares should be created so that they remain squares when the sides of the triangle are moved.

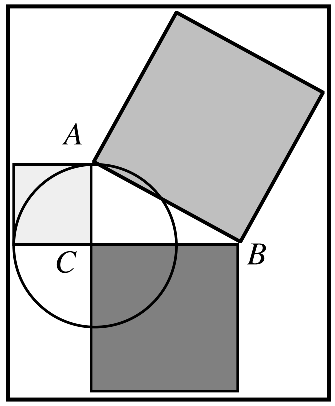


Figure 15: A right triangle with squares on its sides

b. Record the area of each square in Part **a** and the measure of $\angle ACB$.

c. Compare the area of the square constructed on $\overline{AB}$ to the sum of the areas of the other two squares.

d. Move point *A* to several different locations on the circle, repeating Parts **b** and **c** at each location. **Note:** Save this construction for use in Exploration **2**.

Discussion 1

a. **1.** In general, what is the relationship between the area of the square constructed on $\overline{AB}$ to the sum of the areas of the other two squares when the measure of $\angle ACB$ is greater than 90˚?

2. What is this relationship when the measure of $\angle ACB$ is less than 90˚?

b. Describe your generalizations from Part **a** of Discussion **1** in terms of the type of triangle—right, obtuse, or acute— and the lengths of its sides.

c. Recall that the converse of a statement in the form "If A, then B" is the statement "If B, then A." The converses of the generalizations you made in Part **a** also are true. How could you use these statements, along with the converse of the Pythagorean theorem, to classify a triangle as acute, obtuse, or right knowing only the measures of its sides?

Exploration 2

In Exploration **1**, you found that in an obtuse triangle, the square of the length of the side opposite the obtuse angle is greater than the sum of the squares of the lengths of the other two sides. You also found that in an acute triangle, the square of the length of one side is less than the sum of the squares of the lengths of the other two sides. In this exploration, you discover a more precise way to describe these generalizations.

a. Remove the squares from the construction you created in Exploration **1**. Move point A so that $\angle CAB$ is obtuse.

b. Construct an altitude from vertex A to the opposite side $\overline{BC}$ to form two right triangles. Label the sides of ΔABC, the altitude, and the lengths of the two segments formed on side $\overline{BC}$ as shown in Figure **16**.

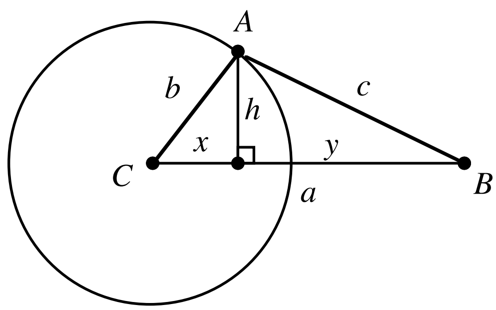


Figure 16: Labeled construction

c. Use trigonometry to express x in terms of b and $\angle ACB$.

d. Use the expression for x found in Part **c** and the Pythagorean theorem to write an expression for h^2 in terms of b and $\angle ACB$.

e. Use the Pythagorean theorem to express h^2 in terms of c and y.

f. Express y in terms of x and a.

g. Substitute the expression for y from Part **f** and the expression for x from Part **c** into the expression for h^2 from Part **e** to write h^2 in terms of the sides of ΔABC and $\angle ACB$.

h.

1. Since the expressions from Parts **d** and **g** are both equal to h^2, they are equal to each other. Set these two expressions equal to each other and solve for c^2.
2. How does the equation from Step **1** appear to be related to the Pythagorean theorem?

Mathematics Note

The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other two sides, minus twice the product of the lengths of these sides and the cosine of the angle included between them. In Figure **17**, for example, $c^2 = a^2 + b^2 - 2ab\cos\angle C$.

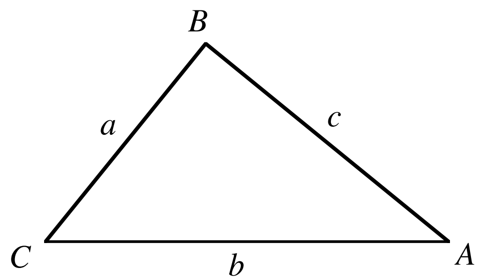


Figure 17: Triangle *ABC*

The law of cosines can often be used to determine an unknown length or angle measure in a triangle. For example, if $a = 10$ cm, $b = 12$ cm, and $m\angle ACB = 35°$, then the law of cosines can be used to find c as follows:

$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$
$$c^2 = 10^2 + 12^2 - 2(10)(12)\cos 35°$$
$$c^2 \approx 47.40$$
$$c \approx 6.89 \text{ cm}$$

Discussion 2

a. Compare the equation you wrote in Part **h1** of Exploration **2** to the law of cosines described in the mathematics note.

b. **1.** When $\angle ACB$ is acute, what is the sign of its cosine?

2. When $\angle ACB$ is obtuse, what is the sign of its cosine?

c. Describe how you could use the law of cosines to find a in Figure **18**.

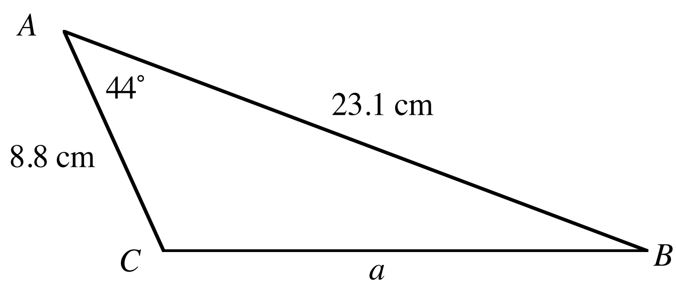


Figure 18: Triangle with an unknown side

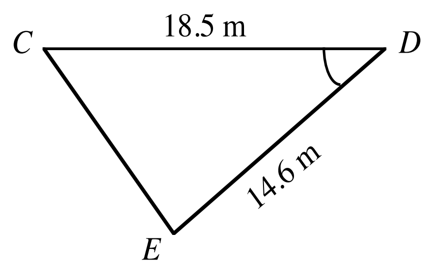
d. Given the information in Figure **18** and the value of a, how could you determine the measure of one of the remaining angles using:

1. the law of cosines?
2. the law of sines?

e. The law of cosines works with acute triangles and obtuse triangles. Does it also work for right triangles? Justify your response.

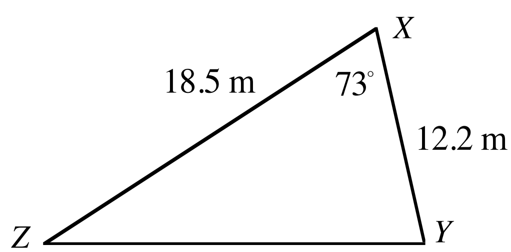
Assignment

3.1 **a.** Find the length of $\overline{CE}$ in the map below.



b. Using your response to Part **a**, describe how to find the measures of the other two angles of the triangle.

3.2 The diagram below shows a map of the Martin family's garden plot.



Use the measurements given to find each of the following:

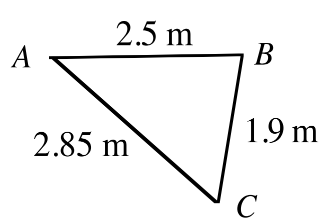
a. YZ

b. $m\angle XZY$

c. $m\angle XYZ$

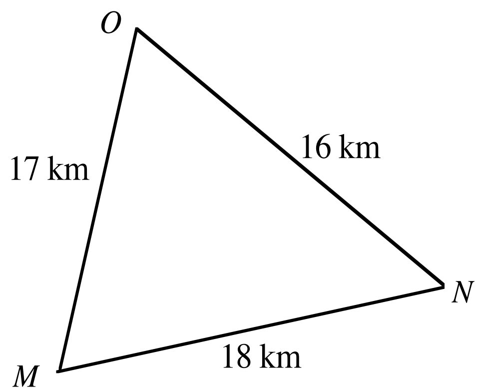
d. the area of the triangle

3.3 **a.** Use the law of cosines to find $m\angle BAC$ in the triangle below.



b. Given $m\angle BAC$, describe two methods for determining $m\angle ABC$ and $m\angle ACB$.

3.4 The following sketch shows some measurements taken during a survey of a triangular plot of land.



Use the information given to determine each of the following:

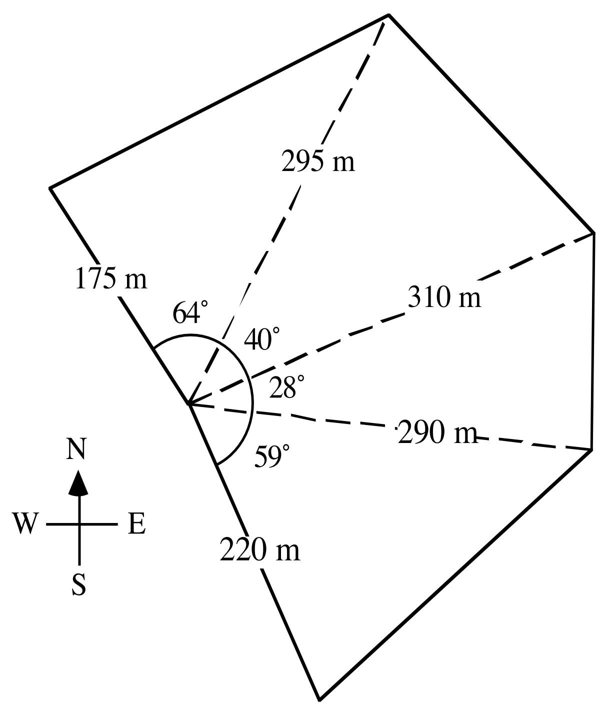
a. $m\angle NMO$

b. $m\angle MNO$

c. $m\angle MON$

d. the area of the triangle

3.5 The following map shows the data collected during a field survey.



Using the information recorded on the map, determine:

a. the length of each segment along the perimeter of the property

b. the total area of the property.

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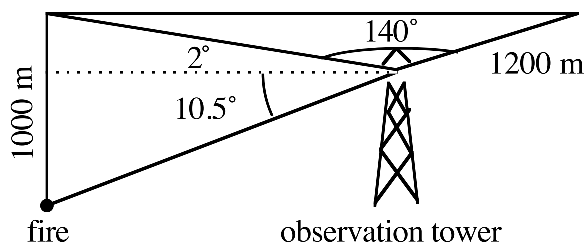
3.6 Two ships leave San Francisco Bay at the same time. As they pass under the Golden Gate Bridge, one ship is approximately 0.5 km due north of the other, cruising at 21 knots (approximately 39 km/hr) at a survey bearing of S 70° W. The other ship is traveling due west at 24 knots (approximately 44 km/hr). How far apart are the two ships after 3 hr?

3.7 Fire lookouts live and work in small, one-room observation towers high in the mountains. Part of their job involves reporting forest fires and helping to coordinate firefighting efforts.

During one fire, a plane delivering a load of fire retardant passes 1000 m above the flames. The lookout measures the angle of elevation to the plane and the angle of depression to the fire. At that moment, a second plane flying at the same altitude reports its location as 1200 m from the observation tower. Given the measurements shown in the diagram below, how far is the second plane from the first plane?



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Research Project

Obtain a survey map of a property that interests you. Verify that the areas given for each plot of land on the survey are correct and that the sum of all the angles of each plot of land produces a closed polygon. In your report, sketch a map of the property and include a detailed description of how you completed your calculations.

Summary Assessment

1. Use the following steps to complete a survey of a plot of land specified by your teacher.
   - **a.** Devise a method of dividing the plot into triangles that will minimize the number of measurements required.
   - **b.** Using a compass, measure and record the necessary bearings.
   - **c.** Determine the lengths of all boundaries.
   - **d.** Find the total area of the plot of land.
2. Write a summary of the procedure you used to survey the plot, including:
   - **a.** a field sketch showing your preliminary measurements
   - **b.** a scale drawing of the completed survey with all appropriate information
   - **c.** descriptions of where and how you used trigonometric ratios, the law of sines, and the law of cosines to complete the survey.

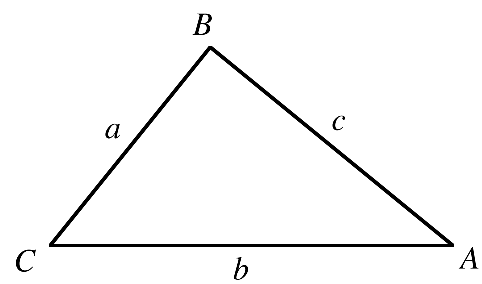
Module Summary

- An **azimuth** can be described as the angle formed by rotating a ray clockwise from the ray representing north.
- A **surveying bearing** is given as the number of degrees measured either to the east or west from a north or south reference. Since surveying bearings always measure acute angles, they are always greater than 0° and less than or equal to 90°.
- The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles. In a triangle *ABC*,

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

- The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of the lengths of these sides and the cosine of the angle included between them. In a triangle *ABC*,

$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$



- The area of any triangle can be calculated if the measure of two sides and the included angle are known. In the triangle ΔABC,

$$Area = \frac{1}{2}ab\sin C$$

where a and b are the length of the sides and C is the measure of the included angle.

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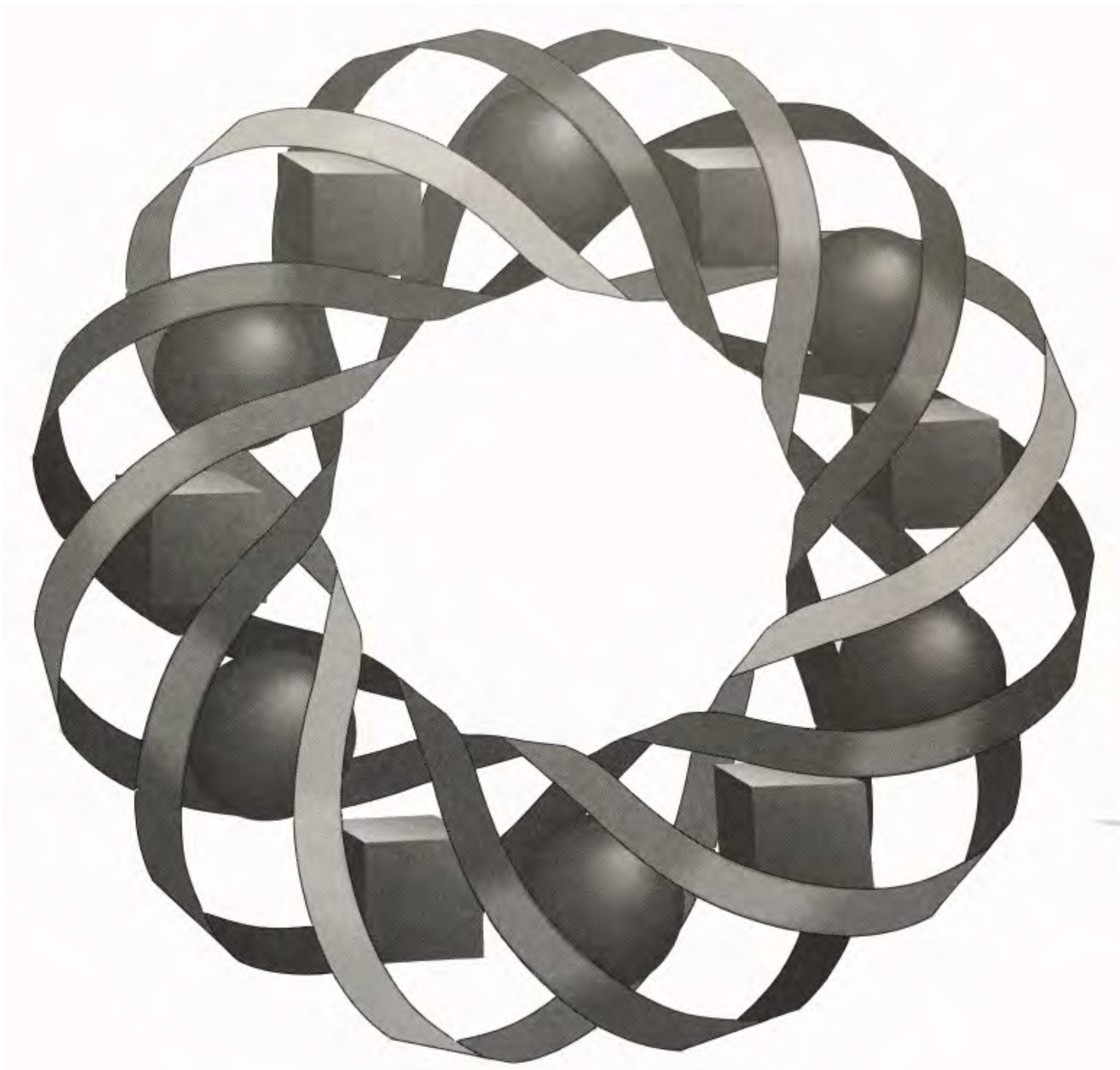
Taste Test



What's your favorite soft drink? Do you drink it because of its taste, or for some other reasons? In this module, you explore the bewildering number of choices involved in marketing a new soft drink.

Tom Teegarden • Deanna Turley

Danny Jones • Laurie Paladichuk

Taste Test

Introduction

Soft-drink manufacturers and other food companies often conduct taste tests to try out new product lines. Marketers know that placement of products on tables and the order in which products are presented may influence consumers' responses. Depending on the number of possibilities, however, it may not be practical to consider all the different ways to arrange products.

Activity 1

Imagine that you are the director of marketing at a soft-drink company. As part of a new advertising campaign, you decide to conduct a taste test using four of your company's products. The four soft drinks you've selected are 6-Down, Dr. Salt, Valley Fog, and Branch Tea. How should you organize the taste test?

Exploration

During the taste test, samples of each soft drink will be served at specially designed stations to a panel of judges. The serving stations will be placed on tables in a large television studio.

Each table must have at least one serving station, but can accommodate no more than three. To heighten visual impact, the tablecloths should be one of the five colors in the company's logo and brand labels: red, white, blue, black, or green.

a. List all the different ways you could design a table according to the above specifications. For example, one possible design places two serving stations on a table with a white tablecloth.

b. Organize your list to verify that no possible design has been overlooked.

c. Recall that a **tree diagram** can be used to show all the possible outcomes of an experiment. Create a tree diagram that shows all the possible table designs for the taste test.

d. Compare the number of possible designs indicated in your tree diagram with the number you determined using the list from Part **b**.

e. Suppose that two different types of tables are available: rectangular or circular. Create a tree diagram that shows how these new choices affect the number of possible designs.

f. Determine a method for finding the number of possible designs in Part **e** without making a list or a tree diagram.

Discussion

a. Describe the method you devised in Part **f** of the exploration.

b. How is this method reflected in the organization of the tree diagram you produced in Part **e**?

Mathematics Note

The **fundamental counting principle** states that if one selection can be made in h ways, and for each of these ways a second selection can be made in k ways, then the number of different ways the two selections can be made is $h \bullet k$.

For example, the number of outfits that can be made with 2 different pairs of pants and 3 different shirts is $2 \bullet 3 = 6$.

c. Describe how the fundamental counting principle can be used to determine the number of possible table designs considering the list of choices below.

- You must choose a table from n possible shapes.
- You must choose a tablecloth from m possible colors.
- You must select a specific number of serving stations per table, from at least 1 to no more than s.
- You must decide whether to test 2, 3, or 4 different soft drinks at each station. (Assume that each station must offer the same number of soft drinks.)

Assignment

1.1 Imagine that you are planning to conduct a taste test at your school. You have decided to use 7 rooms, with 3 tables in each room. Each table has 4 serving stations and each station has 3 unlabeled soft-drink dispensers.

a. Determine a method for identifying each soft-drink dispenser with a distinct label.

b. How many different labels are needed to identify all the dispensers?

1.2 Consider a taste test that involves 20 different soft drinks. In how many different ways can the judges select first-place and second-place winners?

1.3 A sporting goods store stocks ice chests in 3 different sizes: small, large, and jumbo. Each size comes in 5 different colors: red, blue, green, purple, and gray. Each ice chest also can be purchased with or without an electric cooling unit.

a. Create a tree diagram that shows all the possible ice chests.

b. How many different ice chests does the store stock?

1.4 A restaurant offers three different sizes of pizza with thin or thick crust and 5 choices of toppings. How many different kinds of single-topping pizzas can the restaurant make?

1.5 In the United States, each radio station is identified by a set of 4 call letters. These call letters traditionally begin with either the letter K or the letter W. Assuming that the remaining three letters can be selected from any letter of the English alphabet, how many different sets of call letters can be created before a new naming system must be devised?

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1.6 Guida has 3 pairs of dress pants: red, blue, and green. She also has 2 vests—red and blue—along with 3 silk shirts: red, green, and white.

a. Make a tree diagram that shows all the possible outfits that include pants, a vest, and a shirt.

b. How many different outfits can Guida create?

c. If Guida picks an outfit at random, what is the probability that the outfit will be all red?

1.7 A local ice cream parlor features 7 different flavors of ice cream, 5 different toppings, and 3 different types of cones.

a. How many different single-flavor ice cream cones with one topping are possible?

b. How many different single-flavor desserts are possible if a customer may choose to have ice cream in a dish instead of a cone and may choose not to have a topping?

1.8 Professional sports stadiums typically seat thousands of people. To allow reserved seating, each seat must be identified by a unique label—using numbers, letters, or both. Suggest an identification system for an arena with 18,000 seats.

1.9 Imagine that you are chief of security at an art museum. The security code for the electronic lock on the museum's main door contains four digits. Each digit may range from 1 through 9. Because of a recent security violation, you have been asked to assign a new code to the door.

a. From how many different codes can you choose?

b. Suppose that a burglar has enough time to try 10 different security codes before an alarm sounds. What is the probability that the burglar will discover the correct code?

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Activity 2

As the director of marketing at a soft-drink company, you know that a product's packaging can be as important to sales as the product itself. To help boost the popularity of a new beverage, you decide to explore some new color schemes on the can's label.

Exploration

The company's design staff has determined that the most appealing colors to potential consumers are red, blue, white, and silver. Your plan is to produce some model cans using these colors, then conduct a survey to determine which mix of colors is most popular.

a. Imagine that the can design will have four elements: background, company logo, soft-drink name, and highlighting.

1. The first color selected will be used for the label's background. The second color selected will be used for the company logo. This color must be different from the background color.

   Determine how many cans must be made to show all possible arrangements of first and second colors.

2. The third color selected will be used for the name of the soft drink. This color must be different from both the background color and the logo color. Determine how many cans are necessary to show every possible arrangement of first, second, and third colors.

3. The fourth color selected will be used for highlighting. This color must be different than the previous three colors. Determine how many cans are necessary to show every possible arrangement of first, second, third, and fourth colors.

4. Justify your response to Step **3** by creating a tree diagram that shows all the possible arrangements of the four colors among the four design elements.

b. How would the number of cans necessary to show all the possible arrangements of colors among the four design elements change if you added a fifth color—black—to the original set? Hint: To answer this question, repeat Part **a** using a set of five colors.

c. Suppose that the marketing team decides to incorporate a fifth design element in the can.

1. Determine the number of cans necessary to show all the possible arrangements of five colors among five design elements.
2. Express your response to Step **1** using factorial notation.

d. Suppose that the marketing team had identified a set of *n* different colors. If no color can be used more than once in a design, how many cans would be necessary to show all the possible arrangements of colors for each of the following:

1. a design with 1 element
2. a design with 2 elements
3. a design with 3 elements
4. a design with *r* elements, where $r < n$
5. a design with *n* elements (express your response using factorial notation).

Mathematics Note

A **permutation** is an ordered arrangement of items from a set. For example, two permutations of three colors that could be used on the cans described in the exploration are shown in Figure **1**.

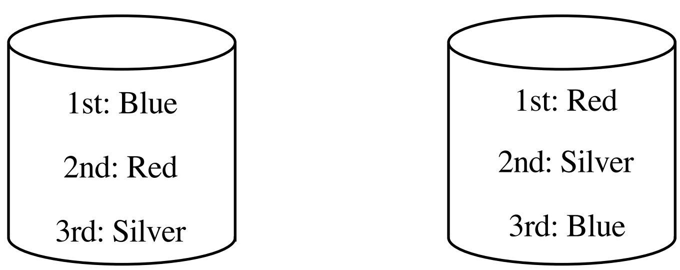


Figure 1: Two possible cans

The total number of permutations of *r* items chosen from a set of *n* items can be denoted by $P(n,r)$ or ${}_nP_r$.

For example, the number of ways to select the first, second, and third colors from a set of 5 colors can be denoted by $P(5,3)$, or "the number of permutations of 5 items taken 3 at a time." This number of permutations may be found using $5 \bullet 4 \bullet 3 = 60$.

e. Use the notation $P(n,r)$ to express the number of possible designs for each case considered in Part **d**.

Discussion

a. Describe how the fundamental counting principle could be used to determine the number of possible designs in each case considered in Part **d** of the exploration.

b. Use your response to Part **a** above to suggest a formula for calculating $P(n,r)$.

c. Consider a can with n design elements in which no color can be used more than once. Use factorial notation to describe the number of designs possible given a set of n colors.

d. Consider a can with $(n-r)$ design elements in which no color can be used more than once. Use factorial notation to describe the number of designs possible given a set of $(n-r)$ colors.

e. What does the product of your responses to Parts **b** and **d** of the discussion represent?

f. Use your response to Part **e** to determine an equation for calculating $P(n,r)$.

Mathematics Note

The number of permutations of n items taken r at a time can be calculated using the following formulas:

$$P(n,r) = \overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r\text{ terms}} \text{ or } P(n,r) = \frac{n!}{(n-r)!}$$

For example, the number of possible rankings of 12 soft drinks into first, second, and third places can be found as follows:

$$P(12,3) = \overbrace{12 \bullet 11 \bullet 10}^{3\text{ terms}} = 1320 \text{ or } P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 1320$$

g. How do the formulas described in the mathematics note compare with the ones you suggested in Parts **b** and **f** of the discussion?

h. Use an example to explain why both formulas in the mathematics note yield the same result.

i. The permutation of n objects arranged n at a time, $P(n,n)$, is $n!$ Use this fact, along with one of the formulas for $P(n,r)$, to suggest a definition for $0!$

Assignment

2.1 Find the value of each of the following:

a. $P(8,3)$

b. $P(8,4)$

c. $P(8,7)$

d. $P(8,8)$

2.2 Use factorials to write an expression for each of the following:

a. $P(12,4)$

b. $P(8,3)$

c. $P(8,7)$

2.3 Suppose that you decide to use 5 tables in the soft-drink taste test described in Activity **1**. To boost publicity for this promotion, you plan to station a television celebrity at each table.

How many different arrangements can you create if you may choose from each of the following?

a. a set of 5 celebrities

b. a set of 8 celebrities

2.4 You and 7 other members of your marketing staff plan to have a group picture taken at the taste test. In how many ways can the group be lined up in a row for the photograph?

* * * * *

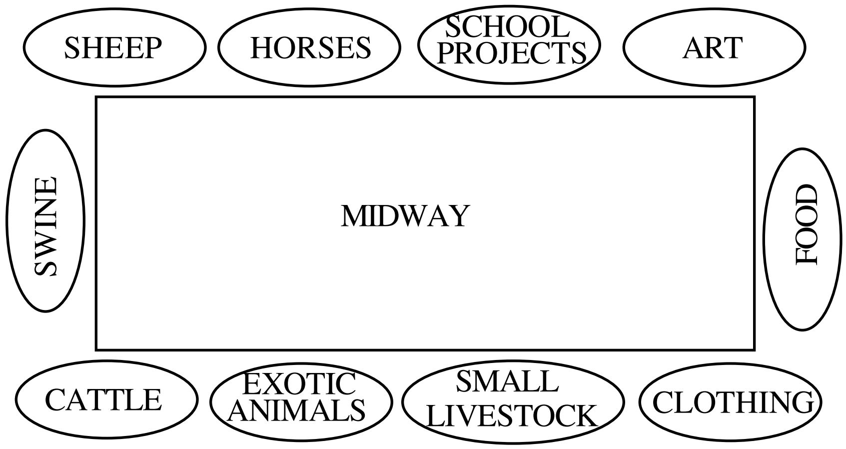
2.5 Imagine that you coach a children's baseball team. While selecting the batting order for the next game, you decide to avoid any appearance of favoritism by randomly drawing the names of 9 players out of a hat. Players will bat in the order in which their names were drawn.

a. If there are 12 players on the team, how many different batting orders are possible?

b. When you announce the batting order, the father of the player who is batting ninth accuses you of favoritism. He claims that if the selection process were actually random, then the chances of his child batting ninth are almost 0. How many different batting orders are possible in which his child is picked ninth?

c. You explain to the father that, since names were drawn at random, every child had the same chance of being picked ninth. Use your responses to Parts **a** and **b** to determine the probability that any particular child would be picked 9th in a random drawing.

2.6 As shown in the diagram below, there are 10 exhibit areas at a county fair. To determine which areas are most popular, you survey fairgoers leaving the exhibit grounds and ask them to rank each location from 1st to 10th.



a. How many different rankings are possible for 1st and 2nd place?

b. How many different rankings are possible for 1st through *r*th place?

c. Suppose you ask people to rank only the exhibit areas they actually visited. How many different responses are possible then?

2.7 A local radio station is holding a random drawing for three prizes—one first prize, one second prize, and one third prize.

a. There are 550 entries in the contest and each player can win only one prize. How many different arrangements are possible for the first-, second-, and third-prize winners?

b. Imagine that your name is included in the 550 entries.

1. What is the probability that you will win the first prize?

2. What is the probability that you will win one of the three prizes?

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Activity 3

As marketing director, you must determine how to conduct a taste test for four different soft drinks. Should you test two soft drinks at each serving station? Or three? Or all four? How many different stations would have to be used to test all possible combinations of the soft drinks?

Exploration

In this exploration, you develop a method for calculating the number of different combinations which can be tested at each serving station.

a. You have four soft drinks to test: 6-Down, Dr. Salt, Valley Fog, and Branch Tea. Imagine that each serving station has two dispensers, labeled 1 and 2.

1. Record each possible permutation for testing 2 of 4 soft drinks at a given station on a separate slip of paper.
2. Rearrange the paper slips into groups so that each permutation in the group contains the same two soft drinks.
3. Record both the number of different groups and the number of permutations in each group.
4. Express the number of permutations in each group as a factorial.
5. What is the product of the number of different groups and the number of permutations in each group?

> ### Mathematics Note
> A **combination** is a collection of items from a set in which order is not important.
>
> For example, the members of a committee with no designated chairperson is a combination. The committee composed of Nicholas and Alexandra is exactly the same as the committee consisting of Alexandra and Nicholas.

b.

1. Suppose that the soft-drink dispensers at each serving station are unlabeled. List all of the possible combinations for testing 2 of 4 soft drinks at a given station.
2. Compare the number of combinations in your list to the number of groups you recorded in Part **a**.
3. Describe the relationship among the number of combinations, the total number of permutations you listed in Part **a**, and the number of permutations in each group in Part **a**.

c. Suppose that each serving station has three drink dispensers, labeled 1, 2, and 3. Repeat Parts **a** and **b** assuming that 3 of the 4 soft drinks are tested at each station.

d. Suppose that each serving station has four drink dispensers, labeled 1, 2, 3, and 4. Repeat Parts **a** and **b** assuming that all 4 soft drinks are tested at each station.

e. Determine a general equation that relates the number of possible combinations of r items selected from a set of n items and the number of possible permutations of r items from a set of n items.

Discussion

a. What might a combination describe in terms of sets?

b. Explain how to determine the number of possible combinations of r items selected from a set of n items.

Mathematics Note

The number of different combinations, or subsets, of r items selected from a set of n items can be denoted by $C(n,r)$, ${}_nC_r$, or

$$\binom{n}{r}$$

These symbols are often read as "the combinations of n things taken r at a time" or "n choose r."

Using the fundamental counting principle, the relationship between $C(n,r)$ and $P(n,r)$ can be expressed as follows:

$$C(n,r) \bullet P(r,r) = P(n,r)$$

This equation can be used to write the following formulas:

$$C(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

For example, the number of different combinations of 3 people chosen from a group of 20 can be denoted by $C(20,3)$, ${}_{20}C_3$, or

$$\binom{20}{3}$$

Using one of the formulas given above,

$$C(20,3) = \frac{20!}{(20-3)!\,3!} = \frac{20 \bullet 19 \bullet 18}{3!} = 6840$$

c. Which of the formulas given in the mathematics note corresponds best with the explanation you gave in Part **b** of the discussion?

d. How is the quantity $r!$ in the formulas related to the number of permutations in each group found in Part **a** of the exploration?

e. Express the formulas given for combinations in terms of the notation used for permutations.

f. The value of $C(n,n)$ is 1. Use the formulas given in the mathematics note to explain this result.

Assignment

3.1 **a.** Consider a race involving six runners. In how many different ways can the runners place first, second, and third?

b. The top three runners in the race will qualify for the next round of competition. How many different subsets of runners can finish in the top three?

c. Given a subset of three runners who qualify for the next round, in how many different ways can they place first, second, and third?

d. Write an equation that describes the relationship among your responses to Parts **a**, **b**, and **c**.

3.2 Is the following equation true or false? Justify your response.

$$\frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}$$

3.3 Determine whether each of the following statements is true or false. Justify your responses.

a. $C(7,2) = C(7,5)$

b. $C(10,6) = C(10,4)$

c. $C(n,r) = C(n,n-r)$

3.4 Imagine that you decide to conduct a taste test involving 5 different soft drinks. Assuming that you test only 2 soft drinks at each station, how many different stations would you need to match each soft drink against every other one?

3.5 Suppose that you decide to test 3 of the 5 soft drinks in Problem **3.4** at each station.

a. How many different stations would you need to conduct all the possible three-way competitions simultaneously?

b. Dr. Salt is one of the 5 soft drinks in the taste test. How many different stations will have Dr. Salt as 1 of their 3 soft drinks?

3.6 Of the first 25 people who volunteered for the taste test, 10 are male and 15 are female. You plan to randomly select 5 of them to go from station to station and judge the drinks.

a. In how many different ways can the 5 judges be selected?

b. In how many different ways could the 5 judges turn out to be all males? all females?

c. What is the probability that the 5 judges will turn out to be all males? all females?

3.7 Concerned that the random selection process described in Problem **3.6** might produce a group of 5 judges who are all male or all female, you decide at the last minute to choose 3 of the females and 2 of the males from the 25 volunteers.

a. In how many different ways can the 2 males be selected? In how many different ways can the 3 females be selected?

b. Based on your responses to Part **a**, how many different ways are there to select the panel of 3 females and 2 males? (Hint: Use the fundamental counting principle.)

c. Compare your responses to Problem **3.6a** and **3.7b**. How do you explain the difference?

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3.8 Imagine that you own 200 compact discs, but only have room to store 75 of them. How many different subsets of 75 compact discs can you select?

3.9 Sam's Chinese Restaurant has 12 different main entrees. You and 5 friends are planning a "Taste of China" evening with videos, dinner, and music. You agree to order 6 different entrees and share them at the meal. From how many different groups of 6 entrees can you and your friends select?

3.10 Fifteen points are evenly spaced on a circle. Imagine that you want to create a design by connecting each point to each of the other points.

a. How many different line segments must you draw to complete the design?

b. How many different triangles with each vertex on one of the circle's 15 points does your design contain?

3.11 One of the booths at a fair offers the chance to win your choice of 14 different prizes—if you can toss a softball into the mouth of a milk can. Your friend Jack wins twice. From how many sets of 2 different prizes can Jack choose?

3.12 At basketball camp, the players are divided randomly into 14 teams of 5 players each.

a. In how many different ways can the first team of 5 be selected?

b. In how many different ways can the second team of 5 be selected?

c. The coaches promise that each team will play every other team once before the week is over. How many games will this require?

3.13 Consider the set of numbers $\{1, 2, 3, 4, 5\}$.

a. How many subsets of 3 numbers can be chosen from this set?

b. How many of the possible subsets of 3 numbers include the number 5?

c. How many of the possible subsets of 3 numbers do not include the number 5?

d. How is the sum of your responses to Parts **b** and **c** related to your response to Part **a**?

e. Generalize the relationship you described in Part **d** for subsets of *r* elements chosen from a set of *n* elements.

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Research Project

Imagine that you have been asked to design the schedule for an annual cookie baking competition. There are 13 entrants. Each entrant must bake 36 chocolate chip cookies.

During each round of the competition, the 6 judges each sample 2 different cookies. Each entrants' cookies must be compared with every other entrants' cookies and no entrant may have more than 1 cookie sampled during any round.

a. Design a schedule for the competition and describe it in a report. Your report should address each of the following questions.

1. How many rounds must be held during the competition?
2. How many cookies must each judge sample?
3. Have the entrants baked enough cookies for the competition?
4. How will the winning entrant be determined?

b. Draw a diagram to help others interpret the schedule. Include an explanation of how your diagram works.

Summary Assessment

When selecting the members of its student council, Cooperative High School prides itself on giving an equal voice to each class. The positions are filled by a random drawing of the names of all interested students who pass the minimum school-attendance requirement. Once the council has been selected, another random drawing from among the council members determines the positions of president, vice-president, secretary, and treasurer.

Because representation is proportional to enrollment, this year's freshman and sophomore classes each receive 4 representatives, while the junior and senior classes each receive 3. The chart below shows the number of students, by class, whose names were included in the random drawing.

Class	**Freshman**	**Sophomore**	**Junior**	**Senior**
Female	14	10	9	8
Male	6	7	9	7
Total	20	17	18	15

1. Complete Parts **a–e** below for each class.

a. How many different selections of class representatives are possible for each class?

b. How many possible selections of class representatives consist entirely of females for each class?

c. How many possible selections of class representatives consist entirely of males for each class?

d. What is the probability that all the representatives for a given class will be female?

e. What is the probability that all the representatives for a given class will be male?

2. Use the fundamental counting principle and your responses to Problem **1** to answer each of the following:

a. How many different school councils are possible, before the selection of officers?

b. How many of the possible school councils are all female?

c. How many of the possible school councils are all male?

d. What is the probability that the school council will be all female?

e. What is the probability that the school council will be all male?

3. This year's selection of president, vice president, secretary, and treasurer is causing some concern among the juniors and seniors. As one senior noted, "There's a good chance that all four offices will go to freshmen and sophomores!"

a. In how many different ways can the four offices be filled once the representatives to the council have been selected?

b. In how many different ways can the four offices be filled using only freshmen and sophomores?

c. What is the probability that all four offices will go to freshmen and sophomores?

d. Are the concerns of the senior class justified?

Module Summary

- The **fundamental counting principle** states that if one selection can be made in h ways, and for each of these ways a second selection can be made in k ways, then the number of different ways the two selections can be made is $h \bullet k$.
- A **permutation** is an ordered arrangement of items from a set. The total number of permutations of r items chosen from a set of n items can be denoted by $P(n,r)$ or ${}_nP_r$.
- The number of permutations of n items taken r at a time can be calculated using the following formulas:

$$P(n,r) = \overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r \text{ terms}} \text{ or } P(n,r) = \frac{n!}{(n-r)!}$$

- A **combination** is a collection of items from a set in which order is not important. The number of different combinations, or subsets, of r items selected from a set of n items can be denoted by $C(n,r)$, ${}_nC_r$, or

$$\binom{n}{r}$$

  These symbols are often read as "the combinations of n things taken r at a time" or "n choose r."

- Using the fundamental counting principle, the relationship between $C(n,r)$ and $P(n,r)$ can be expressed as follows:

$$C(n,r) \bullet P(r,r) = P(n,r)$$

  This equation can be used to write the following formulas:

$$C(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

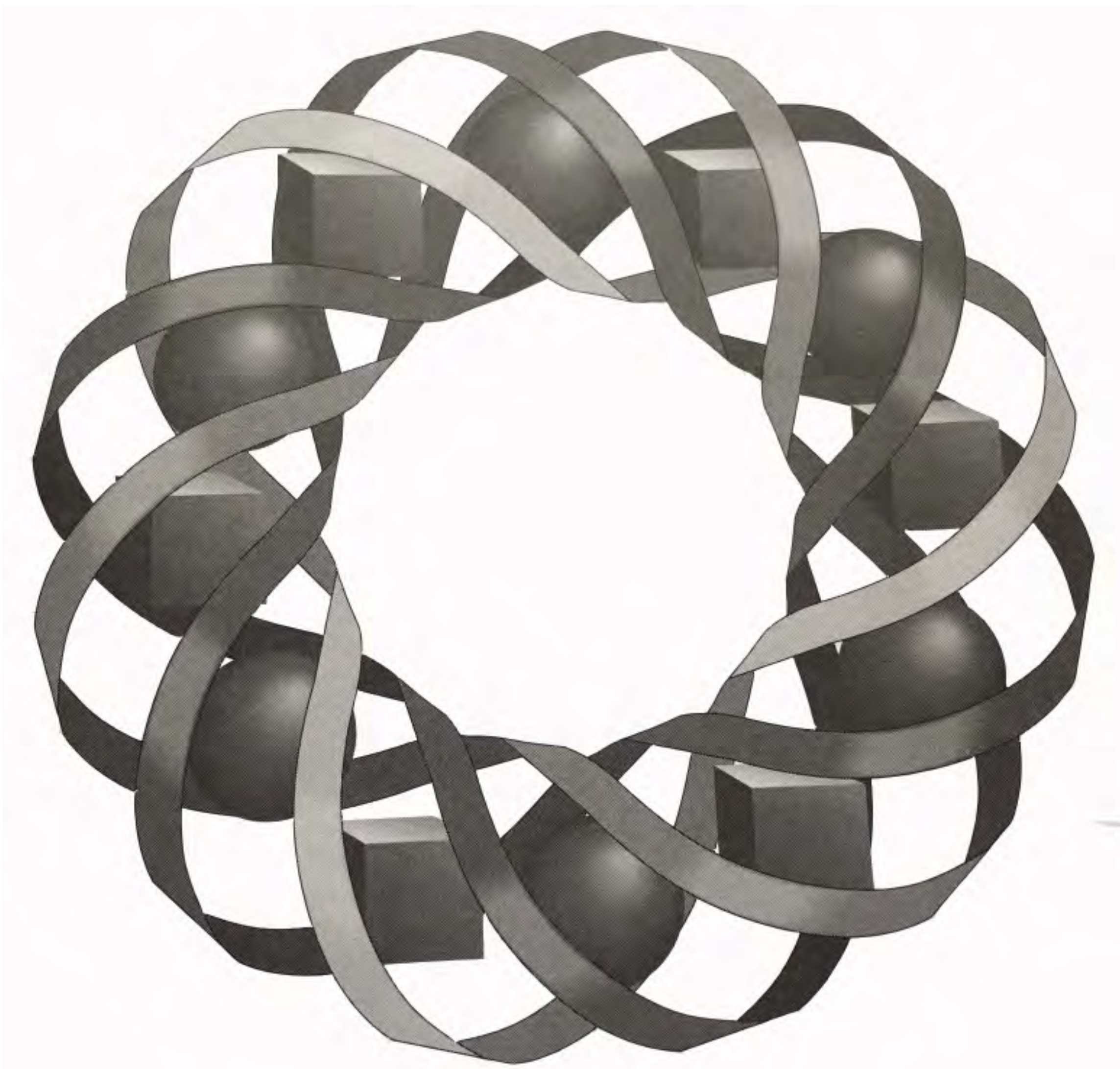
$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Selected References

Johnson, J. "Using Dominoes to Introduce Combinatorial Reasoning." In *Discrete Mathematics across the Curriculum, K–12*. Ed. by M. J. Kenney. Reston, VA: National Council of Teachers of Mathematics, 1991.

Evered, L., and B. Schroeder. "Counting with Generating Functions." In *Discrete Mathematics across the Curriculum, K–12*. Ed. by M. J. Kenney. Reston, VA: National Council of Teachers of Mathematics, 1991.

Classical Crystals



What do diamonds, quartz, and salt all have in common? In this module, you'll explore the properties of some familiar crystals.

Paul Swenson • Teri Willard

Classical Crystals

Introduction

Common table salt, also known as halite, is a mineral composed of sodium and chlorine atoms. Molecules of table salt form crystals that have a cubic shape. (You may want to use a magnifying glass or microscope to get a better look at them.)

Chemists sometimes describe a crystal as a body consisting of a single type of mineral. In a crystal, the forces that bind atoms together cause them to combine in a definite shape. This shape is determined by the number and types of atoms in the mineral. Many minerals other than table salt also form crystals. Some are too small to see with an ordinary microscope, others can be several meters in diameter.

Large crystals are formed by many, many repetitions of small "building blocks" called unit cells. A **unit cell** is the smallest piece of a crystal that possesses the same properties as a larger crystal of the same mineral.

For example, a molecule of table salt consists of one sodium atom and one chlorine atom bonded together. The unit cell of a salt crystal is composed of four molecules of salt—in other words, four sodium atoms and four chlorine atoms.

Figure **1** compares a salt molecule to the unit cell of a salt crystal and to a larger crystal of salt. The larger crystal shows how unit cells join together. One interesting property of the larger crystal is that its shape and symmetry are the same as that of the unit cell.

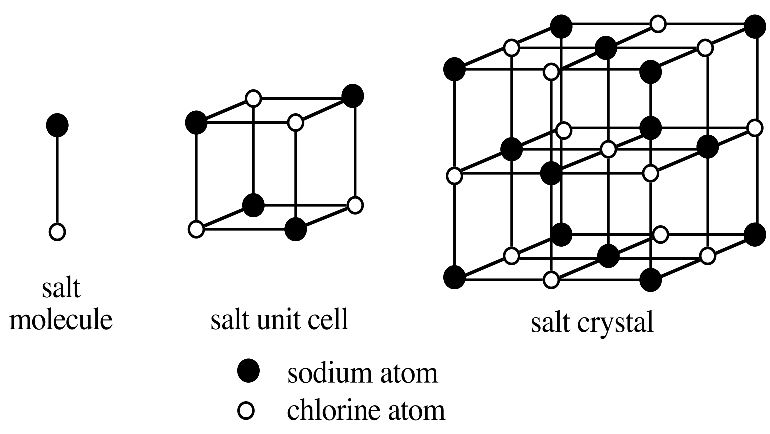


Figure 1: Structure of a salt crystal

The symmetrical properties of crystals have fascinated mathematicians and scientists for centuries. Although crystals may have many different shapes, the more symmetrical a crystal, the more interest it attracts. In fact, the symmetry of certain crystals—like diamonds or emeralds—can even make them more valuable.

Activity 1

A cube is one of the **regular polyhedra**, first described in the sixth century B.C. by a society of Greek mathematicians known as the Pythagoreans. In this activity, you investigate some of the characteristics of regular polyhedra.

Mathematics Note

A **polyhedron** is a simple three-dimensional closed surface made up of faces that are polygons. The plural of polyhedron is **polyhedra** or polyhedrons.

A **regular polyhedron** is a three-dimensional convex solid in which all the faces are congruent regular polygons and the same number of faces meet at each vertex.

The cube in Figure **2** is one example of a regular polyhedron. All of its faces are congruent regular polygons (each a 3-cm square) and the same number of faces (3) meet at each vertex.

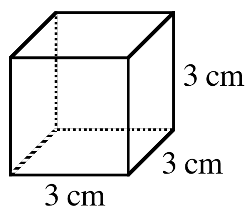


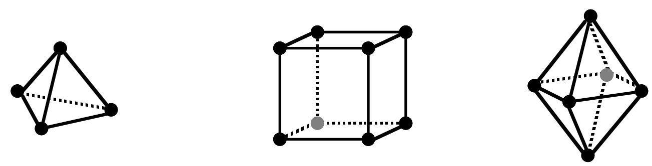
Figure 2: A 3-cm cube

The regular polyhedra were named the **Platonic solids** after the Greek scholar Plato, who lived around 400 B.C.

About 300 B.C., Euclid—another Greek mathematician—discussed the Platonic solids in his book, the *Elements*. Many people attributed mystical powers to these shapes. The Pythagoreans, for example, thought they represented the elements of the universe—fire, earth, air, and water—as well as the universe itself. They were not entirely wrong. Platonic solids do occur in nature. For example, crystals of both table salt and pyrite (an iron-containing mineral) have the outward appearance of a cube. Crystalline gold can take the shape of an octahedron. Diamond crystals occur in many different shapes, including that of tetrahedrons, octahedrons, and dodecahedrons.

Exploration

Figure **3** shows three different Platonic solids. The name of each solid appears below it, along with the names of some minerals that have crystals with that shape. The distances between adjacent vertices are equal.



tetrahedron	**hexahedron**	**octahedron**
tetrahedrite tennantite chalcopyrite	pyrite halite galena flourite	diamond magnetite spinel

Figure 3: Three Platonic solids

a. Build models of the three different Platonic solids shown in Figure **3**. Use toothpicks of equal lengths for the edges, and miniature marshmallows (or some other soft material) for the vertices.

b. Record the characteristics of each solid.

Discussion

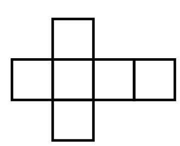
a. What do you think the prefixes *tetra-*, *hexa-*, and *octa-* mean? Explain your responses.

b. Which polygon makes up the faces of the tetrahedron? of the hexahedron? of the octahedron?

c. Describe some of the similarities and differences you observe among the three polyhedra in the exploration.

Assignment

1.1 A net is a two-dimensional pattern that can be folded into a three-dimensional solid. The net for a cube (or Platonic hexahedron) consists of six squares. Many different combinations of six squares will form a cube. Sketch a net for a cube different than the one shown below**.**



1.2 Sketch nets for the octahedron and tetrahedron you built in the exploration.

1.3 **a.** On a sheet of stiff paper, draw a net for a cube with an edge length of 6 cm.

b. Use your net to build the cube. **Note:** Save your cube for use later in the module.

c. Describe the characteristics of a cube that make it a Platonic solid.

1.4 **a.** On a sheet of stiff paper, draw a net for a regular tetrahedron with an edge length of 6 cm.

b. Use your net to build the tetrahedron.

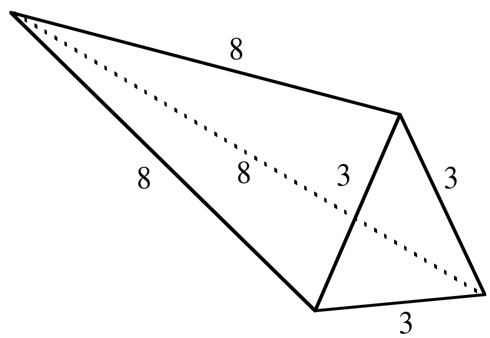
c. Describe the characteristics of a regular tetrahedron that make it a Platonic solid.

1.5 Place your tetrahedron from Problem **1.4** on a mirror.

a. Describe the solid formed by the tetrahedron and its image.

b. Is this solid a Platonic solid? Justify your response. **Note:** Save your tetrahedron for use later in the module.

1.6 The sketch of a tetrahedron below displays the measurements of its edges. Is this tetrahedron a Platonic solid? Explain your response.



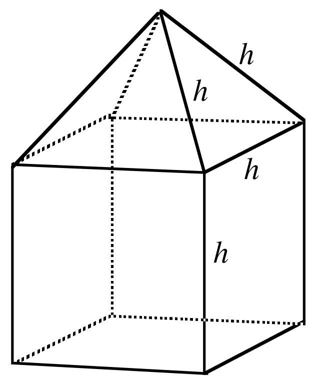
* * * * *

1.7 Using toothpicks and marshmallows (or some other soft material), build a model of a pyramid with a square base. All the toothpicks should be the same length. **Note:** Save your pyramid for use later in the module.

a. Is your pyramid a Platonic solid? Give at least two arguments supporting your answer.

b. Place the square base of the pyramid on a mirror. Does the figure formed by the pyramid and its image represent a Platonic solid? Explain your response.

1.8 Consider the polyhedron that results when a pyramid is constructed on one face of a cube, as shown in the diagram below. The faces of the pyramid are all equilateral triangles with sides of length h. If a pyramid is constructed on each of the other faces of the cube, will the resulting polyhedron be a Platonic solid? Justify your response.



1.9 **a.** Determine the surface area of a cube with an edge length of 10 cm.

b. Describe an efficient way to determine the surface area of any of the Platonic solids.

* * * * * * * * * *

Activity 2

The three Platonic solids you built in the previous exploration—the tetrahedron, the hexahedron, and the octahedron—all occur naturally in certain mineral crystals. Do you think that these are the only Platonic solids that occur in nature?

Exploration

In this exploration, you explore the possible existence of other Platonic solids.

a. Cut six congruent equilateral triangles out of stiff paper.

b. Select one triangle. Label an angle X, as shown in Figure **4** below, and record its measure.

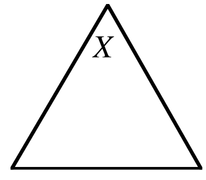


Figure 4: An equilateral triangle

c. Tape two triangles together along one edge, as shown in Figure **5**. Record the sum of the measures of angle *X* and the angle adjacent to it.

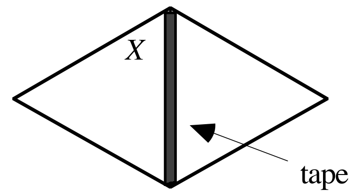


Figure 5: Two triangles taped together

d. Fold the two triangles along the tape. Could these triangles form a vertex of a three-dimensional solid?

e. Tape a third triangle to the others so that all three share a single vertex, as shown in Figure **6**. Record the sum of the measures of the angles at the shared vertex.

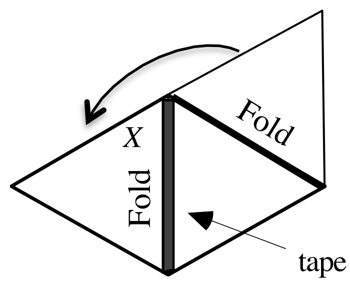


Figure 6: Three triangles taped together

f. Fold the triangles along the tapes. Could these three triangles form a vertex of a three-dimensional solid?

g. Continue taping additional triangles to the others (up to six) so that all share a single vertex. Record the number of triangles and the sum of the measures of the angles at the shared vertex, and note whether or not the combination will form a vertex of a solid.

h. Repeat the procedure described in Parts **a–g** using four congruent squares, four congruent regular pentagons, and four congruent regular hexagons.

Discussion

a. What conditions appear to be necessary to form a vertex of a regular polyhedron?

b. Recall that a tessellation is a repeated pattern that covers an entire plane without gaps or overlaps. Figure **7** shows a portion of a tessellation that uses isosceles triangles to cover the plane.

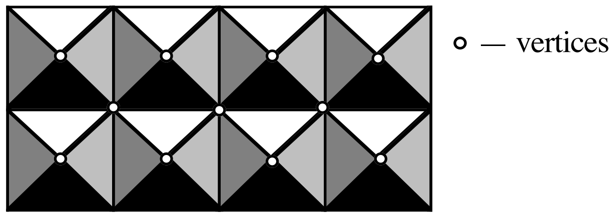


Figure 7: Tessellation using isosceles triangles

1. What is the sum of the measures of the angles at each vertex in this tessellation?
2. Do you think it is possible to create a polyhedron in which the sum of the measures of the angles at each vertex is 360˚? Explain your response.

c. Describe the numbers and types of the regular polygons in the exploration that can be used to form a vertex of a solid.

d. Do you think that any other regular polygons could be used as the faces of a Platonic solid? Explain your response.

e. The problem you investigated in the exploration also interested the Greek scholar Theaetetus (419–369 B.C.). Theaetetus is given credit for determining the exact number of Platonic solids. How many do you think there are? Explain your response.

Assignment

2.1 A regular **dodecahedron** has 12 regular pentagonal faces. Draw a regular pentagon. Using your pentagon as a template, cut 12 congruent pentagons out of stiff paper. Use these pentagons to build a regular dodecahedron.

2.2 Sketch a net that will produce a regular dodecahedron.

2.3 A regular **icosahedron** has 20 regular triangular faces. Fold the net provided by your teacher to produce an icosahedron.

a. Describe the characteristics of an icosahedron.

b. Determine the surface area of your model.

2.4 Crystals of boron take the shape of an icosahedron. The length of one edge of the unit crystal for boron is $8.5 \bullet 10^{-9}$ cm. What is the surface area of a unit crystal of boron?

* * * * *

2.5 Consider a jewel cut in the shape of a dodecahedron with an edge length of 0.5 cm. What is the surface area of this jewel?

2.6 A typical die is a cube. Each face of the cube represents one of the integers from 1 to 6. Recall that on a fair die, each face has the same chance of turning up. Identify the Platonic solid that should be used to make a fair die in which each face represents one of the integers from 1 to 10. Justify your selection.

2.7 In the exploration in Activity **2**, you discovered that the sum of the measures of the angles at a vertex of a Platonic solid must be less than 360˚. Examine several non-Platonic convex solids, such as containers and packaging boxes. Make a conjecture about the sum of the measures of the angles at a vertex of any convex solid.

* * * * * * * * * *

Activity 3

Another distinguishing characteristic of the Platonic solids is their symmetry. In this activity, you investigate the symmetry of some Platonic solids by first examining the symmetries of regular polygons.

Mathematics Note

A **line of symmetry** divides a two-dimensional figure into two congruent parts, each a mirror image of the other.

For example, Figure **8** shows two polygons and their lines of symmetry.

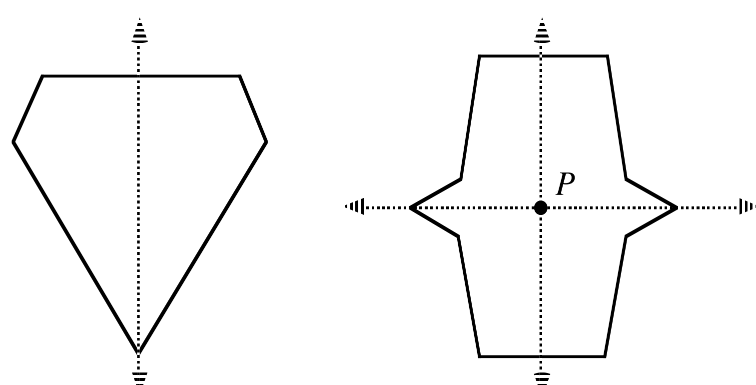


Figure 8: Lines of symmetry for two polygons

Exploration 1

Some regular polygons have several lines of symmetry. The relationships among these lines and the sides of the polygon can be used to determine basic shapes that make up crystals.

a. By placing a mirror perpendicular to the plane containing a polygon and on the polygon's line of symmetry, you should be able to see that each side of the polygon is a reflection of the other side.

Use a mirror to find the lines of symmetry of a regular triangle. Sketch these lines on the triangle.

b. Record any relationships you observe among the lines of symmetry and the sides and angles of the triangle.

c. Two mirrors can be placed on the lines of symmetry in Figure **9** so that the shaded region combines with its reflections to form the entire polygon. The smallest region that can be reflected to form the entire polygon is a **reflecting polygon**.

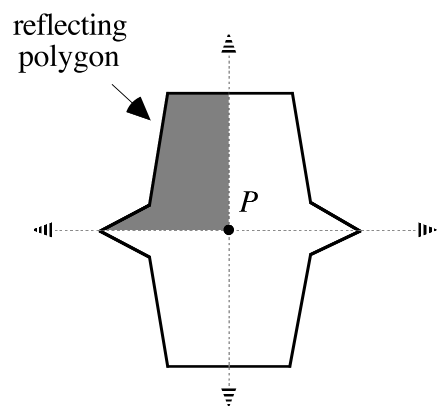


Figure 9: A reflecting polygon

Using the triangle from Part **a**, place two mirrors along two different lines of symmetry to find the reflecting polygon.

d. Measure the angle formed by the mirrors.

e. Sketch the reflecting polygon of the triangle.

f. Repeat Parts **a–e** using a regular quadrilateral, a regular pentagon, and a regular hexagon.

Discussion 1

a. Compare the reflecting polygon for a regular triangle to those for a regular quadrilateral, pentagon, and hexagon.

b. For each regular polygon, how many images of a reflecting polygon are needed to recreate the original?

c. How is the measure of the angle formed by the two mirrors in the exploration related to the number of sides in the regular polygon?

d. Recall that the measure of an interior angle of a regular polygon can be found using the formula below, where m is the measure of each interior angle and n is the number of sides in the polygon:

$$m = \frac{180(n - 2)}{n}$$

How does the measure of an interior angle of a regular polygon compare with the measure of the angle between the two mirrors used to find a reflecting polygon?

Exploration 2

In Exploration **1**, you examined lines of symmetry for regular polygons. In this exploration, you investigate **planes of symmetry** for Platonic solids.

Mathematics Note

A **plane of symmetry** divides a three-dimensional object into two congruent three-dimensional objects, each a mirror image of the other. For example, Figure **10** shows a rectangular prism and one of its planes of symmetry.

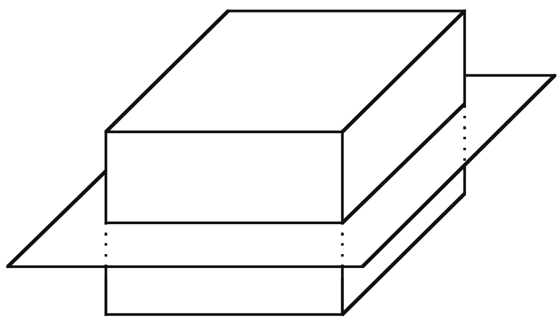


Figure 10: A rectangular solid and a plane of symmetry

Just as lines of symmetry divide polygons into congruent parts, planes of symmetry divide polyhedrons into congruent parts. Crystals shaped like Platonic solids have several planes of symmetry. These planes may be located using mirrors.

To find a plane of symmetry, place a single mirror on one face of the polyhedron. Hold the mirror perpendicular to the face. If the mirror is located in the plane of symmetry, the visible part of the polyhedron and its reflection will form a polyhedron congruent to the original. As shown in Figure **11**, it may be necessary to view the reflection from above the mirror to see the polyhedron formed.

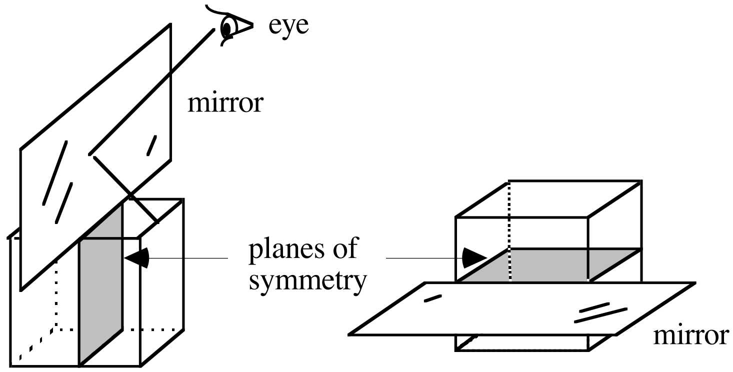


Figure 11: Using mirrors to find planes of symmetry

a. Obtain the cube you made in Problem **1.3**. Use a single mirror to find as many planes of symmetry as you can for the cube.

b. Record the position of each plane of symmetry.

c. Repeat Parts **a** and **b** using the regular tetrahedron you built in Problem **1.4**.

Discussion 2

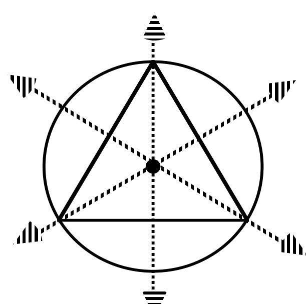
a. How many different planes of symmetry are there for a cube? Where is each one located?

b. How many different planes of symmetry are there for a regular tetrahedron? Where is each one located?

c. How many planes of symmetry do you think there are for a regular octahedron? Where would these planes of symmetry be located?

Assignment

3.1 The point at which a regular polygon's lines of symmetry intersect is the center of the circle that circumscribes the polygon. For example, the diagram below shows a regular triangle, its lines of symmetry, and the circle that circumscribes it.



Describe how you could use a circle to construct a regular polygon with *n* sides, as well as all of the polygon's lines of symmetry.

3.2 **a.** Make a table with headings like those in the table below. Complete the table for three different regular polygons.

No. of Sides	Measure of an Interior Angle (I)	Measure of Angle Formed by Two Adjacent Lines of Symmetry (S)	Ratio $I:S$

b. Using the patterns you observe in the table, describe the relationship between the measure of an interior angle of a regular polygon and the measure of an angle formed by the intersection of two adjacent lines of symmetry.

c. Complete the table for a regular polygon with n sides.

3.3 **a.** Describe how to use two mirrors to locate a reflecting polygon for any regular polygon.

b. Test your method using a regular polygon with more than six sides.

3.4 Determine the number of different planes of symmetry for each of the Platonic solids you have constructed in this module. **Note:** Make sure to count each plane only once.

3.5 Do you observe any relationship between the symmetry of the face of a Platonic solid and the symmetry of the solid itself? Explain your response.

3.6 Describe a relationship among the sides, vertices, and lines of symmetry for regular polygons with:

a. an odd number of sides

b. an even number of sides.

* * * * *

3.7 Consider the relationships you described for lines of symmetry of regular polygons in Exploration **1** of Activity **3**. How could you use these relationships to help locate lines of symmetry using a ruler rather than a mirror?

3.8 Consider a pyramid with a square base, such as the one you constructed in Problem **1.7**. Describe the planes of symmetry for such a pyramid.

3.9 **a.** Describe the locations of the planes of symmetry for a sphere.

b. How many planes of symmetry are there for a sphere? Explain your response.

* * * * * * * * * *

Activity 4

The mineral galena is composed of lead and sulfur. Also known as lead ore, it has been our most important source of lead for centuries. In the past, lead was used extensively in paints, gasoline, metal alloys, batteries, and glass. Because of its toxic effects on humans and the environment, however, the use of lead is now much less widespread.

As shown in Figure **12**, crystals of galena occur in several different shapes: cubes, truncated cubes, cuboctahedrons, and octahedrons.

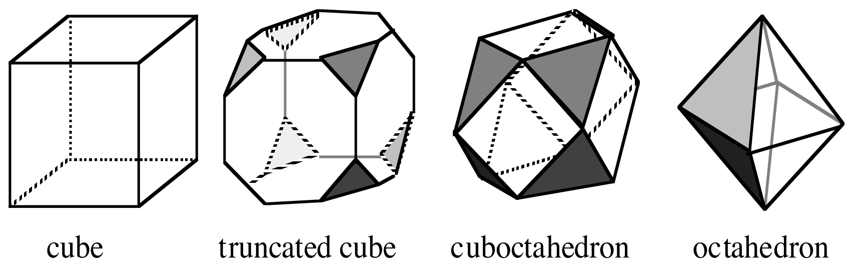


Figure 12: Different shapes of galena crystals

A **truncated cube** can be thought of as a cube with its "corners" cut off. A **cuboctahedron** is a special kind of truncated cube. Its corners have been "cut" so that the remaining faces are squares and equilateral triangles. The length of the sides of the squares equals the length of the sides of the triangles.

A cuboctahedron also can be visualized as an octahedron with each of its vertices cut off in the same way. Each square face is joined to four triangular faces, and each triangular face is joined to three square faces. Both the truncated cube and the cuboctahedron are classified as **Archimedean solids**, after the Greek mathematician Archimedes (287–212 B.C.).

Mathematics Note

Archimedean solids, or **semiregular polyhedra**, are solids whose faces consist of two or three different types of congruent regular polygons. Each vertex is formed by the intersection of the same numbers and types of these polygonal faces.

For example, the faces of a truncated cube are equilateral triangles and regular octagons. Two octagons and one triangle meet at each vertex.

Exploration

a. Determine the number of squares and equilateral triangles that make up the faces of a cuboctahedron in Figure **12**.

b. Build a model of a cuboctahedron.

c. Draw a net for a cuboctahedron.

Discussion

a. How many squares and how many equilateral triangles are required to create the faces of a cuboctahedron?

b. How do you think the name *cuboctahedron* was derived?

c. How many faces intersect to form a vertex of a cuboctahedron? Is this number the same at every vertex?

d. What is the sum of the measures of the interior angles at each vertex?

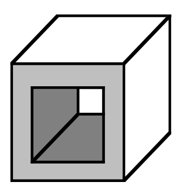
Assignment

4.1 How do the planes of symmetry of a cuboctahedron compare with the planes of symmetry of a cube and an octahedron?

4.2
- **a.** Create a table that shows the number of vertices, faces, and edges for each Platonic solid.
- **b**. Using the table from Part **a**, find the sum of the number of faces and the number of vertices for each polyhedron.
- **c**. How does the sum of the numbers of faces and vertices compare to the number of edges?
- **d**. Write a formula that describes the relationship among the number of faces (F), vertices (V), and edges (E) of the polyhedron. This relationship is known as **Euler's formula**.

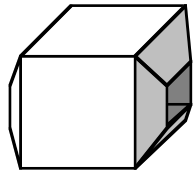
4.3 The relationship you discovered in Problem **4.2**, Euler's formula, is true for any polyhedron. In Parts **a** and **b** below, you determine if this relationship also applies to solids with holes in them.

- **a.** Consider the solid shown in the diagram below, a cube with a square hole through its middle.



  1. Does Euler's formula also apply to this solid? Why do you think this is so?
  2. Is this solid a polyhedron? Explain your response.

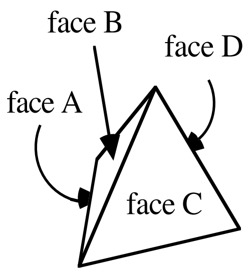
b. Like the solid in Part **a**, the following solid also has a hole through it.



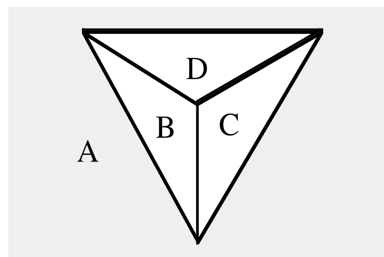
1. Explain why Euler's formula is not true for this solid.

2. Does this prove that Euler's formula does not apply to all solids with holes? Justify your response.

4.4 In a **planar map** of a polyhedron, there is a one-to-one correspondence between the faces of the polyhedron and the regions of the planar map. In addition to this correspondence, the regions which represent faces that share an edge also share a boundary. For example, the diagram below shows a planar map of a tetrahedron.



tetrahedron



planar map of tetrahedron

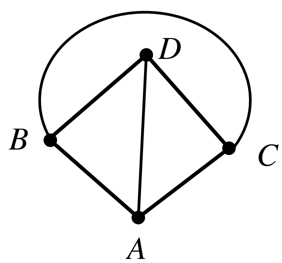
a. Does Euler's formula apply to the planar map of a tetrahedron? Explain your response.

b. Create a planar map of a cube.

c. Does Euler's formula also apply to the planar map of a cube?

4.5 In a **graph** of a planar map, there is a one-to-one correspondence between the regions of the map and the vertices of the graph. In addition to this correspondence, the vertices which represent regions that share a boundary are joined by an edge.

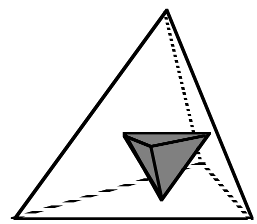
If it is possible to draw such a graph so that the edges intersect only at vertices, then the graph is a **planar graph**. For example, the diagram below shows a planar graph of the map of a tetrahedron from Problem **4.4**.



Draw a planar graph for the map of a cube from Problem **4.4b**.

4.6 A **dual** of a polyhedron is the solid formed by the segments joining the center of each face of the polyhedron to the center of each adjacent face. A dual is therefore **inscribed** in the original solid.

The diagram below shows a tetrahedron with its inscribed dual. The dual of a tetrahedron is also a tetrahedron.



a. What is the dual of a regular octahedron?

b. A cuboctahedron also has a dual. How many vertices, faces, and edges does this dual have?

4.7 As noted in the introduction to this activity, a cuboctahedron can also be thought of as an octahedron with its vertices cut off. Two squares and two equilateral triangles meet at each vertex of the cuboctahedron. The lengths of the sides of both polygons are equal. What is the height of the corners cut from the regular octahedron to form the regular cuboctahedron? Explain your response.

* * * * *

4.8 The surface of a soccer ball appears to be made up of regular pentagons and regular hexagons. Each pentagon is surrounded by five hexagons. If each face were actually a polygon, would the pattern used on a soccer ball result in an Archimedean solid? Explain your response.

4.9 A cuboctahedron has 14 faces. Could this solid be used to make a fair die in which each face represents an integer from 1 to 14? Explain your response.

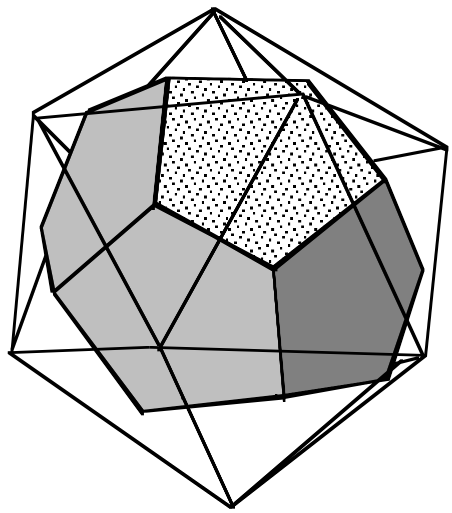
* * * * * * * * * *

Research Project

Write a report that describes each Archimedean solid and explains its relationship to one of the Platonic solids. When appropriate, include models or drawings of these solids as part of your report.

Summary Assessment

It is possible to position a Platonic solid inside another Platonic solid. These special pairs of "solids within solids" are referred to as **nested solids**. Every Platonic solid can be "nested" in every other Platonic solid—including itself. The figure below shows a dodecahedron nested in an icosahedron.



In a pair of nested solids, the vertices of the inner solid must be points of the outer solid. In some—such as the pair shown in the diagram above—the vertices of the inner polyhedron are the centers of the faces of the outer solid. The inner solid may even share the same vertices as the outer solid.

Build a model or make a sketch of a pair of nested polyhedra (other than a solid within itself or a solid and its dual). Write a description of the nested solids, explaining where the vertices of the inner solid lie. Compare the lengths of the edges of both solids and explain how you determined these lengths.

Module Summary

- A **unit cell** is the smallest piece of a crystal that possesses the same properties as a larger crystal of the same mineral.
- A **line of symmetry** divides a two-dimensional figure into two congruent parts, each a mirror image of the other.
- A **polyhedron** is a simple three-dimensional closed surface made up of faces that are polygons. The plural of polyhedron is **polyhedra** or polyhedrons.
- A **regular polyhedron** is a three-dimensional convex solid in which all the faces are congruent regular polygons and the same number of faces meet at each vertex. These solids are referred to as the **Platonic solids**.
- A regular **tetrahedron** has four faces and four vertices. Three regular triangular faces intersect at each vertex.
- A **cube** or regular **hexahedron** has six faces and eight vertices. Three square faces intersect at each vertex.
- A regular **octahedron** has eight faces and six vertices. Four regular triangular faces intersect at each vertex.
- A regular **icosahedron** has 20 faces and 12 vertices. Five regular triangular faces intersect at each vertex.
- A regular **dodecahedron** has 12 faces and 20 vertices. Three regular pentagonal faces intersect at each vertex.
- A **plane of symmetry** divides a three-dimensional object into two congruent three-dimensional objects, each a mirror image of the other.
- A **dual** of a polyhedron is the solid formed by the segments joining the center of each face of the polyhedron to the center of each adjacent face.
- Each vertex of an **inscribed** polyhedron is a point of the outer solid.
- **Archimedean solids**, or **semiregular polyhedra**, are solids whose faces consist of two or three different types of congruent regular polygons. Each vertex is formed by the intersection of the same numbers and types of these polygonal faces.

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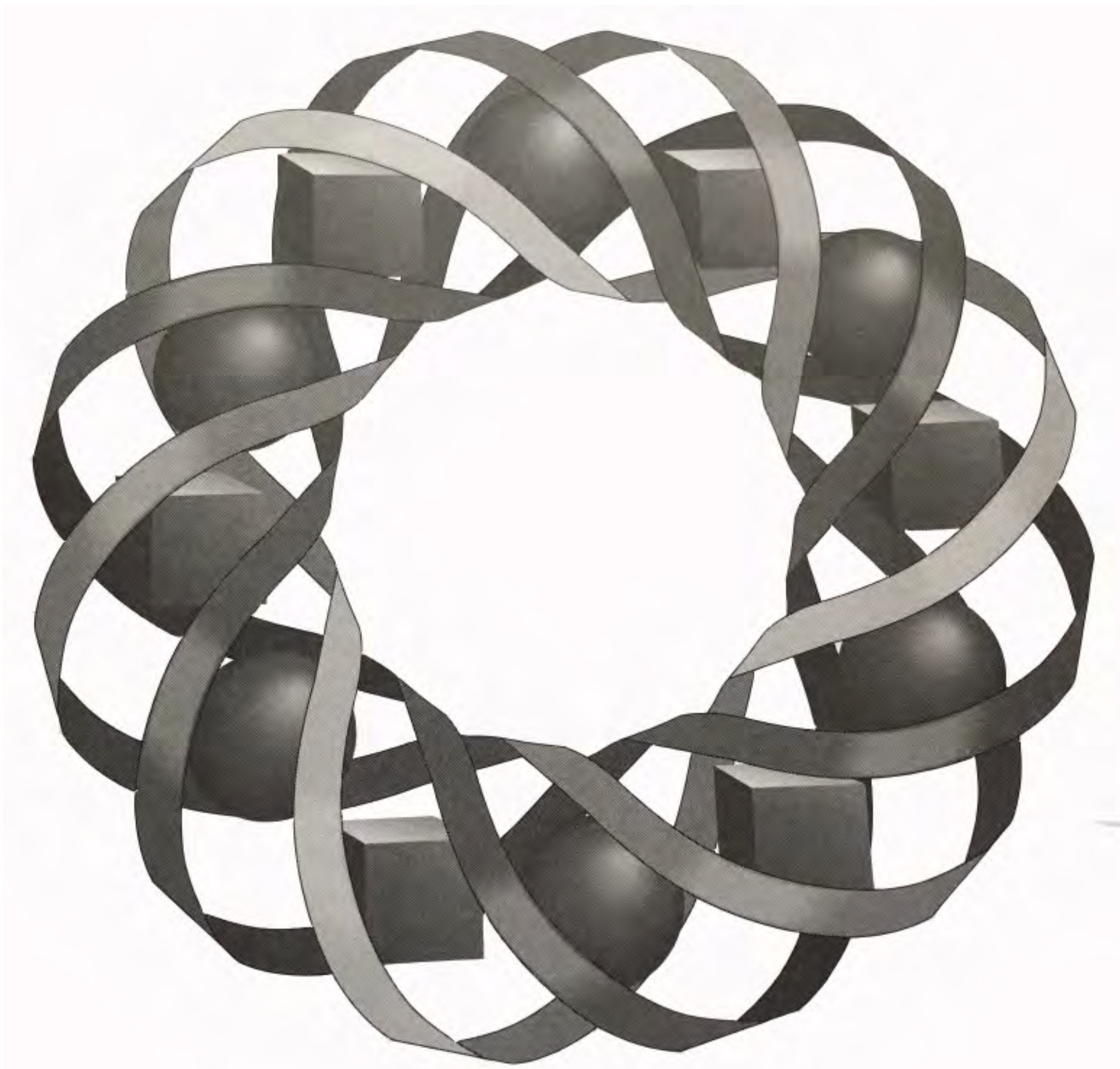
Strive for Quality



The quality of the products we buy affects our lives in many different ways. What is quality and how can we control it? In this module, you examine some statistical methods for evaluating quality.

Anne Merrifield • Pete Stabio

Strive for Quality

Introduction

As a consumer, you expect every product you buy to meet certain standards of quality. You want your food to be free from contamination, your clothes to be stylish and durable, and your stereo system to sound clear and lifelike.

To monitor product quality, manufacturers typically use a process called **quality control**. This process involves four steps:

- defining the characteristics of a quality product
- monitoring the product for these characteristics
- using statistics to analyze the results obtained from monitoring
- making needed adjustments in production to improve quality.

In this module, you investigate the mathematics involved in the third step: using statistical techniques to analyze samples taken during the manufacturing process.

Mathematics Note

All the members of a group can be referred to as a **population**. A **sample** is a subset of a population. Typically, a sample includes only some members of the population, not all of them.

A **parameter** is a numerical characteristic of a population. A **statistic** is a numerical characteristic of a sample. Statistics are used to estimate the corresponding parameters of the population.

For example, the junior class in your school can be considered a population. The mean number of courses taken by all juniors is a parameter of that population. You can estimate the population mean by selecting a sample of the junior class and determining the mean number of courses taken by the students in the sample.

Sampling is the process of choosing a subset of a population. In a **simple random sample**, each member of the population must have the same chance of being included in the sample.

For example, suppose that the name of each student in the junior class is written on a slip of paper. If the slips are placed in a container and mixed thoroughly, then each name would have the same chance of being drawn from the container. By drawing 30 names from the container, you could obtain a simple random sample of 30 students from the population. **Note:** For the remainder of this module, each mention of a "random sample" refers to a simple random sample.

Exploration

Manufacturers often monitor product quality by taking samples, then testing those samples to determine if an appropriate number of products pass inspection. Imagine that you work for a company that makes paper coffee filters. Your most popular product is a circular filter 20 cm in diameter.

a. Using only a sheet of paper, a ruler, a pencil, and scissors, simulate the production of a coffee filter by completing Steps **1–3** below.

1. Draw two approximately perpendicular diameters of a filter on the sheet of paper.
2. Using the diameters from Step **1** as a guide, draw an outline of the filter as shown in Figure **1**.

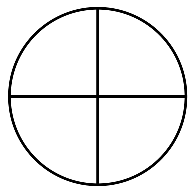


Figure 1: Outline of coffee filter

3. Cut out the filter with scissors.

b. Repeat Part **a** four more times to produce a total of five coffee filters.

c.

1. On each filter, draw several diameters at intervals of approximately 30°.
2. In order for a filter to pass inspection, every diameter must be within 5 mm of the desired length of 20 cm. Inspect each filter you manufactured and determine the number of filters that fail inspection.

d. The president of your company has decided to improve the quality of the firm's products. The diameters of all new filters must now fall within 2 mm of the desired 20 cm. To meet the new requirement, you may change the manufacturing process by adding new equipment and additional steps.

1. Describe the steps you would use to enhance the manufacturing process.
2. Manufacture five filters using the new process described in Step **1**.
3. Exchange filters with another manufacturer. Inspect the filters by drawing and measuring diameters as described in Part **c**. Each diameter must now fall within 2 mm of the desired 20 cm. Identify filters that fail inspection by writing the word "defective" on them.
4. Return the filters to the manufacturer, along with a record of the number that failed inspection.

e. Collect the class data from Part **d**, compile the results, and determine the percentage of failures for the entire class.

f. How many filters do you think would fail inspection in a sample of 10,000 produced by your class?

Discussion

a.
1. What characteristic did you use to measure quality in the exploration?

2. What other criteria might a coffee-filter manufacturer use to determine quality?

b.
1. Was it difficult to produce filters that would pass inspection using the process described in Part **a** of the exploration? Why or why not?

2. Do you think that it is reasonable to allow a variation of 5 mm if the desired diameter is 20 cm? Why or why not?

c. What factors might account for any differences in the numbers of failures observed in Part **d** of the exploration?

d. How could you improve the manufacturing process used in Part **d** of the exploration?

e.
1. Do you think the failure rate for the class data would be acceptable to a real manufacturer?

2. What factors should a company consider when making this decision?

f. In Part **f** of the exploration, how did you determine the number of filters you expected to fail?

g. Why is it often impractical to test every item produced for quality?

h. If your company manufactured 10,000 filters a day, describe a process you might use to monitor the quality of the filters.

Activity 1

Analyzing random samples of their products can provide manufacturers with important information about product quality. For example, the mean number of defective items that occur over many samples can help companies determine how many defective items to expect in any one sample.

If more than the expected number fail inspection, this may mean that the production process should be reviewed or modified. In this activity, you examine one model used to analyze such situations.

Exploration

A manufacturing company has found that the percentage of defective items produced by its factory is about 30%. As a quality control specialist, you decide to verify this figure by inspecting some samples.

The company produces 1000 items a day. To monitor product quality, you select one item from this population, inspect it, determine whether or not it is defective, then remove it from the population. To obtain a sample of 4 items, you repeat this process 4 times. This process involves **conditional probability**.

Mathematics Note

Conditional probability is the probability of an event occurring given that an initial event has already occurred. The probability that event B occurs, given that event A has already occurred, is denoted $P(\text{B}|\text{A})$.

For example, consider a paper bag containing three chips: two red and one blue. Suppose that you draw one chip from the bag. The probability that this chip is blue is $1/3$. However, suppose that you obtain a red chip on your first draw. Without replacing the first chip, you then draw a second. Since only two chips remain in the bag, the conditional probability of drawing a blue chip, given that a red one has already been drawn, is $1/2$.

In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A has already occurred:

$$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$$

For example, again consider the paper bag containing three chips: two red and one blue. Suppose that you draw one chip from the bag. Without replacing the first chip, you then draw a second. The tree diagram in Figure **2** shows all the possible outcomes in this situation, along with their probabilities. In this case, the probability of drawing a red chip followed by a blue chip, or $P(\text{RB})$, is $1/3$.

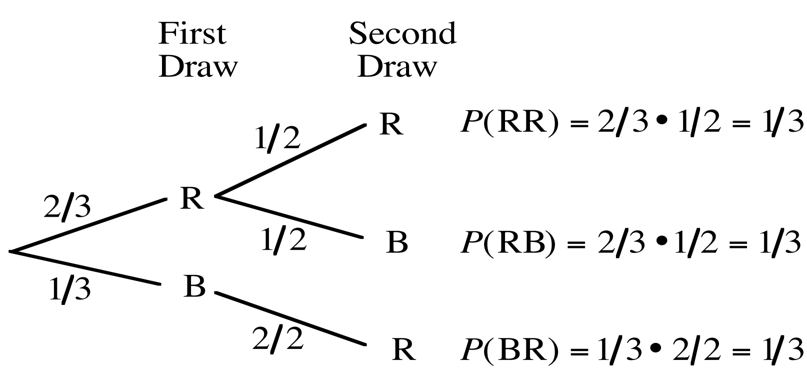


Figure 2: Possible Outcomes and Probabilities

a. Assuming that the percentage of defective items is 30%, how many defective items would there be in a population of 1000?

b. Create a tree diagram that shows all the possible outcomes for a sample of 4 items taken from this population of 1000 items. Determine the probability of each outcome.

Mathematics Note

Two events are **mutually exclusive** if they cannot occur at the same time in a single trial. If A and B are mutually exclusive events, then $P(\text{A and B}) = 0$.

When two events are mutually exclusive, the probability that one or the other occurs is the sum of the probabilities of the individual events. This can be written symbolically as follows: $P(\text{A or B}) = P(\text{A}) + P(\text{B})$.

For example, each outcome shown in the tree diagram in Figure **2** is mutually exclusive of the others. The probability of obtaining BR *and* RB in a single sample of 2 chips is 0, since one excludes the other. However, the probability of obtaining either BR *or* RB—in other words, one red chip and one blue chip, regardless of order—is $1/3 + 1/3 = 2/3$.

c. Determine the probabilities of obtaining each of the following numbers of defective items in a sample of 4 items from the company's daily production.

1. 0
2. 1
3. 2
4. 3
5. 4

d. Recall that two events A and B are **independent** if

$$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B})$$

Three events A, B, and C are independent if each pair of events is independent and $P(\text{A and B and C}) = P(\text{A}) \bullet P(\text{B}) \bullet P(\text{C})$. This definition can be extended to any number of independent events.

When each item drawn in a sample is replaced in the population before drawing another, the draws are independent events. Repeat Parts **b** and **c** assuming that each item is returned to the population after inspection.

e. Compare your results from Parts **c** and **d** and record your observations.

Discussion

a. What differences did you observe in the probabilities calculated in Parts **c** and **d** of the exploration?

b. In which situation were the expressions representing the probabilities easier to write? Explain your response.

Mathematics Note

A **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure.
- The probability of a success remains the same from trial to trial.
- The total number of successes is observed.

For example, consider an experiment that consists of tossing a six-sided die 10 times and observing the number of times that a 6 appears. In this case, there is a fixed number of trials, 10. For each trial, there are only two possible outcomes: either a 6 or not a 6. The probability that a 6 appears remains constant for each toss, and the result of one toss does not influence the result of any other. Therefore, this represents a binomial experiment.

c. Which of the samplings in the exploration can be considered a binomial experiment? Explain your response.

d. Why do you think that quality control specialists typically take samples without replacing the items?

e. Refer to the tree diagram you created in Part **d** of the exploration.

1. Describe the different outcomes in which exactly 2 defective items occur in a sample of 4.
2. What is the theoretical probability of each of these outcomes?
3. How did you determine the total probability of obtaining 2 defective items in a sample of 4?
4. How did you determine the total probability of obtaining 3 defective items in a sample of 4?

f. Even though sampling without replacement does not fit the definition of a binomial experiment, quality control specialists typically use binomial experiments to model their inspections.

1. Based on your observations in the exploration, why do you think this is acceptable?
2. When do you think this would not be acceptable?

g. Given that A and B are independent events, how could you demonstrate that $P(B|A) = P(B)$?

Assignment

1.1 As part of its quality control procedure, a compact disc (CD) maker uses the following process: A CD is randomly selected from the population, tested to determine whether or not it meets a certain standard, then returned to the population. The manufacturer repeats this process 10 times.

a. What is a trial in this experiment?

b. Are the trials independent events? Explain your response.

c. What could be considered a success in this experiment?

d. Is this process a binomial experiment? Explain your response.

1.2 A computer firm produces 5000 computer chips in one day. Of that total, 1050 are defective.

a. What is the probability that a chip selected at random from the day's production will be defective?

b. If the chip selected in Part **a** is removed from the population, what is the probability that a second chip selected at random will be defective?

c. Suppose that you select a sample of 10 chips from the day's production and determine that 4 are defective. If this sample is not replaced, what is the probability of selecting a defective chip from the remaining population?

d. Considering your responses to Parts **a–c**, write a paragraph evaluating the following statement: "If a small random sample is taken without replacement from a large population, then the probability of selecting a defective item is essentially unchanged from trial to trial and the experiment can be modeled by a binomial experiment."

1.3 Determine whether or not each of the following procedures is a binomial experiment. If the procedure is not a binomial experiment, can it be reasonably modeled by one? Explain your responses.

a. You select a random sample of 6 computer chips from a batch of 20. You replace each chip before selecting the next one.

b. You select a random sample of 6 computer chips from a batch of 20. You do not replace the sample.

c. You select a random sample of 500 American teenagers and determine their favorite brand of tennis shoes from a list of 10 brands.

d. You select a random sample of 5 electric motor shafts from an assembly line that produces 2000 shafts a day and determine if the shaft diameter is between 1.71 cm and 1.73 cm.

e. You roll a pair of dice 5 times and determine the number of times their sum is greater than 8.

* * * * *

1.4 Rindy rolls a fair die 10 times and records the number of times the result is greater than 4.

a. What is a trial in this experiment?

b. Are the trials independent? Explain your response.

c. What could be considered a success in this experiment?

d. What is the probability of a success?

e. Is this a binomial experiment? Explain your response.

1.5 Design a binomial experiment in which the probability of success does not equal the probability of failure. Describe how your design meets the criteria required for a binomial experiment.

1.6 Consider a box of 30 light bulbs in which 5 bulbs are defective.

a. If sampling from the box is done without replacement, what is the probability that in a sample of 30 bulbs,

1. exactly 4 bulbs will be defective?

2. exactly 5 bulbs will be defective?

b. If sampling from the box is done with replacement, the probability that 4 bulbs will be defective in a sample of 30 bulbs is approximately 0.18. The probability that 5 bulbs will be defective in a sample of 30 is approximately 0.19.

How do these values compare with your responses in Part **a**?

c. What must be true of the relative sizes of a sample and a population if the sampling process is to be reasonably modeled by a binomial experiment?

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Activity 2

As a quality-control consultant, you have been hired to help a computer disk manufacturer evaluate its production process. Due to vigorous competition, the company wants to reduce manufacturing costs and market a less expensive product. However, the company's executives worry that cutting costs may also reduce the quality of their disks.

They want you to determine the percentage of defective disks produced by the current manufacturing process and estimate the percentage of defects that will occur if they modify that process. Using your findings, they will decide whether or not to make any changes.

Discussion

a. Describe the difficulties you might encounter in using a tree diagram to show all the possible outcomes in a sample of 10 disks.

b. In the exploration in Activity **1**, you created a tree diagram of all the possible outcomes for a sample of 4 items from a population of 1000. For example, one possible outcome is DNDD, where D represents a defective disk and N represents a disk that is not defective.

1. If the rate of defective disks is 30%, how could you express the theoretical probability for this outcome using exponents?
2. What do the exponents represent?
3. How many other possible outcomes have exactly 3 defective disks?
4. Do each of these have the same probability as the outcome DNDD? Explain your response.
5. Describe how you could use combinations to count the number of possible outcomes that contain exactly 3 defective disks.
6. Using your responses to Steps **1** and **5** above, describe how to determine the theoretical probability of obtaining exactly 3 defective disks in a random sample of 4 disks.

Mathematics Note

The **binomial formula** can be used to determine the probability of obtaining r successes in n trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

For example, if 25% of a population of computer disks are defective, then $(1-25\%) = 75\%$ are not. The theoretical probability that exactly 4 defective disks will occur in a sample of 10 is:

$$\begin{aligned} P(4 \text{ successes in } 10 \text{ trials}) &= C(10,4) \bullet (0.25)^4 \bullet (0.75)^{10-4} \\ &= 210 \bullet (0.25)^4 \bullet (0.75)^6 \\ &\approx 0.15 \end{aligned}$$

c.

1. In the binomial formula, what does the quantity $(1-p)$ represent? Explain your response.
2. What does the expression $p^r \bullet (1-p)^{n-r}$ represent?

d. If the average rate of defective disks is 40%, describe how to determine the theoretical probability of obtaining each of the following in a random sample of 10 disks:

1. exactly 7 defective disks
2. at least 7 defective disks.

Assignment

2.1 Suppose that 30% of the computer disks manufactured by a company are defective. A single sample of 8 disks, recorded as NNDNDDNN, indicates that 3 disks are defective and 5 disks are not defective.

a. Give an example of a different sample in which 3 of 8 disks are defective.

b. Describe how to determine the probability of the outcome you listed in Part **a**.

c. Use combination notation, $C(n,r)$, to express the number of ways in which exactly 3 defective disks can occur in a sample of 8 disks.

d. Write an expression that represents the probability of obtaining exactly 3 defective disks in a sample of 8 disks.

e. Determine the probability of obtaining 3 defective disks in a sample of 8 disks.

2.2 By modifying the production process (and increasing its production costs), a manufacturer can reduce the rate of defective disks to 5%.

a. What is the probability that a disk selected at random from this population will pass inspection?

b. What is the probability that a disk selected at random from this population will not pass inspection?

c. Consider a sample of 100 disks drawn from this population. In how many ways can 20 defective disks occur in a sample of 100 disks?

d. In how many different ways can a sample of 100 disks contain exactly 80 disks that pass inspection?

e. Determine the probability that a sample of 100 disks consists of 20 defective disks and 80 nondefective disks.

2.3 A consulting firm has determined that an average of 20% of a manufacturer's computer disks are defective.

a. If you select a random sample of 4 disks from this population, what is the theoretical probability that:

1. none of the disks is defective?

2. exactly 1 of the disks is defective?

3. exactly 2 of the disks are defective?

4. exactly 3 of the disks are defective?

5. exactly 4 of the disks are defective?

b. Determine the sum of the probabilities in Part **a**. Explain why this sum occurs.

c. What is the theoretical probability that at least 3 of the disks are defective?

2.4 In a sample of 20 chips from a manufacturing process with a defect rate of 10%, is it possible for all the chips to be defective? If so, determine the probability that this occurs. If not, explain why not.

2.5 When a CD production line is working properly, 90% of the compact discs pass inspection. Determine the theoretical probability of obtaining at least 18 discs that pass inspection in a random sample of 20.

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2.6 A census of the student population at a large school revealed that 75% support the opening of a concession booth at football games. If you select a random sample of 5 students from this population, what is the probability that:

a. the majority of those sampled are in favor of opening the booth?

b. none of those sampled are in favor of opening the booth?

c. all of those sampled are in favor of opening the booth?

d. less than half of those sampled are in favor of opening the booth?

2.7 During basketball season, Clyde makes 80% of his free throws. Assuming this rate continues and that free throws are independent events, what is the probability that Clyde will make:

a. at least 9 of his next 10 free throws?

b. at least 8 of his next 10 free throws?

* * * * * * * * * *

Activity 3

As a quality control specialist, it is your job is to determine if a company's quality control procedures are working well. After you have inspected some samples and compiled the results, how do you decide what action to take? In this activity, you investigate how to interpret the results of your inspections.

Mathematics Note

A **probability distribution** is the assignment of probabilities to each possible outcome of an experiment. If the set of outcomes is either finite or can have a one-to-one correspondence with the natural numbers, the distribution is a **discrete distribution**.

For example, Table **1** below shows the discrete probability distribution for the numbers of heads that can occur when flipping two fair coins.

Table 1: A probability distribution

No. of Heads	Theoretical Probability
0	0.25
1	0.5
2	0.25

Exploration

Imagine that you have been hired by Compuquartz Corporation, a computer chip manufacturer, to test their chips for defects. The company believes that an average of 35% of their chips are defective.

a. Table **2** shows the probabilities of finding 0, 1, or 2 defective chips in a sample of 20 chips. Complete this discrete probability distribution table for up to 20 defective chips.

Table 2: Theoretical probability of defective chips

No. of Defective Chips	Theoretical Probability
0	0.0002
1	0.0020
2	0.0100

b. **1.** Use your completed table to create a histogram that shows probabilities versus the numbers of defective chips.

2. Using your histogram, determine the number of defective chips most likely to occur in a sample of 20 chips. Record the probability of this occurrence.

c. In previous modules, you calculated the expected value of an experiment by multiplying each possible outcome by its corresponding probability, then adding all of the resulting products. Use this method to calculate the expected number of defective chips in a sample of 20.

Mathematics Note

The **expected value** of a binomial experiment is the theoretical mean number of successes in n trials.

If a binomial experiment consists of n trials and p is the theoretical probability of success on any trial, then the expected value (or expected number of successes) is $n \bullet p$.

For example, consider an experiment that consists of flipping a fair coin 40 times. If a head is considered a success, the expected number of successes for this experiment is $40 \bullet 0.5 = 20$.

d. Use the formula for the expected value of a binomial experiment described in the mathematics note to calculate the expected number of defective chips in a sample of 20. Compare this value to the one determined in Part **c**.

e. For large sample sizes, the probability that a specific number of defective items will occur tends to be small. For this reason, quality control specialists often focus on the probability that the number of defective items in a sample will fall in a certain interval of values.

Identify the interval that describes each of the following:

1. the numbers of defective chips within 1 of the expected number

2. the numbers of defective chips within 2 of the expected number

3. the numbers of defective chips within 3 of the expected number

4. the numbers of defective chips within 4 of the expected number

f. Determine the probability that the number of defective chips in a sample will fall in each interval you identified in Part **e**.

g. To help interpret the statistics generated by sampling, quality control specialists often use graphs and charts. For example, the chart shown in Figure **3** below shows an **expected region** for the number of defective items in a sample.

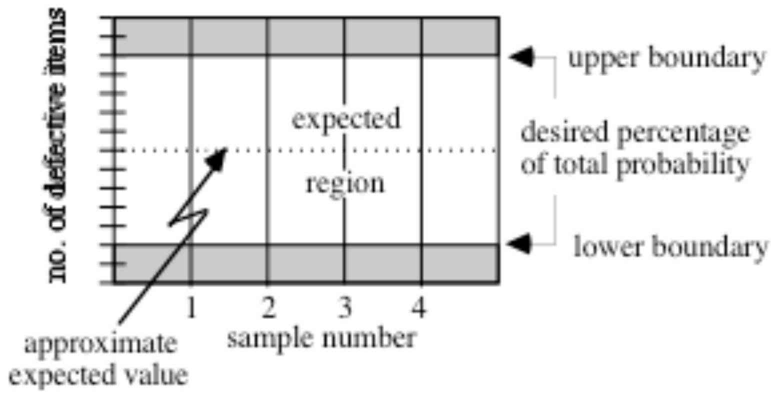

Figure 3: A chart showing an expected region

The boundaries of an expected region enclose a desired percentage of the total probability in an experiment. The dotted line drawn through the middle of the region represents the whole number of defective items closest to the expected value.

1. Which interval from Part **e** encloses approximately 90% of the probability in this experiment?
2. Use this interval, along with the expected number of defective chips, to draw a quality-control chart that shows a 90% expected region. Allow enough room on your chart to plot the results of four samples.

h. In quality control, the number of defective items in a sample is reasonable if the number falls within the expected region. If a sample statistic falls outside the expected region, this may indicate a need to examine the assumed defect rate.

1. Suppose that you take four samples of 20 chips from the population and obtain the following numbers of defective chips: 2, 9, 6, and 12, respectively. Plot this data on your quality-control chart.
2. Based on these results, what recommendations would you make to the company?

i. Compuquartz finds a 35% defect rate in their chips unacceptable. After adjusting their manufacturing process, they obtain a new defect rate of 13%. Repeat Parts **a–h** using the company's new defect rate.

Discussion

a. Describe how you would determine the expected number of defective chips in a sample of size x given a defect rate of y.

b. Describe any similarities or differences you observe in the probability histograms for the 35% and 13% defect rates. What defect rate do you think would correspond with a perfectly symmetrical histogram?

c. Can you identify the expected value for an experiment by examining its probability histogram? Defend your response.

d. Consider an experiment that involves rolling a single die one time. The probability of each face occurring is $1/6$. There are 6 faces. Explain why the expected value is not $6 \bullet 1/6 = 1$.

e. If the number of chips sampled had been 1000, what would have happened to the probability of obtaining a specific number of defective chips in the probability distribution table?

f. What percentage of the total probability is represented by the shaded regions in Figure **4** below?

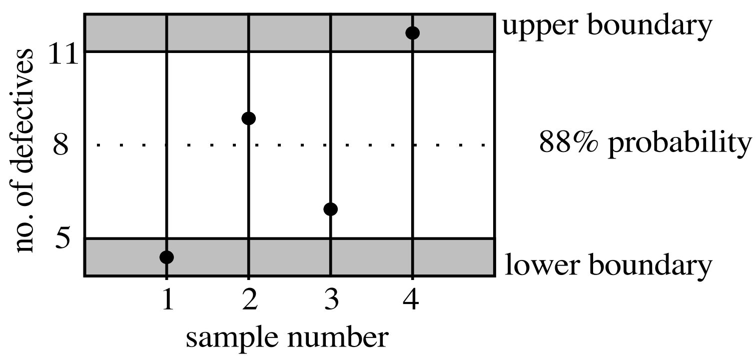

Figure 4: A completed quality-control chart

g. What is indicated by a sample in which the number of defective items falls above the upper boundary of the expected region?

h. What recommendations did you make to Compuquartz about their manufacturing process in Part **i** of the exploration?

Assignment

3.1 As part of its quality-control procedure, a company samples 100 ball bearings to see if their masses fall within an acceptable range.

a. Assuming the population has a 10% defect rate, determine the expected number of defective ball bearings in the sample.

b. Let d be the expected number of defective ball bearings found in Part **a**. Determine the probability that the number of defective ball bearings in the sample will fall in the interval $[d-1, d+1]$.

c. Determine the probability that the number of defective ball bearings will fall in the interval $[d-2, d+2]$.

d. If your sample contained more than 24 defective ball bearings, would you question the assumed defect rate of 10%? Why or why not?

3.2 Imagine that a manufacturer of coffee filters has hired you to establish a quality control procedure. The manufacturer thinks the defect rate in the manufacturing process is about 20%. You plan to inspect a random sample of 25 filters.

a. Assuming a defect rate of 20%, create a table and histogram showing the probability distribution for defective filters in the sample.

b. Determine the expected number of defective filters in the sample. Mark this value on the histogram.

c. According to your probability distribution, what are the chances that your random sample will have more than 8 defective filters?

d. If your sample contained more than 8 defective filters, would you question the assumed defect rate of 20%? Why or why not? Create a quality control chart to support your claim.

3.3 In a study on brand recognition, a marketing firm determined that one brand of tennis shoes, Rocket Shoes, was recognized by 60% of consumers.

a. Given a sample of 30 people, what is the probability that from 13 to 23 of them will recognize Rocket Shoes?

b. What interval represents an 80% expected region in this situation?

* * * * *

3.4 While writing an article on women in the workplace, a reporter for the local newspaper discovers the following statistic: According to the *Statistical Abstract of the United States*, 33.2% of the mathematical and computer scientists in 1992 were female. To see how the local situation compares to the national one, the reporter decides to collect data from a sample of 50 local scientists.

a. In a random sample of 50 mathematical and computer scientists, how many would you expect to be female?

b. Find an interval that corresponds to an expected region that contains approximately 90% of the probability in this situation. Create a chart of this region.

c. Use a simulation to generate data for 3 samples of 50 scientists each, assuming that 33.2% of the population of scientists is female. Plot the results of each sample on the chart from Part **b**. Report what these samples indicate.

3.5 As the old saying goes, "one bad apple spoils the whole bushel." Assume that there are 50 apples in a bushel. The probability of any individual apple going bad before the bushel is sold is 0.001. What is the probability that a bushel of apples will spoil before it is sold?

* * * * * * * * * *

Summary Assessment

In 1993, a nationally known clothing manufacturer found that only 248,000 of the 303.6 million items it sold were defective.

1. In a shipment of 10,000 items sold by this company in 1993, how many would you expect to be defective?

2. The following table shows an incomplete probability distribution for the number of defective items in a shipment of 10,000.

No. of Defective Items	Probability
0	0.00033439
1	0.00267726
2	0.01071656
$\vdots$	$\vdots$
20	0.00015799

a. Complete the table above by determining the probabilities of obtaining 3, 4, 5, …, 19 defective items.

b. What is the sum of the probabilities for obtaining from 0 to 20 defective items? Explain what this sum indicates.

3. Given a shipment of 10,000 items, what is the probability that the number of defective items falls in each of the following intervals:

a. [0, 4]

b. [20, 10,000]

4. **a.** Create a histogram that shows the probabilities of finding from 1 to 20 defective items in a shipment of 10,000.

b. What numbers of defective items appear to be most probable?

c. Is there a chance that 1% of the shipment of 10,000 items could be defective? Explain your response.

5. Considering a sample of 10,000 items, how many defective items would indicate to you that the company should examine its manufacturing process? Defend your response.

Module Summary

- All the members of a group can be referred to as a **population**. A **sample** is a subset of a population. Typically, a sample includes only some members of a population, not all of them.
- A **parameter** is a numerical characteristic of a population. A **statistic** is a numerical characteristic of a sample. Statistics are used to estimate the corresponding parameters of the population.
- **Sampling** is the process of choosing a subset of a population.
- In a **simple random sample**, each member of the population must have the same chance of being included in the sample.
- **Conditional probability** is the probability of an event occurring given that an initial event has already occurred. The probability that event B occurs, given that event A has already occurred, is denoted $P(\text{B}|\text{A})$.

  In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A has already occurred:

  $$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$$
- Two events are **mutually exclusive** if they cannot occur at the same time in a single trial. If A and B are mutually exclusive events, then $P(\text{A and B}) = 0$.
- If two events are mutually exclusive, then the probability that one or the other occurs is the sum of the probabilities of the individual events. This can be written symbolically as follows: $P(\text{A or B}) = P(\text{A}) + P(\text{B})$.
- Two events A and B are **independent** if $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B})$. Three events A, B, and C are independent if each pair of events is independent and $P(\text{A and B and C}) = P(\text{A}) \bullet P(\text{B}) \bullet P(\text{C})$. This definition can be extended to any number of independent events.
- A **binomial experiment** has the following characteristics:
  1. It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
  2. The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
  3. Each trial has only two possible outcomes: a success or a failure.
  4. The probability of a success remains the same from trial to trial.
  5. The total number of successes is observed.

- The **binomial formula** can be used to determine the probability of obtaining r successes in n trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

- The **expected value** of a binomial experiment is the theoretical mean number of successes in n trials. If a binomial experiment consists of n trials and p is the theoretical probability of success on any trial, then the expected value (or expected number of successes) is $n \bullet p$.
- A **probability distribution** is the assignment of probabilities to each possible outcome of an experiment. If the set of outcomes is either finite or can have a one-to-one correspondence with the natural numbers, the distribution is a **discrete distribution**.

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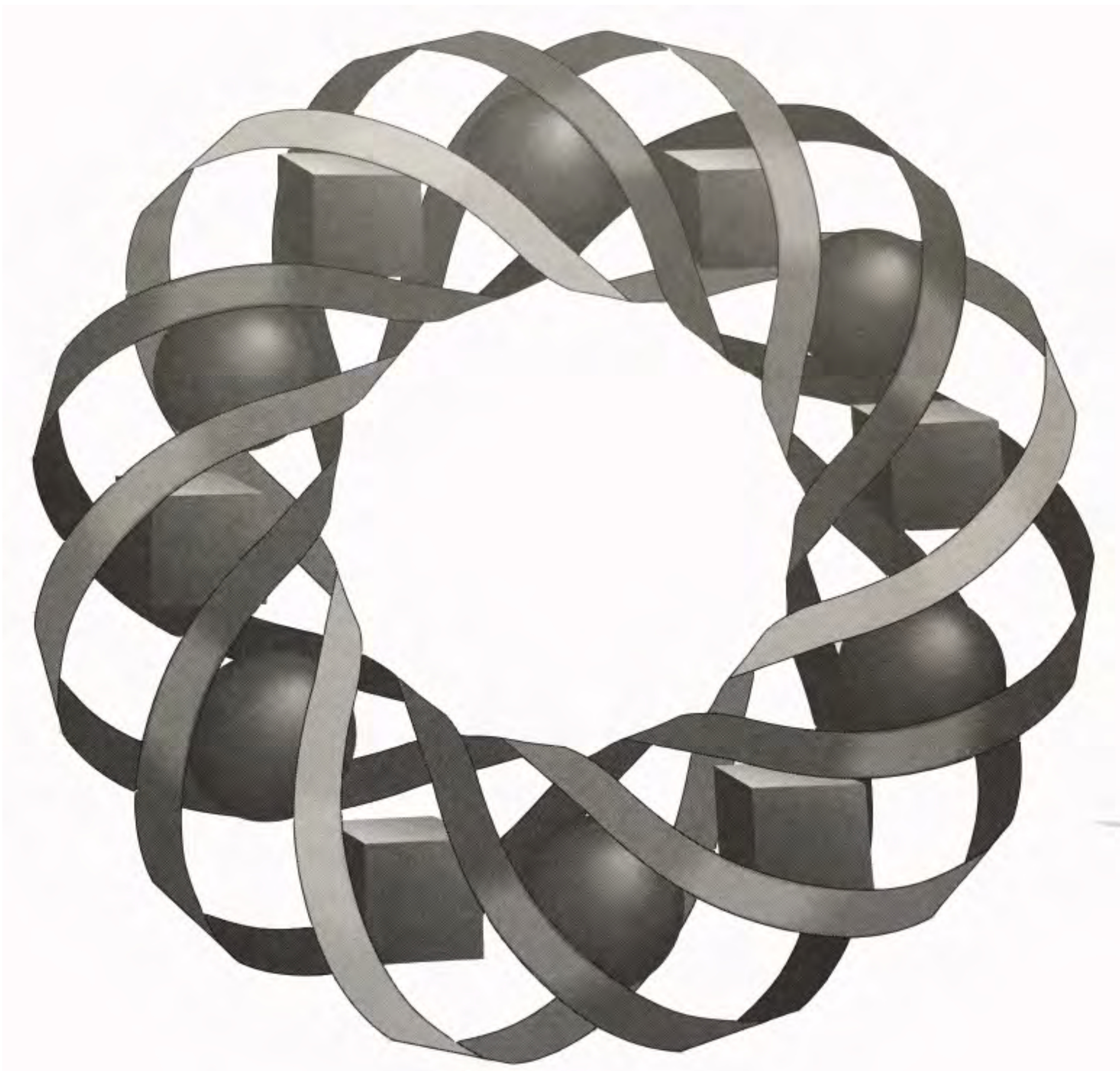
Kvanli, A. H., C. S. Guynes, and R. J. Pavur. *Introduction to Business Statistics*. St. Paul, MN: West Publishing Co., 1986.

Moore, D. S., and G. P. McCabe. *Introduction to the Practice of Statistics*. New York: W. H. Freeman and Co., 1989.

Whipkey, K. L., M. N. Whipkey, and G. W. Conway, Jr. *The Power of Mathematics*. New York: John Wiley & Sons, 1978.

Graphing the Distance



Scientists and mathematicians often use graphs to help predict the outcomes of real-life situations from experimental results. In this module, you'll examine the use of graphs in modeling motion.

Maurice Burke • Jeff Hostetter

Graphing the Distance

Introduction

Have you ever wondered how scientists at the National Aeronautics and Space Administration (NASA) plan the launch of a space shuttle? Or how traffic officers reconstruct the scene of an accident? In this module, you investigate objects in motion and the relationship between distance traveled and time.

Activity 1

One of the tools scientists use to analyze motion is a **distance-time graph**. A distance-time graph displays the distance between two objects (or an object and a fixed point) as a function of time. Typically, time is represented on the x-axis and distance on the y-axis. By comparing the distance-time graphs of different kinds of motion, you can observe many interesting and useful patterns.

Exploration

A **sonar range finder** uses sound waves to measure the distance from itself to another object. In this exploration, you use a sonar range finder to collect data, then use that data to create distance-time graphs.

a. Connect a sonar range finder to a science interface device as demonstrated by your teacher. As shown in Figure **1**, hold the range finder parallel to the plane of a wall or other flat surface. As a person holding the range finder moves, the science interface device collects data. This data can then be transferred to a graphing utility.

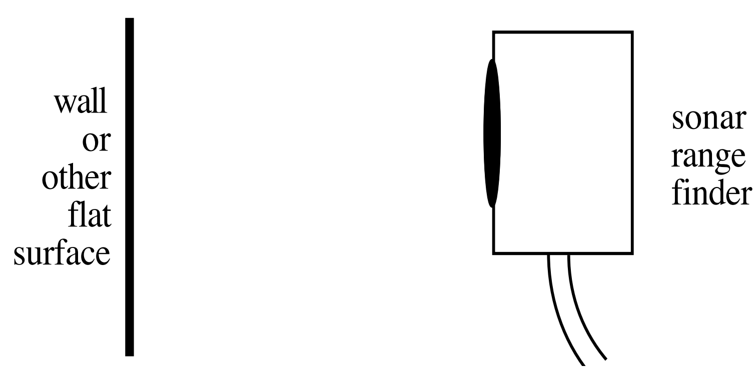
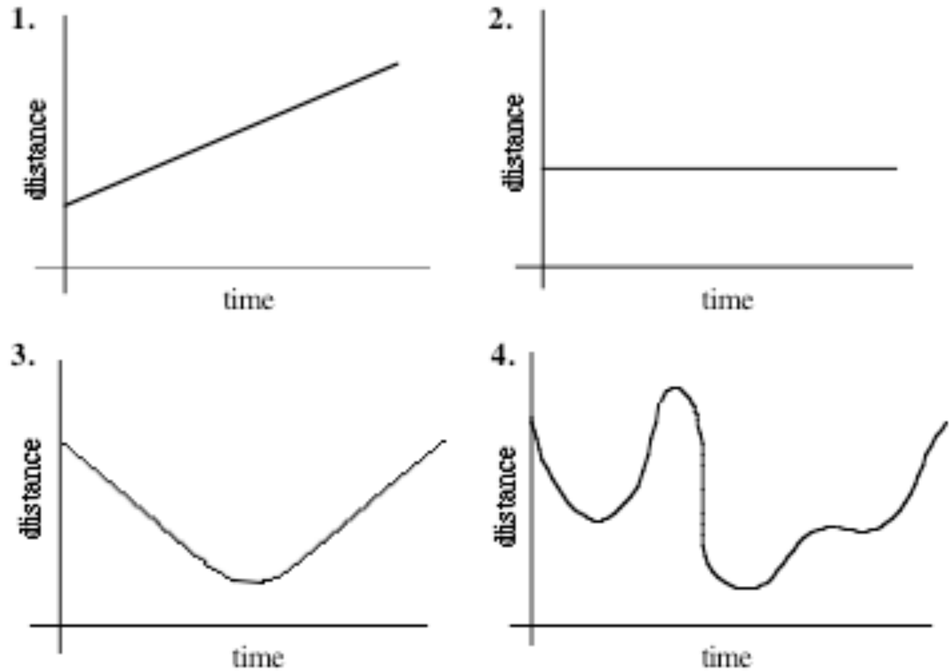


Figure 1: Positioning the sonar range finder

b. Practice using a range finder, science interface, and graphing utility to generate distance-time graphs. Move the range finder toward the wall, then away from it, and observe the resulting graphs. Draw one of the graphs on a sheet of graph paper.

c. Use the range finder to create distance-time graphs that match the ones shown below. (This may take a few trials.) Record the method you used to create each graph.



d. Figure **2** below shows a distance-time graph of data collected during the launch of a model rocket, where the distance is the rocket's height above the ground. After the rocket's engine ignited, it flew straight up. A few seconds after the engine burned out, it began to fall straight back toward the ground. Later, its parachute opened and slowed its descent.

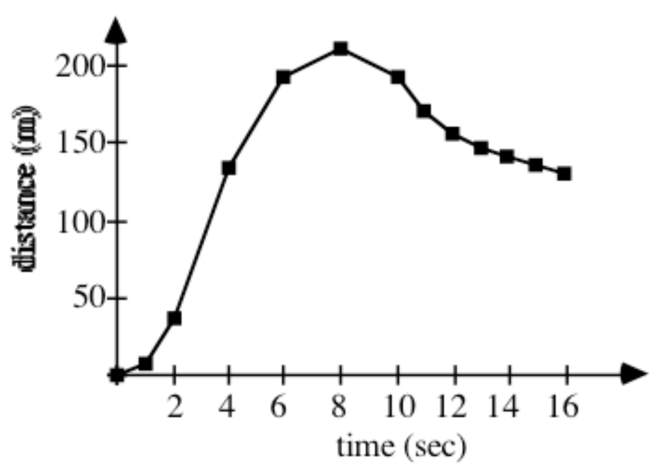


Figure 2: Distance-time graph for a model rocket

Point the range finder at the floor. By raising and lowering the range finder along a vertical path, create a distance-time graph whose shape resembles the graph in Figure **2**.

Discussion

a. In Part **b** of the exploration, you sketched a scatterplot on a sheet of paper. What do the points on the graph represent?

b. On a distance-time graph, what do the x- and y-intercepts represent?

c. How does the motion of the range finder in Part **b** of the exploration affect the resulting distance-time graph?

d. Describe how you moved the range finder to obtain each of the graphs in Part **c** of the exploration.

e. Figure **3** shows a distance-time graph generated by moving a range finder toward and away from a wall. Describe what is happening to the range finder at points P, Q, and R.

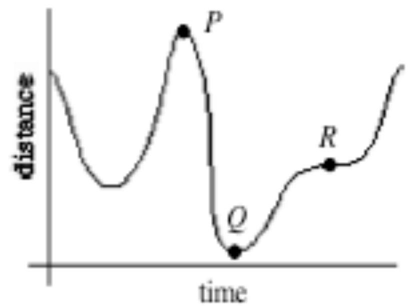


Figure 3: A distance-time graph

f. How does the kind of motion represented by a "curved" section of a distance-time graph differ from the motion represented by a "straight" section?

g. The average speed of an object during a time interval can be calculated by dividing distance traveled by time.

Table 1: Distance-time data for a model rocket

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0	0	11	170.8
1	7.4	12	155.4
2	36.9	13	147.7
4	134.2	14	141.9
6	192.9	15	136.3
8	212.3	16	130.8
10	192.9		

1. Using the data from Table **1**, determine the total distance traveled by the model rocket during the interval [6, 12].
2. What is the rocket's average speed during the interval [6, 12]?

h. Given the current location of an object moving at constant speed, what information would you need to predict the object's location in the future?

Science Note

Displacement is a change in the position of an object. It has both magnitude and direction.

For example, consider the distance-time data for the model rocket in Table **1**. At $t = 2$ sec, the rocket is 36.9 m above the ground. At $t = 6$ sec, it is 192.9 m above the ground. Its displacement during this time is $192.9 - 36.9 = 156.0$ m .

From $t = 6$ sec to $t = 12$ sec, however, the rocket's position changes from 192.9 m above the ground to 155.4 m above the ground. Its displacement over this period is $155.4 - 192.9 = -37.5$ m . In this case, positive values for displacement indicate movement away from the ground, while negative values indicate movement toward the ground.

i. **1.** Using the information in Table **1**, determine the displacement of the model rocket during the time interval [6, 12].

2. Compare this displacement to the distance you determined in Part **g** of the discussion.

Science Note

Velocity is the rate of change in position with respect to time. In other words, an object's velocity is its speed in a specific direction.

The **average velocity** of an object can be calculated by dividing its displacement by the change in time. For example, the model rocket's average velocity between $t = 2$ sec and $t = 6$ sec can be found as follows:

$$\frac{192.3 \text{ m} - 36.9 \text{ m}}{6 \text{ sec} - 2 \text{ sec}} = \frac{156.0 \text{ m}}{4 \text{ sec}} = 39 \text{ m/sec}$$

j. **1.** Use the data in Table **1** to determine the rocket's average velocity during the time interval [6, 12].

2. Compare this average velocity to the average speed you determined in Part **g** of the discussion.

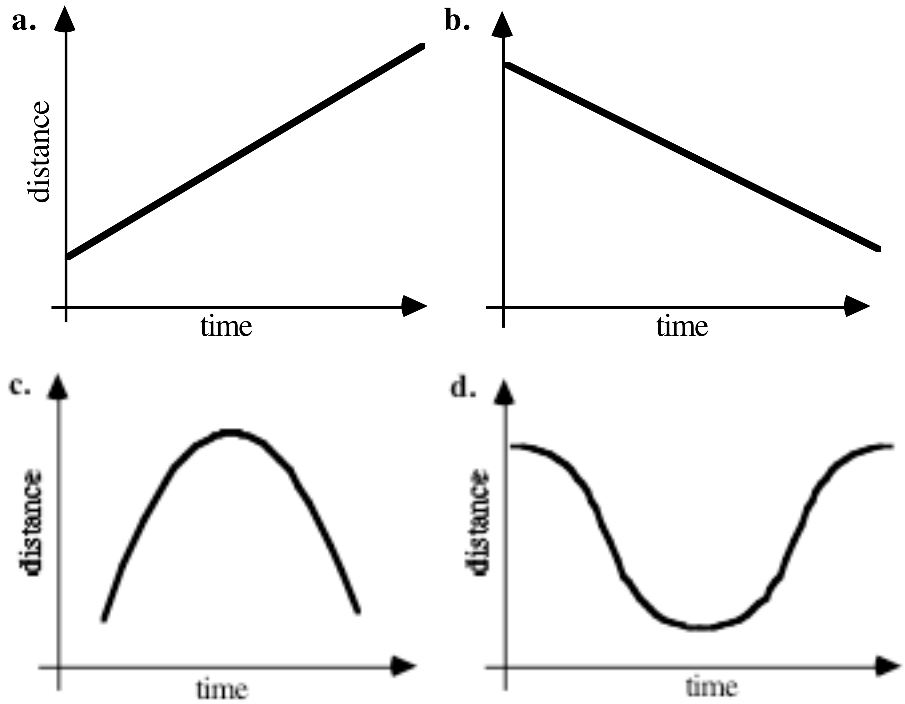
k. Using only your graphs from Parts **c** and **d** of the exploration, how can you tell when the range finder was moving at the greatest velocity?

l. What does a negative value for average velocity indicate about the motion of the rocket?

m. The **instantaneous velocity** of an object is its velocity at a particular instant in time. Describe how you could approximate the instantaneous velocity of the rocket at $t = 11$ sec.

Assignment

1.1 Describe a real-world motion that could be represented by each of the distance-time graphs below.



1.2 Sketch a copy of the distance-time graph in Figure **2.** On your copy, indicate the points at which you think the following events occurred:

a. the rocket takes off

b. the rocket's engine burns out

c. the rocket reaches its highest altitude

d. the parachute opens.

1.3 The table below contains data collected during the flight of a model rocket. Use the table to complete Parts **a–c.**

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0	0	11	170.8
1	7.4	12	155.4
2	36.9	13	147.7
4	134.2	14	141.9
6	192.9	15	136.3
8	212.3	16	130.8
10	192.9		

a. Determine the rocket's average speed during the time interval [6, 10].

b. Determine the average velocity of the rocket during the same interval.

c. Explain why your responses to Parts **a** and **b** are different.

1.4 **a.** Determine the average velocity of the rocket in Problem **1.3** during the interval from $t = 6$ sec to $t = 8$ sec.

b. Estimate the instantaneous velocity of the rocket at $t = 3$ sec. Describe how you determined your estimate.

1.5 The table below shows a space shuttle's distance from earth at various times during its initial vertical ascent.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
24	1791	136	53,355
48	7274	160	66,809
72	15,539	184	78,374
96	27,920	208	88,117
120	43,326		

Source: Johnson Space Center, Houston, Texas.

a. Create a distance-time graph of this data.

b. Based on this data, over what time interval does the space shuttle appear to start slowing down? Justify your response.

c. What is the average velocity of the space shuttle from $t = 24$ sec to $t = 120$ sec?

d. **1.** Estimate the shuttle's instantaneous velocity, in meters per second, 195 sec after liftoff and describe how you determined your estimate.

2. Express your response to Step **1** in kilometers per hour.

* * * * *

1.6 Sketch a distance-time graph that illustrates the motion of Little Red Riding Hood in the following paragraph:

> Little Red Riding Hood left home and walked briskly down the road towards Grandmother's house. Along the way, she stopped to pick some violets growing beside the road. The Wolf saw her picking flowers and offered to show her a shortcut. He led Little Red Riding Hood on a winding path through the woods. After the path crossed the road for the third time at the place where she had picked the flowers, Little Red Riding Hood realized that she'd been tricked. She got back on the road and ran the rest of the way to Grandmother's house.

1.7 The table below shows the distances between an object and a fixed point over time.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
1	2	14	303
2	5	15	303
3	33	16	313
4	55	17	323
5	94	18	333
6	160	19	104
7	273	20	41
8	283	21	20
9	293	22	11
10	303	23	7
11	303	24	5
12	303	25	3
13	303		

a. Create a distance-time graph of this data.

b. Which ordered pair (t, d), where d represents distance and t represents time, corresponds to the moment when the object first stopped moving away from the fixed point? Explain your response.

c. **1.** When did the object start moving back toward the fixed point?

2. How did its velocity change at this time?

d. Calculate the average velocity of the object during the time interval [4, 10]. What does this value tell you about the object's motion?

e. Calculate the average velocity of the object during the time interval [18, 23]. What does this value tell you about the object's motion?

* * * * * * * * * *

Activity 2

In the months before each launch, NASA engineers determine a space shuttle's longitude, latitude, altitude, and weight for every 0.04 sec of the flight. How are they able to predict these values with such accuracy and confidence?

At least some of the credit for scientists' ability to make such predictions must go to the English mathematician and physicist, Sir Isaac Newton (1642–1727). Using three concise laws of motion, Newton described the rules that govern the movement of objects both on earth and in space.

Science Note

A **force** is a physical quantity that can affect the motion of an object. Two familiar forces are gravity and friction.

According to Newton's **first law of motion**, an object in a state of rest or moving in a straight line at a constant speed will continue in that state unless acted on by a force.

Exploration

When the distance between an object and a fixed point changes at a constant rate, the distance-time graph can be modeled by a linear equation. In this exploration, you use a range finder to explore the movement of a ball at a constant velocity.

a. Obtain a track, a ball, and the range-finder apparatus from your teacher. As shown in Figure **4**, place the track on a level surface and position the range finder at one end. Place the ball on the track approximately 0.5 m from the range finder.

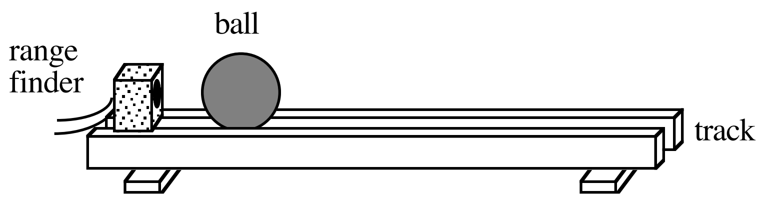


Figure 4: Ball on track with range finder

b. Push the ball away from the range finder just hard enough so that it rolls to the end of the track. Collect distance-time data as the ball rolls.

c. Repeat Part **b** several times, increasing the force of the initial push each time. Observe how changing the ball's speed affects the resulting distance-time graphs.

d. Select a data set and graph from Part **c** that appears to accurately describe the motion of the ball. Determine the average velocity of the ball over the time interval represented by the graph.

e. In the Level 2 module "If the Shoe Fits . . .," you used technology to find a linear regression equation to model data. Determine a linear regression equation that models the graph you chose in Part **d**. **Note:** Save your data, graph, and equation for use in the assignment.

Discussion

a. How did the ball's speed affect the graphs in Part **c** of the exploration?

b. In Parts **d** and **e** of the exploration, you found the ball's average velocity and determined a linear function to model its distance-time data.

1. What does the slope of the line indicate about the ball's movement?
2. What does the y-intercept of the line represent?
3. Should the line pass through the origin? Why or why not?

c. If you placed the ball at the far end of the track and pushed it toward the range finder, what would the resulting distance-time graph look like?

d.

1. How can you tell if an equation provides a good model of a distance-time graph?
2. If the function that models the ball's distance-time data is written in the form $f(x) = mx + b$, which part of the equation represents the ball's average velocity? Explain your response.
3. How could you determine the ball's instantaneous velocity in this situation?

e. Suppose that after collecting distance-time data for a ball on a ramp, the resulting graph can be modeled by a function of the form $d(t) = c$, where c is a constant. Describe the motion of the ball.

f. Describe some real-world situations that could be modeled with linear graphs of distance versus time.

Assignment

2.1 Describe how distance, velocity, and time are related when a ball is moving along a straight track at a constant rate.

2.2 Use the linear equation you found in Part **e** of the exploration to answer the following questions.

a. How far was the ball from the range finder after 1.7 sec?

b. When was the ball 0.75 m away from the range finder?

c.
1. If the track were long enough, how far from the range finder would the ball be after 10 min?
2. Do you think your prediction is accurate? Why or why not?

2.3 The following equation describes the motion of a ball on a level track, where $d(t)$ represents distance in meters from a range finder and t represents time in seconds:

$$d(t) = 0.75t + 0.5$$

a. Make a table showing the ball's distance from the range finder after 0 sec, 1 sec, 2 sec, and 3 sec.

b. Add a column to your table that shows the average velocity of the ball during the following time intervals: [0, 1], [1, 2], and [2, 3].

c. What is the instantaneous velocity of the ball at $t = 2$ sec?

d. How do your responses to Parts **b** and **c** relate to the equation that describes the ball's motion?

2.4 Describe a function that could be used to model the distance-time graph of each of the following:

a. a ball that is not moving

b. a ball moving at a constant velocity of 1 m/sec.

2.5 The table below shows a space shuttle's distance from earth at some specific times after liftoff.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
72	15,539	136	53,355
96	27,920	160	66,809
120	43,326	184	78,374

Source: Johnson Space Center, Houston, Texas.

a. Create a distance-time graph of this data.

b. **1.** Determine a linear equation that closely models the data.

2. What is the average velocity of the shuttle during the interval [72, 184]?

c. Use the equation you found in Part **b** to predict the shuttle's altitude after 520 sec.

d. Would it be reasonable to use this linear model to predict the shuttle's altitude at any time during its flight? Explain your response.

2.6 The table below shows some distance-time data collected as a model rocket returned to the ground under its parachute.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
12	155.4	15	136.3
13	147.7	16	130.8
14	141.9	17	125.2

a. Determine an equation that closely models the data.

b. Use your equation to predict when the rocket will reach the ground.

c. Do you think your prediction is reasonable? Why or why not?

* * * * *

2.7 Each of the linear equations below models a distance-time data set collected using a range finder:

1. $d(t) = 3.5t + 1.3$
2. $d(t) = -1.2t + 1.3$
3. $d(t) = 3.0t + 1.3$
4. $d(t) = 3.5t + 2.0$
5. $d(t) = 1.2t + 2.0$

a. Which equations represent objects moving at the same average velocity?

b. Which equations represent objects moving at the same average speed?

c. Which equation(s) represents the object which is moving the fastest?

d. Which equation(s) represents the object which is moving the slowest?

e. Which equation(s) corresponds to the object that started nearest to the range finder? farthest from the range finder?

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Activity 3

In the previous activity, you examined distance-time graphs of a ball moving at a constant speed. These graphs could be modeled by linear functions. But how would you model a distance-time graph for an object whose speed was increasing or decreasing? In this activity, you investigate **polynomial functions**, a group of functions that can provide models for other types of motion.

Mathematics Note

A **polynomial** in one variable, x, is an algebraic expression of the general form:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

where n is a whole number and the **coefficients** a_i are real numbers for $i = 0,\ 1,\ 2,\ \ldots,\ n$.

The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression with a non-zero coefficient. A polynomial in the general form shown above has a degree of n, provided that $a_n \neq 0$.

For example, the expression $-5x^2 + 3x - 7$ is a polynomial of degree 2. The expression $7x^3 - \sqrt{3}x^2 + 6x - 0.25$ is a third-degree polynomial.

A function f is a **polynomial function** if $f(x)$ is defined as a polynomial in x. The degree of a polynomial function is the same as the polynomial it contains.

For example, the function $f(x) = -5x^2 + 3x - 7$ is a second-degree polynomial function.

Discussion 1

a. Are linear functions of the forms $f(x) = mx + b$ and $f(x) = c$ also polynomial functions? If so, identify the degree of each. If not, explain why not.

b. If you coasted on a bicycle from the top of a hill to the bottom, would you expect a distance-time graph of your movement to be linear? Explain your response.

c. Distance-time graphs of the motion described in Part **b** can be modeled by second-degree polynomial functions, also known as **quadratic functions**. The graph of a quadratic function is a **parabola**. For example, Figure **5** below shows the graphs of two quadratic functions.

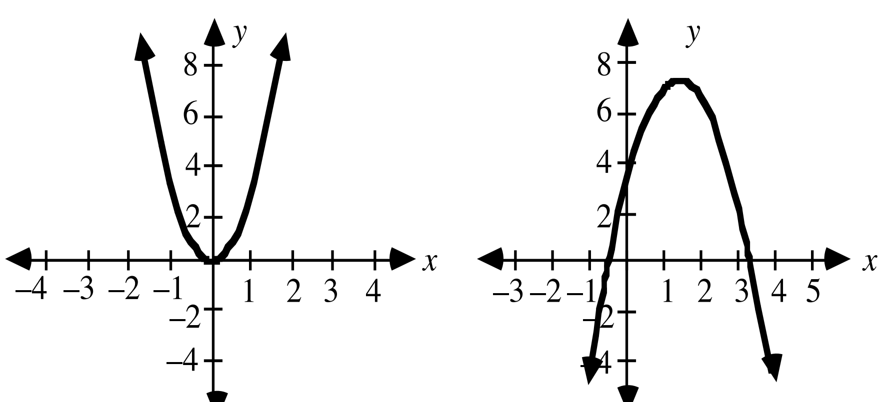


Figure 5: Graphs of two quadratic functions

1. Considering the general form of a polynomial described in the previous mathematics note, what do you think would be the general form of a quadratic function?

2. A parabola is symmetric about a line, known as its **axis of symmetry**. Describe the location of this axis of symmetry.

3. When a parabola opens upward, its **vertex** occurs at the lowest point in the graph; when a parabola opens downward, its vertex occurs at the highest point. Describe how you could locate the vertex of a parabola using the axis of symmetry.

d. A third-degree polynomial function is also known as a **cubic function**. Figure **6** below shows the graphs of two cubic functions.

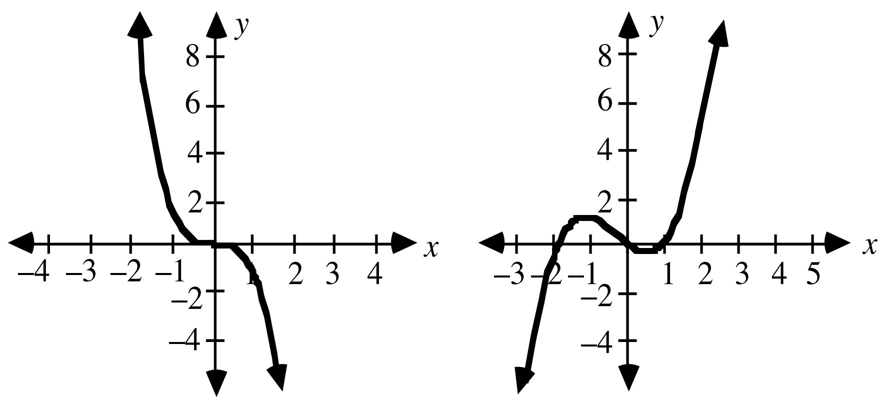


Figure 6: Graphs of two cubic equations

1. Describe the general form of a cubic function.

2. Does the graph of a cubic function appear to be symmetric about a line? Explain your response.

Exploration

Quadratic functions also can be written in the form $f(x) = a(x - c)^2 + d$, where a, c, and d are constants. In the following exploration, you investigate how the values of these constants affect the graph of the functions.

a. Use a graphing utility to create a graph of the quadratic function $f(x) = x^2$.

b. Sketch a copy of the graph on a sheet of graph paper.

c. On the same coordinate system as in Part **b**, sketch the image that results when the graph of $f(x) = x^2$ is translated 1 unit to the right.

d. Use a graphing utility to determine which of the following functions represents the graph you sketched in Part **c**.

1. $f(x) = x^2 + 1$

2. $f(x) = x^2 - 1$

3. $f(x) = (x - 1)^2$

4. $f(x) = (x + 1)^2$

e. Predict the equation of the function that results when $f(x) = x^2$ is translated 2 units to the left.

Verify your prediction using a graphing utility.

f. Compare the graphs of each of the following pairs of functions:

1. $f(x) = x^2$ and $f(x) = -x^2$

2. $f(x) = x^2$ and $f(x) = 3x^2$

3. $f(x) = x^2$ and $f(x) = \frac{1}{3}x^2$

g. Use a graphing utility to compare the graph of the function $f(x) = x^2$ to the graph of each of the following. In each case, experiment with both negative and positive values for the constant. Record your observations.

1. $f(x) = x^2 + d$

2. $f(x) = (x - c)^2$

3. $f(x) = ax^2$

h. Repeat the exploration for a cubic function. To do this, replace the exponent 2 with the exponent 3 in each of the functions in Parts **a–g**.

Discussion 2

a. How does the value of a appear to affect the graphs of functions of the form $f(x) = ax^2$ and $g(x) = ax^3$?

b. How does the value of c appear to affect the graphs of functions of the form $f(x) = (x - c)^2$ and $g(x) = (x - c)^3$?

c. How does the value of d appear to affect the graphs of functions of the form $f(x) = x^2 + d$ and $g(x) = x^3 + d$?

d. Describe how you could use the symmetry of a parabola to determine the translation of the graph of $f(x) = x^2$ that results in the graph of a given quadratic function.

e. Compare the graph of the function $f(x) = -0.5(x - 3)^2 + 4$ to the graph of $f(x) = x^2$.

f. Figure **7** below shows a scatterplot of the data in the table on the left.

What quadratic function would you use to model this data? Justify your response.

x	y
–4	1
–3	2
–2	1
–1	–2
0	–7
1	–14
2	–23
3	–34
4	–47

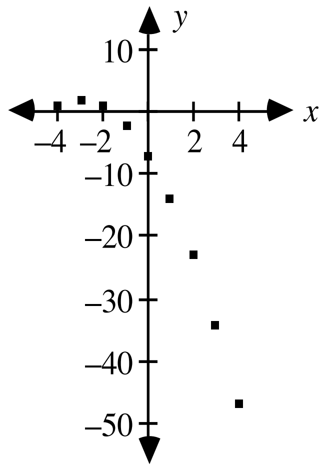


Figure 7: Data and scatterplot

g. Describe how you could rewrite a quadratic function of the form $f(x) = a(x-c)^2 + d$ in the general form $f(x) = a_2x^2 + a_1x + a_0$.

h. Is the expression shown below a polynomial? Justify your response.

$$\frac{x^2 - 1}{x - 1}$$

Assignment

3.1 **a.** What is the degree of the following polynomial: $3x^7 + 1$?

b. In the polynomial in Part **a**, what are the coefficients of each of the following terms: x^7, x^6, x^5, x^4, x^3, x^2, x^1, and x^0?

c. What is the degree of the polynomial below?

$$\frac{x^2}{1000} + 10^6$$

d. Is 6 a polynomial? If so, identify its degree. If not, explain why not.

3.2 Which of the following expressions are polynomials? Justify your response.

a. $\sqrt{x}$

b. $x^{-3} + 6$

c. $x(x+1)(x+2)$

3.3 Determine a function of the form $f(x) = a(x - c)^2 + d$ that represents each of the following transformations of the function $f(x) = x^2$. Use graphs to support your responses.

a. a translation 3.5 units to the right

b. a reflection in the x-axis

c. a vertical "stretch"

3.4 Determine a function of the form $g(x) = a(x - c)^3 + d$ that represents each of the following transformations of the function $g(x) = x^3$. Use graphs to support your responses.

a. a translation $2\frac{2}{3}$ units to the left

b. a translation 4.2 units upward

c. a vertical "shrink"

3.5 **a.** Write the function whose graph results in the following transformations of the graph of $f(x) = x^2$: a translation 3 units to the right, a translation 2 units down, and a reflection in the x-axis.

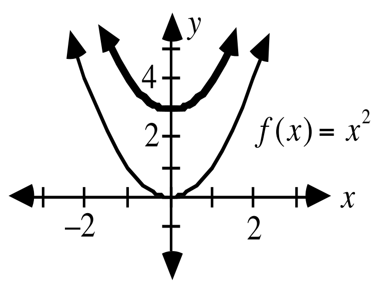
b. To verify your response, graph $f(x) = x^2$ and the function from Part **a** on the same coordinate system.

c. Rewrite the function in Part **a** in the general form of a polynomial.

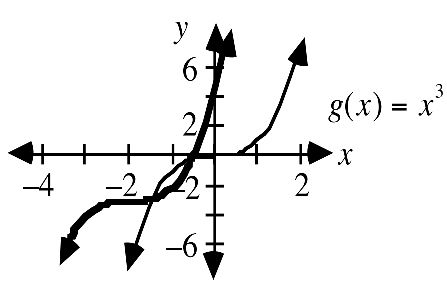
d. Demonstrate that the functions found in Parts **a** and **c** are equivalent.

3.6 In Parts **a–c** below, determine a function of the form $f(x) = a(x - c)^2 + d$ or $g(x) = a(x - c)^3 + d$ whose graph is represented by the bold curve.

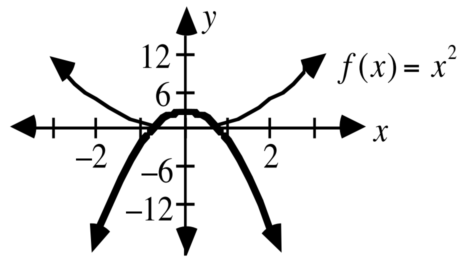
a. Hint: The point (1,4) on the graph is the image of the point (1,1).



b. Hint: The point (–2,–3) on the graph is the image of the point (0,0).



c. Hint: The vertex is at (0,3) and the point (–2,–11) is the image of the point (–2,4).



3.7 The distance-time data in the table below can be modeled by a quadratic function.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	3.0	0.8	7.8
0.2	4.8	1.0	8.0
0.4	6.2	1.2	7.8
0.6	7.2	1.4	7.2

a. Create a scatterplot of the data.

b. Determine a quadratic function that models the data and graph it on the same coordinate system as in Part **a**.

c. Write the function in Part **b** in the general form of a polynomial.

* * * * *

3.8 **a.** Write the function whose graph results in a translation 2 units to the left, a translation 6 units upward, and a reflection in the x-axis of the function $g(x) = x^3$.

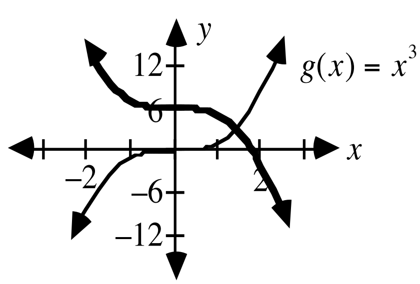
b. To verify your response, graph $g(x) = x^3$ and the function from Part **a** on the same coordinate system.

c. Rewrite the function in Part **a** in the general form of a polynomial.

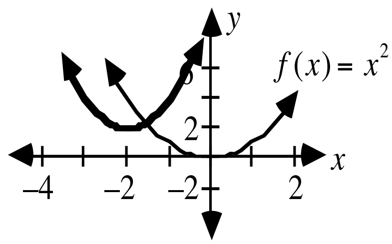
d. Demonstrate that the functions found in Parts **a** and **c** are equivalent.

3.9 In Parts **a** and **b** below, determine a function of the form $f(x) = a(x - c)^2 + d$ or $g(x) = a(x - c)^3 + d$ whose graph is represented by the bold curve.

a. Hint: The point (–1,7) is the image of point (1,1).



b. Hint: The point (–1,3) is the image of point (1,1).



3.10 The data in the table below can be modeled by a cubic function.

x	$f(x)$	x	$f(x)$
0.0	7.00	0.8	2.04
0.2	4.56	1.0	2.00
0.4	3.08	1.2	1.96
0.6	2.32	1.4	1.68

a. Create a scatterplot of the data.

b. Determine a cubic function that models the data and graph it on the same coordinate system as in Part **a**.

c. Rewrite the function in Part **b** in the general form of a polynomial.

* * * * * * * * * *

Activity 4

According to legend, Isaac Newton "discovered" gravity after watching an apple fall from a tree. In this activity, you explore how the **acceleration** due to gravity affects the distance-time graphs of freely falling objects.

Science Note

Acceleration is the rate of change in velocity with respect to time.

For example, consider a car driving along a straight section of highway. Over time, the velocity of the car can increase, decrease, or remain the same. When the car's velocity increases, its acceleration is positive. When the car's velocity decreases, its acceleration is negative. If the car's velocity remains constant, its acceleration is 0.

The average acceleration of an object over a particular time interval can be determined by dividing the change in velocity by the change in time. For example, consider a model rocket launched straight into the air. At $t = 3$ sec, its velocity is 48.65 m/sec. At $t = 5$ sec, its velocity is 29.33 m/sec. The rocket's average acceleration during this period can be estimated as follows:

$$\frac{29.33 \text{ m/sec} - 48.65 \text{ m/sec}}{5 \text{ sec} - 3 \text{ sec}} = -9.65 \text{ m/sec}^2$$

This means that, during the time interval [3, 5], the rocket's velocity decreased by an average of 9.65 m/sec for every second that passed.

Discussion 1

a. When you rolled a ball along a level track in Activity **2**, its velocity remained almost constant over time. If one end of the track were raised, and the ball rolled down the incline, do you think that its velocity also would remain constant? Explain your response.

b. As the ball continues down the track, how would the distances traveled in equal time intervals compare?

c. What do you think a graph of the distance-time data collected for a ball rolling down an inclined track will look like?

Exploration

In this exploration, you collect distance-time data for a ball rolling down an incline. You then use polynomial functions to model this data.

a. Obtain a track, a ball, and the range-finder apparatus from your teacher. Set up the track and range finder as in Activity **2**, then use books or blocks to raise the end of the track with the range finder on it.

b. Place the ball approximately 0.5 m from the range finder and gently release it. (Do not push the ball down the track.) You should begin collecting distance-time data just before the ball is released.

c. Repeat Part **b** several times, then select a data set and graph that you think accurately describes the motion of the ball down the track.

d.
1. Edit your data set so that it contains only information collected as the ball was actually moving, beginning with the moment of its release.
2. Determine a quadratic equation that appears to model this data set.

e. Recall that a **residual** is the difference between an observed value and the corresponding value predicted by a model, and that the sum of the squares of the residuals can be used to evaluate how well a model fits a data set.

Calculate the sum of the squares of the residuals for your model. Adjust the equation until the sum of the squares of the residuals indicates that the model closely approximates the data. **Note:** Save your data set, graph, and equation for use later in the module.

f. In Activity **2**, you used technology to generate linear models called linear regression equations. Many calculators and computer software programs also can generate other types of regression equations, including exponential, power, and polynomial regressions.

Use technology to determine a quadratic regression equation for your data set in Part **d**.

g. If the track were extremely steep, the ball's motion would be virtually a **free fall**. In physics, the term *free fall* refers to an object falling without air resistance and affected only by the force of gravity.

Remove the range finder from the track and point it at the floor from a height of approximately 60 cm. Hold the ball directly beneath the range finder at a height of about 10 cm. Release the ball.

You should begin collecting distance-time data just before the ball is released and stop after the ball hits the ground. Repeat this experiment several times, then select a data set and graph that you think accurately describes the motion of the ball.

h. Repeat Parts **d–f** using the data for the falling ball.

Discussion 2

a.
1. How did raising one end of the track affect the speed of the ball over time?
2. How is this effect displayed on the distance-time graphs?

b.
1. Rewrite the quadratic functions you found in Parts **d** and **h** of the exploration in the general form of a polynomial function.
2. Compare each of these equations to the quadratic regression equation for the same data set.

c. How do the distance-time graphs and equations you found in this exploration compare with those you used to model a ball rolling on a level track in Activity **2**?

d. Use your graphs from this exploration and the one in Activity **2** to answer the following questions.

1. Describe the shape of a distance-time graph when an object's acceleration is 0.
2. What influence does an object's acceleration have on the shape of its distance-time graph?
3. How does the magnitude of the acceleration affect the equations used to model the distance-time data?

Science Note

The acceleration due to gravity is a constant typically denoted by g. On earth's surface, the acceleration due to gravity is about 9.8 m/sec^2 in a direction toward the earth's center. For comparison, the acceleration due to gravity on the moon's surface is about 1.6 m/sec^2.

When an object is acted on only by gravity, its distance from the ground is described by the following function:

$$d(t) = -\frac{1}{2}gt^2 + v_0 t + d_0$$

where $d(t)$ represents the object's distance from the ground after t sec, g is the acceleration due to gravity, v_0 is the object's velocity in the vertical direction at $t = 0$, and d_0 is the object's distance above the ground at $t = 0$.

For example, consider a tennis ball dropped from a height of 10 m. Since the ball is dropped and not thrown, its initial velocity in the vertical direction is 0, or $v_0 = 0$. Since its initial distance above the ground is 10 m, $d_0 = 10$. On earth, the value of g is about 9.8 m/sec^2. To calculate the ball's height above the ground after 1 sec, these values can be substituted into the equation for $d(t)$ as follows:

$$\begin{aligned} d(1 \text{ sec}) &= -\frac{1}{2}\left(9.8 \text{ m/sec}^2\right)(1 \text{ sec})^2 + (0 \text{ m/sec})(1 \text{ sec}) + 10 \text{ m} \\ &= -4.9 \text{ m} + 0 \text{ m} + 10 \text{ m} \\ &= 5.1 \text{ m} \end{aligned}$$

e.

1. Using the general formula described in the previous science note, write a quadratic function that should describe the distance from the ground over time of the falling ball in Part **h** of the exploration.
2. Compare this function to the ones you determined in the exploration. Why do you think there are differences in these equations?

Assignment

4.1 The table below shows the space shuttle's distance above the earth at various times after liftoff.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0	0	72	15,539
24	1791	96	27,920
48	7274		

Source: Johnson Space Center, Houston, Texas.

a. Create a distance-time graph of this data.

b. Find an equation that closely models the data. Graph this equation on the same coordinate system as in Part **a**.

c. Use your model to estimate the shuttle's distance above the earth at each of the following times:

1. 50 sec

2. 600 sec

d. This shuttle will orbit the earth at an altitude of approximately 160 km. Given this fact, do your predictions in Part **c** seem reasonable? Explain your response.

e. Determine the approximate number of seconds required for the shuttle to reach an altitude of 160 km.

4.2 The data in the following table was generated using a ball, a ramp, and a range finder.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0	0.022	1.0	0.426
0.2	0.022	1.2	0.733
0.4	0.022	1.4	1.124
0.6	0.071	1.6	1.600
0.8	0.206	1.8	2.164

a. Determine an equation that models the data collected while the ball was rolling.

b. Predict the ball's position 2 sec after its release.

c. If the ramp were long enough, how long would it take the ball to reach a position 4 m from the range finder?

4.3 Use the data given in Problem **4.2** to complete Parts **a–c** below.

a. Estimate the ball's instantaneous velocity at each of the times listed in the following table.

Time (sec)	Velocity (m/sec)
0.5	
0.7	
0.9	
1.1	
1.3	

b. Use the values found in Part **a** to estimate the ball's average acceleration during each of the intervals listed in the table below.

Time Interval (sec)	Acceleration $\left(\text{m/sec}^2\right)$
$[0.5, 0.7]$	
$[0.7, 0.9]$	
$[0.9, 1.1]$	
$[1.1, 1.3]$	

c. How does the acceleration of the ball appear to change over time?

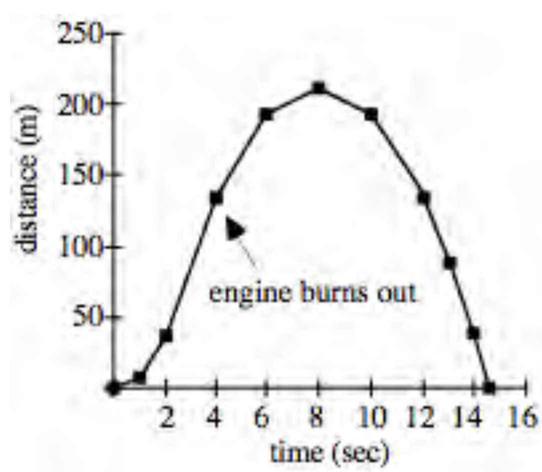
4.4 Sketch a distance-time graph that could represent each of the situations described below.

a. A ball moves away from a range finder with an acceleration of 0.

b. A ball moves away from a range finder with a positive acceleration.

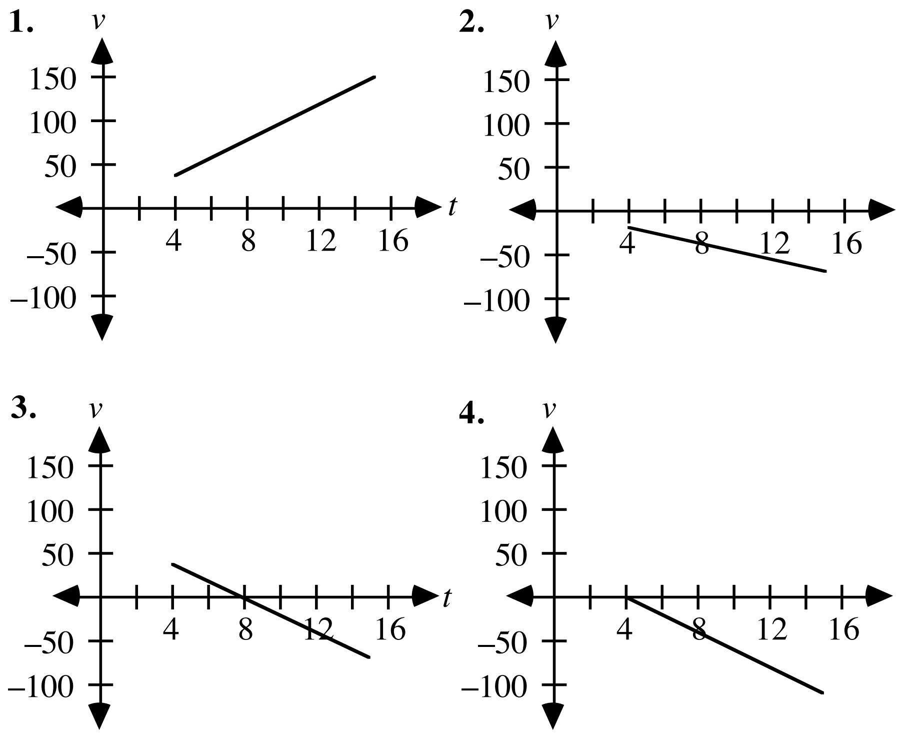
c. A ball moves away from a range finder with a negative acceleration.

4.5 The distance-time graph below shows data collected during the flight of a model rocket. After its engine burns out, the primary force acting on the rocket is gravity.



a. Identify the locations on the graph where the velocity of the rocket is positive, zero, or negative.

b. Based on your responses to Part **a**, which of the graphs below represents a graph of velocity versus time for the interval [4, 14]? Justify your choice.



c. Using the graph you selected in Part **b**, describe a graph of the rocket's acceleration versus time over the same interval. Justify your response.

4.6 The table below shows the distance above the ground at various times for a bouncing ball.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.00	0.0	0.48	0.933
0.08	0.301	0.56	0.878
0.16	0.549	0.64	0.762
0.24	0.736	0.72	0.583
0.32	0.865	0.80	0.342
0.40	0.929	0.88	0.0

a. Find a quadratic equation that closely models this data.

b. According to your model, when does the ball reach its highest point?

c. What is the velocity of the ball at the time it reaches this point?

d. What is the velocity of the ball at $t = 0$?

4.7 When the acceleration of a rocket increases at a constant rate, then the rocket's distance above the ground with respect to time can be modeled by a cubic equation. The data in the table below shows the height above the ground of a model rocket launched straight into the air. Find a third-degree polynomial that closely fits this data.

Time (sec)	**Distance (m)**	**Time (sec)**	**Distance (m)**
0.0	0.0	1.5	17.9
0.5	2.6	2.0	36.9
1.0	7.4		

* * * * *

4.8 Suppose that you have used a range finder to collect distance-time data for a freely falling object. Your data can be modeled by the function $d(t) = 4.9t^2$, where $d(t)$ represents distance in meters and t represents time in seconds.

a. Describe the motion of the object, including its initial velocity and initial distance from the range finder.

b. Use the model equation to estimate when the object was 4 m from the range finder.

4.9 Sir Isaac Newton once said that if he had seen farther than other scientists and mathematicians, this was because he had "stood on the shoulders of giants." One of those giants was Galileo Galilei (1564–1642), who died in the year of Newton's birth. In fact, Newton's first law of motion was actually a variation on Galileo's concept of inertia.

a. Besides describing inertia, Galileo theorized that, in the absence of air resistance, two objects of different sizes and weights dropped from the same height would reach the ground at the same time. What does Galileo's theory predict about the motions of an apple falling from a tree and the ball you dropped in the exploration?

b. On one of the Apollo missions to the moon, an astronaut demonstrated Galileo's theory by dropping a hammer and a feather from the same height. Given that the acceleration due to gravity on the moon is $1/6$ that on earth, what function could be used to describe the two objects' distance from the lunar surface with respect to time?

c. If the hammer and feather were dropped from a height of 2 m, how long would it take them to reach the lunar surface?

d. If this demonstration were conducted on earth, how long would it take the hammer to reach the ground?

4.10 The data in the table below shows a rocket's distance above the ground at various times after launch. At $t = 2$ sec, the rocket's engine burned out. After this time, gravity is the primary force acting on the rocket.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
2	36.9	8	212.3
4	134.2	10	192.9
6	192.9	11	170.8

a. Find a polynomial equation that closely fits the data.

b. Interpret the significance of each coefficient in your equation.

4.11 The table below shows a space shuttle's distance above the ground during the first seconds after liftoff. Find an equation that models this data. Explain why you think your model fits the data well.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.00	0.00	7.68	153.31
1.92	6.10	9.60	249.63
3.84	33.53	11.52	371.86
5.76	81.69	13.44	519.99

Source: Johnson Space Center, Houston, Texas.

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Research Project

Select one of the following topics.

a. In addition to his three laws of motion, Isaac Newton also proposed a law of universal gravitation. Together, these few principles revolutionized the sciences of physics and astronomy. Write a report on Newton's contributions to the study of motion, including an explanation of the relationship among force, mass, and acceleration.

b. Scientists at NASA closely analyze each launch of the space shuttle and make a wealth of information available to the public. Contact the Johnson Space Center regarding a future shuttle flight. Write a report on the planned launch, including an analysis of the flight-path data.

Summary Assessment

1. The distance-time data shown below was obtained by moving a book toward and away from a range finder taped to a desk.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	1.393	2.4	1.856
0.4	1.145	2.8	1.838
0.8	0.851	3.2	1.549
1.2	0.682	3.6	1.308
1.6	0.859	4.0	0.841
2.0	1.328		

a. Create a distance-time graph of this data.

b. Describe what happens to the velocity of the book during the interval from 0 sec to 4 sec.

c. Identify at least three different time intervals in which the book's average velocity is 0.

d. During which 0.4-sec interval is the book moving the fastest?

e. Find an equation that models the motion of the book during each of the following intervals:

1. from 0 sec to 0.8 sec

2. from 0.4 sec to 2.0 sec.

2. The following distance-time data was recorded for a falling ball:

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.00	0.42	0.45	0.84
0.05	0.42	0.50	1.00
0.10	0.42	0.55	1.19
0.15	0.42	0.60	1.40
0.20	0.42	0.65	1.63
0.25	0.44	0.70	1.89
0.30	0.50	0.75	2.17
0.35	0.59	0.80	2.06
0.40	0.70		

a. Create a distance-time graph for this data.

b. Describe the time interval for which the ball was actually falling.

c. Find an equation that models the distance-time graph for this interval.

d. Explain how the terms of the equation you found in Part **c** relate to the movement of the ball.

e. Calculate the average velocity of the ball during each of the following intervals:

  1. $[0.25, 0.35]$

  2. $[0.65, 0.75]$

f. Explain why the two average velocities you found in Part **e** are different.

Module Summary

- A **distance-time graph** displays the distance between two objects as a function of time.
- **Displacement** is a change in the position of an object. It has both magnitude and direction.
- **Velocity** is the rate of change in position with respect to time.
- The **average velocity** of an object can be calculated by dividing its displacement by the change in time.
- The **instantaneous velocity** of an object is its velocity at a particular instant in time. To estimate the instantaneous velocity at a given instant t, one can use the average velocity of the object during a small interval containing t.
- A **force** is a physical quantity that can affect the motion of an object. Two familiar forces are gravity and friction.
- According to Newton's **first law of motion**, an object in a state of rest or moving in a straight line at a constant speed will continue in that state unless acted on by a force.
- A **polynomial** in one variable, x, is an algebraic expression of the general form:

  $$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

  where n is a whole number and the **coefficients** a_i are real numbers for $i = 0,\ 1,\ 2,\ \ldots,\ n$.
- The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression with a non-zero coefficient. A polynomial in the general form shown above has a degree of n, provided that $a_n \neq 0$.
- A function f is a **polynomial function** if $f(x)$ is defined as a polynomial in x. The degree of a polynomial function is the same as the polynomial it contains.
- Second-degree polynomial functions also are known as **quadratic functions**. The graph of a quadratic function is a **parabola**.
- Third-degree polynomial functions also are known as a **cubic functions**.
- **Acceleration** is the rate of change in velocity with respect to time.
- The acceleration due to gravity is a constant typically denoted by g. On earth's surface, the acceleration due to gravity is about 9.8 m/sec^2 in a direction toward the earth's center.

- When an object is acted on only by gravity, its distance from the ground is described by the following function:

$$d(t) = -\frac{1}{2}gt^2 + v_0t + d_0$$

where $d(t)$ represents the object's distance from the ground after t sec, g is the acceleration due to gravity, v_0 is the object's velocity in the vertical direction at $t = 0$, and d_0 is the object's distance above the ground at $t = 0$.

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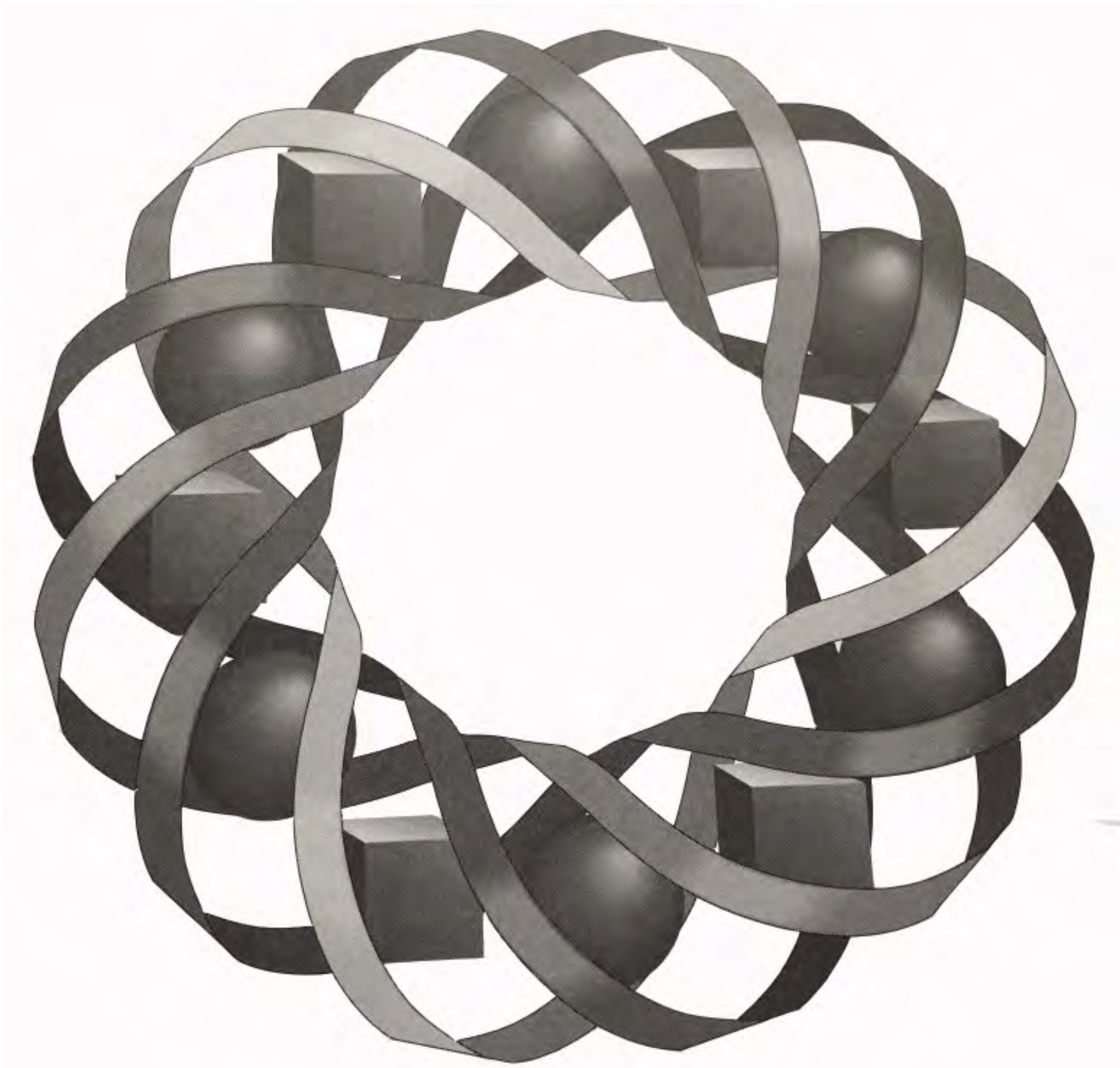
Murphy, J., J. Hollon, and P. Zitzewitz. *Physics: Principles and Problems*. Toronto: Charles Merrill Publishing, 1972.

For more information about the space shuttle program, contact the NASA Teacher Resource Room, Mail Code AP–4, Johnson Space Center, Houston, TX 77058; 713–483–8696.

Fair Is Fair



How would you divide a cake between two hungry people? How would you divide a car among three teenage drivers? And what would you do if everyone in the family wanted the old grandfather clock? In this module, you'll learn that a fair division doesn't always have to be equal.

Anne Merrifield • Pete Stabio • Mike Trudnowski • Lisa Wood

Fair Is Fair

Introduction

Have you ever argued with someone over who should get the last piece of pie? Or bickered over the right to read the only copy of an exciting book? Whenever two or more parties want the same item, the potential for conflict arises.

Dividing inheritances often causes ill will among previously agreeable family members. The distribution of water rights among resource users can be subject to court battles. The problem of dividing land among nations has resulted in bloody wars. In this module, you investigate some methods of obtaining fair—and peaceful—divisions.

Discussion

a. In most situations, there is more than one way to divide items among interested individuals. Some ways are fair while others are not. Describe the characteristics of a division you would consider fair.

b. Give examples of both fair and unfair divisions in each of the following situations. Discuss why each division is fair or unfair.

1. \$25 among 3 people
2. 12 equal slices of pizza among 5 people
3. a sports car valued at \$22,000 and a diamond ring appraised at \$7,000 between 2 people
4. \$100,000 cash and a motorcycle valued at \$12,000 among 3 people
5. a 2-L bottle of mineral water between a large adult and a small child marooned on a desert island for a week

c. In some situations, an item that several people want must be awarded as a single unit. What methods might be used to facilitate a division that is acceptable to all?

Activity 1

In practice, not all divisions are fair. Some people may use their authority to make divisions without concern for fairness. For example, a military officer assigning duties or a construction manager delegating tasks may consider efficiency or safety first, and fairness later. In a bankruptcy court, on the other hand, even a judge who tries to distribute assets fairly among all creditors may not seem completely fair to those involved.

Mathematics Note

A fair division problem exists when individuals must divide a set of items among themselves. A **fair division** occurs when all individuals, by their own assessment, consider the portions they are awarded as fair. "Fairness" depends on each individual's opinion. This opinion may not agree with the opinion of others involved.

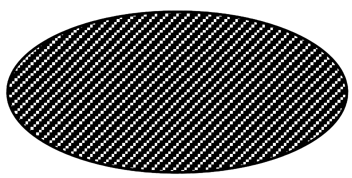
An item is considered **continuous** if it may be awarded in parts in a fair division. For example, a cake is continuous while the plate it is served on is not.

Exploration

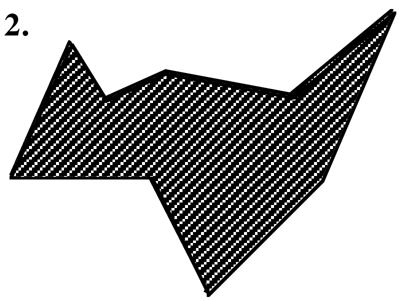
a. One technique for dividing a continuous object between two persons is the **cut-and-choose method**. In this method, one person cuts the object into two shares. The other person then chooses the desired piece.

Create figures similar to those illustrated below. Use the cut-and-choose method to achieve a fair division in each case.

1.

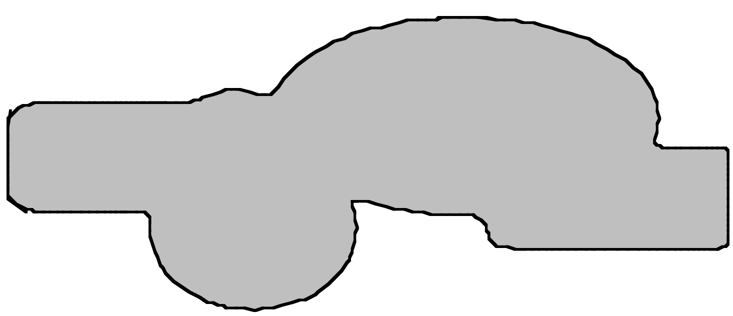


2.



b. A second technique for dividing a continuous object is the **continuous-knife method**. In this technique, a knife is held above the left edge of the object and moved slowly from left to right. Either person can stop the knife to cut off a piece that represents a fair share. The person who stops the knife receives the portion to the left. The second person receives the remaining portion. (The continuous-knife method may also be performed by moving the knife from top to bottom instead of left to right.)

Create a figure similar to the one illustrated below and use the continuous-knife method to achieve a fair division.



Discussion

a. Did the cut-and-choose method result in a fair division? Why or why not?

b. Describe any difficulties that arose when using the cut-and-choose method.

c. Why would you expect the continuous-knife method to result in a fair division?

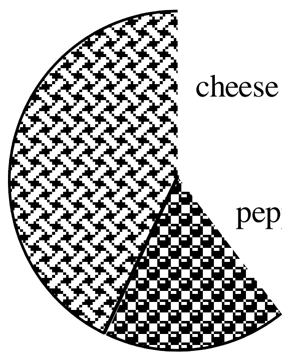
d. What difficulties arose when using the continuous-knife method?

e. What kinds of objects can be divided by the methods described in the exploration?

f. What kinds of items should not be divided using either of these methods? Explain your response.

Assignment

1.1 John and Leticia have some pizza to share. As shown in the diagram below, part of the pizza is cheese and part is pepperoni.



a. Describe how the two friends could divide the pizza fairly and explain how this method results in a fair division.

b. How might the friends change their method if John does not like pepperoni and Leticia does?

1.2 In this problem, you use a geometry utility to model the continuous-knife problem for two people.

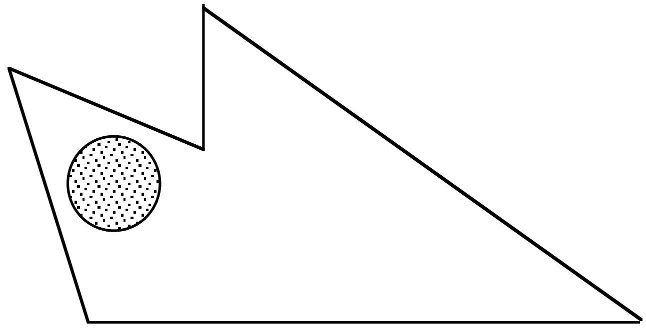
a. Create an irregular polygon. Draw a line segment to act as the "knife," then move the segment across the polygon until it appears to be fairly divided into two portions.

b. Determine the areas of the resulting portions of the polygon.

c. Would this method result in a fair division? Explain your response.

1.3 Xang and Katelyn are willing to share the last submarine sandwich. Describe at least two different methods, other than cut-and-choose or continuous-knife, that might be used to accomplish a fair division. Rank your methods in order of preference and justify your decisions.

1.4 The figure below shows a plot of land with a large circular swimming pool. Describe how two people might divide this plot of land into two fair portions.



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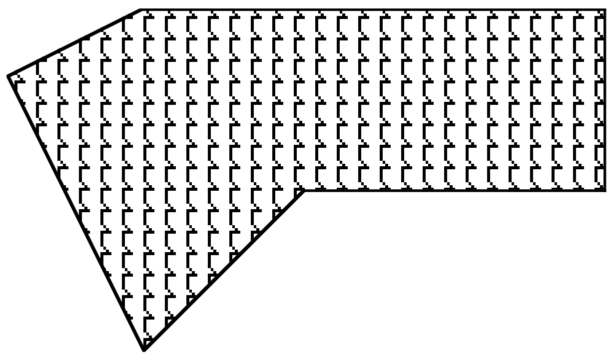
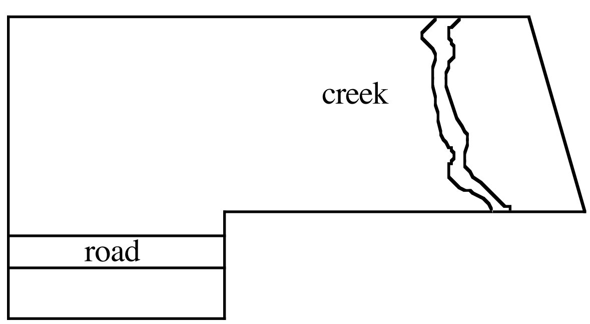
1.5 To complete a home economics project, Willis and Karissa each need enough cloth to sew a pair of shorts. The diagram below shows the piece of material they must divide between them.



a. Describe how this piece of material could be divided fairly.

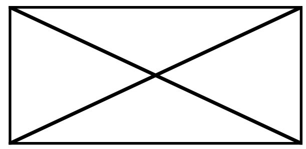
b. After the cloth was divided, Willis was able to finish his project, but Karissa ran into trouble. What problems might Karissa have encountered, even though she originally considered the division a fair one?

1.6 Mr. and Mrs. Summers would like to give the property shown in the map below to their two grown children.

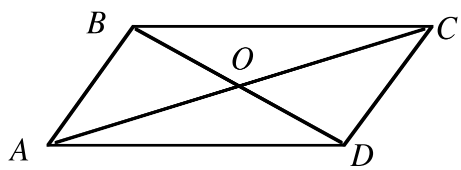


Both children would like to build homes on this land. Describe how the property could be divided fairly.

1.7 A figure has **point symmetry** if it can be rotated 180° about a point and each point in the image coincides with a point in the preimage. For example, the rectangle below has point symmetry about the point where its two diagonals intersect.



In the parallelogram below, point O is the point where the diagonals intersect. Explain why any line through point O divides the parallelogram into two regions with equal areas.



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Activity 2

Not all fair division problems involve only two parties. When three or more individuals are involved, the situation can become much more complicated.

In order to obtain a fair division among several people, some additional assumptions must be made. First, all persons involved must be capable of determining a fair portion. The value that each individual places on a particular portion may depend on more than just its size. Each person should get a fair share based on his or her own assessment.

In the division of land, for example, one person may consider lake frontage more valuable than forests. In such a case, that person may accept a smaller piece of land with lake frontage as a fair share.

Exploration

One way to divide a continuous object among three or more people is the **reduction method**. Parts **a–f** below describe the steps that must be followed to divide an item among three people using the reduction method. Use these steps to simulate the division of a candy bar.

a. The individuals determine the order in which each will participate in the division.

b. The first person "reduces" the item by cutting off a fair share.

c. The second person has a choice: either reduce the portion cut by the previous person or leave it intact. If the second person leaves the portion intact, then the third person has the same choice.

d. The last person to reduce the piece receives that portion and is now out of the process.

e. The remaining two people divide the rest of the item by repeating the reduction method or by using any other method for fairly dividing an object between two people. Any pieces removed through reduction, but not yet awarded, are included in this division.

Discussion

a. In the reduction method, when would a person choose to reduce a share?

b. Who receives the first piece if no one reduces it?

c. Do you think that the reduction method will result in a fair division of a candy bar? Explain your response.

d. Would this method result in a fair division for two people? Why or why not?

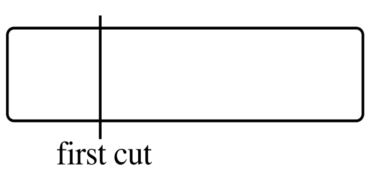
e. In any division of a continuous item, the fractional value assigned to each portion may vary from person to person. The sum of these fractional values, however, is always 1. Why is this true?

f. Could the cut-and-choose method, the continuous-knife method, or the reduction method be used when more than three people are involved in the division of a continuous object? Explain your response.

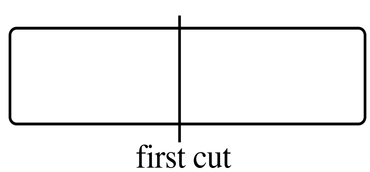
Assignment

2.1 Imagine that you and two friends have decided to share a candy bar using the reduction method. One of your friends makes each of the first cuts shown in Parts **a** and **b** below. In each case, would you reduce the portion on the left or let this cut stand? Explain your responses.

a.



b.



2.2 **a.** Describe a method other than those previously mentioned in this module of dividing a continuous object fairly among three people.

b. Will your method work when five people are involved? If so, explain how. If not, modify your method so that it will work.

2.3 Create an irregularly shaped object out of paper. Describe a fair division of this object among three or more people.

2.4 Chadd, Rusty, and Kristi own a lawn-mowing service. They have been hired to mow a park that contains about 2 acres of grass. The park includes flat areas, hilly regions, and groves of trees—all of which must be mowed.

a. Why would the reduction method fail to provide a fair division in this case? Explain your response.

b. Describe how the park can be divided so that each of the three must mow a fair portion of the grass.

c. Both Chadd and Kristi use push mowers, while Rusty uses a riding mower. How does this fact affect your response to Part **b**?

* * * * *

2.5 Use a geometry utility to model the continuous-knife method of dividing an irregular polygon among four or more people. Describe how the method was altered to accommodate additional people.

2.6 Mr. and Mrs. Estrada have purchased a new compact disc (CD) player for their four children: Gisela, Milo, Hank, and Rozella. Since each of the four likes a different type of music, they must find a way to divide the time with the CD player fairly.

a. Explain why it might not be fair simply to assign each child 6 hr a day with the CD player.

b. Devise a method that fairly divides access to the CD player during one month.

c. If Rozella and Hank suddenly start listening to the same type of music, should that change the method you devised in Part **b**? Explain your response.

* * * * * * * * * *

Activity 3

In the previous activities, you examined the fair division of continuous objects. Some objects, however, are difficult or impractical to divide into parts. For example, how would you divide a car or a boat?

In this activity, you investigate the fair division of **discrete** items. An item is considered discrete if it can only be awarded as an entire unit in a fair division. For example, houses, cars, and boats are all discrete items.

Exploration

a. Along with a partner, choose a discrete item that you would both like to own but must divide fairly.

b Develop a process that could be used to decide who will get the item.

c. Determine how the student who does not get the item should be compensated to make the division fair.

d. Determine the value of this settlement for each person.

Discussion

a. How was the person who did not receive the item compensated?

b. Did your method result in a fair division? Explain why or why not.

c. Is it necessary for the values of the item and the compensation to be equal in order to have a fair division?

d. Could your method be adjusted to accommodate more people and more items?

Mathematics Note

Mathematicians have developed some approaches to fair division that include steps you may not have considered. The **bid-and-divide** method, for instance, involves assigning a cash value to an item through a bidding process. The secret bids represent the amounts that individuals would be willing to pay for the item and determine the value to be divided fairly. The item is awarded to the highest bidder. The individual who receives the item then compensates the others for a fair share.

For example, imagine that you and a friend share a calculator. Your friend is moving to another town. Since the calculator is a discrete item, the two of you must determine who will keep it. Using the bid-and-divide method, you each submit a secret bid that reflects the value you place on the calculator. Suppose that you bid $60 and your friend bids $70. Since $70 is the higher bid, the value to be divided fairly is $70. This is referred to as the **value pool**.

Because there are two of you, your share of the bids is half the value that you placed on the calculator, while your friend's share is half the value that he or she placed on it. Since you bid $60, your share is 0.5($60) or $30. Your friend's share is 0.5($70) or $35. The sum of these shares, $30 and $35, accounts for $65 of the $70 in the value pool. The remainder of the value pool, $5, is the **value-pool balance**. Each person is awarded half this balance, or $2.50.

Since your friend's bid is higher than yours, your friend is awarded the calculator. Once the item is awarded, each person's compensation must be calculated. A chart similar to the one illustrated in Figure **1** may help you keep track of the calculations in the bid-and-divide method.

As shown in Figure **1**, your **total fair share** is the sum of a share of your bid and an equal share of the value-pool balance. In this example, your total fair share is $32.50, while your friend's is $37.50.

Compensation for each individual is determined by calculating the difference between the total fair share and the value of the item received. Your compensation, $32.50, equals the difference between your total fair share and the value of the item awarded—in this case, $0. Your friend's compensation is –$32.50, the difference between his or her total fair share and the value of the calculator.

	You	Friend
Bids	60.00	70.00
Sum of Bid(s)	60.00	70.00
Share of Bid(s)	30.00	35.00
Share of Value-pool Balance	2.50	2.50
Total Fair Share	32.50	37.50
(Item Awarded)	(none)	(calculator)
Total Value Awarded	0.00	70.00
Compensation	+32.50	–32.50
Final Settlement Value	32.50	37.50

Value Pool	
Total of High Bid(s)	70.00
–Total of Shares of Bid(s)	65.00
Value-pool Balance	5.00

Figure 1: Fair division using bid-and-divide method

In a fair division of the calculator by this method, your friend keeps the calculator and compensates you $32.50 for your fair share. The **final settlement value** is the sum of each person's compensation and the value of the item awarded, if any. When a fair division has been properly administered, each person's final settlement value equals that person's total fair share. The sum of the final settlement values equals the total of bids in the value pool. The sum of the compensations equals 0.

e. In the bid-and-divide method, why do you think the high bid is used to determine the value pool?

f. In the final settlement, why does each person always receive more than a fair share of that person's original bid?

g. Why must the sum of the final settlement values equal the value pool?

h. Why must the sum of the compensations equal 0?

Assignment

3.1 Yoshi and Shiho have inherited an ancient Japanese sword. They decide to use the bid-and-divide method to determine who should keep the sword. Yoshi bids $900. Shiho bids $1050.

Complete the table below to determine a fair division in this situation.

	Yoshi	**Shiho**	**Value Pool**	
Bids	900	1050	Total of High Bid(s)	
			– Total of Shares of Bid(s)	
Sum of Bid(s)			Value-pool balance	
Share of Bid(s)				
+ Equal Share of Value-pool balance				
Total Fair Share				
(Item Awarded) Total Value Awarded				
Compensation				
Final Settlement Value				

3.2 Milo and Dena have inherited an antique car from their grandfather. To determine a fair division, they decide to use the bid-and-divide method. Milo bids $24,000 for the car, while Dena bids $32,000.

a. Determine the amount of cash that Dena must pay Milo.

b. What is each person's final settlement value?

3.3 Leif and Neva are the co-winners of an essay contest. The first prize is a bicycle. They decide to use the bid-and-divide method to determine who will keep the bike. Leif bids $120 while Neva bids $130. Determine the value of the final settlements.

3.4 Suppose Neva knows that Leif thinks the bike in Problem **3.3** is worth at least $150. However, she does not think that it is worth more than $175. Neva wants the bike. However, even if she doesn't get the bike, she would like to receive as much compensation as possible. What should she bid? Explain your response.

* * * * *

3.5 Alexi and Norjar share first place in an academic challenge. Later, they realize that the real challenge is how to share the first prize—a new computer—and still remain friends.

They decide to use the bid-and-divide method to find a settlement that allocates possession of the computer to one of them, fair compensation to the other, and lasting friendship to both. If Alexi bids $800 and Norjar bids $900, determine the values of the final settlement.

3.6 By working together, Miranda and Willis have won first place in an essay contest. The prize is an all-expenses paid trip to Europe for one person. Both students contributed equally to the final draft of the essay. Now they must determine a fair method for dividing the prize.

The total dollar value of the trip is $1995. This amount includes travel ($1150), housing ($500), and food ($345).

a. Determine a procedure for dividing the prize.

b. Do you think that your method is fair? Explain your response.

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Activity 4

The division of a family estate can involve several survivors and many items. In this activity, you adapt the bid-and-divide method to handle such a situation.

Exploration

Design a chart similar to the one in Figure **1** to track the division of an estate that involves several items and several survivors. Your chart should be flexible enough for use in a number of different cases.

Discussion

a. What changes did you make to the chart in Figure **1**?

b. How is the value pool determined in your chart?

c. How are the shares of the bids in each column determined?

d. Is it possible to modify your chart so that a spreadsheet can complete all the necessary calculations?

e. Could your chart be used to determine a fair division for two people who must divide a single object?

Assignment

4.1 Dena and Milo must use the bid-and-divide method to distribute an antique car, a coin collection, and the family house. Dena bids \$32,000 for the car, \$6000 for the coins, and \$126,000 for the house. Milo bids \$24,000 for the car, \$5000 for the coins, and \$151,000 for the house.

a. Determine a fair division of the three items.

b. Who must pay cash as a compensation? How much will be paid?

c. Does anyone receive more than a fair share?

d. Explain the significance of the value-pool balance.

e. If Dena changes her bid on the coin collection to \$12,000, how will the final settlement values change?

f. If Milo does not want the car, would it be a good strategy for him to bid \$0? Explain your response.

g. What is the best bidding strategy for someone who does not wish to receive an item but wants the final settlement value to be as large as possible?

4.2 Imagine that a wealthy neighbor has left you and a friend a computer, a valuable painting, and a single \$40,000 scholarship to attend the college of your choice.

a. Use appropriate technology to determine a fair division of the inheritance.

b. Use a chart or spreadsheet to show a fair division in which you receive the computer and the scholarship.

c. Use a chart or spreadsheet to show a fair division in which your friend receives the painting and the scholarship.

4.3 The four Hersey children all want the family grandfather clock. Since they are unable to decide who will keep it, they visit a lawyer. The lawyer asks each of the four to submit a secret bid. All four bids are shown in the table below.

Sibling	Jon	Kris	Anne	Dean
Bid	\$950	\$1000	\$1250	\$1300

Determine the value of the final settlement for each sibling.

4.4 The Hersey children from Problem **4.3** also inherited a home stereo, a color television, and a sports car. Determine how these three items would be divided fairly if each sibling submitted the bids shown in the following table.

Bids				
Items	Jon	Kris	Anne	Dean
stereo	$1500	$1750	$1000	$1200
television	$900	$600	$750	$500
car	$10,000	$12,000	$9500	$11,500

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4.5 In your own home, identify a minimum of four objects of some value which you might want to divide fairly. Ask at least three people (parents, siblings, or friends) to submit bids on these objects, then determine a fair division.

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Summary Assessment

1. Maria, Gisele, and Micah want to divide a continuous object into three parts. They follow the process described below.

- Maria divides the object into two parts.
- Gisele divides each of the two parts into three pieces.
- Micah selects one piece of the object.
- Maria selects one piece.
- Gisele selects two pieces.
- Maria selects one piece.
- Micah is awarded the remaining piece.

a. Explain why this procedure may not result in a fair division.

b. Modify this process to ensure that a fair division occurs.

2. Tawnya and Vasu have received an apple strudel as a gift. Vasu would be content to have $3/8$ of the strudel. Tawnya thinks her fair share is $1/2$ of the strudel. Would a procedure that awarded $1/2$ of the strudel to each of them be a fair division? Explain your response.

3. Miguel, Rolf, and Tristan are graduating from college. In their four years as roommates, they have made several purchases together, including a stereo, a calculator, and a compact disc collection. Now that each will be moving to a different town, they must divide these possessions. Devise a method that the roommates can use to achieve a fair division.

4. The three roommates in Problem **3** also own an old car that they used for grocery shopping and trips to the beach. They decide to use the bid-and-divide method to see who will keep the car. Miguel bids \$400, Rolf bids \$600, and Tristan bids \$375.

a. Determine the value of the final settlement for each person.

b. If Tristan knows the amount of the other two bids and is not interested in keeping the car, how can he maximize the value of his final settlement?

Module Summary

- A fair division problem exists when individuals must divide a set of items among themselves. A **fair division** occurs when all individuals, by their own assessment, consider the portions they are awarded as fair. "Fairness" depends on each individual's opinion. This opinion may not agree with the opinion of others involved.
- An item is considered **continuous** if it may be awarded in parts in a fair division.
- One technique for dividing a continuous object between two persons is the **cut-and-choose** method. In this method, one person cuts the object in two shares. The other person then chooses the desired piece.
- Another technique for dividing a continuous object is the **continuous-knife** method. In this method, a knife is held above the left edge of the object and moved slowly from left to right. Either person can stop the knife to cut off a piece that represents a fair share. The person who stops the knife receives the portion to the left. The second person receives the remaining portion.
- One approach for dividing a continuous object among three or more people is the **reduction** method.
  1. The first person cuts a fair share of the object.
  2. The second person has a choice: either reduce the portion or leave it intact.
  3. The third person has the same choice: either reduce the portion or leave it intact.
  4. The last person to reduce the piece is awarded that portion and is now out of the process.
  5. The remaining two people divide the rest of the object using this or any method for fairly dividing an object between two people.
- An item is considered **discrete** if it can be awarded as a unit in a fair division.
- The **bid-and-divide** method may be used to divide discrete items. The technique involves assigning a cash value to an item through a bidding process. The bids represent the amounts that individuals would be willing to pay for the item and determine the value to be divided fairly. The item is awarded to the highest bidder. The individual who receives the item then compensates the others for a fair share.

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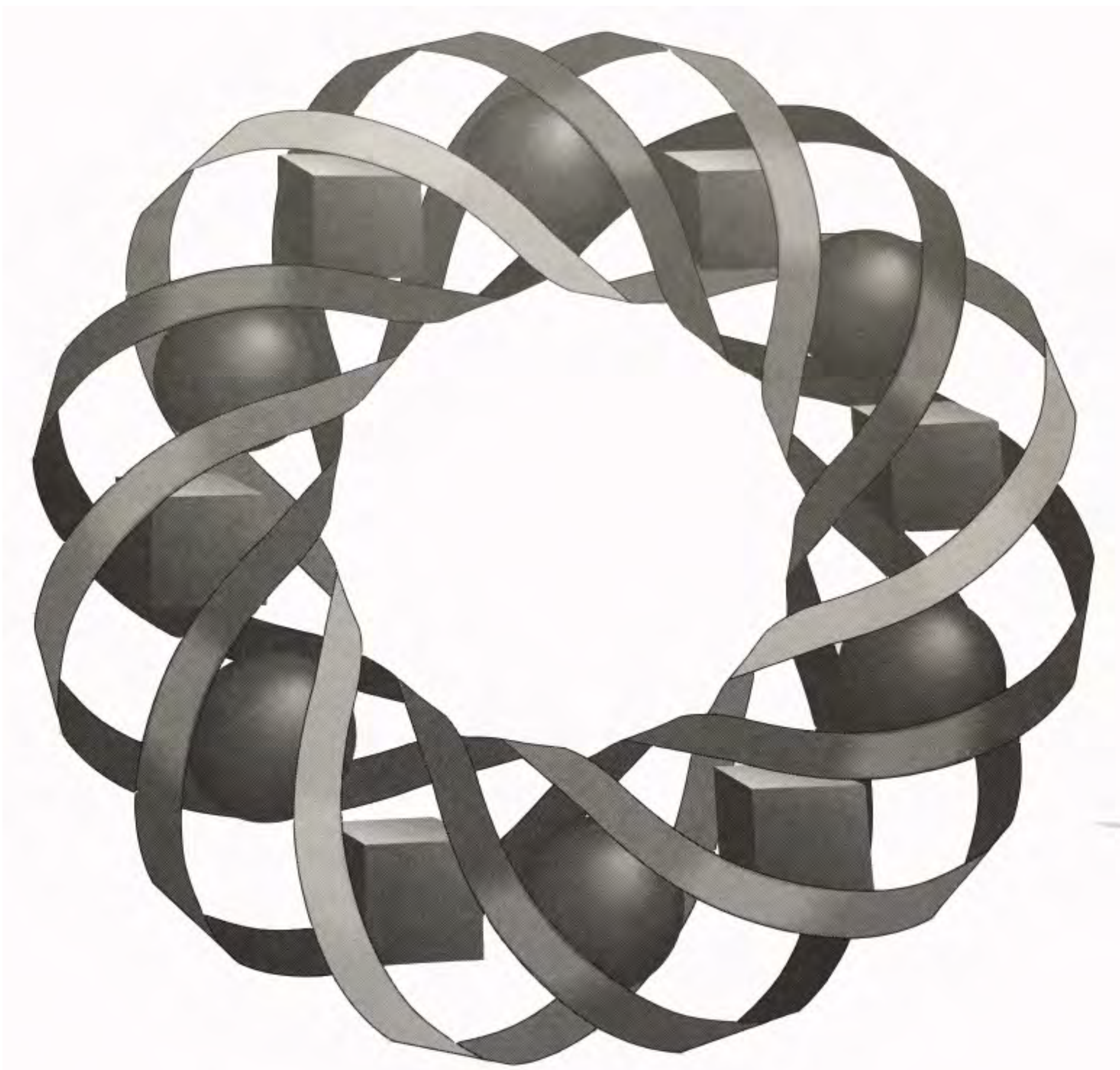
Let the Games Begin



This module uses puzzles and games to introduce logical reasoning and problem-solving strategies.

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Let the Games Begin

Introduction

A man was looking at a picture in a gallery. A woman standing next to him asked, “Who is the man in that picture?” The man, who loved intrigue and puzzles, thought for a moment and replied, “Brothers and sisters have I none, but this man’s father is my father’s son.”

Who is the man in the picture? One way to answer this question is to approach the problem logically.

Activity 1

At one level, logical thinking involves the interpretation of words. Often, the shortest and most familiar words play an important role in determining the meaning of a sentence.

Mathematics Note

A **statement** is a sentence that can be determined to be either true or false, but not both. The truth or falseness of a statement is its **truth value**.

For example, consider the statement, “There are no clouds in the sky today.” Its truth value can be determined by looking at the sky. If there are no clouds, the statement is true. If there are any clouds in the sky, however, the statement is false.

Two statements can be joined into a **compound statement** using the **connectives** *and* and *or*. When a friend describes a new acquaintance, for example, she might say, “Joseph has black hair and wears glasses.” This compound statement consists of the statement, “Joseph has black hair,” connected with the statement, “Joseph wears glasses.”

In mathematics, statements are typically represented using variables. For example, if p represents the statement, “Joseph has black hair,” and q represents the statement, “Joseph wears glasses,” the compound statement described above can be represented by “p and q.”

Exploration

In this exploration, you use a game to examine how logical connectives affect the meaning of statements. The Logic Game is very similar to the game of bingo. It involves matching the numbers on a game board with the numbers that correspond to the compound statements chosen by a caller.

A sample game board is shown in Figure **1** below. (In an actual game, each player has a different board.)

L	**O**	**G**	**I**	**C**
–8	4	–1	9	7
–2	2	3	0	3
8	1	4	–5	–6
5	–3	–4	–7	1

Figure 1: Logic Game board

The object of the Logic Game is to place a token over all the numbers in any row on the game board. The statements used in the game are listed in Table **1**.

Table 1: Statements for the Logic Game

a	The number is –9.
b	The number is less than 4.
d	The number is greater than –1.
e	The number belongs to the set {8, 9}.
f	The number belongs to the set {–9, –8, …, 0, 1}.
h	The number is greater than 2.
j	The number is even.
k	The number is less than –4.
m	The number is odd.
n	The number is negative.
p	The number belongs to the set {5, 6, 7, 8, 9}.
q	The number is positive.

Please read Parts **a–f** before beginning play.

a. Select one person to be the caller and one to be the recorder. Everyone else is a player.

b. To prepare for the game, the caller and the recorder should complete the following steps.

1. Separate the game pieces provided by your teacher into two piles: a Statement pile and a Logic pile.
2. Make a Connector Coin by marking one side of a quarter with the word *and* and the other side with the word *or*.
3. Save the Record Sheet provided for later use.

c. To prepare for the game, each player should complete the steps below.

1. Generate 20 random integers between –9 and 9, inclusive.
2. Write one of the random integers in each of the unshaded regions of the Logic Game board provided by your teacher.

d. The following list describes the caller's duties in the Logic Game.

1. The caller randomly picks two statements from the Statement pile, then flips the Connector Coin to select a logical connective. The caller uses the statements and the logical connective to form a compound statement.
2. The caller randomly selects a letter from the Logic pile, announces the column to which the compound statement applies, then reads the statement.

   For example, consider the two statements *k* and *m*, or "The number is less than –4" and "The number is odd," respectively. If the letter selected from the Logic pile is **L** and the logical connective is *and*, then the caller should announce "Under column **L**, cover any number that is both less than –4 and is odd."
3. The caller returns the selected items to the appropriate piles.
4. The caller continues selecting and announcing compound statements until one player wins.

e. For each announcement by the caller, the recorder writes the variables and logical connectives that correspond with the compound statement on the Record Sheet. In the example given in Part **d**, the recorder would write "*k* and *m*" in the first row under column **L**. A copy of the Record Sheet is shown in Figure **2**.

	L	O	G	I	C
Rule 1	*k* and *m*				
Rule 2					
Rule 3					
Rule 4					
Rule 5					
Rule 6					
Rule 7					
Rule 8					
Rule 9					
Rule 10					
Rule 11					
Rule 12					
Rule 13					
Rule 14					
Rule 15					
Rule 16					
Rule 17					
Rule 18					
Rule 19					
Rule 20					

Figure 2: Record sheet

f. The following list describes the rules players must follow when marking their game boards:

1. Players place a token over each number that satisfies the announcement given by the caller. In the example given in Part **d**, players could place a token over each number in column **L** that is both less than –4 and odd.
2. For each number marked by a token, players should write the corresponding compound statement in the appropriate shaded rectangle. In the example given in Part **d**, players would write "*k* and *m*" below each marked number.
3. To win the game, a player must have tokens covering all the numbers in one row of the game board. When this occurs, that player declares "Logic!"
4. The other players then verify that each number in the row satisfies the compound statement written below it. If so, that player is officially declared the winner. If any numbers do not correctly fit a rule, that player is disqualified until the next game and play continues until a winner is declared.

g. If time allows, play the Logic Game again using a new game board, caller, and recorder.

Discussion

a. When does a number on a Logic Game board satisfy a compound statement that uses the connective *and*? Use an example to support your response.

b. When does a number on the game board satisfy a compound statement that uses the connective *or*? Use an example to support your response.

c. How is the intersection of two sets related to a compound statement formed using the connective *and*?

d. How is the union of two sets related to a compound statement formed using the connective *or*?

Mathematics Note

Venn diagrams are mathematical models that show relationships among different sets of data.

The **intersection** of two sets is the set of all elements common to both sets. For example, the shaded region in the Venn diagram in Figure **3a** shows the intersection of sets A and B, denoted by $A \cap B$.

The **union** of two sets is the set of all elements in one, the other, or both sets. For example, the shaded region in Figure **3b** shows the union of sets A and B, denoted by $A \cup B$.

Disjoint sets have no elements in common. For example, Figure **3c** shows two disjoint sets, B and C.

A set with no elements is the **empty set** or **null set**. A symbol for the empty set is $\varnothing$. For example, the intersection of sets B and C is the empty set. This could be denoted by $B \cap C = \varnothing$.

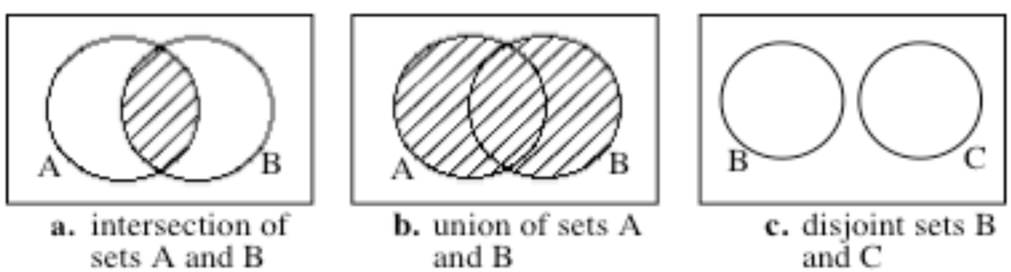

Figure 3: Venn diagrams

For example, suppose set A = {1, 2, 3, 4}, set B = {2, 4, 6}, and set C = {1, 3, 5}. The intersection of set A and set B is {2, 4}, as shown in Figure **4a**. The union of sets A and B is {1, 2, 3, 4, 6}, as shown in Figure **4b**. Sets B and C are disjoint sets because they have no numbers in common (see Figure **4c**). The intersection of set B and set C is the null set, ∅.

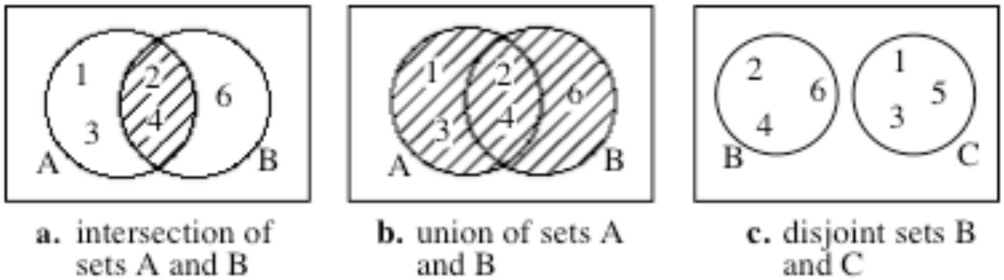

Figure 4: Sample Venn diagrams

e. How could you use Venn diagrams to help you play the Logic Game?

Assignment

Use the statements in the Logic Game from Table **1** to complete Problems **1.1–1.3**.

1.1 Identify the set of integers described by each statement in the Logic Game. Use the corresponding uppercase letter to label each set. For example, label the set of integers described by statement *a* as set A.

1.2 During one Logic Game, a recorder expressed each rule as a union or intersection of sets, labeling each set as in Problem **1.1**. Some players noticed that the recorder had made some mistakes.

Rule 1: $A \cap B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, -9\}$
Rule 2: $F \cap J = \{-8, -6, -4, -2, 0\}$
Rule 3: $N \cup Q = \varnothing$
Rule 4: $P \cap Q = \{5, 6, 7, 8, 9\}$
Rule 5: $E \cap K = \varnothing$
Rule 6: $B \cap K = \{-4, -5, -6, -7, -8, -9\}$
Rule 7: $D \cap K = \{0, 1, 2, 3\}$
Rule 8: $M \cup J = \{-9, -8, -7, \ldots, 7, 8, 9\}$

a. Draw a Venn diagram that corresponds with each set operation in the table, including the appropriate integers from –9 to 9, inclusive, in each set. Shade the appropriate portion of each Venn diagram.

b. Assuming that the rules were recorded correctly, identify all the mistakes in the table and describe the correct sets.

1.3 At the conclusion of another game of Logic, the recorder had made the following entries on the Record Sheet:

Rule 1: B ∩ ___ = {0, 1, 2, 3}
Rule 4: ___ ∪Q = {–8, –6, –4, –2, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
Rule 7: P ___ M = {5, 7, 9}
Rule 8: ___ ∪ ___ = {–9, 3, 4, 5, 6, 7, 8, 9}
Rule 9: D ___ B = {–9, –8, –7, …, 7, 8, 9}
Rule 10: ___ ___ ___ = {–9, –8, –7, –6, –5, 3, 4, 5, 6, 7, 8, 9}

As you can see, the recorder forgot to write down some information. Assuming that the listed numbers correctly satisfy each rule, fill in the missing sets and operations. Verify your responses using Venn diagrams.

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1.4 Triangles are classified by their sides (scalene, isosceles, or equilateral) and by their angles (acute, equiangular, right, or obtuse). Use combinations of classifications, such as "isosceles right triangle," to name all the triangles that satisfy each of the compound statements below.

a. The triangle has two complementary angles and the triangle has two congruent sides.

b. The triangle has three 60° angles or the triangle has no congruent sides.

c. The triangle has at least two congruent sides and the triangle has no obtuse angle.

1.5 Write a compound statement that describes each of the following sets.

a. {1, 9, 25, 49, 81, …}

b. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20}

* * * * * * * * * *

Activity 2

In the Logic Game, you used connectives to describe the intersection or the union of sets. Other types of words can also affect the meaning of statements. For example, the word *not* can be used to change a statement's truth value. In this activity, you use tables to examine the truth values of compound statements.

Exploration

As basketball season approaches, the coach at your school decides that the team needs more talent. When asked to describe the ideal recruit, the coach replies, "The person is fast and the person is tall." While considering all the potential players you know, you make the following notes:

- Smith is slow but tall.
- Lopez is tall and fast.
- Yellowtail is fast but not tall.
- McMurphy is tall but slow.
- Ali is short but not slow.
- Schmidt is slow and short.

a. Using the information given above, complete a copy of Table **2** with the appropriate truth values (true or false).

Table 2: Recruit information

Person	The person is fast.	The person is tall.	The person is fast and the person is tall.
Smith	F	T	F
Lopez			
Yellowtail			
McMurphy			
Ali			
Schmidt			

b. A compound statement that uses the connective *and* is a **conjunction**. Given two statements, there are four different ways in which their truth values may be combined: TF, FT, FF, and TT.

Using the information in Table **2**, which of these ways results in a conjunction with a truth value of true? Which results in a conjunction with a truth value of false?

c. Suppose the coach had said, "The person is fast or the person is tall." Repeat Part **a** using this description of the ideal recruit.

d. A compound statement that uses the connective *or* is a **disjunction**. Repeat Part **b** for a disjunction.

e. A **truth table** shows the truth values of a compound statement for all possible truth values of its individual statements. Using your responses in Parts **b** and **d**, complete a copy of Table **3** to show the truth values for the conjunction "*q* and *r*" and the disjunction "*q* or *r*."

Table 3: Truth table for "*q* and *r*" and "*q* or *r*"

q	*r*	*q* and *r*	*q* or *r*

Discussion

a. How did you determine who would be a good basketball recruit:

1. when the coach used the connective *and*?

2. when the coach used the connective *or*?

b. Using Table **3**, when is the compound statement "*q* or *r*" true? When is it false?

c. When is the compound statement "*q* and *r*" true? When is it false?

Mathematics Note

The compound statement "*p* and *q*" has a truth value of true only when statement *p* and statement *q* are both true.

For example, the compound statement "Joseph has black hair and wears glasses" is true only when the statements "Joseph has black hair" and "Joseph wears glasses" are both true.

The compound statement "*p* or *q*" has a truth value of false only when statement *p* and statement *q* are both false.

For example, the compound statement "Joseph has black hair or wears glasses" is false only when the statements "Joseph has black hair" and "Joseph wears glasses" are both false.

A **negation** of a statement *p* is described as "not *p*," also denoted by $\sim p$. When *p* is true, its negation $\sim p$ must be false. When *p* is false, $\sim p$ must be true.

For example, consider the statement, "My teacher wears glasses." If this statement is false, then its negation—"My teacher does not wear glasses"—is true. If this statement is true, then its negation is false.

d. **1.** Suppose that q represents the statement, "The person is fast," and r represents the statement, "The person is tall." Write a compound statement that could be represented by "~q or r."

2. Compare the sentence you wrote with those of others in your class. Do they all have the same meaning?

e. When is the compound statement "~q or r" true? When is it false?

f. In the introduction to this module, the man in the gallery says, "Brothers and sisters have I none … ." Use your knowledge of logical connectives to interpret this phrase.

Assignment

2.1 Suppose that p represents the statement, "I love math class," and q represents the statement, "I have a pet."

a. Create a truth table for the conjunction "p and q" and the disjunction "p or q."

b. Without using mathematical symbols, write sentences that correspond to each of the four different ways in which the truth values of the individual statements can be combined in the conjunction "p and q."

2.2 **a.** Create a truth table for the compound statement "~p and q."

b. Create a truth table for the compound statement "~p or q."

c. Suppose that p represents the statement, "The light is on," and q represents the statement, "The door is open." Without using mathematical symbols, write sentences that correspond to the compound statements in Parts **a** and **b**.

2.3 Consider statement p, "The light is on," and statement q, "The door is open."

a. Without using mathematical symbols, write a sentence that corresponds to the compound statement ~(p and q).

b. Create a truth table that shows the possible truth values of ~(p and q) using the following column headings:

p	q	p and q	~(p and q)

c. Create a truth table that shows the possible truth values of "~p or ~q" using the following column headings:

p	q	~p	~q	~p or ~q

d. Two statements are **logically equivalent** when they have exactly the same truth values. Are the statements ~(p and q) and "~p or ~q" logically equivalent? Defend your response using Venn diagrams.

e. Using the words *off* and *closed*, write a sentence that has the same meaning as the sentence you wrote in Part **a**.

* * * * *

2.4 **a.** Create a truth table that shows the possible truth values of "(p or q) and r" using the following column headings.

p	***q***	***r***	***p* or *q***	**(*p* or *q*) and *r***

b. Verify your results in Part **a** using Venn diagrams.

2.5 **a.** Determine a logical equivalent to ~(p or q).

b. Verify your response to Part **a** using a truth table or Venn diagram.

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Activity 3

In this activity, you examine compound statements known as **conditionals**. A knowledge of conditionals can help you evaluate statements and develop strategies for solving logical problems.

Mathematics Note

A **conditional statement** is a compound statement that can be written in "if-then" form. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "if p, then q," or by $p \rightarrow q$ (read "p implies q").

For example, given the hypothesis p, "I have my hand raised," and the conclusion q, "I see a red card," the conditional statement $p \rightarrow q$ is "If I have my hand raised, then I see a red card."

Exploration

In this exploration, you play a logic game called Color Card. To start the game, players must be seated so that each one can see all the others. Each player draws one card from a deck of ordinary playing cards, in which each card is either red or black, but not both. Without looking at it, each player holds his or her card face out toward the rest of the players. All players then respond according to the following rules:

- A player who sees a red card raises one hand.
- A player who does not see a red card does not raise a hand.

After observing the number of raised hands, players try to determine the colors of their own cards. Each player writes down one of the following conclusions: "red," "black," or "color cannot be determined." Players then present logical arguments to defend their decisions.

a. Play Color Card with two players. Play several rounds, drawing new cards from the deck each time.

b. List all the possible combinations of red and black cards for two players.

c. Determine whether or not it is always possible for both players to correctly determine the colors of their cards.

d. Repeat Parts **a–c** using three players.

Discussion

a. In a two-person game of Color Card, what conditional statements describe the logic that players should use to determine the colors of their cards?

b. In a three-person game of Color Card, what combinations of colors make it possible for all players to determine the colors of their cards? Explain your reasoning.

c. If the combination of colors in a three-person game is two black and one red, what conditional statements describe the logic that players should use to determine the colors of their cards?

d. The truth value of a conditional statement depends on the truth values of its hypothesis and its conclusion. Table **4** below defines the truth values for a conditional $p \rightarrow q$. Use the table to describe all the cases in which a conditional statement is true.

Table 4: Truth table for a conditional statement

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

e. Give an example of a conditional statement that illustrates each row in the truth table in Table **4**. Justify your responses.

Assignment

3.1 Two players are playing Color Card according to the rules described in the exploration. Given that the hypothesis is true, determine whether each of the conditional statements below is true or false. Justify your responses.

a. If my card is not red, then it is black.

b. If my card is red, then the other player has a hand up.

c. If the other player does not have a hand up, then my card is not red.

d. If my card is not red, then the other player has a hand up.

e. If my card is black, then the other player has a hand up.

f. If the other player does not have a hand up, then my card is black.

g. If my card is not red, then it is not black.

3.2 Three players are playing Color Card. Two of them have red cards and one has a black card. Can any of them identify the colors of their own cards? Explain your reasoning.

3.3 In a three-person game of Color Card, all three players have red cards. Following the rules described in the exploration, all three raise their hands. One player says, "I cannot determine the color of my card." Based on this statement, the other two players identify the colors of their cards. Write a conditional statement that explains the logic used by the two players who identified their cards.

3.4 Three playing cards from an ordinary deck have been placed face down in a row. Use the following clues to identify the face value and suit of each card.

- To the right of a jack there is at least one ace.
- To the left of an ace there is at least one ace.
- To the left of a club there is at least one diamond.
- To the right of a diamond there is at least one diamond.

Mathematics Note

The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and conclusion and negating both of them. The contrapositive of $p \rightarrow q$ can be represented as $\sim q \rightarrow \sim p$, or "if not q, then not p."

For example, consider the conditional statement, "If it is hot outside, then the snow outside is melting." The contrapositive of this conditional is, "If the snow outside is not melting, then it is not hot outside."

3.5 Write the contrapositive of each of the following statements.

a. If I study hard, then I earn good grades.

b. If I drive carefully, then I do not get into accidents.

c. If I do not use sugar on my cereal, then I stay calm.

d. If I do not lie, then I do not feel guilty.

3.6 **a.** Create a truth table for the contrapositive $\sim q \rightarrow \sim p$.

b. Explain how Table **4** in Part **d** of the previous discussion and the truth table in Part **a** above could be used to demonstrate that a conditional and its contrapositive are logically equivalent.

* * * * *

3.7 In the riddle described in the introduction to this module, a man in a picture gallery says, "Brothers and sisters have I none, but this man's father is my father's son."

Use conditional statements to solve this riddle.

3.8 Suppose that p represents the statement, "The light is on," and q represents the statement, "The door is open."

a. Use the appropriate symbols to express the conditional statement, "If the light is on, then the door is open."

b. What are all the possible circumstances that would make the conditional statement in Part **a** false?

c. Complete a truth table for the conditional statement, "If the light is on, then the door is open."

d. How does your truth table in Part **c** compare to the truth table for "$\sim p$ or q" shown below?

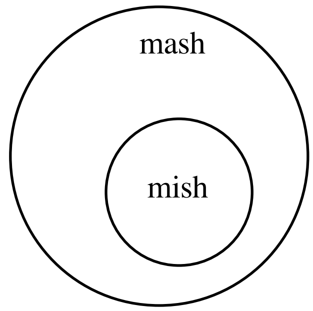
p	q	$\sim p$	$\sim p$ **or** q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

3.9 **a.** Create a truth table for the conditional statement "If $x = 3$, then $x < 5$."

b. Create a truth table for the contrapositive of the statement in Part **a**.

c. How do the truth tables indicate that the conditional in Part **a** and its contrapositive in Part **b** are logically equivalent?

3.10 Conditionals are sometimes illustrated using Venn diagrams. For example, the Venn diagram for the true conditional, "If an object is a mish, then it is a mash," is shown below. The inner circle represents the objects that satisfy the hypothesis, while the outer circle represents objects that satisfy the conclusion.



Use the Venn diagram above to determine a logical conclusion for each of the following hypotheses.

a. If Micah is a mish, . . .

b. If Fermi is a mash, . . .

c. If Pretty Eagle is not a mish, . . .

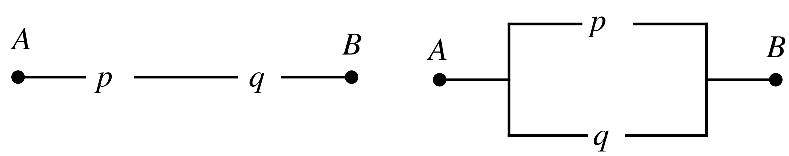
d. If Vetrovsky is not a mash, . . .

* * * * * * * * * *

Summary Assessment

1. The ability to reason logically is an important asset in many professions. Electricians, for example, often face complicated wiring problems that would be difficult to solve without logical reasoning skills.

There are two basic types of circuits, **series** and **parallel**, as shown in the diagram below. The figure on the left represents part of a series circuit. The one on the right represents part of a parallel circuit.



series circuit **parallel circuit**

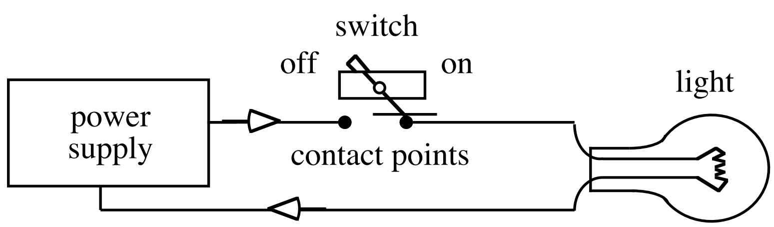
In both figures, p and q represent switches that can be turned either on or off. For electricity to flow from point A to point B, it must have an uninterrupted path between the two points.

a. Let the number 1 represent a switch in the "on" position and the number 0 represent a switch in the "off" position. Create a table that describes all the possible positions for the two switches in the series circuit and shows whether or not electricity can flow from A to B. (In this situation, a "1" in the table indicates that electricity can flow from A to B, while a "0" indicates that it cannot.)

b. Repeat Part **a** for the parallel circuit.

c. Which type of circuit corresponds with the connective *and* and which corresponds with the connective *or*? Use truth tables to justify your response.

2. The diagram below shows a light bulb controlled by a single switch.

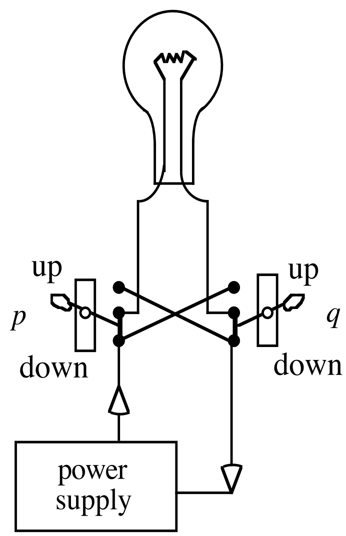


The switch in this diagram is in the "off" position. The bulb is not lit because electricity cannot flow through the switch. The bulb lights only when electricity has an uninterrupted path from the power supply to the bulb and back again. Flipping the switch to the "on" position connects the two contact points and completes the circuit.

In many homes, a single light can be controlled from two different switches. The light can be turned on or off from either switch, regardless of the position of the other switch. Because these switches have three contact points, they are called three-way switches. In each switch, there is an "up" position and a "down" position, as shown below.

The switch is up. The switch is down.

The diagram below shows a circuit in which the light can be controlled from either three-way switch. The large "X" in the diagram represents the wires that connect the two switches.



a. Draw a picture of all the possible sets of switch positions and determine whether the light is on or off in each one.

b. Write a conditional statement that describes the positions of switches p and q necessary to turn the light on.

c. As in Problem **1**, create a table that describes all the possible positions for the two switches and shows whether or not the light is on in each case.

d. Is the circuit shown above logically equivalent to a parallel circuit, a series circuit, or neither? Justify your response.

Module Summary

- A **statement** is a sentence that can be determined to be either true or false, but not both. The truth or falseness of a statement is its **truth value**.
- Two statements can be joined into a **compound statement** using the **connectives** *and* and *or*.
- A compound statement that uses the connective *and* is a **conjunction**.
- A compound statement that uses the connective *or* is a **disjunction**.
- **Venn diagrams** are mathematical models that show relationships among different sets of data.
- The **intersection** of two sets is the set of all elements common to both sets. The intersection of set A and set B can be denoted by $\mathrm{A} \cap \mathrm{B}$.
- The **union** of two sets is the set of all elements in either set or in both sets. The union of set A and set B can be denoted by $\mathrm{A} \cup \mathrm{B}$.
- **Disjoint sets** have no elements in common.
- The **empty set** or **null set** is a set that contains no elements. The symbol for the empty set is $\varnothing$.
- A **truth table** shows the truth values of a compound statement for all possible truth values of its individual statements.
- The compound statement "*p* and *q*" has a truth value of true only when statement *p* and statement *q* are both true. The truth table for "*p* and *q*" is shown below.

p	***q***	***p* and *q***
T	T	T
T	F	F
F	T	F
F	F	F

- The compound statement "*p* or *q*" has a truth value of false only when statement *p* and statement *q* are both false. The truth table for "*p* or *q*" is shown below.

p	***q***	***p* or *q***
T	T	T
T	F	T
F	T	T
F	F	F

- A **negation** of a statement p is described as "not p," also denoted by $\sim p$. When p is true, its negation $\sim p$ must be false. When p is false, $\sim p$ must be true. The truth table for "not p" is shown below.

p	**not *p***
T	F
F	T

- Two statements are **logically equivalent** when they have exactly the same truth values.
- A **conditional statement** is a compound statement that can be written in "if-then" form. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "if p, then q," or by $p \to q$ (read "p implies q"). The truth table for the conditional $p \to q$ is shown below.

p	***q***	$\boldsymbol{p \to q}$
T	T	T
T	F	F
F	T	T
F	F	T

- The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and conclusion and negating both of them. The contrapositive of $p \to q$ can be represented as $\sim q \to \sim p$, or "if not q, then not p."
- A conditional statement and its contrapositive are logically equivalent.
- A conditional statement $p \to q$ has the same truth value as the compound statement $(\sim p)$ or q. In other words, $(\sim p)$ or q is false exactly when p is true and q is false.

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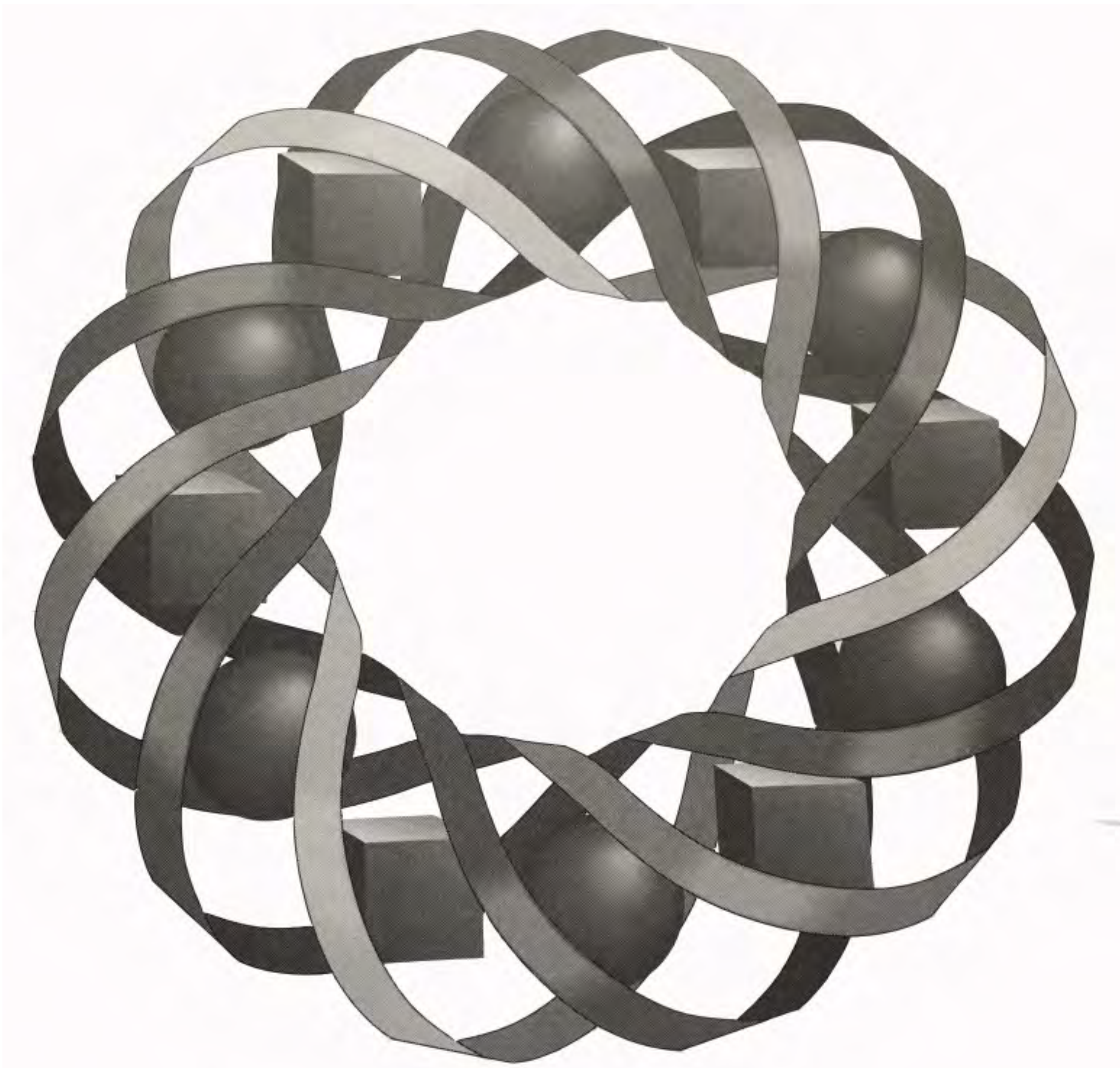
What’s Your Orbit?



How in the world do astronomers predict the paths of planets? In this module, you’ll examine some planetary data and model it mathematically.

Masha Albrecht • Deanna Turley

What's Your Orbit?

Introduction

Italian astronomer Galileo Galilei (1564–1642) was also a mathematician, physicist, musician, painter, and inventor. He built his first telescope in 1609 and is credited with many astronomical discoveries, including the first observations of sunspots, lunar mountains and craters, Saturn's rings, and four of Jupiter's moons.

During Galileo's time, most scientists and philosophers believed that the earth was the center of the universe. When his findings cast doubt on this theory, he was branded a heretic.

Modern astronomers now understand that the earth revolves around the sun—and that our sun is only one star in an immense galaxy of stars. Our galaxy, in turn, is just one galaxy among countless other galaxies in the universe. Furthermore, most scientists agree that the universe itself has been expanding for some time.

Activity 1

Although new innovations like the space shuttle and the Hubble telescope have greatly increased our knowledge of the universe, much remains unknown. In order to make predictions about the characteristics of planets and stars, scientists must rely on their abilities to develop mathematical models of current data. In this module, you focus on three types of models: power equations of the form $y = ax^b$, exponential equations of the form $y = ab^x$, and polynomial equations of degree 1, 2, and 3.

Exploration 1

Before you model data mathematically, you should be aware of the characteristics of the graphs of potential models. In this exploration, you examine graphs of power equations of the form $y = x^b$. With proper restrictions on the domain and the values of b, it is possible to make some generalizations about the graphs of these equations.

a. Choose a non-integer, rational value for b greater than 1.

1. Express the value of b as a fraction m/n in lowest terms.
2. Determine a decimal approximation of b.

b.

1. Using the fractional representation of b in Part **a**, graph the power equation $y = x^b$ over the domain [–10, 10].
2. On a separate coordinate system but using the same domain, graph the power equation $y = x^b$ using the decimal approximation of b.
3. Compare the two graphs created in Steps **1** and **2**.

c. Repeat Parts **a** and **b** for another value of b greater than 1.

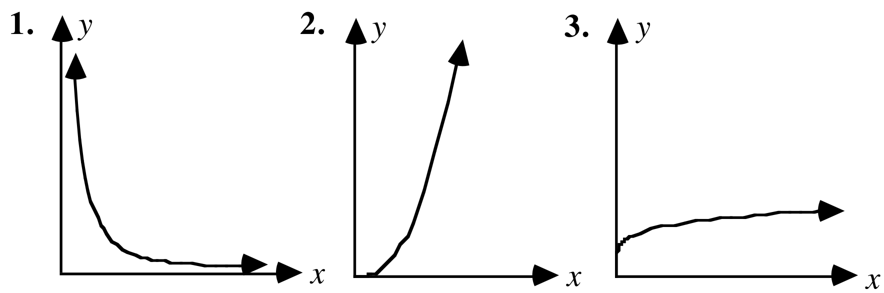
d. Repeat Parts **a** and **b** for two rational values of b between 0 and 1.

e. Repeat Parts **a** and **b** for two rational values of b less than 0.

Discussion 1

a. Judging from your investigations in Exploration **1**, how does the graph of a power equation using a fractional representation of b compare with the graph of the power equation using the corresponding decimal approximation?

b. Describe some possible values of b for each of the following graphs of equations of the form $y = x^b$.



c. In the Level 2 module "Atomic Clocks Are Ticking," you examined the following properties of exponents.

- If d is a real number greater than 0, m and n are positive integers, and m/n is in lowest terms,

$$d^{m/n} = \left(d^{1/n}\right)^m = \left(\sqrt[n]{d}\right)^m = \sqrt[n]{d^m}$$

- If d is a nonzero real number and n is an integer,

$$d^{-n} = \frac{1}{d^n}$$

- If d is a real number greater than 0,

$$\left(d^m\right)^n = d^{m \bullet n} \text{ and } \frac{d^m}{d^n} = d^{m-n}$$

1. How could you express the equation $y = x^{-1/3}$ without using a negative exponent?

2. How could you express the equation $y = x^{-1/3}$ using a radical sign?

3. How could you express the equation $y = x^{0.3}$ using a radical sign?

d. **1.** Why does a graph of $y = x^{1/2}$ only show values in the first quadrant?

2. Why would you expect the graph of $y = x^b$, where b is the decimal approximation of a rational number, to only show values in the first quadrant?

e. Why is the expression x^b, when $b < 0$, undefined for $x = 0$?

f. In the Level 3 module "Graphing the Distance," you examined the effect of various values of *a* on polynomial equations of the form $y = a(x - c)^2 + d$ and $y = a(x - c)^3 + d$. How do you think the value of *a* will affect graphs of equations of the form $y = ax^b$?

g. For what values of *b* are power equations of the form $y = ax^b$ also polynomial equations?

Exploration 2

In this exploration, you investigate the relationship between the volume and circumference of a balloon. One way to measure the volume of a balloon is to count the number of "breaths" it contains. Before beginning this experiment, practice taking several even breaths.

a. Inflate your balloon one breath at a time. After each breath, measure the balloon's circumference. Record a minimum of six data points.

b. Create a scatterplot of your data.

c. One possible model for the data is a power equation. Suggest appropriate values for *a* and *b* in a model of the form $y = ax^b$.

d. Graph your suggested model on the same coordinate system as the scatterplot in Part **b**.

e. Create a spreadsheet with headings like those in Table **1** below.

Table 1: Balloon spreadsheet

No. of Breaths (x)	Circumference (y)	$y = ax^b$

1. Enter your experimental data in the appropriate columns.

2. In the right-hand column, enter a spreadsheet formula that will calculate the approximate circumference given the number of breaths using an equation of the form $y = ax^b$. **Note:** Make sure to use this formula in each cell of the column.

f. Recall that a **residual** is the difference between an observed value and the corresponding value predicted by a model, and that the sum of the squares of the residuals can be used to evaluate how well a model fits a data set.

1. Use the spreadsheet to calculate the sum of the squares of the residuals for your model.
2. Adjust the values of a and b in your model to identify a power equation that closely approximates the data.
3. Record the corresponding sum of the squares of the residuals.

Discussion 2

a. Describe how you determined an appropriate value for a in Part **c** of Exploration **2**.

b.
1. Do you think that anyone in the class found the power equation with the least possible sum of the squares of the residuals for their data set? Explain your response.
2. If the power equation that produces the least possible sum of the squares of residuals could be found, how well do you think it would fit the data?

c. What types of information might help you evaluate the appropriateness of mathematical models?

Assignment

1.1 For each value of b below, sketch the graph of a power equation of the form $y = x^b$ over the domain (0, 10].

a. 5/6

b. 7/4

c. −10/3

d. 0.6667

e. 2

1.2 The table below contains information about the planets in our solar system. The mean distance represents the average distance of a planet from the sun during its orbit. The period represents the time, measured in earth days, required for a planet to complete one orbit around the sun.

Planet	**Mean Distance from Sun (millions of km)**	**Period (earth days)**
Mercury	57.9	87.97
Venus	108.2	224.70
Earth	149.6	365.26
Mars	227.9	686.98
Jupiter	778.3	4331.87
Saturn	1427.0	10,760.27
Uranus	2869.6	30,684.65
Neptune	4496.6	60,189.55
Pluto	5899.9	90,468.77

a. Create a scatterplot of this data.

b. Determine an equation of the form $y = ax^b$ that models the data.

c. Is the model you selected a good one? Explain your response.

1.3 Imagine that it is the year 2039. Earth's inhabitants have colonized the moon. There are five lunar colonies, each housed in a pressurized hemispherical dome. The table below shows the diameter and volume of each dome.

Colony	**Diameter (km)**	**Volume (km^3)**
Alpha	5.0	32.73
Beta	6.4	68.63
Gamma	5.6	45.98
Delta	8.2	144.35
Epsilon	7.6	114.92

a. Considering the relationship between the diameter and volume of a sphere, would a linear, exponential, or power equation provide the best model for this data?

b. Find an equation of the type identified in Part **a** to model the data.

c. Use the equation determined in Part **b** to predict the volume of a hemispherical dome with a diameter of 7.0 km.

1.4 The table below shows the population of Beta colony during its first 20 years.

Year	Population	Year	Population
1	1000	11	6192
2	1200	12	7430
3	1440	13	8916
4	1728	14	10,699
5	2074	15	12,839
6	2488	16	15,407
7	2986	17	18,488
8	3583	18	22,186
9	4300	19	26,623
10	5160	20	31,948

a. Make a scatterplot of this data.

b. What type of equation do you think would provide a good model for the population growth of Beta colony?

c. Find an equation that describes the trend in the data.

d. Graph your equation on the same coordinate system as the scatterplot from Part **a**.

e. The population of Beta colony cannot exceed 1000 people per km^3. Use your model, along with the dimensions of Beta colony's dome from Problem **1.2**, to determine how long it will take the population to reach this limit.

1.5 Find at least five round, flat objects of different sizes.

a. Measure and record their diameters and circumferences.

b. Based on the relationship between the diameter and circumference of a circle, what type of model would you expect to closely approximate the data collected in Part **a**?

c. Create a mathematical model of this data.

d. Use your model to predict the circumference of a disk with a radius of 1 m.

* * * * *

1.6 A chemistry class at Centerville High School conducted an experiment to determine how much potassium nitrate would dissolve in 100 mL of water at various temperatures. The results of their experiment are shown below.

Temperature (˚C)	Amount of Potassium Nitrate Dissolved (g)
5	27
20	54
40	77
60	94
80	109

a. Determine a power equation that closely models the data.

b. How much potassium nitrate do you think will dissolve in 100 mL of water at 70˚ C? Explain your response.

c. Do you think that it is reasonable to use your model to predict the amount of potassium nitrate that will dissolve in 100 mL of water at 110˚ C? Explain your response.

1.7 The table below shows the atomic radii and melting points for a family of elements known as the alkali metals. Because atoms are so small, their radii are often measured in angstroms, where 1 angstrom equals $1 \bullet 10^{-10}$ m. Melting point refers to the temperature at which a solid turns into a liquid.

Alkali Metals	Atomic Radius (angstroms)	Melting Point (˚C)
Lithium	1.52	180
Sodium	1.86	98
Rubidium	2.44	39
Cesium	2.62	29

a. Create a scatterplot of melting point versus atomic radius.

b. Does there appear to be a positive or a negative association between atomic radius and melting point? Explain your response.

c. Find an equation that closely models this data and explain why you chose this type of model.

d. Potassium is also an alkali metal. Its atomic radius is approximately 2.31 angstroms. Use your model from Part **c** to predict the melting point of potassium.

e. The actual melting point of potassium is approximately 64˚ C. Compare this value with your prediction in Part **c** and suggest a possible explanation for any difference that occurs.

* * * * * * * * * *

Activity 2

In the module "Graphing the Distance," you used technology to find linear and quadratic regression equations to model data. In this activity, you examine power and exponential regressions and investigate another method for evaluating the appropriateness of models.

Exploration 1

Table **2** below shows data collected during a balloon experiment like the one described in Activity **1**.

Table 2: Balloon experiment data

No. of Breaths	Circumference (cm)
1	31
2	42
3	53
4	58
5	65
6	70

a. Use technology to find a power regression equation for the data in Table **2**.

b. Determine the sum of the squares of the residuals for this regression equation.

c. Describe how well the graph of the regression equation fits a scatterplot of circumference versus number of breaths.

d. Repeat Parts **a–c** using an exponential regression.

Discussion 1

a. Which regression equation—power or exponential—appears to provide a better model for the data? Explain your response.

b. Would you use this model to predict the circumference of a balloon after 50 breaths? Explain your response.

c. If asked to make a prediction based on a given data set, what steps would you take to find an appropriate model?

d. What other criteria might help you select the most appropriate model from several possibilities?

Exploration 2

In this exploration, you examine another tool for evaluating mathematical models: the **residual plot**. (You may recall the use of residual plots from the Level 2 module, “If the Shoe Fits . . . .”)

a. Most of the asteroids in our solar system lie between the orbits of Mars and Jupiter. Table **3** shows the orbital period and mean distance from the sun for nine of these asteroids, listed in order of discovery. Use this data to make a scatterplot of period versus mean distance from the sun.

Table 3: Orbital period of nine asteroids

Name	Mean Distance from Sun (millions of km)	Period (earth years)
Ceres	411.20	4.60
Pallas	411.84	4.61
Juno	396.48	4.36
Vesta	350.88	3.63
Astraea	382.88	4.14
Hebe	360.32	3.78
Iris	354.24	3.68
Flora	327.04	3.27
Metis	354.72	3.69

b. Select three of the following five types of equations: linear, quadratic, cubic, exponential, and power. For each type you select, determine a corresponding regression equation to model the data in Table **3**.

c. Create a graph of each regression equation on the same coordinate system as the scatterplot from Part **a**. To simplify comparison, use the same scales on the axes of all graphs.

Mathematics Note

A **residual plot** is a scatterplot created using the ordered pairs (x-value of the data, residual). If the sum of the squares of the residuals is relatively small, a residual plot in which the points are randomly scattered above and below the x-axis typically indicates that a reasonable model has been selected.

For example, Table **4** shows the x- and y-coordinates of a set of data points, the corresponding y-values predicted by the linear regression model $y = 0.75x + 1.14$, and the value of the residual for each data point.

Table 4: Data points, predicted *y*-values, and residuals

x-value	y-value	Predicted y-value	Residual
1	2	1.89	0.11
2	2	2.64	–0.64
3	4	3.39	0.61
4	5	4.14	0.86
5	4	4.89	–0.89
6	5	5.64	–0.64
7	7	6.39	0.61

Figure **1** shows the corresponding residual plot. Since no pattern appears to exist, this regression equation may be a reasonable model for the data.

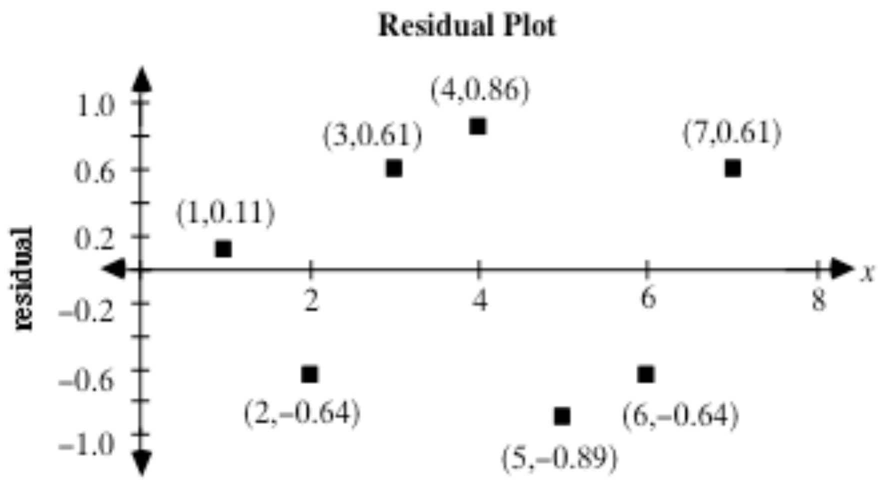


Figure 1: A residual plot

d. Create a residual plot for each model selected in Part **b**.

Discussion 2

a. Of the three regression equations you determined in Exploration **2**, which one appears to provide the best model of the data? Defend your choice.

b. Suppose that a new asteroid is discovered with a mean distance from the sun of $3.50 \bullet 10^8$ km.

1. Describe how you would use your model to predict the asteroid's orbital period.
2. How confident would you feel about the accuracy of this prediction? Explain your response.

c. Another known asteroid orbits the sun at a mean distance of $5.90 \bullet 10^9$ km.

1. Using the model you selected in Part **a** of the discussion, predict the asteroid's orbital period.
2. How confident do you feel about the accuracy of your prediction?

d. German astronomer Johannes Kepler (1571–1630), who lived during the same time as Galileo, also studied the motion of planets. He developed a mathematical description of planetary motion that included three basic laws.

Kepler's third law stated that the ratio of the cube of the mean distance (r) from the sun to the square of the period (p) is a constant for every planet in the solar system. This can be represented algebraically as follows:

$$\frac{r^3}{p^2} = k$$

How does this information affect your choice of an appropriate model for the data in Table **3**?

Assignment

2.1 A meteorite is a chunk of space debris that falls to the earth's surface. The force of impact sometimes forms a crater. The table below shows the depth and diameter of seven meteorite craters on earth.

Name of Crater	Diameter (m)	Depth (m)
Barringer	1240	210
Herault	230	50
Odessa 1	160	40
Odessa 2	21	5
Explosion 1	120	27
Explosion 2	47	14
Explosion 3	32	6

a. Create a scatterplot of crater depth versus diameter.

b. Select a regression equation to model the data. Use the sum of the squares of the residuals and a residual plot to support your choice.

c. Graph your equation on the same coordinate system as the scatterplot in Part **a**.

d. How closely does your model approximate the data? Explain your response.

e. **1.** How confident would you be in predicting the depth of a crater with a diameter of 600 m? Explain your response.

2. How confident would you be in predicting the depth of a crater with a diameter of 5000 m? Explain your response.

f. The Wolf Creek crater has a diameter of approximately 820 m and a depth of approximately 30 m.

1. What depth would your model predict for a crater with a diameter of 820 m?

2. Compare this prediction with the actual depth of the Wolf Creek crater and suggest some possible explanations for any difference that occurs.

2.2 The distances involved when considering objects in the solar system are immense. For example, the mean distance from the sun to the earth is approximately $1.496 \bullet 10^8$ km. Even though light travels at an incredible speed, it still takes several minutes for light from the sun to reach earth.

The following table shows the approximate time required for light from the sun to reach each of the other planets in our solar system, as well as each planet's mean distance from the sun.

Planet	Mean Distance from Sun (millions of km)	Time for Light to Reach Planet (sec)
Mercury	57.9	195
Venus	108.2	360
Mars	227.9	763
Jupiter	778.3	2601
Saturn	1427.0	4757
Uranus	2869.6	9562
Neptune	4496.6	14,991
Pluto	5899.9	19,670

a. Find a regression equation that fits this data. Defend your selection.

b. Describe how well the equation models the data.

c. Use your model to determine the approximate time required for light from the sun to reach earth.

d. Light travels at a speed of approximately $3 \bullet 10^8$ m/sec. Given this information, did your model provide a reasonable prediction of the time required for light from the sun to reach earth?

2.3 Planets travel along their orbital paths at high speeds. The following table shows the mean distance from the sun and the orbital speed of eight planets in our solar system. Use this data to estimate earth's orbital speed, given that its mean distance from the sun is approximately $1.496 \bullet 10^8$ km. Defend your prediction.

Planet	Mean Distance from Sun (millions of km)	Orbital Speed (km/sec)
Mercury	57.9	47.8
Venus	108.2	35.0
Mars	227.9	24.1
Jupiter	778.3	13.1
Saturn	1427.0	9.6
Uranus	2869.6	6.8
Neptune	4496.6	5.4
Pluto	5899.9	4.7

2.4 The farther a planet is from the sun, the colder its surface temperature is likely to be. The surface temperature of planets, as well as other extreme temperatures, are usually measured in degrees Kelvin. **Note:** The Kelvin temperature scale was invented by Sir William Thomson (1824–1927), also known as Lord Kelvin. The relationship between the Kelvin and Celsius scales is approximately $K = {}^\circ\text{C} + 273$.

The table below shows the mean surface temperature for seven planets, along with their mean distances from the sun.

Planet	Mean Distance from Sun (millions of km)	Mean Surface Temperature (Kelvin)
Mercury	57.9	373
Mars	227.9	250
Jupiter	778.3	123
Saturn	1427.0	93
Uranus	2869.6	63
Neptune	4496.6	53
Pluto	5899.9	43

a. Use the information in the table to determine a model for predicting a planet's surface temperature given its mean distance from the sun. Defend your choice of models.

b. Because of their many similarities, Earth and Venus are often referred to as "sister" planets.

1. Use your model to predict the surface temperature of the two planets, given that Earth's mean distance from the sun is approximately $1.496 \bullet 10^8$ km, while Venus' is approximately $1.082 \bullet 10^8$ km.

2. Do you think that your predictions are reasonable? Explain your response.

c. Earth's actual mean surface temperature is approximately 295 K while Venus' is approximately 753 K. This difference can be attributed to several factors other than the mean distance from the sun, including the composition of each planet's atmosphere.

Given these facts, discuss the dangers of making predictions about complex phenomena using models that only describe the relationship between two quantities.

* * * * *

2.5 A weak solution of hydrogen peroxide is a common household antiseptic. When heated, hydrogen peroxide decomposes to form water and oxygen gas. The table below shows the change in concentration over time for a heated solution of 1% hydrogen peroxide.

Time (min)	Percent Concentration
2	0.90
5	0.78
10	0.60
20	0.37
30	0.22
40	0.13
50	0.08

a. Find a regression equation that models this data. Justify your choice.

b. Use your model to predict the percentage of hydrogen peroxide that remains after 2 hr.

c. How confident are you in the prediction you made in Part **b**?

2.6 The pressure on a fixed amount of gas can be measured using a column of mercury. At a pressure of 500 mm of mercury, a certain amount of gas occupies 10 L. The information in the following table shows how the volume of the gas decreases as the pressure increases.

Volume (L)	Pressure (mm of mercury)
10	500
9	556
8	625
7	714
6	833
5	1000
4	1250
3	1667
2	2500

a. Find a regression equation that models this data. Justify your choice.

b. Use your model to predict the pressure on the gas when its volume is 7.33 L.

c. Do you think that your model will provide good predictions for the volumes of gas at very high pressures? Explain your response.

* * * * * * * * * *

Research Project

Through taxes and user fees, the U.S. government collects money to pay for national defense, health care, road construction, and numerous other goods and services. These programs often cost more than the government's annual revenue. To make up the difference, the government must borrow money. The total amount owed is known as the national debt.

Conduct some research on the national debt from about 1980 to the present. (Make sure to record the source of the data you collect.) Determine a reasonable model for this data.

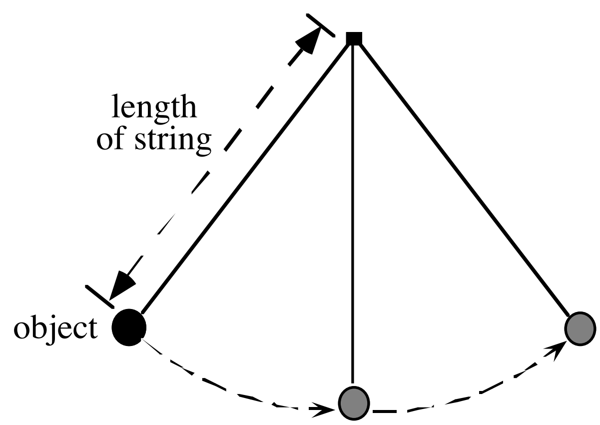
Use your model to predict the size of the national debt in 20 years from now. Describe some of the consequences that may occur if the debt continues to follow the trend described in your model. Finally, discuss why it might be risky to make long-term predictions based on your model.

Summary Assessment

Galileo's inventions and observations helped other astronomers make sense of the motion of the planets and stars. He also made important contributions to the study of other types of motion, including balls rolling on inclined planes, freely falling objects, and swinging pendulums.

While at the chapel of the University of Pisa, Galileo noticed that one of the chandeliers was swinging. Using his own heartbeat as a timer, he measured the time required for the chandelier to complete one swing. Upon returning to his room, he performed a series of experiments that resulted in "the law of the pendulum."

As shown in the diagram below, use a length of string to suspend an object from a fixed location.



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1. A pendulum completes one swing when it returns to the same side as its initial release. The time required for a pendulum to complete one swing is its **period**.

a. Determine a method for measuring one period of your pendulum.

b. Record the length of the string to the nearest 0.01 m and the period of the pendulum to the nearest 0.1 sec.

c. Change the length of the string significantly. Record the new string length and the period for this pendulum.

d. Repeat Part **c** for six different lengths of string.

2. **a.** Find a regression equation that fits the data and explain why you chose this type of model.

b. What does the equation you chose in Part **a** reveal about the motion of pendulums?

c. Describe some of the limitations of your equation for modeling the motion of pendulums.

3. In general, the relationship between the period p of a pendulum and the length l of its string can be described by $p = 2\pi\sqrt{l/g}$, where g is the acceleration due to gravity (about $9.8\ \text{m/sec}^2$). How does the relationship expressed by your model compare to this one?

Module Summary

- If d is a real number greater than 0, m and n are positive integers, and m/n is in lowest terms,

$$d^{m/n} = \left(d^{1/n}\right)^m = \left(\sqrt[n]{d}\right)^m = \sqrt[n]{d^m}$$

- If d is a nonzero real number and n is an integer,

$$d^{-n} = \frac{1}{d^n}$$

- If d is a real number greater than 0,

$$\left(d^m\right)^n = d^{m \bullet n} \text{ and } \frac{d^m}{d^n} = d^{m-n}$$

- A **residual** is the difference between the y-coordinate of a data point and the corresponding y-value predicted by a model.
- A **residual plot** is a scatterplot created using the ordered pairs (x-value of the data, residual). If the sum of the squares of the residuals is relatively small, a residual plot in which the points are randomly scattered above and below the x-axis typically indicates that a reasonable model has been selected.
- When trying to determine a mathematical model for a data set, you may wish to follow these steps:
  1. Create a scatterplot of the data.
  2. Use the shape of the scatterplot and the situation in which the data was collected to identify appropriate types of model equations.
  3. Determine the corresponding regression equations.
  4. Evaluate each potential model using the sum of the squares of the residuals and a residual plot.

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Our Town

In this module, you use one community's bridges and traffic patterns to explore the applications of graph theory to city planning.

Masha Albrecht • Patricia Bean • Peter Rasmussen

Our Town

Introduction

The old city of Königsberg (now Kaliningrad, Russia) was situated along the banks of the Pregol River. As shown in Figure **1**, two islands in the river were connected to each other and to the rest of the city by a system of seven bridges.

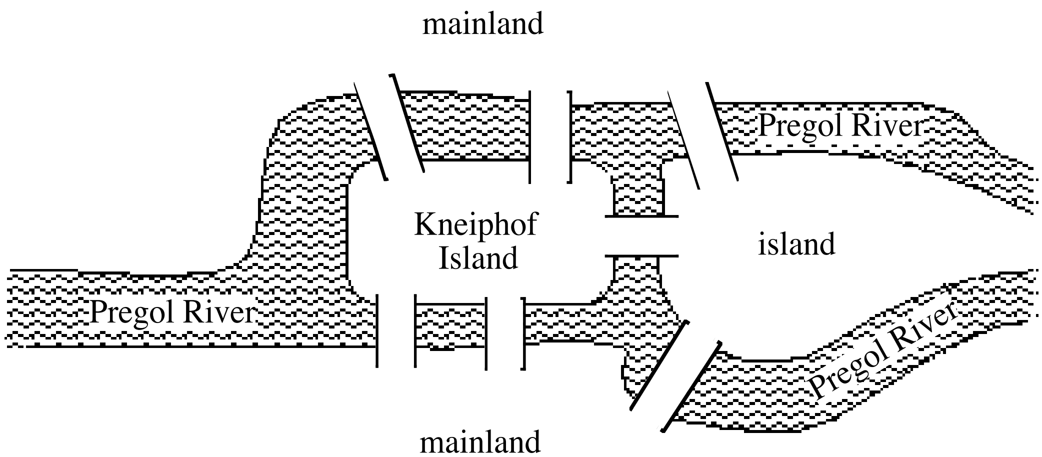


Figure 1: The bridges of Königsberg

The bridges of Königsberg inspired a famous mathematics problem: Is it possible to plan a route through the city that crosses each of the seven bridges exactly once before returning to the starting point?

To solve this problem, Swiss mathematician Leonhard Euler (1707–1783) created a simplified map or graph of Königsberg by representing each area of the city as a point and each bridge as an arc. Today, Euler is considered the founder of **graph theory**.

In this module, you use graph theory to investigate some planning problems in the imaginary community of Our Town.

Activity 1

Welcome to Our Town! The town council has called a meeting to discuss the construction of a new bridge. As a newcomer, you will shake hands with many of your fellow citizens for the first time today. Exactly how many handshakes depends on the number of people who attend the meeting. In this activity, you use some of the basic concepts of graph theory to model this situation.

Exploration

Since the new bridge has become a hot topic in the past week, the town meeting is likely to involve some spirited debate. Hoping to begin the discussion on a friendly basis, the council has organized a cooperative activity. Each person in attendance has been assigned to a small group. Group members will shake hands with each other and tell a little bit about themselves.

a. Determine the total number of handshakes for a group of two people.

b. Suppose a third person joins the group in Part **a**. Determine the total number of handshakes that will occur if each member shakes hands with every other member of the group.

c. Determine the total number of handshakes that will occur in a group containing each of the following numbers of people:

1. 4 people
2. 5 people
3. 6 people
4. 7 people
5. 10 people.

Mathematics Note

A **graph** is a non-empty set of **vertices** and the **edges** that connect them. A **loop** is an edge that connects a vertex to itself. A pair of vertices may be connected by more than one edge.

A **simple graph** is a graph without loops in which any pair of vertices has at most one connecting edge.

For example, the two graphs in Figure **2** consist of the same set of vertices. Since the graph on the right has no loops and each pair of vertices is connected by no more than one edge, it is a simple graph.

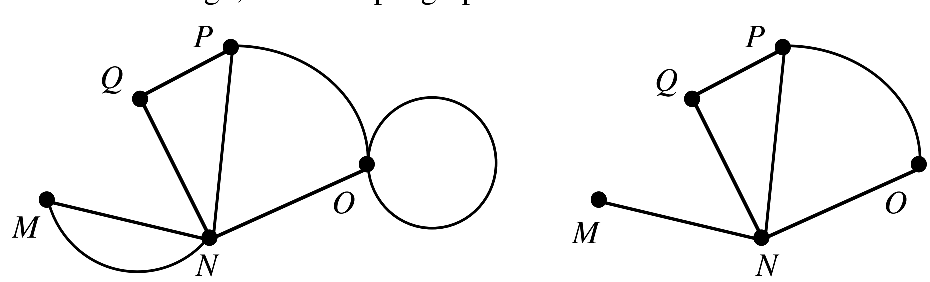


Figure 2: Two graphs with five vertices each

d. Draw graphs to model the situations in three of the examples in Part **c**. Use vertices to represent people and edges to represent handshakes.

e. Develop a method you could use to find the number of handshakes that would occur in a group of 100 people.

f. Write a formula for the number of handshakes that would occur in a group of n people.

Discussion

a. In your graphs in Part **d** of the exploration, how are the numbers of vertices and edges related to the numbers of people and handshakes?

b. How can you determine when a graph represents all possible handshakes?

c. Compare your graphs with those of others in the class. Describe two examples of graphs that look different, but still model the same situation.

d. How do your graphs support the formula you wrote in Part **f** of the exploration?

Mathematics Note

A **complete graph** is a simple graph in which every distinct pair of vertices is connected by exactly one edge. For example, Figure **3** shows two graphs with the same set of vertices. (Note that the intersection of two edges is not necessarily a vertex.) The graph on the left is complete, while the graph on the right is not.

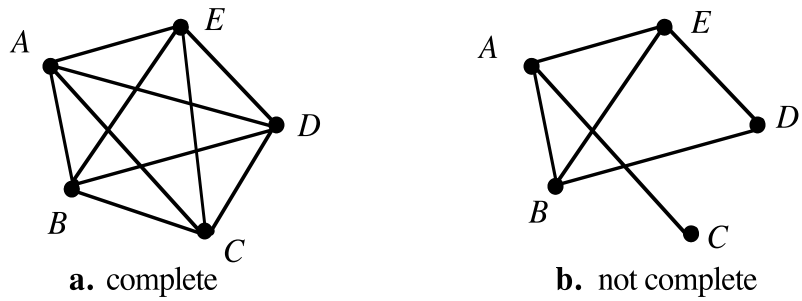


Figure 3: Two graphs with the same set of vertices

The **degree** of a vertex is the number of edges that meet at that vertex. If the number of edges that meet at a vertex is even, then the vertex has an **even degree**. If the number of edges that meet at a vertex is odd, then it has an **odd degree**.

For example, the degree of each vertex in the graph in Figure **3a** is 4. The degree of vertex A in the graph in Figure **3b** is 3. In Figure **2**, the degree of vertex O is 4. (Without the loop, the degree of vertex O would be 2.)

e. Are the graphs you created in the exploration complete graphs? Justify your response.

Assignment

1.1 **a.** To determine the relationships among the number of vertices, the degree of each vertex, and the number of edges in a complete graph, complete the following table.

No. of Vertices	Sketch of Complete Graph	Degree of Each Vertex	Total No. of Edges	No. of Odd Vertices	No. of Even Vertices
2					
3					
4					
5					
6					

b. Determine the degree of each vertex in a complete graph with *n* vertices.

c. Determine the number of edges in a complete graph with *n* vertices.

1.2 In a complete graph, what is the relationship between the number of edges and the sum of the degrees of all vertices? Justify your response.

1.3 Can a complete graph contain exactly one vertex with an odd degree? Justify your response.

1.4 **a.** Draw five vertices and label them with the names of people you know. Draw an edge between any pair of vertices that represents two people who know each other.

b. Is your graph a complete graph? Explain your response.

c. Find the degree of each vertex and state whether it is odd or even.

Mathematics Note

A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices can occur more than once in a path.

For example, Figure **4** shows a graph with five vertices: *A*, *B*, *C*, *D*, and *E*. The sequence of vertices *E*–*B*–*A*–*D* is a path.

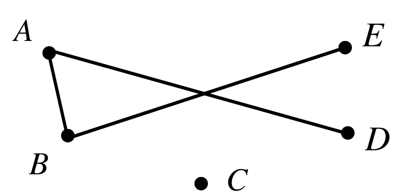


Figure 4: A graph with five vertices and three edges

A graph is **connected** if a path exists from each vertex to every other vertex in the graph. Figure **5** shows an example of a connected graph.

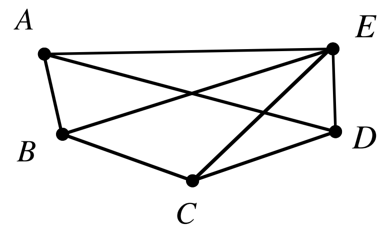


Figure 5: A graph with five vertices and seven edges

The graph in Figure **4** is not connected, because there is no path to vertex *C* from any other vertex.

A **closed path** is a path that begins and ends at the same vertex.

A **circuit** is a closed path in which no intermediate vertex is repeated. A circuit may or may not contain all the vertices of a graph.

In Figure **5**, for example, closed path *D–C–B–A–D* is a circuit. Closed path *A–E–B–C–E–D–A*, however, is not a circuit.

A **Hamiltonian circuit** is a circuit in which every vertex in a graph is visited exactly once. In Figure **5**, for example, the path *A–B–C–D–E–A* is a Hamiltonian circuit.

1.5 **a.** Obtain a map of your school, town, or state. Select five places on the map that you have visited. Draw a vertex to represent each of these locations. Indicate any route you have traveled between locations by drawing an edge between the appropriate pair of vertices.

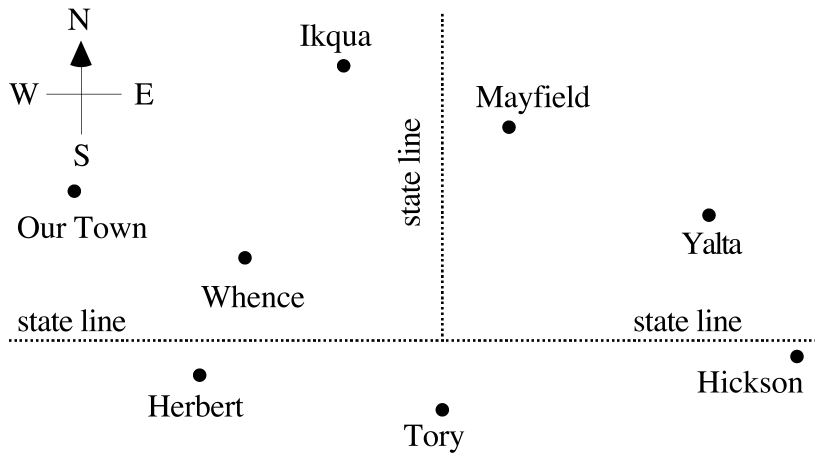
b. Is your graph a complete graph? Justify your response.

c. Does your graph contain any circuits? If so, describe one. If not, explain why not.

d. Does your graph contain a Hamiltonian circuit? If not, add the edge(s) needed to create one.

* * * * *

1.6 A map of the region around Our Town is shown below:



The airlines that serve the region offer the following connections:

- Our Town and Ikqua
- Ikqua and Mayfield
- Yalta and Whence
- Yalta and Tory
- Tory and Hickson
- Our Town and Yalta
- Mayfield and Yalta
- Whence and Herbert
- Hickson and Herbert

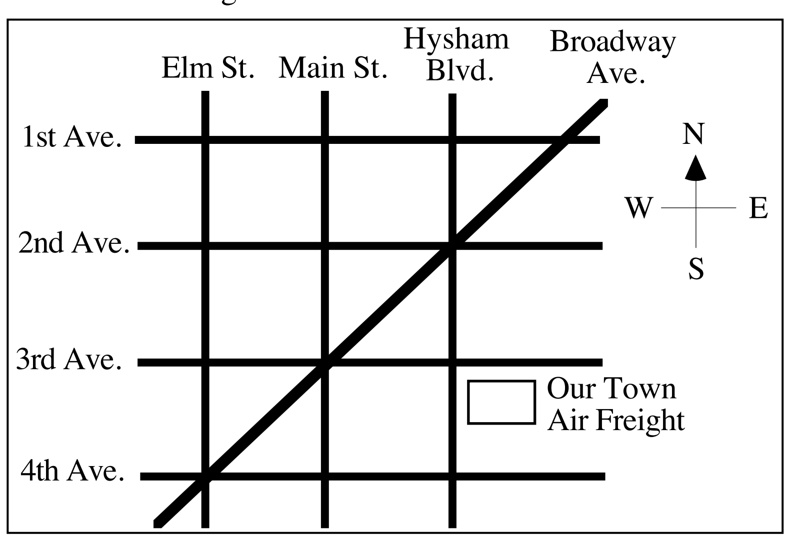
a. Draw a graph that models these flight connections.

b. Is your graph a complete graph? Explain your response.

c. Imagine that you must fly on a business trip that includes Our Town, Mayfield, Yalta, Whence, and Ikqua. Is it possible for you to follow a path that forms a Hamiltonian circuit? Explain your response.

d. If a Hamiltonian circuit does exist, should your business trip follow that path? Why or why not?

1.7 The Our Town Air Freight Company offers pick-up and delivery service in the neighborhood shown below.



The freight truck begins its route at the company's office at the corner of Third and Hysham. The driver's scheduled stops are outlined in the following table. In the left-hand column, the letter P represents a pick-up, while the letter D represents a delivery.

Pick-up or Delivery	Time	Corner
P	8:15 A.M.	Third & Elm
D	after 2:00 P.M.	Broadway & Hysham
P	9:00 A.M.	Main & Fourth
D	anytime	Broadway & Hysham
P	11:00 A.M.	Third & Hysham
P	11:45 A.M.	Second & Main
D	anytime	Third & Main
D	anytime	First & Elm
D	before 10:00 A.M.	Main & Broadway
D	anytime	Fourth & Hysham

a. Draw a graph that models the driver's stops.

b. Is the graph a complete graph? Explain your response.

c. Is the graph connected? Explain your response.

d. Plan an efficient path for the driver to follow. Use terms from graph theory to explain why your route is the best one.

e. Is the route you planned in Part **d** a Hamiltonian circuit? Explain your response.

1.8 Identify each of the following statements as true or false and explain how you determined your response.

a. Every polygon forms a complete graph.

b. Every polygon forms a connected graph.

c. Every polygon contains a Hamiltonian circuit.

* * * * * * * * * *

Research Project

Hamiltonian circuits are named after Sir William Rowan Hamilton (1805–1865). Find out more about Hamilton's life and his contributions to graph theory.

Activity 2

Like the old city of Königsberg, Our Town was built on the banks of a river and contains two islands. As shown in Figure **6**, however, Our Town is connected to its two islands by only five bridges.

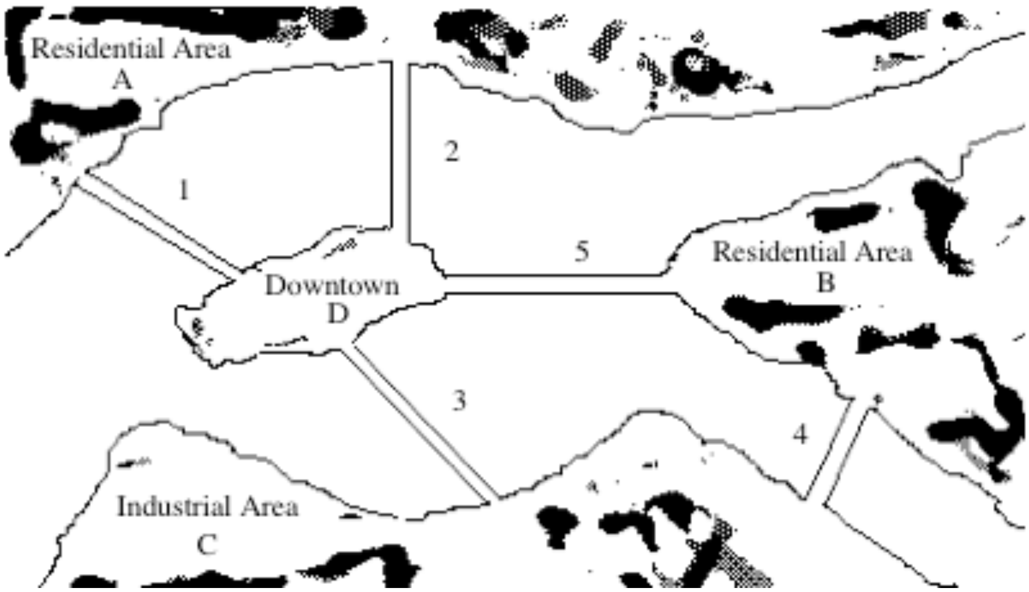


Figure 6: The bridges of Our Town

Exploration 1

The students of Our Town have challenged each other to select a starting point, then follow a path that crosses each bridge exactly once before returning to that point.

a. Draw a graph to represent the map of Our Town in Figure **6**.

b. Select an area of the town as your starting point. Beginning from this point, is there a closed path that crosses each bridge exactly once? If so, describe this path. If no such path exists, explain why not.

c. To accommodate Our Town's growing population, city planners have decided to build two more bridges, bringing the total to seven. These bridges (labeled 6 and 7) are shown in Figure **7**.

Draw a graph to represent a map of the town that includes all seven bridges, then repeat Part **b**.

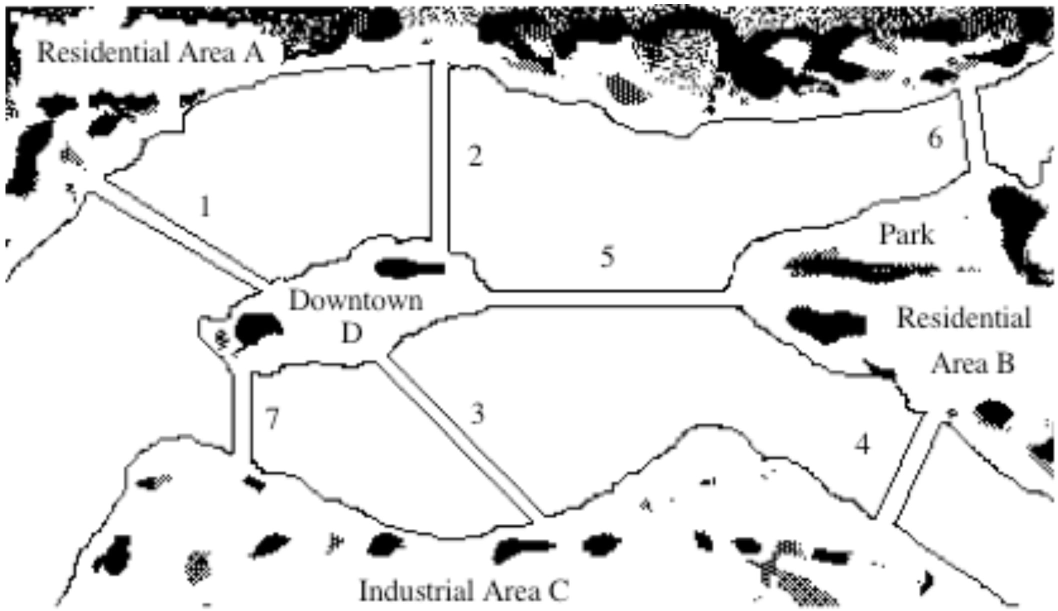


Figure 7: Our Town with seven bridges

Discussion 1

a. Are the graphs you created in Exploration **1** simple graphs? Explain your response.

b. How did the number of bridges affect your ability to find a closed path that crosses each bridge exactly once?

c.
  1. Is either of the graphs you created in Exploration **1** a complete graph? Explain your response.
  2. Is either graph a connected graph? Explain your response.

Mathematics Note

A connected graph is **traversable** if it is possible to traverse every edge of the graph exactly once.

For example, Figure **8** shows a graph with two edges between vertices *B* and *D*. Since path *A–B–D–B–C–D–A* traverses every edge exactly once, this graph is traversable.

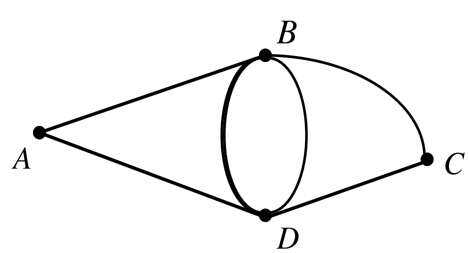


Figure 8: A traversable graph

d. Are all graphs that have a Hamiltonian circuit traversable?

e. Do all traversable graphs have a Hamiltonian circuit?

Exploration 2

While studying the problem of the bridges of Königsberg, Euler discovered a relationship between the degrees of the vertices of a graph and its traversability. In this exploration, you investigate some of the reasoning behind his discovery.

Figure **9** shows eight different graphs. Use a copy of these graphs supplied by your teacher to complete Parts **a–d**.

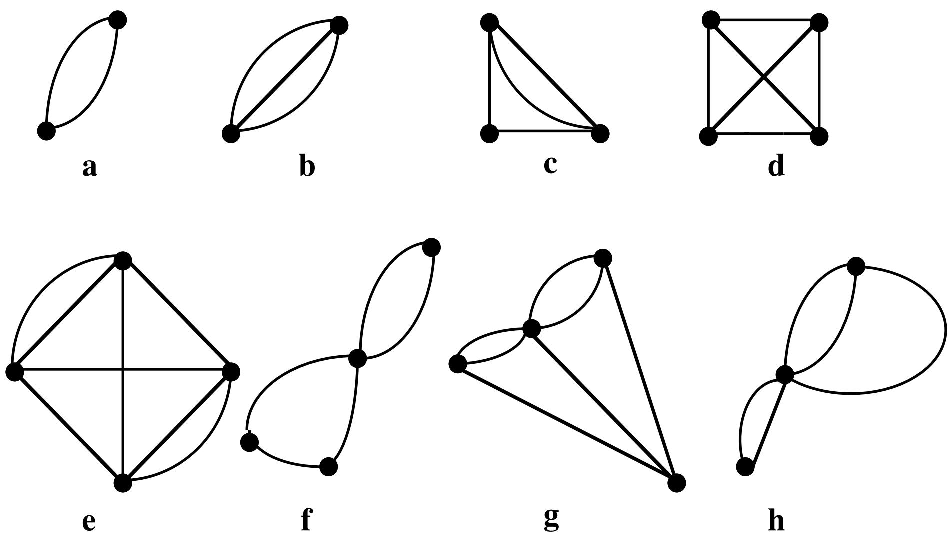


Figure 9: Eight graphs

a. **1**. Specify the degree of each vertex in the graphs in Figure **9**.

2. Determine if there is a path that traverses each graph.

b. Use your responses to Part **a** to complete Table **1** below.

Table 1: Degrees of vertices and traversability of graphs

Graph	No. of Odd Vertices	No. of Even Vertices	Traversable? (Yes or No)
a			
b			
c			
d			
e			
f			
g			
h			

c. Use Table **1** to find a rule that describes when a graph is traversable.

Discussion 2

a. What relationship did you find between the traversability of a graph and the degrees of its vertices?

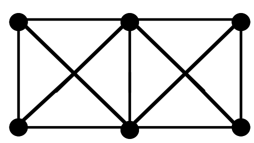
b. Using the relationship identified in Part **a** of Discussion **2**, determine if the graph of each of the following is traversable:

1. the system of five bridges in Our Town
2. the system of seven bridges in Our Town.

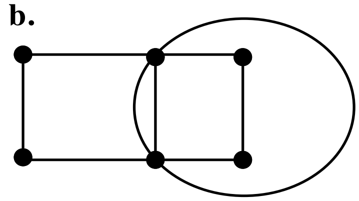
Assignment

2.1 Determine whether or not each of the following graphs is traversable. Justify your responses.

a.



b.



2.2 Determine if a complete graph with each of the following numbers of vertices is traversable. Explain your responses.

a. 5

b. 6

c. 101

d. n

2.3 Draw a traversable graph with five vertices that is not a complete graph.

2.4 Because of its seven beautiful bridges, Our Town has become a popular tourist destination. To show visitors some of the community's other attractions, the city council has asked you to design a tour that begins and ends downtown and visits all four areas of the city.

a. Using the map of Our Town in Figure **7**, design a tour and model it with a graph.

b. Is your graph in Part **a** traversable? Justify your response.

c. Is the tour you designed in Part **a** a Hamiltonian circuit? Justify your response.

2.5 **a.** Design a tour that begins downtown and visits all seven bridges in Our Town. Model the tour with a graph.

b. Is your graph in Part **a** traversable? Justify your response.

c. Is the tour you designed in Part **a** a Hamiltonian circuit? Justify your response.

2.6 Select four locations in your town or county. Use these locations as the vertices of a graph. Connect the vertices with edges representing major travel routes. Use this graph to show an efficient snowplowing, mail delivery, or garbage removal route. Explain why your route is an efficient one.

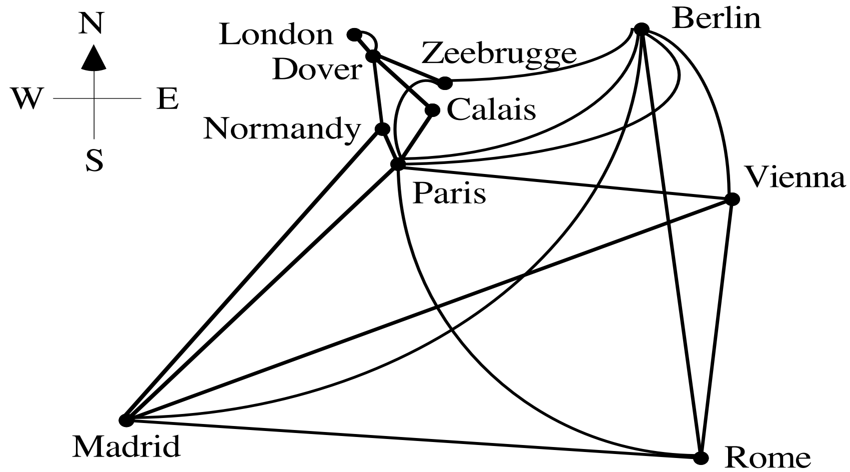
* * * * *

2.7 The map below shows the six cities included in a European tour sponsored by Our Town Travel.

a. Each of the six cities is connected to every other one by a direct flight. Draw a graph to model these connections.

b. Use your graph from Part **a** to design a tour that is a Hamiltonian circuit.

2.8 Malcolm lives in London and would like to visit the same cities offered in the tour in Problem **2.7**. His travel agent suggests that he take a bus from London to Dover, then take a ferry across the English Channel—either to Normandy, Calais, or Zeebrugge. Once on the European continent, he can then visit the other six cities by train. The available connections are shown in the following graph.



Malcolm would like to experience each connection on the graph, without repeating any one of them, if possible. Is it possible to create a tour that begins and ends in London and traverses each connection exactly once? If so, describe such a tour. If not, explain why not.

* * * * * * * * * *

Research Project

Select a city built along one or more rivers, such as Pittsburgh, St. Louis, or New York. Find a map of the city and draw a graph that represents the city's system of bridges. Demonstrate the traversability of the bridge system to the class and discuss the system in terms of Hamiltonian circuits. Describe how the system changes if some of the bridges allow only one-way traffic.

Activity 3

To speed the flow of traffic in Our Town, planners have restricted movement on some of the city's bridges. As shown in Figure **10**, bridges 1, 2, 3, and 7 allow only one-way traffic. Bridges 4, 5, and 6 allow two-way traffic. When travel on any bridge is blocked, access to different parts of the city can be severely affected. In this activity, you use graphs to determine routes that avoid possible blockages.

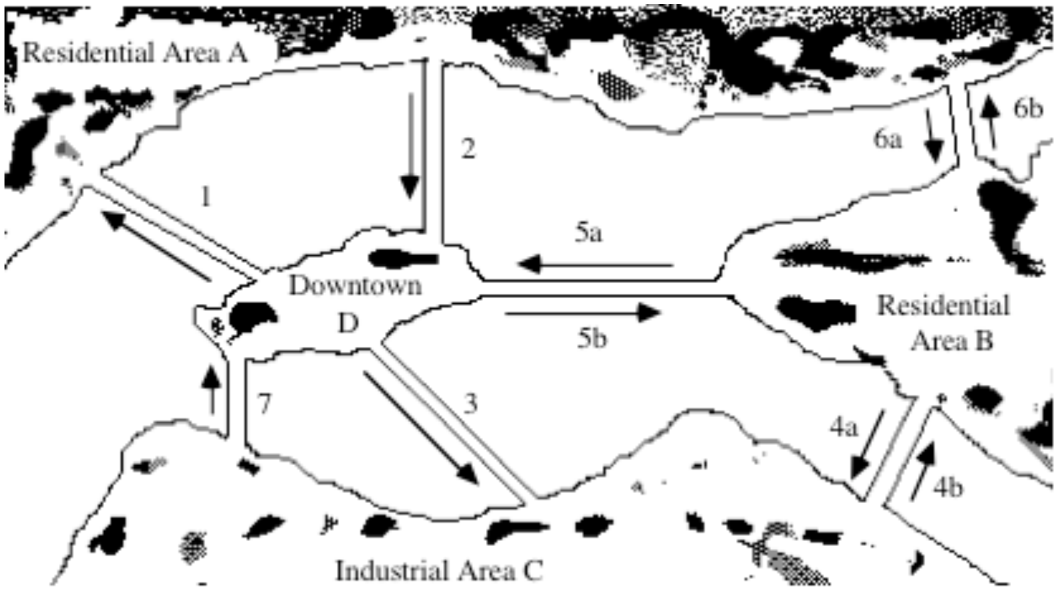


Figure 10: One-way and two-way bridges of Our Town

Exploration

The high school in Our Town is located on the downtown island. On your way home from school one afternoon, you discover that the bridge connecting your neighborhood to the downtown area is closed due to an accident. What alternate route would you use to get home as quickly as possible? In this exploration, you use matrices to examine the possibilities.

Mathematics Note

A directed graph or **digraph** is a graph in which each edge indicates a single direction.

Figure **11** shows one example of a digraph. Notice that vertices *A* and *C* are connected by paths in both directions.

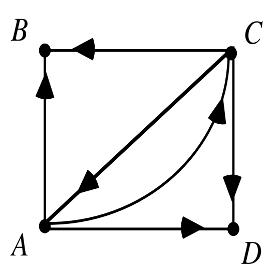


Figure 11: A digraph

a. Using a copy of the map in Figure **10** supplied by your teacher, create a digraph of the bridges of Our Town showing the possible directions of traffic. Let each vertex represent one of the four areas of Our Town.

b. A one-bridge route requires only one bridge crossing to get from one area to another. Table **2** represents a partially completed matrix of one-bridge routes from each area of Our Town to every other area of the town. For example, the element in column B, row A indicates that from area A to area B there is 1 one-bridge route. Use your digraph to complete the matrix.

Table 2: Matrix of one-bridge routes (all bridges open)

		to			
	Area	**A**	**B**	**C**	**D**
	A	0	1	0	1
from	**B**		0		
	C			0	
	D				0

c. Suppose that bridge 5 is closed for repairs. In this case, it would be impossible to drive directly from area D to area B. Using bridge numbers, describe all alternate routes from area D to area B that cross:

1. two bridges
2. three bridges.

d.
1. Delete edge 5b from your digraph in Part **a** to create a digraph representing the traffic flow when bridge 5b is closed.
2. Create a matrix like the one in Part **b** to represent the one-bridge routes between areas of Our Town after bridge 5b has closed.

e.
1. Use your digraph from Part **d** to count the number of two-bridge routes between each area of Our Town. Use these values to create a matrix representing the two-bridge routes. **Note**: A two-bridge route requires exactly two bridge crossings. The two crossings can be over the same bridge.
2. In your two-bridge matrix, identify the element that represents the number of two-bridge routes from D to B. How does this number compare to your response to Part **c1**?

f.
1. Square the matrix you created in Part **d2**.
2. Compare the information in the squared matrix with the values in your matrix from Part **e1**.

g.
1. Cube the matrix created in Part **d2**.
2. Identify the element in row D, column B of the cubed matrix. How does this number compare to your response to Part **c2**?

Discussion

a. If the matrix of one-bridge routes in Table **2** were squared, what information would be obtained?

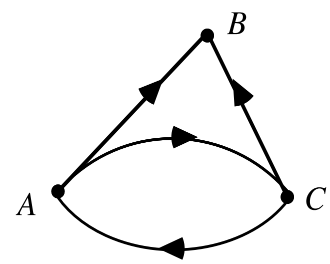
b. If the matrix of one-bridge routes were cubed, what information would be obtained?

c. If the matrix of one-bridge routes were raised to the nth power, what information would be obtained?

d. In Part **g** of the exploration, you cubed the matrix of one-bridge routes after bridge 5b had closed. The elements in the resulting matrix represent the number of three-bridge routes when bridge 5b is closed. Are each of these routes also paths?

Assignment

3.1 **a.** Create a matrix of one-edge routes for the digraph below.



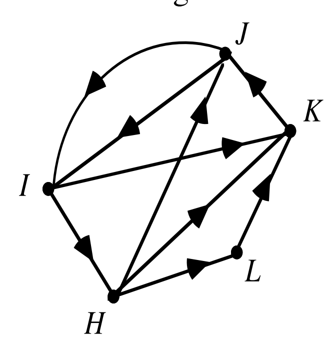
b. Create a matrix of two-edge routes for this digraph.

c. List all the possible two-edge routes from C to B.

d. Create a matrix of three-edge routes for the digraph.

e. List all the possible three-edge routes from C to B.

3.2 **a.** Create a matrix of one-edge routes for the digraph below.



b. Create a matrix of two-edge routes for this digraph.

c. List all the possible two-edge routes from J to J.

d. Create a matrix of three-edge routes for this digraph.

e. List all the possible three-edge routes from J to J.

3.3 Our Town is planning to build a new hospital. Before selecting a site, city planners must analyze the traffic flow to each potential location. Since traffic moves more slowly over bridges, they would like to maximize the number of one-bridge routes to the hospital. However, since individual bridges are often blocked or congested, they would also like to maximize the number of two-bridge routes.

a. Imagine that you are a city planner in Our Town. Using a copy of the map in Figure **10** supplied by your teacher, recommend an area to build the new hospital.

b. Prepare a presentation to the town council that supports your recommendation.

3.4 As suggested in the exploration, bridge 5b, the oldest bridge in Our Town, is often closed for repairs. With this in mind, the city planner must consider how hospital access would be affected if bridge 5b were closed.

a. Using a copy of the map in Figure **10** supplied by your teacher, recommend a site for the new hospital for which access will be least affected when bridge 5b is closed.

b. Prepare a presentation to the town council that supports your recommendation.

3.5 Draw a digraph that corresponds with the matrix of one-edge routes below.

$$\text{from}\ \begin{array}{c} \\ H \\ I \\ J \\ K \end{array}\!\!\begin{array}{c} \text{to} \\ \begin{array}{cccc} H & I & J & K \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

* * * * *

3.6 Use the matrix and digraph from Problem **3.5** to complete Parts **a–d**.

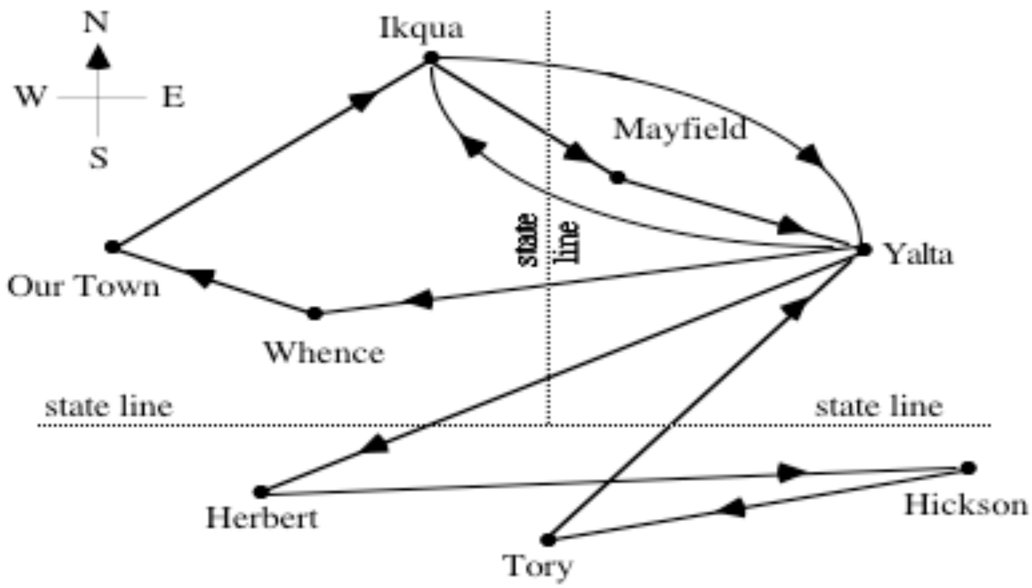
a. Describe all the two-edge routes from I to J.

b. Describe all the two-edge routes from J to H.

c. Determine the number of three-edge routes that exist between each pair of vertices.

d. Describe all the three-edge routes from J to I.

3.7 As illustrated in the digraph below, Friendly Skies Airlines offers commuter service to and from the cities near Our Town.



a. Create a matrix of the one-edge routes in this digraph.

b. Which city has the most direct flights to the other cities? Describe how your matrix supports your response.

c. A company new to the region around Our Town must decide where to locate its corporate headquarters. During a typical week, each member of its sales staff travels to a different city each day from Monday to Wednesday, then returns home on Thursday night. On Friday, all company employees must report to the home office.

1. Select the best location for the corporate headquarters and explain how you made your choice.

2. Using the location you recommended in Step **1**, create a schedule that shows the possible weekly itineraries for the members of the company's sales staff. The schedule should include the day, the town from which the person is leaving, and the town to which the person is traveling.

* * * * * * * * * *

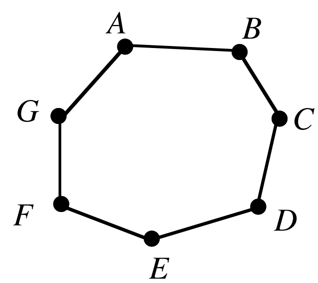
Summary Assessment

1. Imagine that you are the president of a computer company in Our Town. A law firm has hired your company to install a computer network for their seven attorneys. Using the network, each attorney must be able to communicate with every other member of the firm.

a. Draw a graph showing the necessary connections between the seven lawyers.

b. Why does the graph of the connections in Part **a** need to be a complete graph?

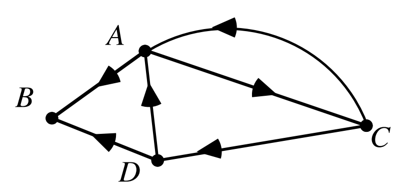
c. The actual wiring of a computer network may differ from the communication paths. The graph below shows a possible wiring diagram for a seven-computer network.



Does this graph contain a Hamiltonian circuit? Explain your response.

d. Is the graph in Part **c** traversable? Explain your response.

2. **a**. Create a matrix of the one-edge routes between each pair of vertices in the digraph below.



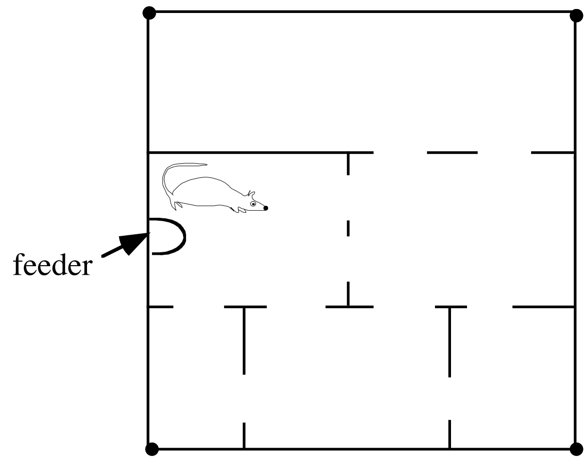
b. How would you determine the number of two-edge routes that exist between each pair of vertices?

c. How would you determine the number of three-edge routes that exist between each pair of vertices?

d. Beginning from vertex A, is there a closed path that traverses every edge exactly once? If not, add one or more directed edges so that the graph does contain such a path.

e. Does the graph contain a Hamiltonian circuit that begins at vertex *A*? If not, add one or more directed edges so that the graph does contain such a circuit.

3. To study the intelligence of mice, a biologist has designed the maze shown in the diagram below.



a. In one experiment, a mouse starts at the feeder. It is rewarded only if it passes through every doorway exactly once before returning to the feeder. What type of path must the mouse traverse to earn a reward?

b. Is it possible for the mouse to earn a reward in the experiment described in Part **a**? Explain your response.

c. In another experiment, the mouse starts at the feeder and is rewarded only if it passes through every room exactly once before returning to the feeder. What type of path must the mouse traverse to earn a reward?

d. Is it possible for the mouse to earn a reward in the experiment described in Part **c**? Explain your response.

e. By adding or subtracting doors, create a maze in which it is possible for the mouse to earn a reward in the experiment in Part **c** but not possible in the experiment in Part **a**.

Module Summary

- A **graph** is a non-empty set of **vertices** and the **edges** that connect them. A **loop** is an edge that connects a vertex to itself. A pair of vertices may be connected by more than one edge.
- A **simple graph** is a graph without loops in which any pair of vertices has at most one connecting edge.
- A **complete graph** is a simple graph in which every distinct pair of vertices is connected by exactly one edge.
- The **degree** of a vertex is the number of edges that meet at that vertex. If the number of edges that meet at a vertex is even, then the vertex has an **even degree**. If the number of edges that meet at a vertex is odd, then it has an **odd degree**.
- A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices can occur more than once in a path.
- A graph is **connected** if paths connect each vertex to every other vertex in the graph.
- A **closed path** is a path that starts and stops at the same vertex.
- A **circuit** is a closed path in which no intermediate vertex is repeated.
- A **Hamiltonian circuit** is a circuit in which every vertex in a graph is visited exactly once.
- A connected graph is **traversable** if it is possible to traverse every edge of the graph exactly once.
- A directed graph or **digraph** is a graph in which each edge indicates a single direction.

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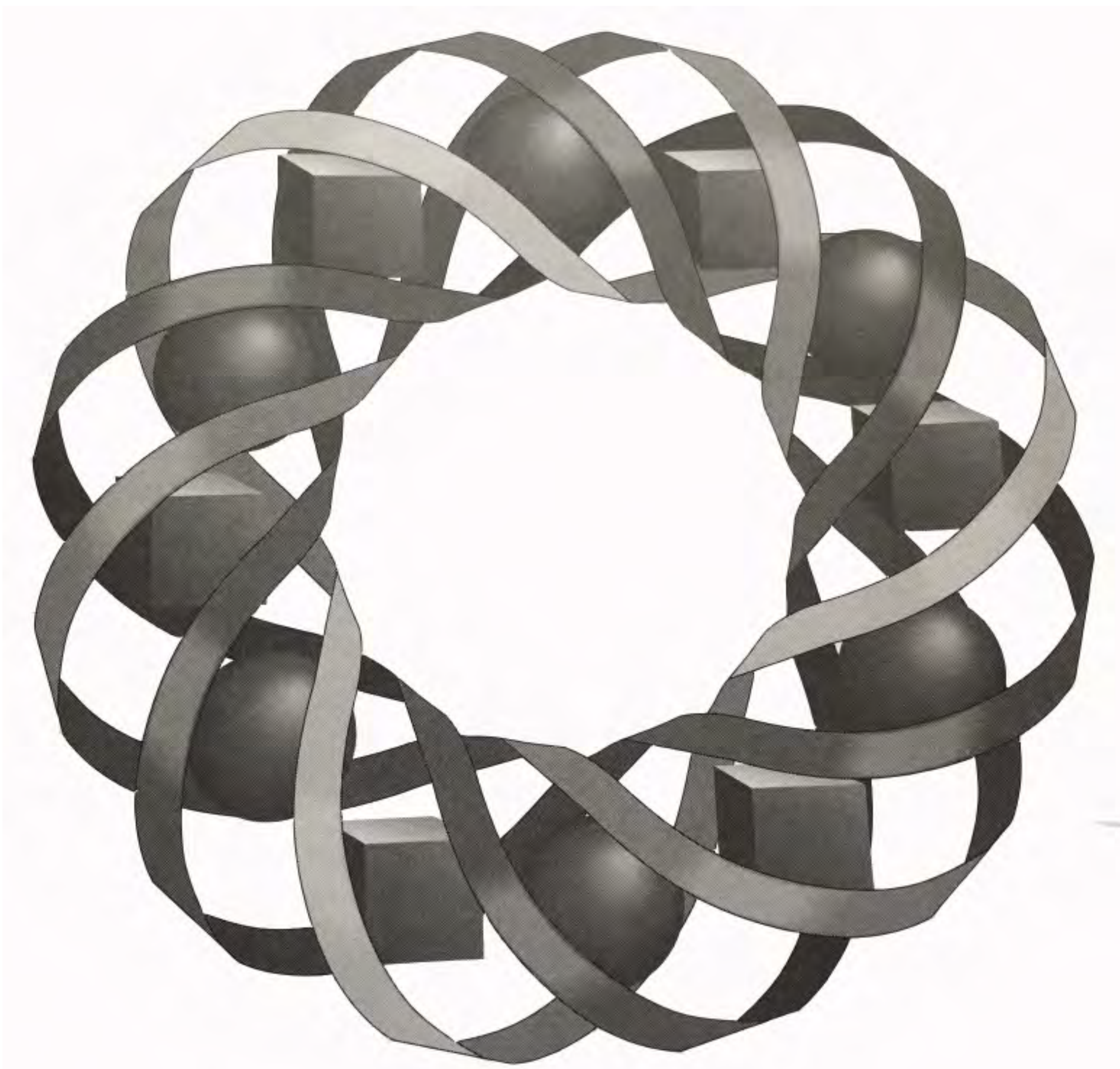
One Dish and Two Cones



Shapes based on conic sections are commonplace in our technological society. In this module, you learn to recognize conic shapes and investigate some of their properties.

Wendy Driscoll • Mary Ann Miller • Deanna Turley

One Dish and Two Cones

Introduction

In today's fast-paced world of telecommunications, satellite dishes have become common sights. They sprout from backyards and office buildings, roam the streets on mobile transmission trucks, and bring a world of information to schools and communities. How does the shape of a satellite dish affect how the dish works? You can answer this question by investigating **conic sections**.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a right circular cone. Depending on the slope of the plane, the intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**, as shown in Figure **1** below.

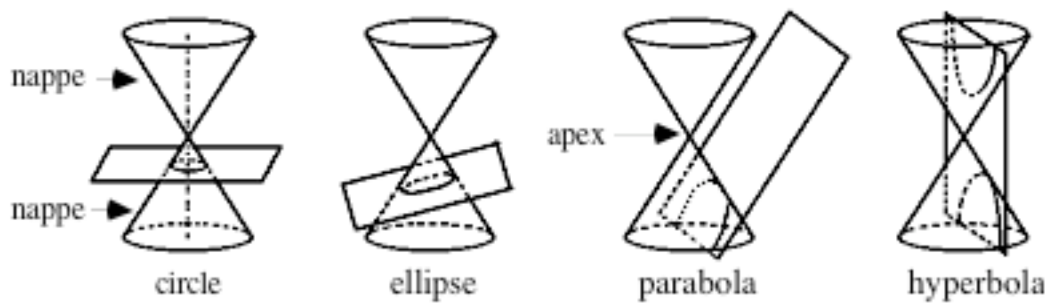


Figure 1: Conic sections

Mathematicians have written about these shapes for well over 2000 years. In fact, the Greek geometer Appollonius of Perga wrote eight books on conic sections in the third century B.C. Since that time, mathematicians have continued to investigate and explore the special properties of these curves.

Exploration

A flashlight produces a partial cone of light. As shown in Figure **2**, the light defines a shape when it strikes a flat surface.

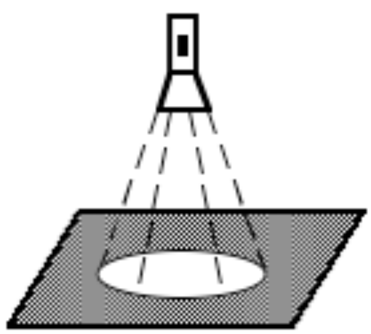


Figure 2: Flashlight, flat surface, and cone of light

a. Use a flashlight to create a partial cone of light. Holding the flashlight still, position a flat surface so that the light forms a circle.

b. Record the relationship between the cone of light and the position of the flat surface.

c. Repeat Parts **a** and **b** to form an ellipse, a parabola, and one-half of a hyperbola.

Discussion

a. Describe how to create each of the following conic sections using a flashlight and a flat surface:

1. a circle
2. an ellipse
3. a parabola
4. one-half of a hyperbola.

b. When the intersection of a plane and a cone contains the cone's apex, geometric figures other than a circle, an ellipse, a parabola, or a hyperbola are formed. These other intersections are **degenerate conic sections**.

1. Describe the geometric shapes of the three degenerate conic sections.
2. For each one, describe the location of the plane relative to the cone.

Activity 1

Conic sections can be found in many real-world situations and are used in many practical applications. The orbits of the planets in our solar system, for example, are elliptical. The shape of a parabola appears in automobile headlights, telescopes lens, microwave transmitters, and satellite dishes. In this activity, you continue your exploration of conics through paper folding.

Exploration 1

a. Cut out the circle template provided by your teacher.

b. Mark a point P inside the circle but not at the center.

c. Select a point on the circle and label it P_1. Label the remaining points P_2 through P_{20}.

d. As shown in Figure **3**, fold and crease the paper so that P_1 coincides with P.

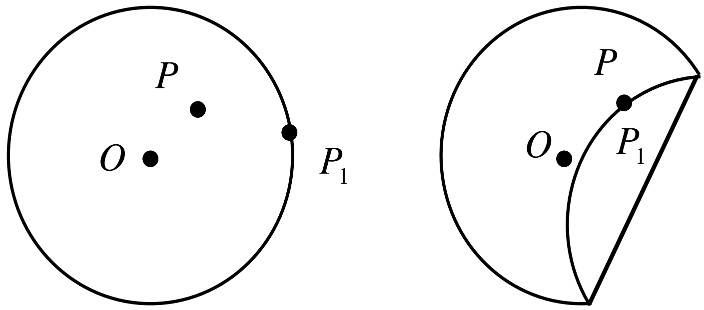


Figure 3: Folding paper so that P_1 coincides with *P*

e. Unfold the paper. As shown in Figure **4**, use a straightedge to mark a point where the crease intersects $\overline{OP_1}$, a radius of circle *O*. Label this point X_1.

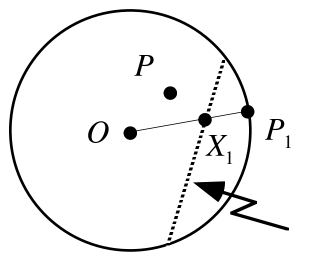


Figure 4: Marking point X_1

f. Repeat Parts **d–e** for each labeled point on the circle. This creates the points X_2, X_3, X_4, …, X_{20}.

g. Draw a smooth curve by connecting X_1 through X_{20}. **Note:** Save your construction for reference later in the module.

Discussion 1

a. What conic section results from the process described in Exploration **1**?

b. Would you expect this conic to be exactly the same for everyone in the class? Explain your response.

c. What happens to the shape of the conic when *P* is moved closer to the center of the circle?

d. In Part **c** of Exploration **1**, you formed a crease by folding the paper so that P_1 on the circle coincides with *P*. Describe how the line represented by the crease is related geometrically to $\overline{PP_1}$.

Mathematics Note

Each conic separates a plane into three regions: the conic itself, an "interior," and an "exterior." In Figure **5** below, the interior of each conic is shaded. Note that the boundaries are not included in the interior, and that a hyperbola has two branches.

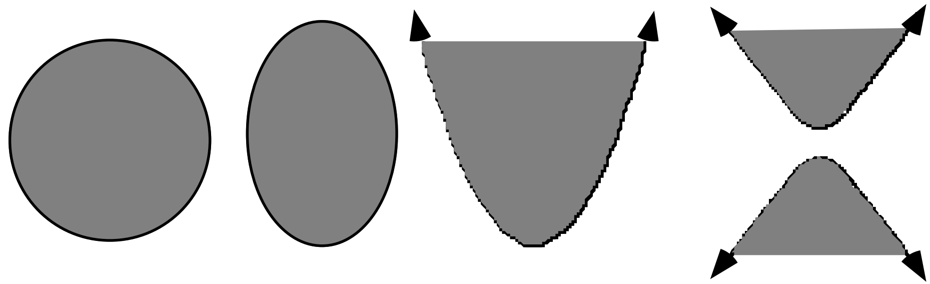


Figure 5: Interiors of a circle, ellipse, parabola, and hyperbola

A **tangent line** to a conic is a line in the plane of the conic that intersects the curve at exactly one point and contains no points in the interior. The point at which the conic and the tangent line intersect is the **point of tangency**.

For example, Figure **6** shows a line tangent to a circle. A radius drawn to the point of tangency on a circle is perpendicular to the tangent line.

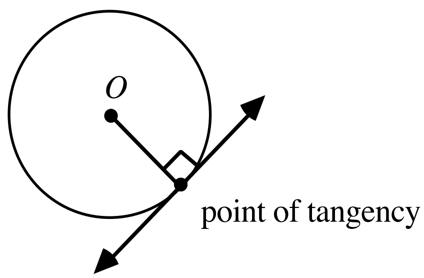


Figure 6: A tangent line to circle *O*

e. How are the lines represented by creases in the paper related to the conic section?

f. What do the points $X_1, X_2, X_3, \ldots, X_{20}$ represent in relation to the lines described in Part **e** of Discussion **1**?

Exploration 2

In this exploration, you use a geometry utility to create three of the four conics. This method of constructing conics is directly related to the paper-folding you did in Exploration **1**.

a. Construct a large circle and label its center O. Construct a point P_1 on the circle so that it moves freely around the circle without changing the circle's size.

b. Construct a point P in the interior of the circle, but not at the center.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct $\overleftrightarrow{OP_1}$.

f. Mark the intersection of the perpendicular bisector from Part **d** and $\overleftrightarrow{OP_1}$. Label this point X. Your construction should now resemble the one shown in Figure **7**.

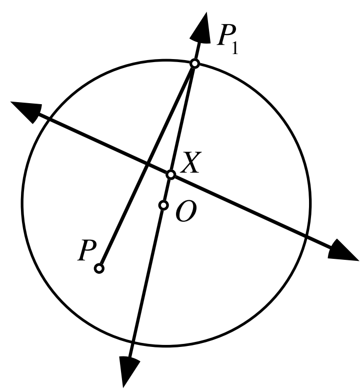


Figure 7: Beginning conic construction

g. Trace the path of X as P_1 is moved around the circle. Note which conic is formed by the path.

h. Move P to a different location inside the circle and repeat Part **g**. Experiment with several other locations of P, including the center of the circle as well as points on and outside the circle. **Note:** Save this construction for use in Activity **2**.

Discussion 2

a. Which step in Exploration **2** corresponds with forming a crease in the paper in Exploration **1**? Explain your response.

b. Which conic is formed when P is at each of the following locations?

1. inside the circle but not at the center

2. at the center of the circle

3. outside the circle

4. on the circle

c. In Exploration **2**, where are O and P located in relation to the interiors of each conic?

Assignment

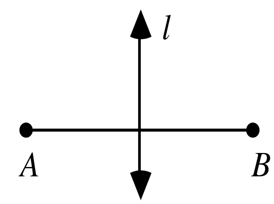
1.1 Complete Parts **a–c** below for each conic generated in Exploration **2**.

a. Describe or draw any axes of symmetry.

b. The **center** of each of these conics is its point of rotational symmetry. Describe the locations of the center in relation to the lines of symmetry.

c. Describe where O and P are located in relation to the center of each conic and its axes of symmetry.

1.2 In the diagram below, line l is the perpendicular bisector of $\overline{AB}$.



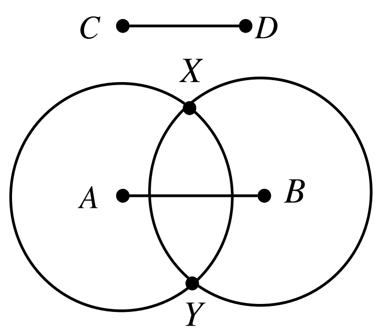
a. Draw a copy the diagram above. Select a point on line l and label it C.

b. Draw $\overline{AC}$ and $\overline{BC}$.

c. What conclusions can you make about the measures of $\overline{AC}$ and $\overline{BC}$? Explain your response.

d. What conclusions can you make about the measures of $\angle CAB$ and $\angle CBA$? Explain your response.

1.3 **a.** Use a geometry utility to create the diagram below. Read Steps **1–5** before beginning the construction.



1. Construct a line segment and label the endpoints A and B.
2. Construct another line segment and label the endpoints C and D.
3. Construct two different circles centered at A and B with radii CD.
4. Label the two intersections of the circles X and Y.
5. Trace the path of points X and Y as the length of $\overline{CD}$ changes.

b. The line created by tracing X and Y has certain properties in relationship to $\overline{AB}$. Describe these properties.

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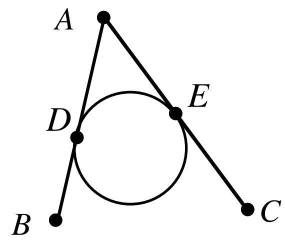
1.4 In the following diagram, $\overline{AB}$ and $\overline{AC}$ are tangent to the circle at points D and E, respectively.



a. Describe how you could use the tangents to locate the circle's center.

b. Describe the relationship between $\overline{AD}$ and $\overline{AE}$.

1.5 Consider a circle with center at point O. A line intersects the circle at points P and Q.

a. Can $\overline{QP}$ lie on a tangent to the circle? Explain your response.

b. Can $\overline{OP}$ be perpendicular to $\overline{QP}$? Explain your response.

* * * * * * * * * *

Activity 2

One way to define the conics is to consider each shape in geometric terms. In the following exploration, you use a geometry utility to investigate the relationship between the distances from some specific points not on the conics to points on the conics themselves.

Exploration

a. Using your construction from Exploration **2** of Activity **1,** begin with point P in the interior of the circle but not the center, as shown in Figure **8** below. Measure the radius $\overline{OP_1}$.

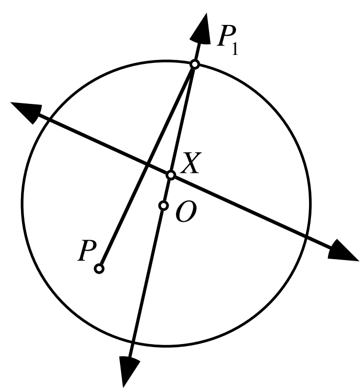


Figure 8: Beginning conic construction

b. Measure the distances from O to X and from P to X.

1. Find the sum of OX and PX.
2. Find the absolute value of the difference of OX and PX.

c. Retrace the path of X as P_1 is moved around the circle. Note the conic formed and describe what happens to the sum of OX and PX and the absolute value of the difference of OX and PX.

d. Repeat Parts **b** and **c** when P is located outside the circle.

e. Change the size of the circle, then repeat Parts **a–d**. Note any changes you observe in the sum of OX and PX and the absolute value of the difference of OX and PX as the conics are generated.

Discussion

a. As P_1 is moved around the circle with P located inside the circle, which value remains constant: the sum of OX and PX or the absolute value of the difference of OX and PX? How could you use this constant value to describe the conic generated?

b. When P is located at the center of circle in the exploration, the conic generated by the path of X is a circle. In this situation, what would be true about the sum of OX and PX and the absolute value of the difference of OX and PX?

c. Why do you think a circle can be thought of as a special case of an ellipse?

d. As P_1 is moved around the circle with P located outside the circle, which value remains constant: the sum of OX and PX or the absolute value of the difference? How could you use this constant value to describe the conic generated?

Mathematics Note

An **ellipse** is a set of all points in a plane such that the sum of the distances from any point in the set to two fixed points is a constant. Each of these fixed points is a **focus** (plural **foci**). Figure **9** shows one example of an ellipse. Points F_1 and F_2 are the foci, and $F_1X + F_2X$ is a constant (AB) for any point on the ellipse.

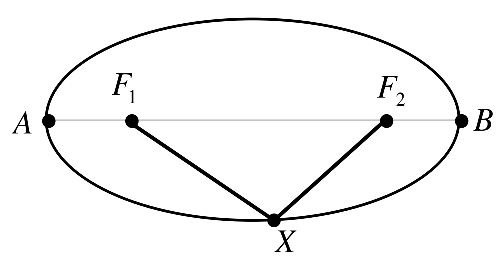


Figure 9: An ellipse

A **circle** is a set of all points in a plane equidistant from a given fixed point, the center. A circle is a special case of an ellipse, where both foci are located at the center of the circle.

A **hyperbola** is a set of all points in a plane such that the absolute value of the difference in the distances from any point in the set to two fixed points (the foci) is a constant. Figure **10** shows one example of a hyperbola. Points F_1 and F_2 are the foci, and $|F_1X - F_2X|$ is a constant (AB) for any point on the hyperbola.

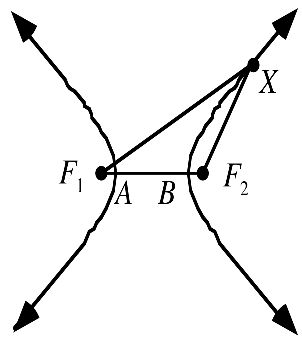


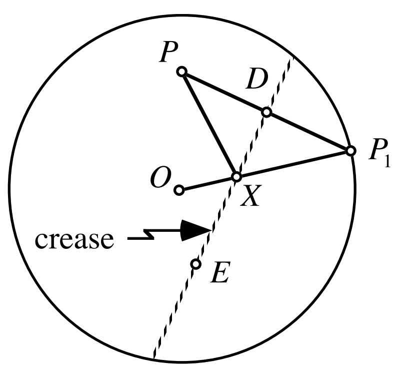
Figure 10: A hyperbola

e. Which points in the exploration correspond to the foci of an ellipse?

f. Which points correspond to the foci of a hyperbola?

Assignment

2.1 In Activity **1**, you used paper folding to model the construction of an ellipse, as shown in the diagram below. Suppose that the crease line in this diagram represents a mirror. As you may recall from the Level 1 module "Reflect on This," when a light ray strikes a flat mirror, the outgoing angle is congruent to the incoming angle.



Describe the path of a light ray passing through O and striking the mirror at X. Justify your response.

2.2 In the diagram given in Problem **2.1**, X is a point on the perpendicular bisector of $\overline{PP_1}$.

a. What is the relationship between the lengths of $\overline{PX}$ and $\overline{P_1X}$?

b. How would this relationship change if X were moved to another location along the crease? State your conclusion as a generalization about all points on the perpendicular bisector of a segment.

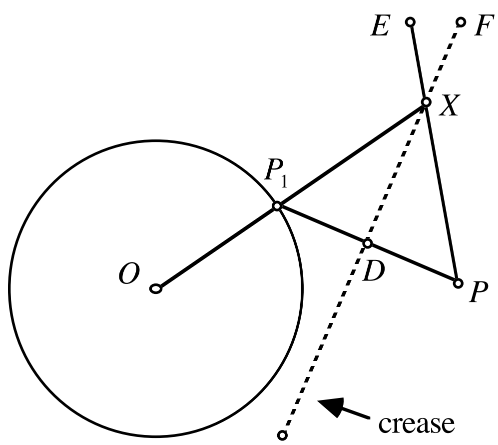
2.3 **a.** Paper folding can be used to demonstrate many geometric relationships in the conics. Use paper folding to model the construction of a hyperbola by completing Steps **1–5** below.

1. Draw a circle on a sheet of paper and label its center O.

2. Mark a point P outside the circle.

3. Select one of the points on the circle and label it P_1.

4. Fold and crease the paper so that P_1 coincides with P as shown in the following diagram.



5. Unfold the paper. As shown in the diagram, mark a point X where the crease intersects $\overleftrightarrow{OP_1}$ and a point D where the crease intersects $\overline{PP_1}$.

b. Write an argument showing that the crease is the perpendicular bisector of $\overline{PP_1}$. (Your argument must demonstrate that the crease bisects $\overline{PP_1}$ and is perpendicular to $\overline{PP_1}$.)

c. Write an argument showing that $\angle PXD$ is congruent to $\angle EXF$.

d. Suppose that the crease line represents a mirror. Describe the path of a light ray passing through E and striking the mirror at X. Justify your response.

2.4 Like the rest of the planets in our solar system, the earth's orbit is approximately an ellipse with the sun located at one focus. When the earth is located at its closest point to the sun, or at its farthest point from the sun, it lies along a line containing the foci of the elliptical path. At the closest point in its orbit, the earth is about $1.47 \bullet 10^8$ km from the sun. At the farthest point, the earth is about $1.52 \bullet 10^8$ km from the sun.

a. Determine the length of the segment that contains the foci of the ellipse and whose endpoints are on the ellipse.

b. What is the distance between the sun and the other focus?

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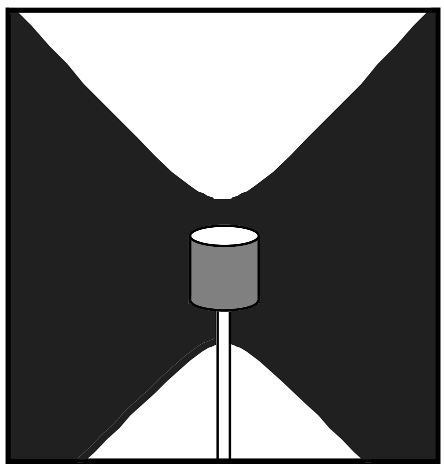
2.5 The shadow created by the lamp in the figure below forms a conic section. Identify this conic and justify your response.



2.6 According to Boyle's law, the volume (V) of a fixed amount of gas at constant temperature varies inversely with the pressure (P) on it. This relationship can be expressed as $PV = c$, where c is a constant.

a. A certain amount of oxygen gas occupies a volume of 500 cm^3 at a pressure of 0.978 atmospheres (atm). Find the constant c for this amount of oxygen.

b. Use the constant c from Part **a** to graph the equation $PV = c$. Represent volume on the vertical axis and pressure on the horizontal axis. Indicate an appropriate domain and range for this setting.

c. The graph in Part **b** is a conic section. Identify this conic and justify your response.

d. Determine the volume of the oxygen when the pressure is 1.5 atm.

e. Determine the volume of oxygen when the pressure is 0.75 atm.

f. If you double the pressure, what is the effect on the volume?

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Activity 3

As you observed in the previous activity, circles, ellipses, and hyperbolas can all be generated in a similar manner. In this activity, you examine the geometric properties of a parabola using a different approach.

Exploration

a. Using a geometry utility, construct a long, horizontal segment. Construct a point P_1 on the segment.

b. Construct a point P not on the segment.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct a line perpendicular to the horizontal segment through P_1.

f. Construct a point at the intersection of the line from Part **e** and the perpendicular bisector of $\overline{PP_1}$. Label this intersection X. Your construction should now resemble the one shown in Figure **11**.

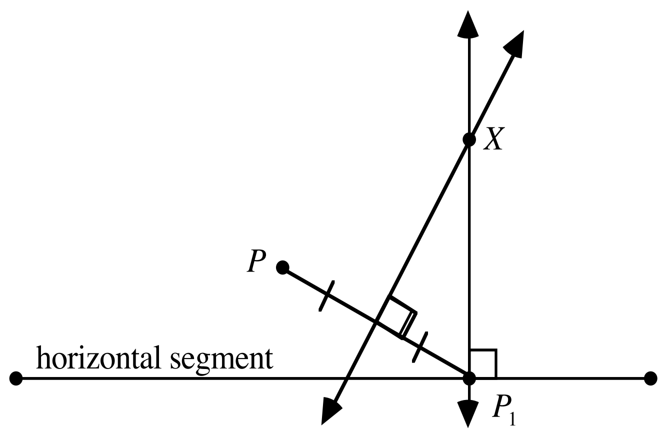


Figure 11: Completed construction

g. Trace the path of X as P_1 is moved along the horizontal segment from one endpoint to the other. Note the conic formed. As the path is traced, collect the coordinates of point X at 10 locations.

h. Measure the distance from P to X and from X to P_1. Observe how these distances compare as P_1 moves along the horizontal segment.

i.
1. Create a scatterplot of the data collected in Part **g**.
2. Recall that the graph of a quadratic function is a parabola. Determine a function that models the data.

j. Relocate point P and repeat Parts **g** and **i**. Observe how the location of P affects the shape of the conic formed. **Note:** Save your work for use in the assignment and in the Exploration in Activity **4**.

Discussion

a. Describe the differences between the construction that results in a parabola and the one that results in an ellipse.

b. How is the perpendicular bisector of $\overline{PP_1}$ related to the parabola?

c. Describe the relationship among the perpendicular bisector of $\overline{PP_1}$, point X, and the parabola.

d. In the exploration, $\overline{PX}$ and $\overline{P_1X}$ are congruent. Why is this true?

Mathematics Note

A **parabola** is the set of all points in a plane equidistant from a line and a point not on the line. The line is the **directrix** of the parabola. The point is the **focus** of the parabola.

In Figure **12**, for example, the directrix of the parabola is line l and the focus is point F. The distances from any point P on the parabola to F and l are equal.

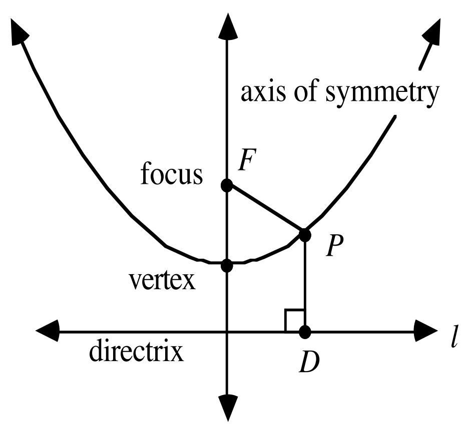


Figure 12: A parabola

e. Describe any symmetries you observe in the parabola in Figure **12**.

f. The **vertex** of a parabola occurs at the point where the axis of symmetry intersects the parabola. How could you use the coordinates of the vertex, the equation of the axis of symmetry, and the distance from the focus to the directrix to locate the focus and directrix?

Assignment

3.1 Use your construction from the exploration to complete Parts **a** and **b**.

a. Describe what happens to the parabola when the focus is moved closer to the directrix.

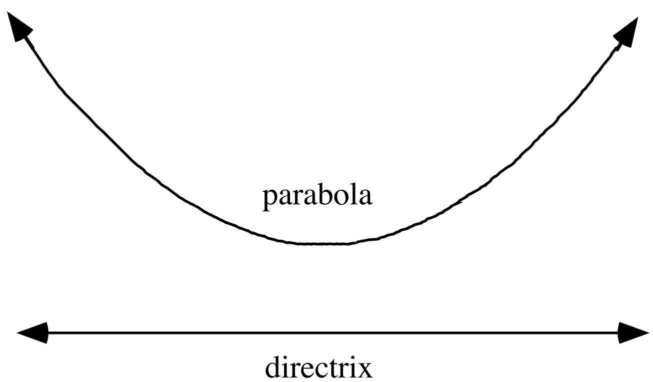
b. Describe what happens to the parabola when the focus is moved away from the directrix.

3.2 **a.** Adapt the paper-folding process from Activity **1** to construct a parabola with a sheet of paper. Describe the new procedure.

b. Describe the axis of symmetry of the parabola you created in Part **a**.

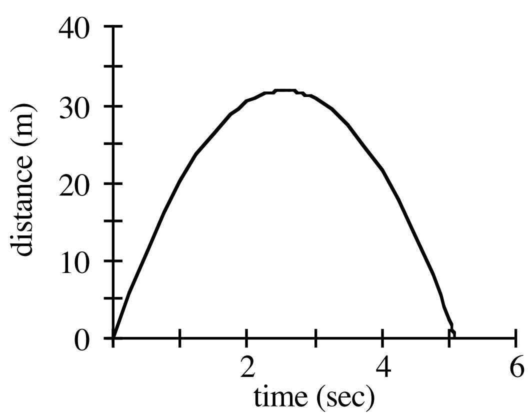
c. Where are the focus and the directrix located in relation to the axis of symmetry?

3.3 The diagram below shows a parabola and its directrix. Describe how you could locate the focus of this parabola.



* * * * *

3.4 When a ball is thrown into the air, the graph of its distance above the ground versus time is a parabola. The graph below shows the distance-time graph of a ball thrown straight upward with an initial velocity of 25 m/sec. The ball returns to the ground after approximately 5.1 sec.



a. What is the equation of this parabola's axis of symmetry? Justify your response.

b. Estimate the coordinates of the parabola's vertex. What does this point represent in terms of the motion of the ball?

c. Compare the location of the focus and the directrix for this parabola with those of the parabola in Problem **3.3**.

d. Write an equation that describes the parabola's distance above the ground as a function of time.

* * * * * * * * * *

Activity 4

Many of the practical applications of conics are a result of their reflective properties. When light reflects off a flat mirror, as shown in Figure **13**, the outgoing angle is congruent to the incoming angle.

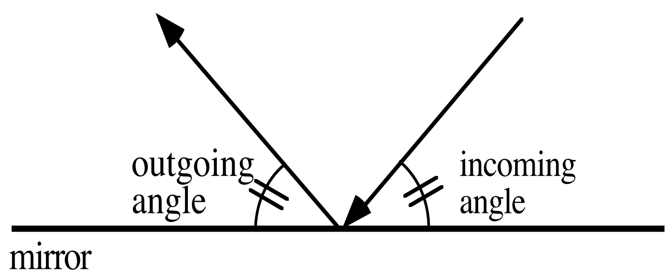


Figure 13: Light reflecting off a flat mirror

But how does light reflect off curved surfaces? In this activity, you examine the reflective characteristics of curved surfaces whose shapes are based on conic sections.

Exploration

In this exploration, you examine a method for determining the angle of reflection for curved surfaces.

a. Using a geometry utility, construct a circle with center at point O.

b. Mark a point X on the circle. Construct and measure $\overline{OX}$.

c. A tangent line to a circle is perpendicular to the radius at the point of tangency. Construct the tangent line to the circle at X.

d. Use the geometry utility to simulate a magnified view of the circle and tangent line near X.

e. Observe how the circle and the tangent line compare near the point of tangency. Record your observations.

f. Again magnify the view of the circle and tangent line near X. Record your observations.

Discussion

a. Imagine that both the tangent line and the circle are mirrors and that a ray of light strikes them very near the point of tangency, as shown in Figure **14**. Describe how you think the light will reflect off each surface.

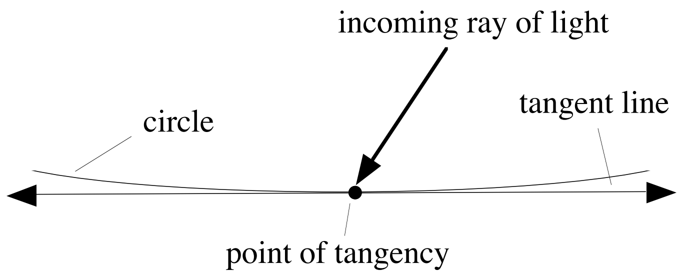


Figure 14: Light striking two mirrors

Mathematics Note

As illustrated in Figure **15**, a portion of a curve near a point of tangency has the characteristics of its tangent line. This property is called **local linearity**. The reflective properties of a point on a curve are the same as the reflective properties of a line tangent to the curve at that point.

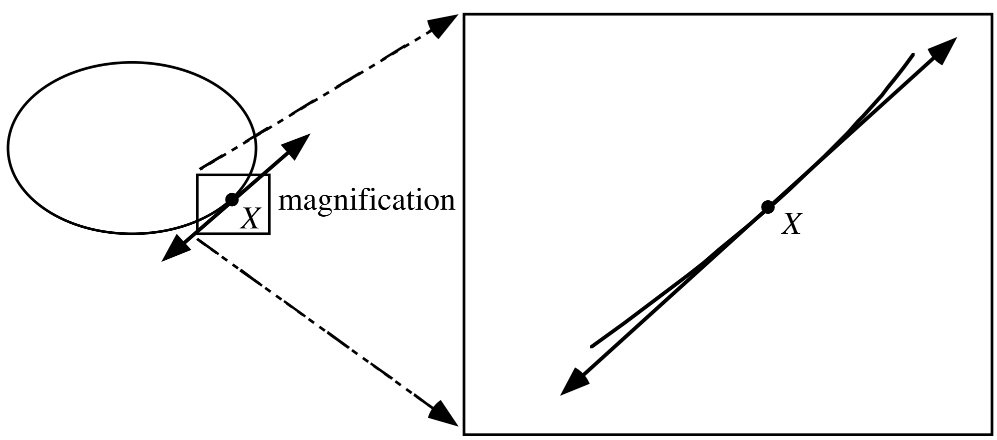


Figure 15: Magnification of a curve and tangent line

b. **1.** Imagine that the parabola in Figure **16** is a cross section of a curved mirror and that a ray of light passing through A strikes the parabola at X. Describe where you think the ray of light will be reflected.

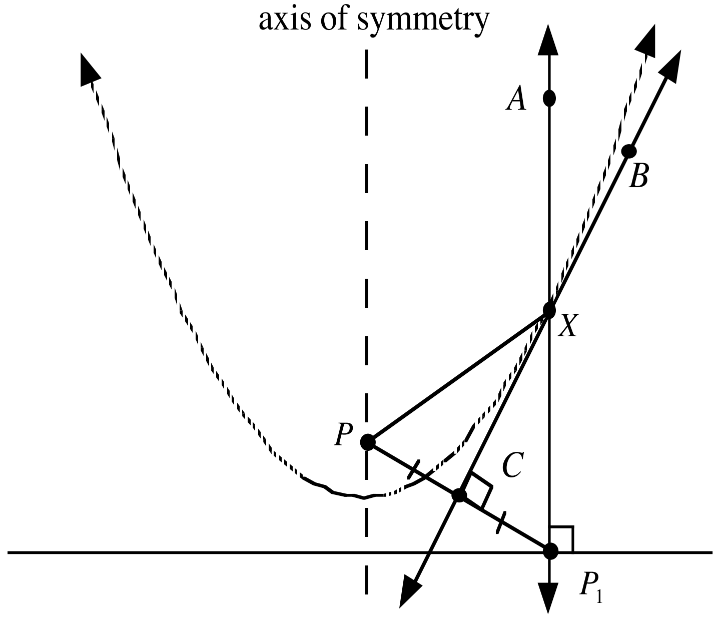


Figure 16: Cross section of a parabolic mirror

2. Suppose that a light bulb is positioned at point P in Figure **16**. What do you think will happen to the light rays emanating from the bulb?

c. In Activity **1**, you used a circle with center at point O to construct an ellipse, as shown in Figure **17**.

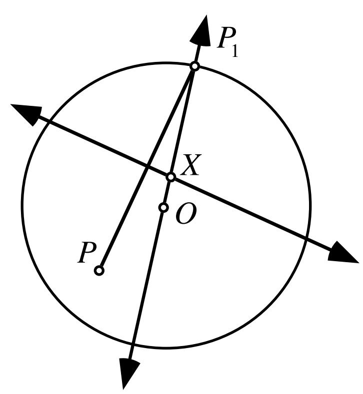


Figure 17: Construction for an ellipse

Suppose that this circle were extremely large.

1. Up close, what would a small piece of this circle look like?

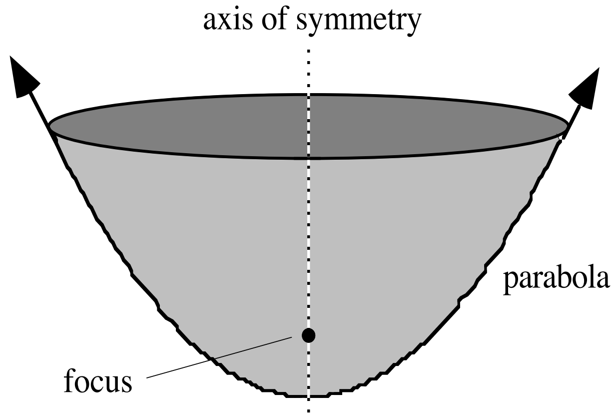
2. How would $\overline{OP_1}$ appear to be related to the circle?

3. What would a tangent line to the circle at P_1 look like?

4. As P_1, moves around the circle, what conic would appear to be formed by the path of point X?

Assignment

4.1 **Conicoids** are three-dimensional objects produced by rotating a conic around the axis of symmetry that contains the focus or foci. As shown in the figure below, a **paraboloid** is created by rotating a parabola around its axis of symmetry. Many satellite dishes are paraboloids.



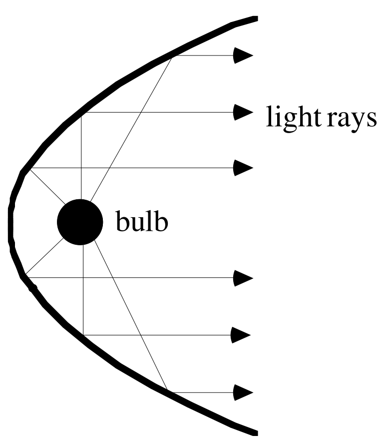
a. What do you think happens to the location of the focus as the parabola is rotated?

b. Satellite dishes are paraboloids. Based on your knowledge of the reflective properties of a parabola, explain how satellite dishes collect and concentrate television signals. Use a diagram in your explanation.

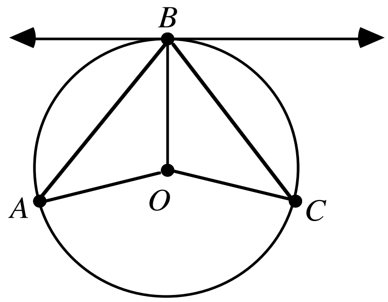
c. If a receiver collects signals sent to the satellite dish, where would you place it in the dish? Explain your response.

d. Why is the direction in which a satellite dish faces important to signal reception?

4.2 Paraboloids are also used in automobile headlights. As shown in the diagram below, a paraboloid around the bulb directs light rays parallel to the axis of symmetry. Use the reflective properties of a parabola to describe the best location for the bulb.



4.3 Consider a mirror whose cross section is a circle, as shown in the diagram below. When a light source at point *A* is directed at point *B*, the ray of light is reflected to point *C*.

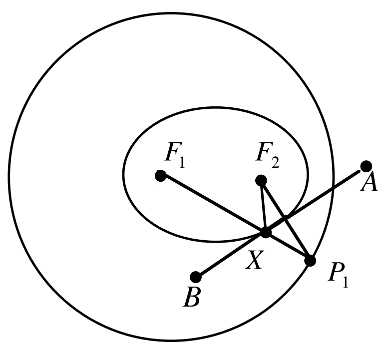


a. Use a geometry utility to investigate the relationship between $m\angle ABC$ and $m\angle AOC$.

b. With the light source at *A*, what must the measure of the incoming angle be so that *A*, *B*, and *C* form the vertices of an equilateral triangle?

c. With the light source at *A*, what must the measure of the incoming angle be to form other regular polygons?

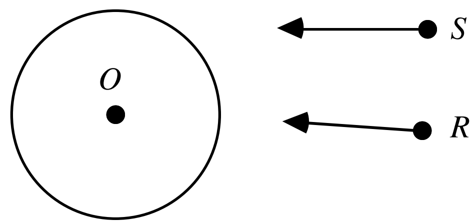
4.4 The following diagram shows the segments used in Activity **2** to construct point *X* on an ellipse. Use this diagram to describe the reflective properties of an ellipse for light emanating from one of the foci, F_1 or F_2.



4.5 In a paragraph, describe some applications for which the reflective properties of an ellipse might be put to use.

* * * * *

4.6 A sphere is a conicoid produced by rotating a circle about a diameter. The figure below shows a cross section of a sphere with a mirrored exterior and two sources of light. The light source at R is pointed toward the center O. The light source at S is not.



a. Describe how each ray of light will reflect off the mirror, including sketches of the incoming and outgoing rays.

b. Imagine that the inside of the sphere also has a mirror. Describe how light emanating from point O would reflect inside the sphere, including a diagram to support your response.

4.7 A regular dinner spoon has some interesting reflective properties. When you look at the concave side of the spoon, for example, your reflection is inverted. When you look at the convex side, however, your reflection is not inverted. Use a diagram to help explain why this occurs.

* * * * * * * * * *

Research Project

In Activity **2**, you used a geometry utility to construct a hyperbola. Use your construction to investigate the reflective properties of a hyperbola. Write a report explaining your findings and describe how these reflective properties might be used in practical applications. Include diagrams where appropriate.

Summary Assessment

1. Conic graph paper is printed with two sets of evenly spaced concentric circles and can be useful for sketching hyperbolas and ellipses. The centers of the circles are used to represent the foci.

Using the definition of a hyperbola and a sheet of conic graph paper, sketch a hyperbola such that the difference in the distances between two fixed points and any point on the hyperbola is:

a. 8

b. 4

2. Using the definition of an ellipse and a sheet of conic graph paper, sketch an ellipse such that the sum of the distances between two fixed points and any point on the ellipse is:

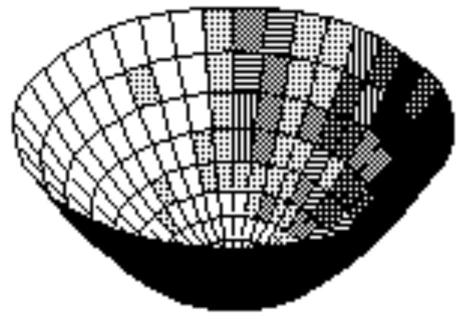
a. 12

b. 16

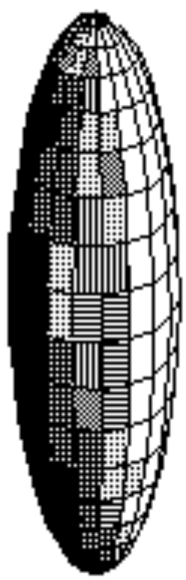
3. Suppose that you were the lighting director of a theater. Describe the shape of the lamp you might use to illuminate an actor's face on the stage. Explain why you chose this particular shape.

4. Write a geometric definition of each conicoid in the diagram below and describe its reflective properties.

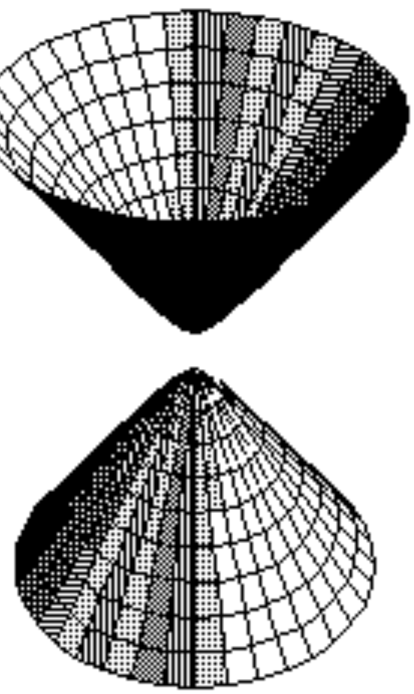
a. Sphere



b. Paraboloid



c. Ellipsoid



d. Hyperboloid

Module Summary

- A **conic section** can be formed by the intersection of a plane with a cone. Depending on the slope of the plane, the intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.
- A **tangent** line to a conic is a line in the plane of the conic that intersects the curve at exactly one point and contains no points in the interior. This intersection is the **point of tangency**.
- The **interior** of a conic is the region or regions of a plane containing the focus (or foci) of the conic.
- A radius drawn to the point of tangency on a circle is perpendicular to the tangent line at that point.
- An **ellipse** is a set of all points in a plane such that the sum of the distances from any point in the set to two fixed points is a constant. Each of these fixed points is a **focus** (plural **foci**).
- A **circle** is a set of all points in a plane equidistant from a given fixed point, the center. A circle is a special case of an ellipse, where both foci are located at the center of the circle.
- A **hyperbola** is a set of all points in a plane such that the absolute value of the difference in the distances from any point in the set to two fixed points (the foci) is a constant.
- A **parabola** is the set of all points in a plane equidistant from a line, the **directrix**, and a point not on the line, the focus.
- A portion of a curve near the point of tangency has the characteristics of its tangent line. This property is called **local linearity**. The reflective properties of a point on a curve are the same as the reflective properties of a line tangent to the curve at that point.
- **Conicoids** are produced by rotating the corresponding conic around the axis of symmetry that contains the focus or foci.

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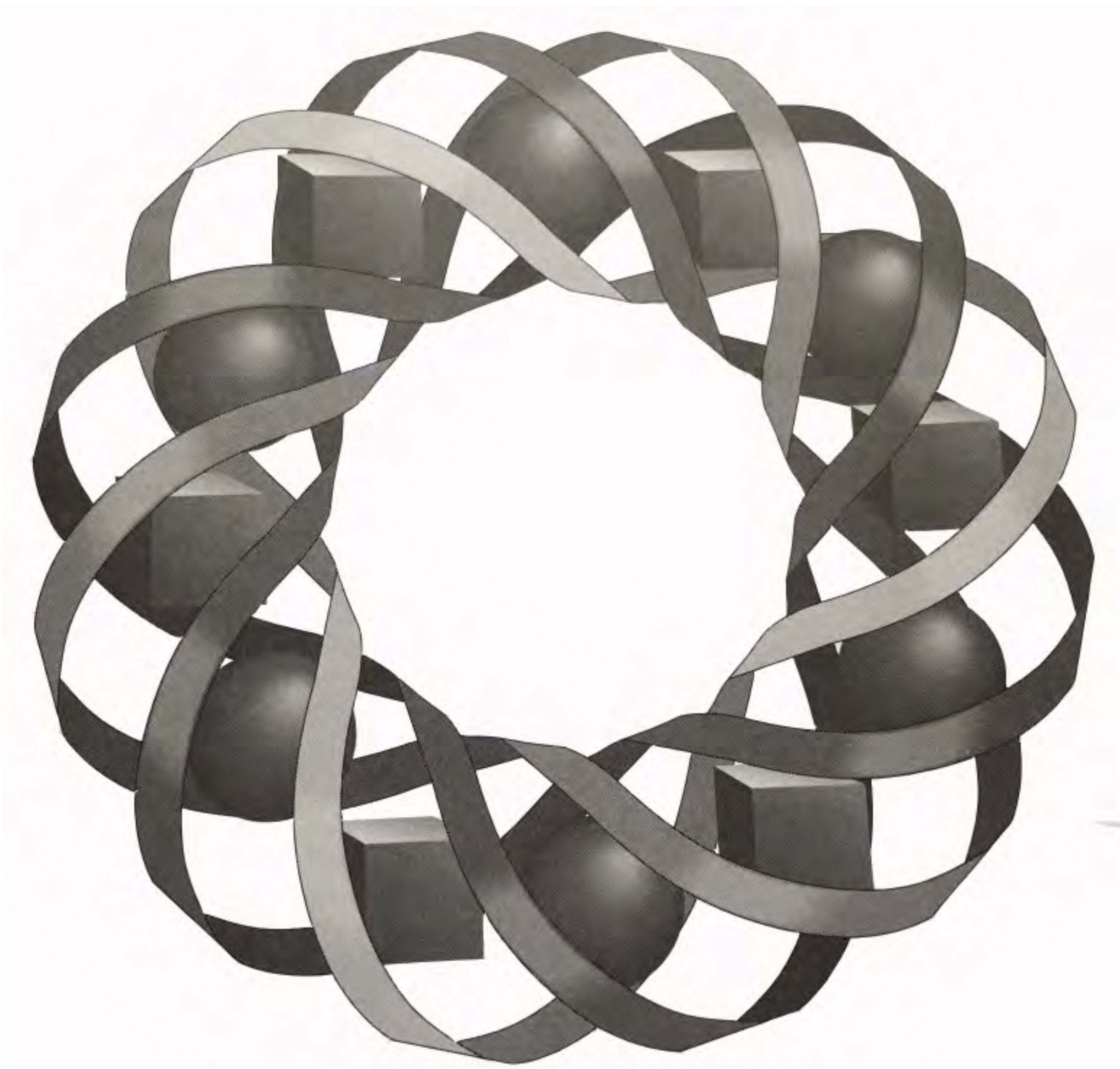
Finding Gold



In this module, you investigate some connections among the golden section, the Fibonacci sequence, and Pythagorean triples.

Tom Teegarden • Deanna Turley

Finding Gold

Introduction

During the period from the sixth century B.C. to the fifth century A.D., Greek scholars raised mathematics to a higher level than ever before in the Middle East or in Europe. One of the legendary figures of Greek mathematics was the philosopher and mystic, Pythagoras (ca. 580–500 B.C.). Historians credit Pythagoras and his followers with many important achievements.

After the fall of the Roman empire, Europe entered a period known as the Dark Ages. During these times, few advances were made in the studies of art, mathematics or science. Not until about 1000 A.D. did widespread interest in mathematical knowledge begin to re–emerge. An Italian merchant named Leonardo of Pisa (ca. 1180–1250), better known as Fibonacci, was an important contributor to this revival.

In this module, you examine some of the mathematical contributions of both Pythagoras and Fibonacci.

Activity 1

Many mathematicians have studied objects and shapes that can be characterized by specific ratios. For example, another Greek mathematician and scientist, Archimedes (ca. 287–212 B.C.), used circles to develop an approximation for π, the ratio of the circumference of a circle to its diameter. In the following exploration, you discover a classical ratio in some rectangles.

Before beginning the exploration however, consider the rectangles in Figure **1**. Which of these rectangles looks "most pleasing" to you? To determine if others share your preference, use the template supplied by your teacher to survey at least 10 other people and record your results.

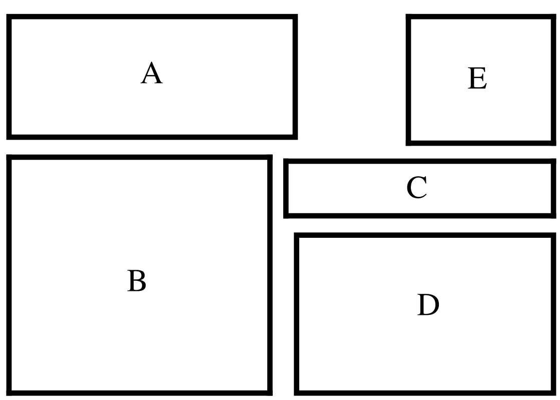


Figure 1: Five rectangles

Exploration 1

a. Draw a rectangle.

b. Measure the rectangle's longer side (l) and its shorter side (s).

c. Calculate the ratio l/s.

d. Use the class data from the rectangle survey to complete Table **1**.

Table 1: Rectangle survey data

Rectangle	Number of People	Percentage of People	Ratio l/s
A			
B			
C			
D			
E			

Discussion 1

a. How does the ratio l/s of the rectangle you drew compare to those of others in the class?

b.
1. According to the class survey, which rectangle was most popular?
2. What is the ratio l/s of this rectangle?
3. What percentage of those surveyed chose the most popular rectangle?

c. How does the ratio l/s of the rectangle you drew compare to that of the most popular rectangle?

Mathematics Note

The **golden ratio** or **golden section**, often denoted by the Greek letter ϕ (phi), is the irrational number $\left(1+\sqrt{5}\right)/2$ which may be approximated as 1.618.

In a **golden rectangle**, the ratio of the lengths of the longer side to the shorter side is the golden ratio ϕ.

d. What general conclusion can you make about the dimensions of the most popular rectangle?

Exploration 2

In 1876, the German psychologist Gustav Fechner conducted a rectangle survey similar to the one you completed in the introduction. About 75% of those surveyed selected a golden rectangle as "most pleasing." The golden ratio is not limited to visually attractive rectangles, however. In fact, it appears in the dimensions of many objects in both art and nature. Some researchers, for instance, believe that the Greek artist Phidias (ca. 490–430 B.C.) used the golden ratio in his sculptures of the human form.

In the following exploration, you investigate the ratios of some human dimensions.

a. Select a sample of students from your class.

b. For each person in your sample, measure the length of one arm from shoulder to fingertips (*a*) and the width of the back from shoulder to shoulder (*b*). Calculate the ratio a/b.

c. For each person in your sample, measure the length of one arm from shoulder to fingertips (*a*) and the length of that same arm from elbow to fingertips (*e*). Calculate the ratio a/e.

d. Compile the class data for the ratios a/b and a/e and calculate the mean for each ratio.

Discussion 2

a. Considering the data you collected in Exploration **2**, would it seem reasonable for Phidias to use ϕ in a sculpture of a person in your class?

b. Identify some other objects that appear to contain the golden ratio.

Assignment

1.1 Golden rectangles have some interesting properties. For example, they can be constructed recursively from other golden rectangles. Recall that a recursive process uses the result of one procedure as the input for the next repetition of the same procedure.

a. **1.** Draw a rectangle in which the ratio of the lengths of the longer side to the shorter side is about 1.6. This rectangle approximates a golden rectangle.

2. Draw a line segment that divides the rectangle into a square and another rectangle.

b. Measure the side of the square.

c. **1.** Calculate the ratio of the longer side to the shorter side of the smaller rectangle.

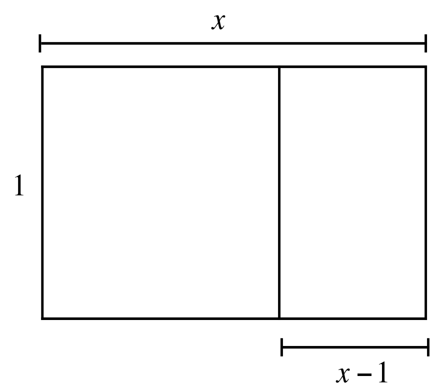
2. Is the smaller rectangle a golden rectangle? Justify your response.

d. Divide the smaller rectangle into a square and another rectangle. Is the new rectangle a golden rectangle? Justify your response.

e. Repeat Parts **c** and **d** three more times.

f. Describe what occurs each time you repeat the procedure in Part **e**.

1.2 The diagram below shows two golden rectangles, the smaller of which was formed by the process described in Problem **1.1**.



a. Use these two rectangles to write a true proportion involving x.

b. Solve this proportion for x.

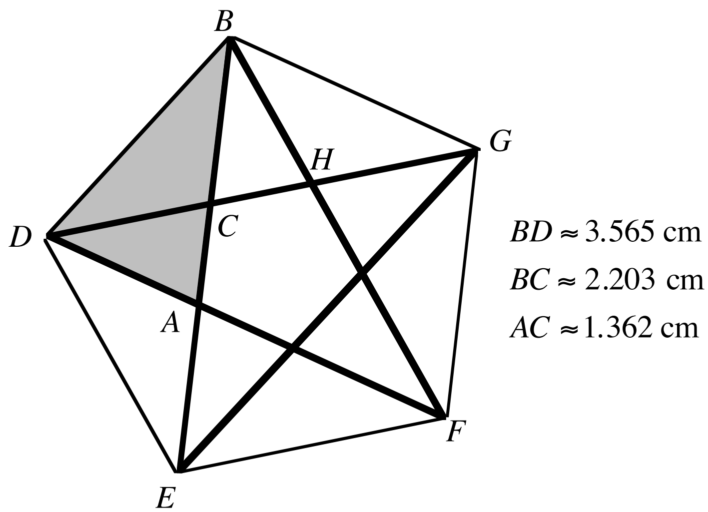
c. Since the two rectangles are golden rectangles, the value of x equals ϕ. Use the value of ϕ to determine the arithmetic relationships that exist between each of the following pairs of values:

1. ϕ and $1/\phi$

2. ϕ and ϕ^2

3. ϕ^2 and $1/\phi$

1.3 The followers of Pythagoras, known as the Pythagoreans, gave special significance to a pentagram, a star formed by connecting each vertex of a regular pentagon with its nonadjacent vertices. The diagram below shows a pentagram inscribed in regular pentagon *BGFED*. (All lengths given are approximate measures.)



a. The ratio BC/AC provides one example of the golden ratio. Find another pair of segments that appear to form the golden ratio.

b. Name each isosceles triangle in the shaded portion of the figure.

c. An isosceles triangle with a leg-to-base ratio of ϕ is a golden triangle. Which triangles in Part **b**, if any, are golden triangles? Justify your response.

d. Determine the approximate measures of the angles in any golden triangle identified in Part **c**. What general conclusion can you make about the angles in a golden triangle?

1.4 Determine whether each statement below is true or false. Provide a counterexample for each false statement.

a. Every isosceles triangle is a golden triangle.

b. Every isosceles triangle with a 36° angle is a golden triangle.

c. Every isosceles triangle with a 36° non-base angle is a golden triangle.

d. Every triangle that contains the golden ratio is a golden triangle.

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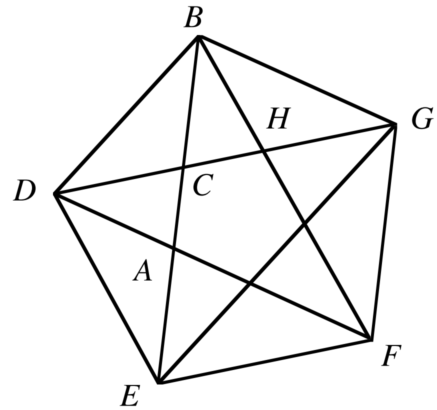
1.5 Use the labeled points in the regular pentagon below to complete Parts **a–c**.

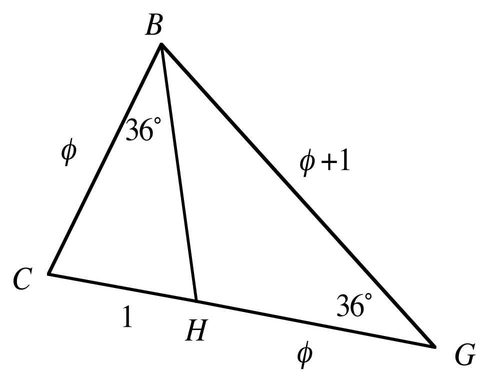


a. Name three pairs of noncongruent, similar triangles.

b. Write a proportion for each pair of triangles in Part **a** that verifies their similarity.

c. Are all golden triangles similar to each other? Explain your response.

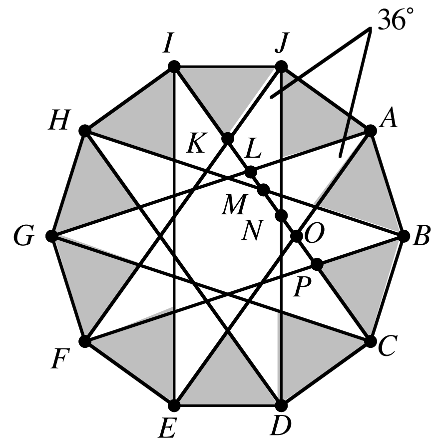
1.6 The diagram below shows a portion of the drawing from Problem **1.5**.



Triangles *CBH* and *CBG* are golden triangles. Use these triangles to verify that

$$\phi = \frac{1+\sqrt{5}}{2}$$

1.7 The unshaded figure below shows a regular "star" polygon formed by connecting every fourth vertex of a regular decagon. (All lengths given are approximate measures.)



$KI \approx 2.0$ cm
$LI \approx 2.8$ cm
$MI \approx 3.3$ cm
$NI \approx 4.0$ cm
$OI \approx 4.5$ cm
$PI \approx 5.3$ cm
$CI \approx 7.3$ cm

a. Using the given distances, find three pairs of line segments whose ratio of lengths appears to form the golden ratio.

b. Find a golden triangle. Give evidence to support your choice.

* * * * * * * * * *

Activity 2

In modern mathematics, Fibonacci is probably best known for the sequence of numbers that bears his name. The first two terms of the **Fibonacci sequence** are both 1. Successive terms are generated by adding the previous two terms. In other words, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, … . Any sequence in which successive terms are formed by adding the previous two terms is referred to as a **Fibonacci-type** sequence.

In this activity, you investigate how Fibonacci-type sequences relate to some of the mathematics of the Pythagoreans.

Exploration 1

a. Generate the first 25 terms of the Fibonacci sequence.

b. Write a recursive definition of the Fibonacci sequence, where F_n is the nth term.

c. **1.** For each pair of consecutive Fibonacci numbers in Part **a**, F_n and F_{n+1}, calculate the following ratios:

$$\frac{F_{n+1}}{F_n} \text{ and } \frac{F_{n+2}}{F_{n+1}}$$

2. For $n = 1, 2, 3, \ldots$, each ratio you calculated in Step **1** also forms a sequence. Recall that the **limit** of a sequence, $k_1, k_2, k_3, \ldots, k_n, \ldots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L.

As n increases, what limit do the two sequences of ratios appear to approach?

d. Using any two nonzero natural numbers of your choice, create a Fibonacci-type sequence of your own. Generate the first 25 terms of the sequence.

e. Repeat Part **c** using your sequence from Part **d**.

f. Create a sequence in which each successive term is the sum of the previous three terms. Begin the sequence with any three nonzero natural numbers. Generate the first 25 terms.

g. For each pair of consecutive numbers in your sequence in Part **f**, F_n and F_{n+1}, calculate the following ratio:

$$\frac{F_{n+1}}{F_n}$$

h. Generate the first 10 terms of the sequence whose explicit formula is shown below:

$$S_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

Discussion 1

a. Considering the terms of a Fibonacci-type sequence, describe what happens to the following ratios as n increases:

$$\frac{F_{n+1}}{F_n}, \text{ and } \frac{F_{n+2}}{F_{n+1}}$$

b. What is the relationship between a Fibonacci-type sequence and ϕ?

c. Describe the sequence generated by the explicit formula in Part **h** of Exploration **1**.

Exploration 2

a. Write any four consecutive terms of the Fibonacci sequence.

b. Calculate the product of the first and fourth terms and set this value equal to a.

c. Calculate twice the product of the second and third terms and set this value equal to b.

d. Perhaps the most famous mathematical relationship associated with Pythagoras is the one that bears his name. The Pythagorean theorem states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

For example, in triangle *ABC* of Figure **2**, $AB^2 = AC^2 + CB^2$, or $5^2 = 3^2 + 4^2$.

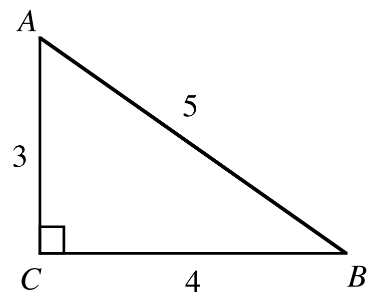


Figure 2: Right triangle *ABC*

Determine the length of the hypotenuse c of the right triangle with legs of lengths a and b as determined in Parts **b** and **c**.

e. Determine if c is a term in the original sequence.

f. Calculate the area of the right triangle with sides a, b, and c.

g. Calculate the product of the four terms in Part **a**. Compare this number with the area of the triangle you calculated in Part **f.**

h.
1. Repeat Parts **b–g** using a different set of four consecutive terms of the Fibonacci sequence.
2. Repeat Parts **b–g** using four nonzero, consecutive terms of a Fibonacci-type sequence.

Discussion 2

a. In Part **h** of Exploration **2**, why were the four consecutive terms of a Fibonacci-type sequence restricted to nonzero numbers?

b. Natural numbers that satisfy the Pythagorean theorem are **Pythagorean triples**. For example, since $3^2 + 4^2 = 5^2$, (3,4,5) is a Pythagorean triple.

Did the values for *a, b*, and *c* in Exploration **2** form a Pythagorean triple when using four successive terms from:

1. the Fibonacci sequence?
2. a Fibonacci-type sequence?

c. Was the length of the hypotenuse found in Part **d** a term of the original sequence when using:

1. the Fibonacci sequence?
2. a Fibonacci-type sequence?

d. What relationship exists between four consecutive terms of any sequence and the area of a right triangle created using the process described in Parts **b–d** of Exploration **2**?

Assignment

2.1 **a.** Using terms in the Fibonacci sequence, generate three Pythagorean triples.

b. Is the length of each hypotenuse (c) a term in the Fibonacci sequence?

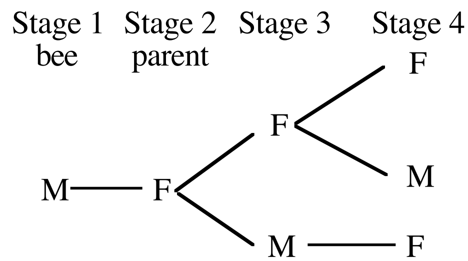
c. For each Pythagorean triple generated in Part **a**, calculate the area of the corresponding right triangle. Describe how you found your solutions.

2.2 **a.** Create three ordered triples (a,b,c) where a is an odd number greater than 1, $b = (a^2 - 1)/2$, and $c = b + 1$. Verify that each of these triples is a Pythagorean triple.

b. Explain why any three numbers a, b, and c that satisfy the constraints given in Part **a** form a Pythagorean triple.

c. Is the value for c in this case always a term in the Fibonacci sequence? Explain your response.

2.3 While female bees have both a female and a male parent, male bees have only a female parent. The tree diagram below shows four ancestral generations of a male bee. Stage 1 represents the bee itself, stage 2 represents the bee's parent, stage 3 represents the bee's grandparents, and stage 4 represents the bee's great-grandparents.



a. Continue the tree diagram for stages 5 and 6.

b. List the first six terms of the sequence $t_1,\ t_2,\ \ldots,\ t_n$, where t_n is the number of bees in stage n.

c. **1.** How many bees are there in stage 9 of the tree diagram?

2. How many bees are there in stage 15 of the tree diagram?

d. How could you find the number of bees in stage n of the tree diagram?

* * * * *

2.4 The **Lucas sequence**, named after French mathematician Edouard Lucas, is a Fibonacci-type sequence with 1 and 3 as the first two terms.

a. Choose any four consecutive terms in the Lucas sequence. Using the method described in Parts **b–d** of Exploration **2**, find values for a, b, and c.

b. Is (a,b,c) a Pythagorean triple? Explain your response.

c. Do you think that c will always be a term in the Lucas sequence? Explain your response.

2.5 Choose any four consecutive terms in the Fibonacci sequence.

a. Using the method described in Parts **b–d** of Exploration **2**, find the value for a hypotenuse c. Which term in the sequence is c?

b. Describe the relationship between the term numbers of the four consecutive terms and the term number of the hypotenuse.

* * * * * * * * * *

Activity 3

To create convincing arguments, mathematicians use a method based on logical reasoning. This process is referred to as **proof**. Although diagrams and pictures may be used to support arguments, a conclusion based solely on observation may or may not be true.

The Pythagorean theorem is one of the most well-known theorems in mathematics. Although the theorem bears the name of Pythagoras, many mathematicians throughout history have developed unique proofs for it. In fact, over 350 different proofs appear in E. S. Loomis' *The Pythagorean Proposition*. In this activity, you examine several proofs of this famous theorem.

Exploration

Many demonstrations of the Pythagorean theorem show, either by adding or subtracting areas, that the area of a square on the hypotenuse of a right triangle is equal to the combined areas of the squares on the legs. In this exploration, you develop the motivation for a proof first written by H. E. Dudeney in 1917.

a. Draw a right triangle.

b. As shown in Figure **3** below, draw a square on each side of the right triangle.

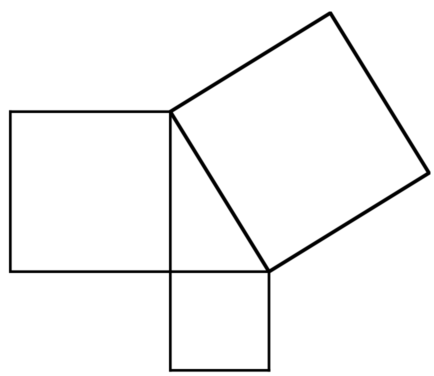


Figure 3: Squares on sides of right triangle

c. Find the center of the square on the longer leg.

d. Through this center, draw lines both parallel and perpendicular to the hypotenuse of the right triangle. These lines divide the square on the longer leg into four congruent quadrilateral regions.

e. Cut out the four quadrilateral regions found in Part **d** and the square on the shorter leg.

f. Arrange the pieces in Part **e** so that they fill the square on the hypotenuse without leaving gaps or overlapping.

g. Repeat Parts **a–f** using a different right triangle.

Discussion

a. How does Figure **3** relate to the Pythagorean formula $a^2 + b^2 = c^2$?

b. What can you conclude about the combined areas of the squares on the legs and the area of the square on the hypotenuse?

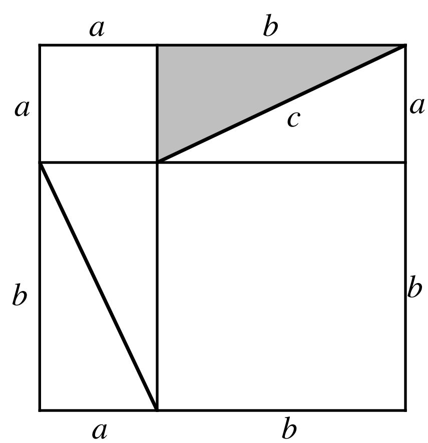
c. Does the procedure you followed in the exploration prove the Pythagorean theorem?

d. H. E. Dudeney's proof of the Pythagorean theorem is described as a "dissection" proof. Why do you think this term is used?

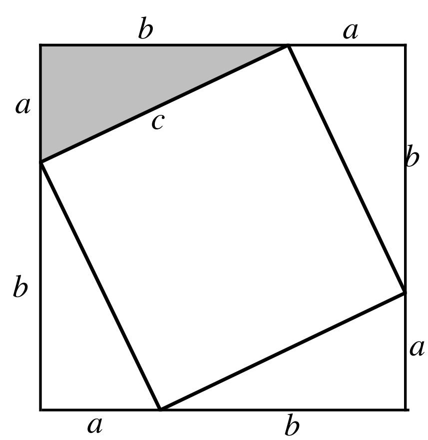
Assignment

3.1 No evidence exists for the original proof of the Pythagorean theorem, but it is generally attributed to Pythagoras himself. According to legend, Pythagoras sacrificed an ox to celebrate the significance of this proof. Hence, its nickname the "ox-killer" proof.

Pythagoras' proof considered two squares, each with side length of $a + b$. As shown below, square 1 is divided into six non-overlapping regions while square 2 is divided into five non-overlapping regions.



square 1

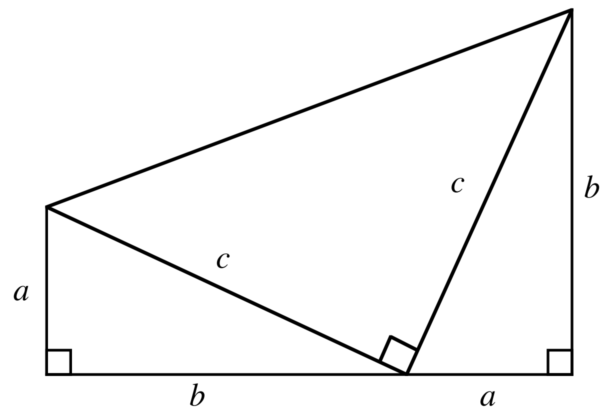


square 2

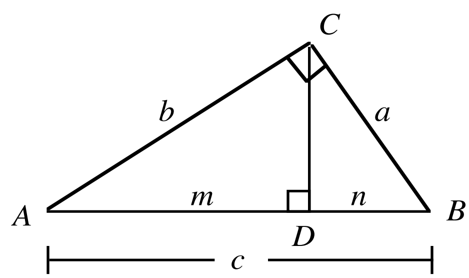
a. What type of quadrilateral is the inner figure in square 2? Justify your response.

b. Use the areas of squares 1 and 2 to prove the Pythagorean theorem. (Hint: Express each square's area as the sum of the areas of its regions.)

3.2 Five years before he became president of the United States, James A. Garfield discovered a creative proof of the Pythagorean theorem. In his proof, he calculated the area of a trapezoid in two ways: by using the area formula for a trapezoid, and by adding the areas of the three right triangles that compose the trapezoid, as shown below. Verify that his method produces the formula $a^2 + b^2 = c^2$.



3.3 In his proof of the Pythagorean theorem, British mathematician John Wallis used similar right triangles, as shown in the diagram below.

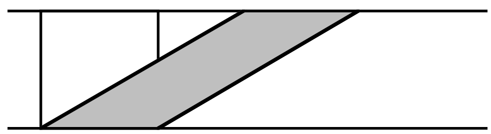


a. Prove that $\Delta ACD \sim \Delta CBD \sim \Delta ABC$.

b. Use measures a, b, m, n, and c to write two true proportions.

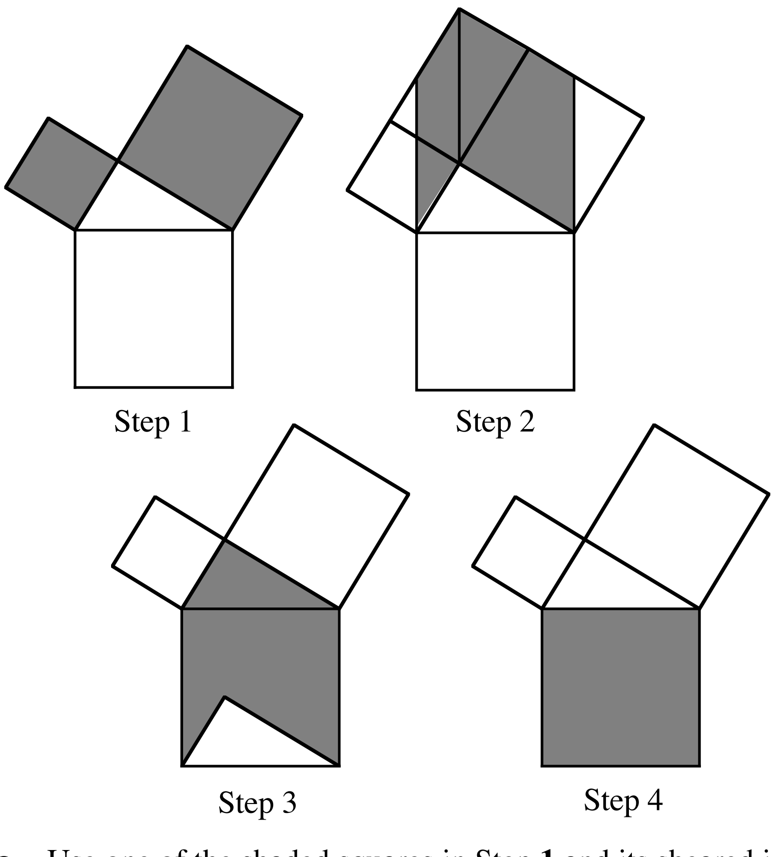
c. Use the proportions in Part **b** to prove the Pythagorean theorem.

3.4 Shearing is a transformation that can be used to create parallelograms of equal area by holding one side of a square fixed and sliding the opposite side along the line containing this side. The diagram below shows one example of this process



In his proof of the Pythagorean theorem, Euclid began with squares on the sides of a right triangle, as in Step **1**. He sheared the squares to obtain the two shaded parallelograms in Step **2**.

The shaded polygon in Step **2** was then transformed to reach Step **3**. Finally, another transformation of a part of the shaded polygon was performed to reach Step **4**.

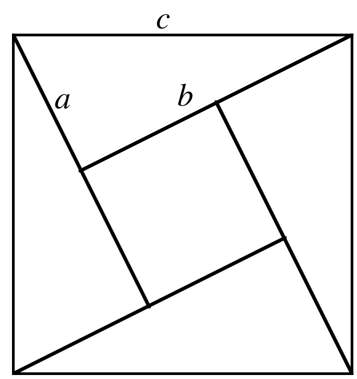


a. Use one of the shaded squares in Step **1** and its sheared image in Step **2** to demonstrate that the areas are equal.

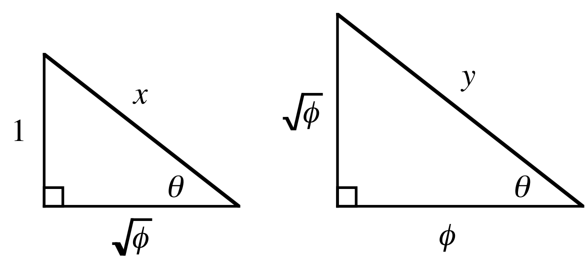
b. Describe in detail the mathematics that occur at each step and explain how this process demonstrates the Pythagorean theorem.

* * * * *

3.5 The Hindu scholar Bhaskara contributed a short, yet elegant, proof of the Pythagorean theorem. For his proof, he drew the following figure with the single word, "Behold!" Calculate the area of the larger square in two different ways, then use the results to prove that $a^2 + b^2 = c^2$.



3.6 Use the following diagram to complete Parts **a–e** below, where ϕ is the golden ratio.



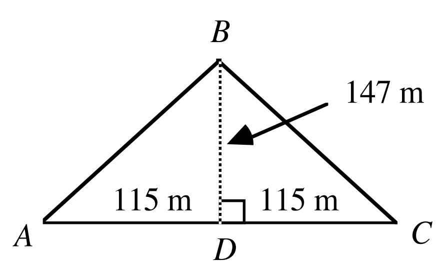
a. Determine the values of *x* and *y*.

b. Explain why the two triangles are similar.

c. Explain why the triangles are not congruent, even though five of their parts are the same.

d. Find tan θ and cos θ and compare their values.

e. The following diagram shows some measurements of a cross section of the Great Pyramid of Giza. Find the ratio of *AB* to *AD*, then compare triangle *ABD* to the two triangles given above.



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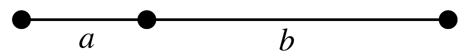
Research Project

In ancient Greece, mathematicians used a compass and straightedge to demonstrate various properties of geometry and find exact measures.

As shown in Figure **4**, a point divides a line segment into the golden ratio when the ratio of the longer segment to the shorter segment is the same as the ratio of the whole line segment to the longer segment.



$$\frac{b}{a} = \frac{a+b}{b}$$

Figure 4: The golden ratio

The ratio of the shorter segment to the longer segment, or a/b, is the reciprocal of the golden ratio. Given that the value of the golden ratio is ϕ, the value of its reciprocal is $1/\phi$ or $\phi - 1$.

Using a compass, a straightedge, and the right triangle in Figure **5** below, construct a segment whose length is equal to the reciprocal of the golden ratio. Describe the process that you used and explain why it works.

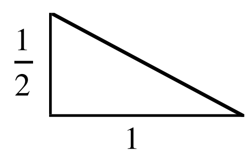


Figure 5: A right triangle

Summary Assessment

1. The following calculator algorithm generates terms of a sequence *S*:

Step 1: Choose a number between 1 and 10 as the first term.
Step 2: Add 1 to the term.
Step 3: Take the reciprocal of the result. This is the next term in *S*.
Step 4: Repeat Steps 2–3 until you have 10 terms of sequence *S*.

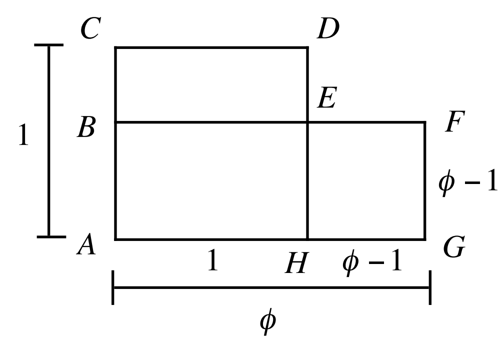
a. Use the algorithm to generate one sequence of 10 terms.

b. Repeat the algorithm using a different value in Step 1.

c. Does the sequence appear to approach a limit? Explain your response.

d. When Step 2 and Step 3 are interchanged in the algorithm, does the sequence appear to approach a limit? Explain your response.

2. Euclid determined that a point divides a line segment into the golden ratio ϕ when the ratio of the longer segment to the shorter segment is the same as the ratio of the whole line segment to the longer segment. Euclid also found a similar relationship in the figure below.



In this diagram, the area of the larger square, *ACDH*, equals the area of the larger rectangle, *ABFG*. Use this fact and the appropriate area formulas to determine the exact value of ϕ.

Module Summary

- The **golden ratio**, or **golden section**, often denoted by the Greek letter ϕ (phi), is the irrational number $\left(1+\sqrt{5}\right)/2$ or approximately 1.618.
- In a **golden rectangle**, the ratio of the measures of the longer side to the shorter side is the golden ratio (ϕ).
- The first two terms of the **Fibonacci sequence** are both 1. Successive terms are generated by adding the previous two terms. In other words, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, … .
- Any sequence in which successive terms are formed by adding the previous two terms is a **Fibonacci-type** sequence.
- The **limit** of a sequence, $k_1,\ k_2,\ k_3,\ \ldots,\ k_n,\ \ldots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L.
- The **Pythagorean theorem** states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse. In other words, if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then:

$$a^2 + b^2 = c^2$$

- Natural numbers x, y, and z that satisfy the Pythagorean theorem are **Pythagorean triples** and may be represented by (x,y,z).
- The reciprocal of the golden section, $1/\phi$, is approximately 0.618.

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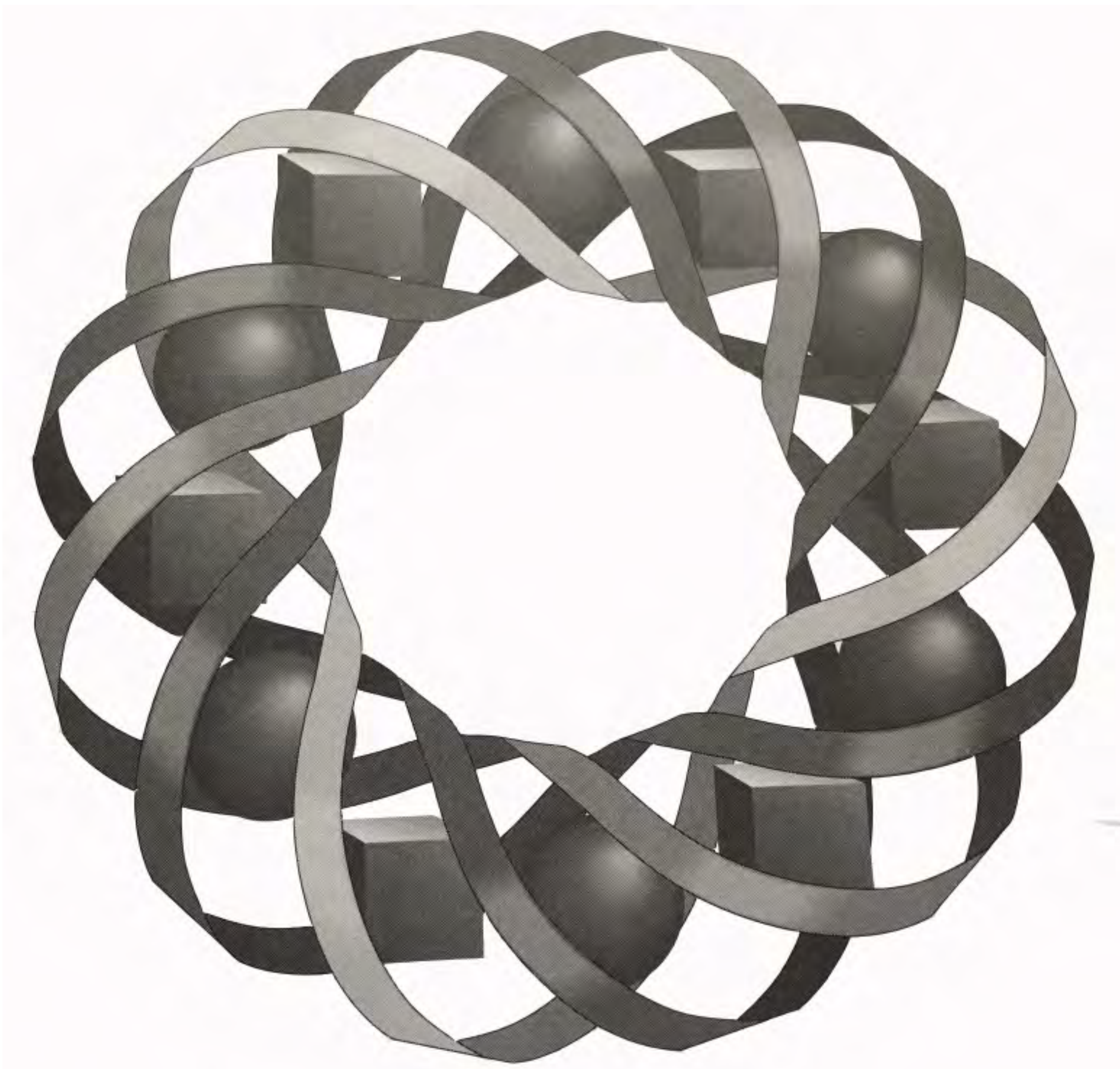
Banking on Life



The price you pay for borrowing money—as well as the amount you earn by saving it—should be of interest to you. In this module, you learn how financial institutions calculate the interest they pay on savings accounts and the interest they charge on consumer loans.

Gary Bauer • Jeff Hostetter

Banking on Life

Introduction

In order to pay for life's major expenses, you have two options: save money or borrow it. If you decide to save money now to make purchases later, your investments may earn interest. If you decide to borrow money now and make payments later, you will owe interest on your loans. In this module, you explore the mathematics of investing and lending.

Activity 1

Principal is the amount of money invested or borrowed. **Interest** is the amount earned on invested money or the amount paid for borrowed money. The amount of interest earned or paid depends on three quantities: principal, interest rate, and time. Interest also varies according to the method used to calculate it. In this activity, you begin to investigate how savings accounts earn money.

Mathematics Note

When you invest money, one method for calculating interest involves **simple interest**. In this case, interest is paid only on the original principal. The formula for calculating simple interest I, where P represents the principal, r represents the interest rate, and t represents time, is:

$$I = Prt$$

To use this formula, t must be expressed in the same units as time in the interest rate r. For example, if the interest rate is 3% per year, then t also must be expressed in years. If \$1000 is invested at an interest rate of 3% per year for 10 yr, the interest earned can be calculated as follows:

$$I = \$1000(0.03)(10) = \$300$$

Exploration 1

Suppose that two new parents begin saving for their child's education on the day of birth. They invest \$1000 earning simple interest at an annual rate of 4%. The interest on this investment is paid once a year, on the child's birthday. In the following exploration, you investigate the income earned from this investment each year and the cumulative value of the original investment plus interest.

a. Calculate the interest earned on \$1000 in 1 yr at an annual interest rate of 4%.

b. Determine the amount of interest earned each year, the total interest earned, and the principal plus total interest for the investment in each of the next 18 years. Record this information in a copy of Table **1** below.

Table 1: Interest by year

Year	Principal	Annual Interest	Total Interest	Principal plus Total Interest
1	\$1000.00			
2	\$1000.00			
3	\$1000.00			
⋮	⋮			
18	\$1000.00			

c. On the same set of axes, create a scatterplot of each of the following:

1. total interest versus time

2. principal plus total interest versus time.

d. Determine an equation to model each scatterplot in Part **c**.

Discussion 1

a. Compare the two scatterplots you created in Part **c** of Exploration **1**.

b. What type of function seems to best model the scatterplots? Explain your response.

c. How do the equations found in Part **d** of Exploration **1** describe the differences in the graphs?

d. What do the slope and *y*-intercept of each graph represent in terms of the investment?

e. The term *account balance* sometimes refers to the total of the principal and the interest earned. When interest is paid only on the original principal using simple interest, the balance B_t of the investment after *t* years can be calculated using the explicit formula below:

$$\begin{aligned} B_t &= P + I \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

where *P* represents the principal, *r* represents the annual interest rate, and *t* represents time in years.

1. Explain why the three expressions given for B_t are equivalent.

2. Use the explicit formula for B_t to determine the balance of the investment after 18 yr. How does the balance compare to the sum of the principal and the total interest after 18 yr?

Exploration 2

If the simple interest earned in Exploration **1** had been added to the original principal each year and no money removed from the investment, most banks would have paid interest on each year's balance. This is **compound interest**. When interest is compounded, the interest earned for each period is calculated using the balance from the previous period. In other words, the balance from one period becomes the principal for the next period.

a. Suppose that the new parents described in Exploration **1** deposited \$1000 in a savings account with an interest rate of 4% per year, compounded annually.

Complete Table **2** for years 3–18. Notice that in this account the balance at the end of each year becomes the principal for the following year.

Table 2: Value of account using compound interest

Year	Principal (P)	Interest (Prt)	Total Interest	Balance ($P + Prt$)
1	\$1000.00	\$40.00	\$40.00	\$1040.00
2	\$1040.00	\$41.60	\$81.60	\$1081.60
3	\$1081.60			
⋮				
18				

b. Write a recursive formula that describes the balance B_t at the end of any year.

c. Create a scatterplot of the account balance versus the year.

d. One way to find an equation that models the scatterplot in Part **c** is to determine an explicit formula for the sequence of values for the account balance.

1. Write an expression for the balance (B_1) at the end of year 1 in terms of the original principal (P_0) and the annual interest rate (r).
2. Write an expression for the balance (B_2) at the end of year 2 in terms of B_1 and r.
3. Substitute the expression for B_1 from Step **1** into the expression you wrote in Step **2**. Rewrite the expression using an exponent.
4. Repeat the process described in Steps **2** and **3** for the balance in years 3, 4, and 5.

e. Generalize the formula found in Part **d** for any balance B_t after t years in terms of P_0, r, and t. Test your formula using the information in Table **2**.

Discussion 2

a. Describe how the difference between simple interest and compound interest affects the value of an $1000 investment after 18 yr.

b. How is the formula for simple interest used in calculating compound interest?

c. How could you determine the interest earned on an investment given the original principal and the final balance?

d. Imagine that you must save $1000 in 5 yr. Your savings account pays 5% interest per year, compounded annually. Describe how you could use a symbolic manipulator to determine the initial deposit you should make.

Assignment

1.1 In Exploration **1**, the child's parents invested $1000 earning simple interest at a rate of 4% per year.

a. Determine the total amount of interest earned after 30 yr.

b. How many years would it take for the sum of the principal and the total interest earned to reach $5000?

c. How much would the parents have had to invest to produce a balance of $15,000 (principal and interest) after 18 yr?

1.2 In Exploration **2**, the child's parents deposited $1000 in a savings account with an interest rate of 4% per year, compounded annually.

a. Assuming that they make no withdrawals, determine the total amount of interest earned in this account after 30 yr.

b. How many years would it take for the account balance to reach $5000 if no withdrawals are made?

c. **1.** How much would the parents have had to deposit to produce a balance of $15,000 after 18 yr?

2. How does this amount compare with your response to Problem **1.1c**?

1.3 Consider the investment account described in Exploration **1**, earning simple interest at a rate of 4% per year. On the child's 10th birthday, an aunt adds $500 to the principal in the account.

a. Considering the original investment of $1000 and all previous interest earned, what is the sum of the principal and interest after 18 yr?

b. Create a scatterplot that shows the effect of this deposit on the sum of the principal and the interest over time. Describe how the graph shows this effect.

* * * * *

1.4 A local department store is going out of business. In order to attract customers, the owners decide to discount everything left in the store 15% each week until all merchandise is sold.

a. If an item regularly sells for \$150, what will its price be during the third week of the sale?

b. During what week will the sale price of this item fall below \$10?

c. Some of the store's customers think that, if they wait long enough, the items they want will be free. Is this true? Justify your response.

1.5 A small museum has 17,000 items in its collection. Of these items, 2000 are on permanent display. The remainder of the collection is placed on display on a rotating basis, 12% every six months. Using this system, how many years will pass before museum visitors have had the opportunity to see the entire collection? Explain your response.

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Activity 2

In most savings accounts, interest is compounded more than once per year. To calculate the balance in such cases, the interest rate must be adjusted to correspond with the compounding period. This adjusted rate is referred to as the **interest rate per compounding period** and equals r/n, where r is the annual interest rate and n is the number of times interest is compounded per year.

For example, if the annual interest rate is 8% and the interest is compounded quarterly (4 times per year), then the interest rate per quarter is $8\%/4 = 2\%$. In this activity, you explore how increasing the number of compoundings affects the account balance over time.

Exploration

a. Imagine that the parents described in Activity **1** had deposited \$1000 in a savings account at an annual interest rate of 4%, compounded monthly. What is the interest rate per compounding period for this account?

b. Assuming that they make no withdrawals, determine the account balance at the end of each month for the next 18 yr.

c. Create a scatterplot of account balance versus time in months.

d. Determine an explicit formula that calculates the balance B_t after t years in terms of the original principal (P_0), the interest rate per compounding period (r/n), and the number (n) of times interest is compounded per year. Use the data from Part **b** to test the formula.

e. On the same graph, plot the equation found in Part **d** using an original principal of $1000, an annual interest rate of 4%, and each of the following numbers of compoundings per year: 5, 10, 100, and 1000.

f. As the number of compoundings increases without bound, the balance after a given time period approaches a limiting value. To investigate what happens when the number of compounding periods becomes very large, consider an investment of $1.00 at an annual interest rate of 100%, compounded n times per year for 1 yr.

Create a spreadsheet with columns similar to those in Table **3**. Use the formula found in Part **d** to complete the spreadsheet.

Table 3: Account balances for different compoundings

No. of Compoundings per Year (n)	Balance at End of Year (in dollars)
1	
10	
100	
1000	
10,000	
100,000	
1,000,000	
10,000,000	
100,000,000	

g. The values in the right-hand column of Table **3** can be thought of as the terms of a sequence. As the number of compoundings per year increases, what limit does this sequence appear to approach?

Mathematics Note

The limit of the following expression, as n approaches infinity, is an irrational number approximately equal to 2.71828:

$$\left(1+\frac{1}{n}\right)^n$$

This irrational number is represented as ***e***, in honor of Swiss mathematician Leonhard Euler.

h. For an initial principal of $1.00 and a period of 1 yr, the formula for account balance when interest is compounded n times a year becomes:

$$B_1=\left(1+\frac{r}{n}\right)^n$$

To investigate how a change in interest rate affects the balance at the end of the year, create and complete a spreadsheet with columns like those in Table **4** below.

Table 4: Balance in dollars for different interest rates

No. of Compoundings per Year (n)	$B_1 = \left(1 + \frac{1}{n}\right)^n$	$B_1 = \left(1 + \frac{2}{n}\right)^n$	$B_1 = \left(1 + \frac{3}{n}\right)^n$
1			
10			
100			
1000			
10,000			
100,000			
1,000,000			
10,000,000			
100,000,000			

i. **1.** Calculate e^2. Compare it to the values in the spreadsheet in Part **h**.

2. Calculate e^3. Compare it to the values in the spreadsheet in Part **h**.

Discussion

a. When interest is compounded annually, an investment of $1000 at an annual interest rate of 4% for 18 years results in a balance of $2025.82. How does this compare with the account balance when interest is compounded monthly?

b. **1.** Describe how the graphs of the four equations in Part **e** of the exploration are related.

2. What do you think a graph of the corresponding equation would look like if interest were compounded 1,000,000 time per year or if interest were compounded continuously?

c. Considering your responses to Part **i** of the exploration, predict the relationship between e^r and the following equation as n becomes extremely large.

$$B_1 = \left(1 + \frac{r}{n}\right)^n$$

d. Using a calculator, a student evaluated the equation in Part **c** above for $n = 10^{14}$. The calculator reported that $B_1 = 1$. Why did this occur?

e. How could you use your response to Part **c** of the discussion, along with the formula from Part **d** of the exploration, to write a formula for account balance when interest is compounded continuously?

Assignment

2.1 **a.** Imagine that you have invested \$1000 at an annual interest rate of 5% and made no withdrawals. Determine the balance after 10 yr when interest is compounded:

1. every year
2. every month
3. every day
4. every hour
5. every second

b. Repeat Part **a** for the balance after 20 yr.

c. How does increasing the number of compoundings per year affect the account balance?

2.2 What has more impact on the balance of a savings account from which the owner makes no withdrawals: the annual interest rate or the number of compoundings per year? Justify your response.

2.3 To help consumers compare earnings among accounts with different compounding periods, the federal government requires financial institutions to report the **annual percentage yield** or **APY** for all accounts that earn interest. Annual percentage yield is the interest rate that the account would earn if interest were compounded annually.

a. Consider an investment of \$1000 at an annual interest rate of 3%, compounded quarterly. Calculate the account balance after 1 yr if no withdrawals are made.

b. To find the annual percentage yield for this account, you can solve the formula for compound interest for r

$$B_t = P\left(1 + \frac{r}{n}\right)^{nt}$$

where B_t equals the value determined in Part **a**, $P = 1000$, $n = 1$, and $t = 1$. Determine the APY for the 3% annual interest rate when it is compounded quarterly.

c. Determine the APY for a savings account with an annual interest rate of 5%, compounded monthly.

2.4 **a.** After 1 year, will the balance of an account be significantly more if interest is compounded every hour rather than every day? Use an example to support your response.

b. In general, what effect does increasing the number of compoundings per year have on the account balance?

2.5 For small values of n, how do the values of e^r and the following expression compare?

$$\left(1+\frac{r}{n}\right)^n$$

* * * * *

2.6 One general equation used to model the growth or decay in a quantity is $N_t = N_0 e^{nt}$, where N_t represents the final amount, N_0 represents the initial amount, n represents some constant, and t represents time. When $n > 0$, the equation can be used to model growth; when $n < 0$, the equation can be used to model decay.

A population of bacteria has a constant n of 0.538 when t is measured in days. How many days will it take an initial population of 8 bacteria to increase to 320?

2.7 A population of bacteria grows at a rate of 5% per hour. If the initial population is 100, what is the population after 10 hr?

2.8 A population of song birds is decreasing at a rate of 8% per year. If this trend continues, how many years will it take for the current population of 1000 birds to reach 100 birds?

2.9 The number of employees at Sky High, a local parachute company, has increased exponentially in the past three years, from 25 to 115. What was the annual growth rate for the company during this time?

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Activity 3

To make large purchases such as a car or a home, many people use a portion of their savings as a down payment, then borrow the rest. In this activity, you investigate the methods banks use to calculate the monthly payments on loans.

Business Note

Loans are generally repaid over a period of time by making regular monthly payments, or **installments**. **Amortization** is the process of repaying a loan by installments. Each month, the borrower pays the interest on the unpaid balance of the loan as well as a portion of the principal.

The **annual percentage rate** or **APR** is used to calculate the interest paid on an installment loan. (Note that APR refers to the interest rate paid on loans while APY refers to the interest rate earned on savings accounts.)

Exploration

A high school senior has approximately \$2000 in a savings account. The student decides to use \$1000 for a down payment on a used car and \$1000 for the first semester of college. Since the car costs \$5000, the student must borrow an additional \$4000.

a. A local credit union offers car loans at an APR of 9% with 24 monthly installments. Estimate the size of the monthly payment.

b.

1. Part of each monthly installment pays the interest on the balance of the loan. Determine the interest for 1 month on a principal of \$4000 at an APR of 9%. This is the interest due for the first month of the loan.
2. The difference between the monthly installment and the interest due is applied to the principal. Use the estimated payment from Part **a** to calculate the amount the principal will be reduced after the first month.
3. The balance of the loan remaining is the difference of the previous balance and the amount applied to the principal. Determine the balance of the loan remaining after one installment. This balance is then used to continue the process.

c. Create a spreadsheet with headings like those in Table **5** below. Use the spreadsheet to determine the size of the monthly installment necessary to pay off the loan in 24 months.

Note: Write the formulas for individual cells so that when you change the estimated payment, the APR, or the amount borrowed, the spreadsheet will automatically update the remaining cells.

Table 5: Monthly payment for car loan

Amount Borrowed	**APR**	**Estimated Payment**	
4000	0.09		
Payment No.	**Loan Balance after Payment**	**Interest Payment**	**Principal Payment**
	4000		
1			
2			
3			
⋮			
24			

d. The amount of interest paid over the life of the loan is referred to as the **finance charge**.

1. Determine how much the student will pay in finance charges on a loan of \$4000 at an APR of 9% for 24 months.
2. Including the down payment, find the total costs for this car.

e. With the salary from a part-time job, the student can afford a monthly car payment of no more than \$100. Use the spreadsheet from Part **c** to determine the maximum amount of money the student can borrow for 24 months at 9%.

f. After shopping for a less expensive model, the student decided to buy the \$5000 car anyway. Using a monthly payment of \$100, how many months will it take to repay a \$4000 loan at an APR of 9%?

Discussion

a. How did you use the formula for simple interest in Part **b** of the exploration?

b. How much interest would the student pay over 24 months on a loan of \$4000 at an APR of 9%?

c. What can the student do to reduce the size of the monthly payments?

d. How does increasing the time to repay a loan affect the finance charges?

Mathematics Note

The **monthly payment** M for an installment loan can be calculated using the explicit formula below, where r is the monthly interest rate $(\text{APR}/12)$, P is the principal of the loan, and t is the time in years:

$$M = P\left[\frac{r}{1-\left(\frac{1}{1+r}\right)^{12t}}\right]$$

For example, the monthly payment for a \$10,000 loan at an APR of 8% over 4 yr can be calculated as follows:

$$M = \$10{,}000\left[\frac{\frac{0.08}{12}}{1-\left(\frac{1}{1+\frac{0.08}{12}}\right)^{12\cdot 4}}\right] = \$244.13$$

The balance B_n remaining after n monthly payments, where r is the monthly interest rate, can be found using the following recursive formula:

$$B_n = B_{n-1} + rB_{n-1} - M,\ n > 1$$

or by using the explicit formula below:

$$B_n = P(1+r)^n - M\left[\frac{(1+r)^n - 1}{r}\right]$$

Using the example given above, the balance remaining after the first month's payment can be calculated recursively as shown below:

$$B_1 = \$10,000 + \frac{0.08}{12} \bullet \$10,000 - \$244.13 = \$9822.54$$

Using the explicit formula, the balance remaining after 18 monthly payments can be calculated as follows:

$$B_{18} = \$10,000\left(1 + \frac{0.08}{12}\right)^{18} - \$244.13\left[\frac{\left(1 + \frac{0.08}{12}\right)^{18} - 1}{\frac{0.08}{12}}\right] = \$6618.05$$

e. How does the recursive formula for the balance remaining on a loan compare to the recursive formula for compound interest below?

$$B_t = B_{t-1} + rB_{t-1},\ t > 1$$

Assignment

3.1 Since the costs of buying a home are so high, many people finance their purchases over a period of 30 yr.

a. What is the monthly payment on a \$50,000 loan for 30 yr at an APR of 8%?

b. Determine the total amount of interest paid on the loan in Part **a**.

c. Calculate the balance remaining after 5 yr.

3.2 A typical home loan can be repaid over 15, 20, or 30 years. Imagine that you could afford a monthly house payment of \$450.

a. How much money could you borrow for 30 yr at an APR of 8%?

b. How much could you borrow for 15 yr at an APR of 8%?

c. How much could you borrow for 30 yr at an APR of 16%?

3.3 **a.** Lisa plans to borrow \$70,000 to purchase a home. If she can afford a \$500 monthly payment, how long will it take her to repay an installment loan at an APR of 8%?

b. Banks usually limit the repayment period to 30 yr. How much would Lisa's monthly payment have to be in order to repay the loan in 30 yr?

c. Lisa can reduce the size of the loan by increasing the size of her down payment. How much money should Lisa use as a down payment in order to reduce her monthly payment to \$500?

3.4 The Jakes family wants to borrow \$45,000 for 20 yr. After considering their other expenses, they decide that they can make monthly payments of no more than \$350. What annual percentage rate (to the nearest 0.1%) do the Jakes need to be able to afford the loan?

3.5 Imagine that you plan to borrow \$65,000 at an APR of 7.5%. The bank will allow you to borrow the money either for 15 yr or for 30 yr.

a. Calculate the monthly payment for each loan.

b. Compare the interest paid on each loan.

c. Determine the amount of money you will still owe after 15 yr if you take the 30-yr loan.

* * * * *

3.6 **a.** Wynette plans to buy a new car that costs \$22,000. Her bank will loan her up to 80% of the purchase price. What is the maximum amount of money she can borrow for the car?

b. The APR for a 36-month loan is 8%. If Wynette borrows the maximum amount, determine:

1. the monthly payment
2. the total interest cost
3. the total cost of the car.

c. The APR for a 60-month loan is 9.5%. Repeat Part **b** for this loan option.

3.7 One loan option available at some financial institutions involves a **balloon payment**. In this type of loan, the borrower pays relatively small installments for a certain time, then pays the balance remaining at the end of the loan in one large payment (the balloon).

For example, the monthly payments on a loan of $5000 for 2 yr at an APR of 8% are $226.14. In a loan involving a balloon payment, the consumer might pay only $150 a month for 2 yr, then pay the remaining balance in one payment at the end of the 2 yr.

a. Imagine that you have borrowed $5000 for 2 yr at an APR of 8%. If you make monthly payments of $150 on this loan for 2 yr, what will be the value of your balloon payment?

b. Compare the finance charges on the balloon-payment loan in Part **a** to the finance charges on a traditional installment loan for the same amount. Explain any differences you observe.

c. One drawback to a loan with a balloon payment is the large unpaid balance at the end of the repayment period. Describe some situations in which you think this type of loan might be useful to a borrower, despite this drawback.

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Research Project

Compare the interest rates and compounding periods for savings accounts at several financial institutions in your community or region. Also compare the interest rates and repayment times offered for loans on both new and used cars. Use examples to illustrate the long-term effects of any differences you observe.

Summary Assessment

1. Whenever consumers use credit cards to make purchases, they are borrowing money. If the account is not paid in full each month, the credit-card company charges interest on any balance remaining.

Since the balance in a credit-card account changes whenever a payment is made or a purchase occurs, credit-card companies typically use the **average daily balance** to calculate interest charges. The average daily balance is determined by recording the account balance at the beginning of each day in a month, totaling these daily balances, and then dividing the total by the number of days in the month.

Calculate the average daily balance for the account described in the table below. Assume that each purchase or payment does not affect the average daily balance until the day after it occurs.

Balance at Beginning of Month: $250			
Month	**Day**	**Purchases ($)**	**Payments ($)**
Sept.	4	15.00	
Sept.	5	150.00	
Sept.	7	42.00	
Sept.	8		50.00
Sept.	11	50.00	
Sept.	21	58.00	
Sept.	22	25.00	
Sept.	23	37.00	
Sept.	24	98.00	
Sept.	29	125.00	

2. The APR for the account in Problem **1** is 18%. Use the average daily balance to calculate the finance charge on the account for the month of September.

3. Many banks do not use the amortization formula to calculate the monthly payment for credit-card balances. Instead, the minimum payment is 2% of the balance due or $50, whichever is greater. In this case, the balance due is the average daily balance plus the finance charges for the month.

a. Calculate the minimum payment due on the account in Problem **1**.

b. Calculate the balance due necessary for the minimum payment to be greater than $50.

4. When the balance due on a credit-card account is higher than the amount calculated in Problem **3b**, the minimum monthly payment is calculated using the 2% rule. After the balance due decreases to the amount calculated in Problem **3b**, the minimum monthly payment is \$50. Use this method of repaying the loan to complete the following.

a. If no other purchases are made, how many years would it take to pay off a credit-card balance of \$7500 with an APR of 18%?

b. What is the total interest paid while repaying \$7500 in this manner?

5. If a credit-card company used the amortization formula to determine monthly payments, the payments would be the same each month. Recall that the monthly payment M for an installment loan can be calculated using the explicit formula below, where r is the monthly interest rate $(\text{APR}/12)$, P is the principal, and t is the time in years:

$$M = P\left[\frac{r}{1-\left(\frac{1}{1+r}\right)^{12t}}\right]$$

a. Using 2% of \$7500 as the monthly payment, how many years would it take to pay off a credit-card balance of \$7500 with an APR of 18%?

b. What is the total interest paid while repaying \$7500 in this manner?

c. Compare the finance charges in Problems **4** and **5**. Which method would you rather use to pay off a loan?

Module Summary

- **Principal** is the amount of money invested or borrowed.
- **Interest** is the amount earned on invested money or the amount paid for borrowing money.
- **Simple interest** is paid only on the original principal. The formula for calculating simple interest *I*, where *P* represents the principal, *r* represents the interest rate, and *t* represents time, is:

  $$I = Prt$$

  When interest is **compounded**, the interest earned for each period is calculated using the balance from the previous period. In other words, the balance from one period becomes the principal for the next period.
- The **interest rate per compounding period** equals r/n, where *r* is the annual interest rate and *n* is the number of times interest is compounded per year.
- When interest is compounded, the balance B_t after *t* years can be calculated using the following explicit formula, where *P* is the original principal, *r* is the annual interest rate, and *n* is the number of compoundings per year:

  $$B_t = P\left(1 + \frac{r}{n}\right)^{nt}$$
- The limit of the following expression, as *n* approaches infinity, is an irrational number approximately equal to 2.71828:

  $$\left(1 + \frac{1}{n}\right)^{n}$$

  This irrational number is represented as ***e***, in honor of Swiss mathematician Leonhard Euler.
- The **annual percentage yield** or **APY** is the interest rate that an account would earn if interest were compounded annually.
- Loans are generally repaid over a period of time by making regular monthly payments or **installments**. **Amortization** is the process of repaying a loan by installments. Each month, the borrower pays the interest on the unpaid balance of the loan as well as a portion of the principal.
- The **annual percentage rate** or **APR** is used to calculate the interest paid on an installment loan.
- The amount of interest paid over the life of the loan is referred to as the **finance charge**.

- The **monthly payment** M for an installment loan can be calculated using the explicit formula below, where P is the principal of the loan, r is the monthly interest rate $(\text{APR}/12)$, and t is the time in years:

$$M = P\left[\frac{r}{1-\left(\frac{1}{1+r}\right)^{12t}}\right]$$

- The balance B_n remaining after n monthly payments on an installment loan can be found using the following explicit formula, where P is the principal, r is the monthly interest rate $(\text{APR}/12)$, and M is the monthly payment:

$$B_n = P(1+r)^n - M\left[\frac{(1+r)^n - 1}{r}\right]$$

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