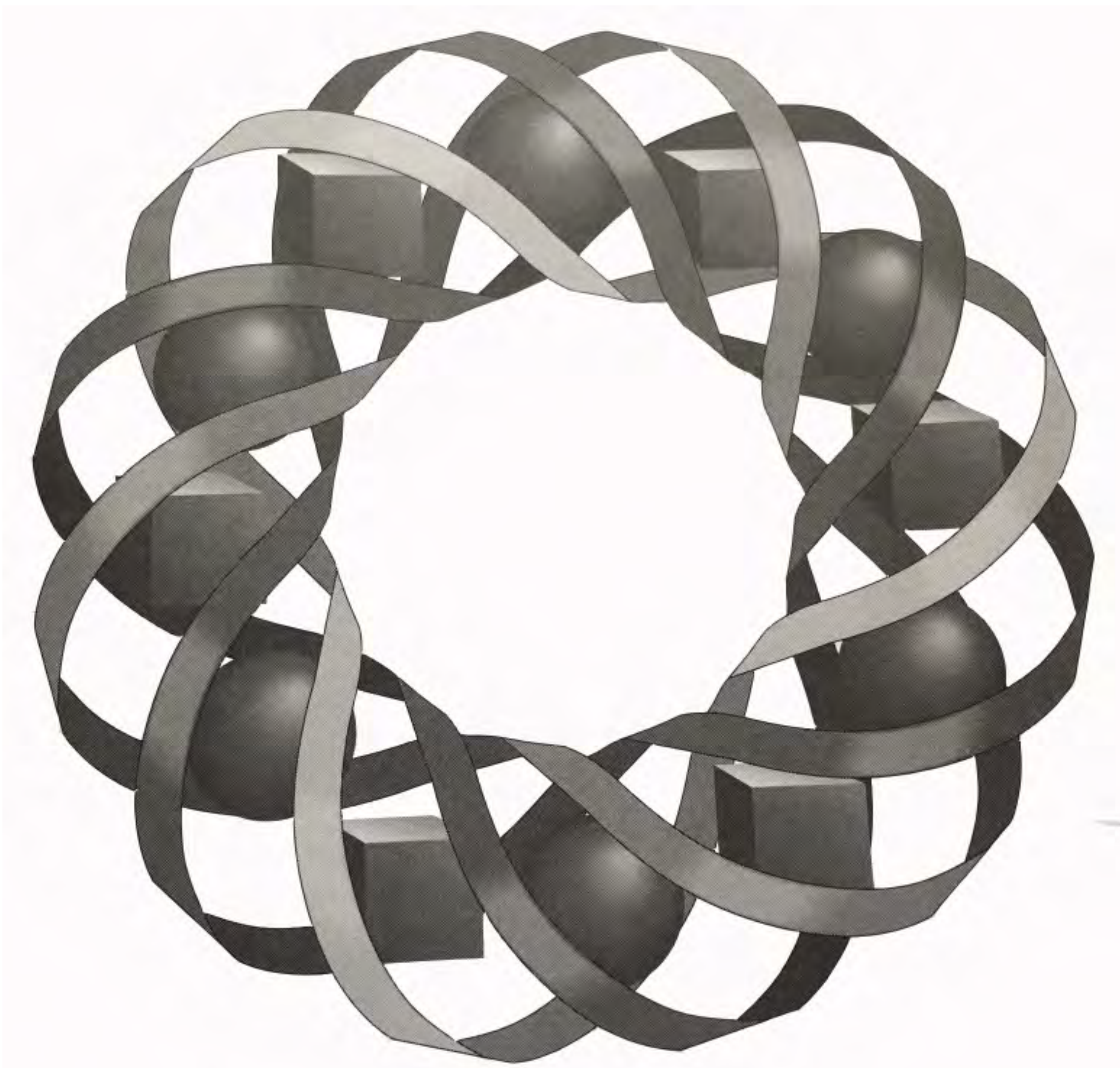
SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 4 Volumes 1-3

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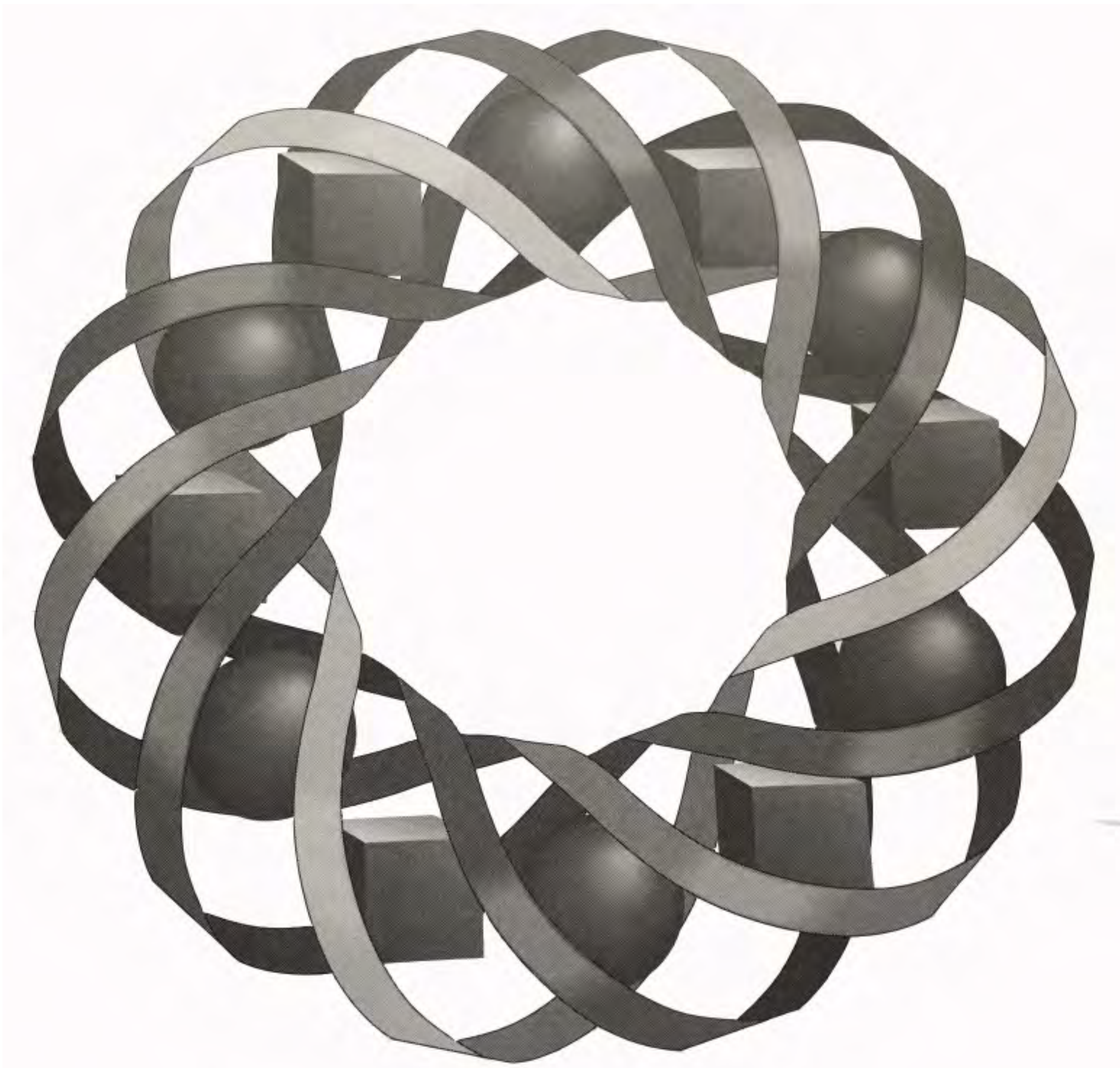
Colorful Scheduling



Have you outgrown coloring? Mathematicians haven't. Mathematicians use coloring theory to solve problems in scheduling, organization, and cartography.

Masha Albrecht • Darlene Pugh

Colorful Scheduling

Introduction

As a cartographer, you wish to color a map of the western United States. Any states that share a common border cannot be the same color. States that touch only at one corner, however, can be the same color.

Exploration

Figure **1** shows a map of the western part of the continental United States. Color the map as suggested in the introduction, using as few colors as possible.

Figure 1: Map of the western United States

Discussion

a. How many different colors did you need to color the map in Figure **1**?

b. Using the number of colors identified in Part **a**, do you think you could color a map of the entire United States in the same fashion?

c. What is the least number of colors you would need to color a map of all the world's countries?

Activity 1

Mathematicians have puzzled over map-coloring problems since the 19th century. In this activity, you create several maps and determine the minimum number of colors needed to color each one, provided that regions which share a border cannot be the same color.

Exploration

a. On a blank sheet of paper, draw at least seven straight, nonparallel lines. Extend each line entirely across the paper.

b. Imagine that each region enclosed by intersecting lines (or the edge of the paper) is a country on a map. Given that countries which share a border cannot be the same color, color the map using the least possible number of colors. Record this number.

Mathematics Note

The **chromatic number** of a map is the least number of colors required to color the map. For example, the chromatic number of the map in Figure **2** is 3.

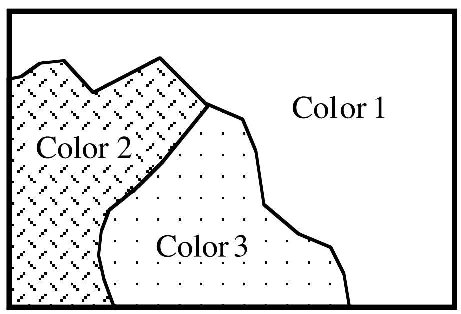


Figure 2: Map with chromatic number 3

Note: In this module, it is assumed that regions which share a border cannot be the same color and that each region is contained in one continuous border.

c. **1.** Design a map of four regions that has a chromatic number of 3.

2. Design another map of four regions that has a chromatic number of 4.

d. Predict the chromatic number you think would be sufficient to color any map drawn on a flat surface.

Discussion

a. **1.** How does the chromatic number of your straight-line map compare with those of others in the class?

2. How do you think the chromatic number would change if you drew a map using 6 straight lines? 15 straight lines? *n* straight lines?

b. Cartographers have long believed that four colors are enough to color any map of the earth. Do you think that a chromatic number of 4 is large enough for any map drawn on paper? Explain your response.

Assignment

1.1 Why does a checkerboard have a chromatic number of 2?

1.2 Using templates provided by your teacher, determine the chromatic number for maps of each of the following continents:

a. South America **b.** Australia

1.3 **a.** If a map has a chromatic number of 2, what is the minimum number of regions it can have?

b. If a map has a chromatic number of 2, what is the maximum number of regions it can have? Draw an example to illustrate your answer.

1.4 **a.** If a map has a chromatic number of 4, what is the minimum number of regions it can have? Draw an example to illustrate your answer.

b. If a map has a chromatic number of 4, what is the maximum number of regions it can have?

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1.5 Would you expect all regions on a map to have simple closed curves as boundaries? Explain your response.

1.6 **a.** Determine the maximum number of colors required to color a map consisting of 14 regions formed by intersecting straight lines. Justify your response.

b. Determine the minimum number of colors required to color the map described in Part **a**. Justify your response.

1.7 Tie a length of string into a simple loop. Drop the loop onto a flat surface so that it forms several regions. If the separate regions formed by the string represent countries on a map, what is the map's chromatic number? Sketch and color one example to support your response.

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Activity 2

Besides map making, **coloring theory** can be applied to other real-world situations. For example, the New York City Department of Sanitation uses coloring theory to schedule routes for garbage trucks, while computer scientists use it to analyze printed circuit boards.

Mathematics Note

A **graph** is a non-empty set of points or **vertices** V, along with a set of **edges** E. Each edge is a one- or two-element subset of V.

For example, the graph in Figure **3** is the set of points $\{M, N, O, P, Q\}$ and the set of edges $\{\{M,N\}, \{M,P\}, \{M,Q\}, \{N,O\}, \{N,P\}, \{N,Q\}, \{O\}, \{O,P\}, \{P,Q\}\}$. Notice that the edge represented by $\{O\}$ is a **loop**.

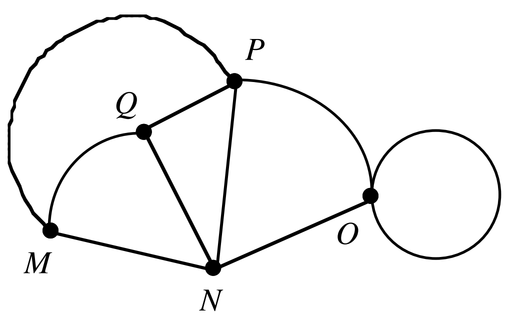


Figure 3: A graph with five vertices and nine edges

The **degree** of a vertex is the number of edges that meet at that vertex. In Figure **3**, the degree of vertex M is 3, the degree of vertex N is 4, the degree of vertex O is 4, the degree of vertex P is 4, and the degree of vertex Q is 3. Note that the loop contributes 2 to the degree of vertex O.

Exploration

Graphs may be used to represent many different types of information. For example, graphs can model computer networks, transportation routes, or molecular structures. In such graphs, the vertices might represent individual computers, cities, or atoms, respectively, while the edges represent their connections.

A map can also be represented by a graph. In this case, vertices correspond to regions, while edges correspond to shared borders. For example, Figure **4a** shows a map of six regions, R_1, R_2, R_3, R_4, R_5, and R_6. In Figure **4b**, these regions are represented by the vertices. The edges of a graph join vertices that represent regions with a common border.

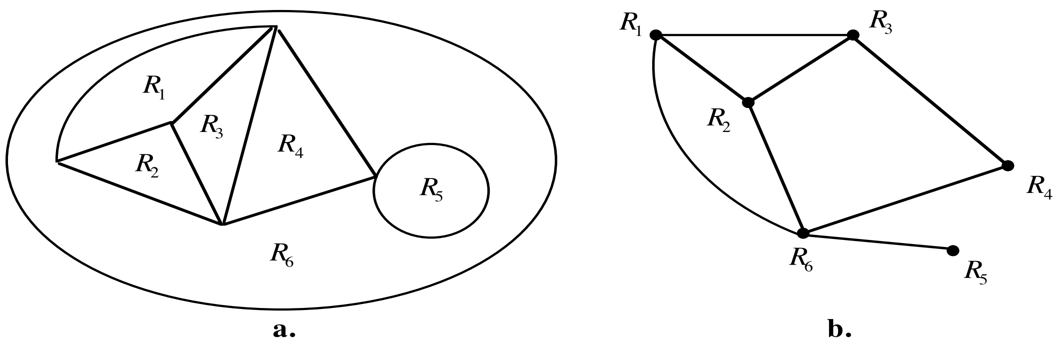


Figure 4: A map with its corresponding graph

On a map, no two regions that share a border, such as R_2 and R_3, can have the same color. Since the edges on the corresponding graph represent shared borders, no two vertices connected by an edge can have the same color. As a result, the chromatic number of the map in Figure **4a** is the same as the chromatic number of the graph in Figure **4b**. In this exploration, you use graphs to investigate a map's chromatic number.

a. **1.** Obtain a map of South America from your teacher. Draw a single dot inside each country's borders.

2. If two countries share a border, connect the two dots that represent them with a curve or a segment. The result is a graph of South America.

b. **1.** Compare your graph with those of others in the class.

2. Devise a method for determining when two graphs are equivalent.

c. Use your graph to determine the chromatic number of the map of South America.

Discussion

a. In your graph of South America, which vertex has the greatest degree? Which vertex has the least degree?

b. Is your graph equivalent to those of others in the class? Explain your response.

c. When using graphs to depict maps and their colors, would you ever need to use a loop as shown in Figure **3**? Explain your response.

Assignment

2.1 Obtain a map of the counties, parishes, or other subdivisions of your state or local area.

a. Make a graph of the map.

b. Label each vertex with its corresponding degree.

c. Determine the chromatic number of the map.

2.2 Imagine that you are the music director at your school. You must schedule half hour practice sessions for five musical ensembles. Only one student plays each instrument. Because some students play in more than one group, some groups cannot practice at the same time. The table below shows the instruments in each ensemble.

Duo	Trio	Quartet	Quintet	Jazz Ensemble
violin	tuba	violin	flute	clarinet
piano	trumpet	cello	clarinet	saxophone
	piano	viola	oboe	drums
		bassoon	piano	guitar
			bassoon	

A graph can help solve your scheduling problem. Complete Parts **a–d** below to determine how many different practice times are necessary.

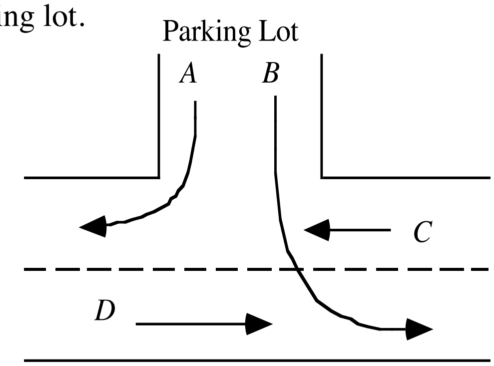
a. Draw a graph in which each vertex represents one of the ensembles. If any two ensembles contain the same instrument, connect the vertices that represent them with an edge. This indicates that the two groups should not be scheduled to rehearse at the same time.

b. Determine the chromatic number of the graph created in Part **a**.

c. The chromatic number of your graph is the minimum number of practice times necessary in the music schedule. Explain why this is true. Hint: Before showing that a number n is a minimum value, you must first demonstrate that it satisfies the constraints of the problem. You must then show that $n-1$ does not satisfy these constraints.

d. Devise a schedule of possible rehearsal times for the five groups.

2.3. The diagram below shows the intersection of a two-lane, two-way street with the exit from a school parking lot. The arrows in the diagram indicate the flow of traffic through the intersection. Cars leaving the parking lot can turn either to the left or to the right. Cars on the through street cannot enter the parking lot.



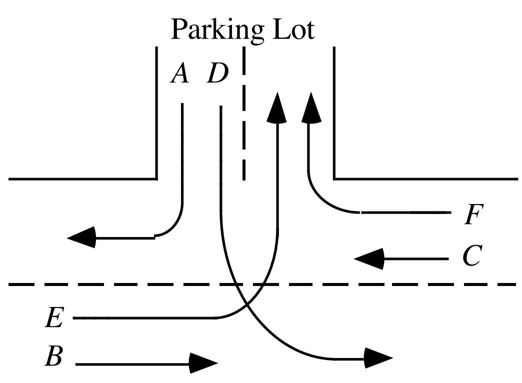
The city plans to install a new traffic light at this intersection. To determine how many settings the light requires, complete Parts **a–c**.

a. Draw a graph in which each vertex represents a direction of traffic. If two cars can collide while following the flow of traffic, connect the vertices that represent these directions with an edge.

b. To prevent collisions, how many different settings should the new traffic light have? Hint: Determine the chromatic number of the graph.

c. Determine which directions of traffic can move at the same time without resulting in collisions. Describe how you identified these directions.

2.4 Due to an increase in school enrollment, city planners have changed the exit from the parking lot in Problem **2.3** to both an entrance and an exit. The city has also added lanes for left and right turns to the through street, thus creating six possible directions of traffic.



a. Use the directions in Problem **2.3a** to create a graph of this situation.

b. How many traffic-light settings should be used at this intersection? Justify your response.

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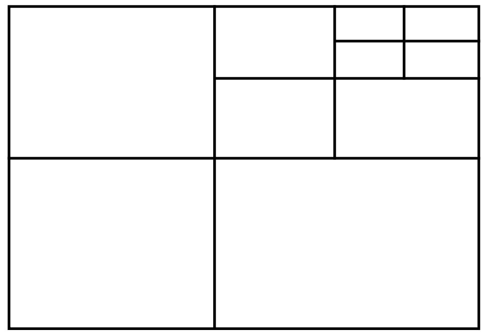
2.5 Color the following map using only three colors, given that regions which share a border cannot be the same color.



2.6 Use graphs to show that in a party of six people, three of the people either know each other or are complete strangers.

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Activity 3

Topology is a branch of mathematics that includes the study of the properties of shapes. In this activity, you investigate some connections between topology and coloring theory.

Mathematics Note

If one graph, surface, or shape can be stretched, shrunk, or distorted into another graph, surface, or shape without cutting, tearing, folding, or overlapping, then the two are **topologically equivalent**. When graphs are topologically equivalent, there is a one-to-one correspondence between the vertices so that corresponding edges connect corresponding vertices.

For example, the four graphs in Figure **5** are all topologically equivalent because corresponding edges connect corresponding vertices. **Note:** Edges that appear to overlap or intersect, as in Figure **5d**, do not indicate a vertex unless marked by a dot.)

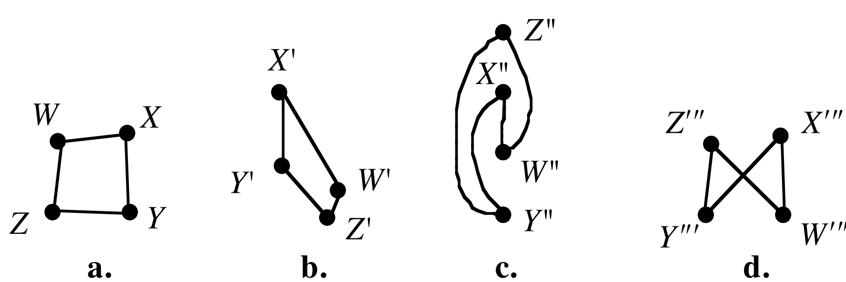


Figure 5: Four topologically equivalent graphs

A graph is **planar** if it is topologically equivalent to a graph drawn on a plane with no overlapping edges.

For example, the graph in Figure **6a** is planar because it is drawn in a plane with no pair of edges crossing or overlapping. The graph in Figure **6b** also is planar because it is topologically equivalent to the one in Figure **6a**.

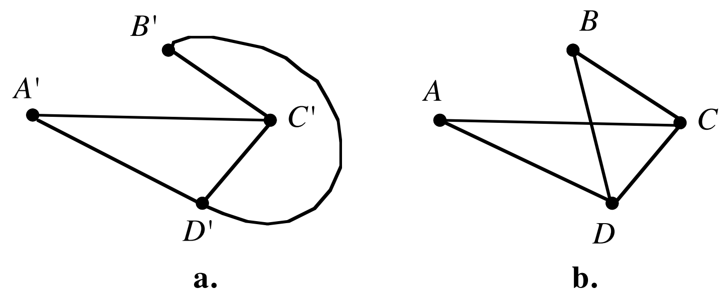


Figure 6: Two planar graphs

Discussion 1

a. Is the graph in Figure **5a** topologically equivalent to the one in Figure **6b**? Explain your response.

b. What changes could you make in the graph in Figure **5c** so that the resulting graph is topologically equivalent to the one in Figure **6a**?

Exploration

In this exploration, you distort graphs by stretching and shrinking them.

a.
1. Using a geometry utility, construct a graph with exactly five vertices and eight edges.
2. Determine whether or not your graph is planar.
3. Make a copy of the graph. Place the copy near the original graph.
4. Distort the copy by stretching or shrinking edges and moving vertices, while retaining five vertices and eight edges.
5. Determine whether or not the original graph and the distorted copy are topologically equivalent.

b.
1. Construct a set of four topologically equivalent graphs each of which has exactly 7 vertices and 10 edges.
2. Distort the graphs so that none of them look alike. Delete or add an edge in one or two of the graphs (leaving at least two of the graphs topologically equivalent).
3. Challenge another member of your class to identify the graphs that are topologically equivalent.

Discussion 2

a. Compare your strategy for identifying topologically equivalent graphs with those of others in the class.

b. Why are graphs no longer topologically equivalent when edges are added or deleted?

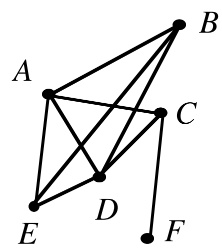
c. What characteristics might make it hard to determine whether or not two graphs are topologically equivalent?

d. Explain why graphs of the same map are always topologically equivalent.

e. Is the graph of any flat map a planar graph? Explain your response.

Assignment

3.1 Draw a graph that is topologically equivalent to the graph below, but has no overlapping edges (except at the vertices).



3.2 Consider two graphs that have the following characteristics in common: each has six vertices—three with a degree of 3, two with a degree of 2, and one with a degree of 1. Would two such graphs always be topologically equivalent? Use an example to support your response.

3.3 **a.** Consider maps of South America, Australia, and the continental United States. Are the graphs of these maps planar? Explain your response.

b. On the U.S. map, outline a subset of states whose graph is topologically equivalent to the graph of Australia.

c. Is it possible to outline a subset of states whose graph is topologically equivalent to the graph of South America? Justify your response.

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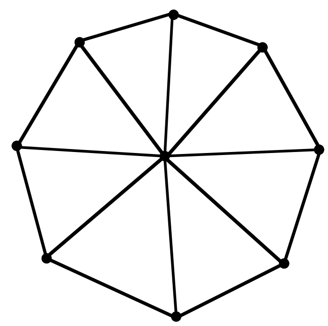
3.4 Describe an example of a one-to-one correspondence between the map of the subset of states in Problem **3.3b** and its corresponding graph.

3.5 **a.** On a map of the United States, outline a subset of states whose graph is topologically equivalent to the graph shown below.



b. On a copy of the graph in Part **a**, label each vertex using the two-letter abbreviation of the corresponding state.

3.6 Describe several ways that a graph of a map of the United States, including Alaska and Hawaii, differs from a graph of a map of South America.

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Activity 4

In the following activity, you investigate the chromatic numbers of various graphs. You also examine why it was so difficult to prove that a chromatic number of 4 is sufficient for any map drawn on a plane.

Exploration 1

In some computer video games, when an object moves off one side of the screen, it reappears on the opposite side. Likewise, when the object moves off the top of the screen, it reappears on the bottom. In this exploration, you determine the chromatic number for such computer screens by examining **complete graphs**.

Mathematics Note

In a **complete graph**, every pair of vertices is joined by exactly one edge. For example, the graph in Figure **7** is complete because every vertex is connected to every other vertex by exactly one edge.

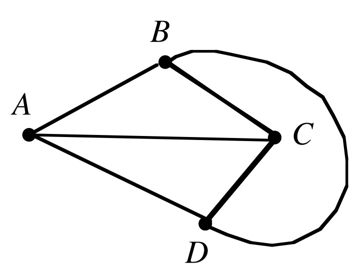


Figure 7: A complete graph

a.
1. Use pieces of string to create a complete graph with three vertices.
2. Determine whether or not your complete graph is planar.
3. Is every complete graph with this number of vertices planar? Justify your response.
4. Is it possible to create a map whose graph is equivalent to your complete graph? If so, draw one example.
5. Determine the chromatic number of the resulting map, if applicable.

b. Repeat Part **a** for a complete graph with four vertices.

c. Repeat Part **a** for a complete graph with five vertices.

d. The map in Figure **8** appears on a computer game screen. Players can move off the screen from region B and reappear at region A. They also can move off the screen from region C and reappear at region D.

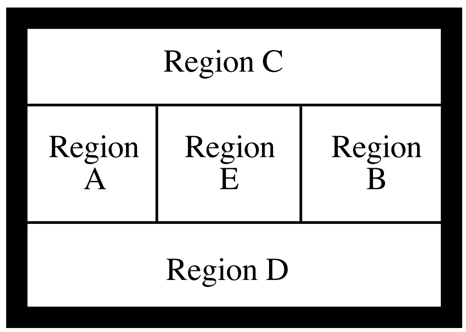


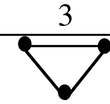
Figure 8: Map on a computer game screen

Since regions A and B have a common border, they cannot share the same color. Likewise, since regions C and D have a common border, they cannot share the same color. What is the chromatic number of this map?

e. Use pieces of string to create a complete graph of the map in Figure **8** with no overlapping edges.

f. Table **1** displays some characteristics of a complete graph with three vertices. The column heading "Possible Surfaces" refers to the surfaces on which it is possible to draw a particular graph without overlapping edges. Complete this table for graphs with one, two, four, and five vertices.

Table 1: Characteristics of complete graphs

No. of Vertices	1	2	3	4	5
Sample Graph					
Chromatic Number			3		
Possible Surfaces			any surface		

g. Draw a map that represents each sample graph you created in Table **1**.

Discussion 1

a. Describe any difficulties you encountered in Part **c** of Exploration **1**.

b. If the graph of a map is complete, what do you know about the countries on that map?

c. When is the graph of any flat map a planar graph?

d. What characteristics describe complete graphs that are also planar graphs?

e. Why can you model a complete graph with five vertices and no intersecting edges on a computer screen, but not on a flat surface?

f. Is there any relationship between the number of vertices in a complete graph and the chromatic number of the corresponding map? Explain your response.

g. Describe how your work in Exploration **1** helps show that four colors will always be sufficient to color any flat map.

Mathematics Note

The **four-color theorem** states that a chromatic number of 4 is sufficient to color any map drawn on a flat surface in which each region is contained in one continuous border and where regions that share a border must be a different color. In other words, planar maps never require more than four colors.

To prove this theorem, mathematicians searched for a planar graph that required five colors. After proving that a complete graph with five vertices could never be planar, however, they still had to consider whether or not other graphs that require five colors could be drawn in a plane.

In 1976, Kenneth Appel and Wolfgang Haken, two mathematicians from the University of Illinois, used a computer to check all the other possible graphs. The computer verified that no planar graph has a chromatic number greater than 4. It took thousands of diagrams and 1200 computer hours to prove the theorem.

Because the computer algorithms used in the proof have been questioned, Appel and Haken's work remains controversial. Other mathematicians are still searching for a simpler, more elegant, proof.

Exploration 2

In this exploration, you investigate the chromatic number of maps drawn on spherical surfaces.

a. Predict the chromatic number sufficient to color any map drawn on a globe in which each region is contained in one continuous border.

b. **1.** Use an inflated balloon to model a globe. On your balloon, draw a complete graph with four vertices and no intersecting edges.

2. Determine the graph's chromatic number.

3. Draw a map on your balloon that corresponds to the graph you constructed in Step **1**.

c. Is it possible to draw a complete graph with five vertices and no overlapping edges on a balloon? If so, draw one example.

Discussion 2

a. Describe some surfaces that are topologically equivalent to a balloon.

b. Describe any difficulties you encountered when you tried to draw a complete graph with five vertices on the balloon.

c. How do you think the chromatic number of maps drawn on a spherical surface compare to the chromatic number of maps drawn on a flat surface?

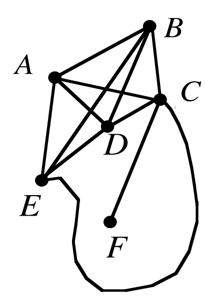
d. On a sphere, when are complete graphs of maps planar?

e. Explain why a sphere does not model a computer game screen like the one described in Exploration **1**.

f. What chromatic number is sufficient for any map drawn on a surface that is topologically equivalent to a sphere? Explain your response.

Assignment

4.1 **a.** Is the graph shown below a planar graph? Explain your response.



b. A non-empty subset of a graph's vertices and edges is a **subgraph**. Identify a subgraph of the graph in Part **a** that forms a complete graph with four vertices.

c. If a subgraph of a graph is not planar, then the graph cannot be planar.

1. Identify a subgraph of the graph in Part **a** that forms a complete graph with five vertices.

2. Is the subgraph in Part **c1** planar? Explain .

3. How can this subgraph help you to explain your response to Part **a**?

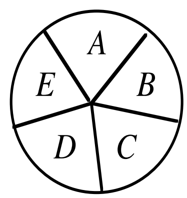
4.2 **a**. Draw a complete planar graph with four vertices.

b. Determine the degree of each vertex.

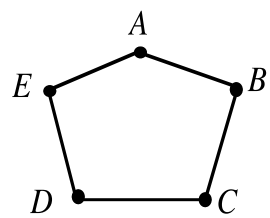
c. Predict the degree of each vertex in a complete graph with n vertices.

4.3 Identify the maximum number of vertices for any complete graph with no overlapping edges drawn on an egg and on a cylindrical surface with both bases. Justify your responses.

4.4 The "pie map" in the diagram below requires three colors—yet its graph does not have a complete subgraph with three vertices.



Pie Map



Graph of Pie Map

Create a map that requires four colors but whose graph does not have a complete subgraph with four vertices.

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4.5 Draw any polygon. Explain whether or not the polygon and all of its diagonals can be considered a planar graph.

4.6 **a**. What is the difference between a planar graph figure that is not planar graph. Explain your response.

b. Give an example of a planar figure that is not a planar graph.

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Research Project

Mathematicians have extended the drawing of maps to surfaces other than planes and spheres. Included in these surfaces are doughnut shapes and pretzel shapes. Research the chromatic numbers of maps drawn on a **torus** (a doughnut shape). Your report should include a description of the equation used to find the chromatic number for a torus, as well as an example of a map that requires the maximum chromatic number for this surface.

Summary Assessment

1. The diagram below shows the arrangement of desks in a classroom.

1	6	11	16
2	7	12	17
3	8	13	18
4	9	14	19
5	10	15	20

To administer a test to a class, a teacher would like to create enough different versions so that no two adjacent students—either vertically, horizontally, or diagonally—have exactly the same test.

a. If one student sits at each desk, how many different versions of the test should the teacher create? Justify your response.

b. On a copy of the diagram above, show which version of the test should be placed at each desk.

Module Summary

- The **chromatic number** of a map is the least number of colors required to color the map.
- A **graph** is a non-empty set of points or **vertices** V, along with a set of **edges** E of one- or two-element subsets of V.
- The **degree** of a vertex is the number of edges that meet at that vertex.
- **Topology** is the branch of geometry concerned with the properties of shapes.
- If a graph, surface, or shape can be stretched, shrunk, or otherwise distorted into another graph, surface, or shape without cutting, tearing, folding, or overlapping, the two graphs are **topologically equivalent**.
- A graph is **planar** if it is topologically equivalent to a graph drawn on a plane with no overlapping edges.
- In a **complete graph**, every pair of vertices is joined by exactly one edge.
- The **four-color theorem** states that a chromatic number of 4 is sufficient to color any map drawn on a flat surface in which each region is contained in one continuous border and regions that share a border must be a different color. In other words, planar maps never require more than four colors.

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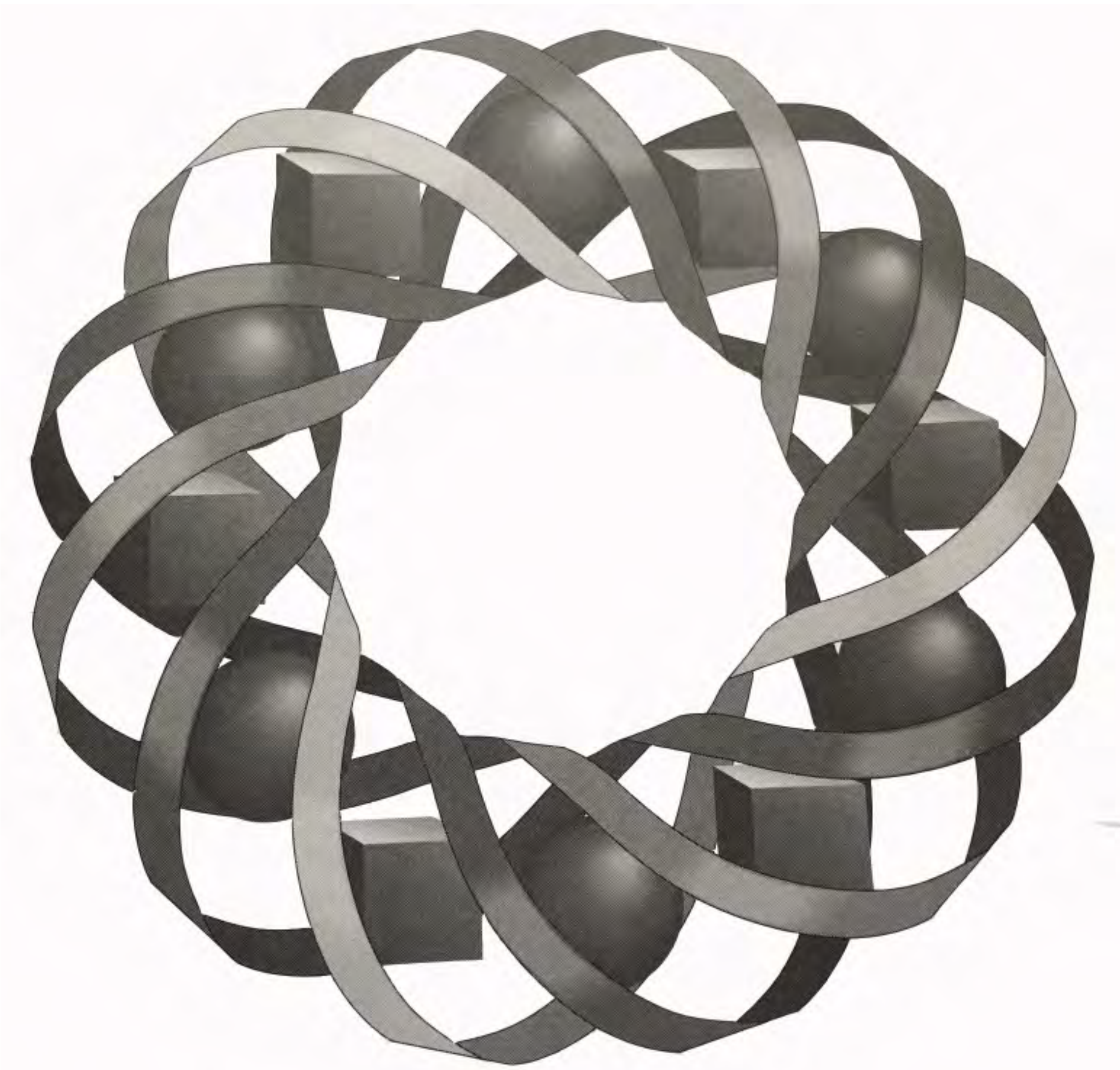
Can It!



How can you model such diverse phenomena as the height of tides in the ocean, the flow of electricity in a circuit, and the hours of daylight in a year? In this module, you explore functions that can model events which occur over and over again.

Margaret Plouvier • Todd Robins • Teri Willard

Can It!

Introduction

Mathematicians have long used functions to model real-world situations. Many of these situations involve events that repeat over time. For example, the positions of the hands on the face of a clock, the rise and fall of ocean tides, and the hours of sunlight in the days of the year all occur in predictable cycles. In this module, you investigate the functions required to model these and other **cyclic** events.

Activity 1

Consider the cyclic behavior of the second hand as it moves around the face of a clock. As the hand sweeps around the face, every point on the dial is visited again and again. Given the distance that the second hand moves in each interval of time, it is possible to model its motion and predict its location at any particular time.

Exploration

Although time is measured on a linear scale, its passing can also be represented by the circular motion of a second hand. In this exploration, you simulate the "wrapping" of a linear scale onto a circle.

a.

1. Use a can to trace a circle on a sheet of graph paper.
2. Identify the center of the circle.
3. As shown in Figure **1**, create a coordinate system with its origin at the center of the circle. Let the radius of the circle represent 1 unit. A circle with a radius of 1 unit is a **unit circle**.

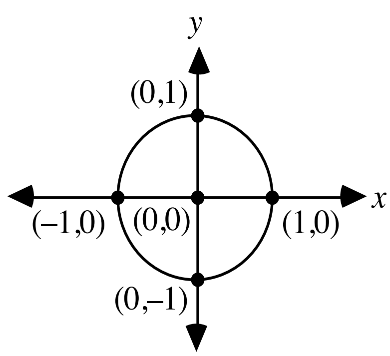


Figure 1: Coordinate system and unit circle

b.

1. Cut a paper strip with a length of approximately 10 units, where 1 unit equals the radius of the can in Part **a**.
2. Create a number line on one side of the strip, beginning with 0 on the left. Label the points on the number line that represent the whole numbers from 0 to 9, where 1 unit equals the radius of the can.
3. Label the approximate location of each of the following real numbers on the number line: $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$, $7\pi/4$, 2π, $9\pi/4$, $5\pi/2$, and 3π.
4. Label the back of your paper-strip number line with the additive inverses of the labels on the front. Align the labels so that 0 is directly behind 0, $-\pi/4$ is directly behind $\pi/4$, -1 is directly behind 1, and so on.
5. Tape the left-hand end of your number line to the bottom of the can, as shown in Figure **2**.

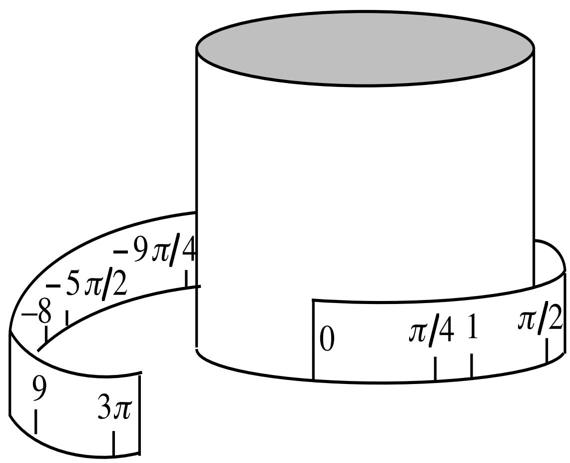


Figure 2: Placement of number line

c. Position the can on your drawing of a unit circle from Part **a** so that the 0 on the number line corresponds with the point (1,0). Use the number line to mark the points on the circle that correspond to labeled points on the number line. This process simulates the function that pairs each point on the real number line with a location on the unit circle, often referred to as a **wrapping function**.

Label the point on the circle that corresponds with 0 on the number line as $W(0)$, the point that corresponds with $\pi/4$ as $W(\pi/4)$, and so on. In this case, *W* indicates that the point has been assigned to the real number by the wrapping function *W*. If any point on the circle corresponds to more than one point on the number line, mark it with all corresponding labels.

d. The location of each point on the unit circle is identified by an ordered pair of coordinates. Approximate the ordered pair that corresponds to each labeled point on the circle. Record them in a spreadsheet with headings like those in Table **1** below.

Table 1: Real numbers and corresponding *xy*-coordinates

Real Number	*x*-coordinate	*y*-coordinate
0	1	0
$\pi/4$	0.7	0.7
1	0.5	0.8
⋮	⋮	⋮

e. Draw a ray from the origin through the point that corresponds to $W(1)$.

f. Leaving the left-hand end of the number line taped to the can, wrap the number line so that the negative values are showing. Repeat Parts **c** and **d** for these values.

g. Create two scatterplots: one in which the domain is the real numbers and the range is the *x*-coordinates, and another in which the domain is the real numbers and the range is the *y*-coordinates.

Note: Save your number line, drawing of the unit circle, and spreadsheet for use throughout the module.

Discussion

a. In the exploration, each labeled point on your number line was matched with an ordered pair on the circle. If the number line were continued indefinitely, how many real numbers would you expect to be paired with each point on the circle?

b.
1. What is the domain of the wrapping function used in the exploration?
2. What is the range of this function?

c. Describe the scatterplots you created in Part **g** of the exploration.

d. Using the wrapping function, what are the greatest and least values that result for each of the following:
1. the *x*-coordinate?
2. the *y*-coordinate?

e. Consider the length of the arc from (1,0) to the point $W(1)$. What is the ratio of this arc length to the radius of the unit circle?

Mathematics Note

On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**. In general, the measure of a central angle in radians is the ratio of the length of the intercepted arc to the radius of the circle.

In Figure **3**, for example, the measure of angle θ is 1 radian because the ratio of the arc length to the radius is $r/r = 1$.

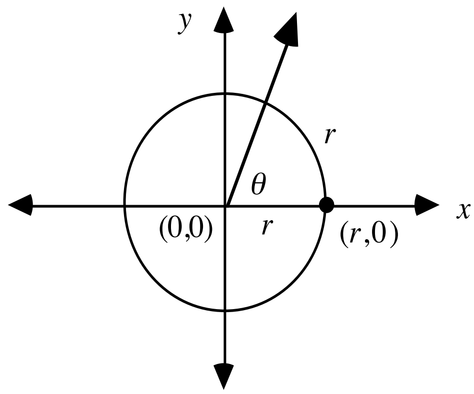


Figure 3: An angle with a measure of 1 radian

f. **1.** What is the measure, in radians, of the central angle that defines the circumference of a circle?

2. Why doesn't this value change as the size of the circle changes?

g. Based on your responses to Part **f** of the discussion, how are radian measures and degree measures related?

Assignment

1.1 **a.** Describe the location of the point (0,1) on the unit circle using a real number.

b. Describe the measure of the central angle whose sides intercept the unit circle at (1,0) and (0,1) in each of the following units:

1. degrees

2. radians

1.2 Consider a point on the unit circle that is paired with the real number 3. Identify a second real number that is paired with this point and describe how you determined this number.

1.3 Describe the radian measures of two central angles, one positive and one negative, whose sides intercept the unit circle at (1,0) and each of the following points:

a. (1,0)

b. (–1,0)

c. a point one-fourth of the way cloclwise around the circle

d. a point four-thirds of the way clockwise around the circle

1.4 Two children are sitting on a merry-go-round. One child is 1 m from the center and the other is 1.5 m from the center.

a. How many radians has each child rotated through after 3 complete rotations of the mery-go-round?

b. Determine the total distance traveled by each child after 3 complete rotations of the merry-go-around.

1.5 Consider a circle with center at the origin and a radius of 5 units. Assume a point starts at (5,0) and moves around the circle at a constant rate of 5 units per second.

a. How many seconds will it take the point to complete one revolution?

b. How long will it take the point to reach (–5,0)?

1.6 The earth's orbit around the sun is roughly circular, with a radius of approximately $1.496 \bullet 10^{8}$ km.

a. Assuming that the earth moves at a constant speed, how far does it travel in 1 hr?

b. How far does the earth travel between April 1 and July 1?

* * * * *

1.7 **Angular velocity** describes the rate of change in a central angle over time. Consider a wheel that makes 150 revolutions per minute. What is the wheel's angular velocity in radians per second?

1.8 A rotating water sprinkler makes one revolution every 20 sec. The water reaches a distance of 10 m from the sprinkler.

a. What is the arc length of the sector watered when the sprinkler has rotated through $5\pi/2$ radians?

b. What is the area of the sector watered when the sprinkler has rotated through $\pi/4$ radians?

1.9 A clock maker is designing the face of a watch. The watch face is a circle with a radius of 1 cm. To help specify the location of each element of the design, the clock maker has placed the center of the face at the origin of a two-dimensional coordinate system.

a. Determine the coordinates of the points where the hours 3, 6, 9, and 12 are located. Include a diagram with your response.

b. **1.** Determine the coordinates of the point where 11 o'clock is located.

2. Describe three other points on the face of the watch whose coordinates have the same absolute values as the coordinates of the point associated with 11 o'clock. Identify the hour that corresponds with each pair of coordinates.

c. Determine the coordinates of the hour 33 hours after 12 o'clock.

* * * * * * * * * *

Activity 2

In Activity **1**, you used a wrapping function to assign each real number to a location on a unit circle with center at the origin. This allowed you to determine a corresponding
x-and y-coordinate for each real number. This wrapping function provides the basis for several other mathematical functions. Because of their association with the unit circle, these functions are often referred to as **circular functions**. In this activity, you examine the graphs and identify some of the characteristics of three circular functions.

Discussion 1

a. Figure **4** shows a right triangle ABC constructed in a unit circle with center at the origin.

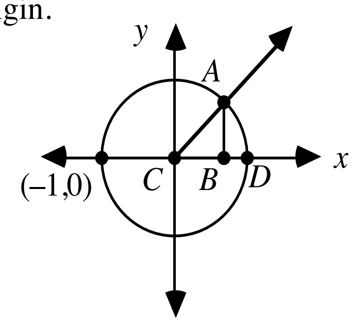


Figure 4: Construction of a triangle in a unit circle

1. Use right-triangle trigonometry to describe the relationship between AB and $\angle ACB$. Remember $AC = 1$.
2. Use right-triangle trigonometry to describe the relationship between BC and $\angle ACB$.

Mathematics Note

The **sine function**, $f(t) = \sin(t)$, uses the wrapping function to assign a real number t to the y-coordinate of the corresponding point on a unit circle with center at the origin.

The **cosine function**, $f(t) = \cos(t)$, uses the wrapping function to assign a real number t to the x-coordinate of the corresponding point on a unit circle with center at the origin.

For example, the point on the unit circle assigned to $t = 0$ by the wrapping function has coordinates (1,0). Since the x-coordinate of this point is 1, $\cos(0) = 1$. Since the y-coordinate of this point is 0, $\sin(0) = 0$.

b. When a real number is assigned by the wrapping function to a point on the unit circle, the point can be located in one of four possible quadrants.

1. In which quadrants are values of $\sin(t)$ positive? In which quadrants are they negative?
2. In which quadrants are values of $\cos(t)$ positive? In which quadrants are they negative?

c. Figure **5** shows the line tangent to the unit circle at point D.

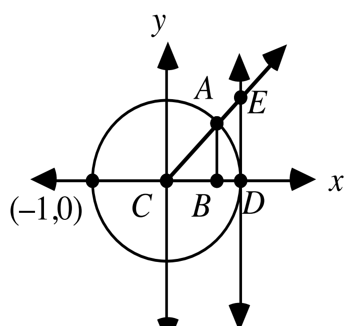


Figure 5: Unit circle with a tangent line

1. What is the measure of $\angle CDE$? Explain your response.
2. Why is $\Delta ABC \sim \Delta EDC$?
3. Using similar triangles, how could you show that $DE = AB/BC$?

d. Considering your responses to Part **a** of the discussion, what would you expect to be the relationship between DE and $\tan \angle ACB$?

Mathematics Note

The **tangent function**, $f(t) = \tan(t)$, where t is any real number except an odd multiple of $\pi/2$, is the ratio of the y-coordinate to the x-coordinate of the point assigned to t by the wrapping function of the real number line around a unit circle with center at the origin.

For example, the point on the unit circle assigned to $t = 0$ by the wrapping function has coordinates (1,0). Since the x-coordinate for this point is 1 and the y-coordinate is 0, $\tan(0) = 0/1 = 0$.

e. In the mathematics note, values of t that are odd multiples of $\pi/2$ are excluded from the domain of the tangent function. Explain why this restriction exists.

f. In which quadrants are values of $\tan(t)$ positive? In which quadrants are they negative?

Exploration

In Activity **1**, you used a number line to mark points on a unit circle. In this exploration, you use these points to create a graph of one of the circular functions.

a. Choose one of the three circular functions: sine, cosine, or tangent.

b.
1. Draw a two-dimensional coordinate system on a sheet of freezer paper.
2. Use your number line from Activity **1** to label the x-axis for each marked point on the number line from -3π to 3π.
3. Label the following additional points on your number line from Activity **1**: 0.25, 0.5, 0.75, 1.25, 1.5, 1.75.
4. Use the number line to label the y-axis in increments of 0.25 for at least 2 units above and 2 units below the origin.

c. On your drawing of the unit circle from Activity **1**, construct a line tangent to the circle at the point $(1,0)$ and $(-1,10)$.

d.
1. Draw a ray from the origin to the first mark on the unit circle (the point that corresponds with $\pi/4$).
2. Construct a perpendicular to the x-axis from the point where the ray intersects the circle. Your drawing should now resemble Figure **6**.

   Note: If you have chosen the tangent function, you must use the intersection of the ray and the tangent through $(-1,0)$ for measures greater than $\pi/2$ radians and less than $3\pi/2$ radians.

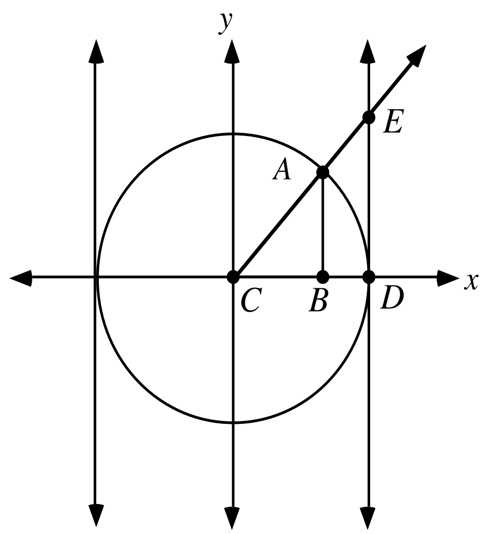


Figure 6: Construction on a unit circle

3. Cut a paper strip whose length corresponds to $f(\pi/4)$, where f represents the circular function you have chosen to plot. (Recall that in Figure **6**, CB corresponds with $\cos\angle ACB$, BA with $\sin\angle ACB$, and DE with $\tan\angle ACB$.)
4. Glue the paper strip onto the coordinate system you created in Part **b**, so that its vertical axis of symmetry aligns with the point on the x-axis that corresponds to $\pi/4$.

   If the value of the function is positive, glue the strip above the x-axis. If the value is negative, glue the strip below the x-axis.

   For example, Figure **7** shows a paper strip representing $\sin(\pi/4)$ glued in the appropriate position.

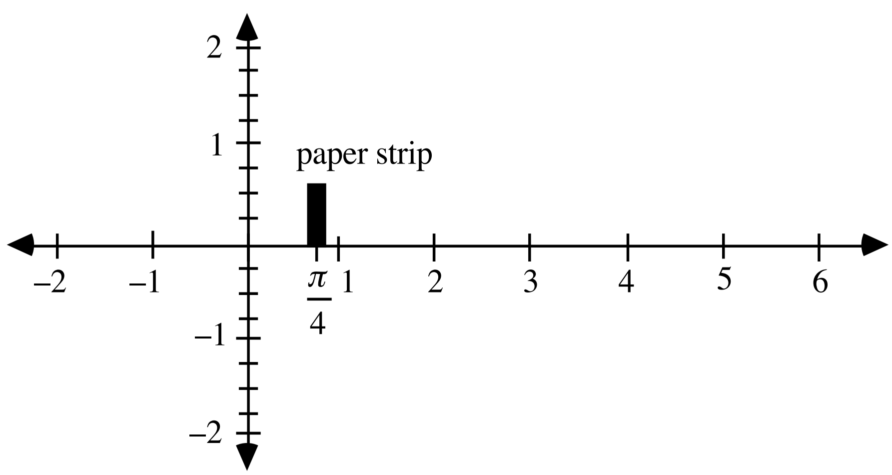


Figure 7: Position of paper strip on coordinate system

e. Repeat the process described in Part **d** for each mark on the unit circle.

f. The center of the top of each paper strip represents a point on the graph of the function. Sketch a smooth curve connecting these points, including points where the value of the function is 0.

g. Using a graphing utility, graph each of the following functions on a separate coordinate system. Describe the characteristics of each graph.

1. $y = \sin x$
2. $y = \cos x$
3. $y = \tan x$

Discussion 2

a. Display your paper-strip graph to the class and describe its characteristics.

b. What do the coordinates of each point on the smooth curves drawn in Part **f** represent?

c. Compare the paper-strip graphs, the scatterplots created in Activity **1**, and the graphs created using a graphing utility in Part **g** of the exploration.

d.
1. What interval of the domain was graphed before the values of each function began to repeat?
2. How are these intervals related to the unit circle?

e.
1. Identify the domain and range for each of the three circular functions: sine, cosine, and tangent.
2. How does the unit circle determine the range values for each of these functions?

Mathematics Note

A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest interval of the domain over which the function repeats.

The **absolute maximum** of a function is the greatest value of the range. The **absolute minimum** is the least value of the range.

If both an absolute maximum and an absolute minimum exist in a periodic function, the **amplitude** of the function is half the distance between them. If M represents the absolute maximum and m represents the absolute minimum, the amplitude can be found by the following formula:

$$\left|\frac{M - m}{2}\right|$$

For example, Figure **8** shows a periodic function with a period of 2. In other words, the function completes 1 cycle for every interval of 2 units in the domain.

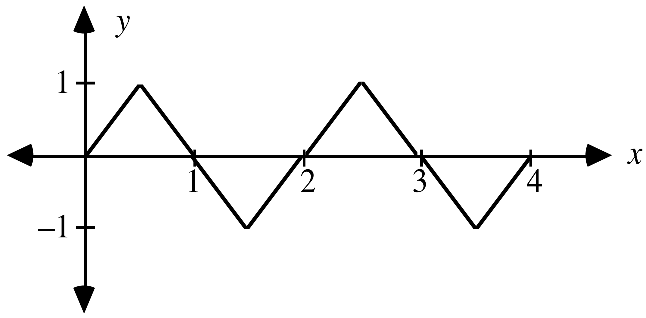


Figure 8: A periodic function

Since the absolute maximum is 1 and the absolute minimum is –1, the amplitude of this function can be found as follows:

$$\left|\frac{1-(-1)}{2}\right| = 1$$

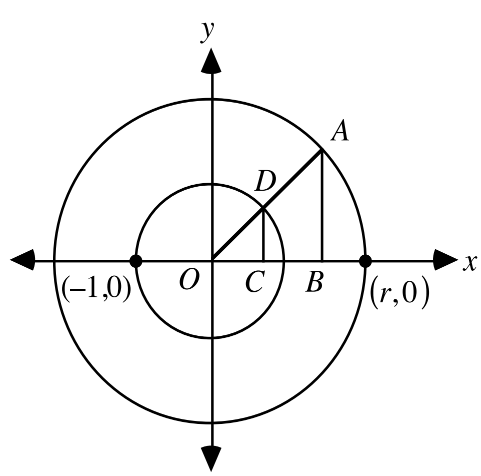
f. Identify the period and amplitude of the sine, cosine, and tangent functions.

g. How does the relationship below explain the characteristics of the graph of the tangent function?

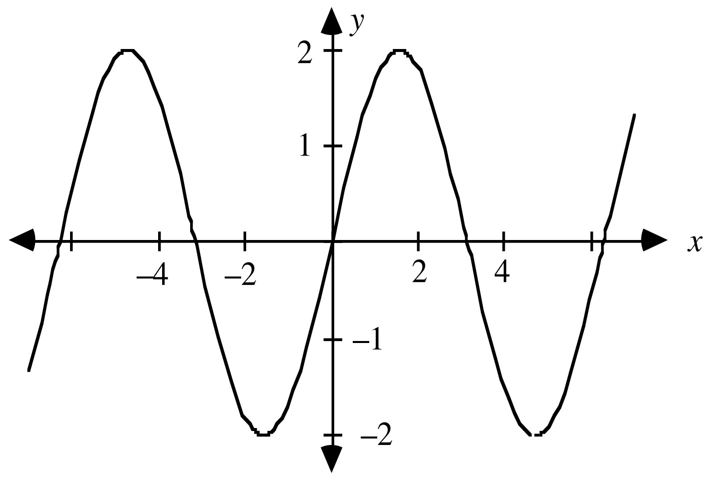
$$\tan x = \frac{\sin x}{\cos x}$$

Assignment

2.1 The figure below shows a unit circle with center at the origin and a circle of radius *r* with center at the origin. As you have seen in the previous activity, the coordinates of point *D* are $(\cos\angle DOC, \sin\angle DOC)$. What are the coordinates of point *A*? Justify your response using similar triangles.



2.2 Use the graph below to answer the following questions.



a. What is the period of the curve?

b. What is the amplitude of the curve?

c. Sketch a curve that has the same period as the given curve but twice its amplitude.

d. Sketch a curve that has the same amplitude as the given curve but twice its period.

e. Sketch a curve that has twice the amplitude and twice the period of the given curve.

2.3 **a.** Graph $y = \sin x$ over the interval $[0, 4\pi]$ and $y = 0.5$ on the same coordinate system.

b. Determine the values of *x* for which $\sin x = 0.5$ and explain how you identified these values.

c. If you had graphed $y = \sin x$ over the interval $[-4\pi, 4\pi]$, for how many values of *x* would $\sin x = 0.5$? Justify your response.

2.4 **a.** An **identity** is an equation that is true for all real numbers. For example, the equation $0 \bullet x = 0$ is an identity since it is true for any real number *x*. There are many identities involving circular functions, including the three shown below. Verify that each of these equations is true for several values of θ.

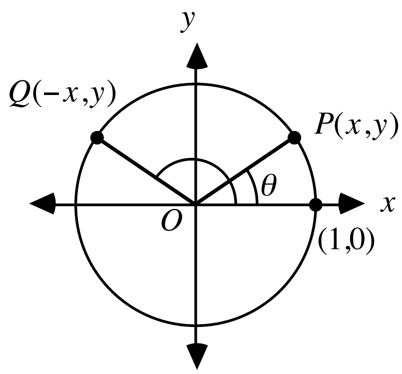
1. $\sin\theta = \sin(\pi - \theta)$

2. $\cos\theta = \cos(-\theta)$

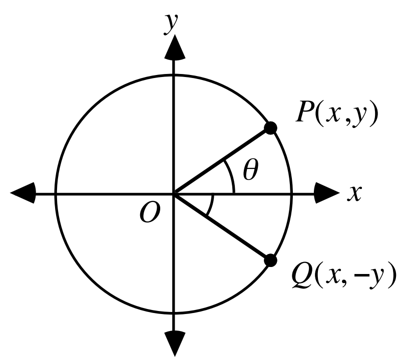
3. $\tan\theta = \tan(\pi + \theta)$

b. The diagrams below show a unit circle with center at the origin and two points on the circle, *P* and *Q*. Also, $\angle QOR$ is equal to θ in each diagram. Use these figures to explain why each identity is true.

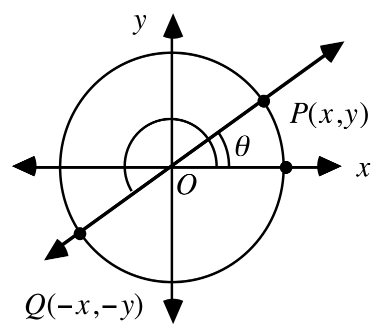
1. $\sin\theta = \sin(\pi - \theta)$



2. $\cos\theta = \cos(-\theta)$



3. $\tan\theta = \tan(\pi + \theta)$



c. How could you use the identities listed in Part **a** to determine two real numbers that have the same sine, cosine, or tangent?

2.5 Chronobiology is the study of biological rhythms. Scientists in this field have found that seasonal changes in the length of the day can influence the behavior and metabolism of animals. The following table lists the number of daylight hours for specified dates in Boston, Massachusetts.

Date	Day	Daylight Hours	Date	Day	Daylight Hours
12/01/91	1	9.30	06/28/92	211	15.25
12/15/91	15	9.10	07/01/92	214	15.23
12/28/91	28	9.10	07/15/92	228	14.98
01/01/92	32	9.13	07/28/92	241	14.60
01/15/92	46	9.43	08/01/92	245	14.47
01/28/92	59	9.85	08/15/92	259	13.92
02/01/92	63	10.00	08/28/92	272	13.33
02/15/92	77	10.58	09/01/92	276	13.17
02/28/92	90	11.18	09/15/92	290	12.50
03/01/92	92	11.23	09/28/92	303	11.88
03/15/92	106	11.88	10/01/92	306	11.75
03/28/92	119	12.52	10/15/92	320	11.10
04/01/92	123	12.72	10/28/92	333	10.52
04/15/92	137	13.37	11/11/92	337	10.35
04/28/92	150	13.93	11/15/92	351	9.80
05/01/92	153	14.07	11/28/92	364	9.40
05/15/92	167	14.60	12/01/92	367	9.30
05/28/92	180	14.98	12/15/92	381	9.10
06/01/92	184	15.07	12/28/92	394	9.10
06/15/92	198	15.27			

Source: *The Old Farmer's Almanac*, 1992.

a. Use a graphing utility to create a scatterplot of the data.

b. Does the resulting graph appear to be related to the sine function? Explain your response.

c. What is the amplitude of the curve? Explain your response.

d. What is its period? Explain your response.

* * * * *

2.6 In Problem **2.4**, you examined three trigonometric identities. There are many, many more. The three identities listed in Part **a**, for example, show relationships between the sine and cosine functions.

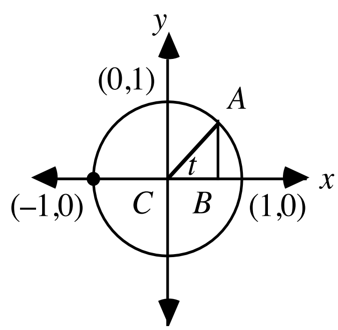
a. Verify that the following equations are true for several values of t.

1. $(\sin t)^2 + (\cos t)^2 = 1$

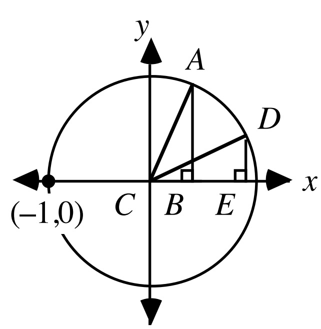
2. $\sin t = \cos\left(\dfrac{\pi}{2} - t\right)$

3. $\cos t = \sin\left(\dfrac{\pi}{2} - t\right)$

b. Use right-triangle trigonometry, the Pythagorean theorem, your knowledge of the circular functions, and the figure below to argue that $(\sin t)^2 + (\cos t)^2 = 1$ is an identity for all real numbers that can be represented in the first quadrant on a unit circle.



c. In the following diagram, $m\angle DCE = (\pi/2) - m\angle ACB$. Use this relationship, along with right-triangle trigonometry, similar triangles, and your knowledge of the circular functions, to argue that $\sin t = \cos((\pi/2) - t)$ is an identity for all real numbers that can be represented in the first quadrant on a unit circle.



2.7 A carousel has a radius of 6 m and completes 6 revolutions in 1 min.

a. Through how many radians does the carousel turn in 1 min?

b. A horse named Moonlight is located 5 m from the center of the carousel. Assuming that the center of the carousel is located at the origin of a two-dimensional coordinate system, and that Moonlight's initial position is (5,0), what are Moonlight's coordinates after 3 min?

c. A horse named Red Ribbon is located 3.5 m from the center of the carousel, next to Moonlight. What are Red Ribbon's coordinates after 3 min?

d. Determine the coordinates for the locations of Moonlight and Red Ribbon after $13\pi/4$ radians.

2.8 Some people believe that biological rhythms influence our personal lives. In its simplest form, the biorhythm theory states that from birth to death, each of us is influenced by three internal cycles: the physical, the emotional, and the intellectual.

The 23-day physical cycle affects a broad range of bodily functions, including resistance to disease and strength. The 28-day emotional cycle governs creativity, sensitivity, and mood. The 33-day intellectual cycle regulates memory, alertness, ability to learn, and other mental processes.

According to this theory, each of the cycles starts at a neutral baseline or zero point on the day of birth. From that zero point, the three cycles enter a rising, positive phase, during which the energies and abilities associated with each one are high, then gradually decline. Each cycle crosses the zero point midway through its period, entering a negative phase in which physical, emotional, and intellectual capabilities are somewhat diminished, and energies are recharged.

Since the three cycles have different periods, the highs, lows, and baseline crossings rarely coincide. As a result, people are usually subject to a mix of biorhythms and their behavior—from physical endurance to academic performance—is a composite of these varying influences.

a. Sketch your first physical, emotional, and intellectual cycles on the same set of axes. On this graph, let the origin represent birth.

b. What is the period for each of the three cycles?

c. How does the graph of a sine curve compare with the graph of the physical cycle? the emotional cycle? the intellectual cycle?

d. According to biorhythm theory, the most vulnerable days occur when each cycle crosses the baseline, switching from positive to negative or vice versa. These are "critical days." Mark and label the critical days for each of the three cycles on the graph.

e. Determine your biological rhythms for today.

f. In an average human life span of 70 years, when will the three cycles coincide on the baseline?

* * * * * * * * * *

Activity 3

In order to approximate cyclic events with circular functions, it is often necessary to modify $y = \sin x$ or $y = \cos x$. In this activity, you investigate the general forms of the circular functions and explore how to transform the shapes of their graphs.

Exploration

The general form of the sine function is $y = a(\sin b(x + c)) + d$, where a, b, c, and d are real-number parameters. In this exploration, you investigate the transformations that result from using different values for these four parameters.

a.
1. Using a graphing utility, graph the functions $y = \sin x$, $y = 2\sin x$, $y = 4\sin x$ where x is a real number, on the same coordinate system. **Note:** Set your graphing utility to report angle measures in radians.
2. On a second coordinate system, graph $y = \sin x$, $y = -2\sin x$, and $y = -4\sin x$.
3. Make a prediction about the role of a in the graph of $y = a\sin x$.
4. Check your prediction by graphing $y = \sin x$ and $y = a\sin x$ for various values of a.

b.
1. On the same coordinate system, graph $y = \sin x$ and $y = \sin bx$ for a chosen value of b.
2. Repeat Step **1** for two other values of b.
3. Make a prediction about the role of b in the graph of $y = \sin bx$
4. Check your prediction by graphing $y = \sin x$ and $y = \sin bx$ for various values of b.

c. Using a process similar to the one described in Part **b**, determine the role of c in the graph of $y = \sin(x + c)$. Hint: Examine the graphs for the following values of c: $\pi/4$, $\pi/2$, π, $-\pi/4$, $-\pi/2$, and $-\pi$.

d. Using a process similar to the one described in Part **b**, determine the role of d in the graph of $y = \sin x + d$.

e. **1.** Based on your results in Parts **a–d**, sketch a graph of $y = 3\sin\left(2(x - \pi)\right) - 5$ on graph paper.

2. Check the accuracy of your sketch by graphing the equation on a graphing utility.

Discussion

a. Describe the role of each of the four parameters in the graph of the equation $y = a\left(\sin b(x + c)\right) + d$.

b. Defend the following statement: "The graph of the cosine function is a transformation of the graph of the sine function."

Mathematics Note

Given a function of the form $y = a\left(\sin b(x + c)\right) + d$, the parameters a, b, c, and d transform the graph of $y = \sin x$ in the following manner:

- The amplitude of the graph is $|a|$.
- If a is negative, the graph is a reflection of $y = \sin x$ in the x-axis.
- The period of the graph is $2\pi/b$.
- If b is negative, the graph is a reflection of $y = \sin x$ in the y-axis.
- The graph is a horizontal translation of $y = \sin x$ of $-c$ units.
- The graph is a vertical translation of $y = \sin x$ of d units.

For example, the graph of $y = -3\left(\sin 2(x + \pi/2)\right) + 1$ has an amplitude of 3 and a period of π. As shown in Figure **9**, the graph also represents a horizontal translation of $-\pi/2$ units, a vertical translation of 1 unit, and a reflection in the x-axis of a graph of $y = \sin x$.

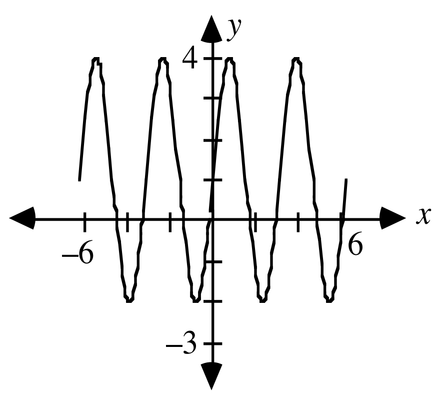
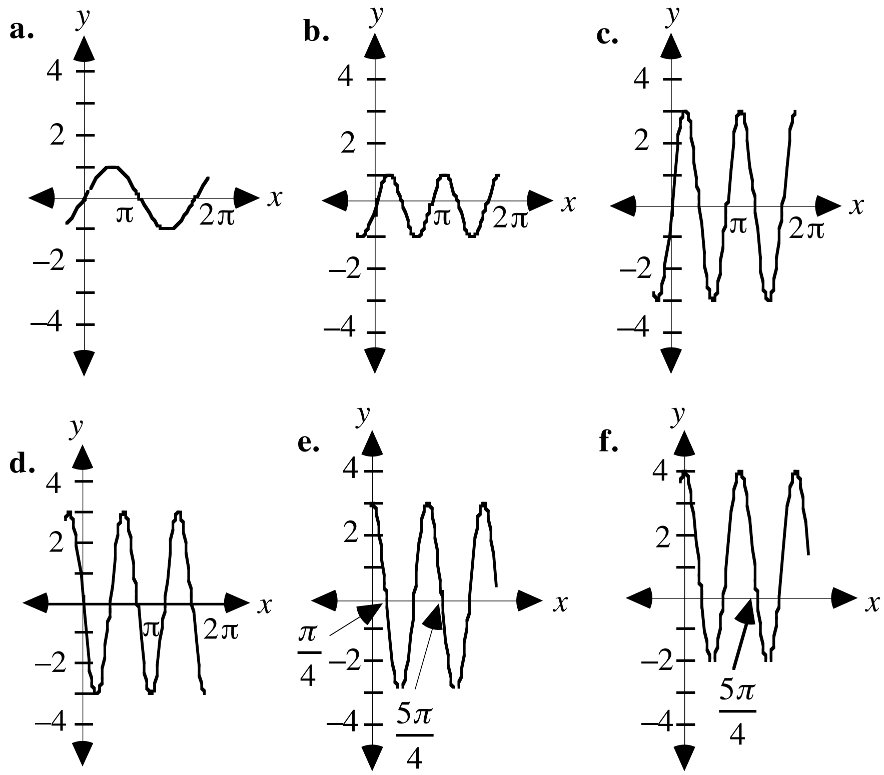


Figure 9: Graph of $y = -3\sin\left(2(x + \pi/2)\right) + 1$

These parameters play the same roles in the graph of $y = a\left(\cos b(x + c)\right) + d$.

Assignment

3.1 The graph of $y = -3\sin\left(2(x - \pi/4)\right) + 1$ can be obtained from the graph of $y = \sin x$ using the sequence of transformations shown in Parts **a–f** below. Determine the equation of each graph.

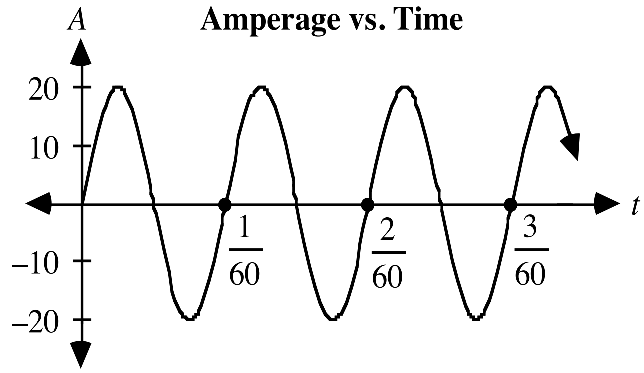


3.2 The flow of electrical current is measured in **amperes** (A). In the United States, ordinary household circuits use alternating current of 20 A. An alternating current flows in one direction during part of a generating cycle, and in the opposite direction during the rest of the cycle.

The rate at which the current alternates is its **frequency**. Frequency is measured in cycles per second. A frequency of 1 cycle per second equals 1 **hertz** (Hz). The frequency of ordinary household current is 60 Hz.

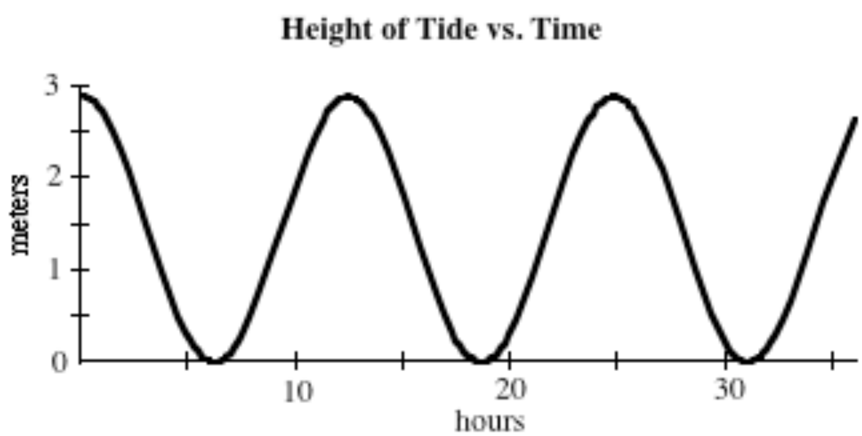
Because of the way in which the flow of alternating current changes over time, it can be modeled by a sine function, as shown in the following graph.

Determine the equation of the curve below using A to represent amperage and t to represent time. (Hint: The period of this function equals the reciprocal of the frequency.)



3.3 In the ocean near Boston, Massachusetts, the average high tide exceeds the level of the water at low tide by 2.9 m. The tide comes in and goes out every 12.4 hr. This fluctuation in height can be roughly approximated by a cosine curve, as shown in the following graph.

Determine an equation for the curve below, using h to represent the height of the water above low tide and t to represent the number of hours since high tide. Assume that the water is at high tide at $t = 0$.



3.4 The Bay of Fundy in eastern Canada has the highest tides in the world. The tides there rise and fall by as much as 15 m. The average high tide is approximately 8 m and the tidal cycle takes 12.4 hr.

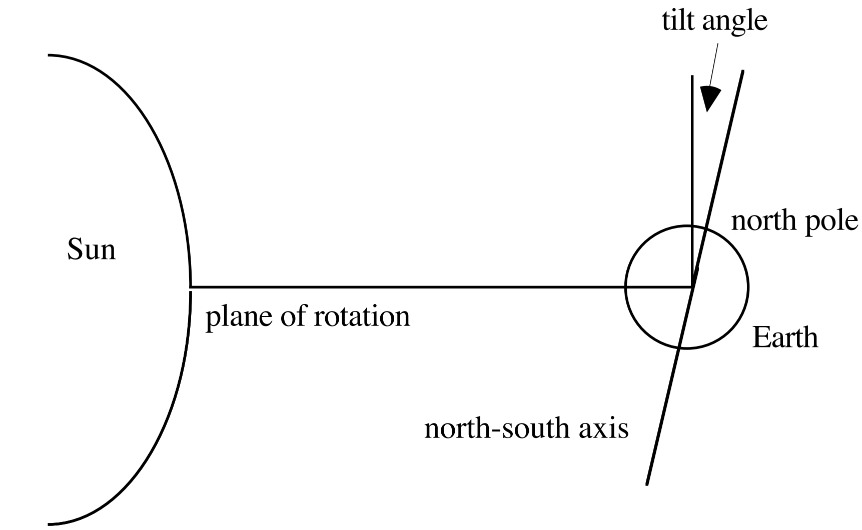
a. **1.** Determine the equation of a cosine curve that models the tides at this bay if the water is at high tide at $t = 0$.

2. Calculate the change in the height of the tide from $t = 3$ to $t = 4$, where t represents hours after high tide.

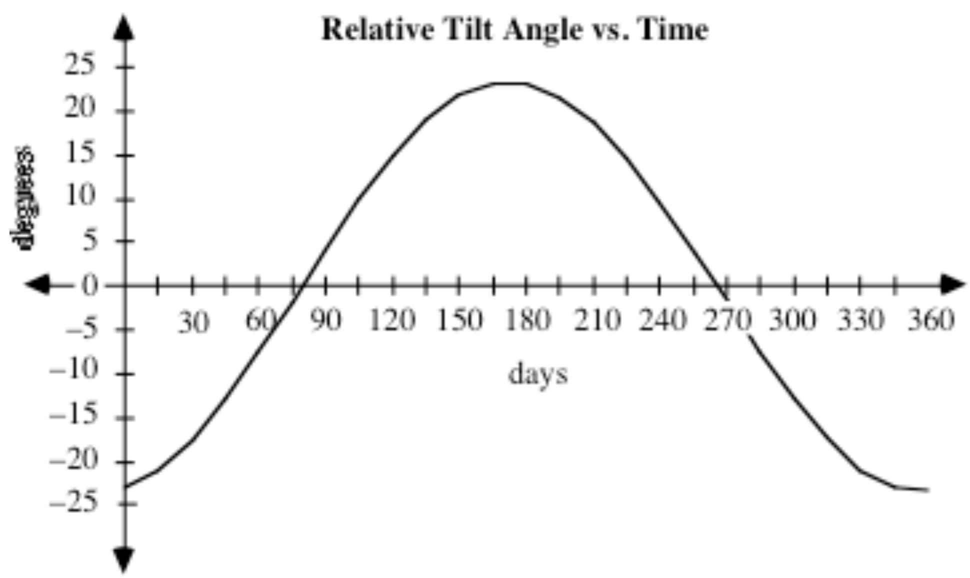
b. **1.** Determine the equation of a sine curve that models the tides at this bay if the water is 4 m above low tide and rising at $t = 0$.

2. Calculate the change in the height of the tide from $t = 3$ to $t = 4$

3.5 As shown in the following diagram, the earth's north-south axis is tilted relative to a line perpendicular to its plane of rotation about the sun.



As the earth rotates about the sun, this relative angle changes. The following graph shows the change in the earth's relative 'tilt' for 360 days.



a. What are the period and amplitude of this graph?

b. Which curve, the sine or the cosine, would you use to model this graph? Justify your response.

c. How would you alter the function selected in Part **b** to model the graph more closely?

* * * * *

3.6 The original Ferris wheel was built in Chicago in 1893. Named after its inventor, George W. Ferris, the wheel had 36 cars, each of which could seat 60 people. The diameter of the wheel was about 76 m. It completed 5 revolutions every 6 min.

a. Imagine that you are riding the original Ferris wheel. At $t = 0$, the height of your chair above the ground is 0 m. Determine a circular function that models the vertical motion of your chair on the wheel.

b. How are the values for the period and amplitude represented in your model equation?

c. Describe the vertical or horizontal translations, if any, in your model equation.

d. Near the end of the ride, the operator slows the Ferris wheel to 0.5 revolutions per minute. Determine a circular function that best models the motion of your chair at this rate.

3.7 A home furnace turns on whenever the room temperature drops below the thermostat setting. The furnace stays on until the temperature reaches a specific number of degrees above the thermostat setting.

a. Estimate how often this cycle of heating and cooling occurs and select reasonable values for the maximum and minimum room temperatures.

b. Model the room temperatures over time with a circular function.

c. What errors are possible when using a sine or cosine function to model the temperatures in a room?

* * * * * * * * * *

Activity 4

One of the real-world events that can be modeled by a circular function is the motion of a mass on a spring. For example, Figure **10** shows an object suspended from a spring. When the object is not moving, its position is referred to as its equilibrium point. As the object bounces up and down, its distance in centimeters above or below the **equilibrium point** can be modeled by the equation $d = -3\cos t$, where t represents time in seconds.

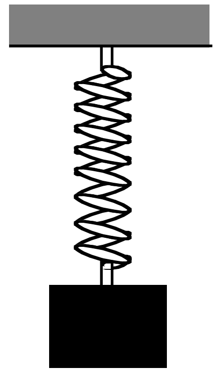


Figure 10: Object on a spring

Given what you have learned about the cosine function, you can use $d = -3\cos t$ to determine the object's distance from the equilibrium point for any time t. However, what if you had to determine the times at which the mass would be 2 cm above its equilibrium point? To do this, you must identify the values of t that correspond with $d = 2$. To find these values, it may help to consider the concept of **inverses**.

> ### Mathematics Note
> The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.
>
> For example, consider the relation $\{(0,2), (1,3), (4,-2), (-3,-2)\}$. The inverse relation is $\{(2,0), (3,1), (-2,4), (-2,-3)\}$.
>
> If the inverse of a function f is also a function, it is denoted by f^{-1}.
>
> For example, consider the function $g = \{(0,2),\ (1,3),\ (4,-2)\}$. The inverse of this function is the function $g^{-1} = \{(2,0),\ (3,1),\ (-2,4)\}$.

Exploration 1

In this exploration, you examine the inverse of a circular function.

a. In Activity **2**, you learned that the sine of a real number is the y-coordinate of a point on a unit circle with center at the origin assigned to that real number by a wrapping function. Likewise, the cosine is the x-coordinate and the tangent is the ratio of the y-coordinate to the x-coordinate.

1. Modify the headings of your spreadsheet from Part **d** of the exploration of Activity **1**, replacing "y-coordinate" with "sine" and "x-coordinate" with "cosine."
2. Add another column to the spreadsheet to calculate the tangent of each real number you marked on the unit circle.

b. Select either the sine or cosine function. Create a scatterplot of the inverse of the chosen function.

c. Using the scatterplot, determine if the inverse is also a function.

Discussion 1

a. Describe how the scatterplot of the function you selected in Exploration **1** compares with the scatterplot of its inverse.

b.
1. Judging from the values in the spreadsheet, what do you think a graph of the inverse of the tangent function will look like?

2. Do you think the inverse tangent relationship is a function? Explain your response.

c. Are the inverses of the sine and cosine functions also functions?

d. The graph in Figure **11** shows the distances above and below the equilibrium point, over time, for the mass in Figure **10**. Describe what each function in the graph represents and how you would use the graph to approximate the times when the object is 2 cm above and 2 cm below its equilibrium point.

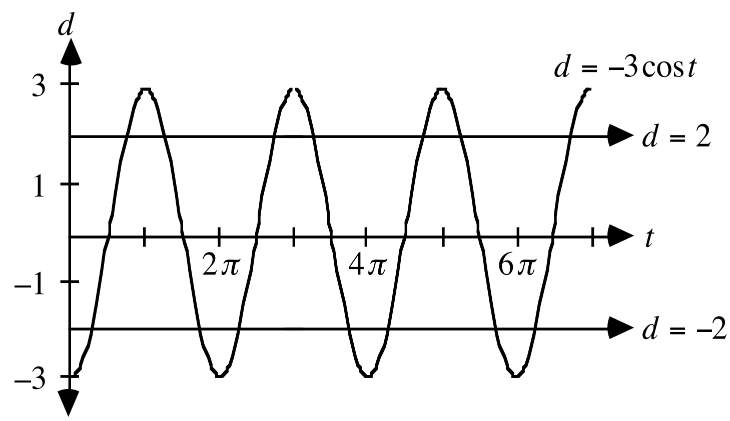


Figure 11: Graph of distance from equilibrium point versus time

Exploration 2

a. Select one of the three circular functions: sine, cosine, or tangent. Restrict its domain so that the inverse of the restricted relationship represents a function. This inverse function should include all range values of the original circular function.

b. The inverse functions for sine, cosine, and tangent can be denoted as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively. Using a graphing utility, graph the inverse of the function selected in Part **a**.

c.
1. Identify the range of the inverse function graphed by the utility.

2. Determine the corresponding restrictions made on the domain of the original function.

d.

1. Select a value, x_1 within the restricted domain identified in Part **c** for the sine function.
2. Evaluate $\sin(x)$.
3. Evaluate the inverse $\sin(x)$ noted as $\sin^{-1}(\sin x)$
4. Compare $\sin^{-1}(\sin x)$ to the original x value selected in Step **1**.
5. Repeat Steps **2–4** for a value outside the domain identified in Part **c** and record your observations.
6. Repeat Steps **1-5** for cosine and tangent.

e. An inverse function can be used to determine when the mass in Figure **10** will be 2 cm below its equilibrium point by solving the equation $-2 = -3\cos t$ for t.

After dividing both sides of the equation by –3, the value of t can be found using the inverse cosine as shown below:

$$-2 = -3\cos t$$
$$-2/-3 = \cos t$$
$$\cos^{-1}(2/3) = \cos^{-1}(\cos t)$$

The inverse function "undoes" the original function, the resulting equation is:

$$\cos^{-1}(2/3) = t$$

1. Determine the value of t in the equation $\cos^{-1}(2/3) = t$.
2. Determine when the mass in Figure **10** will be 2 cm above its equilibrium point by solving the equation $2 = -3\cos t$ for t.

Discussion 2

a. Describe how you determined the appropriate restrictions on the domain of the function in Part **a** of Exploration **2**.

b. How did these restrictions compare with the restrictions used by the graphing utility in Part **c** of Exploration **2**?

c. In Part **d** of Exploration **2**, you used the inverse of a function to "undo" the function.

1. What results did you observe using values within the restricted domain of the function?
2. What results did you observe using values outside the restricted domain?

3. Explain why you think these results occurred.

d. In Part **d** of Discussion **1**, you used the graph of $d = -3\cos t$ to determine when the mass on the spring would be 2 cm above or below its equilibrium point. How did the solutions found in Part **e** of Exploration **2** compare to those determined in Discussion **1**? Explain why this occurs.

e. Using the inverse cosine function to solve $2 = -3\cos t$ for t yields only a single solution. In Problem **2.4**, you were introduced to the following trigonometric identities: $\sin\theta = \sin(\pi - \theta)$, $\cos\theta = \cos(-\theta)$, and $\tan\theta = \tan(\pi + \theta)$.

Describe how these identities, along with the periodic nature of the circular functions, can help you determine all possible solutions for $2 = -3\cos t$, including those not found using the inverse cosine function.

f. What is the result of $\sin^{-1}(\sin(3x + 1))$?

Assignment

4.1 **a.** Solve each of the following equations without using a symbolic manipulator, identifying at least two possible values of x.

1. $-\sin(x) = 0.75$

2. $2\tan(x) = 6.5$

3. $\cos(3x) = 0.15$

4. $4\sin(2x) + 1 = 1.75$

5. $-2\cos(3(x - 1)) + 5 = -4.32$

b. Use technology to check your responses in Part **a**.

4.2 The hours of light in each day changes with the seasons. At locations near 40° N latitude, the hours of daylight range from a minimum of about 9 hr to a maximum of approximately 15 hr.

Assuming that the mean number of daylight hours occurs on March 21 (and that it is not a leap year), the number of daylight hours on any given day can be modeled by the following equation:

$$h = 12 + 3\sin\left(\frac{2\pi}{365}d\right)$$

where h represents the number of daylight hours and d represents the number of days after March 21.

a. **1.** Based on this model, what is the mean number of daylight hours in a year?

2. When do the longest and shortest days of the year occur?

b. During what dates would you expect to have at least 12 hr of daylight?

c. The model described above is based on the sine function. To apply a similar model to a location at 40° S latitude, the graph must be reflected in the line $y = 12$. What equation would you use to model the daylight hours for a location at 40° S latitude?

d. Use your model from Part **c** to determine when you would expect to have at least 12 hr of daylight in a location at 40° S latitude.

4.3 As the harbormaster at a seaport, you must be aware of the change in depth that occurs due to the rise and fall of the tides. To enter or leave the port, each ship requires a minimum depth of water. The required depth varies with each ship and depends on whether it is loaded with cargo or not. The depth of the entrance can be modeled by the equation $d = \cos(0.51t) + 5.2$, where d is the depth of the water and t is the number of hours after 12:00 noon today.

a. At 1:30 P.M., a ship that requires 4.4 m of water asks to enter the harbor. How much time does the ship have before the entrance becomes too shallow?

b. If the ship enters the harbor at the latest possible time identified in Part **a**, how soon after this time could the ship leave the port?

c. The ship in Part **a** must be back at sea no more than 26 hours after entering the harbor. Determine the latest possible time the ship can leave the port.

4.4 As a playground swing moves back and forth, its horizontal velocity can be modeled by a circular function. Consider a swing that has a horizontal velocity of 0 twice in each second and reaches a maximum horizontal velocity of 5 m/sec.

a. Determine an equation that models the swing's horizontal velocity over time.

b. On a graph of your model, identify the times when the swing's height above the ground is the greatest. Justify your choices.

c. At what times will the swing be traveling at a horizontal velocity of 2 m/sec? Justify your response.

* * * * *

4.5 The horizontal distance between a pendulum and a motion detector can be modeled by the function $d = 1.1\cos(20.9(t - 8.5)) + 3.2$, where d is the distance in centimeters and t is the time in seconds.

a. What is the greatest horizontal distance between the pendulum and the motion detector? Explain your response.

b. At what times will the pendulum be as far away from the motion detector as possible? Explain your response.

c. At what times during its swing will the pendulum be hanging straight down? Explain your response.

4.6 As noted in Problem **3.2**, ordinary household circuits carry alternating current. This current can be modeled by the function $i = I_{\max} \sin\theta$, where i is the current, $I_{\max}$ is the maximum current, and θ is the angle of rotation measured in the generator.

a. Write a function that could be used to model alternating current with a maximum of 20 A.

b. Graph the function in Part **a** over the interval $[0,\ 4\pi]$.

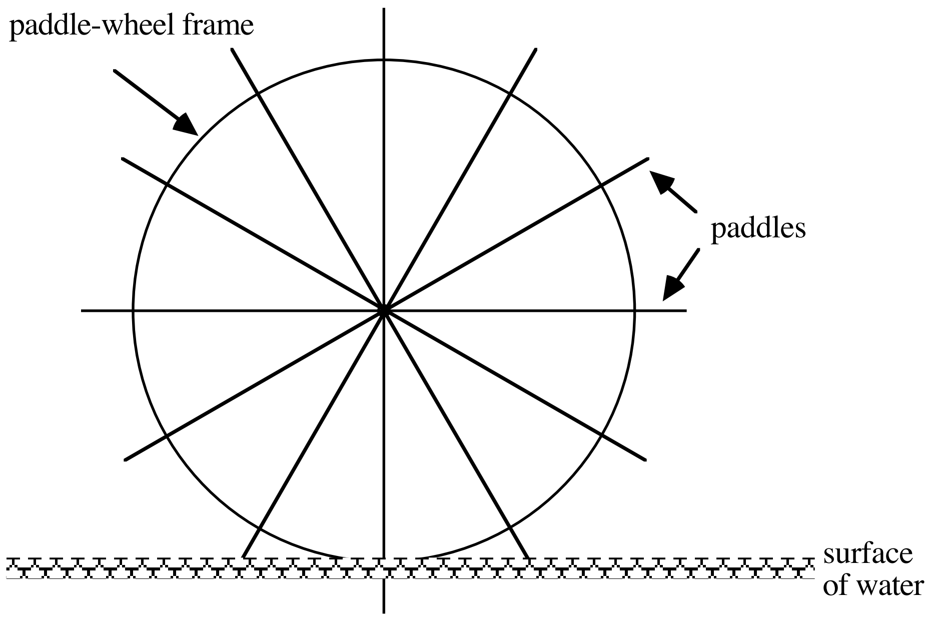
c. The effective value of an alternating current equals $0.707 \bullet I_{\max}$. Determine the values of θ when i equals the effective current.

* * * * * * * * * *

Summary Assessment

A paddle wheel on the back of a riverboat measures 1.91 m from the center of the wheel to the end of each paddle. The circular frame of the wheel has a diameter of 3.3 m. The wheel rotates so that the circular frame is tangent to the water level. It completes one rotation every 36 sec.

1. Graph the distance (height) from the surface of the water to the tip of one of the paddles with respect to time. Indicate which paddle you used and any assumptions you made.



a. What is the amplitude of the graph?

b. What is the period of the graph?

c. Write the cosine function that models the graph.

d. Write the sine function that models the graph.

2. **a.** How many radians does one paddle rotate through in 6 sec?

b. How long does it take one paddle to rotate through $15\pi/4$ radians?

c. Determine the angular velocity of a paddle in radians per minute.

3. The paddles on the wheel are spaced so that when one paddle is entering the water, a second paddle is in the water, while a third is exiting the water. Determine the number of evenly-spaced paddles on the paddlewheel.

Module Summary

- A **unit circle** has a radius of 1 unit.
- On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**.
- In general, the measure of a central angle in radians is the ratio of the length of the intercepted arc to the radius of the circle.
- The **wrapping function** pairs each point on the real number line with a location on the unit circle.
- The **sine function**, $f(t) = \sin(t)$, uses the wrapping function to assign a real number t to the y-coordinate of the corresponding point on a unit circle with center at the origin.
- The **cosine function**, $f(t) = \cos(t)$, uses the wrapping function to assign a real number t to the x-coordinate of the corresponding point on a unit circle with center at the origin.
- The **tangent function**, $f(t) = \tan(t)$, where t is any real number except an odd multiple of $\pi/2$, is the ratio of the y-coordinate to the x-coordinate of the point assigned to t by the wrapping function of the real number line around a unit circle with center at the origin.
- The tangent function is equivalent to the ratio of the sine function to the cosine function:

$$\tan x = \frac{\sin x}{\cos x}$$

- A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest interval of the domain over which the function repeats.
- The **absolute maximum** of a function is the greatest value of the range. The **absolute minimum** is the least value of the range.
- If both an absolute maximum and an absolute minimum exist in a periodic function, the **amplitude** of the function is half the distance between them. If M represents the absolute maximum and m represents the absolute minimum, the amplitude can be found by the following formula:

$$\left|\frac{M-m}{2}\right|$$

- Given a function of the form $y = a(\sin b(x + c)) + d$, the parameters a, b, c, and d transform the graph of $y = \sin x$ in the following manner:
  - The amplitude of the graph is $|a|$.
  - If a is negative, the graph is a reflection of $y = \sin x$ in the x-axis.
  - The period of the graph is $2\pi/b$.
  - If b is negative, the graph is a reflection of $y = \sin x$ in the y-axis.
  - The graph is a horizontal translation of $y = \sin x$ of $-c$ units.
  - The graph is a vertical translation of $y = \sin x$ of d units.

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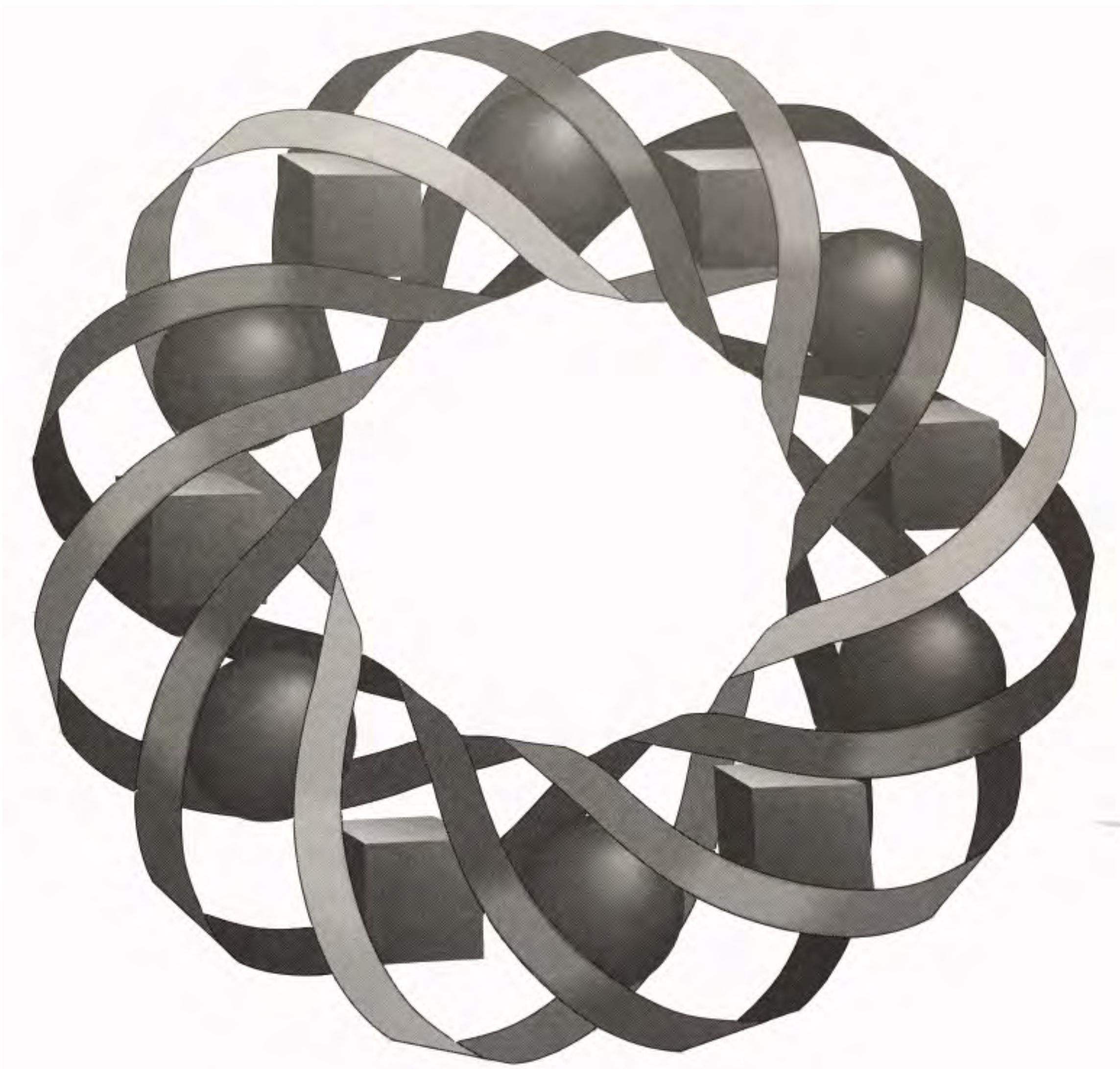
Motion Pixel Productions



Thanks to computer animation, you can now practice landing an airplane, synthesizing dangerous chemicals, even performing medical surgery. And all without fear of mistakes or injury. In this module, you use matrices to explore the applications of transformational geometry in computer graphics.

Gary Bauer • Franklin Lund • Margaret Plouvier

Motion Pixel Productions

Introduction

Many industries use computer animation to simulate unsafe, unreal, or otherwise extraordinary situations. For example, flight schools use computer simulators to allow pilots to practice dangerous landing situations. Chemical engineers use animation to study the molecular structure of experimental drugs. And Hollywood producers create everything from dinosaurs to starships with computer-generated special effects.

Computer animation requires more than just an understanding of art and motion. Animators must use computer software to describe each frame of a scene in a language the computer can understand. In some types of software, each point in a frame is given coordinates in three dimensions; the entire frame is then represented by matrices. The computer manipulates these matrices to build a vivid, three-dimensional world in the two-dimensional space of a video screen.

Discussion

a. What are some recreational uses of computer animation?

b. Describe some computer-generated graphics that you have seen recently on television or in magazines.

c. Computer-aided design (or CAD) software helps individuals create graphics on computers. What types of businesses might use CAD in designing or marketing their products?

Activity 1

To create the illusion of motion, animators once produced thousands of frames of film by hand, changing the positions of each figure slightly from frame to frame. When these frames were displayed in sequence at 24 frames per second, they produced the impression that the figure actually moved.

Each change in position can be thought of as a transformation. Recall that a **transformation** is a function, a one-to-one correspondence whose domain is a plane and whose range is the same plane. Under a transformation, each point in a **preimage** is paired with a point in the **image**. In order for a computer to display these images, the transformations may be described using matrices.

Discussion 1

a. Consider an animated scene that contains a figure shaped like a triangle. The coordinates of the vertices of the original figure, or preimage, can be represented in a matrix. Describe the possible dimensions of this matrix.

b. The figure created by a transformation is an image. If you performed a transformation on the triangle in Part **a**, what would be the dimensions of the image matrix?

c. Since multiplication of matrices is not commutative, order must be considered. If a preimage is represented in a 2×3 matrix, describe the order of multiplication with a 2×2 transformation matrix that will result in the desired image.

Mathematics Note

A **dilation** is a transformation that pairs a point C, the center, with itself and any other point P with a point P' on ray CP so that $CP'/CP = r$, where r is the **scale factor**. A dilation with center at point C and a scale factor of r is denoted as $\mathrm{D}_{C,r}$. For example, a dilation with center at point $E(1,4)$ and a scale factor of 2 can be represented as $\mathrm{D}_{E,2}$.

A **rotation** is a transformation that pairs one point C, the center, with itself and every other point P with a point P' that lies on a circle with center C such that $m\angle PCP'$ is the magnitude of the rotation. A rotation of θ degrees with center at point P is denoted as $\mathrm{R}_{P,\theta}$. The value of θ is positive for counterclockwise rotations and negative for clockwise rotations. For example, a counterclockwise rotation of 90° with center at point $C(-3,2)$ can be represented as $\mathrm{R}_{C,90^\circ}$. A clockwise rotation of 45° with the same center can be denoted by $\mathrm{R}_{C,-45^\circ}$.

A **reflection** in a line m, denoted by r_m, is a transformation that pairs each point on the line with itself and each other point P with a point P' so that m is the perpendicular bisector of $\overline{PP'}$. For example, a reflection in the line l whose equation is defined as $y = -x$ can be represented as r_l.

A **translation** is a transformation that pairs every point $P(x,y)$ with an image point $P'(x+h, y+k)$. A translation with point P as the preimage and point P' as the image is denoted as $\mathrm{T}_{P,P'}$.

For example, a translation with $B(-5,2)$ as the preimage and $B'(1,-2)$ as the image can be represented as $\mathrm{T}_{B,B'}$. This translation represents a horizontal movement of $1-(-5)=6$ units and a vertical movement of $-2-2=-4$ units. Using the Pythagorean theorem and right-triangle trigonometry, this is equivalent to a movement of $\sqrt{6^2+(-4)^2}=\sqrt{52}$ units at a direction angle of $\tan^{-1}(-4/6) \approx -33.7^\circ$.

d. Describe how the preimage is changed under each of the four transformations described in the mathematics note. (You examined these four transformations in the Level 2 module "Crazy Cartoons.")

e. A preimage and its image have the same **orientation** if they have the same sequence of corresponding vertices when read either clockwise or counterclockwise.

In Figure **1**, for example, ΔABC and $\Delta A'B'C'$ do not have the same orientation. If the vertices of ΔABC are read counterclockwise starting with vertex A, the resulting sequence is A, B, C. Following the same procedure for $\Delta A'B'C'$ provides the sequence A', C', B'. The two figures ΔABC and $\Delta A''B''C''$, however, do have the same orientation.

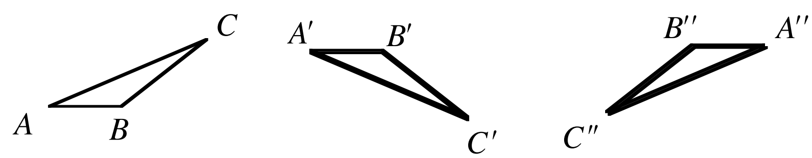


Figure 1: Three triangles

For which transformations is the orientation of the image the same as the orientation of the preimage?

f. Consider a transformation in which the image and the preimage are identical in both shape and size and located in the same position. What would be an appropriate name for a transformation of this kind?

Exploration 1

To generate a transformation using matrix multiplication, you must first determine the necessary **transformation matrix**. To find this transformation matrix, it may help to examine the transformation of a simple shape with easily identifiable coordinates.

For example, suppose that you wanted to perform a reflection, rotation, or dilation on a figure. One convenient method for determining the desired transformation matrix involves performing the transformation on a triangle like the one shown in Figure **2**.

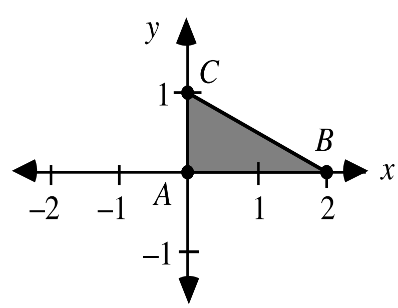


Figure 2: Right triangle in the first quadrant

This triangle can be represented by the following matrix:

$$\begin{array}{c|ccc} & A & B & C \\ \hline x & 0 & 2 & 0 \\ y & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The transformation matrix can be represented as shown below:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The product of these two matrices represents the image under the desired transformation:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$$

a. **1.** Graph and label ΔABC from Figure **2** on a sheet of graph paper.

2. Place a sheet of clear acetate film over the graph paper and trace ΔABC. Label the corresponding vertices A', B', and C'. This will serve as the image of ΔABC.

b. Select a transformation from the following list:

- r_x where x is the x-axis
- r_y where y is the y-axis
- r_l where l is the line $y = x$
- r_m where m is the line $y = -x$.

c. Apply the chosen transformation to ΔABC using the sheet of acetate. Determine the coordinates of the image by locating $\Delta A'B'C'$ in the appropriate position.

d. Represent the coordinates of the image in a matrix.

e. Write a matrix equation for the transformation of the preimage ΔABC to its image $\Delta A'B'C'$ in the following form, where the product matrix is the matrix you wrote in Part **d**:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$$

f. Determine the values of a, b, c, and d. These are the elements of the 2×2 transformation matrix that will transform any figure in the manner chosen in Part **b**.

g. Verify your solution to Part **f** by multiplying the transformation and the preimage matrices.

h. Repeat Parts **c–g** using a different transformation from Part **b**.

Discussion 2

a. When any point (or figure) is transformed in a plane, the entire plane is transformed in the same manner. How is this demonstrated in Part **c** of Exploration **1**?

b. Describe how you determined the values of a, b, c, and d in Part **f**.

c. Describe how to use ΔABC to determine the transformation matrix for any of the transformations listed in Part **b**.

d. How would you determine the matrix equation necessary to reflect the figure represented by the matrix below in the x-axis?

$$\begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & 5 \end{bmatrix}$$

Exploration 2

In Exploration **1**, you determined the 2×2 transformation matrices necessary to reflect a figure in four different lines. In this exploration, you use a similar process to determine the matrix necessary to rotate an object a desired number of degrees. You also examine the relationship between the elements of this matrix and the sine and cosine functions.

a. Using graph paper and acetate as in Exploration **1**, transform the triangle in Figure **2** using one of the following rotations: $R_{O,90°}$, $R_{O,180°}$, or $R_{O,270°}$, where O is the origin. Record the coordinates of the vertices of the image in a matrix.

b. Using the process described in Exploration **1**, determine the elements of the transformation matrix required to accomplish the selected rotation.

c. The 2×2 transformation matrix that produces a rotation about the origin is of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. Verify that the transformation matrix found in Part **b** is of this form.

d. Repeat Parts **a–c** using any angle of rotation.

Discussion 3

a. How did the elements in the rotation matrices compare to the sine and cosine of the angles of rotation?

b. Would the rotation matrix be affected if the angle of rotation were measured in radians rather than degrees? Explain your response.

Mathematics Note

The 2×2 transformation matrix that results in a rotation of angle r about the origin can be represented as shown below:

$$\begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix}$$

For example, the image of the triangle in Figure **2** under a rotation of $\pi/3$ radians about the origin can be determined as follows:

$$\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0 & 1.0 & -0.9 \\ 0 & 1.7 & 0.5 \end{bmatrix}$$

c. Consider the rotation of a figure about the origin. How could you determine the angle of rotation given only the coordinates of the preimage and the image?

Assignment

1.1 **a.** Determine the transformation matrix that produces a rotation of 90° about the origin.

b. Write a matrix expression that results in a rotation of 90° about the origin of the triangle with vertices $A(1,1)$, $B(1,5)$, and $C(4,1)$.

c. Determine the vertices of the image $A'B'C'$.

d. Verify that the image is correct by performing the rotation on graph paper.

1.2 Determine a matrix that produces each of the following transformations. Use appropriate notation to represent each one.

a. a reflection in the line x, the x-axis

b. a rotation of 270° about the origin, O

c. a rotation of -270° about the origin, O

d. a reflection in the line y, the y-axis

e. a reflection in the line l with equation $y = -x$

f. a rotation of 26° about the origin, O

g. a rotation of -78° about the origin, O

1.3 **a.** Use a series of at least four transformations to spin and flip a simple figure. Record the transformation matrix necessary to create each image.

b. Graph the result of each individual transformation on a separate index card. Use the same coordinate system on each card.

c. Arrange the cards in the appropriate order. Flip through the cards to create a simple animation of your figure.

1.4 **a.** Graph a simple figure on a sheet of graph paper. Plot and record the coordinates of the image under a dilation by a scale factor of 2 with center at the origin.

b. Using the method described in Explorations **1** and **2**, determine the 2×2 transformation matrix required to perform this dilation by matrix multiplication.

1.5 Computer animators and special-effects artists often use transformations to create the illusion of motion. For example, when objects on the screen that appear to be small and distant quickly become very large, viewers may feel the sensation of rapid flight.

a. Describe the geometric transformations that a special-effects artist might perform to make an asteroid appear to move toward the viewer.

b. Describe the geometric transformation that a special-effects artist might perform to make an asteroid appear to move away from the viewer.

* * * * *

1.6 As part of an advertising presentation, a graphic artist must enlarge a 4×6 photograph to 12×18. Using the lower left-hand corner of the photograph as the origin, the location of a book in the original photo can be denoted by the vertices (1.5,2), (1.9,3.1), (2.5,2), and (2.9,3.1).

a. Create a matrix that will enlarge the 4×6 photograph appropriately.

b. Represent the vertices of the book in the enlargement in a matrix.

c. What is the ratio of the area of the book in the enlargement to the area of the book in the original?

1.7 When a slide projector displays an image on a screen, the image is a dilation by a designated scale factor with center located at the projector. Consider the preimage of a picture that can be denoted, in part, by the coordinates (–3,0), (–1.5,7), (0,6), (1.5,7), (3,0), and (0,3). If the scale factor is 15, what are the corresponding coordinates of the image?

* * * * * * * * * *

Activity 2

To make a character's movements appear life-like, computer animation programs require an efficient means of representing transformations and calculating new coordinates. In Activity **1**, you used 2×2 matrices and matrix multiplication to perform reflections, rotations, and dilations. It is not possible, however, to use this method to perform translations. In this activity, you examine a technique that allows the use of matrix multiplication to represent all transformations.

Exploration

The coordinates of the point (x, y) can be represented in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

If this point is translated *a* units horizontally and *b* units vertically, its image can be represented by the matrix below:

$$\begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

a. Determine the dimensions of the transformation matrix, denoted by $[\ \ ]$, required to perform the multiplication in the equation below:

$$[\ \ ] \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

b. Verify that a matrix with the dimensions you determined in Part **a** cannot be used to produce a translation of *a* units horizontally and *b* units vertically.

Mathematics Note

When represented as a matrix, the **homogeneous form** of a point (x, y) is:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This form allows the coordinates of a point in two dimensions to be represented in a 3×1 matrix. For example, the homogeneous form of the point (2,3) can be represented in the following matrix:

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

c. When represented in homogeneous form, the image of a point (x, y) translated a units horizontally and b units vertically results in the following matrix:

$$\begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

Determine the dimensions of the transformation matrix, denoted by $[\quad]$, required to perform the multiplication in the equation below:

$$[\quad] \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

d. Find the transformation matrix that satisfies the equation in Part **c**.

e. When the point (x, y) is reflected in the x-axis, the coordinates of the image are $(x, -y)$.

1. Use the homogeneous form of the points to write a matrix equation that represents this transformation.
2. Determine the transformation matrix that produces the desired image matrix.

f. When the point (x, y) is dilated by a factor of n with center at the origin, the coordinates of the image are (nx, ny). Repeat Part **e** for this transformation.

Discussion

a. Describe the 3×3 matrix that allows a translation to be represented using matrix multiplication.

b. In the translation matrix you described in Part **a** of the discussion, what is the significance of the portion below?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. How does the 3×3 matrix that produces a reflection in the x-axis compare with the 2×2 matrix for a reflection in the x-axis?

d. How does the 3×3 matrix that produces a dilation by a factor of 2 with center at the origin compare to the 2×2 matrix that produces the same result?

e. What 3×3 transformation matrix do you think rotates a figure θ degrees about the origin? Defend your response.

f. The **identity transformation** preserves the position of a figure. What 3×3 matrix produces the identity transformation?

g. What advantages are there in representing transformations using 3×3 matrices?

Assignment

2.1 **a.** Consider ΔABC with vertices at A (0,0), B (2,0), and C (0,1). Use a matrix to represent the vertices of the triangle in homogeneous form.

b. Determine the 3×3 transformation matrix that produces a dilation of the triangle by a scale factor of 4 with center at the origin.

c. Compare this transformation matrix with the one you created in Part **f** of the exploration for a point (x,y).

2.2 The 3×3 transformation matrix that produces a rotation of angle θ with center at the origin is shown below:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. Use this matrix to determine the image of the triangle described in Problem **2.1** under a rotation of 30° about the origin.

b. Verify your results in Part **a** by performing the transformation on a sheet of graph paper.

2.3 Determine the 3×3 matrix that produces each of the following transformations:

a. a dilation by a scale factor of k with center at the origin

b. a reflection in the line $y = x$

2.4 Determine the 3×3 transformation matrix for each of the following transformations:

a. reflection in the y-axis

b. reflection in the line $y = -x$

2.5 **a.** Consider the triangle with vertices (0,0), (2,0), and (0,1). On a sheet of graph paper, reflect this preimage in the line $y = 1$.

b. Determine a 3×3 transformation matrix that results in a reflection in the line $y = 1$.

* * * * *

2.6 **a.** Consider the triangle from Problem **2.5**. Use a 3×3 transformation matrix to reflect this preimage in the x-axis.

b. Write an expression using matrix multiplication that describes a translation of the image from Part **a** 2 units vertically.

c. **1.** Multiply the transformation matrix from Part **a** on the right by the transformation matrix from Part **b**.

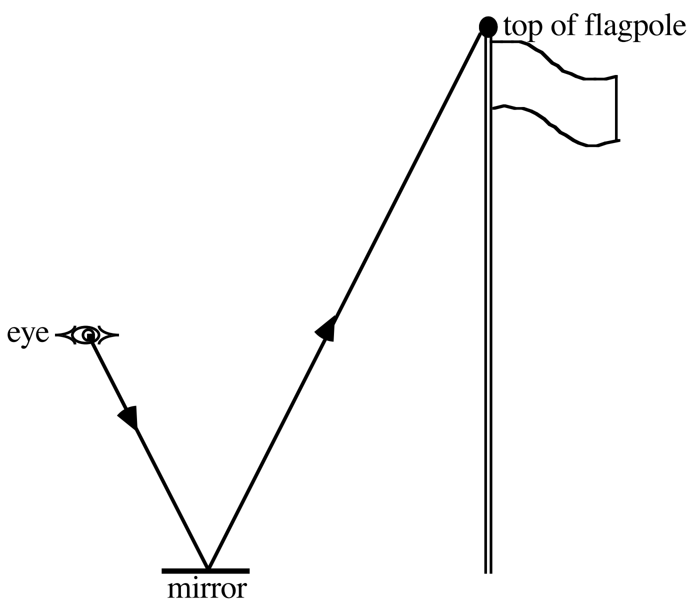
2. The resulting matrix represents the same transformation matrix as the one obtained in Problem **2.5**. Why do you think this occurs?

2.7 Computer animators frequently work in three-dimensional space, where the coordinates of points are represented by ordered triples in the form (x,y,z).

a. What do you think the homogeneous form of the point (x,y,z) would look like?

b. Based on your understanding of the 3×3 translation matrix for two dimensions, determine a matrix that would result in a translation of a units along the x-axis, b units along the y-axis, and c units along the z-axis.

2.8 To find the height of a flagpole, a student positioned a mirror on the ground. She then stood so that she could see the reflection of the top of the flagpole when she looked in the mirror, as shown below.



a. The line of sight from the top of the flagpole to the mirror and the line of sight from the student's eye to the mirror represent the hypotenuses of two similar right triangles. Sketch and label these triangles.

b. What transformations of the triangle in which the student's eye is a vertex result in the triangle in which the flagpole is a vertex?

c. Suppose that the mirror is placed at the origin of a rectangular coordinate system and the coordinates of the location of the student's eye are (–2,1.7). The distance from the mirror to the flagpole's base is 3 times the distance from the mirror to the student's feet.

1. Use 3×3 matrix multiplication to perform one of the transformations identified in Part **a** on the triangle in which the student's eye is a vertex.
2. Perform the second transformation identified in Part **a** on the image created in Step **1**. Find the coordinates representing the top of the flagpole.
3. Multiply the 3×3 transformation matrix used in Step **1** on the left by the 3×3 transformation matrix from Step **2**.
4. Multiply the preimage by the matrix found in Step **3**. Compare the coordinates for the top of the flagpole to those found in Step **2**.

* * * * * * * * * *

Activity 3

An image on a video screen is made up of many small squares or dots called **pixels**. To simulate motion, the color or brightness of each pixel changes as the need arises. For each change in an animated scene, the computer must complete an enormous number of calculations. Because this process takes time, some computer animation may produce movements that are jerky and unrealistic.

To perform the thousands of transformations required for complicated animations, a computer may use multiple matrix operations. In the previous activity, you used 3×3 matrices to perform all transformations by matrix multiplication. In this activity, you examine another way to enhance the efficiency of the computer by combining multiple transformations into a single matrix.

Exploration

In many cases, a single transformation can produce the same image as a combination of two or more transformations. When more than one transformation is performed on a figure, the result is a composite transformation, or a **composition** of transformations. In this exploration, you investigate the composition of reflections in intersecting lines.

a. **1.** Use appropriate technology to create an x-axis and a y-axis that intersect at the origin, O.

2. Create the lines $y = x$ and $y = -x$.

3. To represent a preimage, draw a scalene triangle somewhere on the coordinate plane. Label the vertices A, B, and C.

4. Reflect ΔABC in the line $y = x$ to obtain its image, $\Delta A'B'C'$.

5. Reflect $\Delta A'B'C'$ in the y-axis to obtain $\Delta A''B''C''$.

6. Compare $\Delta A''B''C''$ to the original preimage, ΔABC.

7. Record the measures of $\angle AOA''$, $\angle BOB''$, and $\angle COC''$.

8. Find a single transformation that maps ΔABC onto $\Delta A''B''C''$. Use technology to test your conjecture.

Mathematics Note

A **composition** B of transformations B_1 and B_2 is a function whose domain is the domain of B_1 and whose range is the range of B_2. In other words, the composition B is a one-to-one correspondence that maps a preimage point P in the domain of B_1 to an image point P'' in the range of B_2. This composition can be denoted by $B = B_2 \circ B_1$, (read "B equals B_2 composed with B_1").

For example, if l represents the line $y = x$ and y represents the y-axis, the transformation $C = r_y \circ r_l$ is the composition of a reflection in the line $y = x$ followed by a reflection in the y-axis.

b. **1.** Record the 3×3 transformation matrix that produces each reflection in Part **a**. Use appropriate notation to represent each reflection.

2. Using appropriate notation, record the 3×3 matrix that represents the single transformation which maps ΔABC onto $\Delta A''B''C''$.

3. Multiply the two transformation matrices recorded in Step **1** in the order that represents the composition in Part **a**.

4. Compare the product in Step **3** with the matrix you identified in Step **2**.

c. **1.** Delete the two images created in Part **a**. Your graph should now consist of the x- and y-axes, the lines $y = x$ and $y = -x$, and the preimage ΔABC.

2. Reflect ΔABC in one of the four lines to make its image, $\Delta A'B'C'$.

3. Reflect $\Delta A'B'C'$ in one of the remaining lines to get $\Delta A''B''C''$.

4. Compare $\Delta A''B''C''$ to the original preimage, ΔABC.

5. Record the measures of angles $\angle AOA''$, $\angle BOB''$, and $\angle COC''$.

6. Find a single transformation that maps ΔABC onto $\Delta A''B''C''$. Use technology to test your conjecture.

d.
1. Record the 3×3 transformation matrix that produces each reflection in Part **c**. Use appropriate notation to represent each reflection.
2. Using appropriate notation, record the 3×3 transformation matrix that represents the single transformation which maps ΔABC onto $\Delta A''B''C''$.
3. Multiply the two transformation matrices recorded in Step **1** in the order that represents the composition in Part **c**.
4. Compare the product in Step **3** with the matrix you identified in Step **2**.

e. Use the method described in Parts **c** and **d** to investigate the transformations that result from the following compositions:

1. any two dilations with the same center
2. any two rotations with the same center
3. any two translations.

Discussion

a. Do you believe that order is important in the composition of reflections? Explain your response.

b. In the exploration, you observed that the composition of two reflections in intersecting lines is equivalent to a single rotation, where the center of rotation is the point of intersection.

1. Do you think that this will hold true for any even number of reflections in intersecting lines? Explain your response.
2. Do you think this will hold true for an odd number of reflections? Explain your response.

c.
1. What is the result of a composition of two reflections in the same line? What single transformation is the same as this composition?
2. What transformation would result from a composition of two reflections in parallel lines?

d.
1. What transformation results from the composition of two translations?
2. Is order important in the composition of translations? Explain your response.

e.
1. What transformation results from the composition of two dilations with the same center?
2. Is order important in the composition of dilations? Explain your response.

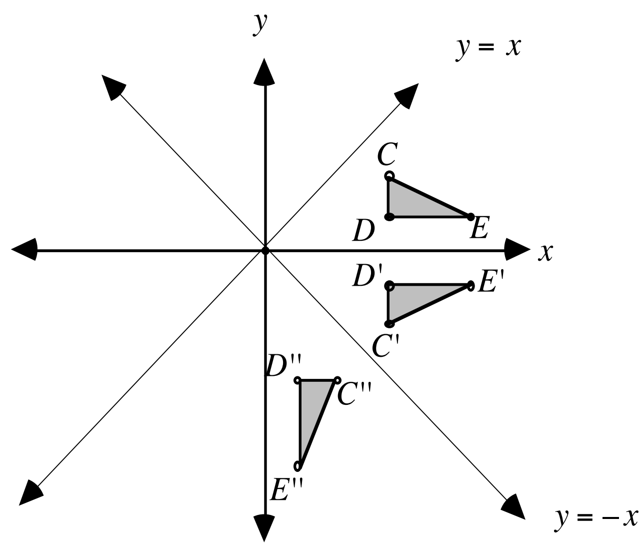
f. **1.** What transformation results from the composition of two rotations with the same center?

2. Is order important in the composition of rotations? Use an example to justify your response.

g. How do you think the composition of functions might affect the efficiency of computer animation programs?

Assignment

3.1 The diagram below shows the transformation of a scalene right triangle represented by ΔCDE.



a. What two consecutive transformations map ΔCDE onto $\Delta C''D''E''$? Justify your response by assigning coordinates to the vertices of ΔCDE and evaluating the appropriate matrix equation to find the coordinates of $\Delta C''D''E''$.

b. What single transformation maps ΔCDE onto $\Delta C''D''E''$? Justify your response by evaluating the appropriate matrix equation to find the coordinates of $\Delta C''D''E''$.

c. Write a matrix equation that shows that the single transformation you found in Part **b** equals the composition of two transformations described in Part **a**.

3.2 Complete the following chart by writing the single transformation that is equivalent to the composition of $B_2 \circ B_1$ in the appropriate cell. Let O represent the origin, x represent the x-axis, y represent the y-axis, l represent the line $y = x$, and m represent the line $y = -x$.

For example, the correct entry for $r_x \circ r_l$ is shown below. Note that the first transformation performed is drawn from the row headings, while the second transformation is drawn from the column headings. Be sure to use correct notation.

B_2

$\circ$	$\mathbf{r}_x$	$\mathbf{r}_l$	$\mathbf{R}_{O,90^\circ}$	$\mathbf{R}_{O,180^\circ}$	$\mathbf{R}_{O,270^\circ}$
$\mathbf{r}_x$					
$\mathbf{r}_l$	$R_{O,270^\circ}$				
B_1 $\mathbf{R}_{O,90^\circ}$					
$\mathbf{R}_{O,180^\circ}$					
$\mathbf{R}_{O,270^\circ}$					

3.3 Consider the figure described by the matrix **M** below.

$$\mathbf{M} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

a. Find the product of the following three matrices using the order of multiplication indicated below:

$$\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \bullet \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

b. Find the product of the same three matrices using the following order of multiplication:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bullet \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

c. Compare your results in Parts **a** and **b**. What conjecture might be made concerning the grouping of matrix multiplication?

d. In Parts **a** and **b**, what composition of transformations was performed on matrix **M**?

3.4 **a.** Create a simple, non-symmetric "stick" figure to represent a cartoon character.

b. Graph several points of your cartoon character and determine the coordinates of each point.

c. If points A, B, C, and D have coordinates (0,0), (4,2), (6,7), and (7,4), respectively, what transformation results from $\mathrm{T}_{A,B} \circ \mathrm{T}_{C,D}$?

d. Write a matrix expression that represents this composition.

e. Determine the image matrix of your cartoon character under the transformation $\mathrm{T}_{A,B} \circ \mathrm{T}_{C,D}$.

3.5 Find the image matrix of your original cartoon character from Problem **3.4** under the transformation $\mathrm{r}_x \circ \mathrm{R}_{O,285^\circ}$.

3.6 **a.** Consider ΔABC with vertices $A(3,1)$, $B(6,1)$, and $C(6,2)$. Using ΔABC as the preimage, determine the coordinates of the image $\Delta A'''B'''C'''$ under three transformations of your choice.

b. Determine a single 3×3 matrix that represents the composition of the three transformation matrices in Part **a**. Use matrix operations to verify that your matrix is the correct one.

3.7 A reflection can be performed in any line, not just the x- and y-axes and the lines $y = x$ and $y = -x$. One method for determining the coordinates of an image under such a reflection involves transforming the plane so that the desired line of reflection coincides with one of the axes. After the reflection matrix for the axis is determined, the plane is transformed so that the line is returned to its original position. The composition of these three transformations results in the reflection of the object in the desired line.

a. Consider ΔABC with vertices at (0,0), (2,0), and (0,1), respectively. Suppose you wish to reflect ΔABC in the line $y = 2x$. To transform the plane so that the line $y = 2x$ coincides with the x-axis, determine the measure of the angle formed by the line $y = 2x$ and the x-axis using right-triangle trigonometry.

b. Determine the 3×3 transformation matrix that will rotate the line $y = 2x$ so that it coincides with the x-axis. This is the first transformation.

c. Since the line $y = 2x$ now coincides with the x-axis, the second transformation is a reflection in the x-axis. Determine the matrix for this reflection.

d. The final transformation should return the plane to its original position. Determine the rotation matrix that will result in this transformation.

e. Find the single 3×3 transformation matrix that represents the composition of the transformations in Parts **b–d**. Multiply this transformation matrix by the preimage matrix to determine the location of the image.

f. Verify your solution by reflecting ΔABC in the line $y = 2x$ on a sheet of graph paper and estimating the coordinates of the image.

3.8 A rotation can have any point as its center, much like a reflection can be in any line. The process for identifying the required transformation matrix and the coordinates of the image is very similar to the one described in Problem **3.7**. In this case, however, you must translate the plane so that the center of rotation coincides with the origin, perform a rotation of the desired angle, then translate the plane back to its original position.

a. Use ΔABC from Problem **3.7** to determine the composite matrix that results in a 45° rotation about the point (–2,3).

b. Determine the coordinates of the image under the composition in Part **a**.

c. Verify your solution using graph paper.

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3.9 Reflecting an object in a line that does not pass through the origin, such as $y = -3x + 5$, requires an additional transformation to get the line to coincide with the x-axis. In this case, the line must first be translated –5 units vertically, then rotated to coincide with the axis.

a. Use ΔABC, with vertices at (0,0), (2,0), and (0,1), respectively, to determine the composite matrix that results in a reflection in the line $y = -3x + 5$.

b. Determine the coordinates of the vertices of the image under the composition in Part **a**.

c. Verify your response using graph paper.

3.10 Designer patterns on wallpaper, floor tiles, and kitchen countertops often use transformational geometry. Use a geometry utility to create your own tile or wallpaper pattern. Print a copy of your design, and write a short paragraph describing the geometric transformations you used to create it.

* * * * * * * * * *

Activity 4

In previous activities, you examined the use of compositions to find a single transformation matrix that produces the same result as multiple transformations. One type of transformation that you have not yet examined is itself a composition. This transformation is a **glide reflection**.

Mathematics Note

A **glide reflection** is the composition of a reflection and a translation parallel to the line of reflection. For example, Figure **3** shows the image $\Delta A'B'C'$ of ΔABC under a glide reflection. It is the result of a reflection in line *m* and a translation parallel to line *m* along vector **t**.

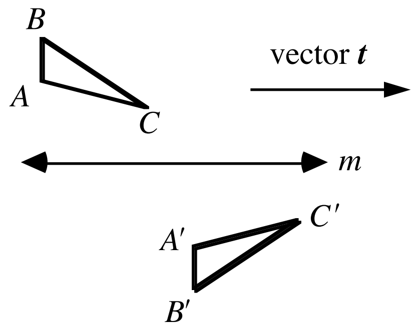


Figure 3: Glide reflection of ΔABC

Exploration 1

In this exploration, you examine one of the properties of the glide reflection.

a. Create ΔABC. Transform the triangle by translating it, then reflecting the image in a line parallel to the translation.

b. Find the image of ΔABC using the same transformations in Part **a**, but in reverse order. In other words, reflect the triangle in the line, then translate the image parallel to the line of reflection. Compare the result to the one obtained in Part **a**.

Discussion 1

a. In a glide reflection, does the order of the translation and reflection affect the image? Explain your response.

b. How does the orientation of the preimage compare to that of the image under a glide reflection? Why is this true?

c. The translation in a glide reflection can be expressed as the composition of two reflections.

1. Describe the relationship between these two lines of reflection.
2. Describe the relationship between these two lines and the original line of reflection in the glide reflection.

Exploration 2

In the previous discussion, you found that a glide reflection could be expressed as a composition of reflections. Is this true only for glide reflections? In this exploration you will attempt to determine if it is possible to express other transformations as compositions of reflections.

a. Create a quadrilateral *ABCD*.

b. Select and perform one of the following transformations on quadrilateral *ABCD*:

1. a reflection in a line
2. a rotation about a point
3. a translation
4. a glide reflection.

c. Figure **4** shows several of the steps used in the process of determining if a selected transformation can be expressed as a composition of reflections. In this case, quadrilaterals $ABCD$ and $A'B'C'D'$ represent the preimage and image under a glide reflection.

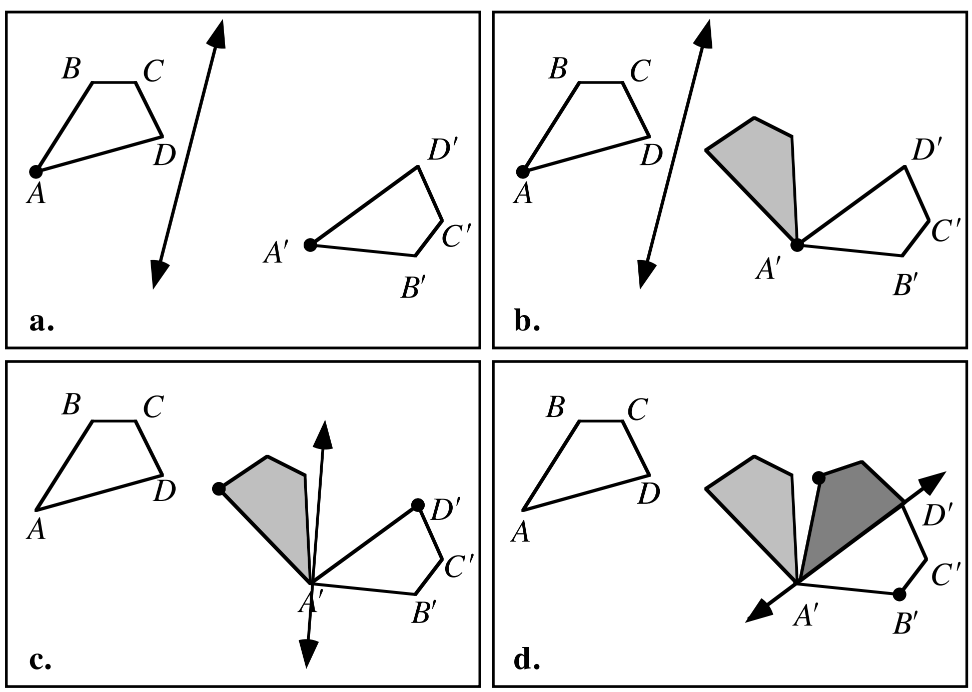


Figure 4: Quadrilateral $ABCD$ under a composition of reflections

Complete the following steps to determine if your selected transformation can be expressed as a composition of reflections.

1. Select a vertex in the image and its corresponding vertex in the preimage. Determine the line of reflection for these two vertices. For example, Figure **4a** shows the line of reflection for A and A'.
2. Reflect the preimage quadrilateral $ABCD$ in this line of reflection. (In Figure **4b**, the resulting figure is shaded.) If this reflection coincides with image quadrilateral $A'B'C'D'$, then the process is complete.
3. If the process is not complete, select another vertex on image quadrilateral $A'B'C'D'$ and its corresponding vertex on the image created in Step **2**. For example, the vertices chosen in Figure **4c** are D' and its corresponding vertex on the shaded quadrilateral. Repeat Steps **1** and **2** for these two points. (The result in Figure **4d** is the more densely shaded quadrilateral.)
4. If the process is not complete, repeat Step **3**, choosing a new vertex in the image quadrilateral $A'B'C'D'$ each time. In Figure **4d**, for example, the vertices chosen are B' and its corresponding point on the more densely shaded quadrilateral. The line of reflection between these two points contains A' and D'.

   If the last image created after repeating Step **3** for each of the remaining vertices does not coincide with the image quadrilateral $A'B'C'D'$, it is not possible to express the transformation as a composition of reflections.
5. If it is possible to express your chosen transformation as a composition of reflections, record the number of reflections necessary.

d. Repeat Parts **b** and **c** for a different transformation.

e. Graph a congruent image of your quadrilateral in any location on the plane and with any orientation. Repeat Part **c** for this new transformation.

Discussion 2

a. Which transformations can be expressed as a compositions of reflections?

b. Was it possible to express the transformation in Part **e** of Exploration **2** as a composition of reflections?

c. If a preimage and its image under a transformation in a plane are congruent, the transformation is either a rotation, a reflection, a translation, or a glide reflection. Given this fact, make a general statement concerning the expression of transformations as a composition of reflections and the number of reflections necessary.

d. Can all transformations be expressed as a composition of reflections?

e. Why might it be helpful for a computer programmer who creates software that simulates motion to express various transformations as a composition of reflections?

Assignment

4.1

a. Consider the triangle with vertices $A(-1,-3)$, $B(4,0)$, and $C(2,1)$. Graph and label this figure on a coordinate plane.

b. On the same set of axes, graph the image of ΔABC under the reflection r_y, where y is the y-axis. Label the image $\Delta A'B'C'$.

c. On the same set of axes, graph the image of $\Delta A'B'C'$ under the translation $\mathrm{T}_{P,P'}$, where point P has coordinates (13,5) and P' has coordinates (12,8). Label the resulting image $\Delta A''B''C''$ and record the coordinates of its vertices.

d. Use composition notation to describe the transformation of ΔABC to $\Delta A''B''C''$.

e. Write a matrix equation that illustrates the relationship between the matrix for ΔABC, the transformation matrices, and the matrix for $\Delta A''B''C''$. Compare the resulting matrix with the coordinates found in Part **c**.

4.2

a. Use graph paper and a straightedge to complete Steps **1–5** below.

1. Graph a simple, non-symmetrical stick figure and label it C.
2. Determine the coordinates of several vertices of the figure.
3. Graph the image of the stick figure under the composition $\mathrm{T}_{R,R'} \circ \mathrm{r}_x$, where x is the x-axis, point R has coordinates (–5,10), and point R' has coordinates (–3,11). Refer to the image as C′.
4. Graph the image of the original figure C under the composition $\mathrm{r}_x \circ \mathrm{T}_{R,R'}$. Refer to this image as C″.
5. Explain whether or not the order of the transformations makes a difference in the final image.

b. Is the composition $\mathrm{T}_{R,R'} \circ \mathrm{r}_x$ a glide reflection? Justify your response.

4.3

a. Graph the image of your stick figure from Problem **4.2** under the composition $\mathrm{T}_{S,S'} \circ \mathrm{r}_x$, where x is the x-axis, point S has coordinates (–5,5), and point S' has coordinates (–3,5).

b. Graph the image of the stick figure under the composition $\mathrm{r}_x \circ \mathrm{T}_{S,S'}$.

c. Explain whether or not the order of the transformations makes a difference in the final image.

d. Write matrix equations that generate matrices for the final images obtained in Parts **a** and **b**.

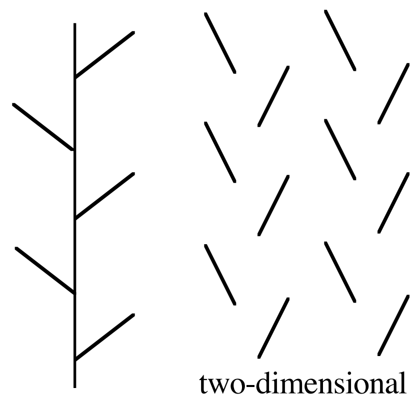
4.4 As you found in the exploration, a glide reflection can be expressed as a composition of three reflections.

a. Make a sketch of three lines of reflection for which the final image is a glide reflection of the preimage.

b. Describe the properties of these three lines and explain how these properties guarantee that the resulting transformation is a glide reflection.

* * * * *

4.5 Some two-dimensional patterns in crystals or the twigs of trees have an especially appealing sense of symmetry. Use glide reflections and a geometry utility to recreate one of the following patterns. Include a short paragraph describing the geometric transformations in your design.



tree twig   two-dimensional crystallographic pattern

4.6 Interior decorators often use a patterned border to accent the walls or ceiling of a room. Design a pattern for a border using glide reflections. Create a copy of your design using a geometry utility, then write a short paragraph describing the geometric transformations in your design.

* * * * * * * * * *

Summary Assessment

1. Create a flip-card animation of a simple figure by plotting each frame onto a coordinate system that can be taped or glued to a 3×5 index card. Your flip cards should use all of the transformations studied in this module.

2. Express each transformation used as a 3×3 matrix.

3. Identify the coordinates of each image.

4. Determine a single 3×3 transformation matrix that will transform the initial preimage into the final image.

Module Summary

- An **image** is the figure created by a transformation of a **preimage**.
- A **dilation** with center at point *P* and a scale factor of *r* is denoted as $\mathrm{D}_{P,r}$.
- A **rotation** of θ degrees with center at point *P* is denoted as $\mathrm{R}_{P,\theta}$. The value of θ is positive for counterclockwise rotations and negative for clockwise rotations.
- A **reflection** in the line *m* is denoted as r_m.
- A **translation** with *P* as the preimage and P' as the image is denoted as $\mathrm{T}_{P,P'}$.
- The transformation B that produces the same image as a transformation B_1 followed by a transformation B_2 is the **composition** of B_1 and B_2. This composition is denoted $\mathrm{B} = \mathrm{B}_2 \circ \mathrm{B}_1$, (read "B equals B_2 composed with B_1 "). This notation implies that transformation B_2 is performed after transformation B_1.
- A preimage and its image have the same **orientation** if they have the same sequence of corresponding vertices when read either clockwise or counterclockwise.
- When represented as a matrix, the **homogeneous form** of a point (x,y) is:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- The **identity transformation** preserves the position of a figure.
- A **glide reflection** is the composition of a reflection and a translation parallel to the line of reflection.
- If a preimage and its image under a transformation in a plane are congruent, the transformation is either a rotation, a reflection, a translation, or a glide reflection.
- If a preimage and its image under a transformation in a plane are congruent, the transformation can be expressed as a composition of reflections in no more than three lines.

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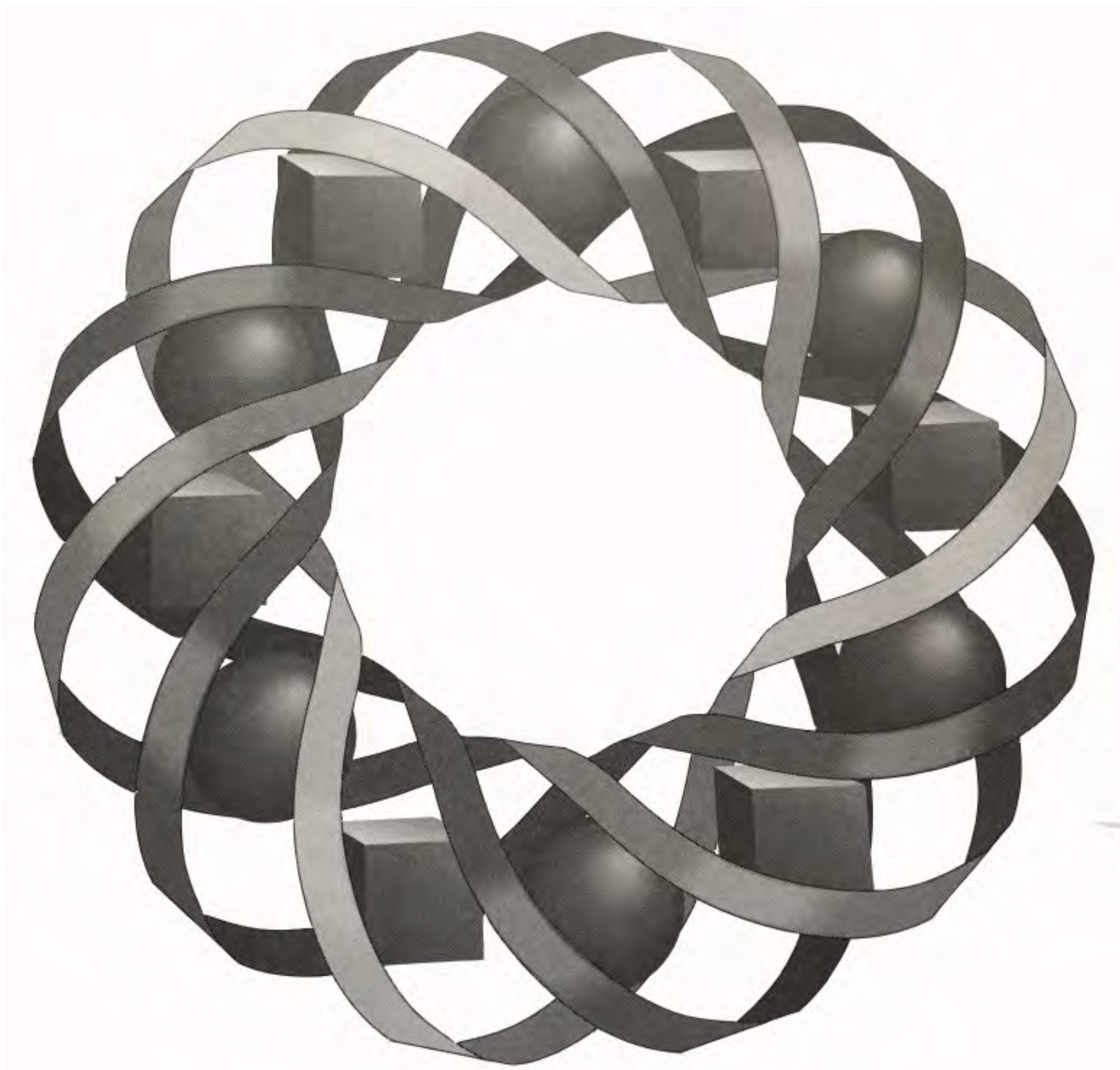
North Carolina School of Science and Mathematics, Department of Mathematics and Computer Science. *Matrices*. Reston, VA: National Council of Teachers of Mathematics, 1988.

Wedge, J. C. "Role of the Animator in the Generation of Three-Dimensional Computer-Generated Animation." ERIC, 1985. ED 282512.

Drafting and Polynomials



Many computer-generated images are defined by polynomial curves. In this module, you use polynomial functions and their corresponding graphs to study some elements of graphic design.

Masha Albrecht • Tom Teegarden

Drafting and Polynomials

Introduction

Builders learned long ago that drawing preliminary sketches makes the construction process easier and more exact. Turning that first sketch into a set of detailed plans can simplify the process even more.

Drafting accurate plans once required special drawing equipment. In recent years, however, drafting tools have changed dramatically. Computer-aided design has made precise drawing much easier and faster. For example, Figure **1** shows the front view of a Viking ship originally built around 900 A.D. The curves in the hull were simulated using a computer graphics program.

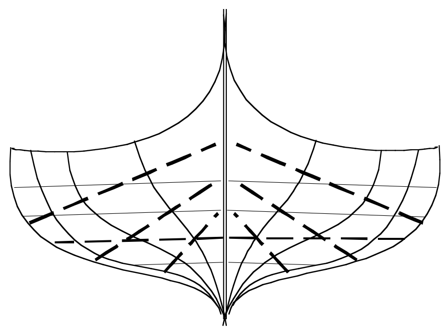


Figure 1: Computer representation of a Viking ship

Computer software typically represents complicated curves by piecing together parts of several simpler curves. These simpler curves are normally defined by functions made up of **polynomials**.

Mathematics Note

A **polynomial** in one variable, x, is an algebraic expression of the general form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

where n is a whole number and the **coefficients** a_i are real numbers for $i = 0,\ 1,\ 2,\ \ldots,\ n$.

The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. A polynomial in the general form shown above has a degree of n, provided that $a_n \neq 0$.

For example, the expression $2x^5 + 4x^4 - 2x^3 - 3x^2 + 5x^1 + 6$ is a polynomial in one variable with a degree of 5. In this expression, the coefficients are 2, 4, –2, –3, 5, and 6. The expression $2x^5 + 6$ is also a fifth-degree polynomial. In this case, the coefficients are 2, 0, 0, 0, 0, and 6.

Some additional examples of polynomials are shown in Table **1** below.

Table 1: Some polynomials and their degrees

Polynomial	Degree
5	0
$2x+4$	1
$3.6x^2 - 5x + 2$	2
$-5x^3 + x$	3
$\frac{3}{7}x^6 + 3x^5 - \frac{2}{9}x^2 - 11$	6

There are many practical applications for curves described by polynomials. For example, the outlines of many letter designs, or fonts, are created using polynomial functions. In fact, the letters of the words you are reading right now were defined using the graphs of polynomials.

Activity 1

Imagine that you are a graphics designer who has been asked to connect several points in a drawing with a smooth curve. One way to accomplish this task involves using a two-dimensional coordinate system to identify the coordinates of each point. You could then sketch a curve that passes through the points, determine a mathematical equation that models the curve, and enter that equation in a graphing utility.

Determining the equation that describes a curve, however, can be a complicated task. In the following exploration, you examine the simple case of connecting two points with a curve and determining the corresponding equation.

Exploration

a. Create an *xy*-coordinate system on a sheet of graph paper.

b. Plot two points with different *x*-coordinates on the graph paper: one above the *x*-axis, and one below it. Label each point with its coordinates.

c. A line is the simplest and smoothest continuous curve that contains two points. Draw the line that contains the two points in Part **b**.

d. Determine the equation of the line in Part **c**.

e. Estimate the coordinates of the point where the line intersects the x-axis. The x-coordinate of this point is the ***x*-intercept**.

This x-coordinate is also a **root** or **zero** of the function that describes the curve. It is called a *zero* because, when substituted for x in the function, the value of the function is 0.

Mathematics Note

Factoring is the process of using the distributive property to represent a mathematical expression as a product.

For example, the expression $2x + 6$ can be factored into the equivalent expression $2(x + 3)$. Similarly, the expression $2x^2 + 3x - 5$ can be expressed as $(2x + 5)(x - 1)$.

f. Using the distributive property, a linear equation in the form $y = mx + b$ can be rewritten as:

$$y = m\left(x + \frac{b}{m}\right)$$

1. Express the equation from Part **d** in the form above.
2. Describe how the zero identified in Part **e** relates to this form of the equation.

Discussion

a. Is the graph you drew in the exploration a function? Explain your response.

b. What is the degree of a linear function?

c. If for a given function g, $g(7) = 0$, what does this imply about the graph of g?

d. Consider the graph of a line that intersects the x-axis at 12. What are some possible equations for this line?

e. The form of a linear equation described in Part **f** of the exploration,

$$y = m\left(x + \frac{b}{m}\right)$$

can be rewritten by replacing m with a and b/m with $-c$. This results in an equation of the form $y = a(x - c)$.

1. What role does the value of a play in the graph of $y = a(x - c)$?
2. How is the equation of a line in the form $y = a(x - c)$ related to its equation in slope-intercept form? Explain your response.

f. In this module, you will model curves by examining their zeros. What advantages are there to writing the equation of a line in the form $y = a(x - c)$?

Mathematics Note

A function *f* is a **polynomial function** if $f(x)$ is defined as a polynomial in x.

When the roots or zeros of a non-zero polynomial function are known, and all of these roots are real numbers, the function can be written as a product of factors, as shown below:

$$f(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$$

In general, if $f(c) = 0$, c is a zero of $f(x)$ and $(x - c)$ is a factor of $f(x)$.

For example, Figure **2** shows a graph of $f(x) = x^2 + 5x - 6$. From the graph, $f(-6) = 0$ and $f(1) = 0$. Therefore, $(x - (-6)) = (x + 6)$ and $(x - 1)$ are factors of $f(x)$. The function can be written in factored form as $f(x) = (x + 6)(x - 1)$.

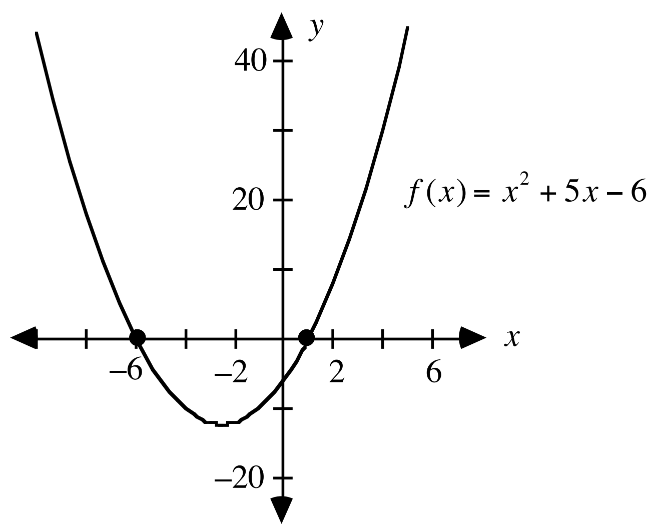


Figure 2: Graph of a polynomial function $f(x) = x^2 + 5x - 6$

When a function is expressed in factored form, its zeros can be determined from the factors. For example, if $g(x) = 4(x - 3)(x - 5)(x + 2)$ then 3, 5, and –2 are zeros of $g(x)$. Likewise, if $h(x) = (x - 3)(x - 5)(x + 2)$, then 3, 5, and –2 are its zeros.

Figure **3** shows the graphs of these two functions. Notice that although the graphs of the two functions are different, their zeros are the same.

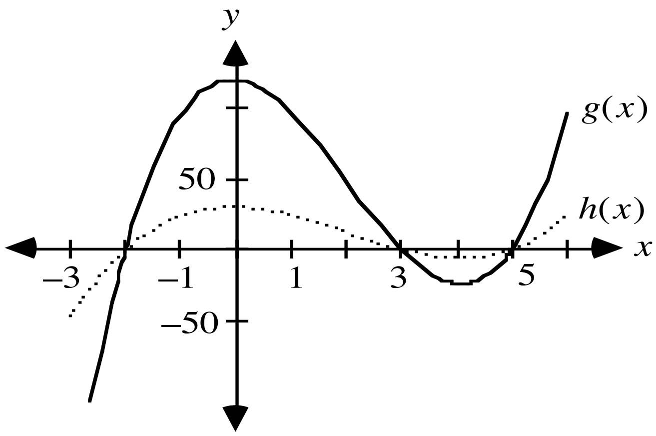


Figure 3: Graphs of two polynomial functions with same roots

Assignment

1.1 **a.** Using the method described in the exploration, find the equation of a first-degree polynomial function whose graph contains the points (1,2) and (3,6). Express the equation in factored form.

b. Identify the root(s) of this polynomial function.

1.2 **a.** Predict the zeros for each of the following polynomial functions. The function $m(x)$ is said to have a **double root** because the factor $(x-2)$ appears twice.

1. $f(x) = 2(x + 3)$
2. $h(x) = x$
3. $g(x) = -0.2(x - 3)(x + 2)(x - 7)$
4. $m(x) = (x - 2)(x - 2)$

b. Create a graph of each function in Part **a** and estimate the zeros.

c. To verify your estimates in Part **b**, substitute the value of each predicted zero for x in the corresponding function.

1.3 **a.** Select four different real numbers. Write a polynomial function whose roots are these numbers.

b. Determine a different polynomial function with the same roots.

c. Compare the graphs of the two functions in Parts **a** and **b**.

1.4 **a.** The distributive property can be used to find the product of 8 and 16 by expressing 16 as $(10+6)$ and distributing the 8 as follows:

$$\begin{aligned} 8(10+6) &= 80+48 \\ &= 128 \end{aligned}$$

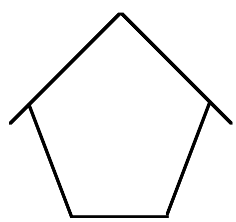
Use the distributive property to multiply $3(x+2)$.

b. The product of 20 and 16 can be found by expressing 21 as $(20+1)$ and 16 as $(10+6)$, then using the distributive property as follows:

$$\begin{aligned} (20+1)(10+6) &= 20(10+6)+1(10+6) \\ &= (200+120)+(10+6) \\ &= 336 \end{aligned}$$

Use the distributive property to multiply $(x+7)(x-4)$.

1.5 The birdhouse design shown in the following diagram was created using polynomial functions. The functions were graphed on the same coordinate system, then segments of each graph were used to form the birdhouse. Determine a set of polynomial functions that could be used in this process, including the necessary restrictions on the domain of each one.



1.6 Determine a set of polynomial functions, along with their respective domains, that could be used to create the word WAX in capital letters on a computer screen.

* * * * *

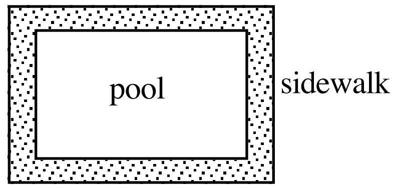
1.7 **a.** Use the distributive property to multiply the factors of each function below. Identify the degree of each resulting polynomial.

1. $f(x) = 2(x+3)$
2. $g(x) = -0.8(x-4)$
3. $h(x) = (x-2)(x-2)$
4. $m(x) = 0.2(x-3)(x+2)(x-7)$

b. Use a symbolic manipulator to verify the products you found in Part **a**.

c. Use the products in Part **a** to graph each polynomial function. Compare the zeros of each function with its factors.

1.8 A local swimming pool is 20 m wide and 30 m long. As shown in the diagram below, the sidewalk surrounding the pool has a constant width.



a. Let w represent the width of the concrete sidewalk. Represent the area of the sidewalk using a polynomial function in w.

b. Use the distributive property to express this polynomial function in a simplified form.

* * * * * * * * * *

Activity 2

Drafting by hand can be a challenging, painstaking task. Although a straightedge works fine for connecting points with straight lines, drawing shapes that are not straight can be more difficult. One instrument used for tracing smooth, precise curves by hand is a spline. Modern splines are made of flexible plastic or metal which allows them to retain their shape when bent. Curves drawn using this method are known as spline curves.

To draw smooth, precise shapes, computer-aided design programs frequently make use of both spline curves and Bézier curves. Bézier curves are named after the French mathematician Pierre Bézier, who pioneered computer modeling of curved surfaces for Renault, the automobile manufacturer. Both types of curves actually contain many polynomial curves pieced together mathematically. Figure **4** shows the profile of a boat hull created on a computer graphics program. This drawing consists of a Bézier curve and its reflection.

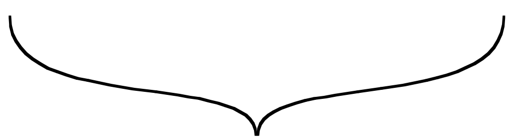


Figure 4: A boat hull

To create a similar drawing by hand, a boat builder might plot four points on a grid, then use a spline to draw the curve. In this activity, you experiment with polynomial curves that connect noncollinear points.

Exploration

A draftsperson or graphic designer usually attempts to draw curves as smoothly and simply as possible. Figure **5** shows two smooth polynomial curves that connect the same four points. Since the polynomial on the left connects the points with a simpler curve, it would typically be the preferred model.

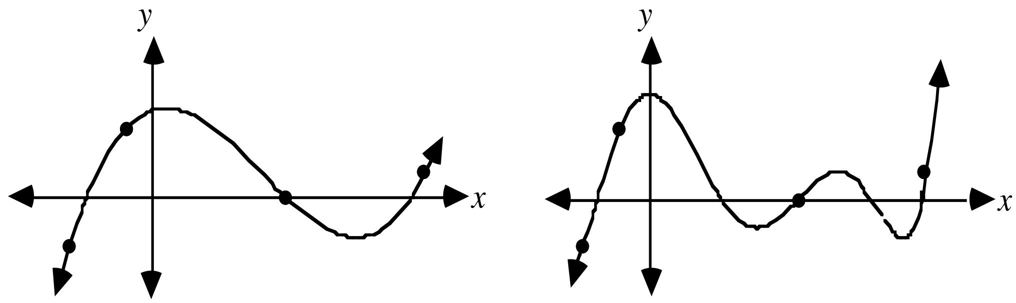


Figure 5: Connecting four points with two different curves

In this exploration, you use a spline to create a polynomial curve containing three noncollinear points.

a. Create an *xy*-coordinate system on a sheet of graph paper.

b. Draw three noncollinear points on the graph so that as the *x*-coordinates increase in value, the *y*-coordinates alternate between positive and negative values. This will result in two points on one side of the *x*-axis, and one on the other side. The three points selected should satisfy the conditions of a function. Label each point with its coordinates.

c. Fit a model spline to the points so that the spline crosses the *x*-axis the fewest possible number of times. The curve formed by the spline also should represent a function.

d. Trace the curve formed by the spline.

e. Estimate the zeros of the function that describes the curve.

f. Use the zeros from Part **e** to determine the factors of a polynomial with the same *x*-intercepts as the spline curve.

g. Use the factors to write a polynomial function in the form below that appears to approximate the curve.

$$f(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$$

h. Plot the three points from Part **b** and the function from Part **g** on a graphing utility. If necessary, modify the function in order to obtain a better approximation of the three points.

Check your approximation by calculating the sum of the squares of the residuals. (Recall that a smaller sum indicates a better approximation.)

Discussion

a. When is it appropriate to use a line as the smoothest, simplest curve connecting a set of points?

b. What is the degree of your polynomial from the exploration?

c. Is it possible to connect three points with a polynomial curve which has a lesser degree than the one you found? Explain your response.

d. How could you use the coordinates of the point of the function you used in Part **b** of the Exploration to find the value of *a* for the function you wrote in Part **g** of the Exploration?

e. **1.** How can you change the equation of a polynomial function without changing its zeros?

2. How do these changes affect the function's graph?

Mathematics Note

A second-degree polynomial function is a **quadratic** or **parabolic function**. Its graph is a **parabola** that opens either upward or downward.

Every parabola is symmetric about a line known as the **axis of symmetry**. The point of intersection of a parabola and its axis of symmetry is the **vertex** of the parabola. For example, Figure **6** shows a graph of the quadratic function $f(x) = (x-2)(x+4)$. This parabola's axis of symmetry is the line $x = -1$; its vertex is the point (–1,–9).

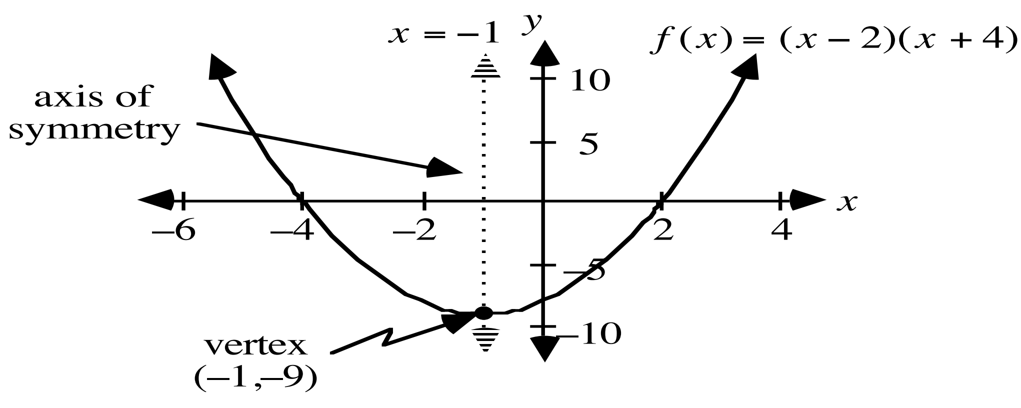


Figure 6: A parabola

f. Describe some situations outside your mathematics classroom in which you have encountered parabolic curves.

Assignment

2.1 **a.** Find the coordinates of the vertex of the parabola described by the equation $y = -x(x - 10)$.

b. Consider a parabola that intersects the *x*-axis at two points. Explain how the zeros of the corresponding function can be used to find the vertex of the parabola.

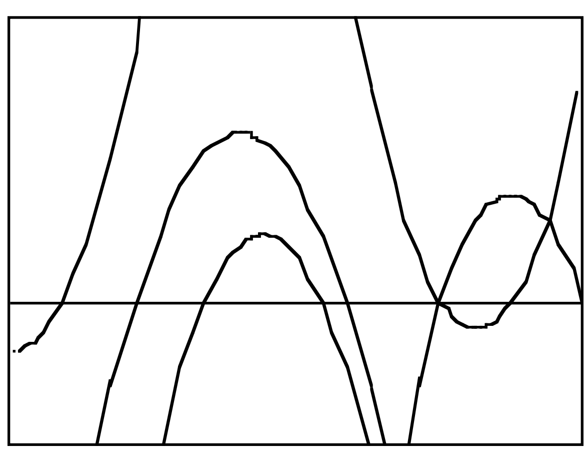
c. Test your theory from Part **b** using another parabolic function. Describe your test and report on its outcome.

2.2 As mentioned in the introduction, the outlines of the letters in a type font can be described by polynomial functions. Although defining the different parts of a letter’s outline may require as many as 20 polynomial equations, graphic artists can easily change the size or shape of letters with the help of design software. For example, the three different G’s in the following diagram show how a letter created using Bézier curves can be altered on a drawing program.

Use two parabolic functions and two line segments to design a simplified version of the letter U. Describe how your design differs from the letter printed below.

U

2.3 The diagram below shows a portion of a stained glass window 12 cm wide and 9 cm high. Determine a set of polynomial functions, along with their respective domains, that could be used to describe the curves in the window.



2.4 A graphic artist is producing a poster 40 cm wide and 55 cm long. The poster will feature a rectangular design surrounded by a border of uniform width.

a. Write a polynomial function *f* which represents the possible area of the rectangular design in terms of the width of the border.

b. Explain what each variable in the function represents.

c. If the border must represent no more than half of the total area of the poster, determine an appropriate domain and range for the function.

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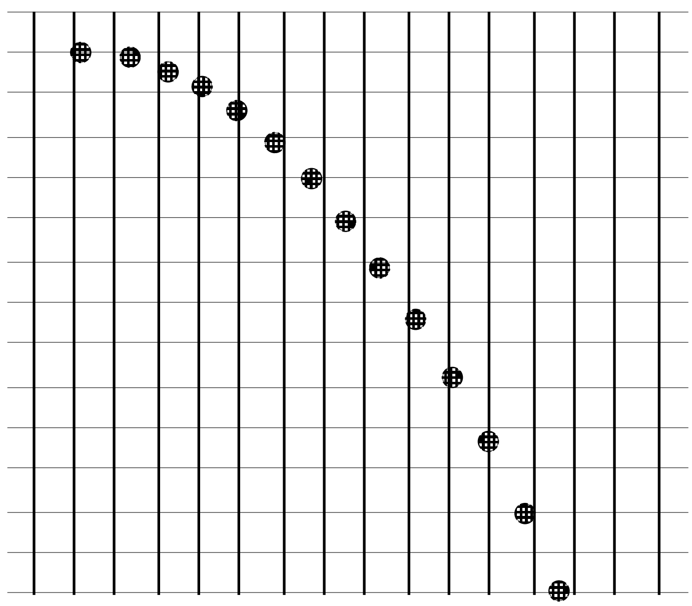
2.5 One application of polynomial functions is the study of projectile motion. A projectile is any object thrown or projected into the air. The path of a projectile is its trajectory.

The drawing below was made from a flash photograph of the trajectory of a golf ball over a portion of its flight. The flashes were made $1/30$ of a second apart. The vertical axis represents the height of the ball above the ground in meters; the horizontal axis represents distance in meters.



a. Identify a location for the origin in the figure above and determine the approximate coordinates for five of the images of the golf ball.

b. Plot these points on a graphing utility and find a polynomial function that models them.

2.6 While playing at a city park, Coretta kicks a soccer ball high into the air. The ball hits the ground 20 m away. Someone watching from a nearby window claims that the ball went as high as the window ledge, about 5 m above the ground.

a. Sketch a graph of the flight of the ball. Let the *x*-axis represent distance from the spot where it was kicked, and the *y*-axis represent distance above the ground.

b. Identify the zeros of the function that describes your sketch and estimate the coordinates of the vertex.

c. Find an equation that describes the path of the ball.

d. Explain what each variable in your equation represents.

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Activity 3

Given any set of points, it is possible to model them with a variety of polynomial functions. This fact has some important consequences in both computer-aided design and mathematical modeling. Figure **7** shows two polynomial functions of different degrees which model the same set of three points. Although both curves contain all of the points, they vary in simplicity and degree.

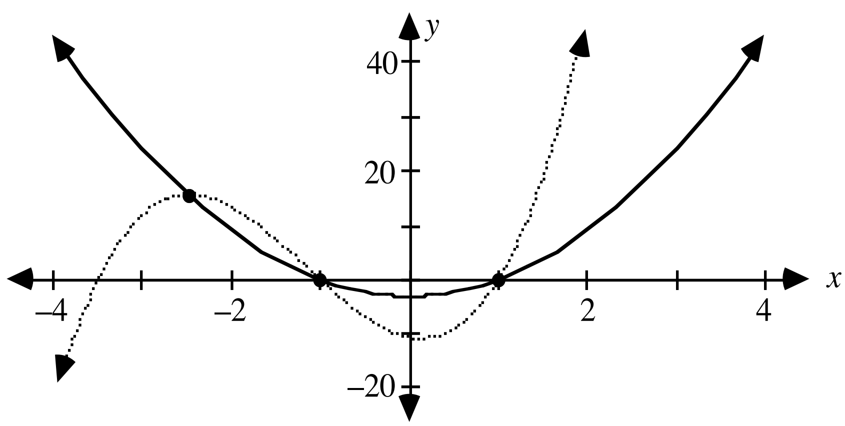


Figure 7: Two polynomial functions fitting the same points

Exploration

In this exploration, you use smooth curves to connect five points. You then model each curve with polynomials of degrees greater than 2.

a. Create an *xy*-coordinate system on a sheet of graph paper.

b. Plot five noncollinear points on the graph paper. Select these five points so that a spline curve containing all the points crosses the *x*-axis a maximum number of times. The five points also should satisfy the conditions of a function. Label each point with its coordinates.

c. Place your model spline so that it connects all five points. The resulting curve should be as smooth and simple as possible.

d. Trace the curve formed by the spline.

e. Estimate the zeros of the function that describes the curve.

f. Use the zeros from Part **e** to determine the factors of a polynomial.

g. Use the factors to write a polynomial function (in factored form) that approximates the graph.

h. Plot the points from Part **b** and the function from Part **g** on a graphing utility. If necessary, modify the lead coefficient in the function to obtain a curve that approximates the points as closely as possible.

Discussion

a. Describe the curve you used to connect the points in the exploration.

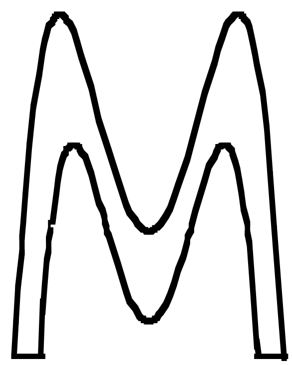
b. What is the degree of the polynomial that models the curve?

c. Is it possible for a polynomial function of the same degree as the one found in the exploration to have fewer *x*-intercepts than the curve in the exploration?

d. Describe some physical structures outside the mathematics classroom in which you have seen curves like the ones described in Part **a**.

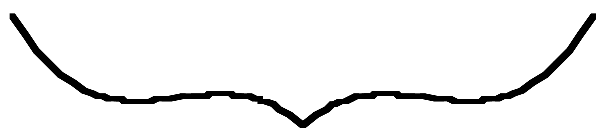
Assignment

3.1 **a.** Determine a set of polynomial functions that could be used to create the outline of the letter M below. Include the appropriate restrictions on the domain of each function.



b. Create the outline of another letter using polynomial functions. Record the functions you used along with the appropriate restrictions on the domain of each one.

3.2 The diagram below shows a cross section of a ship's hull.



a. Determine a polynomial function that could be used to create the curve which defines the right-hand side of the hull. Assume that the axis of symmetry for the entire diagram is the y-axis.

b. Determine a polynomial function that could be used to create the curve which defines the left-hand side of the hull.

c. How are the two functions in Parts **a** and **b** related? Explain your response.

3.3 In Problem **1.6**, you determined a set of first-degree polynomial functions that could be used to produce the letters WAX. Use polynomials with degrees of 2 or higher to produce these same letters. Include the domain and the range for each function used.

3.4 Use polynomial functions to design a picture of your choice. Your response should include a sketch of the picture, a list of the functions and their zeros, and a description of the domain and range used to view the picture on a graphing utility.

* * * * *

3.5 The formula for calculating the total amount in an interest-bearing savings account is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the initial amount invested, r is the annual interest rate in decimal form, n is the number of times interest is calculated per year, and t is the time in years.

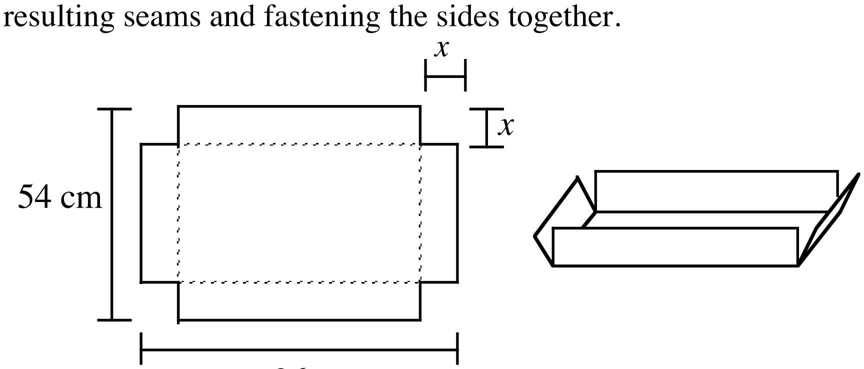
a. Christine's grandparents started a savings account for her on her 13th birthday. They invested \$250 at an annual interest rate of 5.5%. What was the amount in the account after 1 year if interest is calculated yearly and no money was withdrawn during the year?

b. On each of Christine's next five birthdays, her grandparents deposited \$200, \$325, \$450, \$400, and \$675, respectively, into the account. If no money was withdrawn from the account during these five years, what was the balance on Christine's 18th birthday?

c. Letting $x = 1 + r/n$, write a polynomial equation in x that models the amount of money in the saving account on Christine's 18th birthday.

d. If interest is calculated yearly and no money added or withdrawn after Christine's 18th birthday, write a polynomial equation in x that models the amount in the saving account on her 30th birthday.

3.6 A container company plans to create an open-topped box from a sheet of cardboard 54 cm wide and 96 cm long. As shown in the following diagram, a square with length x is cut from each corner of the cardboard. The box is formed by folding the cardboard along the resulting seams and fastening the sides together.



a. **1.** Determine a polynomial function in x that represents the surface area of the open-topped box.

2. What is the domain of this function? Explain your response.

3. Find the zeros of the function and explain what they represent in terms of the box.

b. Repeat Part **a** for a polynomial function in x that represents the volume of the box.

c. Determine the value of x that results in a box with the maximum possible volume.

d. Find the surface area of the box with the maximum volume.

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Activity 4

While polynomials of lower degrees can provide simple curves for graphic design, polynomials of higher degrees can be used to model more complex curves. In this activity, you explore the degrees of polynomials whose graphs have similar characteristics.

Exploration 1

The degree of a polynomial function has an effect on the shape of its graph. For example, consider polynomial functions of the form $y = x^n$ and $y = -x^n$, where n is a positive integer.

a. Let n be even. Using a graphing utility, graph at least three different functions of the form $y = x^n$ on the same set of axes. Compare the graphs and determine the domain and range of each function.

b. Let n be even. Graph some polynomial functions of the form $y = -x^n$. Compare the graphs and determine the domain and range of each function.

c. Compare the domains and ranges of the functions in Part **a** with the domains and ranges of the functions in Part **b**.

d. Let n be odd and greater than 1. Using a graphing utility, graph at least three different equations of the form $y = x^n$ on the same set of axes. Compare the graphs and determine the domain and range of each function.

e. Let n be odd and greater than 1. Graph some polynomial functions of the form $y = -x^n$. Compare the graphs and determine the domain and range of each function.

f. Compare the domains and ranges of the functions in Part **d** with the domains and ranges of the functions in Part **e**.

g. Investigate graphs of functions of the form $y = ax^n$ for various values of a and n. Note the shapes of the graphs and the corresponding domains and ranges.

Discussion 1

a. **1.** What effect does a have on the general shape of the graph of $y = ax^n$?

2. What general statement can you make about the domain and range of the function $y = ax^n$?

b. **1.** Describe the y-values of a graph of $y = x^n$ when n is even, as x increases from –5 to 5.

2. Describe the y-values of a graph of $y = x^n$ when n is odd, as x increases from –5 to 5.

c. As the x-values of a polynomial function increase (or decrease) without bound, the change in the corresponding y-values is referred to as the **end behavior** of the function. What generalizations, if any, can you make about the end behaviors of the graphs in Exploration **1**?

d. Is it possible for the range of a polynomial function with an even degree to contain all the real numbers? Explain your response.

e. Is it possible to predict the range of a polynomial function with an odd degree? Explain your response.

Exploration 2

Earlier in this module, you discovered a relationship between the zeros of a polynomial function and its real-number factors. In this exploration, you investigate this relationship for another group of polynomial functions of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$.

a. Using a graphing utility, graph each of the following equations on a separate coordinate system. Approximate the roots of each function.

1. $f(x) = x^2 - 3x + 2$

2. $f(x) = x^3 - 6x^2 + 11x - 6$

3. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

4. $f(x) = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120$

b. Using technology, factor each polynomial function in Part **a**. Do the results support the approximate roots determined from the graphs?

c. One way to translate the graph of a function involves adding a real number to its equation. Experiment with this technique by adding various real numbers to each function in Part **a** and graphing the results.

d. **1.** Consider the function $f(x) = x^2 - 3x + 2$. Write a new function $g(x)$ that represents a vertical translation of $f(x)$.

2. Graph $g(x)$ and adjust the translation until $g(x)$ has a different number of roots than $f(x)$.

3. Use the graph of $g(x)$ to approximate its roots.

4. Use technology to factor $g(x)$. Do the results support the approximate roots determined from the graph?

e. Repeat Part **d** for each of the remaining functions listed in Part **a**.

Discussion 2

a. When a linear function is translated vertically, how does this transformation affect the roots of the function? Explain your response.

b. Does translating a polynomial function change its degree?

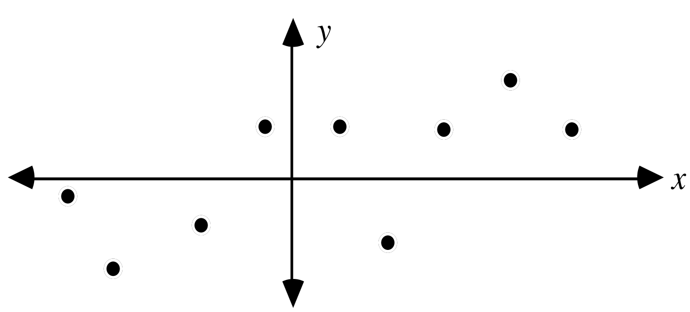
c. In Parts **d** and **e** of Exploration **2**, was it possible to express each vertically translated function as a product of first-degree polynomials? How do the graphs of the vertically translated functions support your response?

d. **1.** How is the number of real roots of a polynomial function of the form $f(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$ related to its degree?

2. Is this same relationship true for all polynomial functions of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$? Explain your response.

e. What can you tell about the degree of a polynomial by examining its graph?

f. **1.** What degree of polynomial function would you recommend to a draftsperson trying to fit a curve to the following set of points? Explain your response.



2. Is this the only degree that the draftsperson could use? Explain your response.

Assignment

4.1 Examine the graphs of $f(x)=(x-2)^n$ for $n=1,\ 2,\ 3,\ 4,\ 5,\ 6$. Describe any patterns you observe.

4.2 **a.** Determine a polynomial function with two zeros whose range is the set of non-negative real numbers.

b. Determine a polynomial function with two zeros whose range is the set of real numbers.

4.3 **a.** Determine a polynomial function with a degree greater than 3 that has exactly two distinct real roots and whose range does not include all real numbers.

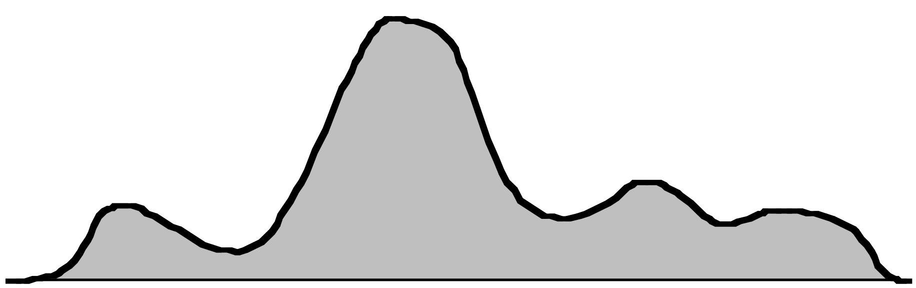
b. Determine a polynomial function with a degree greater than 3 that has exactly three distinct real roots and whose range includes all real numbers.

4.4 **a.** Create a function whose y-values increase over a portion of the interval $[-5, 5]$ for x and decrease over the rest of the interval.

b. Create a function whose y-values decrease as x increases from -5 to 5.

4.5 In Problem **2.2**, you designed a letter U using two second-degree polynomial functions. Redesign your letter U using polynomial functions of degrees higher than 2. Compare the two designs. Do you prefer the new one or the old one? Explain your response.

4.6 Bézier curves are made by splicing together smaller pieces of cubic (third-degree) polynomial curves. The following figure shows a graphical representation of a landscape drawn using Bézier curves.



a. Trace the landscape and estimate where you think the pieces of the different cubic polynomials begin and end. Explain how you determined these locations.

b. Why would a drawing program describe a curve of this kind as a collection of cubic polynomials rather than as a single higher-degree function?

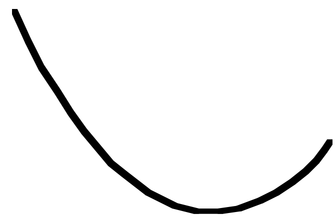
* * * * *

4.7 Consider the function $f(x) = x^4 - 23x^2 + 18x + 40$. Determine the intervals of the domain where:

a. the values of y increase as the values of x increase

b. the values of y decrease as the values of x increase.

4.8 The figure below shows part of the graph of a polynomial function.



Sketch at least two possible extensions of the graph on a coordinate grid. For each version, describe the following:

a. its degree

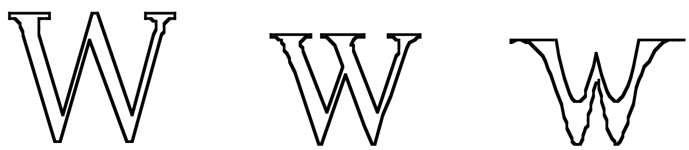
b. its zeros

c. its end behavior.

* * * * * * * * * *

Summary Assessment

The diagram below shows three versions of a capital letter W, along with the names of their respective fonts. The boundaries of each letter were described using polynomials. Design your own capital letter W using polynomials. In your report, include a list of each polynomial function used and its corresponding domain.



New York Times Customized Font

Module Summary

- A **polynomial** in one variable, x, is an algebraic expression of the general form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

where n is a whole number and the **coefficients** a_i are real numbers for $i = 0,\ 1,\ 2,\ \ldots,\ n$.

- The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. A polynomial in the general form shown above has a degree of n, provided that $a_n \neq 0$.
- The x-coordinate of a point where the graph of a function intersects the x-axis is an ***x*-intercept** of the function. It is also referred to as a **root** or **zero** of the function.
- A function f is a **polynomial function** if $f(x)$ is defined as a polynomial in x.
- **Factoring** is the process of using the distributive property to represent a mathematical expression as a product.
- When the real-number roots or zeros of a polynomial function are known, the function can be written in **factored form** as follows:

$$f(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$$

In general, if $f(c) = 0$, $(x - c)$ is a factor of $f(x)$.

- A second-degree polynomial function is a **quadratic** or **parabolic function**. Its graph is a **parabola** that opens either upward or downward.

  Every parabola is symmetric about a line known as the **axis of symmetry**. The point of intersection of a parabola and its axis of symmetry is the **vertex** of the parabola.

- As the x-values of a polynomial function increase (or decrease) without bound, the change in the corresponding y-values is called the **end behavior** of the function.

Selected References

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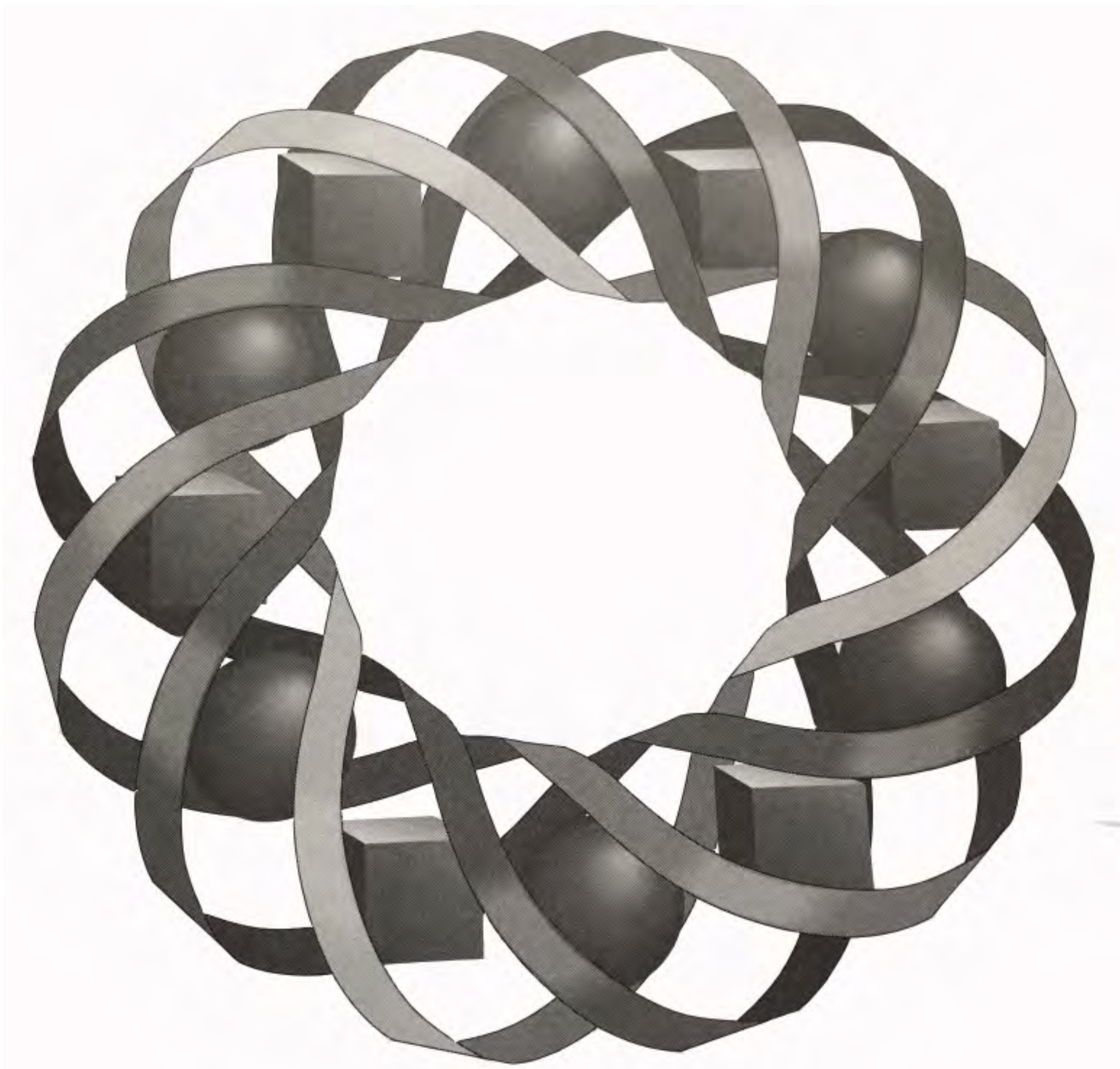
Gilfillan, S. C. *Inventing the Ship*. Chicago: Follett Publishing Co., 1935.

Lewell, John. *A–Z Guide to Computer Graphics*. New York: McGraw-Hill, 1985.

Pokorny, C. K., and C. F. Gerald. *Computer Graphics: The Principles Behind the Art and Science*. Irvine, CA: Franklin, Beedle & Associates, 1989.

Special thanks to the Microsoft and Adobe companies for their assistance with this module.

Log Jam



What do earthquakes, noise levels, and upset stomachs have in common? Data about these phenomena can be difficult to interpret when graphed on a linear scale. In this module, you use logarithms to investigate another type of scale.

Dick Sander • Paul Swenson • Dan West

Log Jam

Introduction

Sheila can’t sleep because of tomorrow's job interview. What questions will she be asked? How should she respond? As Sheila tosses and turns, her stomach grows uneasy. She climbs out of bed and heads for the medicine cabinet.

Her pain may be due to an irritation of the stomach lining caused by an increase in gastric acid. By neutralizing the excess acid with nonprescription antacids, Sheila can temporarily relieve the pain.

Acidity is commonly measured on the pH scale. The *p* comes from the German word for power; the *H* from the chemical symbol for hydrogen. The **pH scale** describes the relative strength of an acid or base in solution by measuring the concentration of hydrogen ions in a solution.

Figure **1** shows the range of common pH values. On the pH scale, a lower value indicates a more acidic solution. The pH of the gastric juices in your stomach is normally about 2.5. Water, which is neutral, has a pH of 7. Any solution which has a pH less than 7 is acidic; any solution which has a pH greater than 7 is basic. A pH of 0 indicates an extremely acidic solution. A pH of 14 indicates an extremely basic solution.

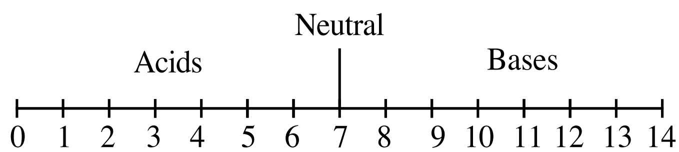


Figure 1: The pH scale

Science Note

An **ion** is an atom (or group of atoms) that has a net positive or negative charge.

An **acidic solution** contains an excess of hydrogen ions (H^+). Many familiar fruits and vegetables, such as lemons, apples, and tomatoes, are weakly acidic. Strong acids, like those found in car batteries, are dangerously corrosive and extremely poisonous.

A **basic solution** contains an excess of hydroxide ions (OH^-). Antacids, some water softeners, and some laxatives contain weak bases. Strong bases, such as those found in lye and drain cleaners, are very caustic and can burn skin.

A solution that is neither acidic nor basic is **neutral**.

Exploration

a. Predict whether each of the following solutions is acidic or basic.

1. a carbonated soft drink

2. milk

3. a solution of baking soda and water

4. vinegar

5. orange juice

b. Blue litmus paper turns pink when dipped in an acidic solution; pink litmus paper turns blue when dipped in a basic solution. Use litmus paper or a pH probe to test your predictions from Part **a**.

Discussion

a. **1.** Which of the solutions in the exploration are acidic?

2. What other common solutions are acidic?

b. **1.** Which of the solutions in the exploration are basic?

2. What other common solutions are basic?

c. What would you expect to happen when an acid and a base are mixed?

Activity 1

After reading the label on the package, Sheila swallows the recommended dosage of antacid and waits for the pain to subside. How does a manufacturer decide how much medication is necessary to relieve the typical symptoms of acid indigestion?

When a basic solution is added to an acidic solution, the excess hydroxide ions react with the excess hydrogen ions to form water. In the following exploration, you observe how the pH of an acidic solution changes with the addition of measured units of a base.

Exploration

a. **1.** Pour about 150 mL of a carbonated soft drink into a 250-mL beaker. Stir or swirl the liquid until most of the carbonation disappears.

2. Measure the pH and record its value.

b. In another beaker, dissolve 2 level teaspoons of baking soda in about 75 mL of water. **Note**: If the baking soda does not completely dissolve, use only the clear solution for Parts **c–d**.

c. **1.** Fill an eye dropper with about 1 mL of the baking soda solution.

2. Add the contents of the dropper to the beaker of soft drink and stir.

3. Measure the pH of this new solution and record its value.

d. Repeat Part **c** 8 to 15 more times.

e. Create a spreadsheet with headings similar to the ones in Table **1** below.

Table 1: Results of pH experiment

Droppers of Baking Soda Solution	pH	Change in pH
0		
1		
2		
3		
⋮		

f. Enter your data in the spreadsheet and determine the change in pH for each consecutive pair of pH values.

g. Determine the number of droppers of basic solution that must be added to the soft drink to produce a change in pH approximately equal to the change in pH obtained when the second dropper was added.

h. Create a scatterplot of pH vs. droppers of basic solution added.

Discussion

a. What happened to the pH as each dropper of baking soda solution was added to the soft drink?

b. How was the change in pH for each consecutive pair of pH values affected as the number of droppers increased?

c. Compare your result in Part **g** of the exploration with those of your classmates. What reasons might there be for any variations in this number?

d. Figure **2** contains a graph of data collected while performing the experiment in the exploration using a sensitive pH monitor. Because this experiment allowed continuous monitoring of the soft-drink solution as droppers of basic solution were added, the graph shows pH versus time in seconds.

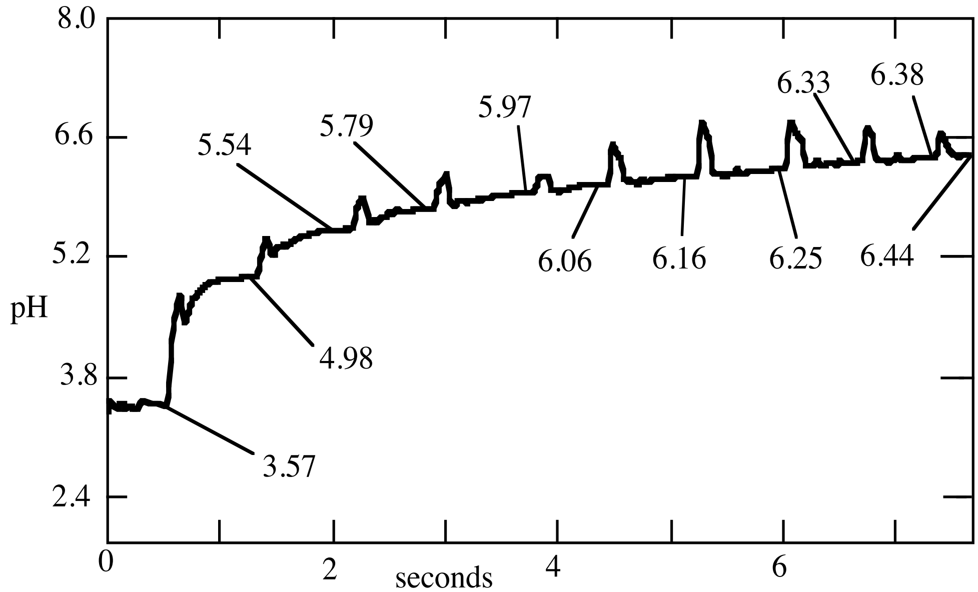


Figure 2: Data from pH experiment

1. What do the spikes on the graph in Figure **2** represent?
2. How does the scatterplot you created in Part **h** of the exploration compare to the graph in Figure **2**?
3. In the graph in Figure **2**, the change in pH produced by the first dropper of basic solution was 1.41. How many more droppers do you think would have to be added to raise the pH another 1.41 units?

Science Note

Acidity can be reported as the concentration of hydrogen ions (H^+) in a solution. On the pH scale, a decrease in 1 unit indicates a change in the concentration of hydrogen ions by a power of 10. For example, the concentration of H^+ in a solution with a pH of 3 is 10 times the concentration in a solution with a pH of 4.

The number of hydrogen ions in a liter of water is about $6 \bullet 10^{16}$. Because the quantities of ions in ordinary volumes of solution are so great, chemists often express concentration in moles per liter. A **mole** of any substance always contains the same number of particles: about $6.02 \bullet 10^{23}$. This number is known as Avogadro's number, in honor of Amadeo Avogadro (1776-1856), an Italian chemist and physicist.

For example, the concentration of hydrogen ions in water is about $1 \bullet 10^{-7}$ moles/L. The relationship between the number of hydrogen ions in water and their concentration in moles per liter is shown below:

$$\frac{6 \bullet 10^{16} \text{ ions}}{1 \text{ L}} \bullet \frac{1 \text{ mole}}{6.02 \bullet 10^{23} \text{ ions}} \approx \frac{1 \bullet 10^{-7} \text{ moles}}{1 \text{ L}}$$

e. Considering the example given in the mathematics note, how does the exponent of the hydrogen ion concentration appear to be related to pH?

f. Which has a greater concentration of hydrogen ions: water or a solution with a pH of 13?

g. Scientific notation is used to express very large or small quantities as the product of a power of 10 and a number greater than or equal to 1 and less than 10. Describe how to simplify each of the following expressions using the properties of exponents.

1. $\left(1 \bullet 10^{-7}\right)\left(6.02 \bullet 10^{23}\right)$

2. $\left(5.4 \bullet 10^{5}\right)/\left(2.2 \bullet 10^{-2}\right)$

h. What is the change in hydrogen ion concentration from a pH of 3 to a pH of 7? Explain your response using a property of exponents.

i. The concentration of hydrogen ions in a solution with a pH of 2.5 can be described as $1 \bullet 10^{-2.5}$ moles/L. What is the mathematical meaning of $10^{-2.5}$?

j. **1.** Decimal values for pH correspond with hydrogen ion concentrations expressed using decimal exponents. Compare the mathematical meanings of $10^{5.97}$ and $10^{597/100}$.

2. What problems arise when treating $10^{1/3}$ as $10^{(0.333\ldots)}$?

3. Since $10^{(0.333\ldots)}$ is equal to $10^{(0.3+0.03+0.003+\cdots)}$, it can be treated as shown below:

$$10^{0.3} \bullet 10^{0.03} \bullet 10^{0.003} \bullet \cdots$$

Given this fact, describe a relationship between $10^{1/3}$ and $10^{(0.333\ldots)}$.

k. Consider an irrational number n (such as π) that cannot be expressed in the form a/b. How could you evaluate 10^{n}?

Assignment

1.1 **a.** What is the concentration of hydrogen ions in a solution with a pH of 4?

b. What is their concentration in a solution with a pH of 11?

c. If a solution contains a hydrogen ion concentration of $1 \bullet 10^{-12}$ moles/L, what is its pH?

1.2 **a.** Determine the approximate number of hydrogen ions in 1 L of a solution with:

1. a pH of 3

2. a pH of 8

3. a pH of n.

b. If 1 L of a solution contains 10^{-n} moles of hydrogen ions, what is its pH?

1.3 **a.** Use ratios to compare the concentration of hydrogen ions in solutions with a pH of 3, a pH of 4, and a pH of 8, respectively.

b. How is the decrease in pH related to the increase in the concentration of hydrogen ions?

1.4 Create a table with the headings shown below.

pH	Concentration of Hydrogen Ions (Moles/L)	No. of Hydrogen Ions per Liter
1		
2		
3		
⋮		
14		

a. Complete the table for pH values from 1 to 14.

b. Make a scatterplot of concentration of hydrogen ions versus pH.

c. Make a scatterplot of number of hydrogen ions per liter versus pH.

d. Compare the shapes of the two graphs from Parts **b** and **c**.

e. Given only the graphs in Parts **b** and **c** (without the table of values), how could you estimate the concentration or number of ions per liter for solutions with a pH greater than 3? Are these graphs useful? Explain your response.

* * * * *

1.5 The Richter scale is used to measure the intensity of earthquakes. An increase in magnitude of 1 unit on this scale corresponds to a 10-fold increase in intensity. According to the Richter scale, an earthquake of magnitude 4, although mild, can be felt by most people. About how many times stronger than an earthquake of magnitude 4 is an earthquake of magnitude 7?

1.6 The magnitude of the historic San Francisco earthquake of 1906 is estimated at 8.3 on the Richter scale.

a. How does its intensity compare to the 1989 San Francisco quake, which measured 6.9 on the Richter scale?

b. Explain this difference in intensity using the following property of exponents: $a^m \bullet a^n = a^{m+n}$.

c. **1.** Give a mathematical interpretation of $10^{8.3}$ using a radical.

2. Give a mathematical interpretation of $10^{8.3}$ using the property of exponents described in Part **b**.

3. Describe how to use a calculator to evaluate each interpretation in Steps **1** and **2**.

* * * * * * * * * *

Activity 2

The numbers of atoms, molecules, or ions involved in chemical analysis often cover an extremely wide range of values. As a result, it can be difficult to interpret such data graphically using a familiar linear scale. In this activity, you examine another useful type of scale: a **logarithmic** scale.

Exploration 1

In the Level 2 module "Atomic Clocks Are Ticking," you investigated some of the properties of exponents. In this exploration, you use technology to reexamine decimal exponents.

a. Create a graph of the function $y = 10^x$.

b. Determine the value of y that corresponds to each of the following values of x: 0, 1, 2, and 3.

c. Use the trace feature to find the value of x (to the nearest 0.01) that corresponds to each value of y listed in Table **2**.

Table 2: Values of *x* and *y* for the equation $y = 10^x$

x	y
0	1
	2
	3
	4
	5
	6
	7
	8
	9
1	10

d. Using the data in Table **2**, create a scatterplot of the y-values versus the corresponding x-values.

e. Using the data in Table **2**, create a scatterplot of the x-values versus the corresponding y-values.

Discussion 1

a. What is the relationship between the graph you created in Part **d** of the exploration and the one you created in Part **e**?

b. Is there a value of x such that $10^x = 0$? Why or why not?

c. A **linear scale** is based on an arithmetic sequence of units in which the difference between successive units is always the same.

1. What type of sequence is formed by the set of y-values in Table **2**?
2. Would a scale based on the x-values in Table **2** form a linear scale? Explain your response.

Mathematics Note

The inverse of the exponential equation $y = b^x$ is the logarithmic equation $x = \log_b(y)$. In other words, a relationship of the form $b^n = m$ can also be expressed as follows: the **logarithm** of m to the base b is n. This can be denoted symbolically as $\log_b(m) = n$. The base b must be greater than 0 and unequal to 1.

On a **logarithmic scale**, the number m corresponding to the number n on a linear scale is the power of the base such that $b^n = m$.

For example, Figure **3** shows a logarithmic scale in which the base is 10. On this scale, 10 corresponds to 1 on the linear scale because $10^1 = 10$ or $\log_{10}(10) = 1$. Similarly, 1 on the logarithmic scale corresponds to 0 on the linear scale because $10^0 = 1$ or $\log_{10}(1) = 0$.

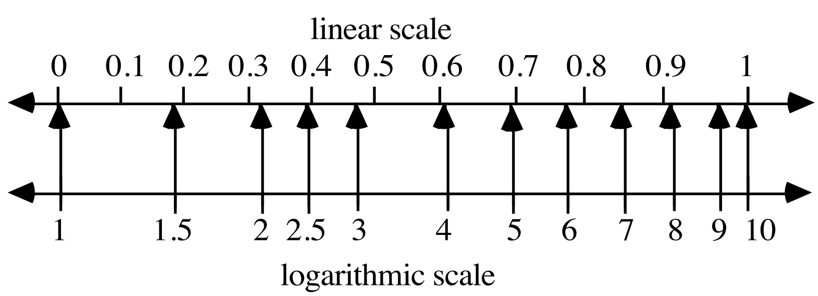


Figure 3: Logarithmic and linear scales

Logarithms of base 10 are referred to as **common logarithms**. The common logarithm of x may be written as $\log_{10} x$, but is usually condensed to $\log x$.

A coordinate system with only one axis marked in a logarithmic scale is a **semilog coordinate system**. When an exponential function of the form $y = b^x$ is graphed on a semilog coordinate system in which the vertical axis is a logarithmic scale, the graph is a straight line.

When both axes are marked in logarithmic scales, they constitute a **log-log coordinate system**.

d. As described in the previous mathematics note, the logarithm of x to base b can be written as $\log_b x$ and has restrictions of $b > 0$, $b \neq 1$, and $x > 0$.

1. Explain why the number 1 cannot be a base.

2. Explain why the base b cannot be negative.

e. Why don't negative numbers have logarithms?

f. The relationship $2^3 = 8$ can be expressed as $\log_2 8 = 3$. Describe how to express each of the following using logarithms:

1. $25 = 5^2$

2. 3^3

3. 81

Exploration 2

a. In Figure **4** below, a logarithmic scale has been labeled with successive powers of 10. Determine the common logarithm of each power of 10.

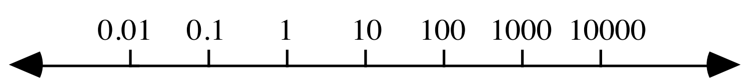


Figure 4: A logarithmic scale

b. The concentration of hydrogen ions in the normal range of pH values may vary from 1 mole/L to $1 \bullet 10^{-14}$ moles/L. Figure **5** shows a scatterplot of this data on a coordinate system that uses linear scales on both axes.

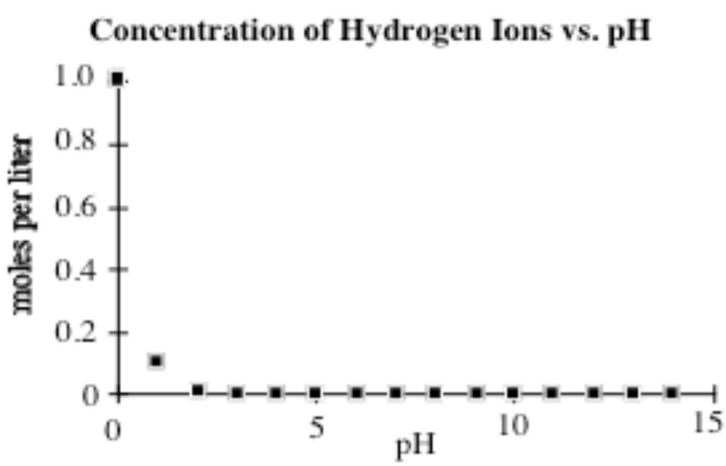


Figure 5: Concentration of hydrogen ions versus pH

How do you think this graph would be affected by changing the scale on the vertical axis to the logarithmic scale in Figure **4**? Record your prediction.

c. Table **3** shows the concentrations of hydrogen ions, in moles per liter, that correspond with pH values from 0 to 14. **Note:** The data in the middle column is a mixture of scientific and regular notation. All data could be written in either form.

Table 3: pH and concentration of hydrogen ions

pH	Concentration of Hydrogen Ions (Moles per Liter)	log(Concentration of Hydrogen Ions)
0	1	
1	0.1	
2	0.01	
3	0.001	
4	0.0001	
5	0.00001	
6	0.000001	
7	0.0000001	
8	0.00000001	
9	$1 \bullet 10^{-9}$	
10	$1 \bullet 10^{-10}$	
11	$1 \bullet 10^{-11}$	
12	$1 \bullet 10^{-12}$	
13	$1 \bullet 10^{-13}$	
14	$1 \bullet 10^{-14}$	

1. Complete the right-hand column of Table **3** using the common log of the concentration of hydrogen ions for pH values from 0 to 14.
2. Create a semilog coordinate system using a base 10-logarithmic scale like the one in Figure **4** for the *y*-axis.
3. Using the semilog coordinate system from Step **2**, create a scatterplot of the concentration of hydrogen ions versus pH.

Discussion 2

a. Compare the graph in Figure **5** with the graph of the same information plotted on a semilog coordinate system in Part **c** of Exploration **2**.

b. The ratio of the concentrations of hydrogen ions in solutions with a pH of 5 and a pH of 10 is 10^5. How is this ratio illustrated by the graph in Figure **5**? How is this ratio illustrated by the graph in Part **c**?

c. Describe some advantages to graphing the data in Table **3** on a semilog coordinate system.

d. Consider the logarithmic equation $3 = \log(x + 2)$. How would you determine the value of *x* that makes this a true statement?

Assignment

2.1 Why is the number 10 a convenient base for a semilog coordinate system?

2.2 Rewrite each of the following equations using logarithmic notation:

a. $6^4 = 1{,}296$

b. $5^7 = 78{,}125$

c. $2^{10} = 1{,}024$

2.3 Find each of the following common logarithms:

a. $\log\left(10^{-2}\right)$

b. $\log 10{,}000$

c. $\log 0.0001$

2.4 The notation commonly used in chemistry to represent the concentration of hydrogen ions in moles per liter is $[H^+]$. Using this notation, which one of the following formulas expresses pH in terms of hydrogen ion concentration? Explain why you believe your choice is correct.

a. $\mathrm{pH} = \log\left(-\left[H^+\right]\right)$

b. $\mathrm{pH} = \log[H^+]$

c. $\mathrm{pH} = -\log[H^+]$

2.5 Use technology and the relationship between hydrogen ion concentration and pH you identified in Problem **2.4** to calculate the pH of these common liquids.

	Solution	**Concentration of Hydrogen Ions (Moles per Liter)**
a.	lemon juice	0.005
b.	vinegar	0.0016
c.	carbonated water	0.001
d.	milk	$2.51 \bullet 10^{-7}$
e.	blood	$3.98 \bullet 10^{-8}$

* * * * *

2.6 Imagine that a country's annual steel production, in millions of tons, can be modeled by the equation $y = 96\log(x + 3)$, where $x = 0$ represents the year 1955.

a. Graph this logarithmic function on a graphing utility, using appropriate intervals for the domain and range. Describe the shape of the graph.

b. Use the graph to estimate the country's steel production in 1980 and in 1995.

c. Using this model, during what year do you predict the steel production will reach 200 million tons?

2.7 A greeting card company spends $300,000 per year on advertising. The marketing department has determined that if the amount of money spent on advertising is increased by *p* dollars, the total sales *S* can be modeled by the equation $S = 132{,}000 \log(p + 300{,}000)$.

a. Graph this logarithmic function on a graphing utility, using appropriate intervals for the domain and range. Describe the shape of the graph.

b. Determine the company's current total sales.

c. Use this model to predict total sales if the company doubles its advertising budget.

d. Based on your results in Parts **b** and **c**, do you think the company should double its advertising budget? Explain your response.

* * * * * * * * * *

Activity 3

In previous activities, you used graphs to discover the logarithmic relationship between the pH of a solution and its hydrogen ion concentration. In this activity, you explore some algebraic methods for solving problems involving logarithms.

Exploration

Because logarithms are exponents, it is possible to apply your knowledge of exponents to problems involving logarithms. In this exploration, you use technology to help determine a property of logarithms that corresponds to each of the following properties of exponents:

- $b^x \bullet b^y = b^{x+y}$
- $b^x / b^y = b^{x-y}$
- $(b^x)^y = b^{x \bullet y}$

a. Create a spreadsheet with headings similar to the ones shown in Table **5**.

Table 5: Logarithmic formulas

x	y	$\log x$	$\log y$	$\log(x \bullet y)$
1	25			
2	24			
3	23			
$\vdots$	$\vdots$			
24	2			
25	1			

b. Complete the spreadsheet for $\log x$, $\log y$, and $\log(x \bullet y)$. Round these values to the nearest 0.001.

c. Note any relationships you observe between values in the column labeled "$\log(x \bullet y)$" and the values in any of the other four columns.

d. **1.** Add a column to your spreadsheet to calculate $\log(x/y)$.

2. Repeat Part **c** for $\log(x/y)$.

e. Repeat Part **d** for $\log x^y$, $\log 10^x$, and $10^{\log x}$.

f. Repeat Parts **a–e** using a base other than 10.

Discussion

a. Do you think the relationships you found for common logarithms are true for logarithms of any base? Justify your response.

Mathematics Note

The following properties are true for logarithms of any base b, where $b > 0$ and $b \neq 1$.

- product property: $\log_b(x \bullet y) = \log_b x + \log_b y$, for $x > 0$ and $y > 0$
- quotient property: $\log_b(x/y) = \log_b x - \log_b y$, for $x > 0$ and $y > 0$
- power property: $\log_b x^y = y\log_b x$, for $x > 0$
- inverse property: $\log_b b^x = x$ and $b^{\log_b x} = x$, for $x > 0$

For example, using the product property of logarithms:

$$\log_2\left(2^2 \bullet 2^{0.5}\right) = \log_2\left(2^2\right) + \log_2\left(2^{0.5}\right) = 2 + 0.5 = 2.5$$

Using the quotient property, $\log_2\left(2^2/2^{0.5}\right) = \log_2\left(2^2\right) - \log_2\left(2^{0.5}\right) = 2 - 0.5 = 1.5$.
Using the power property, $\log_2\left(2^3\right) = 3\log_2(2) = 3$.

b. Compare the properties of logarithms described in the mathematics note to the properties of exponents.

c. When is $\log_b x = \log_b y$?

Assignment

3.1 Use the properties of logarithms to determine the value of *x* that makes each of the following statements true.

a. $6^x = 234$

b. $5^x = 1000$

c. $2^x = 50$

3.2 When graphed on a semilog coordinate system, the relationship between hydrogen ion concentration $[H^+]$ in moles per liter and pH can be modeled by the equation:

$$[H^+] = \frac{1}{10^{pH}}$$

Use the properties of logarithms to show that this equation is equivalent to $pH = -\log[H^+]$.

3.3 Imagine that you are a cell biologist. For your next experiment, you must grow 100,000 bacteria of a particular type. You begin the bacterial culture with 1 cell. Through a process known as mitosis, that cell divides into 2. The 2 new cells split again, creating 4 cells. The bacterial population quadruples every day.

a. Predict how many days it will take to produce enough cells for the experiment.

b. Write an equation that represents the relationship between the number of days and the number of cells.

c. Using appropriate technology, estimate the number of days (to the nearest 0.1 days) required for the cell population to reach 100,000. Explain the process you used to find this approximation.

d. Using algebraic methods, determine the number of days required for the cell population to reach 100,000. Compare this solution with your estimate in Part **c**.

e. Use a symbolic manipulator to verify your solution in Part **d**.

3.4 A *Los Angeles Times* article reported that the intensity of the 1971 Sylmar quake which rocked the northern San Fernando Valley was one-fourth that of the 1989 San Francisco quake. Since the magnitude of the 1989 quake was 6.9 on the Richter scale, the equation that models this situation is $1 \bullet 10^{6.9} = 4 \bullet 10^x$, where *x* represents the magnitude of the Sylmar quake. Find the magnitude of the Sylmar quake.

3.5 If log n represents the magnitude of an earthquake on the Richter scale, then $\log(2n)$ represents the magnitude of an earthquake with twice the intensity.

a. **1.** Use the properties of logarithms to determine the increase in magnitude when the intensity of an earthquake is doubled.

2. Does the original intensity affect the increase in magnitude? Justify your response.

b. What is the increase in magnitude on the Richter scale when the intensity of an earthquake is tripled? Justify your response.

* * * * *

3.6 The properties of logarithms can be used to solve many problems involving exponents. For example, the formula for calculating the total amount in an interest-bearing savings account is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

In this formula, A is the total in the account, P is the original investment or principal, r is the annual interest rate in decimal form, n is the number of times per year that the interest is calculated or compounded, and t is the time in years that the account is active.

a. Imagine that you have deposited \$100 in a savings account at 6% annual interest, compounded monthly. Estimate how long the principal and interest must be left in the account for the total to reach \$450.

b. Using appropriate technology, determine a more precise answer to Part **a**. Describe the process that you used.

c. To determine an answer to Part **a** algebraically, you must substitute the known values into the formula above and solve for t. The solution follows Steps **1–10** below. Describe what has been done in each step.

1. $450 = 100\left(1 + (0.06/12)\right)^{12t}$

2. $450 = 100(1.005)^{12t}$

3. $450/100 = (1.005)^{12t}$

4. $4.5 = (1.005)^{12t}$

5. $\log 4.5 = \log(1.005)^{12t}$

6. $\log 4.5 = 12t \log(1.005)$

7. $\log 4.5 / \log 1.005 = 12t$

8. $301.57 \approx 12t$

9. $301.57/12 \approx t$

10. $t \approx 25.13$

d. Which property of logarithms is necessary in this solution? In which step is the property used?

e. Why is it necessary to use the property identified in Part **d**?

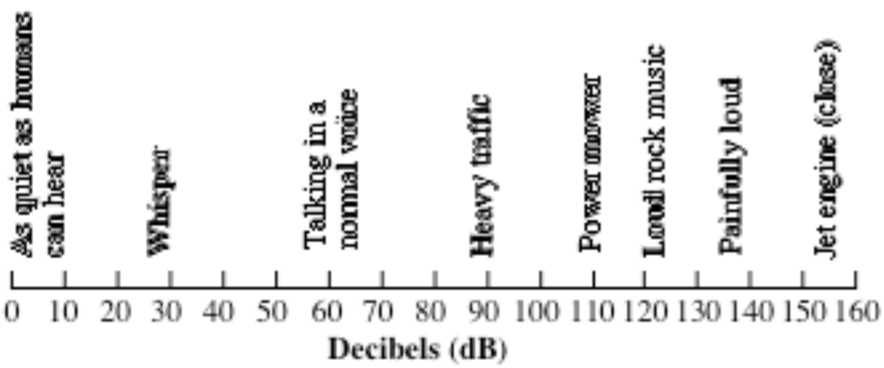
* * * * * * * * * *

Research Project

Several large earthquakes occur each year around the world. Find an article in a magazine or newspaper that describes one of these quakes. In your report, identify the location and magnitude of the earthquake and compare it to the 1989 San Francisco earthquake. Your comparison should mention the differences in magnitude, intensity, and amount of damage done by each quake and explain any reasons for those differences.

Summary Assessment

1. One measure of the relative intensity of sound is the decibel (dB). The decibel chart below shows the magnitude of sound generated by some common objects or situations.



Extremely loud sounds can exert enough pressure on the human ear to cause pain. The following table gives the amount of pressure on the ear, measured in newtons per square meter (N/m^2), at various decibel levels. The pressure at 0 dB, a sound too quiet to be detected by the human ear, is 0.00002 N/m^2. This value is the reference pressure, P_r.

Decibel Level (dB)	Pressure on Ear (N/m^2)
0	0.00002
10	0.0002
20	0.002
30	0.02
40	0.2
50	2
60	20
70	200
80	2,000
90	20,000
100	200,000
110	2,000,000
120	20,000,000
130	200,000,000
140	2,000,000,000
150	20,000,000,000
160	200,000,000,000

a. Use the data given to find a relationship between the decibel level (d) of a sound and the pressure (P) created by that sound.

b. Show that the following equation is equivalent to the relationship you found in Part **a**, where d is the decibel level, P is the pressure created by the sound, and P_r is the reference pressure of 0.00002 N/m^2:

$$d = 10\log\left(\frac{P}{P_r}\right)$$

c. Using the properties of logarithms, solve the equation you found in Part **a** for pressure, P.

2. **a.** What is the pressure on the eardrum caused by music at a loud rock concert (118 dB)? How many times greater is this pressure than the pressure created by normal conversation (54 dB)?

b. A sound creates a pressure on the eardrum of $1 \bullet 10^3 \text{ N/m}^2$. Is this sound audible to the human ear? If so, is this sound painful?

3. Swimming underwater also affects the pressure on your eardrums. For each 1 m of depth, the pressure increases by $10,000 \text{ N/m}^2$.

a. At what depth is the pressure on your ears equivalent to the pressure created by the sound of heavy traffic?

b. If you dove to a depth of 100 m, what would be the pressure on your ears? What is the magnitude of the sound (in decibels) required to create an equivalent amount of pressure?

Module Summary

- An **ion** is an atom (or group of atoms) that has a net positive or negative charge.
- A **basic solution** contains an excess of hydroxide ions (OH^-).
- An **acidic solution** contains an excess of hydrogen ions (H^+).
- A solution that is neither acidic nor basic is **neutral**.
- A **mole** of any substance always contains the same number of particles: about $6.02 \bullet 10^{23}$. The number $6.02 \bullet 10^{23}$ is known as **Avogadro's number**.
- The **pH scale** describes the relative strength of an acid or base in solution. The pH of a solution is equal to the negative log of the hydrogen ion concentration in moles per liter.
- The inverse of the exponential equation $y = b^x$ is the logarithmic equation $y = \log_b(x)$. In other words, a relationship of the form $b^n = m$ can also be expressed as follows: the **logarithm** of *m* to the base *b* is *n*. This can be denoted symbolically as $\log_b(m) = n$. The base *b* must be greater than 0 and unequal to 1.
- Logarithms of base 10 are referred to as **common logarithms**. A common logarithm may be written as $\log_{10} x$, but is usually condensed to $\log x$.
- A **linear scale** is based on an arithmetic sequence of units in which the difference between successive units is always the same.
- On a **logarithmic scale**, the number *m* corresponding to the number *n* on a linear scale is the power of the base such that $b^n = m$.
- A coordinate system with only one axis marked in a logarithmic scale is a **semilog coordinate system**. When both axes are marked in logarithmic scales, they constitute a **log-log coordinate system**.
- The following properties are true for logarithms of any base *b*, where $b > 0$ and $b \neq 1$.

product property:	$\log_b(x \bullet y) = \log_b x + \log_b y$, for $x > 0$ and $y > 0$
quotient property:	$\log_b(x/y) = \log_b x - \log_b y$, for $x > 0$ and $y > 0$
power property:	$\log_b x^y = y \log_b x$, for $x > 0$
inverse property:	$\log_b b^x = x$ and $b^{\log_b x} = x$, for $x > 0$

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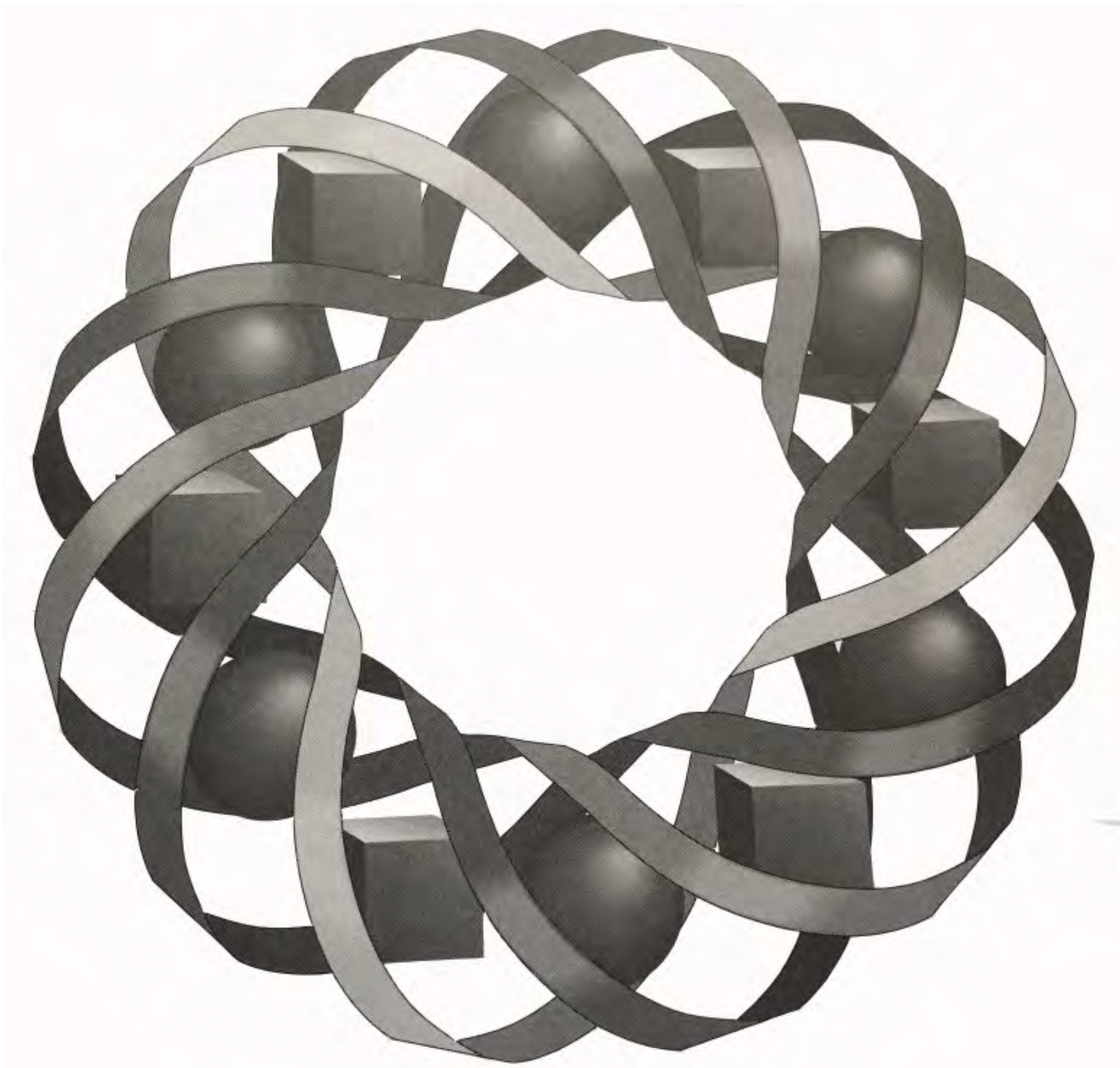
More or Less?



What do car engines and compact discs have in common? When manufacturing either product, quality control engineers use inequalities to represent tolerance and precision intervals.

Kyle Boyce • Ruth Brocklebank • Mark Lutz
Pete Stabio • Tom Teegarden

More or Less?

Introduction

Manufacturers monitor product quality by comparing performance with predetermined standards. If they find unacceptable variations from these standards, they may adjust the manufacturing process in order to improve the product. This process, known as **quality control**, affects the performance of everything from toasters to televisions.

Quality control does not mean that all products will be perfect, however. For example, all computer chips manufactured for a particular model are not exactly the same. Some of the variations that occur during manufacturing or in component materials are beyond physical control. Fortunately, slight differences in a product do not usually affect its performance. The acceptable error in a product is often expressed as a **tolerance interval**.

The tolerance interval, in turn, is determined by the accuracy or precision of the components that make up the product. In this module, this level of accuracy is referred to as a **precision interval**.

Discussion

a. Why is quality control important to consumers?

b. Why is quality control important to manufacturers?

c. What are some products that you associate with high quality?

d. How does competition between manufacturers of similar products result in better products?

e. What factors make it impossible for manufacturers to produce identical products all of the time?

f. What can consumers do when they are not satisfied with the performance of a product?

Activity 1

Most familiar products, from milk cartons to microwave ovens, are manufactured within certain tolerance levels for a variety of specifications. For example, when producing compact discs, the manufacturer must monitor the quality of the recording, the reliability of the materials, and the size of the disc itself. If the dimensions of each disc are not reasonably consistent, some may fail to work in a customer's compact disc player. Because it is not possible to produce hundreds of thousands of disks with exactly the same dimensions, the manufacturer must determine an acceptable tolerance interval.

Exploration

Imagine that you are the quality control manager for a manufacturer of compact discs. The desired circumference of your product is 37 cm.

a. **1.** To model the manufacturing process, cut out five paper circles with the desired circumference of 37 cm.

2. By inspection, determine whether or not your five model discs are all exactly alike.

3. Use a piece of string to measure the circumference of each disc. Record your measurements.

Mathematics Note

An **interval** of real numbers is a set containing all numbers between two given points, the **endpoints** or **bounds** of the interval. An interval may contain one endpoint, both endpoints, or neither endpoint. In addition, some intervals may have only one endpoint, or no endpoints. For example, the set of real numbers can be represented as the interval $(-\infty,\ \infty)$, while the set of real numbers greater than or equal to a can be represented as $[a,\ \infty)$.

Some intervals may be expressed as conjunctions of inequalities. A **conjunction** combines two mathematical statements with the word *and*. For example, the conjunction $2.5 < x$ and $x < 3.9$ describes the interval of real numbers between 2.5 and 3.9, or (2.5, 3.9). This conjunction can be written as $2.5 < x < 3.9$. A conjunction is true only if both statements are true. If one or both of the statements is false, the conjunction is false.

Intervals and associated inequalities also can be expressed using set notation. For example, the inequality $-23.7 \geq x \geq -28.9$ may be written as $x \in [-28.9, -23.7]$, which means "x is an element of the set of numbers in the interval [–28.9, –23.7]."

b.

1. Determine the least interval that contains all five measurements from Part **a**, as well as the desired circumference of 37 cm.
2. Obtain a copy of template A from your teacher. The first number line on the template resembles Figure **1** below. Label this number line *c* to represent the circumference of the discs, then use it to illustrate the tolerance interval.

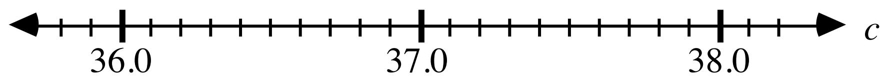


Figure 1: Number line for tolerance

3. Express the tolerance interval as an inequality and using set notation.

c.

1. Find the radius *r* that corresponds to the desired circumference of a compact disc.
2. Determine a precision interval for the radius of compact discs that corresponds to the tolerance interval from Part **b**.
3. The second number line on template A resembles Figure **2** below. Label this number line *r* to represent radius, then use it to illustrate the precision interval.

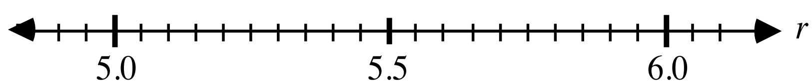


Figure 2: Number line for precision

4. Express the precision interval using set notation.

d. As shown in Figure **3**, draw an arrow from several values for the radius of a compact disc on line *r* to the corresponding values for the circumference on line *c*. This type of model is a **mapping diagram** showing a function from *r* to *c*.

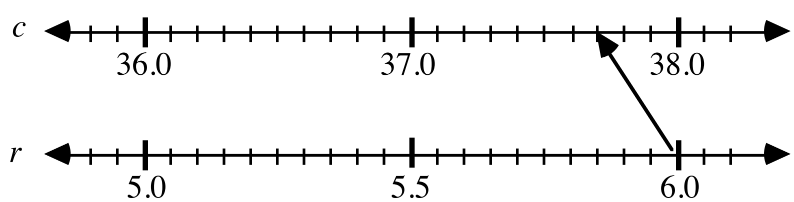


Figure 3: A mapping diagram

e. Compile the class data from Part **a**.

f.

1. Determine a tolerance interval for circumferences and a precision interval for radii that includes all the class data. Express these intervals as inequalities.
2. Represent these intervals graphically using the second set of number lines on template A.

Discussion

a. Were all of the model discs exactly the same size? If not, what factors might explain the differences in size?

b. What function relates the radius of a compact disc to its circumference?

c. Describe the tolerance interval and the precision interval created for the class data.

d. How do your intervals from Parts **b** and **c** of the exploration compare with the class values?

e. In Part **c** of the exploration, you determined the precision interval for the radius of your five compact discs using the tolerance interval of the desired circumference. Describe this relationship using an if-then sentence.

f. Given any tolerance interval for a desired circumference c, how would you determine the corresponding precision interval for the radius r?

g. Describe how to represent an interval indicating each of the following sets of numbers:

1. all non-negative real numbers
2. all negative real numbers.

Assignment

1.1 The tolerance interval for the mass M (in grams) of three cookies in a Snack Pack is $113.4 < M < 114.6$. If x represents the desired mass of each cookie, then $M = 3x$. The tolerance interval may then be expressed as the following conjunction of inequalities: $113.4 < 3x < 114.6$.

In order to determine the corresponding precision interval, it is necessary to solve this expression for x. To accomplish this, the conjunction of inequalities can be written as $113.4 < 3x$ and $3x < 114.6$.

a. Find the values of x for which $113.4 < 3x$ by solving for x. Describe the steps you used and substitute a few solution values into $113.4 < 3x$ as a partial check.

b. Find the values of x for which $3x < 114.6$ by solving for x. Describe the steps you used and substitute a few solution values into $3x < 114.6$ as a partial check.

c. Solve the two inequalities using a symbolic manipulator and compare the results with your solutions in Parts **a** and **b**.

d. Write the solutions found in Parts **a** and **b** as intervals.

e. Determine the values of x that satisfy both inequalities and express these values both as a conjunction of inequalities in the form $a < x < b$ and using interval notation.

1.2 As quality control manager of a compact disc manufacturer, you have received several complaints about discs that do not fit properly in some players. To answer these complaints, your assistant proposes a new tolerance interval of [36.94, 37.06] for the disc circumference.

a. Write a conjunction of inequalities that represents this interval in terms of r, the radius of the disc.

b. Express the precision interval for Part **a** in interval notation.

c. Explain how decreasing the size of the tolerance interval affects the accuracy necessary for the machines that produce compact discs.

1.3 A variety pack of breakfast cereal contains 6 boxes. The tolerance interval for the total mass of a variety pack, including 30 g of packaging material, is $237 \le M + 30 \le 243$, where M is the mass of the cereal in grams.

a. If x is the mass of the cereal in each box, express the tolerance interval in terms of x.

b. Express the precision interval for Part **a** in interval notation.

c. Describe the steps you used to solve the inequality in Part **a**.

1.4 **a.** **1.** Express the relationship between the numbers 3 and 5 as a true inequality.

2. Multiply both sides of the inequality by 5.

3. Determine whether or not the resulting inequality is true. If it is not, adjust the inequality sign so that the statement is true.

b. Repeat Part **a**, multiplying both sides of the inequality by –5.

c. Based on your observations in Parts **a** and **b**, what conjecture can you make about the effects of multiplying the terms of an inequality by a constant?

d. Use at least four different examples, two of which involve negative numbers, to test your conjecture.

e. In general, if $a < b$, then $a + c = b$ where c is a positive number. Given that n is a negative number, use algebra to show that $na > nb$.

1.5 **a.** Do you think that the generalization you made for multiplication in Problem **1.4c** would also be true for division? Explain your response.

b. Support your answer to Part **a** using at least four examples.

1.6 A tent manufacturer uses nylon rope for several components in its tents. One section of rope 630 cm long is cut into 7 pieces. Six of the pieces are used as guylines for each tent, while the remaining piece becomes the drawstring for the storage bag.

a. The drawstring must be between 30 cm and 36 cm long. Find an inequality that expresses the tolerance interval for the drawstring where x is the length of each guyline.

b. Determine the precision interval for the length of each guyline. Express your answer using both inequality notation and interval notation.

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1.7 The snow leopard is one of many species of mammals on the endangered list. Imagine that a wildlife biologist is nursing a sick snow leopard back to health. The animal currently weighs 43 kg.

A healthy leopard should weigh more than 70 kg. The biologist estimates that a special diet will allow the cat to gain an average of 0.5 kg per week. Write and solve an inequality expression to determine when the snow leopard's weight will exceed 70 kg.

1.8 Steel expands when heated and contracts when cooled. Because of this phenomenon, both bridges and railroad tracks contain expansion joints to allow for variations in the lengths of their steel beams.

As the temperature of a steel beam changes, the resulting length of the beam l, in meters, can be described by the following equation

$$l = l_0 + \left(1.05 \bullet 10^{-5} x\right) l_0$$

where l_0 is the original length and x is the change in temperature in degrees Celsius. The value of x may be either positive or negative.

a. Imagine that you are designing a steel bridge to be built in a region where the temperature generally varies between –15° C and 35° C. Express the possible change in temperature as an interval.

b. Express the possible length l of the beam as an inequality involving l_0.

c. If the original length l_0 of a steel beam is 300 m, what is the possible variation in the length of the beam?

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Activity 2

In Activity **1**, you investigated how a tolerance interval for the circumference of a compact disc affected the range of acceptable values for the radius of a disc. In manufacturing and other applications, tolerance intervals are often described in terms of the distance between each endpoint and the desired value.

For example, if the desired circumference of a CD is 37 cm, and e is the allowable error in the circumference, the tolerance interval can be written as $(37 - e, 37 + e)$. The corresponding precision intervals vary according to the error selected for the tolerance interval.

Exploration

In this exploration, you use a graphical model to examine how changing the allowable error in a tolerance interval affects the corresponding precision interval.

a. Figure **4** below shows line l, the graph of a linear function $f(x)$.

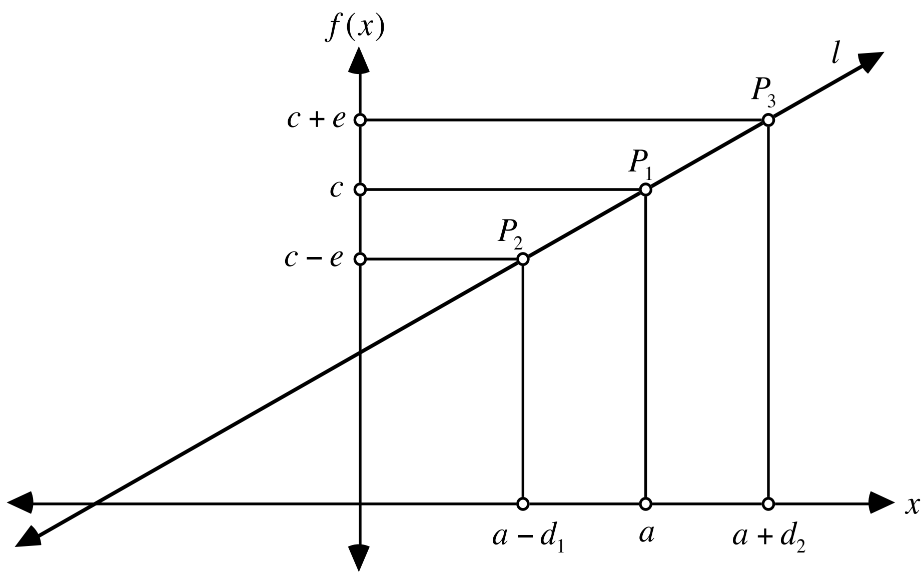

Figure 4: An arbitrary linear function

Using a geometry utility, create a construction like the one shown in Figure **4**. Your construction should meet the following conditions.

1. Line l is oblique to the $f(x)$-axis and represents a relationship between a desired measure c and its corresponding x-value, a.
2. The point with coordinates $(0, c - e)$ on the $f(x)$-axis is a moveable point. The interval $(c - e, c + e)$ represents a tolerance interval for c.

3. The segment from point P_1 to the point with coordinates $(0, c)$ is perpendicular to the $f(x)$-axis. The segment from point P_1 to the point with coordinates $(a, 0)$ is perpendicular to the x-axis.

4. The point with coordinates $(0, c+e)$ is the reflection of the point with coordinates $(0, c-e)$ in the segment from point P_1 to the point with coordinates $(0, c)$.

5. The segment from the point with coordinates $(0, c-e)$ perpendicular to the $f(x)$-axis intersects l at P_2. The segment from point P_2 to the point with coordinates $(a-d_1, 0)$ is perpendicular to the x-axis.

6. The segment from the point with coordinates $(0, c+e)$ is perpendicular to the $f(x)$-axis and intersects l at P_3. The segment from point P_3 to the point with coordinates $(a+d_2, 0)$ is perpendicular to the x-axis.

b. Measure the appropriate segments in order to determine the following distances:

1. e
2. d_1
3. d_2

c. While moving the point with coordinates $(0, c)$ on the $f(x)$-axis, observe the values of the three distances listed in Part **b**.

d.
1. Use an inequality to describe the tolerance interval in terms of $f(x)$.
2. Use an inequality to describe the precision interval in terms of x.

Discussion

a. In the exploration in Activity **1**, you modeled the manufacturing process for compact discs. In that setting, which quantity could be represented by c in Figure **4**? Which quantity could be represented by a in Figure **4**?

b. In Part **c** of the exploration, how did a change in e affect the values of d_1 and d_2?

c. Describe the effect that the size of the tolerance interval has on the size of the precision interval.

d. If $P(x_1, 0)$ is a point on the line segment with endpoints $(a-d_1, 0)$ and $(a+d_2, 0)$, where would you expect the point with coordinates $\left(0, f(x_1)\right)$ to be located?

e. In this case, once a tolerance interval is selected, how would you determine the corresponding precision interval?

Mathematics Note

A limit is used to examine the behavior of a function close to, but not at, a particular point. For example, consider the value of $f(x)$ as x approaches a.

If the value of $f(x)$ gets arbitrarily close to c as x gets close to a, then c is the **limit** of the function as x approaches a. This can be denoted as follows:

$$\lim_{x \to a} f(x) = c$$

Mathematically, this is true if, for every real number e, there exists a corresponding positive real number d so that $c - e < f(x) < c + e$ whenever $a - d < x < a + d$.

In Figure **5**, c is the limit of $f(x)$ as x approaches a when, no matter how small the distance from $c - e$ to $c + e$, there is an interval from $a - d$ to $a + d$ small enough so that every point x in the interval $(a - d, a + d)$ has an $f(x)$ in the interval $(c - e, c + e)$.

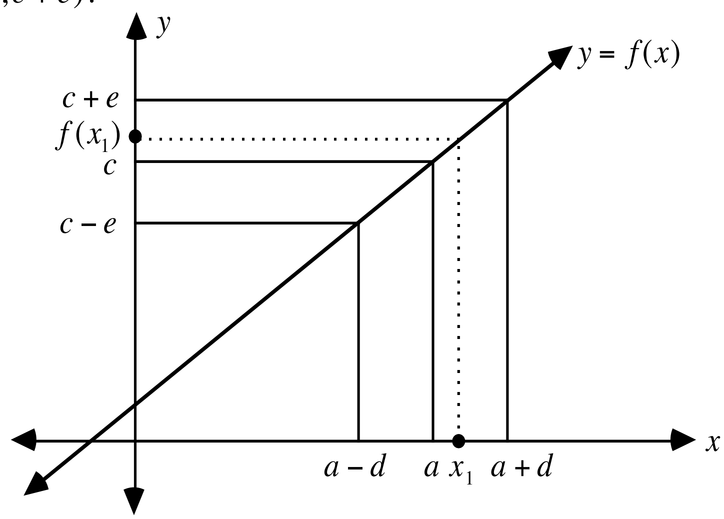

Figure 5: Graphical representation of a limit

In the exploration in Activity **1**, the function $f(r) = 2\pi r$ relates the radius r of a compact disc to its circumference. For example, suppose that $2\pi(5.9)$ cm is the desired circumference for a disk of radius 5.9 cm. If the tolerance interval is determined by an error of $e < 0.4$, then the precision interval can be found as follows:

$$2\pi(5.9) - 0.4 < 2\pi r < 2\pi(5.9) + 0.4$$

$$5.9 - \frac{0.4}{2\pi} < r < 5.9 + \frac{0.4}{2\pi}$$

In this case, d must be less than $0.4/2\pi$, or approximately 0.063. Because any value of r in the interval $(5.9 - 0.063, 5.9 + 0.063)$ yields a circumference in the required tolerance interval, then any other interval determined by $(5.9 - d, 5.9 + d)$, where $d < 0.063$, will also work. In other words, once a precision interval is determined, any "more precise" interval also is acceptable.

f. Describe some other situations, besides the manufacturing of compact discs, in which a product may have an infinite number of acceptable values within a tolerance interval.

Assignment

2.1 Consider the function $f(x) = 2x + 3$ with the set of real numbers as its domain.

a. Find a value for x so that $f(x) = 12$.

b. Determine the interval for x that corresponds with the following interval for $f(x)$: (11.8, 12.2).

c. Express the interval $11.8 < f(x) < 12.2$ in the form $(c - e,\ c + e)$.

d. Determine the interval for x that corresponds with the following interval for $f(x)$:

$$\left(c - \frac{e}{2},\ c + \frac{e}{2}\right)$$

2.2 Consider the function $f(x) = 4x - 7$.

a. What would you expect the limit c of $f(x)$ to be when x approaches 4.5?

b. Determine the interval for x that corresponds with the following interval for $f(x)$: $(c - 1.2,\ c + 1.2)$.

c. Draw a sketch that illustrates the relationship between the two intervals described in Part **b**.

2.3 Consider the function $f(x) = 3x + 1$. The limit c of $f(x)$ when x approaches a is 13.

a. Determine the interval for x that corresponds with the following interval: $13 - e < f(x) < 13 + e$.

b. Write the interval for x you determined in Part **a** in the form $(a - d,\ a + d)$.

2.4 A candy manufacturer markets chocolate candies in a paper package. Each package contains 56 candies. The packaging material has a mass of 2 g.

a. If x represents the mass in grams of each candy, determine a function $f(x)$ that represents the total mass of a package of candy.

b. The desired total mass of each package is 47.9 g, with an allowable error of no more than e grams. Write the tolerance interval for the total mass as an inequality.

c. Determine an inequality to express the largest possible precision interval meeting the conditions in Part **b**.

d. Find a value for a so that $f(a) = 47.9$.

2.5 A food processing company markets a box containing 8 whole-grain snack bars. The desired mass of the 8 bars, including the box, is 296 g, with an allowable error of less than 4 g. The box itself has a mass of 12 g.

a. If x represents the desired mass of each snack bar, write an expression for $f(x)$, the total mass of the product including 8 bars and the box.

b. Determine the tolerance interval for $f(x)$ that meets the above conditions.

c. Determine the largest possible precision interval for x yielding the tolerance interval in Part **b**.

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2.6 A commuter airplane flies 270 km from city A to city B, remains on the ground for 30 min, then returns to city A. To allow passengers to connect with other flights, the plane must complete the round-trip in less than 3.5 hr but more than 3.25 hr. Determine an acceptable interval for the average speed of the plane while it is in the air.

2.7 How close to 2 must x be for x^3 to be within 0.1 of 8? Explain your response.

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Activity 3

In previous activities, you described tolerance intervals and precision intervals using both interval notation and inequalities. A third way of representing intervals involves the use of **absolute values**.

Mathematics Note

The **absolute value** of a real number x, denoted by $|x|$, is defined as follows:

$$|x| = x, \text{ if } x \geq 0$$

$$|x| = -x, \text{ if } x < 0$$

For example, $|2| = 2$, $|-4| = 4$, $|-2/3| = 2/3$, $|0| = 0$ and $|\sqrt{2}| = \sqrt{2}$.

Exploration

In this exploration, you investigate some properties of the absolute-value function.

a. Graph $f(x) = |x|$.

b. Examine the relationship between the x- and y-coordinates of points on the graph.

c. Use the graph of $f(x)=|x|$ to solve each of the following:

1. $|x|=5$

2. $|x|=0$

3. $|x|=-5$

d. **1.** Solve the inequality $|x|< 5$ by inspecting the graphs of $f(x)=|x|$ and $g(x)=5$. Use inequalities to describe the solution set.

2. Solve the inequality $|x|> 5$. Use inequalities to describe the solution set.

3. Express the solution to the inequality in Step **2** as the union of two intervals.

e. **1.** Solve the equation $|x-2|=5$ by inspecting the graphs of $h(x)=|x-2|$ and $g(x)=5$.

2. Graph these solutions on a number line.

3. Describe the locations of the solutions in terms of their distances from 2 on the number line.

4. Express the solution using set notation.

f. Repeat Part **e** for $|x-a|=b$, where *a* and *b* are positive integers, for several values of *a* and *b*.

g. **1.** Solve the inequality $|x-2|<5$.

2. Graph these solutions on a number line.

3. Describe the locations of the solutions in terms of their distances from 2 on the number line.

4. Describe the solution using interval notation.

h. Repeat Part **g** for $|x-a|< b$, where *a* and *b* are positive integers, for several values of *a* and *b*.

Discussion

a. Describe the graph of $f(x)=|x|$.

b. What are the domain and range of the absolute-value function $f(x)=|x|$?

c. The expressions $|x|$ and $|x-0|$ represent the same distance. Describe this distance.

d. What distance is represented by $|x-2|$? by $|x+2|$?

e. Using absolute value, describe an expression that represents the interval $5<x<9$.

f. Describe the general solution to each of the following inequalities, where $a \geq 0$ and $b \geq 0$:

1. $|x - a| < b$

2. $|x - a| > b$

Assignment

3.1 Graph the solution to each of the following inequalities on a number line.

a. $|x - 11| < 4$

b. $|x - 7| > 4$

c. $|x + 15| < 10$

3.2 Express each of the following intervals using absolute values.

a. $-5 < x < 7$

b. $-7 < x < -4$

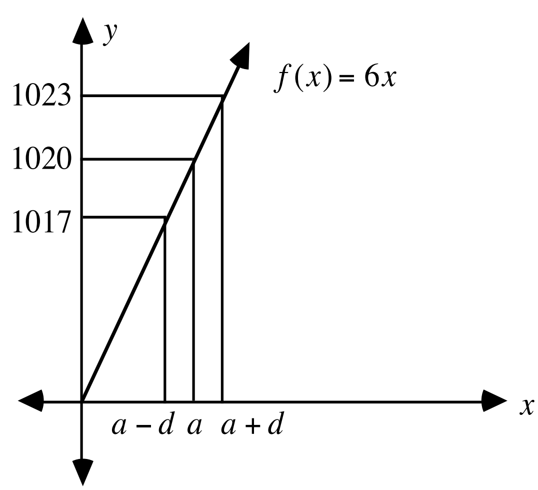
c. $x < 24$ or $x > 28$

d. $[-6,3]$

e. $(-\infty,12) \cup (14,\infty)$

3.3 A tolerance interval for the volume of soda in a soft-drink container can be expressed as $|x - 354| < 2$, where x is measured in milliliters. Solve this inequality.

3.4 The relationship between the tolerance interval for the total volume in milliliters of 6 bottles of fruit juice, and the largest corresponding precision interval for x, the volume of each bottle of juice, is illustrated below.



a. Express the tolerance interval as an inequality using absolute values.

b. Express the precision interval as an inequality using absolute values.

3.5 The total mass M in grams, of 8 slices of lunch meat, not including the package, must be less than 3 g from the desired mass of 224 g.

a. Express the tolerance interval for M in absolute-value form.

b. Determine a function $f(x)$ for the total mass of 8 slices of lunch meat if x is the mass of 1 slice.

c. Determine the largest possible precision interval for x.

d. Determine a, the desired mass of 1 slice, and d, the allowable error in the mass of 1 slice for this interval.

e. Using your responses to Part **d**, express the precision interval for x in absolute-value form.

f. Draw a diagram similar to the one in Problem **3.4** to illustrate the relationship between the tolerance interval and the precision interval.

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3.6 The total mass in grams, including packaging material, of a 4-pack of pudding is $4x + 6$, where x represents the mass of 1 serving of pudding.

a. The tolerance interval for the total mass can be described by the following inequality: $|(4x + 6) - 452| < 12$. Express this inequality without using absolute values.

b. Determine the largest possible precision interval for the mass of 1 serving of pudding.

3.7 If the solution to an inequality in the form $|x - a| \leq b$ is $[-13, 86]$, what is the solution to $|x - a| > b$?

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Activity 4

The precision intervals you explored in the first three activities were determined by solving inequalities involving linear functions. Many manufacturing processes, however, are modeled by nonlinear functions.

Exploration

In this exploration, you discover when the magnitude of the area of a geometric figure exceeds the magnitude of its perimeter.

a. Using a geometry utility, construct a square so that when you drag any vertex, the polygon remains a square.

b. Calculate the length of one side of the square, the perimeter of the square, and the area of the square.

c. Drag one of the vertices of the square until the numbers representing the perimeter and area have approximately the same value. Record the resulting length of the side of the square.

d. Let x represent one side of the square.

1. Write a function $f(x)$ that represents the area of the square. Identify the domain of the function.
2. Write a function $g(x)$ that represents the perimeter of the square and identify its domain.
3. Graph $f(x)$ and $g(x)$ on a graphing utility, using appropriate domains and ranges.
4. Write an inequality that describes the values of x for which the magnitude of the square's area is less than the magnitude of its perimeter.
5. Inspect the graphs from Step **3** to determine the interval for x in which $f(x) < g(x)$.

e.
1. Express the difference of $f(x)$ and $g(x)$ as an inequality in the form $f(x) - g(x) < 0$. Use a symbolic manipulator to factor the inequality.
2. Determine the values of x for which the factors equal 0.
3. How are these values related to the interval you determined in Part **d**?
4. The values found in Step **2** separate a number line into three intervals. Determine the sign of each of the expressions, x, $x - 4$, and $x(x - 4)$ for values of x in each of the intervals.

f. Repeat Parts **a–e** using a circle, along with its radius, circumference, and area.

Discussion

a.
1. What type of graph represents the perimeter of geometric figures?
2. What type of graph represents the area of geometric figures?

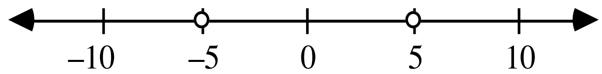
b. How does inspecting the graphs of the functions in Part **d** of the exploration help determine the intervals of x for which:

1. the magnitude of a figure's area exceeds the magnitude of its perimeter?
2. the magnitude of a figure's area is less than the magnitude of its perimeter?

c.
1. Describe how to use graphs to solve an inequality of the form $x^n > k$, where k is a constant and n is a positive integer.
2. Describe how to use graphs to solve an inequality of the form $x^n < k$, where k is a constant and n is a positive integer.

Assignment

4.1 Use a number line like the one shown below to complete Parts **a–e**.



a. **1.** Place a positive sign (+) above the portions of the number line where $x-5$ is positive.

2. Place a negative sign (–) above the portions of the number line where $x-5$ is negative.

b. Repeat Part **a** for $x+5$.

c. Identify values on the number line for which $(x-5)(x+5)>0$.

d. Identify values on the number line for which $(x-5)(x+5)<0$.

e. How do the values in Parts **c** and **d** relate to the solutions of the inequalities $x^2<25$ and $x^2>25$?

4.2 **a.** Solve the inequality $x^2-9<0$ and graph the solution set on a number line.

b. Use your response to Part **a** to write an inequality using absolute values that is equivalent to $x^2<9$.

c. Solve the inequality $x^2>9$. Graph the solution set on a number line.

4.3 **a.** Solve the inequality $x^2<16$. Graph the solution set on a number line.

b. Solve the inequality $4<x^2$. Graph the solution set on a number line.

c. Use your responses to Parts **a** and **b** to solve the conjunction of inequalities $4<x^2<16$. Graph the solution set on a number line.

d. **1.** Graph $f(x)=x^2$, $g(x)=4$, and $h(x)=16$ on a sheet of graph paper.

2. Use the graph to illustrate the relationship between $4<x^2<16$ and the solution set from Part **c**.

4.4 Solve $(x-2)(x-3)(x+4)<0$.

4.5 A container manufacturer is designing a small cylindrical jar made from cut glass. The jars will be used to package caviar. Because cut glass is expensive, the surface area of the jar for a given volume should be kept as small as possible.

a. If the jar's height is 4 cm, determine the minimum radius for the jar's base for which the magnitude of the volume exceeds the magnitude of the surface area. (The surface area of the jar does not include the lid.)

b. Determine the volume of the jar for which the magnitude of the volume is approximately equal to the magnitude of the surface area.

4.6 A packaging company plans to produce a cardboard milk carton. The carton will be a rectangular prism 20.1 cm high, with a square base. The desired volume of the carton is 1 L, or 1000 cm^3.

a. What is the desired measurement for the sides of the square base?

b. An acceptable container must hold within 5% of the specified capacity. What is the tolerance interval for the volume of the container?

c. If the manufacturing process can consistently produce a carton with a height of 20.1 cm:

1. determine an interval of acceptable measures for the sides of the square base

2. find the largest acceptable error for the sides of the square base.

4.7 Imagine that you are an engineer at a container company. A customer has requested a cardboard box with a rectangular base and a volume of 120 cm^3. The width of the box must be 3 cm and—for aesthetic reasons—the customer would like the ratio of length to height to be the golden ratio: $\left(1+\sqrt{5}\right)/2$, or approximately 1.618. The acceptable tolerance for the volume is less than 1%.

a. Write a function that describes the volume in terms of height.

b. What is the desired length of the box? What is the desired height?

c. What is the tolerance interval for the volume of the box?

d. What are the precision intervals for the length and height of the box?

e. If the tolerance for volume is reduced to less than 0.5%, how would this affect the precision intervals for length and height?

f. Describe what happens to the precision interval when the size of the tolerance interval is decreased.

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4.8 As a design engineer, you have been assigned to create a cylindrical container with a capacity of 300 L. The container's height must be 3 times the radius of its base. The volume of the container must be within 1% of the specified capacity.

a. Write a function for the volume in terms of the radius.

b. Find the desired radius of the base.

c. Determine acceptable intervals for each of the following:

1. volume

2. radius

3. height.

4.9 A skydiver leaps from a plane flying 3000 m above the earth's surface. Disregarding air resistance, her distance from the ground is described by the function $f(t) = 3000 - 4.9t^2$, where t is the time in seconds after the start of the jump. The skydiver would like to open her parachute when her distance from the ground is approximately 1200 m. Her skydiving instructor has given her a safety margin of less than 300 m. Determine the appropriate time interval in which she can open her parachute safely, and express your answer as an inequality.

4.10 Like most metals, aluminum expands when heated and contracts when cooled. The volume v of an aluminum container, in cubic meters, after a temperature change of x degrees Celsius is

$$v = v_0 + \left(6.9 \bullet 10^{-5} x\right) v_0$$

where v_0 is the volume before the temperature change.

a. An aluminum vat is used to store milk. Its maximum volume is 40 m^3 and temperatures vary by 36° C. Find the minimum volume of the vat.

b. Express the interval of volumes for the vat, to the nearest 0.001 m^3 in interval notation.

c. If the vat is a cylinder and its height equals its diameter, determine an interval, in centimeters, for the height of the vat. Express the heights to the nearest 0.1 cm.

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Summary Assessment

A packaging manufacturer is planning to retool its factory to produce cylindrical metal containers. As an engineer for the company, you have been asked to determine the precision necessary for the machinery which will stamp out the sides of the container. In order to make this determination, you first must identify an acceptable interval for the container's height.

Using the available equipment, you know that circular bases with the desired 8-cm radius can be produced within a 1% tolerance. The capacity or volume of each container should be 1 L (or 1000 cm^3) with a tolerance of less than 2.5%.

Write a report to the manager of the factory describing an appropriate precision interval for the height of the container. Explain how you determined this interval and verify that it will allow the production of containers within the desired 2.5% tolerance for volume.

Module Summary

- The process of measuring product performance, comparing those measurements with predetermined standards, and then acting on the difference is called **quality control**.
- A **tolerance interval** represents the acceptable error in a product.
- A **precision interval** represents the acceptable error in a component of a product.
- An **interval** of real numbers is a set containing all numbers between two given points, the **endpoints** or **bounds** of the interval. An interval may contain one endpoint, both endpoints, or neither endpoint. In addition, some intervals may have only one endpoint, or no endpoints. For example, the set of real numbers can be represented as the interval $(-\infty,\ \infty)$, while the set of real numbers greater than or equal to a can be represented as $[a,\ \infty)$.
- Some intervals may be expressed as a conjunctions of two inequalities. A **conjunction** combines two mathematical statements with the word *and*. A conjunction is true only if both statements are true. If one or both of the statements is false, the conjunction is false.
- Intervals and associated inequalities can be expressed using set notation.
- A limit is used to examine the behavior of a function close to, but not at, a particular point. If the value of $f(x)$ gets arbitrarily close to c as x gets close to a, then c is the **limit** of the function as x approaches a. Mathematically, this is true if, for every real number e, there exists a corresponding positive real number d so that $c - e < f(x) < c + e$ whenever $a - d < x < a + d$.
- The **absolute value** of a nonzero real number x, denoted by $|x|$, is defined as follows:777

$$|x| = x,\ \text{if } x \geq 0$$
$$|x| = -x,\ \text{if } x < 0$$

- The absolute value of $x - a$, where x and a are real numbers, represents the distance between x and a.
- The inequality $|x - a| < b$, where b is a positive number, is satisfied by the real numbers x whose distance from a is less than b units. These real numbers constitute the interval $(a - b, a + b)$.
- The inequality $|x - a| > b$, where b is a positive number, is satisfied by the real numbers x whose distance from a is more than b units. These real numbers constitute the interval $(-\infty, a - b)\ \cup\ (a + b, \infty)$.

Selected References

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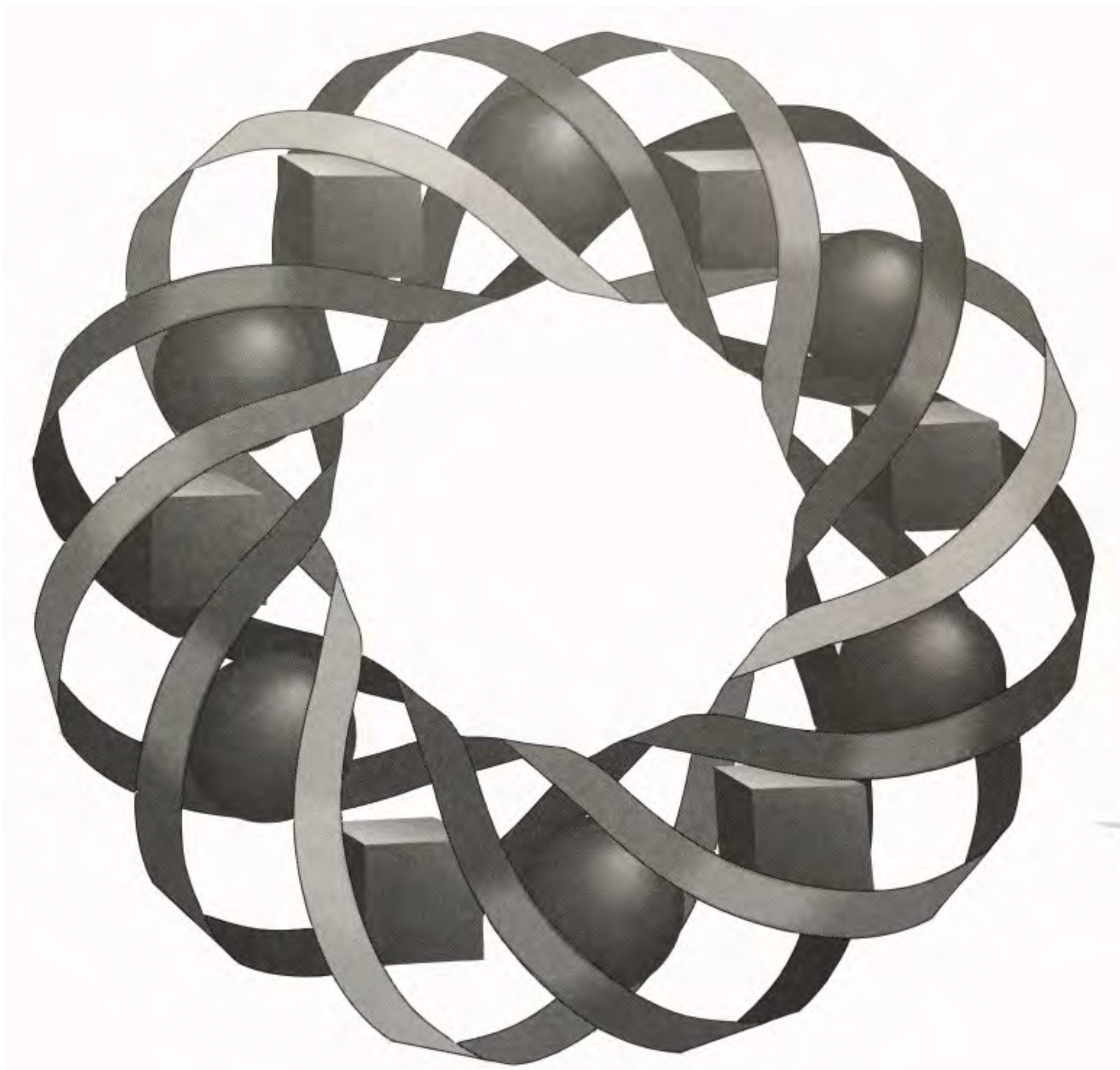
Nearly Normal



Would you recognize a normal curve if you saw one? In this module, you'll explore the mathematics of normal while examining the historic patterns of rainfall in a Northwest town.

Lee Brown • Roger Patterson • Karen Umbaugh

Nearly Normal

Introduction

The year is 2093. You have just been selected as the climate-control officer for New Bernoulli, a small city in the northwestern United States. Built near the site of the original city of Bernoulli, it is one of the first communities in America to be completely enclosed by a dome.

The dome allows for the control of New Bernoulli's air temperature, wind speed, and rainfall. Recently, the citizens have grown tired of warm, cloudless weather day after day. They miss the changing seasons. That's why they've hired you to monitor the climate.

Your job is to ensure that the following conditions are met.

- Changes in season should occur at the times of the earth's solstices and equinoxes.
- The temperature should never drop below –7˚ C and never exceed 27˚ C.
- Rain or snow showers should occur at random, although citizens must receive notice of any precipitation 24 hours in advance.
- Levels of precipitation should resemble the normal 20th-century levels.

The first three conditions seem easy to an experienced climate-control officer like yourself. The last one, however, sounds challenging. First you'll have to research the historic weather patterns for the old city of Bernoulli. Then you'll need to clarify exactly what the citizens mean by a "normal" amount of rain.

Activity 1

Your first task as climate-control officer is to determine the amount of precipitation experienced by the old city of Bernoulli in each month of a typical year. Fortunately, the National Oceanographic and Atmospheric Administration (NOAA), a federal agency that keeps records of weather patterns, had a weather station at Bernoulli throughout the 20th century.

Table **1** shows the total amount of precipitation recorded each May during the 20th century. From looking at the data, can you tell how much rain the citizens of New Bernoulli would like to receive in May of 2093?

Table 1: Recorded precipitation (in cm) for May, 1900–1999

Year	Precip.	Year	Precip.	Year	Precip.	Year	Precip.
1900	12.34	**1925**	19.67	**1950**	12.86	**1975**	16.56
1901	13.32	**1926**	11.81	**1951**	11.48	**1976**	12.44
1902	12.77	**1927**	14.53	**1952**	9.61	**1977**	7.45
1903	14.63	**1928**	9.43	**1953**	11.11	**1978**	20.49
1904	9.63	**1929**	17.43	**1954**	8.21	**1979**	5.75
1905	9.13	**1930**	11.96	**1955**	13.32	**1980**	11.07
1906	16.09	**1931**	10.68	**1956**	9.05	**1981**	16.01
1907	15.98	**1932**	9.22	**1957**	20.34	**1982**	16.11
1908	13.89	**1933**	12.41	**1958**	11.84	**1983**	10.09
1909	14.72	**1934**	18.16	**1959**	9.99	**1984**	9.49
1910	16.04	**1935**	8.59	**1960**	6.80	**1985**	13.19
1911	12.74	**1936**	11.13	**1961**	12.21	**1986**	11.34
1912	11.31	**1937**	9.19	**1962**	13.43	**1987**	12.30
1913	10.93	**1938**	14.63	**1963**	13.60	**1988**	10.22
1914	17.61	**1939**	17.78	**1964**	19.56	**1989**	13.64
1915	12.69	**1940**	13.98	**1965**	16.40	**1990**	11.19
1916	11.83	**1941**	19.85	**1966**	11.21	**1991**	18.07
1917	18.52	**1942**	21.00	**1967**	14.09	**1992**	13.39
1918	16.01	**1943**	17.86	**1968**	15.05	**1993**	11.67
1919	14.99	**1944**	11.58	**1969**	14.20	**1994**	14.56
1920	15.15	**1945**	19.85	**1970**	16.07	**1995**	16.23
1921	7.18	**1946**	13.52	**1971**	14.01	**1996**	17.89
1922	10.35	**1947**	13.65	**1972**	16.48	**1997**	19.01
1923	11.13	**1948**	13.95	**1973**	14.33	**1998**	8.89
1924	14.63	**1949**	14.02	**1974**	16.63	**1999**	8.01

Mathematics Note

The **frequency** of an item in a data set is the number of times that item is observed.

The **relative frequency** of an item is the ratio of its frequency to the total number of observations in the data set.

A **relative frequency table** includes columns that describe a data item or interval, its frequency, and its relative frequency. For example, Table **2** shows a relative frequency table for 20 observations.

Table 2: Frequency table of May precipitation (1904–1923)

Precipitation (cm)	Frequency (yr)	Relative Frequency
$[4, 6)$	0	$0/20 = 0.00$
$[6, 8)$	1	$1/20 = 0.05$
$[8, 10)$	2	$2/20 = 0.10$
$[10, 12)$	5	$5/20 = 0.25$
$[12, 14)$	3	$3/20 = 0.15$
$[14, 16)$	4	$4/20 = 0.20$
$[16, 18)$	3	$3/20 = 0.15$
$[18, 20)$	1	$1/20 = 0.05$
$[20, 22)$	1	$1/20 = 0.05$
$[22, 24)$	0	$0/20 = 0.00$

A **frequency histogram** consists of bars of equal width whose heights indicate the frequencies of intervals.

A **frequency polygon** is formed by the line graph that connects the set of points (x, y), where x is the midpoint of an interval and y is the frequency of the interval. The base of a frequency polygon is the x-axis. For example, Figure **1** shows a frequency histogram and polygon created using the data in Table **2**.

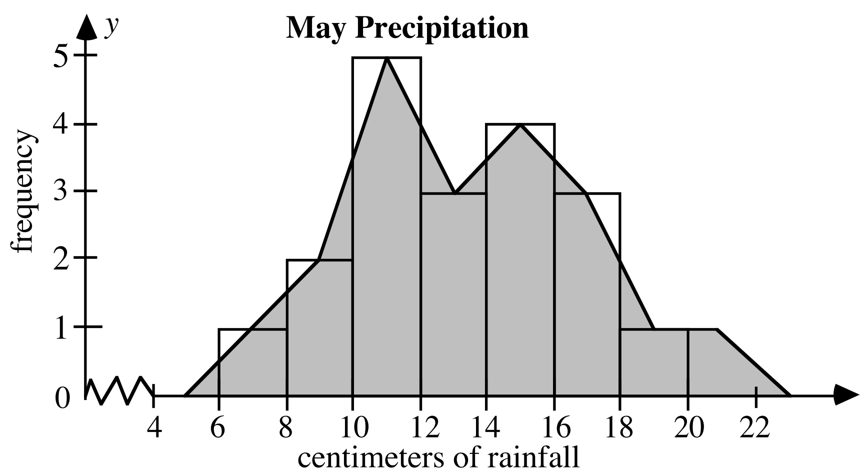


Figure 1: Frequency histogram and polygon

Exploration

In the following exploration, you organize the data in Table **1** to determine an appropriate range of rainfall amounts for New Bernoulli. **Note:** Save the data and graphs from this exploration for use later in the module.

In statistics, a **population** is the entire group of items or individuals about which information is desired. A **sample** is a portion of the population used to gather information about the whole population. Since the amounts of precipitation in years prior to 1900 are unavailable, the historic data in Table **1** may be considered a population rather than a sample.

a. Find the mean, median, and standard deviation of the data in Table **1**.

b.
1. Create a relative frequency table of Bernoulli's May precipitation from 1900 to 1999. Use intervals of 2 cm, where the first interval is $[4, 6)$.
2. Calculate the sum of the frequencies and the sum of the relative frequencies.

c.
1. Create a frequency histogram of the data.
2. Create a frequency polygon of the data.

d.
1. In a relative frequency histogram, the heights of the bars represent relative frequencies instead of frequencies. Create a relative frequency histogram of the data.
2. In a relative frequency polygon, the y-values represent relative frequencies instead of frequencies. Create a relative frequency polygon of the data.

e. On the x-axis of each graph from Parts **c** and **d**,
1. identify the x-value where the mean occurs. Label this point M.
2. identify the x-values 1 standard deviation on either side of the mean. Label the right-most value 1σ. Label left-most value -1σ
3. identify the x-values 2 standard deviations on either side of the mean. Label the right-most value 2σ. Label left-most value -2σ

Discussion

a.
1. Which interval in your table from Part **b** has the highest frequency?
2. Should this interval contain the mean and median of the data? Explain your response.

b. Weather forecasters often mention levels of precipitation that are above or below normal. What do they mean by this use of the word *normal*?

c. Describe the shapes of the frequency polygon and the relative frequency polygon of the May precipitation data.

d. Suppose that the May precipitation pattern from 1800 to 1899 was similar to that from 1900 to 1999.

1. How do you think that the shape of the frequency polygon would change if you included the May precipitation data from 1800 to 1899?
2. How do you think that the shape of the relative frequency polygon would change?

e.

1. What is the probability that the May precipitation in New Bernoulli was in the interval [12, 14) during the 20th century? Explain your response.
2. How is the relative frequency of an interval of rainfall related to its probability?

f. What percentage of the May rainfall amounts in Table **1** are:

1. within 1 standard deviation of the mean?
2. within 2 standard deviations of the mean?

g. How could the relative frequency polygon help you determine the amount of rainfall New Bernoulli should receive in May?

Assignment

1.1 A resident of New Bernoulli has suggested randomly selecting one of the values in Table **1** for the May precipitation amount in the year 2093. Use your relative frequency table to answer the following questions.

a. What is the probability that the precipitation will be between 10 cm and 12 cm?

b. What is the probability that the precipitation will be greater than or equal to 16 cm?

c. What is the probability that the precipitation will be less than 8 cm?

1.2 What is the sum of the relative frequencies in your table? Describe how this sum is related to the probabilities associated with these relative frequencies.

1.3 Obtain a table of the annual precipitation for the past 50 years for a city near your home.

a. Calculate the mean, median, and standard deviation of the data.

b. Create a frequency table of the data, using an appropriate width for the intervals of rainfall.

c. Determine the relative frequency of each interval.

d. Create a frequency histogram of the data.

e. Create a frequency polygon of the data.

1.4 **a.** On the x-axis of your graphs from Problem **1.3**, identify and label the values for the mean, 1 standard deviation on either side of the mean, and 2 standard deviations on either side of the mean as you did in the exploration.

b. Using the data from Problem **1.3**, determine the percentage of precipitation amounts that are within:

1. 1 standard deviation of the mean

2. 2 standard deviations of the mean.

* * * * *

1.5 The following table contains the selling prices of 35 homes in New Bernoulli during the month of October.

$97,000	$126,500	$180,000	$110,000	$78,000
$134,000	$165,000	$190,000	$153,500	$118,500
$210,300	$108,000	$99,500	$85,000	$128,500
$136,000	$203,000	$116,000	$117,500	$141,900
$133,900	$217,000	$169,000	$209,900	$127,000
$181,500	$126,500	$150,000	$112,500	$119,000
$89,900	$131,400	$72,000	$137,800	$162,000

a. Calculate the mean, median, and standard deviation of the data.

b. Create a frequency table of the data, using an appropriate width for the intervals.

c. Determine the relative frequency of each interval.

d. Create a frequency histogram of the data.

e. Create a frequency polygon of the data.

1.6 **a.** On the x-axis of your graphs from Problem **1.5**, identify the x-value where the mean occurs.

b. Using the data from Problem **1.5**, determine the percentage of selling prices that are within:

1. 1 standard deviation of the mean

2. 2 standard deviations of the mean.

c. What is the probability that a house sold in October cost more than $130,000? Explain your response.

* * * * * * * * * *

Activity 2

The climate in New Bernoulli is controlled by computers. In order to recreate a rainfall pattern like that of the 20th century, you have decided to program the computers to perform a simulation.

The model should produce a set of relative frequencies that closely follows Bernoulli's historic rainfall patterns. In order to accomplish this, it may help to examine its **discrete probability distribution**.

Mathematics Note

A **probability distribution** is the assignment of probabilities to a specific characteristic that belongs to each possible outcome of an experiment. If the set of outcomes is either finite or can have a one-to-one correspondence with the natural numbers, the distribution is **discrete**.

For example, an experiment that involves tossing 3 fair coins has 8 equally likely outcomes: TTT, TTH, THT, THH, HTT, HTH, HHT, and HHH, where H represents a head and T represents a tail. One observable characteristic is the number of heads that occur. From the list of 8 possible outcomes, 1 outcome results in 0 heads, 3 outcomes result in 1 head, 3 outcomes result in 2 heads, and 1 outcome results in 3 heads.

Table **3** shows the probability distribution of the number of heads when tossing 3 fair coins (or tossing 1 fair coin 3 times). Since the number of outcomes is finite, this probability distribution is discrete.

Table 3: Probability distribution table for number of heads

No. of Heads	Frequency	Probability
0	1 (TTT)	$1/8 = 0.125$
1	3 (HTT, THT, TTH)	$3/8 = 0.375$
2	3 (HHT, HTH, THH)	$3/8 = 0.375$
3	1 (HHH)	$1/8 = 0.125$

Figure **2** shows a histogram of the probabilities associated with this experiment.

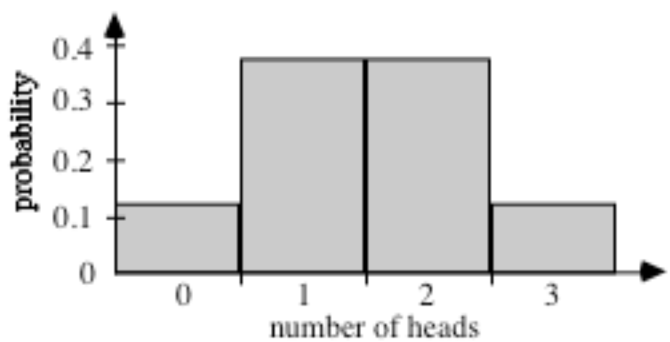


Figure 2: Probability histogram for number of heads

Exploration 1

In the following explorations, you experiment with coin tossing to create a simulation for Bernoulli's rainfall data.

a. Complete Table **4** to represent the probability distribution for the number of heads when tossing 2 fair coins.

Table 4: Probability distribution for number of heads when tossing 2 fair coins

No. of Heads	Frequency	Probability
0		
1		
2		
Total		

b. Complete Table **5** to represent the probability distribution for the number of heads when tossing 4 fair coins.

Table 5: Probability distribution for number of heads when tossing 4 fair coins

No. of Heads	Frequency	Probability
0		
1		
2		
3		
4		
Total		

c. When 2 coins are tossed, there are 4 possible outcomes. When 3 coins are tossed, there are 8 possible outcomes. When 4 coins are tossed, there are 16 possible outcomes. Use this pattern to determine the number of possible outcomes when n coins are tossed, where $n \geq 2$.

d. Complete Table **6** to show the frequencies for each number of heads when tossing 1 to 6 coins. Use the information you have already gathered in Parts **a–c** to begin the table. For example, the third row contains the frequencies when tossing 3 coins (as described in the mathematics note and in Table **3**).

Table 6: Frequencies for number of heads

No. of Coins	**No. of Heads**						
	0	**1**	**2**	**3**	**4**	**5**	**6**
1							
2							
3	1	3	3	1			
4							
5							
6							

e. Create a frequency histogram for each row in Table **6**. Record any similarities you observe among these histograms.

f.
1. When tossing 3 coins, the 8 possible outcomes can be listed in terms of numbers of heads. The set that represents this characteristic for the 8 possible outcomes is $\{0, 1, 1, 1, 2, 2, 2, 3\}$. Determine the mean and the standard deviation of this data set.
2. As described in Step **1**, list the possible outcomes in terms of number of heads when tossing 1, 2, 4, 5, and 6 coins.
3. Find the mean and standard deviation of each data set in Step **2**.

Discussion 1

a. Describe a pattern in Table **6** that would allow you to determine the frequencies of heads when tossing any number of coins.

b. What similarities did you observe among the histograms in Part **e** of Exploration **1**?

c. Compare the shape of the frequency histogram for tossing six coins to the frequency histogram for the rainfall data from Activity **1**.

d. Do you think that tossing one coin 6 times would produce the same probability distribution for numbers of heads as tossing six coins at once? Explain your response.

Mathematics Note

A **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure.
- The probability of a success remains the same from trial to trial.
- The total number of successes is observed.

For example, consider an experiment that consists of tossing a six-sided die 10 times and observing the number of times that a 6 appears. In this case, there is a fixed number of trials, 10. For each trial, there are only two possible outcomes: either a 6 or not a 6. The probability that a 6 appears remains constant for each toss, and the result of one toss does not influence the result of any other. Therefore, this represents a binomial experiment.

The probability distribution for a binomial experiment is a **binomial distribution**. The mean of a binomial distribution is the product of the number of trials and the probability of a success. In other words, the mean μ can be found as follows, where n is the number of trials and p is the probability of a success:

$$\mu = np$$

The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

For example, consider the experiment that consists of tossing a six-sided die 10 times and observing the number of times that a 6 appears. In this case, $n = 10$ and the probability of a success is $1/6$. Therefore, the mean of the corresponding binomial distribution is $10(1/6) = 10/6 \approx 1.67$. The standard deviation is $\sqrt{10(1/6)(5/6)} = \sqrt{50/36} \approx 1.18$.

e. Can an experiment that involves tossing three fair coins and counting the number of heads be considered a binomial experiment? Explain your response.

f. Consider a binomial experiment that involves tossing a fair coin six times and counting the number of heads.

1. What is n, the number of trials, in this experiment?
2. What is p the probability of a success?
3. Using the formula $\mu = np$, what is the mean of the probability distribution for this experiment?
4. How does this mean compare with the mean for tossing six coins that you calculated in Part **f** of Exploration **1**?

g. As described in the previous mathematics note, the standard deviation of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

1. Why is the probability of a failure $(1 - p)$?
2. Using the formula above, what is the standard deviation for an experiment that involves tossing a fair coin six times and counting the number of heads that appear?
3. How does this value compare with the standard deviation for tossing six coins that you calculated in Part **f** of Exploration **1**?

h. Table **5** shows the theoretical probability distribution for number of heads when tossing four fair coins. If you tossed four coins 100 times, do you think that the experimental probability distribution would be the same as the theoretical one? Explain your response.

Exploration 2

The shapes of the frequency histograms for tossing coins and the shape of the frequency histogram for Bernoulli's historic rainfall data are very similar. To take advantage of this fact, you decide to use a coin tossing simulation to help determine amounts of rainfall for New Bernoulli.

a. In Activity **1**, you created a relative frequency table of the May precipitation using thirteen 2-cm intervals. To model these 13 intervals, the simulation will require 12 coins.

Place 12 coins in a container. Shake the container, pour the coins onto a table, and count the number of heads that appear.

b. In the simulation, each possible number of heads is associated with an interval of precipitation. Repeat the process described in Part **a** until you have obtained 100 observations for this experiment (to represent 100 years of data).

Record your results in Table **7** and calculate the relative frequencies for each interval.

Table 7: Simulating May precipitation using 12 coins

No. of Heads	Precipitation (cm)	Frequency	Relative Frequency
0	$[0,2)$		
1	$[2,4)$		
2	$[4,6)$		
3	$[6,8)$		
4	$[8,10)$		
5	$[10,12)$		
6	$[12,14)$		
7	$[14,16)$		
8	$[16,18)$		
9	$[18,20)$		
10	$[20,22)$		
11	$[22,24)$		
12	$[24,26)$		

c. Determine the mean, median, and standard deviation of the data in terms of number of heads.

d. Use the information in Table **7** to draw a relative frequency histogram and a relative frequency polygon.

e. On the *x*-axis of each graph from Part **d** above:

1. identify and label the *x*-value where the mean occurs
2. identify and label the *x*-values 1 standard deviation on either side of the mean
3. identify and label the *x*-values 2 standard deviations on either side of the mean.

f. Using the pattern described in Part **a** of Discussion **1**, determine the theoretical probability distribution for 12 coins. Create a relative frequency polygon of this distribution.

g. Use the formulas $\mu = np$ and $\sigma = \sqrt{np(1-p)}$ to calculate the mean and standard deviation of the binomial distribution for tossing a coin 12 times and counting the number of heads that appear. Compare these values to the ones you determined in Part **c**.

Discussion 2

a. Compare the shapes of the relative frequency polygons you created in Exploration **2** with the one you created for the historic rainfall data in Activity **1**.

b. Consider a random number generator that generates integers from 0 to 12. Why shouldn't this simulation be used to model the probabilities associated with the number of heads when tossing 12 coins?

c. By examining your data from the exploration, what would you consider a "typical" interval of precipitation for May?

d. Do you think that the simulation from the exploration provides a reasonable model of Bernoulli's historic May precipitation? Explain your response.

Assignment

2.1 **a.** In Exploration **2**, what percentage of the coin-tossing data fell

1. within 1 standard deviation of the mean?
2. within 2 standard deviations of the mean?

b. In Activity **1,** Discussion Part **f**, you found that 67% of the May rainfall intervals were within 1 standard deviation of the mean and 96% were within 2 standard deviations of the mean. How do these percentages compare with your responses in Part **a**?

2.2 **a.** Use the information you recorded in Table **7** to answer the following questions.

1. What is the probability that the precipitation is between 10 cm and 12 cm?
2. What is the probability that the precipitation is greater than or equal to 16 cm?
3. What is the probability that the precipitation is less than 8 cm?

b. In Activity **1**, Problem **1.1**, you found that the historic probability of Bernoulli receiving between 10 cm and 12 cm of rainfall was 20%. The probability of receiving greater than 16 cm of rainfall was 27%, while the probability of receiving less than 8 cm of rainfall was 4%.

1. How do these values compare to your responses to Part **a**?
2. Discuss some possible explanations for any similarities or differences you observe.

2.3 An experiment that involved tossing 3 fair coins and counting the numbers of heads was repeated 50 times. The table below shows the relative frequencies that were observed.

No. of Heads	Frequency	Relative Frequency	Probability
0		0.12	
1		0.42	
2		0.28	
3		0.18	

a. What are the corresponding probabilities?

b. What are the corresponding frequencies?

c. What are the mean and the median of the data?

d. What values would you expect the probabilities to approach for a large number of simulations? Explain your response.

e. What is the expected value, in number of heads, for the experiment?

f. **1.** What is the mean of the binomial distribution for an experiment that consists of tossing a fair coin three times and counting the number of heads that appear?

2. How does this mean compare with the expected value calculated in Part **e**?

* * * * *

2.4 In major league baseball's World Series, the first team to win 4 games is declared the champion. Based on historic data, the probability that the World Series will last 4 games is 0.120, the probability that it will last 5 games is 0.253, the probability that it will last 6 games is 0.217, and the probability that it will last 7 games is 0.410.

a. Design a simulation to model the next 50 World Series.

b. Determine the mean and standard deviation for the data generated by the simulation.

c. Compare the mean and standard deviation of the data to the historic values.

* * * * * * * * * *

Activity 3

Although your simulation appears to produce rainfall amounts similar to those in Bernoulli's historic rainfall data, some citizens are still concerned about the model. One critic argues that the simulation generates only 13 different outcomes, while the real amounts of Bernoulli's May precipitation can take on an infinite number of outcomes in the interval from 0 cm to 26 cm.

To address this criticism, you decide to modify the simulation by treating the probability distribution representing amounts of rainfall as a **continuous probability distribution**.

Discussion 1

a. What is the sum of all the relative frequencies in the coin-tossing simulation? Explain your response.

b. If all 13 intervals of precipitation were equally likely, how could you determine a relative frequency for each interval?

c. How many different amounts of precipitation can there be in each interval?

Mathematics Note

A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.

In this situation, the probabilities of these outcomes can be represented graphically by the area enclosed by the *x*-axis, a specific real-number interval, and a distribution curve. The sum of the non-overlapping areas that cover the entire interval is 1.

Figure **3** shows one example of a continuous probability distribution. The probability that a value falls in the interval [0.5, 1.5] is equal to the area of region A, or 0.55. The probability that a value falls in the interval [1.5, 2.5] is equal to $1 - 0.55 = 0.45$, or the area of region B.

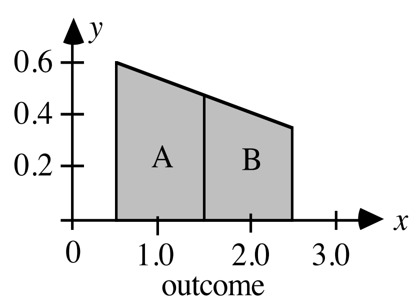


Figure 3: A continuous probability distribution

A **uniform probability distribution** is a continuous probability distribution in which all the probabilities over intervals of equal length are equal.

For example, consider the distribution of real numbers in the interval [0, 10], as shown in Figure **4**. The probability that a randomly selected real number will fall in the interval [4, 6] is $2 \bullet 0.1 = 0.2$ or 20%. Since this is a uniform probability distribution, the probability that a randomly selected real number will fall in any interval with a length of 2 is also 20%.

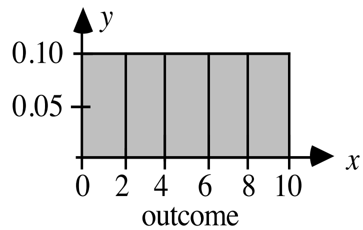


Figure 4: A uniform probability distribution

d. Why must the height of the uniform probability distribution in Figure **4** be 0.1?

e. Given an interval of real numbers, describe the height of its corresponding uniform probability distribution.

Exploration

a. The random number generator on many calculators produces numbers between 0 and 1. Assuming every number between 0 and 1 is equally likely to occur, draw a graph that you think represents the probability distribution of the possible outcomes.

b. Divide the numbers between 0 and 1 into intervals of equal lengths. Mark these intervals on your graph from Part **a**.

c. Select one interval on your graph. Shade the area that represents the probability of generating a random number that falls in this interval. Determine the area of the shaded region.

d. Generate 200 random numbers between 0 and 1. Record how many of these numbers fall in the interval you selected in Part **c**.

e. Determine the percentage of numbers generated that fell in the interval you selected in Part **c**. Compare this to the area you calculated in Part **c**.

f. One of the intervals in the rainfall simulation in Activity **2** represents rainfall values greater than or equal to 10 cm but less than 12 cm. Draw the probability distribution for the outcomes of a random number generator that produces numbers in the interval [10,12). Assume that all numbers in this interval are equally likely to occur.

Discussion 2

a. Does the experiment you conducted in Part **d** of the exploration represent a binomial experiment? Explain your response.

b. A uniform probability distribution, like all continuous probability distributions, shows the probability of all possible outcomes of an experiment. Given this fact, what is the total area in a uniform probability distribution?

c. How could you determine the probability that a randomly selected number between 0 and 1 will fall in the interval $[0.4, 0.5)$?

d. What is the probability that a randomly selected number between 0 and 1 will fall in each of the following intervals?

1. $[0.4, 0.45)$

2. $[0.4, 0.41)$

3. $[0.4, 0.401)$

4. $[0.4, 0.4001)$

e. **1.** If the pattern in your responses to Part **d** above continues, what would be the probability of obtaining exactly 0.4?

2. What do you think is the probability of obtaining a specific number in any continuous probability distribution? Explain your response.

f. **1.** Does the probability distribution shown in Figure **5** below appear to represent a continuous distribution? Explain your response.

2. Is this distribution uniform? Explain your response.

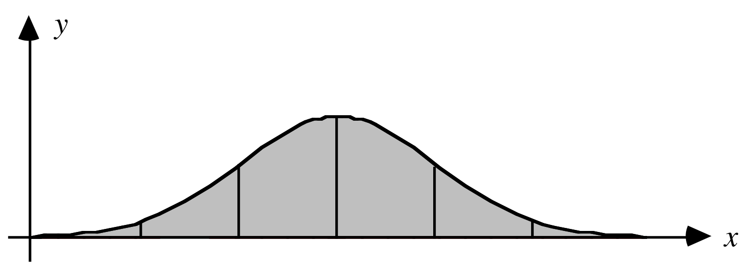


Figure 5: A probability distribution

Assignment

3.1 Given a continuous probability distribution, explain why it makes no difference whether or not an interval's endpoints are included when calculating the probability that a randomly selected value will fall in the interval.

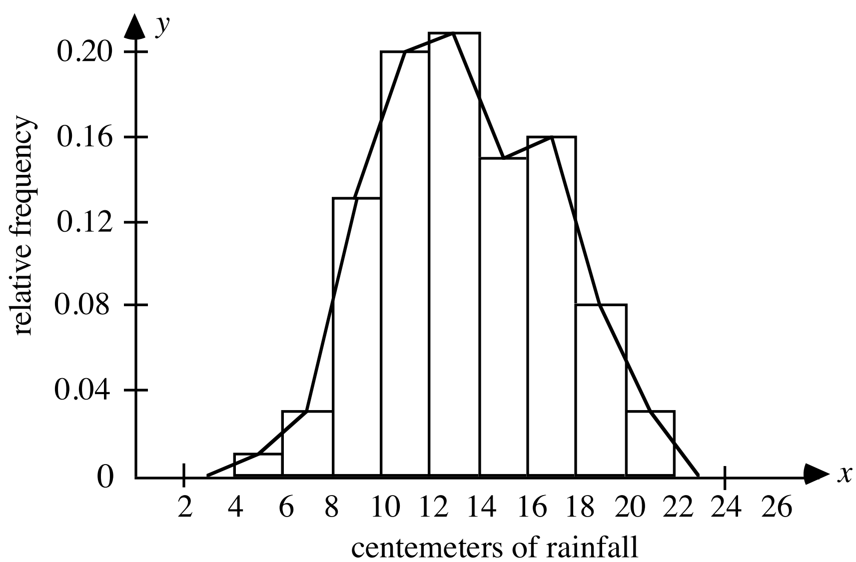
3.2

a. Assume that a random number generator produces numbers in the open interval (–3, 3) so that every number in the interval is equally likely to occur. Draw a graph showing the probability distribution of the possible outcomes.

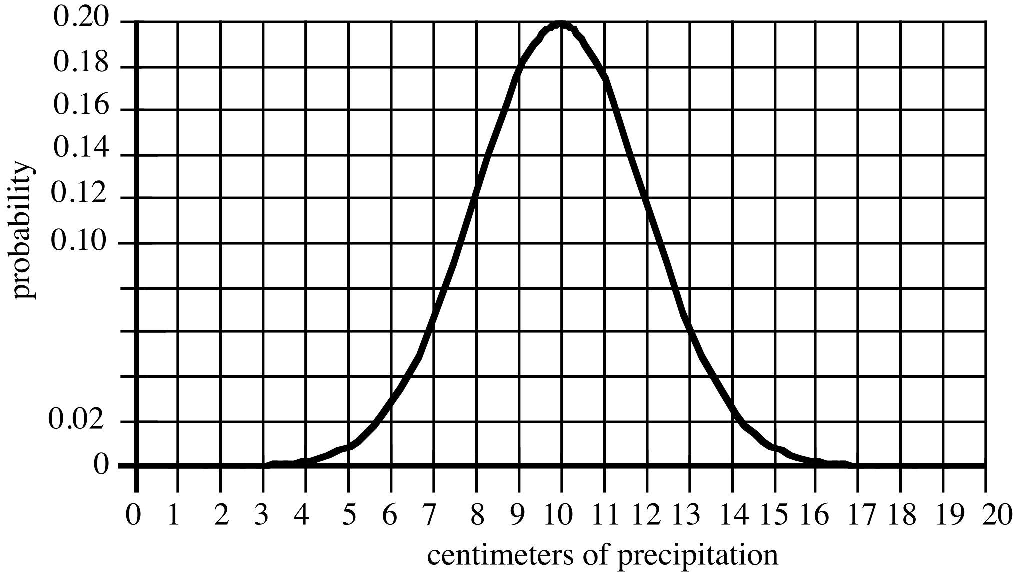
b. Using the random number generator described in Part **a**, what is the probability of obtaining a number in the interval (–1, 1)? Illustrate this probability by shading the appropriate portion of your graph.

c.
1. Create the random number generator described in Part **a**.
2. Generate 200 numbers in the interval (–3, 3).
3. Determine the percentage of those numbers that fall in the interval (–1, 1).
4. Compare this percentage to your response to Part **b**.

3.3 The relative frequency polygon for the precipitation data in Table **1** is shown below. Does this polygon represent a uniform probability distribution? Explain your response.



3.4 In the year 2093, the city of Hypatia has become the largest domed city in the western hemisphere. Hypatia's climate-control officer uses the following continuous probability distribution to determine the May precipitation totals for the city.



a. Determine the area under the curve in this probability distribution. Defend your response.

b. What is the mean amount of precipitation in May for Hypatia? Justify your response.

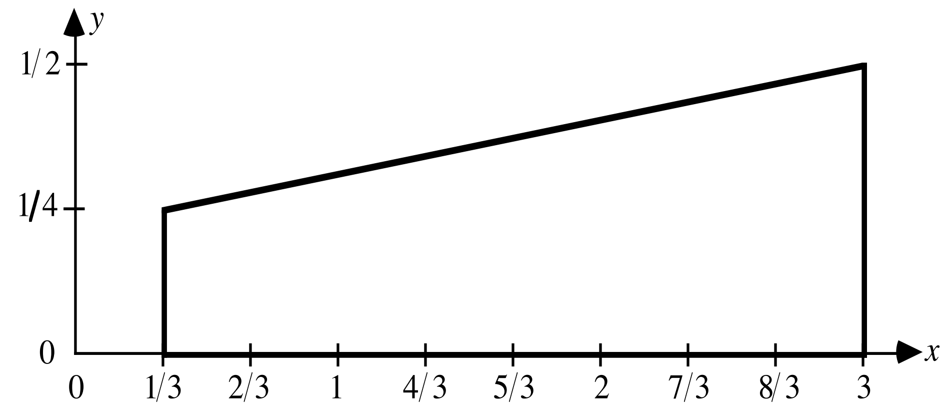
c. Use the graph to estimate the probability that the total precipitation in a given May will be at least 6 cm but less than 14 cm.

d. Hypatia's climate-control officer reports that the standard deviation of May rainfall is 2 cm. Use the graph to estimate each of the following:

1. the probability that the precipitation in a given May will be within 1 standard deviation of the mean
2. the probability that the precipitation in a given May will be within 2 standard deviations of the mean.

* * * * *

3.5 The polygon in the graph below is determined by the intersection of the lines $-3x + 32y = 7$, $x = 1/3$, $x = 3$, and the x-axis.



a. Explain why this polygon and its interior could represent a continuous probability distribution.

b. Could this polygon and its interior represent a uniform probability distribution? Explain your response.

c. Using this distribution, determine the probability that a randomly selected x-value in the interval $[1/3, 3]$ will be:

1. exactly 2
2. less than 2
3. greater than 2.

3.6 **a.** Assuming that a random number generator has a uniform probability distribution for numbers between 0 and 1, what is the probability that a randomly generated number will fall between:

1. 0.6 and 0.8?
2. 0.69 and 0.71?
3. 0.69999 and 0.70001?

b. What do your responses to Part **a** suggest about the probability of obtaining the number 0.7 from the random number generator?

* * * * * * * * * *

Activity 4

The rainfall model you created using a coin-tossing simulation is a discrete probability distribution. However, there is also a continuous probability distribution that can allow you to model Bernoulli's historic May rainfall: the **normal distribution**.

Mathematics Note

A **normal distribution** is a continuous probability distribution. As shown in Figure **6**, the graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.

As in all continuous probability distributions, the total area between the x-axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

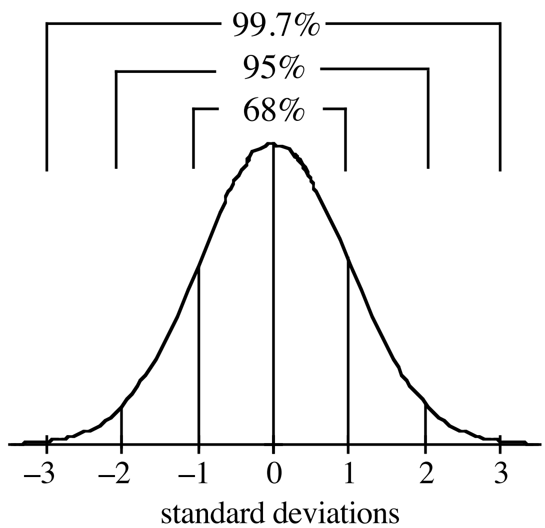


Figure 6: A normal curve and the 68–95–99.7 rule

Normal distributions can be used to model a wide variety of data sets. When this distribution provides a reasonable model, the 68–95–99.7 rule can help you characterize a population. For example, if a population appears to be normally distributed with a mean of 100 and a standard deviation of 10, then you would expect about 68% of the observations to fall between 90 and 110, 95% of the observations to fall between 80 and 120, and 99.7% of the observations to fall between 70 and 130.

Exploration

In Activity **2**, you found that you could approximate the shape of the probability distribution of Bernoulli's historic rainfall data by tossing 12 coins. In this exploration, you investigate the use of a normal curve to model the same data.

a. Obtain a copy of a program that calculates the theoretical probabilities of the outcomes of tossing coins from your teacher. This program plots the midpoints of the tops of the bars in a relative frequency histogram as a scatterplot. Each point in the scatterplot represents the theoretical probability of obtaining a specific number of heads. Based on the mean and standard deviation of the distribution, the program then draws a normal curve to model the data.

1. Use the program to draw the scatterplot and the normal curve for tossing 12 fair coins.
2. Record the mean and standard deviation of the distribution and describe the height and width of the corresponding normal curve.
3. Compare the shape of the curve to the scatterplot.

b. Repeat Part **a** for tossing 20 coins. Compare the mean, standard deviation, and maximum height of the normal curve to the values you obtained in Part **a**.

c. Repeat Part **a** for another value between 20 and 50 coins. Compare the mean, standard deviation, and maximum height of the normal curve to the values you obtained in Parts **a** and **b**.

Discussion

a. Why does a normal curve based on tossing 12 coins appear to be a good model for precipitation amounts in New Bernoulli?

b. How does increasing the number of coins tossed affect the mean and standard deviation of the distribution?

c. How does increasing the number of coins tossed affect the shape of the corresponding normal curve?

d. Using your knowledge of normal distributions, defend the following statement: "As the number of trials in a binomial experiment increases, the resulting probability distribution becomes approximately a normal distribution."

e. The normal curve shown in Figure **7** represents a probability distribution with a mean of 6 and a standard deviation of 1.2.

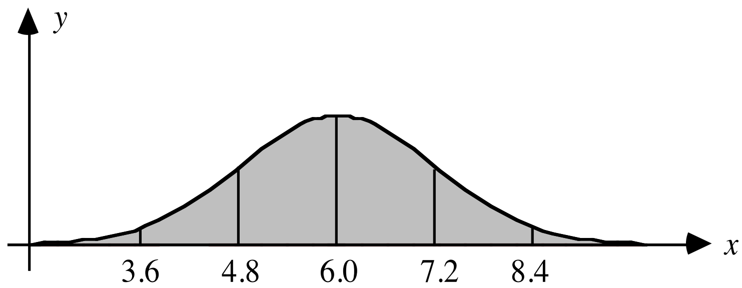


Figure 7: A normal distribution

Using this graph, how could you determine the probability of obtaining each of the following:

1. an x-value between 3.6 and 8.4?
2. an x-value greater than 8.4?
3. an x-value less than 7.2?

Assignment

4.1 In New Bernoulli, less than 3 cm of rain in a month can cause drought, while more than 20.5 cm of rain in a month can trigger floods. To reduce the possibility that the city will experience either drought or flood, the city council has asked you to use a normal distribution with a mean of 13.5 cm and a standard deviation of 3.5 cm to determine precipitation levels for the month of May.

a. Sketch a graph of a normal curve that represents this request. Indicate the mean and standard deviations on the horizontal axis.

b. If you follow the council's recommendations, what is the approximate probability that the May rainfall will be:

1. less than 3 cm?
2. more than 20.5 cm?
3. between 10 and 17 cm?

c. Write a short letter to the city council describing the chances of drought or flood during the month of May. Use probabilities and a discussion of the normal distribution to justify your report.

4.2 The average annual rainfall in a tropical forest is 120 cm, with a standard deviation of 10 cm. If amounts of annual rainfall are normally distributed, what is the probability that the forest will have less than 100 cm of rain this year?

4.3 The city of Pascal has an average annual rainfall of 40 cm, with a standard deviation of 5 cm. The city of Poisson has the same average annual rainfall, with a standard deviation of 2 cm. Levels of annual rainfall for both cities are normally distributed.

a. On the same set of axes, make a sketch of the distribution of rainfall amounts for each town.

b. The mayor of Poisson claims that the annual rainfall in her town is more often between 38 cm and 42 cm than the annual rainfall in Pascal. Is she right? Explain your response.

c. Describe the differences between the two graphs you sketched in Part **a**. Explain why these differences exist.

d. Describe the annual rainfall a resident of each town might expect.

* * * * *

4.4 Imagine that you are conducting an experiment using coins that are not fair. The probability of obtaining a head on any toss of one of these coins is 0.4. Use the program from the exploration in this activity to help you complete this problem.

a. What are the mean and standard deviation of the probability distribution of the number of heads when tossing:

1. 20 coins?

2. 40 coins?

3. 60 coins?

b. Are the formulas for the mean and standard deviation of a binomial distribution appropriate in this situation? Explain your response.

c. Repeat Part **a** when the probability of obtaining a head on any toss is 0.2.

d. Write a summary of your results to Parts **a–c**.

4.5 Kyla and Jon are playing a game in which players take turns rolling two six-sided dice, then moving tokens on a board. The number of squares each player moves is determined by the sum of the faces showing on the two dice.

a. Create a frequency distribution for the sum of the two dice.

b. Determine the mean, median, and standard deviation of the data in Part **a**.

c. Create a theoretical probability distribution for the sum of two dice.

d. Create a relative frequency histogram for the data in Part **c**.

e. Determine the probability that the sum of the faces showing on any given roll are within:

  1. 1 standard deviation of the mean

  2. 2 standard deviations of the mean

  3. 3 standard deviations of the mean.

f. Do you think that the data could be modeled by a normal distribution? Explain your response.

g. On three consecutive turns, Kyla rolls 10 for the sum of the two dice. What is the probability of this occurrence?

* * * * * * * * * *

Research Project

A binomial distribution is also referred to as a **Bernoulli distribution**. During a period of more than two centuries, the Bernoulli family produced many prominent scientists and mathematicians. Research the Bernoullis and their accomplishments. Write a report on their contributions to the study of the normal curve and binomial distributions.

Summary Assessment

One of the radio stations in New Bernoulli offers 45 min of uninterrupted music several times each day. While planning the schedule for next week, the station manager asks you to collect some information on the lengths of popular songs.

a. Collect data on the lengths of at least 50 songs.

b. Assign interval lengths, find the relative frequencies, and construct a histogram of the data.

c. Based on your data, do the lengths of popular songs appear to be normally distributed? Explain your response.

d. Calculate the standard deviation and use it to describe the spread in the data.

e. How many songs would you expect the radio station to be able to play in 45 min? Explain your reasoning.

Module Summary

- The **frequency** of an item in a data set is the number of times that item is observed.
- The **relative frequency** of an item is the ratio of its frequency to the total number of observations in the data set.
- A **relative frequency table** includes columns that describe a data item or interval, its frequency, and its relative frequency.
- A **frequency histogram** consists of bars of equal width whose heights indicate the frequencies of intervals.
- A **frequency polygon** is formed by the line graph that connects the set of points (x,y), where x is the midpoint of an interval and y is the frequency of the interval. The base of a frequency polygon is the x-axis. If the frequency of the interval containing the greatest or least x-value is not 0, a vertex must be added to complete the polygon.
- In a **relative frequency histogram**, the heights of the bars represent relative frequencies instead of frequencies.
- In a **relative frequency polygon**, the y-values represent relative frequencies instead of frequencies.
- A **probability distribution** is the assignment of probabilities to a specific characteristic that belongs to each possible outcome of an experiment. If the set of outcomes is either finite or can have a one-to-one correspondence with the natural numbers, the distribution is **discrete**.
- A **binomial experiment** has the following characteristics:
    - It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
    - The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
    - Each trial has only two possible outcomes: a success or a failure.
    - The probability of a success remains the same from trial to trial.
    - The total number of successes is observed.
- A **binomial distribution** is the probability distribution for a binomial experiment.

- The mean of a binomial distribution is the product of the number of trials and the probability of a success. In other words, the mean μ can be found as follows, where n is the number of trials and p is the probability of a success:

$$\mu = np$$

- The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

- A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.
- A **uniform probability distribution** is a continuous probability distribution in which all the probabilities over intervals of equal width are equal.
- A **normal distribution** is a continuous probability distribution that is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.
- As in all continuous probability distributions, the total area between the x-axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

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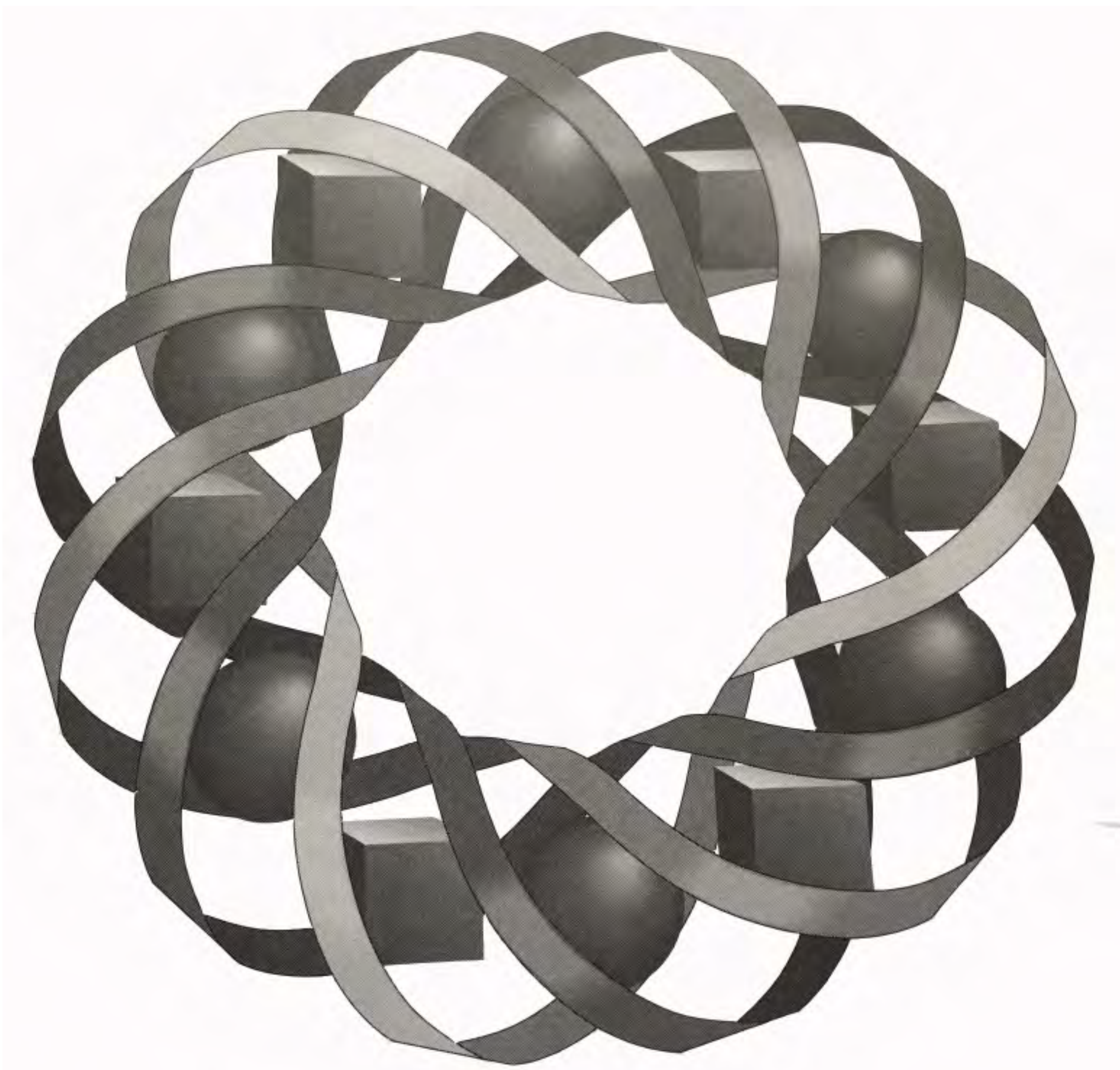
Big Business



How can mathematics help launch a new business? In this module, you'll explore how rational functions and nonlinear inequalities can offer direction to a fledgling cosmetics company.

Sherry Horyna • Jeff Hostetter • Peter Rasmussen

Big Business

Introduction

Americans spend hundreds of millions of dollars annually on products to keep themselves clean, conditioned, made-up, and trimmed down. To keep pace with the wants of consumers, cosmetics companies have entire departments dedicated to research and development. These and other firms also collaborate with the scientific community to create innovative and marketable products.

In the health and beauty industry, developing a new product involves a mix of disciplines, including chemistry, biology, and mathematics. Finding the financial support to launch such a product can be difficult. Most sources of capital require a business plan prior to investment. Business plans may include background information about the product, descriptions of the targeted consumers, sales projections, and advertising strategies.

Exploration

Imagine that you work for a company that wants to market a new home permanent. Permanents (or perms) alter straight hair in order to produce curls. Some types of perms use extremes in pH, either acidic or basic, to curl hair. The pH scale is shown in Figure **1**. A solution with a pH less than 7 is an acid; one with a pH greater than 7 is a base.

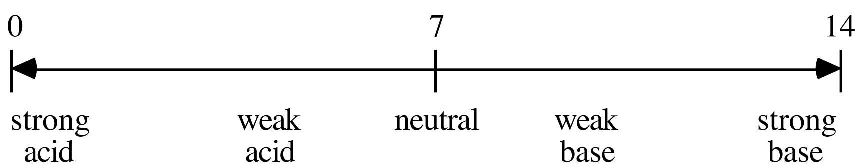


Figure 1: The pH scale

Because chemicals with extreme pH values can damage hair and skin, your company wants the pH of its permanent to be as close to neutral as possible. In this exploration, you simulate the pH balance of a home permanent using ammonia, an actual ingredient in many perms. In water, ammonia forms a solution of ammonium hydroxide, a mild base.

a. In the module "Log Jam," you learned that the pH of a solution is determined by the negative log of the concentration of hydrogen ions in moles per liter.

In a solution of ammonia and water, the concentration of hydrogen ions depends on the concentration of ammonium hydroxide. Determine the pH of a solution of 6.80% ammonium hydroxide.

b. Concentration also may be expressed as a percentage using mass or volume, as shown below:

$$\text{percent concentration} = \frac{\text{amount of pure substance}}{\text{total amount of solution}} \bullet 100$$

One milliliter of 6.80% ammonium hydroxide solution has a mass of 0.993 g. Use this information to calculate the mass (in grams) of pure ammonium hydroxide in 1 mL of this solution.

c. Measure 1 mL of a solution of 6.80% ammonium hydroxide and pour it into a 1-L beaker. Record its pH (determined in Part **a**) in a table with headings like those in Table **1** below.

Table 1: Ammonium hydroxide solution data

Mass of Distilled Water (g)	Total Mass of Solution (g)	pH	Concentration of Ammonium Hydroxide
0	0.993		6.80%
50	50.993		
100			

d. Add 50 mL of distilled water to the ammonium hydroxide solution. At room temperature, 1 mL of water has a mass of 1 g. Therefore, the total mass of the new solution is now 50.993 g, as shown in Table **1**.

1. Measure the pH of the new solution and record it in Table **1**.
2. Calculate the percent concentration of ammonium hydroxide in the new solution and record this value in Table **1**.

e. Add another 50 mL of water to the solution. Record the total mass of the solution, its pH, and the percent concentration of ammonium hydroxide in Table **1**. Continue this process until you have added a total of 1 L of water.

f. Create two separate scatterplots of the data: one that shows pH versus the mass of distilled water added, and one that shows the percent concentration of ammonium hydroxide versus the mass of distilled water added.

g.
1. Write a function that relates percent concentration to the mass of distilled water added. (Hint: Use the relationship for percent concentration described in Part **b** as a guide.)
2. Considering the context, describe an appropriate domain for the function.

h.
1. Use a graphing utility to graph your function from Part **g** over the interval [–5, 5].
2. Describe the domain for this function in general.

Discussion

a. How much water did you have to add to the ammonium hydroxide solution to obtain a pH less than 8?

b. How much water would have to be added to the ammonium hydroxide solution to obtain a pH of 7? Explain your response.

c. How much water would you have to add to reduce the concentration of ammonium hydroxide to 0.01%? Explain your response.

d. Compare the two plots you graphed in Part **f** of the exploration.

e. How does the graph from Part **h** of the exploration compare to the graphs from Part **f**?

f. Compare the domain of the function found in Part **h** with the domain that applies to the context of the exploration.

g. What happens to the graph in Part **h** of the exploration when $x = -0.993$?

Activity 1

In the introduction, you investigated one situation in which mathematics might help the developers of a new home permanent. The function you used to model the concentration of ammonium hydroxide is a **rational function**. Rational functions can be used to model a variety of phenomena from fields as diverse as business, biology, physics, and social science. In this activity, you investigate some of the characteristics of rational functions and their graphs.

Mathematics Note

A **rational function** is in the form:

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomial expressions and $d(x) \neq 0$. The domain of $f(x)$ does not contain values of x for which $d(x)$ equals 0.

For example, consider the function below:

$$f(x) = \frac{2x^3 - 3x + 4}{x + 7}$$

This is a rational function where $n(x) = 2x^3 - 3x + 4$ and $d(x) = x + 7$, with $x \neq -7$. Since $d(x) = 0$ when $x = -7$, the function is undefined at $x = -7$ and the domain of $f(x)$ is the set of all real numbers except –7.

Exploration 1

In some cases, the graph of a rational function can resemble the graph of a polynomial function. This occurs when all the factors of the denominator are also factors of the numerator. Although the two graphs may appear identical, there are important differences. In this exploration, you compare a rational function and a polynomial function with similar graphs.

a. In the module "Drafting and Polynomials," you found that a polynomial function with all real roots can be expressed as the product of factors in the form:

$$p(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$$

where c_n is a real root of the polynomial function $p(x)$.

1. Graph the polynomial function $f(x) = x^3 + x^2 - 14x - 24$.

2. Determine the roots of $f(x)$.

3. Identify the domain of the function.

4. Write $f(x)$ as a product of first-degree factors.

b. Repeat Part **a** for the polynomial function $g(x) = x^2 + 5x + 6$. Graph $g(x)$ on the same coordinate system as $f(x)$.

c. **1.** Create the rational function:

$$r(x) = \frac{f(x)}{g(x)}$$

2. Identify the domain of $r(x)$.

3. Graph $r(x)$ on the same coordinate system as $g(x)$ and $f(x)$.

d. The graph of $r(x)$ appears to be a line. Determine the equation of the line that appears to coincide with the graph of $r(x)$.

Mathematics Note

Two functions $f(x)$ and $g(x)$ are **equivalent** if and only if the domain of $f(x)$ is the same as the domain of $g(x)$ and $f(x) = g(x)$ for all values of x in the domain.

For example, $f(x) = x^2/1$ and $g(x) = x^2$ are equivalent functions since the domain of each is the set of all real numbers and $f(x) = g(x) = x^2$ for all x-values in the domain. The functions $g(x) = x^2$ and $h(x) = x^3/x$, however, are not equivalent, since their domains are not the same. The domain of $g(x)$ is the set of all real numbers; the domain of $h(x)$ is all real numbers except 0.

e. When working with fractions that contain integers in their numerators and denominators, it is possible to express them as equivalent fractions in reduced form. This can be done by expressing the numerator and denominator as the products of their factors, then dividing like factors, as shown below:

$$\frac{12}{6}=\frac{2\bullet 2\bullet 3}{2\bullet 3}=2\bullet\frac{2}{2}\bullet\frac{3}{3}=2\bullet 1\bullet 1=2$$

This process also can be used to rewrite a rational function whose numerator shares the factors of the denominator. For example, consider the following rational function:

$$\begin{aligned}\frac{x^2-x-6}{x-3}&=\frac{(x+2)(x-3)}{x-3}\\&=\frac{(x+2)\cancel{(x-3)}}{\cancel{x-3}}\\&=x+2,\text{ as long as } x\neq 3\end{aligned}$$

Use this process to rewrite the rational function $r(x)$ as a function $t(x)$. Identify any value of x for which the division of like factors is not defined.

f. Use a symbolic manipulator to divide $f(x)$ by $g(x)$. Compare the result with the equation of the line you found in Part **d** and the equation of $t(x)$ from Part **e**.

Discussion 1

a. Where do the graphs of $f(x)$ and $g(x)$ intersect? Why does this occur?

b. Where does the graph of $r(x)$ intersect the x-axis? Does $r(x)$ also intersect with $f(x)$ or $g(x)$ at this point (or points)? Explain your response.

c. How could you use the graphs of two polynomial functions to determine if they have any common factors?

d. Consider a rational function $r(x)$ for which all the factors of the denominator are also factors of the numerator. Describe how to identify a polynomial function whose graph resembles that of $r(x)$.

e.
1. In Part **d** of Exploration **1**, you determined the equation of a line that appeared to coincide with the graph of $r(x)$. Are this line and $r(x)$ equivalent functions? Explain your response.
2. Are $r(x)$ and $t(x)$ equivalent functions? Explain your response.
3. Is the result of the division of $f(x)$ by $g(x)$ found in Part **f** of Exploration **1** equivalent to $r(x)$? Explain your response.

f. Describe how to use a symbolic manipulator to rewrite a rational function as described in Exploration **1**.

Exploration 2

Consider the following two functions: $f(x) = x + 3$ and

$$g(x) = \frac{x^2 + x - 6}{x - 2}$$

a. Identify the domains of $f(x)$ and $g(x)$.

b. Divide the numerator of $g(x)$ by the denominator.

c. Compare the graphs of $f(x)$ and $g(x)$ in a small interval near the value for which the denominator of $g(x)$ equals 0.

d. To investigate $f(x)$ and $g(x)$ near the value for which $g(x)$ is undefined, use appropriate technology to complete the following table.

Table 2: Values of $f(x)$ and $g(x)$

x	$f(x)$	$g(x)$
1.5		
1.6		
1.7		
1.8		
1.9		
2.0		
2.1		
2.2		
2.3		
2.4		
2.5		

e. Determine whether $f(x)$ and $g(x)$ are equivalent.

Discussion 2

a.
1. Which of the functions in Exploration **2**, if any, are polynomial functions?
2. Which, if any, are rational functions?

b. Describe another function that is equivalent to each of the following:
1. $f(x)$
2. $g(x)$

Mathematics Note

A function is **continuous** at a point c in its domain if the following conditions are met:

- the function is defined at c, or $f(c)$ exists
- the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
- the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$

A function is continuous over its domain if it is continuous at each point in its domain.

A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.

For example, a function is discontinuous at $x = c$ if the function is undefined at c, as shown in Figure **2**.

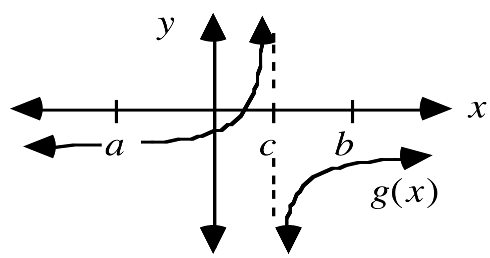


Figure 2: Graph of the discontinuous function $g(x)$

A function also is discontinuous at $x = c$ if the limit of the function does not exist at c, as shown in Figure **3**.

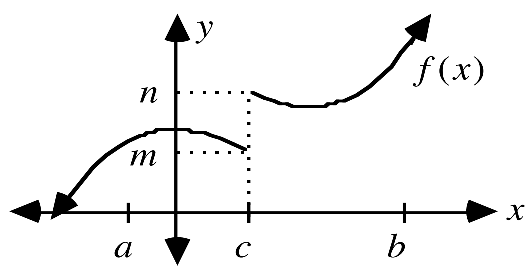


Figure 3: Jump discontinuity in the function $f(x)$

In this case, $f(x)$ approaches m as x approaches c from the left. As x approaches c from the right, $f(x)$ approaches n. Since $m \neq n$, the limit of $f(x)$ as x approaches c does not exist. This kind of discontinuity is referred to as a **jump discontinuity**.

A function also is discontinuous at $x = c$ if the value of the function at c does not equal the limit of the function as x approaches c, as shown in Figure **4**. In this case, the limit of $h(x)$ as x approaches c is m, while $h(c) = n$, and $m \neq n$.

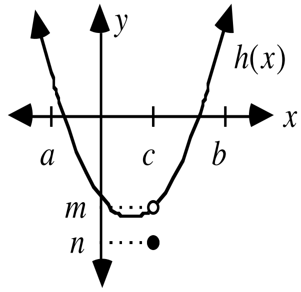


Figure 4: Hole in the graph of the function $h(x)$

c. Consider the rational function below:

$$h(x) = \frac{x^3 - 4x^2 + x + 6}{x - 2}$$

1. Is $h(x)$ a continuous function? Explain your response.
2. Identify the factors of the numerator and the denominator. Use these factors to explain why the graph of $h(x)$ would be similar to the graph of the polynomial function $p(x) = x^2 - 2x - 3$.
3. What differences would exist between the graphs of $h(x)$ and $p(x)$? Explain your response.
4. Identify a discontinuous rational function with a hole at $x = -3$ whose graph would be similar to the graph of $p(x) = x^2 - 2x - 3$. Justify your response.

Assignment

1.1 Identify the domain of each rational function below and describe the values of x at which the function is discontinuous.

a. $f(x) = \dfrac{(x+2)(x-7)}{(x-7)}$

b. $g(x) = \dfrac{x^3 + 4x^2 - x - 4}{x^2 + 5x + 4}$

c. $h(x) = \dfrac{1}{x^2 + 1}$

1.2 Sketch the graph of a function with a hole at $x = -2$.

1.3 Consider the following two functions: $f(x) = x^2 + 2x - 3$ and

$$g(x) = \frac{x^3 + 5x^2 + 3x - 9}{x + 3}$$

a. Divide the numerator of $g(x)$ by the denominator.

b. Are $f(x)$ and $g(x)$ equivalent? Justify your response.

1.4 Identify the domain of each rational function in Parts **a–e**. Sketch a graph of each function and label any holes that occur.

a. $f(x) = (x+2)(x-2)$

b. $f(x) = \dfrac{x^3 - 5x^2 - 2x + 24}{x - 4}$

c. $f(x) = \dfrac{x^3 - 5x^2 - 2x + 24}{x^2 - x - 6}$

d. $f(x) = \dfrac{2x^2 + 1}{5}$

e. $f(x) = \dfrac{5}{2x^2 + 1}$

1.5 Explain why any polynomial function is also a rational function.

* * * * *

1.6 Determine the domain of each rational function below. Sketch a graph of each function and label any holes that occur.

a. $f(x) = \dfrac{x^3 - 10x^2 - 23x + 132}{x - 11}$

b. $f(x) = \dfrac{-4}{x^2 + 3x + 13}$

c. $f(x) = \dfrac{x^5 + 3x^4 - 17x^3 - 15x^2 + 88x - 60}{x^2 + 3x - 10}$

d. $f(x) = \dfrac{x^2 + 0.15x - 0.45}{0.75 + x}$

* * * * * * * * * *

Activity 2

In the previous activity, you examined the graphs of rational functions with common factors in the numerator and the denominator. In this activity, you continue your investigation of rational functions and their graphs by examining functions whose numerators and denominators do not have common factors.

Exploration 1

Not all rational functions have numerators and denominators that share factors. In this exploration, you examine how to express rational functions in much the same way as mixed numbers are used to express improper fractions.

One way to determine the mixed number that is equivalent to an improper fraction is to rewrite the numerator as a sum of two rational numbers. For example, the improper fraction $12/5$ can be rewritten as the mixed number $2\frac{2}{5}$ as follows:

$$\frac{12}{5}=\frac{10+2}{5}=\frac{10}{5}+\frac{2}{5}=2+\frac{2}{5}$$

Using a similar process, any rational expression $n(x)/d(x)$ in which the degree of the numerator is greater than or equal to the degree of the denominator can be written as an expression of the form:

$$\frac{n(x)}{d(x)}=Q(x)+\frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder in the indicated division.

For example, consider the following rational expression:

$$\frac{5x+2}{x+2}$$

One method for rewriting this as a mixed expression is outlined below:

$$\begin{aligned}\frac{5x+2}{x+2}&=\frac{5x+10-8}{x+2}\\&=\frac{5(x+2)-8}{x+2}\\&=\frac{5(x+2)}{x+2}+\frac{-8}{x+2}\\&=5-\frac{8}{x+2}\end{aligned}$$

a. If the resulting mixed expression is equivalent to the original rational expression, then $d(x)\bullet Q(x)+R(x)=n(x)$.

Verify that this relationship is true for the example described above.

b. Rewrite the rational expression below in the form $Q(x) + R(x)/d(x)$.

$$\frac{6x-5}{x+4}$$

c. Verify that the final expression in Part **b** is equivalent to the original expression.

d. Use a symbolic manipulator to determine an expression that is equivalent to the original expression in Part **b**. Compare the result with your solution in Part **b**.

e. Select another rational expression in which the degree of the numerator is greater than or equal to the degree of the denominator. Rewrite the expression in the form $Q(x) + R(x)/d(x)$.

Discussion 1

a. When a rational expression in which the numerator shares all the factors of the denominator is written in the form $Q(x) + R(x)/d(x)$, the value of $R(x)$ is 0. Explain why this is true.

b. **1.** When would the value of $Q(x)$ be 0? Explain your response.

2. When would the value of $Q(x)$ be a real number other than 0?

c. Describe the process required to express a rational expression as a mixed expression using a symbolic manipulator.

Exploration 2

In this exploration, you examine the graph of a rational function in which the numerator and denominator share no common factors.

a. Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x) = 3x + 2$ and $d(x) = x - 1$.

For what value(s) of x does $d(x) = 0$?

b. Identify the domain of $f(x)$.

c. Divide $n(x)$ by $d(x)$. Write your answer in the following form:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder.

d. To explore the values of $f(x)$ near $x = 1$, use appropriate technology to complete the following table.

Table 3: Values of $f(x)$ near $x = 1$

x	$f(x)$
0.9	
0.95	
0.99	
0.999	
1.001	
1.01	
1.05	
1.1	

e. In "Drafting and Polynomials," you examined the graphs of polynomial functions as values in the domain became very large or very small. To explore the values of the rational function $f(x)$ as $|x|$ becomes large, use appropriate technology to complete the following table.

Table 4: Values of $f(x)$ as $|x|$ becomes large

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–10,000			
–1000			
–100			
100			
1000			
10,000			

f. Create a graph of $f(x)$.

g. Use the values in Table **3** to make conjectures about the graph of $f(x)$ as x approaches 1, the value for which $d(x) = 0$.

h. Use the values in Table **4** to make conjectures about the relationship among $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large.

i. Consider the rational expression you rewrote in Part **b** of Exploration **1**, shown below:

$$f(x) = \frac{6x - 5}{x + 4} = 6 - \frac{29}{x + 4}$$

Do the conjectures you made in Part **h** appear to hold true for this function?

Discussion 2

a. Does the graph of $f(x)$ in Part **a** of Exploration **2** have any discontinuities? If so, describe the values at which they occur.

b.
1. Describe the graph of $f(x)$ as x approaches 1 from the left.
2. Describe the graph of $f(x)$ as x approaches 1 from the right.

c. Describe the graph of $f(x)$ as $|x|$ becomes large.

d. What conjectures did you make about the relationship among $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large?

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line approaches 0 as the distance from point P to the origin increases without bound. Asymptotes, like holes, may be the result of discontinuities.

For example, consider the rational function below:

$$f(x) = \frac{x-3}{x-5}$$

By dividing the numerator by the denominator, this function can be rewritten as

$$f(x) = 1 + \frac{2}{x-5}$$

As x approaches 5, $(x-5)$ approaches 0 and $f(x)$ becomes increasingly large or increasingly small. Because $f(5)$ is not defined, a discontinuity occurs at $x = 5$. The result of that discontinuity is a vertical asymptote.

As $|x|$ becomes large, the quantity $2/(x-5)$ approaches 0 and $f(x)$ approaches 1. In this case, the line $y = 1$ is a horizontal asymptote of the function. Figure **5** shows a graph of $f(x)$ and its asymptotes.

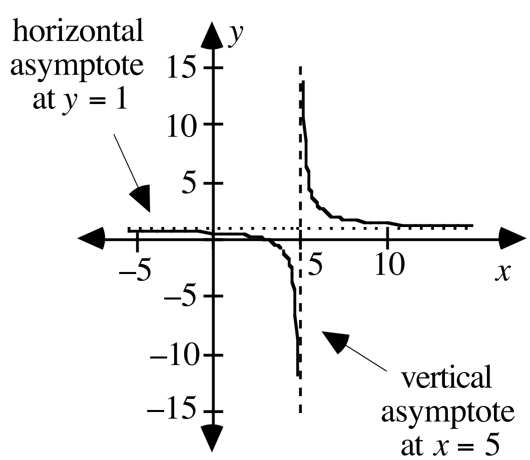


Figure 5: Graph of $f(x) = \frac{x-3}{x-5}$

e. Describe any asymptotes that exist in the graph of the rational function you investigated in Parts **a–f** of Exploration **2**.

f. Does a horizontal asymptote indicate a discontinuity? Explain your response.

g. **1.** Describe a rational function in the following form whose graph includes a vertical asymptote:

$$f(x) = Q(x) + \frac{R(x)}{d(x)}$$

2. Describe a rational function that includes the same expression for $d(x)$ as in Step **1** and has a hole in its graph.

3. Identify the values of x for which the discontinuities appear in each function you described above.

h. Based on your observations, how do rational functions whose graphs include holes compare with those whose graphs include vertical and horizontal asymptotes?

Assignment

2.1 Sketch the graph of a function with a vertical asymptote at $x = -3$ and a horizontal asymptote at $y = 2$.

2.2 Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x) = 2$ and $d(x) = x - 1$.

a. For what value(s) of x does $d(x) = 0$?

b. Identify the domain of $f(x)$.

c. Rewrite $f(x)$ in the following form:

$$Q(x) + \frac{R(x)}{d(x)}$$

d. **1.** Create a graph of $f(x)$ and describe its asymptotes.

2. How are these asymptotes related to your response to Part **c**?

2.3 Identify the domain of each rational function in Parts **a–e**. Sketch a graph of each function and label any holes or asymptotes that occur.

a. $f(x) = 1/x^2$

b. $f(x) = \dfrac{2x - 10}{x - 4}$

c. $f(x) = \dfrac{6}{4 - x^2}$

d. $f(x) = \dfrac{x^3 - x^2 - 8x + 12}{x - 2}$

e. $f(x) = \dfrac{x^3 + 4x^2 - 17x - 60}{x^3 + 11x^2 + 39x + 45}$

2.4 In the introduction tothe module, you explored the relationship between the percent concentration of ammonium hydroxide in a solution and the mass of distilled water added. This relationship can be modeled by the function below, where x represents the mass of distilled water in grams.

$$f(x) = \frac{0.0680}{x + 0.993} \bullet 100$$

a. Identify the domain of $f(x)$.

b. Sketch a graph of $f(x)$.

c. Write the equations of any asymptotes.

2.5 The production manager at your company has determined that the average cost per unit for manufacturing your home perm is approximated by the rational function below, where x is the number of perms produced.

$$C(x) = 27{,}000/(x + 50)$$

a. Find $C(5)$, $C(1000)$, and $C(10{,}000)$.

b. Why do you think that the average cost per unit decreases as the number of units produced increases?

c. Identify the domain of $C(x)$.

d. Sketch a graph of $C(x)$. Label any horizontal or vertical asymptotes.

e. There are usually fixed costs that are required before manufacturing begins. What are the fixed costs for the perm manufacturing process? Justify your answer.

f. Due to the cost of materials, the production manager claims that the lowest possible cost per unit is $2.45, regardless of the number of perms produced. After how many units would this cost be reached?

* * * * *

2.6 Determine the domain of each rational function below. Sketch a graph of each function and label any holes or asymptotes.

a. $f(x) = 5x/(x - 6)$

b. $f(x) = \dfrac{2x^2 - 7x - 4}{4x + 2}$

c. $f(x) = 5/(x - 3)(x - 1)$

d. $f(x) = \dfrac{0.5x^3 + 2}{x + 1}$

e. $f(x) = \dfrac{x^2 - 3x - 4}{x^3 - 7x^2 + 8x + 16}$

2.7 In an electrical circuit, **resistance** is the opposition to the flow of current. When two resistors, r_1 and r_2, are connected in parallel, the total resistance *R* of the circuit (in ohms) is described by the following formula:

$$R = \frac{r_1 \bullet r_2}{r_1 + r_2}$$

Consider a circuit in which $r_1 = 10$ ohms and r_2 is unknown.

a. Let $r_2 = x$. Write a function that could be used to model the total resistance *R* in terms of *x*. Identify the domain of the function.

b. Represent the function in the form $Q(x) + R(x)/d(x)$. Identify any holes or asymptotes.

c. Graph your function from Part **a** and label any holes or asymptotes.

d. Find the value of r_2 that results in a total resistance of 3 ohms.

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Activity 3

In the previous activity, you examined the graphs of rational functions in the form

$$f(x) = Q(x) + \frac{R(x)}{d(x)}$$

and determined where holes, vertical asymptotes, and horizontal asymptotes occur. In this activity, you examine rational functions with **oblique asymptotes** (asymptotes that are neither horizontal nor vertical).

Exploration

Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x) = x^2 + 3x - 2$ and $d(x) = x + 5$.

a. Identify the domain of $f(x)$.

b. Divide $n(x)$ by $d(x)$. Write your answer in the following form:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder.

c. To explore the values of $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large, complete the following table.

Table 5: Values of $f(x)$ as $|x|$ becomes large

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–1000			
–500			
–100			
–50			
50			
100			
500			
1000			

d. Create a graph of $f(x)$ and label any discontinuities.

e. Use your graph from Part **d** and the values in Table **5** to suggest a possible equation for the oblique asymptote to the graph of $f(x)$. Test your equation by graphing it on the same coordinate system as $f(x)$.

Discussion

a. Describe the graph of $f(x)$, including any holes or asymptotes.

b. What happens to the value of $R(x)/d(x)$ as $|x|$ becomes large?

c. What happens to the values of $f(x)$ and $Q(x)$ as $|x|$ becomes large?

Mathematics Note

A rational function $f(x)$ can be rewritten in the following form by dividing the numerator by the denominator:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder. In this form, $Q(x)$, $R(x)$, $n(x)$, and $d(x)$ are polynomial functions. When the degree of $R(x)$ is less than that of $d(x)$, the value of $R(x)/d(x)$ approaches 0 as $|x|$ becomes very large. As a result, its effect on a graph of $f(x)$ can be ignored, producing three general cases.

- If $Q(x)$ equals some constant a, the graph of $f(x)$ has a horizontal asymptote at $y = a$.
- If $Q(x)$ is a linear function in the form $y = mx + b$ with $m \neq 0$, the graph of $f(x)$ has an oblique asymptote described by $y = mx + b$.
- If $Q(x)$ is a polynomial function of degree 2 or greater, $f(x)$ has no horizontal or oblique asymptotes.

For example, consider the rational function

$$f(x) = \frac{2x^2 - 5x + 3}{x - 2}$$

The domain of this function is the set of all real numbers except 2. This discontinuity is represented by a vertical asymptote described by the line $x = 2$. Dividing the numerator by the denominator yields:

$$f(x) = \frac{2x^2 - 5x + 3}{x - 2} = 2x - 1 + \frac{1}{x - 2}$$

In this case, $Q(x) = 2x - 1$, a linear function in the form $y = mx + b$ with $m \neq 0$. Therefore, $f(x)$ also has an oblique asymptote described by the equation $y = 2x - 1$. Figure **6** shows a graph of $f(x)$ and its asymptotes.

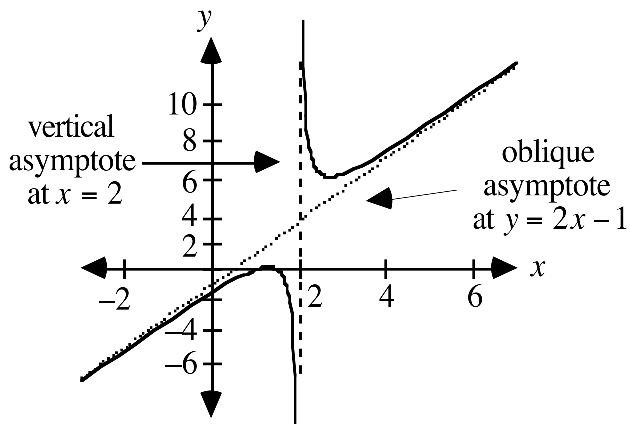

Figure 6: Graph of $\boldsymbol{f(x) = \dfrac{2x^2 - 5x + 3}{x - 2}}$

Assignment

3.1 Sketch a graph of a function with a vertical asymptote at $x = 2$ and an oblique asymptote with the equation $y = x - 1$.

3.2 Write rational functions whose graphs have the characteristics described in Parts **a–c** below. Explain how you selected each function.

a. a hole at $x = -1$

b. a vertical asymptote at $x = -1$ and a horizontal asymptote

c. a vertical asymptote at $x = -1$ and an oblique asymptote

3.3 Identify the domain of each function in Parts **a–d**. Sketch a graph of each function and label all holes and asymptotes.

a. $f(x) = \dfrac{x^2 + 7x + 9}{x + 1}$

b. $f(x) = x + \dfrac{1}{x}$

c. $f(x) = \dfrac{-2x^2 - 11x + 25}{x + 7}$

d. $f(x) = \dfrac{x^3 + 5x^2 + 8x + 9}{x^2 + 2x + 1}$

3.4 The adult dosage for a particular prescription drug is 85 mg. To calculate the appropriate dosage for patients under 16 years old, a pharmacist uses the formula below, where x is the patient's age in years:

$$D(x) = \frac{85x}{x + 6.5}$$

a. Sketch a graph of $D(x)$ and label all asymptotes.

b. Determine the appropriate dosage for a 2-year-old child.

c. Determine the appropriate dosage for a 15-year-old patient.

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3.5 Write rational functions whose graphs have the characteristics described in Parts **a–c** below. Explain how you selected each function.

a. a hole at $x = 5$

b. a vertical asymptote at $x = 5$ and a horizontal asymptote

c. a vertical asymptote at $x = 5$ and an oblique asymptote

3.6 Determine the domain of each function in Parts **a–d**. Sketch a graph of each function and label all holes and asymptotes.

a. $f(x) = \dfrac{-x^2 + 5x + 14}{x + 2}$

b. $f(x) = \dfrac{4x + 3}{2x - 1}$

c. $f(x) = \dfrac{2x^2 - 1}{3x^3 + 2x^2 + 1}$

d. $f(x) = \dfrac{2x^3 + 7x^2 - 4}{x^2 + 2x - 3}$

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Activity 4

After developing a safe and effective home permanent, your company must determine how to package it. Because cylindrical bottles are the most inexpensive form of plastic packaging, your production team has decided to use a 500-mL cylindrical container. (Since the cap does not contain any solution, its volume is not included in the 500 mL.)

Exploration

The formula for the volume of a cylinder is $V = \pi r^2 h$, where *r* represents the radius of the base and *h* represents the height.

a. Recall that 1 mL of water has a volume of 1 cm^3. Substitute 500 cm^3 for *V* in the formula above and solve the resulting equation for *h* in terms of *r*.

b. Using a value of *r* supplied by your teacher, determine the corresponding height of a 500-mL cylindrical container. Construct a model cylinder with these measurements. **Note:** Save your cylinder for use later in this activity.

c. Collect the corresponding values for *r* and *h* from the entire class and graph them as ordered pairs in the form (r, h).

d. Graph the function you determined in Part **a** using a reasonable domain.

e. Graph the function from Part **a** over the interval from –20 to 20. Identify any values for which the function is undefined.

Discussion

a. Which of the cylinders constructed by your class would not provide reasonable containers for a home permanent? Explain your response.

b. If you solve the equation $500 = \pi r^2 h$ for *r* in terms of *h*, is the result a rational function? Explain your response.

c. What would be a reasonable domain for a function that describes the relationship between the height and radius of a cylinder with a volume of 500 mL?

d. How does your graph in Part **e** of the exploration differ from the graph in Part **d**?

e.
1. Over the set of all real numbers, what value of *r* would not satisfy the function you wrote in Part **a** of the exploration?
2. What is the domain of this function?
3. Does this function have any discontinuities? Explain your response.

Assignment

4.1 An employee in the marketing department has suggested that the home permanent bottle's height equal its circumference.

- **a.** Write a function that describes this relationship. What is an appropriate domain for the function in this context?
- **b.** Graph the function over the set of all real numbers.
- **c.** Use technology to find the coordinates (to the nearest 0.01) of the intersection of this function with the one you wrote in Part **a** of the exploration.
- **d.** What is the significance of the point of intersection?
- **e.** What would you conclude if the two functions did not intersect?
- **f.** If your company decides to use a bottle whose height equals its circumference, what will its dimensions be?

4.2 A marketing survey has indicated that consumers prefer bottles that do not easily tip over. After experimenting with the bottle's dimensions, your team has decided that its height should not exceed twice the diameter.

- **a.** Write this constraint on the bottle's height as an inequality.
- **b.** The equation $h = 500/\pi r^2$, where h represents height and r represents the radius of the base, models another constraint on the bottle's height. Graph this equation along with the inequality from Part **a** as a system of relations on a coordinate plane.
- **c.** What is the solution of the system? Explain your response.
- **d.** Use the graph of the system to describe the possible dimensions of the bottle.

4.3 The chief financial officer of your company would like to reduce the cost of packaging the home permanent. In response, your production team has suggested that minimizing the bottle's surface area could help minimize the cost of materials.

- **a.** Without its cap, the bottle can be represented as a cylinder. Write a formula for the surface area of a cylinder.
- **b.** Substitute the expression $500/\pi r^2$, where r represents the radius of the base, for h in your formula for surface area. The resulting equation should describe the surface area of a 500-mL cylinder in terms of its radius.
- **c.** Graph this function using an appropriate domain.
- **d.** What is the minimum surface area for a 500-mL cylindrical bottle?

e. What are the dimensions of that bottle (to the nearest 0.01 cm)?

f. Does the bottle fit the condition that its height should not exceed twice its diameter?

g. Would you use a bottle with the dimensions from Part **e**? Explain your response.

* * * * *

4.4 A new restaurant in town has asked you to design a cylindrical bowl that will hold 300 mL of liquid.

a. Use the formula for the volume of a cylinder to write an equation for the height of the bowl in terms of its radius.

b. The height of the bowl should be one-third its diameter. Write an equation that describes this constraint on height in terms of the radius.

c. What is an appropriate domain for the system of equations in Parts **a** and **b**? Explain your response.

d. Graph this system of equations on a coordinate plane.

e. Find the dimensions of the bowl that satisfies these constraints.

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Activity 5

As you observed in the previous assignment, real-world problems may have more than one acceptable solution. A solution set for an inequality, for example, can consist of a half-plane or a region bounded by curves. Figure **7** shows a solution region bounded on one side by a curve.

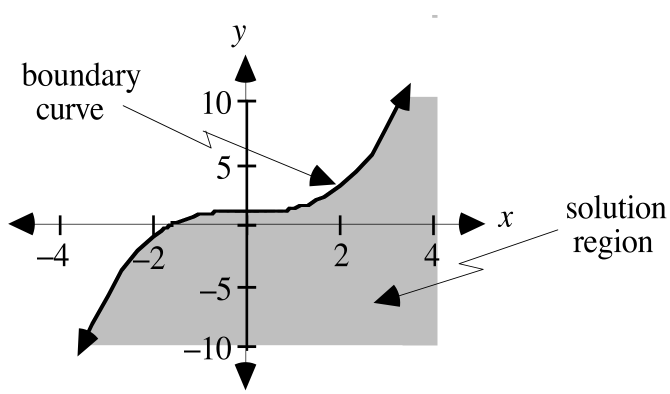

Figure 7: Graph of $y \leq 0.25x^3 + 1$

In this activity, you explore some of the difficulties that can arise in graphing solution sets when a boundary curve is described by a rational function.

Exploration

a. Graph the solution set for $y \geq -0.2x^2 + 3$ by completing Steps **1–4** below.

1. Graph the boundary curve.
2. Decide whether or not the boundary curve is part of the solution set.
3. Decide which side of the boundary curve represents the solution region.
4. Shade the solution region.

b. Repeat Part **a** to graph the solution region for $y < -0.2x^2 + 3$.

c. Describe how the solution regions for $y > -0.2x^2 + 3$ and $y \leq -0.2x^2 + 3$ are related.

Mathematics Note

When graphing inequalities that involve rational functions, a boundary function may contain discontinuities. In such cases, the graph should indicate those portions that are not included in the feasible region as a result of discontinuities.

For example, consider the rational inequality:

$$y \geq \frac{x^3 - 2x^2 - 4x + 8}{x - 2}$$

In this case, the function that defines the boundary of the solution set can be expressed as follows:

$$y = x^2 - 4 + \frac{0}{x - 2}$$

As shown in Figure **8**, there is a hole in the graph of the boundary function at $x = 2$ and the solution set for the inequality contains no points on the line $x = 2$.

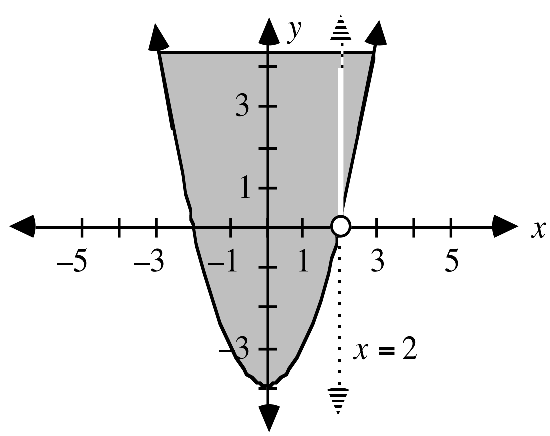

Figure 8: Graph of $y \geq x^2 - 4 + \dfrac{0}{x - 2}$

d. Use a graphing utility to graph the solution sets for each of the following rational inequalities. Label any holes or asymptotes that occur.

1. $y \geq \frac{11}{2x}$
2. $y < \frac{-2x^2 + 7x - 6}{x - 2}$
3. $y \leq \frac{0.3x^3 + x - 1.5}{x + 2}$
4. $y > \frac{15x + 6}{(x - 2)(x + 3)}$

Discussion

a. Describe the strategies you used to graph the solution regions in the exploration.

b. What portions of the coordinate plane are included in the graphs of the inequalities $y > -0.2x^2 + 3$ and $y \leq -0.2x^2 + 3$?

c. Why are there no solutions to a rational inequality on the vertical line marking a hole in the boundary function?

Assignment

5.1 To market its product to salons, your company has decided to offer larger bottles of its permanent solutions. The volume of these bottles should be between 500 mL and 1 L. As with the 500-mL size, the height of the bottle should not exceed twice its diameter.

a. Write a system of inequalities that models these constraints on the height of the bottle. (Recall that $V = \pi r^2 h$.)

b. Graph the system of inequalities.

c. Shade the solution to the system and describe its shape.

d. Describe the meaning of the solution set in terms of the bottle's dimensions.

e. Identify the dimensions of a bottle that satisfies these constraints.

5.2 Graph the solution set for each of the following inequalities. Label any discontinuities that exist.

a. $y \geq \dfrac{x^4 - x}{x - 1}$

b. $y < \dfrac{x^4 - 7x^3 + 10x^2 + x - 2}{x - 2}$

c. $y > 0.0625x^4 - x^3 + 4x^2 - 7$

5.3 Graph each of the following systems of inequalities. Shade the solution region and label any discontinuities that exist.

a. $\begin{cases} y > \dfrac{x^3}{10} \\ y \leq \dfrac{x^2 + 4x - 5}{x - 1} \end{cases}$

b. $\begin{cases} y > \dfrac{15}{(x - 1)(x + 4)} \\ y \leq 0.5x^2 + 1.5x - 5 \end{cases}$

* * * * *

5.4 Graph the solution set for each of the following inequalities. Label any discontinuities that exist.

a. $y > \dfrac{x^5 - 8x^4 + 20x^3 - 25x^2 + 39x - 27}{(x - 1)(x - 3)}$

b. $y \leq x^4 - 3x^3 - 2x^2 - 9x - 8$

c. $y < \dfrac{3x^4 + 3x^3 - 25x - 25}{x + 1}$

5.5 Graph the following system of inequalities. Shade the solution region and label any discontinuities that exist.

$$\begin{cases} y \geq \dfrac{3x^3 - 27}{5} \\ y \leq \dfrac{2x^2 + 6x - 21}{x - 5} \end{cases}$$

* * * * * * * * * *

Summary Assessment

In this module, you have simulated the development of a home permanent and its packaging. However, you have not yet considered the financial aspects of the business. Following the completion of the research and development phases of the project, your company has determined that an additional investment of $65,000 will be required to bring the product to market.

Using some of its real estate as collateral, the company plans to borrow the money from a bank. The board of directors would like to repay the loan in 20 years, with maximum monthly payments of $565. One formula used by banks to calculate the size of monthly loan payments is:

$$P = \frac{Ar(1+r)^n}{(1+r)^n - 1}$$

where P is the size of the monthly payment in dollars, A is the amount borrowed, r is the monthly interest rate written as a decimal, and n is the total number of monthly payments.

1. Use the formula shown above to write a function that describes payment size in terms of the monthly interest rate. Identify the domain of this function.

2. Monthly interest rates are usually calculated by dividing an annual percentage rate (APR) by 12. If the APR does not exceed 24%, what is the maximum monthly rate?

3. What portion of the domain of the function in Problem **1** applies to this setting?

4. Are there any discontinuities in the graph of the function in Problem **1**? If so, how do they affect the company's situation?

5. Determine the interval of monthly interest rates (to the nearest 0.01) that will allow the company to meet its repayment goals.

6. What is the highest annual percentage rate that will allow the company to meet its repayment goals?

Module Summary

- A **rational function** is in the following form, where $n(x)$ and $d(x)$ are polynomial expressions, and $d(x) \neq 0$:

  $$f(x) = \frac{n(x)}{d(x)}$$

  The domain of $f(x)$ does not contain values of x for which $d(x)$ equals 0.

- Two functions $f(x)$ and $g(x)$ are **equivalent** if and only if the domain of $f(x)$ is the same as the domain of $g(x)$ and $f(x) = g(x)$ for all values of x in the domain.

- A function is **continuous** at a point c in its domain if the following conditions are met:
  - the function is defined at c, or $f(c)$ exists
  - the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
  - the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$

- A function is **continuous** over its domain if it is continuous at each point in its domain.

- A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.

- An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line approaches 0 as the distance from point P to the origin increases without bound.

- An **oblique asymptote** is an asymptote that is neither horizontal nor vertical.

- A rational function $f(x)$ can be rewritten in the following form by dividing the numerator by the denominator:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

  where $Q(x)$ is the quotient and $R(x)$ is the remainder. In this form, $Q(x)$, $R(x)$, $n(x)$, and $d(x)$ are polynomial functions. When the degree of $R(x)$ is less than that of $d(x)$, the value of $R(x)/d(x)$ approaches 0 as $|x|$ becomes very large. As a result, its effect on a graph of $f(x)$ can be ignored, producing three general cases.

  - If $Q(x)$ equals some constant a, the graph of $f(x)$ has a horizontal asymptote at $y = a$.
  - If $Q(x)$ is a linear function in the form $y = mx + b$ with $m \neq 0$, the graph of $f(x)$ has an oblique asymptote described by $y = mx + b$.
  - If $Q(x)$ is a polynomial function of degree 2 or greater, $f(x)$ has no horizontal or oblique asymptotes.

- When graphing inequalities that involve rational functions, a boundary function may contain discontinuities. In such cases, the graph should indicate those portions that are not included in the feasible region as a result of discontinuities.

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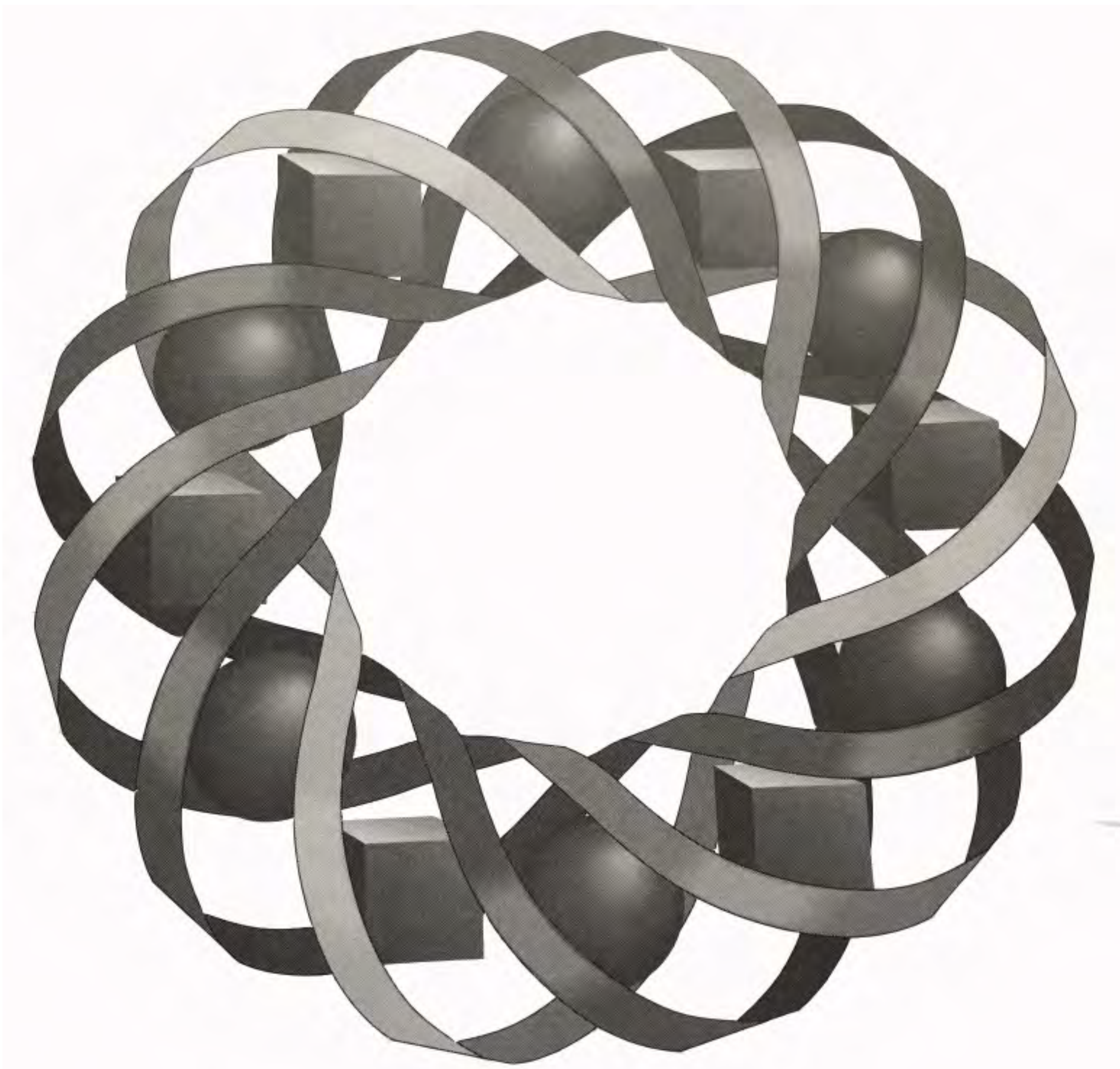
Believe It or Not



How can you convince yourself—and others—that a statement is true or false? In this module, you'll explore some useful tools for developing a reasoned argument.

Terri Dahl • Satinee Lightbourne • Steve A. Yockim

Believe It or Not

Introduction

Children experience the burden of proof early in life. On the playground, friends challenge each other with cries of "Prove it!" In the classroom, teachers require students to support their responses with learned knowledge. Before granting adult privileges, many parents want proof of their teenagers' maturity and responsibility.

The ability to think and reason logically is an important asset in every occupation. Corporate executives, airplane mechanics, office managers, and emergency room nurses must all make decisions based on known facts and sensible assumptions. In this module, you examine the tools for developing convincing arguments.

Activity 1

Convincing arguments are especially vital in a court of law. Under the U.S. judicial system, the accused is presumed innocent until proven guilty. Before making an arrest, detectives examine the evidence to identify the suspect most likely to have committed the crime. In the courtroom, attorneys on both sides use logical arguments to convince the jury of a defendant's innocence or guilt based on certain facts. In this activity, you use logical arguments to defend your own conclusions in "The Murder of Sam Barone."

Exploration

You have been hired as an attorney in a murder case. The following story, "The Murder of Sam Barone," contains some evidence gathered about the crime. Assume that the guilty person is the one who owns the murder weapon and has sufficient motive for the crime. All the statements made by the people involved are true.

a. After reading the story, select the person who you think is guilty.

b. Organize the evidence in the story that supports your verdict.

c. Prepare an argument for presentation to a jury.

“The Murder of Sam Barone”

Sam Barone lived in a residential hotel at 1314 East Labor Street, Hillside, Illinois. A hotel employee found Sam’s body in his room on Sunday, January 4, 1997, at 8:00 P.M. The investigative report revealed that the murder weapon was a rare Egyptian sword purchased that day at an antique shop in Hillside. The estimated time of death was about 7:00 P.M. on the day the body was found.

Police eventually arrested three suspects: two men and one woman. The suspects were Bill and Hank Barone, the brothers of the dead man, and Paula Stewart, a friend of the Barones. Sam’s estranged wife testified that both Bill and Hank hated anybody who owed them money. This hatred was sufficient to motivate either one of them to kill Sam. She also testified that Sam never borrowed money from anyone except a brother. It was a family code.

The following is a transcript of the conversation between police, the three suspects, and the owner of the antique shop.

Bill:	Sure I hated Sam, but it doesn’t mean I killed him.
Hank:	Sam owed me $10,000, but that isn’t reason to kill a man.
Paula:	You seem to be assuming that one of us is guilty. There were plenty of other people in the area on the day of the murder.
Hank:	As for that Egyptian sword, if my name was on the bill of sale, I own it. If you’d look, you’d see my name is not on the bill of sale.
Shopkeeper:	I could never tell those two brothers apart, but I know this: if it ain’t Bill that owns the sword, then it’s Hank. If it ain’t Hank, then it’s Bill. I know I sold it to one of them.
Paula:	The owner of the sword must know a great deal about Egyptian history. Without realizing its true worth, nobody would pay so much money for it.
Bill:	I wrote a book on Egyptian history.
Hank:	I’m sure that the book is accurate, because I personally verified much of the information in it.
Shopkeeper:	The sword was purchased on January 4, 1997, so obviously the owner was in my shop in Hillside on that day.
Hank:	I was in Paris, France, on January 4, 1997, and it would take over 24 hours to reach Hillside from Paris.
Bill:	I was in Aspen, Colorado, skiing on January 2, 1997. Because of a big snowstorm there, nobody in Aspen could leave town for at least three days before or after January 2.

Discussion

a. Present the argument you developed in the exploration to the class.

b. Was each argument believable? Why or why not?

c. Is it possible for any of the statements made by the people involved to be both true and false?

d. Was it necessary to make any invalid or untrue assumptions to support your conclusions?

e. Were any statements reworded to support a conclusion? Did this rewording change the meaning of the statement?

Mathematics Note

In mathematics, a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.

A **conditional statement** is one that can be written in **if-then form**. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "if p, then q," or by $p \rightarrow q$ (read "p implies q").

For example, consider the conditional statement "If an animal is a German shepherd, then the animal is a dog." In this case, the hypothesis is "an animal is a German shepherd." The conclusion is "the animal is a dog."

Conditionals are sometimes illustrated using Venn diagrams. In Figure **1**, the outer circle of the Venn diagram represents the conclusion, while the inner circle represents the hypothesis. If an animal can be placed within the inner circle (a German shepherd), then it is also included within the outer circle (a dog).

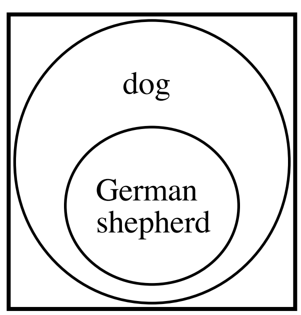


Figure 1: Venn diagram of a conditional statement

f.
1. Using the example given in the mathematics note, describe the information given in the hypothesis of a conditional statement.
2. Describe the information given in the conclusion of a conditional statement.

g. **1.** Write an if-then statement that describes the Venn diagram in Figure **2**.

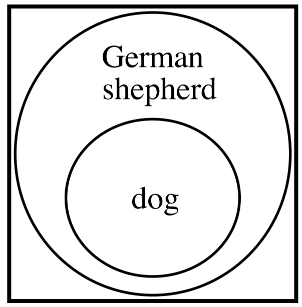


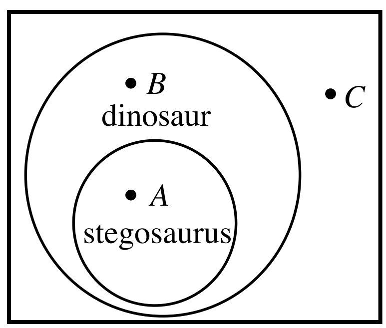
Figure 2: Venn diagram of another conditional

2. Is this conditional statement true or false?

Assignment

1.1 Use the Venn diagram below to complete Parts **a–d**.

- **a.** What can you conclude about an animal at point *A*?
- **b.** What can you conclude about an animal at point *B*?
- **c.** What can you conclude about an animal at point *C*?
- **d.** Write a conditional that is represented by this Venn diagram.



1.2 Draw a Venn diagram that represents the statement, "If students are in art class, then they draw pictures." Place the names of the students mentioned in Parts **a–d** in the appropriate locations in your diagram. In each case, explain why you chose a particular location.

- **a.** Travis is in art class.
- **b.** Natalie draws pictures.
- **c.** Sydney does not draw pictures.
- **d.** C.T. is not in art class.

1.3 Rewrite each of the statements below in if-then form. Underline the hypothesis and place parentheses around the conclusion.

a. All whales are mammals.

b. A triangle is a polygon.

c. In order to be guilty, Hank had to be at the scene of the crime.

d. Bill was not in Hillside if he was snowbound at Aspen.

1.4 Find four statements in "The Murder of Sam Barone" that are not in if-then form. Rewrite each one in if-then form. Underline the hypothesis and place parentheses around the conclusion.

1.5 Draw a Venn diagram for each of the statements in Problem **1.4**.

* * * * *

1.6 Draw a Venn diagram for each of the statements below.

a. Every square is a quadrilateral.

b. All Irish setters are dogs.

c. The number 5 is an integer.

1.7 Rewrite each of the following statements in if-then form.

a. Isosceles triangles have two congruent sides.

b. The square of 3 is 9.

c. All people living in Hawaii live in the United States.

* * * * * * * * * *

Activity 2

Rewriting a conditional statement in if-then form does not always allow us to decide whether the statement is true or false. For example, consider the following conditional statement: "Students who have summer jobs will be able to go to the movies." This statement can be rewritten in if-then form as: "If a student has a summer job, then that student will be able to go to the movies."

Is a conditional statement true if its hypothesis is true? Is the statement false if its hypothesis is false? In this activity, you answer these questions by exploring some mathematical logic.

Exploration

One way to determine the truth value of a conditional statement is to treat it like a promise. For example, consider the conditional "If you study, then you will pass math." This statement can be rewritten as, "I promise you will pass math, if you study." If the promise is kept, or you have no valid reason to complain about the result, then the conditional is considered true. Otherwise, the conditional is false.

a. Write the following sentence as a promise: "If you finish this exploration, then you have permission to go to the dance."

b. In this situation, there are four possible cases to consider. Determine the truth value of each one and record the results in Table **1** below.

Table 1: Record of promises kept

You finished the exploration	You have permission to go to the dance	Is the promise kept? (truth value)
yes	yes	
yes	no	
no	yes	
no	no	

c. Describe any case that leads to a broken promise.

d. The four cases you examined in Part **b** are the same for all conditional statements. As shown in Table **2**, the hypothesis and conclusion of a conditional are evaluated using "true" or "false" instead of "yes" or "no." Use this table to determine the truth value of each case.

Table 2: Truth table for $p \to q$

Hypothesis (p)	Conclusion (q)	Conditional ($p \to q$)
true	true	
true	false	
false	true	
false	false	

e. Describe any case that gives a false conditional.

Discussion

a. Consider the case in Table **1** in which you did not finish the exploration and you had permission to go to the dance. In this case, why is the truth value "true"?

b. Describe the case(s) in which a conditional is a true statement.

c. Use Table **2** to summarize the possible truth values of a conditional.

Mathematics Note

A conditional is false only if its hypothesis is true and its conclusion is false. In all other cases, a conditional is true.

The **negation** of a statement *p* is the statement "It is not the case that *p*" or simply "not *p*." Symbolically, this is written as ~*p*.

In "The Murder of Sam Barone," for example, Hank said, "My name is not on the bill of sale." The negation of this statement is "My name is on the bill of sale."

A statement and its negation have opposite truth values. For example, when statement *p* is true, its negation, not *p*, is false.

d. Consider the conditional, "If you finish this exploration, then you have permission to go to the dance."

1. What is the negation of the hypothesis?
2. What is the negation of the conclusion?

e. Given a statement *p*, what is the truth value of ~(~*p*)?

f. Consider the conditional, "If it is raining, then it is cloudy." Based on this statement, on how many rainy days would you expect it to be cloudy?

Mathematics Note

Quantifiers are words or phrases that indicate the quantity of a particular subject referred to in a statement.

Existential quantifiers provide for the existence of at least one case. For example, *some*, *at least one*, *exactly one*, and *there exists* are existential quantifiers.

Universal quantifiers demand that all cases be considered. For example, *all*, *every*, and *none* are universal quantifiers.

Mathematical statements often contain quantifiers. For example, the following statement uses an existential quantifier: "For some real number x, $2x^2 + 7 = 10$." The statement, "Every rhombus is a parallelogram," uses a universal quantifier.

g. The quantifiers in conditionals are not always stated explicitly. In some cases, the quantifier is implied. What do you think is the implied quantifier in the statement, "A square is a rectangle"?

Assignment

2.1 In each of the following statements, a quantifier is implied. Rewrite each statement using the appropriate quantifier.

a. A mammal is an animal.

b. This person has red hair.

2.2 If possible, write a conditional with a false hypothesis so that:

a. the conditional is true

b. the conditional is false.

2.3 If possible, write a conditional with a true hypothesis so that:

a. the conditional is true

b. the conditional is false.

2.4 Does the negation of both the hypothesis and the conclusion make a true conditional false? Complete the following table, then use the results to justify your response.

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim p \to \sim q$
true	true				
true	false				
false	true				
false	false				

2.5 Under what conditions would the negation of the hypothesis of a false conditional make the conditional true?

* * * * *

2.6 In each of the following statements, identify the quantifier and explain its meaning.

a. Every rhombus is a parallelogram.

b. A possible solution to the equation $x^2 + 3x - 10 = 0$ is $x = -5$.

2.7 Let *p* represent the statement "the sky is not cloudy" and *q* represent the statement "the sun can be seen during the day."

a. Represent each of the following statements symbolically.

1. The sky is cloudy.
2. The sun cannot be seen during the day.
3. If the sky is not cloudy, then the sun can be seen during the day.
4. If the sun cannot be seen during the day, then the sky is cloudy.
5. If the sun can be seen during the day, then the sky is not cloudy.

b. Create a truth table for one of the conditionals in Part **a**.

* * * * * * * * * *

Activity 3

In this activity, you examine three forms of conditional statements: the **converse**, the **inverse**, and the **contrapositive**.

One of your goals will be to determine when a change in form changes the meaning of a statement, and when it does not. For example, consider the following excerpt from Lewis Carroll's *Alice in Wonderland*:

> "You should say what you mean," the March Hare went on.
>
> "I do," Alice hastily replied; "At least—at least I mean what I say—that's the same thing, you know."
>
> "Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
>
> "You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"
>
> "You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"
>
> "It is the same thing with you," said the Hatter, and here the conversation dropped, and the party sat silent for a minute.

Exploration

When Alice, the Hatter, the March Hare, and the Dormouse changed the order of the clauses in their sentences, they also changed the meaning of their statements. In this exploration, you examine how the truth value of a statement changes when the words are rearranged.

a. Write this statement in if-then form: "It is cloudy when it is raining."

b. Complete a truth table like the one in Table **3** for the conditional statement in Part **a**.

Table 3: Truth table for a conditional

Hypothesis	Conclusion	Statement

c. The **converse** of a conditional statement is formed by interchanging its hypothesis and its conclusion. Given the conditional "if p, then q," the converse is, "if q, then p." Symbolically, the converse of the conditional $p \rightarrow q$ is written $q \rightarrow p$.

1. Write the converse of the conditional in Part **a**.

2. Construct a truth table for the converse.

d. The **inverse** of a conditional statement is formed by negating its hypothesis and its conclusion. Given the conditional "if p, then q," the inverse is, "if not p, then not q." Symbolically, the inverse of the conditional $p \to q$ is written as $\sim p \to \sim q$.

1. Write the inverse of the conditional in Part **a**.
2. Construct a truth table for the inverse.

e. The **contrapositive** of a conditional is formed by interchanging its hypothesis and its conclusion and negating both of them. Given the conditional "if p, then q," the contrapositive is, "if not q, then not p." Symbolically, the contrapositive of the conditional $p \to q$ is written as $\sim q \to \sim p$.

1. Write the contrapositive of the conditional in Part **a**.
2. Construct a truth table for the contrapositive.

f. Compare the truth tables you created in Parts **b–e**. What appears to be the relationship among the truth values of this conditional and its converse, inverse, and contrapositive?

Discussion

a. Describe how rewriting the original statement in the exploration changed its truth values.

b. What is the contrapositive of the inverse of a conditional statement? Justify your response.

Mathematics Note

Two statements are **logically equivalent** if one statement is true (or false) exactly when the other statement is true (or false).

For example, Table **4** shows a truth table for the conditional statements $p \to q$ and $\sim q \to \sim p$. Since the two statements have exactly the same truth values, they are logically equivalent.

Table 4: Truth tables for two logically equivalent statements

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim q \to \sim p$
true	true	true	false	false	true
true	false	false	false	true	false
false	true	true	true	false	true
false	false	true	true	true	true

c. Using the truth tables you created in the exploration, which forms of a conditional statement appear to be logically equivalent? Justify your response.

d. In mathematics, a definition is a statement that has a logically equivalent converse. Considering this fact, what appears to be the relationship among a definition and its converse, inverse, and contrapositive? Defend your response.

Assignment

3.1 Choose a television, magazine, or newspaper advertisement that contains a product slogan.

a. Write the product slogan in if-then form and explain what it implies. Underline the hypothesis and put parentheses around the conclusion.

b. Draw a Venn diagram of the conditional.

c. Write the converse of the conditional.

d. Write the inverse of the conditional.

e. Write the contrapositive of the conditional.

f. Which form of the slogan do you think is better for selling the product: the conditional or the contrapositive? Justify your choice.

3.2 **a.** Write the following statement in if-then form, "An equilateral triangle is an isosceles triangle."

b. Complete truth tables for the conditional in Part **a** and its converse, inverse, and contrapositive.

c. Using the truth tables you completed in Part **b**, determine which of the following pairs of statements are logically equivalent.

1. the conditional and the converse
2. the conditional and the contrapositive
3. the converse and the contrapositive
4. the converse and the inverse
5. the conditional and the inverse
6. the inverse and the contrapositive

d. Summarize your findings from Part **c**.

3.3 Write a logically equivalent statement for each of the following statements.

a. If you do not live in the United States, then you do not live in New York.

b. A triangle is a polygon.

Mathematics Note

When a conditional and its converse are both true, they can be written as a single statement using the words **if and only if**. Symbolically, this can be represented as $p \leftrightarrow q$, or p iff q (read "p if and only if q"). All definitions may be written as statements using *if and only if*.

For example, consider the conditional statement for the definition of supplementary angles: "If two angles are supplementary, then the sum of their measures is 180°." The converse of this statement is: "If the sum of the measures of two angles is 180°, then the angles are supplementary." Since both statements are true, they can be combined in the following definition: "Two angles are supplementary if and only if the sum of their measures equals 180°."

3.4 Write a statement using *if and only if* for each definition below.

a. A googol is the numeral 1 followed by one hundred 0s.

b. A regular triangle is an equilateral and equiangular triangle.

* * * * *

3.5 Consider the statement "The number π is an irrational number."

a. **1.** Write this statement in if-then form.

2. Write the converse of the conditional.

3. Write the inverse of the conditional.

4. Write the contrapositive of the conditional.

b. Determine which of the statements in Part **a** are logically equivalent.

3.6 If possible, write a statement using *if and only if* for each of the following conditionals.

a. If a quadrilateral is a rectangle, then it is a parallelogram.

b. If an animal is a canine, then it is a dog.

c. If $a = b$, then $b = a$.

* * * * * * * * * *

Activity 4

Photographs, videotapes, and computer simulations are often admitted as evidence in court. Suppose that the prosecuting attorney in "The Murder of Sam Barone" discovers a picture of one suspect at the crime scene, holding the murder weapon. Is this sufficient evidence to convict that person? Is the fact that a person does not appear in the picture enough to show that the person is innocent?

Exploration

In this exploration, you examine the relationships that exist among the angles formed by lines in the same plane.

a. Sketch all the possible situations that occur when two distinct lines lie in the same plane.

b. Make a conjecture about any of the relationships between the angles formed in your sketches.

c. Use appropriate technology to help support your conjecture.

d. Write generalizations in if-then form about the relationships you found in Part **b**.

e. Repeat Parts **a–d** for three distinct lines in the same plane.

Discussion

a. Compare the relationships you discovered among the angles in the exploration with those observed by your classmates.

b. Describe how your sketches helped you make conjectures.

c. Are your sketches sufficient to convince yourself that these conjectures are true?

d. Does your use of technology prove the conjectures made in Part **b** of the exploration? Explain your response.

Mathematics Note

To create convincing arguments, mathematicians use a method based on logical reasoning. This process is referred to as **proof**. Although diagrams and pictures may be used to support arguments, a conclusion based solely on observation may or may not be true. The method of examining *all* possibilities to prove a statement is **proof by exhaustion**.

For example, consider the statement, “If an integer is between 89 and 97, then it is not prime.” To prove this statement by exhaustion, you must examine the factors of the following integers: 90, 91, 92, 93, 94, 95, and 96. Since each of these integers is divisible by a number other than 1 and itself, each is not prime. Thus, the statement is proven true.

Assignment

4.1 Mandy has a box of 100 marbles. She draws a single green marble from the box and sets it aside. She repeats the process 30 times, each time drawing a green marble.

a. Are you convinced that the next marble Mandy draws will be green?

b. Write a conjecture about the remaining marbles in the box.

c. How could you prove the conjecture in Part **b**?

4.2 Using proof by exhaustion, could you prove that vertical angles formed by two intersecting lines are congruent? Explain your response.

4.3 Consider the following statement: "If I count to 1,000,000 without stopping, then it will take more than a week."

a. Write a convincing argument to support or refute this statement.

b. Could this statement be proved by exhaustion? Explain your response.

c. Would it be practical to prove this statement by exhaustion? Explain your response.

4.4. On a 12-hour analog clock, addition can be performed by "counting on." For example, the sum of 8 and 5 on a clock can be found by starting at 8 o'clock and counting on 5 hours to 1 o'clock.

Recall that the sum of a number n and the additive identity is n. Using proof by exhaustion, prove that the additive identity for clock addition is 12. In other words, show for the numbers n, where $n \in \{1,\ 2,\ 3,\ \ldots,\ 12\}$, that $12 + n = n + 12 = n$.

Mathematics Note

A **counterexample** illustrates a contradiction to a statement. A statement can be proven false by finding only one counterexample.

For example, consider the following conditional: "If x is a real number, then $\sqrt{x^2} = x$." This statement can be proven false using the counterexample $x = -2$ because $\sqrt{(-2)^2} = \sqrt{4} = 2$ and $2 \neq -2$.

4.5 Consider the situation described in Problem **4.1**. What would it take to prove that the statement, "All the marbles in the box are green," is false?

4.6 Spencer examined 10 quadrilaterals, each with four right angles. He observed that all the quadrilaterals were squares. After his observations, he wrote, "I have proven the statement, 'If a quadrilateral has four right angles, then it is a square.' "

a. Explain to Spencer why his method of proof is not correct.

b. Give a counterexample that proves Spencer's statement is false.

c. Suggest a true statement that Spencer might have made.

d. Do Spencer's observations prove the statement you made in Part **c**? Explain your response.

4.7 Find counterexamples in "The Murder of Sam Barone" to prove that each of the following statements is not true.

a. Paula owned the murder weapon.

b. Bill liked Sam.

c. Bill was in Hillside on January 4, 1997.

* * * * *

4.8 The covers of four books are each printed a different color: red, yellow, green, or blue. Prove that if these four books are arranged randomly on a shelf, the green book and the yellow book will be next to each other 50% of the time.

4.9 Identify a counterexample to prove that each of the following statements is false.

a. The sum of an odd number and an even number is always even.

b. Angles with the same measure are vertical angles.

c. If the sidewalk is wet, then it is raining.

* * * * * * * * * *

Activity 5

In Activity **4**, you learned that observations alone are not sufficient to prove a statement true unless you show that every case is true. "The Murder of Sam Barone" contains more than just observations, however. The story also introduces some known facts as evidence. Through logical reasoning, these facts can be used to establish the validity of the statements made by each suspect.

Mathematics Note

Deductive reasoning uses a logical sequence of valid arguments to reach a conclusion. In mathematics, these arguments are often written as a series of if-then statements, each supported by some justification, that yield another valid argument or a conclusion.

For example, suppose that you wanted to prove the statement: "If a triangle is an equilateral triangle, then it is an isosceles triangle." This statement can be proved using the following series of if-then statements.

- If a triangle is equilateral, then it has three equal sides.
- If a triangle has three equal sides, then it has at least two equal sides.
- If a triangle has at least two equal sides, then it is an isosceles triangle.
- In conclusion, if a triangle is an equilateral triangle, then it is an isosceles triangle.

This example illustrates the **transitivity** of if-then statements. Using transitivity, the true statements "if p, then q" and "if q, then r" may be used to form the valid conclusion: "if p, then r."

Exploration

Using the testimony of the characters in "The Murder of Sam Barone," investigators have created the following list of true statements.

- If Bill was snowbound, then he couldn't leave Aspen until January 5.
- If Bill didn't buy the sword, then he didn't own the murder weapon.
- If Bill couldn't leave Aspen until January 5, then he couldn't be in Hillside on January 4.
- If Bill didn't own the murder weapon, then he didn't commit the murder.
- If Bill was in Aspen on January 2, 1997, then he was snowbound.
- If Bill couldn't be in Hillside on January 4, then he couldn't buy the sword.

In the order given, these sentences do not make a convincing argument of Bill's innocence. In the following exploration, you use transitivity to develop a logical sequence for these statements.

a. Cut a copy of the list of statements into six separate slips of paper.

b. Rearrange the sentences so that they lead logically to the following statement: "If Bill was in Aspen on January 2, 1997, then he didn't commit the murder."

Discussion

a. Describe how you determined an order for the statements in Part **b** of the exploration.

b. Why is transitivity important in deductive reasoning?

c. Describe a series of if-then statements that could be used to prove the following conditional: "If $2x - 3 = 5x + 6$, then $x = -3$."

Mathematics Note

A series of if-then statements is only one way of presenting a mathematical proof. For example, the following paragraph is a proof of the conditional, "If *n* is an even integer, then n^2 is an even integer."

> The hypothesis is that *n* is an even integer. By the definition of an even integer, $n = 2b$, where *b* is an integer. When both sides of the equation $n = 2b$ are squared, the result is $n^2 = (2b)^2 = 4b^2 = 2(2b^2)$. By the definition of an even integer, n^2 is an even integer. Therefore, if *n* is an even integer, then n^2 is an even integer.

Notice that each step in this argument includes its own justification. In a mathematical proof, these justifications may consist of definitions, **axioms** or **postulates** (statements accepted as true), or **theorems** (statements that have previously been proven true).

Since it makes direct use of the hypothesis to arrive at the conclusion, the example given above is a **direct proof**. However, not all statements can be proven directly. **Indirect proofs** begin by assuming that the original statement is false. From this assumption, valid arguments are followed until a contradiction to a known fact is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

For example, the conditional "If n^2 is an odd integer, then *n* is an odd integer" cannot be proven using a direct proof. It can, however, be proven indirectly, as shown in the following paragraph.

> Assume that the statement "If n^2 is an odd integer, then *n* is an odd integer" is false. If this conditional is false, then "n^2 is an odd integer" is true and "*n* is an odd integer" is false. If "*n* is an odd integer" is false (and *n* is an integer), then *n* must be an even integer. By the definition of an even integer, $n = 2a$, where *a* is an integer. Squaring both sides of $n = 2a$ results in $n^2 = (2a)^2 = 4a^2 = 2(2a)^2$. By the definition of an even integer, n^2 is an even integer. Since this contradicts the known fact that n^2 is an odd integer, the assumption is false. Therefore, the original statement is true: "If n^2 is an odd integer, then *n* is an odd integer."

d. Why can't the statement, "If n^2 is an odd integer, then n is an odd integer," be proven directly?

e. Consider the conditional "If n is an even integer, then n^2 is an even integer." In an indirect proof of this conditional, what would be your first step?

Assignment

5.1 Prove each of the following conclusions to "The Murder of Sam Barone" using a series of if-then statements.

a. Hank owned the murder weapon.

b. Hank was guilty.

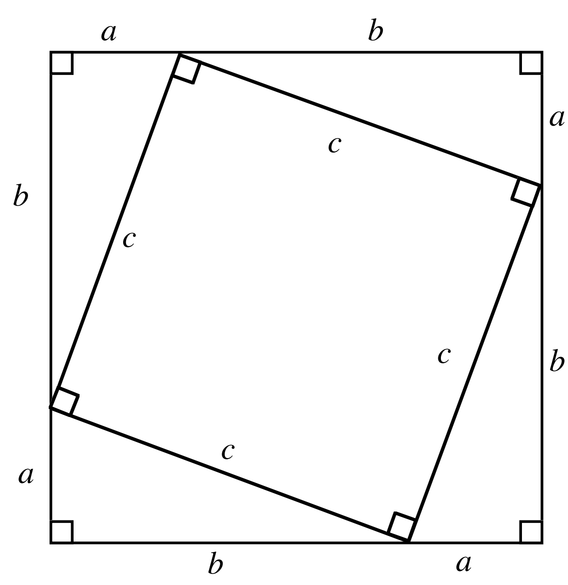
5.2 Prove each of the following using if-then statements. Include justification for each step in your argument.

a. The solution to $5(x-2) = -20$ is $x = -2$.

b. The measure of an angle of $\pi/3$ radians is $60°$.

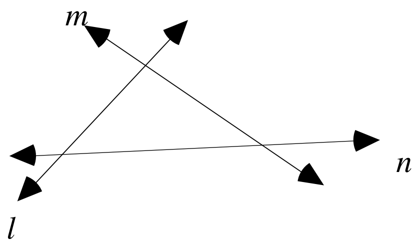
5.3 Changing a conditional to its contrapositive can sometimes make proving a statement easier. Change the following statement to its contrapositive, then prove it: "If $x^2 \neq 4$, then $x \neq 2$."

5.4 The area of a region can be expressed as the sum of the areas of its non-overlapping parts. Use this fact, along with the diagram below, to prove the Pythagorean theorem.



5.5 Use an indirect proof to prove each of the following statements.

a. If lines l, m, and n intersect in three different points, then it is not possible for both l and m to be perpendicular to n.



b. A triangle cannot have three interior angles each with a measure of 70°.

5.6 Using the "The Murder of Sam Barone," prove the following statement indirectly: "If Bill did not own the murder weapon, then he was innocent."

* * * * *

5.7 Prove each of the following using a series of if-then statements. Include justification when necessary.

a. Hank had a motive to murder Sam Barone.

b. All equilateral triangles are isosceles.

5.8 Prove that the supplements of angles with equal measure are congruent.

5.9 Prove the following statement indirectly: "If x and y are integers and $3x + 12y = 450$, then x is an even integer."

* * * * * * * * * *

Summary Assessment

1. In the game "Prove It," players match a secret four-digit number by repeatedly guessing four-digit numbers and adjusting each successive guess based on the clues they receive. The clues are "hot" and "warm." Each hot clue indicates that a digit in the guess is correct and in the appropriate position. Each warm clue indicates that a digit in the guess is correct but improperly positioned.

The table below shows the first seven guesses in one game of "Prove It." Determine the secret four-digit number in this game and explain the logic you used to find it.

Guess	Hot clues	Warm clues
0123	1	1
1234	1	1
2345	0	1
3456	0	0
7899	1	1
8999	1	0
9999	1	0

2. The definition of **divisibility** states that, "Given integers a and b, if b divides a (written $b|a$), then there is an integer c such that $a = b \bullet c$." If $3|15$, for example, then there is an integer 5 such that $15 = 3 \bullet 5$.

Using this definition, classify each of the following statements as true or false. If the statement is true, prove it. If the statement is false, provide a counterexample. In all cases, m and n are integers.

a. If $3|m$, then $3|nm$.

b. If $3|(m+n)$, then $3|m$.

c. For all integers m, $1|m$.

d. If $d|m^2$, then $d|m$.

3. Prove the following statement: "If a figure is a triangle, then it does not have two angles each with a measure of 90°."

4. Prove the following statement: "The equation $2(3x-5)=6x+3$ is never true."

Module Summary

- In mathematics, a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.
- A **conditional statement** is one that can be written in **if-then form**. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "If p, then q," or by $p \rightarrow q$ (read "p implies q").
- A conditional is **false** only if its hypothesis is true and its conclusion is false. In all other cases, a conditional is true.
- The **negation** of a statement p is the statement "It is not the case that p" or simply "not p." Symbolically, this is written as $\sim p$. A statement and its negation have opposite truth values.
- **Quantifiers** are words or phrases that indicate the quantity of a particular subject referred to in a statement.
- **Existential quantifiers** provide for the existence of at least one case. For example, *some*, *at least one*, *exactly one*, and *there exists* are existential quantifiers.
- **Universal quantifiers** demand that all cases be considered. For example, *all*, *every*, and *none* are universal quantifiers.
- The **converse** of a conditional statement is formed by interchanging its hypothesis and its conclusion. Given the conditional statement "if p, then q," the converse of the statement is, "if q, then p." This can be represented symbolically as $q \rightarrow p$.
- The **inverse** of a conditional statement is formed by negating its hypothesis and its conclusion. Given the conditional "if p, then q," the inverse of the statement is, "if not p, then not q." This can be represented symbolically as $\sim p \rightarrow \sim q$.
- The **contrapositive** of a conditional is formed by interchanging its hypothesis and its conclusion and negating both of them. Given the conditional "if p, then q," the contrapositive of the statement is, "if not q, then not p." This can be represented symbolically as $\sim q \rightarrow \sim p$.
- Two statements are **logically equivalent** if one statement is true (or false) exactly when the other statement is true (or false).

- When a conditional and its converse are both true, they can be written as a single statement using the words *if and only if*. Symbolically, this can be represented as $p \leftrightarrow q$, or p iff q (read "p if and only if q"). All definitions may be written as statements using *if and only if*.
- The method of examining all possibilities to prove a statement is **proof by exhaustion**.
- A **counterexample** illustrates a contradiction to a statement. A statement can be proven false by finding only one counterexample.
- **Deductive reasoning** uses a logical sequence of valid arguments to reach a conclusion.
- Using **transitivity**, the statements "if p, then q" and "if q, then r" may be used to form the valid conclusion: "if p, then r."
- In mathematical proofs, definitions, **axioms** or **postulates** (statements accepted as true), and **theorems** (statements that have previously been proven true) are used to justify arguments.
- A **direct proof** makes direct use of the hypothesis to arrive at a conclusion.
- **Indirect proofs** begin by assuming that the original statement is false. From this assumption, valid arguments are followed until a contradiction to a known fact is reached. Once a contradiction is reached, the assumption must be false. Therefore, the original statement is true.

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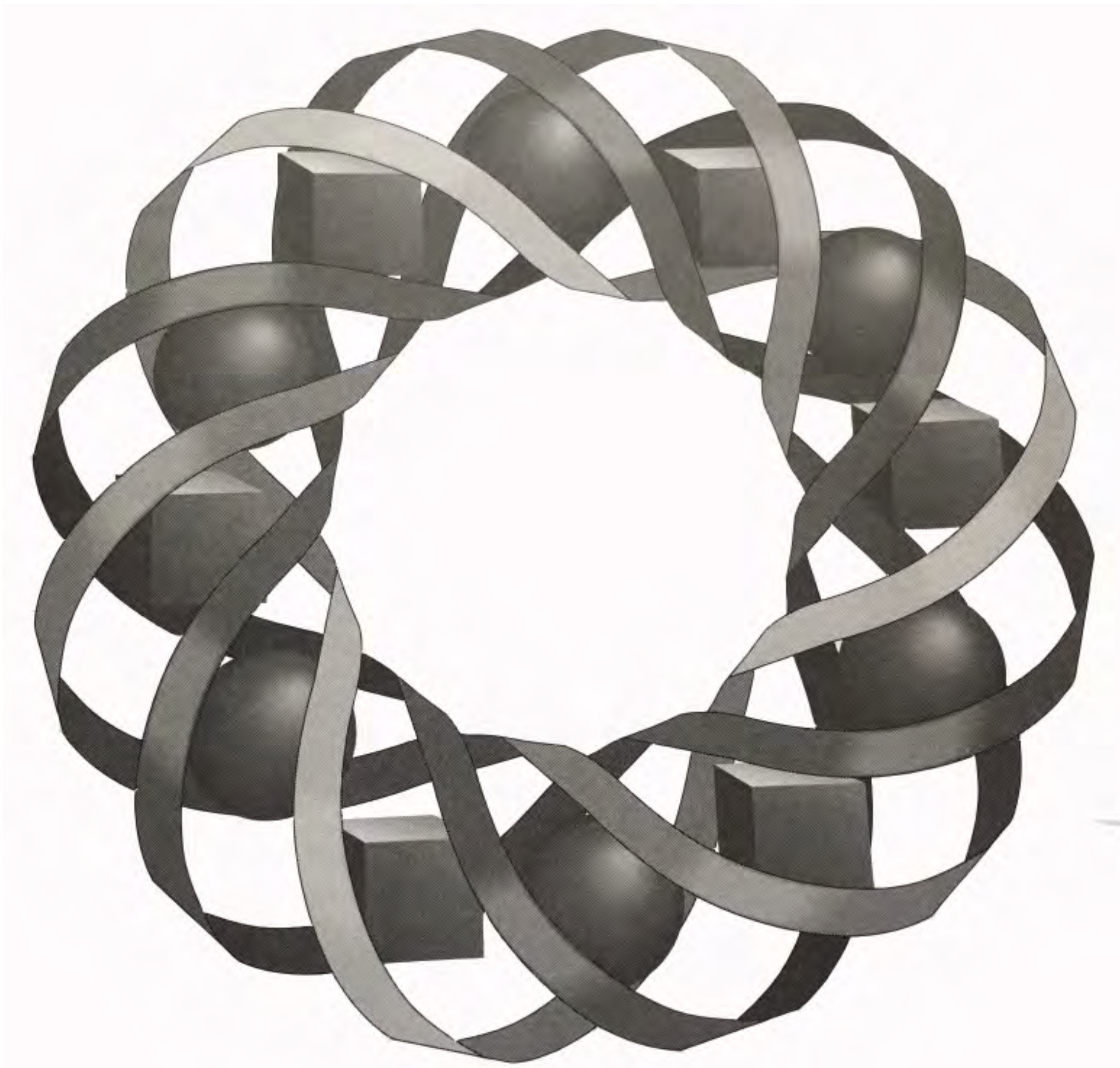
Fly the Big Sky with Vectors



What do swimming across a river, landing a helicopter, and refueling a plane in midair have in common? In this module, you explore how vectors can be used to model these situations and more.

Franklin Lund • Darlene Pugh • Michael Sinclair
Byron Anderson • Janet Kuchenbrod

Fly the Big Sky with Vectors

Introduction

While camping with their parents, Alee and Christopher wander off to fish in a nearby river. They walk downstream from camp for a kilometer or two, cross the river on a bridge, then start walking back upstream, fishing along the way.

When they reach a point on the bank directly opposite their camp, they see their parents sitting down to dinner. Since the afternoon is warm and they are both strong swimmers, they decide to swim across the river. Figure **1** shows a diagram of this situation.

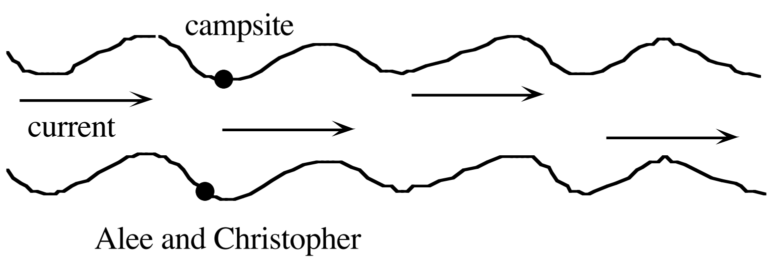


Figure 1: Christopher and Alee on the riverbank

Both Christopher and Alee know that if they try to swim straight to camp, the current will carry them far downstream. In what direction should they swim in order to land near their parents?

Navigators and pilots confront similar predicaments every day. Ocean currents can carry ships off course, while strong winds can force helicopters and planes to stray far from their destinations. In this module, you investigate one method for analyzing such situations.

Exploration

In this exploration, you use a model helicopter to observe how wind affects flight.

a. Using the paper template provided by your teacher, construct a model helicopter like the one shown in Figure **2**.

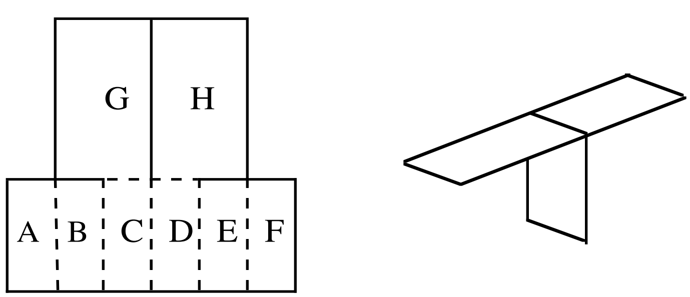


Figure 2: Template and helicopter

b.

1. Place a target on the floor. Drop your helicopter from a height of at least 175 cm. Adjust your position (or that of the target) until the helicopter consistently lands on target.
2. Record your observations of the helicopter's path for three flights, including the horizontal and vertical distance traveled in each flight.

c. Draw a diagram to represent the flight of the helicopter in Part **b**. Use one arrow to indicate the actual path of the helicopter. Use two more arrows to indicate the mean horizontal and vertical distances traveled during the three recorded flights.

d. Use a fan to create a light breeze. Repeat Parts **b** and **c**, recording your observations of the helicopter's flight in windy conditions.

Discussion

a. When you drop the helicopter in Part **b** of the exploration, what forces act on it?

b. If the air were perfectly still, what would you expect the helicopter's path to look like?

c. What causes the helicopter to move in a horizontal direction?

d. How would turning the fan on high speed affect the helicopter's path?

Activity 1

Imagine that you are a passenger on a flight from Butte, Montana, to Salt Lake City, Utah. Midway through the flight, the plane encounters a thunderstorm. To avoid the bad weather, the pilot must change course. What information does the pilot need to dodge the storm, then continue on to Salt Lake City?

An understanding of **vectors** can help you analyze this situation. Vectors can be used to model many types of quantities, including displacement, velocity, and force.

Mathematics Note

A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically represented by a bold, lowercase letter, such as vector **m**. In handwritten work, the same vector can be symbolized by $\vec{m}$. The magnitude of a vector **m** is denoted by $|\mathbf{m}|$.

Displacement is a change in position in a particular direction. For example, a displacement of 5 km to the north can be represented by a vector with a magnitude of 5 km and a direction of north.

The **velocity** of an object is its speed in a specific direction. For example, a car driving to the east at 80 km/hr can be represented by a vector with a magnitude of 80 km/hr and a direction of east.

Force is a push or a pull in a particular direction. One metric unit of force is the newton (N). The force of gravity on a mass of 1 kg, for example, has a magnitude of about 9.8 N and is directed towards the center of the earth.

Exploration

In this exploration, you use vectors to represent flight paths between U.S. cities.

a. On a map provided by your teacher, draw displacement vectors to represent the flights listed below. Each vector should begin at the city in which the flight originates and end at the appropriate destination.

1. Butte, Montana, to Salt Lake City, Utah
2. Butte, Montana, to Seattle, Washington
3. Atlanta, Georgia, to Knoxville, Tennessee
4. Los Angeles, California, to Minneapolis, Minnesota
5. Chicago, Illinois, to Denver, Colorado
6. Tallahassee, Florida, to Atlanta, Georgia
7. Knoxville, Tennessee, to Cincinnati, Ohio
8. Salt Lake City, Utah, to Butte, Montana
9. Tallahassee, Florida, to Cincinnati, Ohio

b. Use appropriate tools to determine the magnitude and direction of each vector in Part **a**.

c. Describe each vector using the measurements you made in Part **b**.

d. Describe a trip from Salt Lake City to Seattle using at least two vectors from Part **a**.

Discussion

a. Why would an instruction such as "fly 3 km" be of little use to a pilot?

b. How do the descriptions of vectors that you wrote in Part **c** of the exploration compare with those of others in your class?

c. A common navigational system defines due north as 0° and measures directions clockwise from north. These measures are referred to as **bearings**. In Figure **3**, for example, vector **m** has a bearing of 100°.

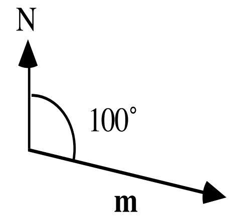


Figure 3: Bearing of 100°

Describe each vector in Part **a** of the exploration with a magnitude and a bearing.

Mathematics Note

Equivalent vectors have the same magnitude and direction. **Opposite vectors** have the same magnitude, but their directions differ by 180°.

In Figure **4**, for example, equivalent vectors **m** and **n** are represented by arrows with the same magnitude and direction. Opposite vectors **p** and **m** are represented by arrows with the same magnitude, but their directions differ by 180°. In this case, the following statements are true: $\mathbf{n} = \mathbf{m}$ and $\mathbf{p} = -\mathbf{m}$.

100 km

Figure 4: Equivalent and opposite vectors

d. Are there any equivalent vectors in your list from Part **a** of the exploration? Explain your response.

e. Are the vectors from Butte to Salt Lake City and from Salt Lake City to Butte opposite vectors? Justify your response.

Mathematics Note

One way to add vectors is the **tip-to-tail method**. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.

The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.

For example, Figure **5** shows the addition of vectors **m** and **n**. Since translation preserves the magnitude and direction of vectors, vector **n** is translated so that its tail coincides with the tip of vector **m**. The resultant vector **r** joins the tail of vector **m** to the tip of vector **n**. This sum can be written as $\mathbf{m} + \mathbf{n} = \mathbf{r}$.

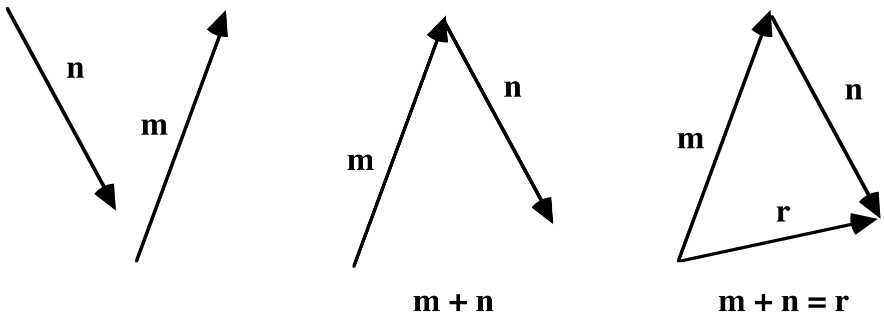


Figure 5: The tip-to-tail method of vector addition

f. Describe how you could use vector addition to represent a trip from Salt Lake City to Seattle.

Mathematics Note

A **scalar** is a real number. For example, 5, –7, π, $\sqrt{3}$, and $3/5$ are all scalars.

The operation $k\mathbf{m}$ denotes the **scalar multiplication** of **m** by the scalar k. As shown in Figure **6**, for example, 5**m** describes a vector in the same direction as **m** with a magnitude 5 times that of **m**. Similarly, –5**m** is a vector 5 times as long as **m**, but in the direction opposite to **m**.

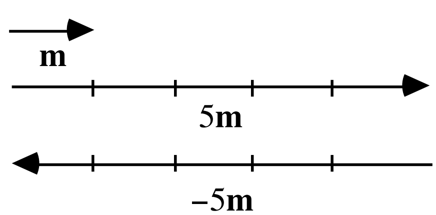


Figure 6: Scalar multiplication

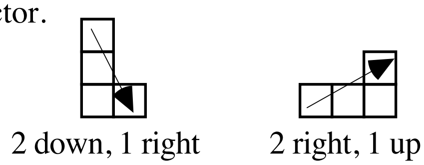
g. How could you use scalar multiplication to describe the vector from Tallahassee to Cincinnati in terms of the vector from Tallahassee to Atlanta?

Assignment

1.1 **a.** Describe three quantities that can be modeled by vectors.

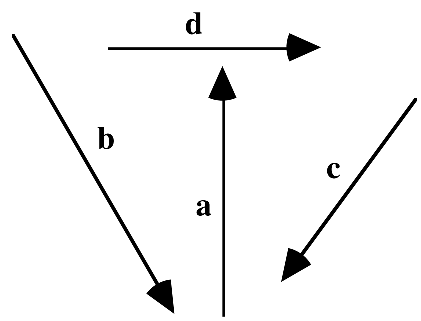
b. Describe three quantities that cannot be modeled using vectors.

1.2 In the game of chess, two players move pieces on a checkerboard according to a set of rules. One of the chess pieces, the knight, can be moved two spaces horizontally then one space vertically, or two spaces vertically then one space horizontally. As shown in the diagram below, a knight's change in position can be modeled by a resultant vector.



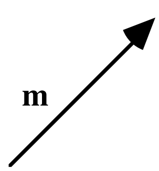
Imagine that a knight is positioned somewhere near the middle of the board. How many different moves are possible? Draw a diagram to support your response, using a resultant vector to represent each possible move.

1.3 **a.** Use the tip-to-tail method to add the vectors below in alphabetical order. Label the resultant vector **e**.



b. Add the vectors in Part **a** in any order you choose. Does changing the order in which vectors are added change the resultant vector?

1.4 The diagram below shows a vector **m**.



a. Using the tip-to-tail method, add **m** to as few other vectors as possible to obtain a resultant that equals each of the following:

1. 3**m**
2. 1**m**
3. –2**m**

b. Describe the vector(s) you added to **m** in each part of Problem **1.4a**.

c. Describe the vector you would add to **m** to obtain a resultant with a magnitude of 0.

1.5 An airplane is flying to the east at 500 km/hr. The plane encounters a wind blowing to the north at 35 km/hr.

a. Draw a vector diagram of this situation.

b. How much is the airplane's velocity affected by the wind?

c. How much is the plane's direction affected by the wind?

1.6 As mentioned in the introduction, Christopher and Alee want to swim across a river to a point directly opposite them on the other bank. The river has a current of 2 km/hr. In still water, Christopher and Alee can swim at a speed of 2.2 km/hr. Sketch a diagram of this situation and use trigonometry to determine the direction in which they should swim.

1.7 Two children are playing with a red wagon. One child pulls on the front of the wagon with a force of 32 N at a bearing of 270˚. The other child pushes on the side of the wagon with a force of 38 N at a bearing of 0˚. What is the resultant force on the wagon and at what bearing is it applied? Use vector addition to support your response.

1.8 Without current, a ferry can cruise at 30 km/hr. While crossing an ocean channel, the ferry encounters a current of 5 km/hr moving at a right angle to its motion.

a. What is the ferry's resultant velocity?

b. How many degrees off its original course does the current push the ferry?

1.9 An airplane is flying 250 km/hr on a bearing of 225˚. The plane encounters a wind blowing at 30 km/hr on a bearing of 180˚. Use a ruler and protractor to create a scale drawing of this situation. Find the magnitude and bearing of the plane's resultant velocity using the tip-to-tail method of vector addition.

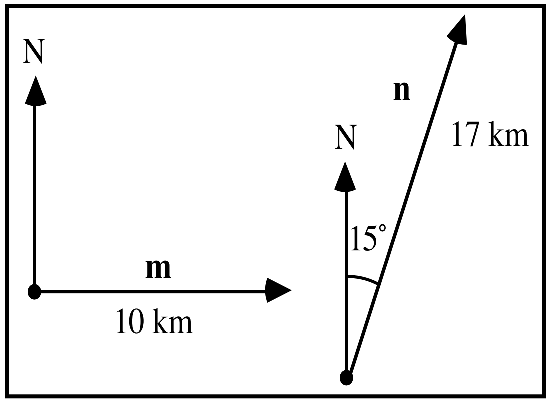
* * * * *

1.10 During its morning hunt, an eagle flies 15 km on a bearing of 55˚. This portion of its flight can be represented by the vector **m**.

a. What would 5**m** represent in this situation?

b. What would –**m** represent in this situation?

1.11 The diagram below shows two displacement vectors. Vector **m** represents a displacement of 10 km due east. Vector **n** represents a displacement of 17 km at a bearing of 15°. Add these two vectors and describe the resultant displacement.



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Activity 2

In Activity **1**, you used scale drawings and the tip-to-tail method to add vectors. In this activity, you discover how the properties of triangles can help you determine the magnitude and bearing of a resultant vector algebraically.

Exploration 1

The Pythagorean theorem states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. As you have seen in previous modules, this relationship can be represented by constructing a square on each side of a right triangle, then comparing the areas of the three squares.

In this exploration, you create a similar model to explore the relationship among these areas when the triangle is not a right triangle.

a.

1. Using a geometry utility, construct a circle with center at point C.
2. Create a moveable point on the circle. Label the point A.
3. Construct $\overline{AC}$. Since point A is on the circle, AC remains constant as A moves around the circle.
4. Construct a right triangle ABC in which $\overline{AB}$ is the hypotenuse.

b. On each side of ΔABC, construct a square whose sides are congruent to the side of the triangle. Create the squares so that when the sides of the triangle are moved, the squares remain squares.

Your construction should now resemble the one shown in Figure **7**.

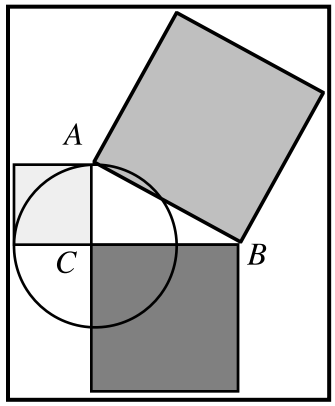


Figure 7: Right triangle with squares on its sides

c. Record the measure of $\angle ACB$, along with the area of each square created in Part **b**.

d. Compare the area of the square constructed on $\overline{AB}$ to the sum of the areas of the other two squares.

e. Move point A to several different locations on the circle and repeat Parts **c** and **d** at each location. **Note:** Save this construction for use in Exploration **2**.

Discussion 1

a. What is the general relationship between the area of the square constructed on $\overline{AB}$ and the sum of the areas of the other two squares in each of the following situations:

1. when $m\angle ACB > 90^\circ$?

2. when $m\angle ACB < 90^\circ$?

b. Describe your generalizations in Part **a** above in terms of the type of triangle—right, obtuse, or acute—and the lengths of its sides.

c. The converses of the generalizations you made in Parts **a** and **b** are also true. How could you use these converses, along with the converse of the Pythagorean theorem, to classify a triangle as acute, obtuse, or right knowing only the lengths of its sides?

Exploration 2

In Exploration **1**, you found that in an obtuse triangle, the square of the length of the side opposite the obtuse angle is greater than the sum of the squares of the lengths of the other two sides. You also found that in an acute triangle, the square of the length of each side is less than the sum of the squares of the lengths of the other two sides. In this exploration, you discover a more precise way to describe these relationships.

a.
1. Delete the squares from the sides of the triangle in your construction from Exploration **1**.
2. Move point *A* so that $\angle BAC$ is obtuse.

b. Let *a* represent the length of the side opposite $\angle BAC$, *b* represent the length of the side opposite $\angle CBA$, and *c* represent the length of the side opposite $\angle BCA$, as shown in Figure **8** below.

c. Construct an altitude from *A* to $\overline{BC}$ to form two right triangles. Let *h* represent the altitude, *x* represent the length of $\overline{CD}$, and *y* represent the length of $\overline{DB}$, as shown in Figure **8**.

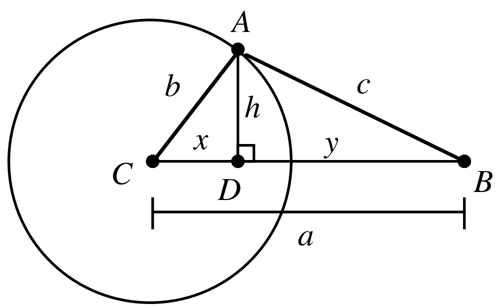


Figure 8: Labeled construction

d.
1. Use trigonometry to express *x* in terms of *b* and $m\angle BCA$.
2. Use this expression for *x*, along with the Pythagorean theorem, to write an expression for h^2 in terms of *b* and $m\angle BCA$.
3. Use the Pythagorean theorem to express h^2 in terms of *c* and *y*.
4. Express *y* in terms of *x* and *a*.
5. To describe h^2 in terms of *a*, *b*, *c*, and $m\angle BCA$, substitute the expression for *y* from Step **4** and the expression for *x* from Step **1** into the expression for h^2 from Step **3**.

e.
1. Since the expressions from Steps **2** and **5** of Part **d** both equal h^2, they are also equal to each other. Use these two expressions to solve for c^2.
2. How does the resulting equation compare with the Pythagorean theorem?

Discussion 2

a. The expression you wrote in Part **e** of Exploration **2** involves the cosine of $\angle BCA$. What is the sign of the cosine in each of the following situations:

1. when $\angle BCA$ is acute?

2. when $\angle BCA$ is obtuse?

Mathematics Note

The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of these lengths and the cosine of the included angle. In Figure **9**, for example, $c^2 = a^2 + b^2 - 2ab\cos\angle C$.

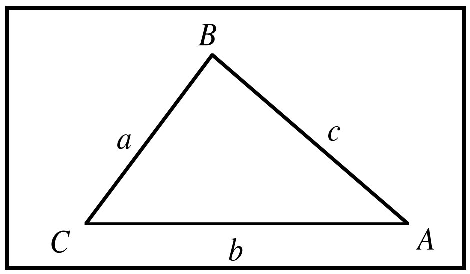


Figure 9: Triangle *ABC*

The law of cosines can be used to determine an unknown length or angle measure in a triangle. For example, if $a = 10$ cm, $b = 12$ cm, and $m\angle C = 35°$, then the law of cosines can be used to find c as follows:

$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$
$$c^2 = 10^2 + 12^2 - 2(10)(12)\cos 35°$$
$$c^2 \approx 47.40$$
$$c \approx 6.89 \text{ cm}$$

b. How does the equation you found in Part **e** of Exploration **2** compare with the law of cosines?

c. Describe how you could use the law of cosines to find the value of a in Figure **10** below.

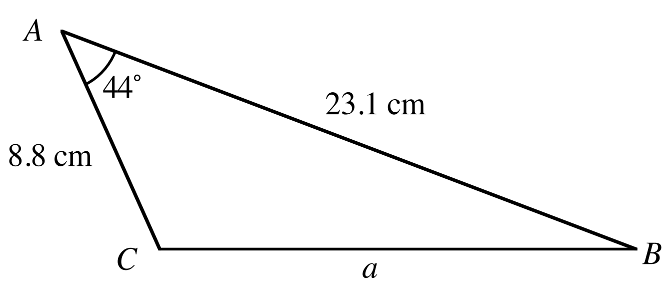


Figure 10: Triangle with an unknown length

d. Once you have found the value of *a* in Figure **10**, how could you use the law of cosines to determine the measure of one of the other angles?

e. The law of cosines is true for both acute triangles and obtuse triangles. It is also true for right triangles? Justify your response.

f. Figure **11** below shows the addition of two vectors, one with a bearing of 20˚ and one with a bearing of 120˚. The bearing of the resultant vector is 105˚.

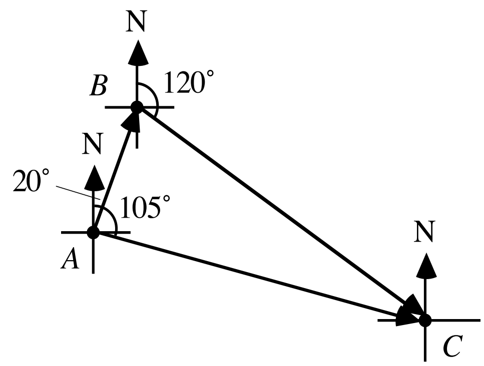
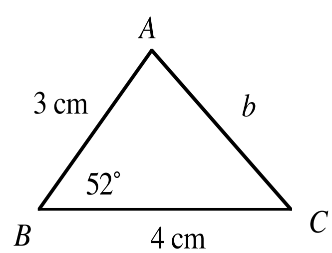


Figure 11: Vector sum with bearings

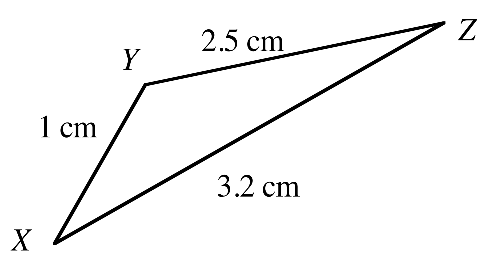
These three vectors form a triangle. Describe how to determine the measures of the angles in this triangle.

Assignment

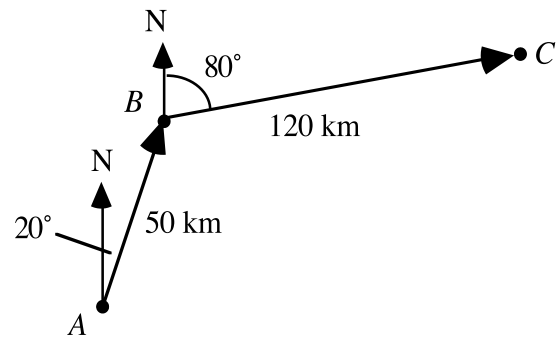
2.1 **a.** Find the value of *b* in the following diagram.



b. Determine the measures of all the angles in the triangle below. Hint: You will need to use inverse trigonometric functions.



2.2 Consider an airplane that flies 50 km at a bearing of 20°, then 120 km at a bearing of 80°. The corresponding displacement vectors are shown in the diagram below.



a. Make a scale drawing of this situation and construct the resultant vector.

b. Determine the magnitude and bearing of the resultant using a ruler and protractor.

c. Determine the magnitude and bearing of the resultant using the law of cosines.

2.3 The speed of a plane or helicopter with respect to the air is its airspeed. But pilots seldom fly in still air. When determining the proper bearings for their destination, they must consider the effect that wind will have on their speed and direction with respect to the ground.

a. An emergency helicopter leaves the hospital on a rescue mission. The patient is located 40 km due east. In still air, the helicopter averages 200 km/hr. If there is no wind, how long will it take the helicopter to reach the patient?

b. As the helicopter lifts off, the wind begins blowing at 15 km/hr at a bearing of 200°.

1. In order to maintain a ground speed of 175 km/hr due east, what should the helicopter's airspeed be?
2. At what bearing should the pilot fly to reach the patient?
3. How long will it take for the helicopter to reach the patient?

c. After the paramedics load the patient on board, the wind changes again. To return to the hospital, the helicopter maintains an airspeed of 200 km/hr at a bearing of 280°. Its ground speed is 210 km/hr due west. What is the wind's magnitude and bearing?

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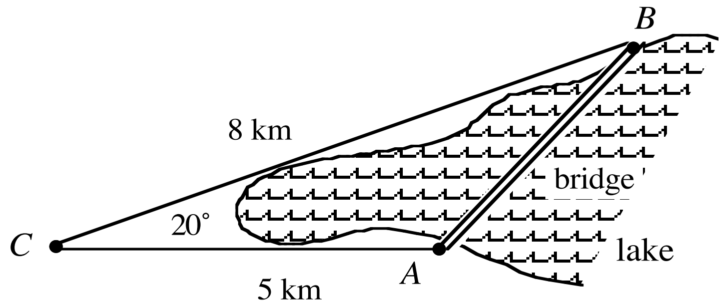
2.4 A construction company plans to build a bridge from point A to point B, as shown on the map below. From point C, a survey crew has measured distances of 5 km to point A and 8 km to point B. The angle between these two measurements is 20°. How long will the bridge be?



2.5 The vertices of ΔFUN are $F(-3,-1)$, $U(3,4)$, and $N(5,-2)$. Make a sketch of this triangle and determine the lengths of its sides and the measures of its angles.

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Activity 3

Imagine that an F-16 fighter jet and an air tanker are both flying due north, 100 km apart. To rendezvous for a midair refueling, the tanker changes its course to a bearing of 32°, while the F-16 turns to a bearing of 340°, as shown in Figure **12**.

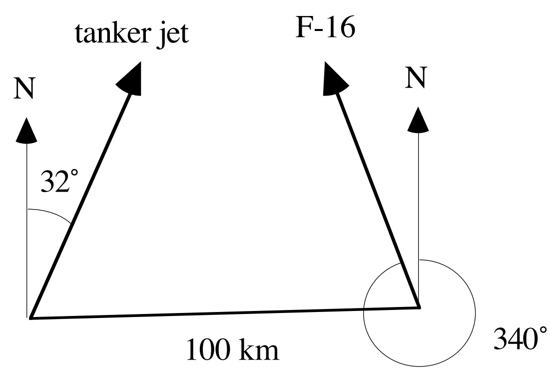


Figure 12: Bearings of F-16 and tanker jet

At the moment when they changed bearings, the two planes were 100 km apart. The pilots must now determine how far each plane must fly to reach the rendezvous point. Without more information, the pilots cannot use the law of cosines to identify these distances. In this activity, you investigate another relationship that could help them analyze this situation.

Exploration

a. Using a geometry utility, construct ΔABC as shown in Figure **13** below.

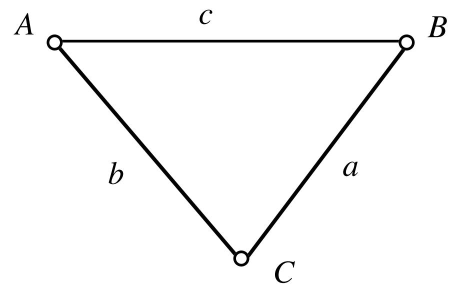


Figure 13: Triangle *ABC*

b. Record the lengths of its sides and the measures of its angles.

c. Construct an altitude from vertex C to the opposite side $\overline{AB}$. Label the altitude h. Notice that the altitude divides the original triangle into two right triangles.

d. Write equations for $\sin\angle CAB$ and $\sin\angle ABC$ in terms of h, b, and a.

e. Solve for h in each equation from Part **d**.

f. Use the equations from Part **e** to calculate h. Compare this value to the distance h measured by the geometry utility.

g. Drag any vertex of the triangle and observe what happens to the value of h in each equation and to the measured altitude.

h. Set the two equations in Part **e** equal to each other and solve for the following ratio:

$$\frac{a}{\sin\angle CAB}$$

i. Using the appropriate sides and angles, repeat Parts **c–h** for an altitude k drawn from vertex B to the opposite side $\overline{AC}$. Compare your results with those obtained for altitude h.

Discussion

a. How does your equation from Part **h** of the exploration compare to those obtained by others in the class?

b. What is the relationship among the ratios you found in Parts **h** and **i** of the exploration?

Mathematics Note

The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles. In the triangle in Figure **14**, for example,

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}$$

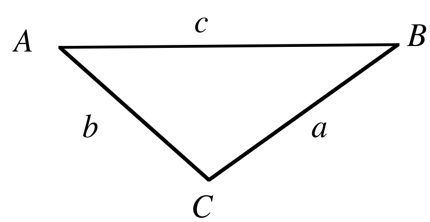


Figure 14: Triangle *ABC*

Like the law of cosines, this relationship can be used to determine unknown lengths or angle measures in triangles. For example, if $a = 7.1$, $m\angle A = 50^\circ$, and $m\angle B = 60^\circ$, then b can be found as follows. By the law of sines,

$$\frac{7.1}{\sin 50^\circ} = \frac{b}{\sin 60^\circ}$$

In this case, $b \approx 8.0$.

c. Describe how the law of sines can be used to find the unknown values in Figure **15**.

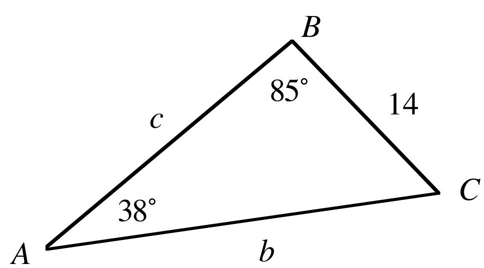


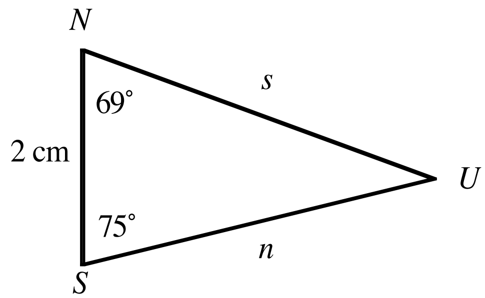
Figure 15: A triangle with some unknown values

d. Describe a triangle with some unknown lengths or angle measures which cannot be determined by the law of sines.

e. Describe a triangle with some unknown lengths or angle measures which can be determined by the law of sines but not by the law of cosines and vice versa.

Assignment

3.1 Find the values of s and n in ΔSUN below.



3.2 Using a geometry utility, make a scale drawing of the rendezvous of the F-16 and the air tanker described in the introduction to this activity.

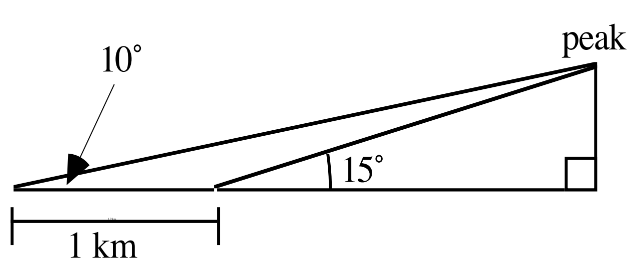
a. Measure the distance that each plane must fly to the rendezvous point. Verify these measurements using the law of sines.

b. If the F-16 maintains an average speed of 500 km/hr after the change in bearings, how long will it take the plane to reach the rendezvous point?

c. At what average speed should the tanker fly to meet the F-16?

3.3 After dinner, Christopher and Alee decide to explore the area around their campsite. To keep in touch with each other, they both carry hand-held radios. Christopher leaves camp at a bearing of 34°, while Alee follows a bearing of 300°. Both hike at about the same speed of 4 km/hr. Their radios have a range of 10 km. How long can they hike without losing radio contact with each other? Use diagrams and calculations to support your response.

3.4 On their way up a mountain, two hikers pause to rest by a lake at an elevation of 3600 m. They have an old map of the area, but the section that describes the elevation of the mountain peak has been torn away. As shown in the diagram below, they estimated an angle of 10° to the peak from the far side of the lake. From the near side, they now estimate an angle of 15° to the peak. The lake is 1 km across. Determine the elevation of the mountain peak.



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3.5 As part of a training mission, a U.S. Coast Guard pilot must fly a helicopter from Kalamazoo to Flint, Michigan.

a. On a map of Michigan, draw the displacement vector from Kalamazoo to Flint. Measure the magnitude and bearing of this vector.

b. In still air, the helicopter can maintain an average speed of 190 km/hr. If there is no wind, how long will it take to complete the flight?

c. As the helicopter leaves Kalamazoo, the wind begins blowing at 40 km/hr on a bearing of 120°. If the pilot wants to reach Flint in the same amount of time as calculated in Part **b**, at what airspeed and bearing should she fly?

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Research Project

In previous activities, you used the law of cosines and law of sines to determine unknown lengths or angle measures in triangles. Figure **16** shows ΔGHK. If you knew h, k, and $m\angle K$, could you find g? Or is more information required?

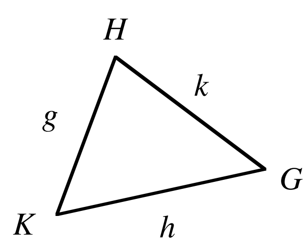

Figure 16: Triangle *GHK*

a. Given ΔGHK, use the law of cosines and the law of sines to explore each of the following situations:

1. $h = 5$ cm, $k = 4.3$ cm, and $m\angle K = 55°$

2. $h = 5$ cm, $k = 2$ cm, and $m\angle K = 55°$

3. $h = 5$ cm, $k = 6$ cm, and $m\angle K = 55°$

4. $h = 5.00000$ cm, $k = 4.09576$ cm, and $m\angle K = 55.0000°$

b. Based on your results in Part **a**, make a conjecture about the possibility of determining the remaining parts of a triangle given the lengths of two sides and the measure of a non-included angle.

Activity 4

Many situations require the addition of multiple vectors to determine a resultant. For example, an airplane in flight is simultaneously acted on by forces of lift, thrust, weight, and drag. To find the resultant force on the plane, you must add the vectors that correspond to the individual forces. In previous activities, you used the tip-to-tail method to find a resultant graphically. In this activity, you investigate a method for adding vectors algebraically.

Exploration 1

a. On a two-dimensional coordinate system, construct a vector **r** whose tail is located at the origin O and whose tip is located at a point P in the first quadrant.

b. Determine the vector's magnitude and direction in terms of the angle formed by the vector and the x-axis.

c. Construct a horizontal vector and a vertical vector whose sum is vector **r**. These vectors, shown in Figure **17** below, are the **horizontal component** and the **vertical component** of vector **r**. Label the intersection of the horizontal and vertical components as point F.

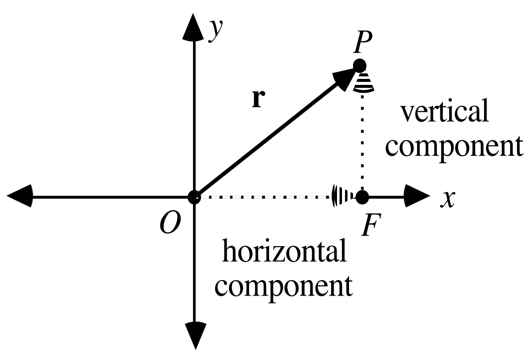

Figure 17: Vector r and its components

d. The three vectors—the horizontal component, the vertical component, and the resultant vector—form a right triangle. Determine the measure of $\angle POF$.

e. Use trigonometry to determine the magnitude of the horizontal and vertical components of vector **r**.

f. Repeat Parts **b–e** for a vector whose tail is located at the origin and whose tip is located in:

1. the second quadrant.
2. the third quadrant
3. the fourth quadrant.

Discussion 1

a. Describe how you found the measure of $\angle POF$ in Part **d** of Exploration **1**.

b. Describe how you found the magnitudes of the horizontal and vertical components in Part **e** of Exploration **1**.

Mathematics Note

The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **m** is denoted by $\mathbf{m}_x$ (read "**m** sub *x*"), while its vertical component is denoted by $\mathbf{m}_y$.

For example, Figure **18** shows a graph of a vector **m** with its tail located at the origin, along with its components.

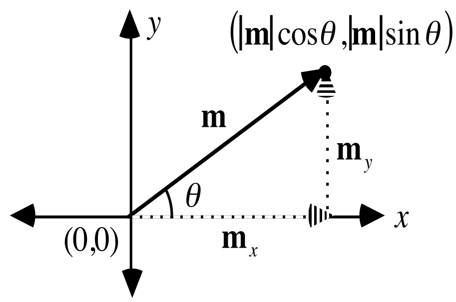

Figure 18: A vector and its components

Recall that the magnitude of vector **m** can be written as $|\mathbf{m}| = \sqrt{(\mathbf{m}_x)^2 + (\mathbf{m}_y)^2}$. Using trigonometric ratios, $\mathbf{m}_x = |\mathbf{m}| \bullet \cos\theta$ and $\mathbf{m}_y = |\mathbf{m}| \bullet \sin\theta$, where θ is an angle measured counterclockwise from the positive *x*-axis.

For example, Figure **19** shows a velocity vector **v** with a magnitude of 10 m/sec and a direction of 240° measured counterclockwise from the positive *x*-axis.

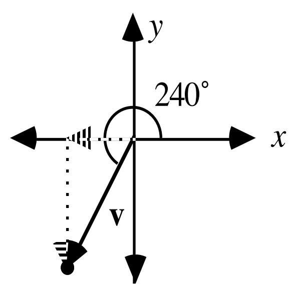

Figure 19: Vector v and its components

The components of vector **v** can be found as shown below:

$$\begin{aligned} \mathbf{v}_x &= |\mathbf{v}| \bullet \cos\theta \\ &= 10 \bullet \cos 240^\circ \\ &= -5 \text{ m/sec} \end{aligned} \qquad \begin{aligned} \mathbf{v}_y &= |\mathbf{v}| \bullet \sin\theta \\ &= 10 \bullet \sin 240^\circ \\ &\approx -8.7 \text{ m/sec} \end{aligned}$$

c. When a vector's tail is located at the origin, how are the coordinates of its tip related to its horizontal and vertical components?

d. Does locating a vector's tail somewhere other than at the origin affect its horizontal and vertical components? Explain your response.

e. The magnitude of a vector is always positive. The components of vector **v** in Figure **19**, however, are $\mathbf{v}_x = -5 \text{ m/sec}$ and $\mathbf{v}_y \approx -8.7 \text{ m/sec}$. In this situation, what do the negative signs indicate?

f. A vector representing the flight of an airplane has a bearing of 260° and a magnitude of 355 km/hr. Describe how you would find the horizontal and vertical components of this vector.

Exploration 2

In Exploration **1**, you learned how to express any vector in terms of its horizontal and vertical components. In this exploration, you observe how these components can be used to add vectors algebraically.

a. Graph a vector **m** with its tail located at the origin of a two-dimensional coordinate system. Add another vector **n** to vector **m** using the tip-to-tail method. Graph the resultant vector **r**.

b. Determine the components of each vector in the sum $\mathbf{m} + \mathbf{n} = \mathbf{r}$. Record the value of each component in a table with headings like those in Table **1** below.

Table 1: Vectors and their components

Vector	Horizontal Component	Vertical Component
m		
n		
r		

c.
1. Describe the relationship between the horizontal components of **m** and **n** and the horizontal component of **r**.
2. Describe the relationship between the vertical components of **m** and **n** and the vertical component of **r**.

d. Repeat Parts **a–c** for a resultant vector **r** that is the sum of three vectors **u**, **v**, and **w**.

Discussion 2

a. What relationship appears to exist between the components of the vectors in a sum and the components of the resultant vector?

b. Describe how you could use this relationship to determine a resultant vector without using the tip-to-tail method of vector addition.

c. Once the components of the resultant vector have been identified, how could you find its magnitude and direction?

Assignment

4.1 Describe the components of a vector with a magnitude of 75 km/hr and a bearing of 50°.

4.2 The forces represented by the following vectors, where direction is measured counterclockwise from the positive x-axis, are acting simultaneously on an object: 20 N at 45°, 15 N at 105°, 35 N at 200°, and 45 N at 300°.

a. What is the sum of the x-components of these vectors?

b. What is the sum of the y-components of these vectors?

c. If the tail of the resultant vector is placed at the origin of a two-dimensional coordinate system, what would be the coordinates of its tip?

4.3 When graphed on a coordinate plane with their tails located at the origin, the tips of four vectors have the following coordinates: (4,5), (–4,13), (7,–4),and (–6,8), respectively. If the tail of the resultant vector is also located at the origin, what are the coordinates of its tip?

4.4 The components of a velocity vector **m** are $\mathbf{m}_x = 15 \text{ m/sec}$ and $\mathbf{m}_y = 20 \text{ m/sec}$. What is the magnitude and direction of vector **m**?

4.5 In an orienteering competition, participants race across an unfamiliar course using a map and compass to navigate between checkpoints. During one race, your path can be described by the following displacement vectors: 350 paces at a bearing of 35°, 1250 paces at a bearing of 275°, and 1000 paces at a bearing of 140°. Use vector addition to describe your final change in position.

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4.6 The forces represented by the following vectors, where direction is measured counterclockwise from the positive x-axis, are acting simultaneously on an object: 25 N at 65˚, 30 N at 340˚, and 10 N at 150˚. What is the resultant force on the object?

4.7 The instructions in a treasure hunt include the following displacement vectors: 50 m at a bearing of 30˚, 14 m at a bearing of 260˚, 15 m at a bearing of 175˚, and 25 m at a bearing of 105˚.

a. What is the sum of the x-components of these vectors?

b. What is the sum of the y-components of these vectors?

c. What are the magnitude and bearing of the resultant vector?

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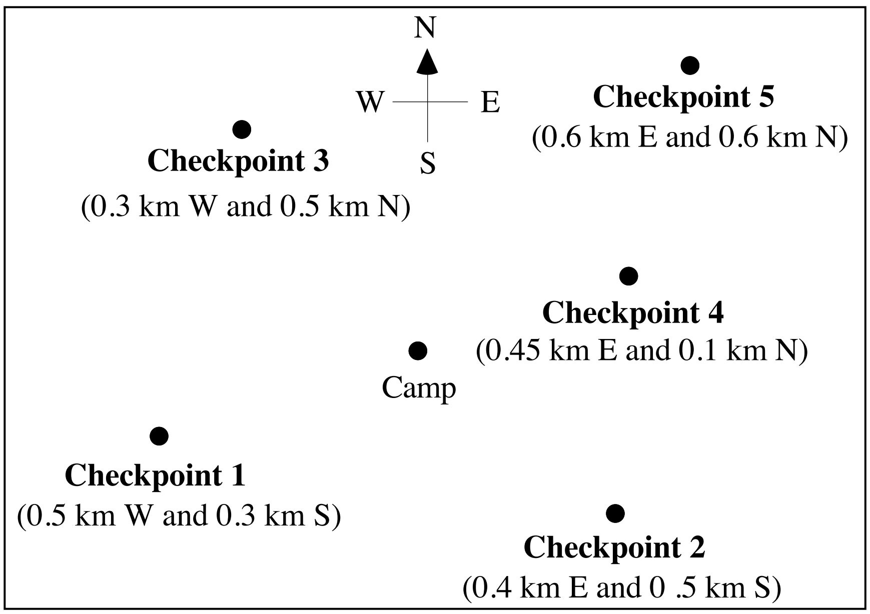
Summary Assessment

1. Imagine that you are a counselor at a summer camp. Your responsibilities include designing an orienteering course. The race begins at camp, proceeds to each of five checkpoints in numerical order, then returns to camp. At each checkpoint, campers should receive instructions on how to find the next checkpoint.

The distances given in the diagram below are the distances from camp. Write a set of instructions for the campers that describes the distance and bearing from each point in the race to the next one.



2. An airplane leaves the airport at a speed of 240 km/hr with a bearing of 160˚. To compensate for the wind, the pilot changes course to a bearing of 165˚ and a speed of 255 km/hr. What is the velocity of the wind?

3. When an earthquake occurs, the movement generates two kinds of waves: primary waves and secondary waves. The speeds at which these waves travel through the earth differ by approximately 3.5 km/sec. With the help of seismographs, geologists can measure the difference in the arrival times of primary and secondary waves. By comparing the differences in arrival times at two or more locations, they can then determine the location of the quake's epicenter.

In the state of Montana, there are seismographs in Missoula and Bozeman. If the seismograph in Missoula shows a difference in arrival times of 146 sec, while the one in Bozeman shows a difference of 122 sec, where is the epicenter of the earthquake? Use the map of Montana supplied by your teacher to justify your response.

Module Summary

- A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically represented by a bold, lowercase letter, such as vector **m**. In handwritten work, the same vector can be symbolized by $\vec{m}$. The magnitude of a vector **m** is denoted by $|\mathbf{m}|$.
- **Displacement** is a change in position in a particular direction.
- The **velocity** of an object is its speed in a specific direction.
- **Force** is a push or a pull in a particular direction. The metric unit of force is the newton (N).
- **Equivalent vectors** have the same magnitude and the same direction.
- **Opposite vectors** have the same magnitude, but their directions differ by 180°.
- One way to add vectors is the **tip-to-tail method**. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.
- The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.
- A **scalar** is a real number.
- The operation $k\mathbf{m}$ denotes the **scalar multiplication** of **m** by the scalar k.
- The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other sides minus twice the product of these lengths and the cosine of the included angle. In a triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

- The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles. In a triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **m** is denoted by $\mathbf{m}_x$ (read "**m** sub x"), while its vertical component is denoted by $\mathbf{m}_y$. Using trigonometric ratios, $\mathbf{m}_x = |\mathbf{m}| \bullet \cos\theta$ and $\mathbf{m}_y = |\mathbf{m}| \bullet \sin\theta$, where θ is a directional angle measured counterclockwise from the positive x-axis.

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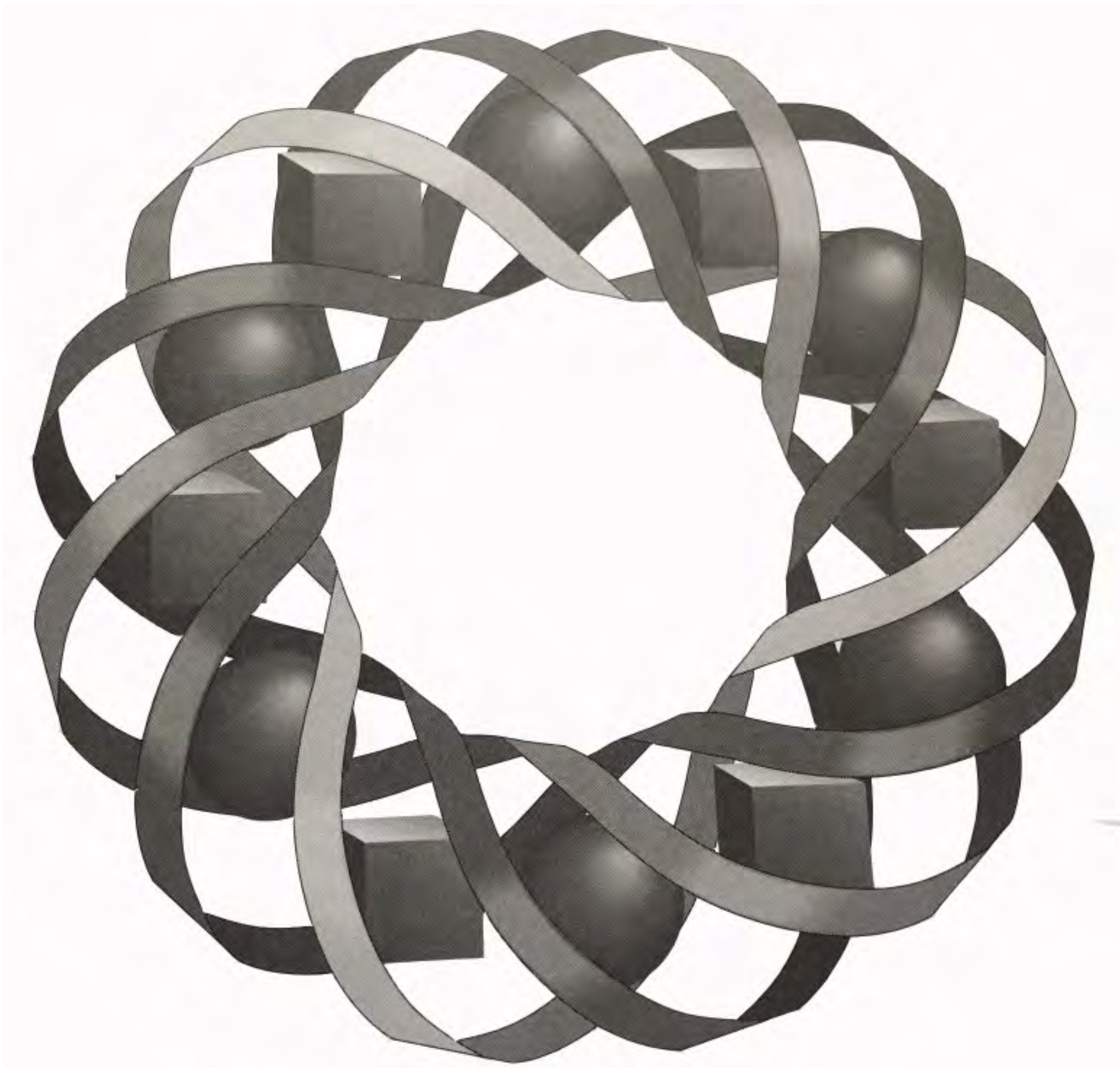
Everyone Counts



The grand opening of a shopping mall promises lots of new choices for customers. How many choices? In this module, you investigate some different ways to count your options.

Art Perleberg • Deanna Turley • Dan West

Everyone Counts

Introduction

After months of intense work, real estate developer Nadia Nelson is exhausted but happy. All construction is complete and spaces have been leased to a dozen businesses. Riverside Mall is ready to open. As owner and manager of the mall, Nadia welcomes her new tenants with the following memo.

Nelson Real Estate Property Management

To: Members of the Riverside Store Managers' Association
From: Nadia Nelson
Re: Opening of Riverside Mall

Congratulations! Your businesses were selected for the mall after a survey of Riverside shoppers. I would like to welcome each of you to Riverside Mall.

Last Chance Beauty Parlor	Les Arts Magnifique
Kuzlowski's Pizza	Almost Human Pets
Riverside Read and Feed	Music Mania
Trendz Clothing	The Zone Sporting Goods
Toddler's Haven Toys	Planet Alpha Arcade
Zuzu's Petals and Gifts	Flix Theaters

Please keep me informed of all activities planned for the mall's grand opening. And if you have any questions, don't hesitate to contact me.

Activity 1

As part of the grand opening festivities, the managers' association is planning a promotion called "Crack the Safe." A large safe will be filled with merchandise contributed by each store. The shopper who manages to open the safe will take home all the prizes.

Nadia Nelson likes the idea, but wonders about the manner in which the promotion will be conducted. She expresses her concerns in another memo.

Nelson Real Estate Property Management

To: Members of the Store Managers' Association
From: Nadia Nelson
Re: "Crack the Safe" Promotion

I am excited about the "Crack the Safe" promotion. You have created an ingenious way to attract shoppers to the mall. I do have some questions, however. Three types of locks are available for the safe: rotary combination, keyed pin, and digital combination. Which type should we use? And how will we make sure that the promotion lasts for several days, yet gives shoppers a reasonable chance to win? Please send your recommendations for these challenges as soon as possible. Opening day is coming!

Exploration

a. As shown in Figure **1**, one type of rotary combination lock consists of a dial with 40 tick marks representing the numbers 0 through 39. The lock is opened by turning the dial to the right, then to the left, then back to the right, stopping each time at the corresponding number of a three-number code.

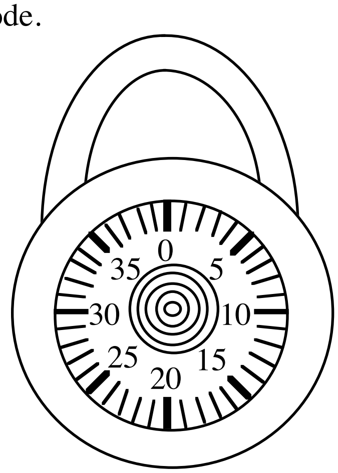


Figure 1: Rotary combination lock

1. Estimate how many different three-number combinations are possible for this type of lock.
2. Using your estimate in Step **1**, determine the probability that a combination chosen at random will open the lock.
3. Develop a systematic plan to find the actual number of different combinations.

Mathematics Note

The **fundamental counting principle** states that if one event can occur in h ways, and for each of these ways a second event can occur in k ways, then the number of different ways in which the two events can occur is $h \bullet k$.

For example, suppose a store sells soft drinks in 2 different sizes and 5 different flavors. In this situation, one event is the choice of size. The second event is the choice of flavor. Using the fundamental counting principle, there are a total of $2 \bullet 5$ or 10 different selections from which to choose.

The fundamental counting principle can be extended to situations involving more than two events. For example, if the store also sells 4 different brands of soft drinks, each of which comes in 2 different sizes and 5 different flavors, the total number of possible selections is $2 \bullet 5 \bullet 4$ or 40.

b. **1.** Use the fundamental counting principle to determine the number of possible combinations.

2. Calculate the probability that any one combination will open the lock.

c. The second type of lock under consideration by the managers' association requires a key. Figure **2** shows a key for a standard pin lock. Depending on the manufacturer, such keys can have either 5 or 6 valleys. Each valley can have from 6 to 10 different depths.

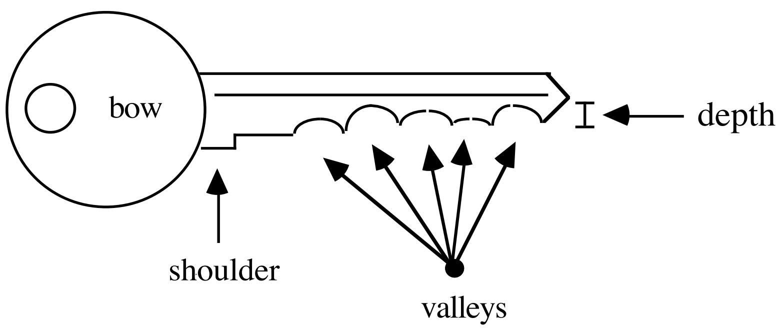


Figure 2: Key for a standard pin lock

1. Draw a key with 5 valleys. Label these valleys with the letters A–E. If 6 different depths are available for each valley in this key, how many different keys are possible?

2. If 10 different depths are available for each valley in a key with 6 valleys, how many different keys are possible?

3. Using the number of keys found in Step **2**, determine the probability that a key chosen at random will open the mall contest lock.

Discussion

a.

1. Describe the strategy you devised in Part **a** of the Exploration for determining the number of possible combinations for the lock.

2. Did your strategy consider combinations with repeated digits? Explain your response.

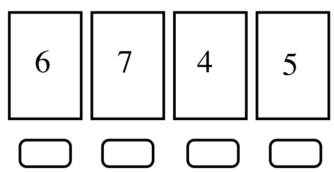
3. Why would it be impractical to use a tree diagram to determine the number of possibilities for the lock?

b. When a visitor to the mall opens the safe, the promotion is over. If you wanted the promotion to last as long as possible, which type of lock would you choose: rotary combination or keyed pin? Discuss the advantages and disadvantages of your choice.

c. Is it possible that the first person who tries to open the safe will be successful?

Assignment

1.1 The third type of lock being considered for the mall promotion is a digital combination lock. As illustrated below, one kind of digital combination lock features four electronic displays, each of which can show a number from 0 to 9. To enter a four-digit code, users press the buttons underneath the displays until the desired number is obtained.



a. Use the fundamental counting principle to determine the number of possible combinations.

b. Determine the probability that a combination chosen at random will open the lock.

1.2 Considering all three types of locks, which would you recommend for the mall's "Crack the Safe" promotion? Justify your choice.

1.3 In addition to selling books and espresso, the manager of Riverside Read and Feed plans to publish works by members of the local community. Before the store can publish its first book, the manager must apply for a Publisher Identifier number for the International Standard Book Number (ISBN).

In most industrialized nations, an ISBN is assigned to each published book. It consists of 10 digits divided into 4 different areas. The Group Identifier represents the language in which the book is written. The Publisher Identifier represents the publishing company. The Title Identifier represents the title of the book. The Check Digit is used to detect errors before shipping. A sample ISBN is shown below.

$$\underbrace{0}_{\text{Group Identifier}} - \underbrace{679}_{\text{Publisher Identifier}} - \underbrace{41295}_{\text{Title Identifier}} - \underbrace{6}_{\text{Check Digit}}$$

The Group Identifier and Check Digit each consist of 1 digit. The numbers of digits in the Publisher and Title Identifiers may vary, but must represent a total of 8 digits.

a. The manager of Riverside Read and Feed hopes to publish 20 books during her first year in business. What is the minimum number of digits needed in the Title Identifier? Explain your response.

b. Considering your response to Part **a**, how many different possibilities would the store have for its Publisher Identifier? Explain your response.

c. Do you think a large publishing company would want more or less digits designated for its Publisher Identifier? Justify your answer.

d. Suppose Riverside Read and Feed receives a three-digit Publisher Identifier. What is the maximum number of books it could publish?

1.4 Businesses often use the letters that correspond to a telephone number to help attract customers. The number for the Coiffure Salon, for example, is 244-4247, or BIG HAIR. As shown below, most touch-tone telephones have 24 letters on the keypad.

1	ABC 2	DEF 3
GHI 4	JKL 5	MNO 6
PRS 7	TUV 8	WXY 9
*	0	#

a. Which letters are missing from the keypad above?

b. The first three digits of a local telephone number represent its numerical prefix. In the past, the letters that correspond with these numbers also represented a prefix. For example, a resident of the Parkway neighborhood might have had the telephone number PAR-2374.

1. Use the fundamental counting principle to determine how many 3-letter prefixes can be formed using the letters on this keypad.

2. Using the number keys that also contain letters, how many 3-digit prefixes can be formed?

3. Compare your answers to Parts **b1** and **b2**. Explain any differences you observe.

c. 1. Suppose that no letter may be repeated in a prefix. In this case, how many 3-letter prefixes can be formed from the keypad?

2. Write your response to Part **c1** as a ratio of factorials.

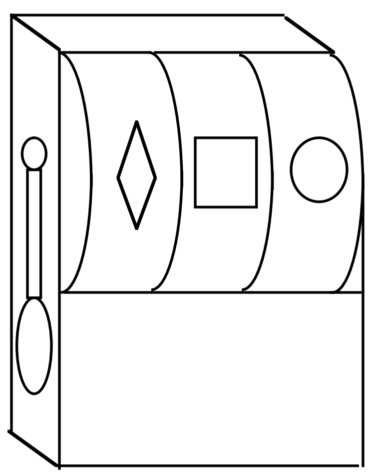
* * * * *

1.5 Some telephone keypads now display all 26 letters. To accommodate the extra letters, two buttons—the $\boxed{7}$ and the $\boxed{9}$—correspond with 4 letters each.

a. If letters may be repeated, how many 3-letter prefixes can be formed from this keypad?

b. How many 3-letter prefixes can be formed if letters may not be repeated?

1.6 The toy store in the mall sells an imitation slot machine with 3 dials. Each dial contains the same sequence of 6 different symbols.



a. How many different arrangements of 3 symbols are possible?

b. What is the probability of getting the same symbol on all 3 dials?

1.7 In 1997, the state of Alaska used two different numbering and lettering systems for its automobile license plates.

a. One system used three letters followed by three digits. Assuming that all digits and letters may be repeated, how many different license plates are possible using this system?

b. The other system consisted of four digits followed by two letters. Assuming that all digits and letters may be repeated, how many different license plates are possible using this system?

c. Suppose that the state planned to use only one numbering and lettering system for the next 10 years. Which one would you recommend? Defend your choice.

d. Would either of these numbering and lettering systems work in the state of California? Write a paragraph explaining your response.

* * * * * * * * * *

Activity 2

Excited by the initial success of Riverside Mall, Nadia Nelson has begun planning a new addition. As shown in Figure **3**, the new wing will house seven more shops.

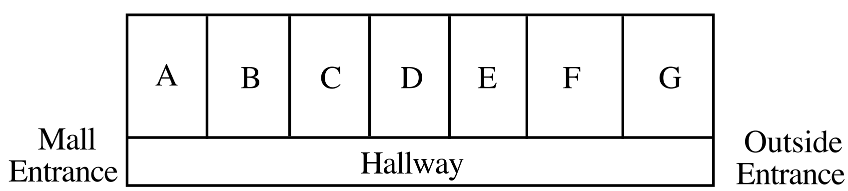


Figure 3: Riverside Mall addition

Exploration

Two business owners are already trying to reserve prime locations in the new wing. In this exploration, you help Ms. Nelson determine the number of different ways in which the stores can be arranged.

a. Draw a tree diagram to determine how many different ways the two businesses can be placed in the seven available locations.

b. Use the fundamental counting principle to verify your Part **a** response.

c. Once the first two stores have been placed, five spaces remain. Determine how many different arrangements are possible for the five stores that plan to rent these spaces.

d. Using the fundamental counting principle, find the total number of possible arrangements if locations are assigned as described in Parts **a** and **c**.

e. **1.** Determine the number of possible arrangements if locations are assigned to seven stores at one time.

2. Compare this number with your response to Part **d**.

f. Using your responses to Parts **c** and **e**, write the number of arrangements you determined in Part **a** as a ratio.

g. Use the process described in Parts **b–f** to write a ratio that represents the number of arrangements of *r* stores in *n* available spaces, given that $n > r$.

Discussion

a. The numerator of the fraction you wrote in Part **g** of the exploration can be expressed as $n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 3 \bullet 2 \bullet 1$. How can this expression be written using factorial notation?

b. How could you use factorial notation to express the denominator of the fraction you wrote in Part **g** of the exploration?

c. How could you use factorial notation to describe the number of ways that *r* stores can be arranged in *n* available spaces?

Mathematics Note

A **permutation** is an ordered arrangement of symbols or objects. The number of permutations of *n* different symbols or objects taken *r* at a time, denoted by $P(n,r)$, is given by the following formula:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Another commonly used notation for permutations is ${}_nP_r$.

For example, suppose that a license plate consists of 7 digits from 1 to 9 in which no digit can be repeated. The number of different license plates that can be made using this system is the number of permutations of 9 symbols taken 7 at a time, or $P(9,7)$. Using the formula given above,

$$P(9,7) = \frac{9!}{(9-7)!} = \frac{9!}{2!} = \frac{9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{2 \bullet 1} = 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 = 181{,}440$$

d. In the formula for $P(n,r)$ given in the mathematics note, why is $n(n-1)(n-2)\cdots(n-r+1)$ equivalent to the following ratio?

$$\frac{n!}{(n-r)!}$$

e. The permutation of *n* objects arranged *n* at a time, $P(n,n)$, is $n!$ Use this fact and the formula for $P(n,r)$ to suggest a definition for $0!$

Assignment

2.1 Before a game of slow-pitch softball, each team must submit a batting order—a list of 10 players that identifies the order in which they bat.

a. Does a batting order represent a permutation? Explain your response.

b. The Mall Misfits have 10 players on their team. How many different batting orders are possible?

c. In how many ways can the first, second, and third batters be chosen from the group of 10 players?

2.2 Music Mania's sound technician is meeting with a local band to discuss the arrangement of songs on the band's upcoming demo tape.

a. Explain why the arrangement of songs on a tape is a permutation.

b. The demo tape will contain 11 songs, 6 on side A and 5 on side B. In how many ways can 6 of the songs be arranged on side A? Write your answer both as a number and in the form $P(n,r)$.

c. After the songs for side A have been selected, in how many ways can the remaining 5 songs be arranged on side B? Write your answer both as a number and in the form $P(n,r)$.

d. The band also plans to issue a compact disc (CD). A CD has only one "side." In how many ways can the 11 songs be arranged on a CD? Write your answer both as a number and in the form $P(n,r)$.

e. Multiply your answers to Parts **b** and **c** and compare the product to your answer to Part **d**.

f. Use your results from Parts **c** and **d** to write an equation for $P(11,6)$.

2.3 As part of the mall's grand opening, Almost Human Pets is holding a prize drawing. The first 400 visitors to the store each receive one ticket. At the end of the week, three ticket numbers will be drawn at random. The person who holds the first ticket drawn wins a St. Bernard. The person who holds the second ticket drawn wins a hamster. The person who holds the third ticket drawn wins a guppy.

a. In how many ways can 3 winners be selected from the 400 ticket holders?

b. After the St. Bernard and the hamster are given away, what is the probability that any one of the remaining ticket holders will win the guppy? Explain your response.

2.4 Each November, the citizens of Riverside hold elections for city council. The council has 3 members. In this year's election, 6 candidates, including all 3 incumbents, are vying for the 3 positions.

Although the candidates' names are supposed to be listed in random order on the ballot, the first 3 names on this year's ballot are all incumbents.

a. In how many different ways can the 3 incumbents be listed in the first 3 positions on the ballot?

b. In how many different ways can the 3 challengers be listed in the last 3 positions on the ballot?

c. How many different ballot arrangements are possible if the incumbents appear in the first 3 positions and the challengers appear in the last 3 positions?

d. How many different arrangements are possible if all 6 names are placed randomly on the ballot?

e. Studies of voter behavior have shown that when there is no strong preference for candidates, those names that appear first on a ballot are more likely to be selected. One of the challengers complains that the ballot was "fixed" to favor the incumbents. Use probability to argue for or against this claim.

2.5 The Store Managers' Association has decided to use a rotary combination lock with 40 numbers for their "Crack the Safe" promotion. One manager suggests telling customers that the secret combination consists of 3 different numbers.

a. Without this hint, how many 3-number combinations are possible?

b. How many are possible with this hint?

c. What percentage of the possible combinations is eliminated by the hint?

d. How is the probability that someone will open the safe with one try affected by knowing the hint? Explain your response.

* * * * *

2.6 Although the Riverside Mall has 12 stores, there are only 8 parking spaces reserved for store managers. If the 8 spaces are always full, in how many ways can they be occupied by the managers' cars?

2.7 In slow-pitch softball, a "rover" or "short fielder" is added to the traditional trio of outfielders. This results in a total of 10 positions on the field. Although Jean would accept any position on the Mall Misfits, he would like to play catcher.

a. Before the season begins, 14 players try out for the Mall Misfits. In how many ways can the team fill the 10 positions, if there are no restrictions on who plays a given position?

b. If Jean must play catcher, in how many ways can the remaining 9 positions be filled?

c. If all 10 positions are filled at random, what is the probability that Jean will be the catcher?

* * * * * * * * * *

Research Project

Prepare a report on the system of numbering used by the U.S. Social Security Administration. Your report should include answers to the following questions:

a. How does the Social Security Administration assign numbers to individuals?

b. Can two people be assigned the same number?

c. How many different social security numbers are possible?

d. At current rates of population growth, are there enough numbers to last until the year 2000? until the year 2500?

Activity 3

As the winter holiday season approaches, Nadia Nelson begins thinking of ways to attract more shoppers to the mall. The decorations, she decides, should be both elaborate and eye-catching. In order to involve the store managers in the decorating process, she circulates the following memo.

Nelson Real Estate Property Management

To: Members of the Store Managers' Association
From: Nadia Nelson
Re: Selection of Decorating Committee

Congratulations again! Although Riverside Mall has been open for only three months, business has exceeded all initial predictions. Just a reminder that the holidays will be here soon and we must begin planning the mall decorations. My hope is that these decorations will reflect the diversity of your membership and provide a strong enticement to holiday shoppers.

The decorations committee consists of three members. Five managers have volunteered to serve. According to the association bylaws, this committee must be selected randomly from all volunteers. The drawing will take place Tuesday evening at the association's regular meeting. See you there!

Exploration

The last names of the 5 volunteers for the decorating committee are Letasky, Milligan, Novotney, Oliphant, and Payne. In this exploration, the name of each volunteer will be represented by its first letter. For example, Letasky will be represented by the letter L.

a. In Activity **2**, you learned that the number of permutations of n distinct items taken r at a time is given by $P(n,r)$. Determine the number of permutations of 5 volunteers taken 3 at a time.

b. In this committee, the order of selection does not matter. In other words, a committee of Letasky, Oliphant, and Payne (LOP) is the same as a committee of Oliphant, Payne, and Letasky (OPL).

1. Determine the number of 3-letter arrangements from Part **a** that include L, O, and P.
2. Express this number of arrangements using permutation notation.

c. All the 3-letter arrangements that include L, O, and P can be considered as one committee. Similarly, all the 3-letter arrangements that include L, M, and O represent another committee. How many different 3-member committees can be selected from the 5 volunteers?

d. Use the fundamental counting principle and your responses to Parts **b** and **c** to write a product that equals the number of permutations of 5 letters chosen 3 at a time, or $P(5,3)$.

e. A collection of symbols or objects in which order is not important is called a **combination**. For example, the number of 3-member committees that can be selected from a group of 5 volunteers is the combination of 5 items taken 3 at a time. This can be written as $C(5,3)$.

Write an equation that relates the permutation of 5 items taken 3 at a time to the combination of 5 items taken 3 at a time.

f. Write $C(5,3)$ in terms of $P(5,3)$ and $P(3,3)$.

Discussion

a. How did the selection process for the arrangements of stores in Activity **2** differ from the selection process for the decorating committee?

b. Explain why the permutation of 5 items taken 3 at a time is not appropriate for counting the number of committees of 3 from a group of 5.

c. What does the permutation of 3 items taken 3 at a time represent in this context?

d. As mentioned in Part **e** of the exploration, a combination is a collection of symbols or objects in which order is not important. Besides its mathematical definition, the word *combination* has several other meanings.

Earlier in this module, for example, you determined the number of different combinations possible for a digital lock. What two elements of the mathematical definition show that the *combination* in a combination lock is not a mathematical one?

Mathematics Note

The number of combinations of *n* different symbols or objects taken *r* at a time, denoted by $C(n,r)$, is given by the following formula:

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{P(n,r)}{r!} = \frac{1}{r!} \bullet \frac{n!}{(n-r)!} = \frac{n!}{r!(n-r)!}$$

Two other common notations for combinations are ${}_nC_r$ and

$$\binom{n}{r}$$

For example, the number of combinations of 7 letters—A, B, C, D, E, F, and G—taken 3 at a time, can be found as shown below:

$$C(7,3) = \frac{P(7,3)}{P(3,3)} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!(4!)} = \frac{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{(3 \bullet 2 \bullet 1)(4 \bullet 3 \bullet 2 \bullet 1)} = 35$$

e. The process described in Parts **a–d** of the exploration leads to the general equation $P(n,r) = C(n,r) \bullet P(r,r)$. Describe how this equation can be used to find the formula for $C(n,r)$ given in the mathematics note.

Assignment

3.1 Using the letters H, I, J, and K to represent four people, find the number of committees of 2 that can be selected from a group of 4. In other words, determine the combination of 4 items taken 2 at a time. Verify your response by listing the committees.

3.2 In the formula for a combination, what does $r!$ represent?

3.3 The narrators for the mall's daily radio ads are selected from a pool of 12 employees, one from each store. As part of management's efforts to build positive relationships among employees, new teams are chosen each day. Ideally, each employee should work with several different teams during the course of a year.

- **a.** What size teams would you recommend for the daily radio ads? Justify your response.
- **b.** James from the pet store and Alice from the flower and gift shop are friends. If the managers' association decides to use teams of 2 for the radio ads, how often can they expect to work as a team?

3.4 The mall's toy store, Toddler's Haven, plans to sell coloring kits that contain 4 crayons, each one a different color. The crayons are available in a total of 12 different colors. Should the store package every possible assortment of 4 crayons or create assortments as they are ordered? Explain your response.

3.5 One of the prizes in the "Crack the Safe" promotion was a deck of cards. Each card in the deck contained the description of an item from one of the stores. After randomly selecting 30 cards from the deck, the winner received each item described on those cards.

In one advertisement for the promotion, store managers claimed "There are more prize combinations possible than there are molecules of air in the mall!"

- **a.** The mall is a rectangular prism 50 m wide, 120 m long, and 5 m high. One cubic centimeter (1 cm^3) contains approximately $3 \bullet 10^{19}$ molecules of air. About how many molecules of air are there in the mall?
- **b.** If the managers' claim was true, what was the minimum number of cards in the deck?

3.6 Kuzlowski's Pizza offers 3 sizes of pizza, 3 types of crust, and 10 different toppings. Create an advertisement for Kuzlowski's that emphasizes the variety of pizzas available. Make your claims more believable by including the following:

a. the number of choices possible considering crust and size only

b. the number of combinations possible when choosing from 0 to 10 toppings [Hint: Find the number of ways that 0 toppings can be selected from 10, then the number of ways that 1 topping can be selected from 10, and so on.]

c. the total number of pizza choices.

3.7 The figure below shows the first five rows of a pattern you may recall from other modules: Pascal's triangle.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

a. Extend the triangle until it includes 10 rows.

b. Find the sum of the terms in the 10th row.

c. Describe how the sum of the terms in the 10th row of Pascal's triangle is related to your response to Problem **3.6b**.

d. What is the relationship between the combinations possible for each number of toppings in Problem **3.6** and the individual terms in the 10th row of Pascal's triangle?

e. Write the 10th row of Pascal's triangle using combination notation.

f. Write the first 5 rows of Pascal's triangle using combination notation.

3.8 **a.** Find the sum of the terms in each row of Pascal's triangle for at least the first 5 rows. Use your results to develop a formula for finding these sums.

b. Use your formula from Part **a** to verify the sum of the terms of the 10th row of Pascal's triangle from Problem **3.7b**.

c. Describe the significance of each number in your formula for the sum of the terms of the 10th row of Pascal's triangle as it applies to pizzas and their toppings.

d. Explain why your formula is directly related to the fundamental counting principle.

* * * * *

3.9 **a.** Predict a pattern for determining how many distinct subsets exist for a finite set.

b. If $R = \{1,\ 3,\ 8,\ 11\}$, how many distinct subsets of R exist?

c. List the subsets of R.

d. Verify your answer to Part **b** by writing the number as a sum of combinations.

3.10 Given 8 coplanar points, no 3 of which are collinear, how many different triangles can you draw?

3.11 In how many ways can a committee of 5 be selected from 20 people?

3.12 In how many ways can a 5-card hand be dealt from a 52-card deck?

* * * * * * * * * *

Summary Assessment

1. Compact disc technology has changed dramatically in the past few decades. The first compact disc (CD) players accepted only one disc at a time and could play music only in the order in which it was recorded.

a. Manufacturers soon provided an option that allowed users to hear songs in random order. Assuming all orders are possible and no songs are repeated, in how many ways could you play a CD with 15 songs?

b. The next generation of CD players could hold more than one disc. Some early models allowed users to play up to 5 discs. Imagine that 2 of these 5 discs each contain 13 songs, while the other 3 discs each contain 16 songs. Assuming all orders are possible and no songs are repeated, in how many ways could you play all the selections on the 5 discs?

2. Riverside Read and Feed plans to stock coffee mugs personalized with first and last initials.

a. How many different mugs would it take to stock every possible arrangement of initials?

b. Would you recommend that the store also stock mugs that include a middle initial? Explain your response.

3. As part of an advertising campaign, the Zone Sporting Goods sponsors a drawing. Each customer is allowed one entry in the drawing, and each entrant can win only one prize. A total of 800 people enter the drawing.

a. Each of the first 20 winners receives the same prize: a discount coupon for 20% off all purchases made on the following Monday. In how many ways can these winners be selected?

b. After the first 20 winners have been selected, each of the next 30 winners also receives an identical prize: 30% off all purchases made on the following Tuesday. In how many ways can these winners be selected?

c. After the first 50 winners have been selected, a final drawing is made from the remaining entrants for a single grand prize: 50% off all purchases for the next year. In how many ways can the grand-prize winner be selected?

d. Imagine that your name is entered in the drawing. You have waited patiently as the first 50 prizes were announced, without winning anything. What is the probability that you will win the grand prize.

Module Summary

- The **fundamental counting principle** provides a method for determining the total number of ways a task can be performed. If an event that can occur in *m* ways is followed by an event that can occur in *n* ways, then the total number of ways that the two events can occur is $m \bullet n$.
- A **permutation** is an ordered arrangement of symbols or objects. The number of permutations of *n* different symbols or objects taken *r* at a time, denoted by $P(n,r)$, is given by the following formula:

  $$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

  Another commonly used notation for permutations is ${}_nP_r$.
- A collection of symbols or objects in which order is *not* important is a **combination**. The number of combinations of *n* different symbols or objects taken *r* at a time, denoted by $C(n,r)$, is given by the following formula:

  $$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{P(n,r)}{r!} = \frac{1}{r!} \bullet \frac{n!}{(n-r)!} = \frac{n!}{r!(n-r)!}$$

  Two other common notations for combinations are ${}_nC_r$ and

  $$\binom{n}{r}$$

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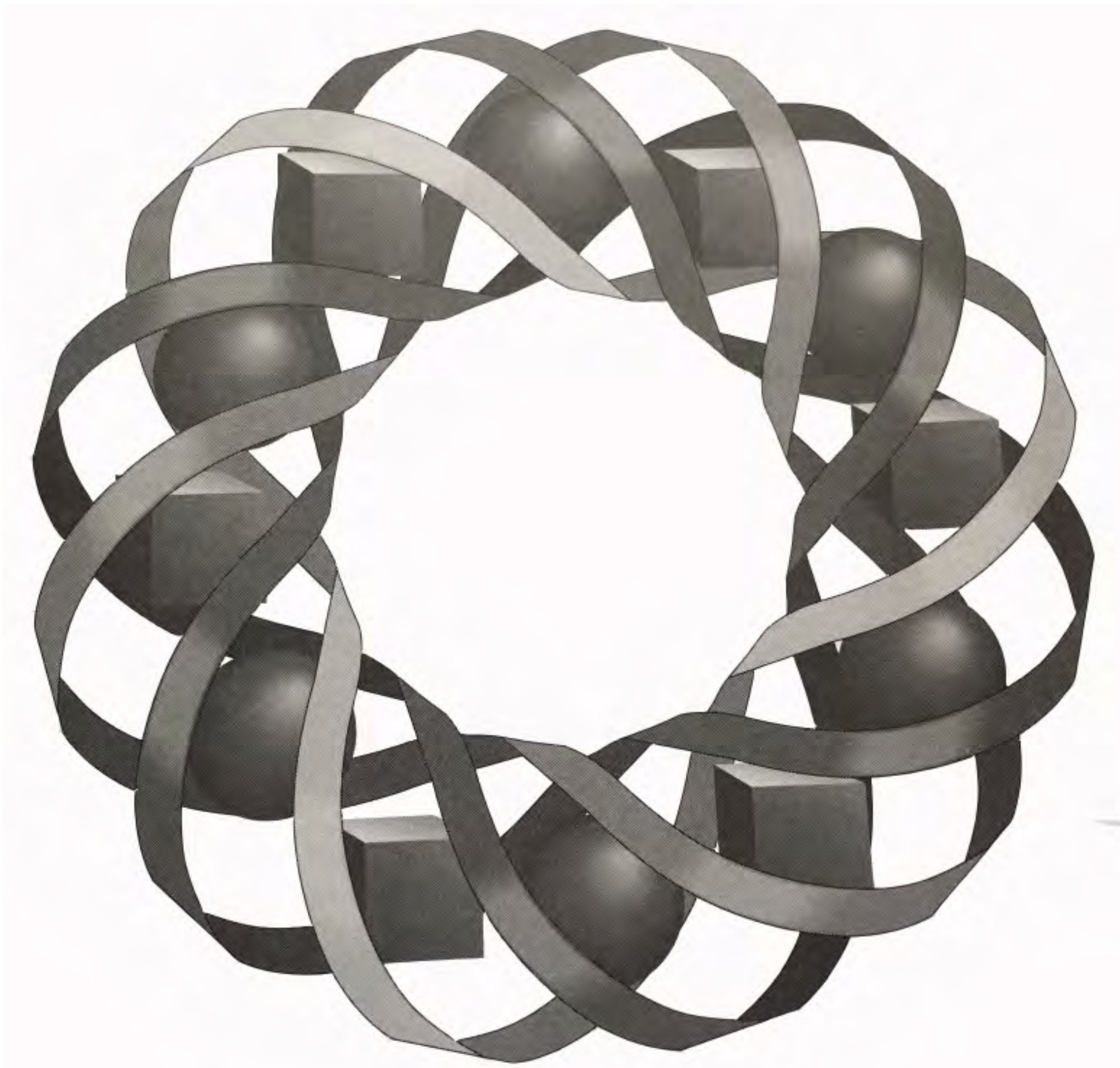
It’s All in the Family



When trying to fit a model to a set of data, it can help you to know the characteristics of different types of mathematical functions. In this module, you examine and compare the graphs of several families of functions.

Masha Albrecht • Gary Bauer • Art Perleberg

It's All in the Family

Introduction

Biologists and botanists have established a classification system that can be used to describe any animal or plant. In this system, organisms with common characteristics are grouped together. For example, crocodiles, lizards, turtles, and snakes belong to the class Reptilia. These animals are all classified as reptiles based on, among other things, the fact that their bodies are covered with scales or bony plates.

As the number of shared characteristics increase, animals can be placed in more restrictive categories, such as orders and families. For example, the several different kinds of tropical crocodiles all belong to the family Crocodilidae.

Relationships among mathematical functions also can be described in terms of shared characteristics or common behaviors. In this module, you investigate some families of functions.

Exploration

Figure **1** shows a bicycle tire with a pebble stuck in its tread. If the tire is rolled to the right on a flat surface, so that the pebble rotates in a clockwise fashion, what would the path of the pebble look like?

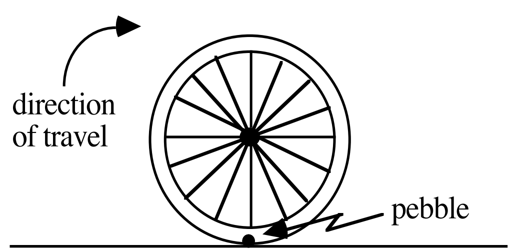


Figure 1: Bicycle tire with pebble

a. Use a cardboard or plastic disk to simulate a bicycle tire.

b. Cut a small notch in the disk and place a piece of chalk in the notch as shown in Figure **2**.

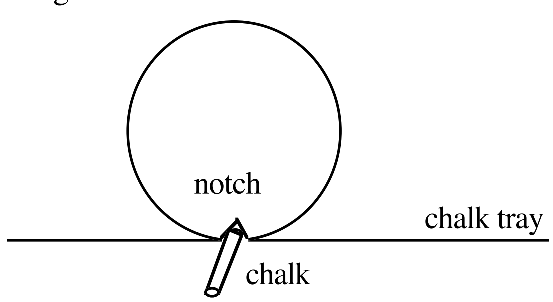


Figure 2: Disk with notch and chalk

c. Place the disk in the chalk tray of a blackboard, with the chalk at the lowest point of the disk and the end of the chalk resting against the board. Holding the chalk carefully in the notch, roll the disk clockwise along the tray. The chalk should trace a path that simulates the path of a pebble imbedded in a tire. Roll the disk until the chalk has completed two full revolutions.

d. To observe how the tire's radius affects the path of the pebble, repeat Parts **b** and **c** using a disk with a radius half that of your original disk.

Discussion

a. Do you think that the curves sketched in the exploration could represent the graphs of functions? Explain your response.

b. In the module "Can It," you used the mathematical terms *period* and *amplitude* to describe circular functions. Use these terms to describe the graphs from the exploration.

c. What characteristics would you use to describe the family of functions which includes the graphs from the exploration?

d. Would you place these graphs in the same class of functions as the sine and cosine functions? Explain your response.

Mathematics Note

A **family** of functions is a set of functions that have a common **parent**. Each family member is generated by performing one or more transformations on the parent function.

For example, consider the family of sine functions, a subset of the periodic functions. The parent function of the family of sine functions is $y = \sin x$. This relationship can be illustrated using the Venn diagram shown in Figure **3**.

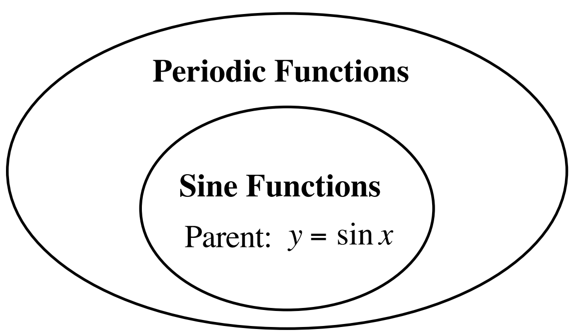


Figure 3: Venn diagram of periodic functions

The family of sine functions has an infinite number of members, all of which represent transformations of $y = \sin x$. Three of these functions are $y = 2\sin x$, $y = \sin x + 3$, and $y = \sin(x - \pi)$.

Activity 1

In the module "Can It," you modified the shapes and locations of graphs of the sine and cosine functions to model real-world data. In this activity, you observe how these modifications affect the graphs of several other families of functions.

Exploration

a. Consider the parent function $y = 2^{-x^2}$. Use a graphing utility to create a graph of this function.

b. Sketch a copy of the graph on a sheet of graph paper.

c. On the same coordinate system as in Part **b**, sketch a graph of the family member that results when the parent function is translated horizontally 1 unit to the right.

d. Use a graphing utility to determine which of the following equations represents the family member you sketched in Part **c**.

1. $y = 2^{-x^2} + 1$

2. $y = 2^{-x^2} - 1$

3. $y = 2^{-(x-1)^2}$

4. $y = 2^{-(x+1)^2}$

e. Predict the equation of the family member that results when the parent function is translated 2 units to the left. Verify your equation using a graphing utility.

f. Compare the graphs of each of the following pairs of functions:

1. $y = 2^{-x^2}$ and $y = 2^{-(3x)^2}$

2. $y = 2^{-x^2}$ and $y = 3 \bullet 2^{-x^2}$

g. Use a graphing utility to investigate the transformations of the parent function created by each of the following forms. In each case, experiment with both negative and positive values for the constant.

1. $y = 2^{-x^2} + d$

2. $y = 2^{-(x-c)^2}$

3. $y = 2^{-(bx)^2}$

4. $y = a \bullet 2^{-x^2}$

Discussion

a. The transformations you examined in the exploration can be applied to many other families of functions. For example, consider the family based on the parent function $y = x^2$.

1. What are the domain and range of the function $y = x^2$?
2. How would you modify this equation to obtain a function that is a vertical translation of the parent by 2 units?
3. How do the domain and range of the transformed function compare with those of the parent?

b.

1. How would you modify the equation $y = x^2$ to obtain a function that is a horizontal translation of the parent by 2 units?
2. How do the domain and range of the transformed function compare with those of the parent?

c.

1. In a function of the form $y = a \bullet x^2$, what transformation of the parent function $y = x^2$ results when $a = -1$?
2. How do the domains and ranges of the two functions compare?

d.

1. How would you modify the equation $y = x^2$ to obtain a function that is a vertical "stretch" of the parent?
2. Select a specific vertical stretch of the parent. How do the domains and ranges of the two functions compare?

e.

1. How would you modify the equation $y = x^2$ to obtain a function that is a horizontal "stretch" of the parent?
2. Select a specific horizontal stretch of the parent. How do the domains and ranges of the two functions compare?

Mathematics Note

For any point (x,y) of a function $f(x)$, adding a nonzero real number d to y results in an image point $(x,y+d)$. When this occurs, the graph is translated d units vertically. The equation of the resulting graph is $y = f(x) + d$.

For any point (x,y) of a function $f(x)$, subtracting a nonzero real number c from x results in an image point $(x-c,y)$. When this occurs, the graph is translated c units horizontally. The equation of the resulting graph is $y = f(x-c)$.

For any point (x,y) of a function $f(x)$, multiplying y by a nonzero real number a results in an image point (x,ay). When this occurs, the graph appears to be vertically "stretched" or "shrunk," depending on the value of a. The equation of the resulting graph is $y = a \bullet f(x)$.

For any point (x,y) of a function $f(x)$, multiplying x by a nonzero real number b results in an image point (bx,y). When this occurs, the graph appears to be horizontally "stretched" or "shrunk," depending on the value of b. The equation of the resulting graph is

$$y = f\left(\frac{1}{b}x\right).$$

For example, the graphs in Figure **4** show four transformations of the parent $f(x) = \sin x$.

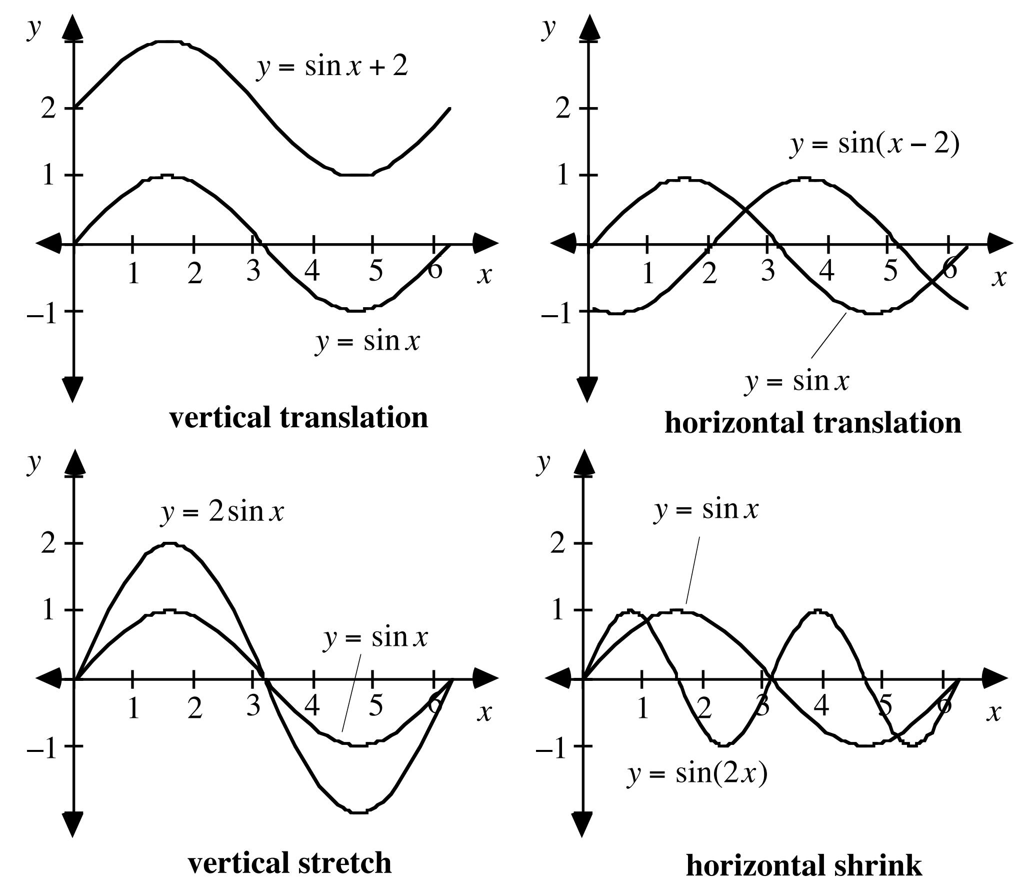


Figure 4: Four transformations of $y = \sin x$

f. As mentioned in the mathematics note, multiplying each x in an ordered pair (x,y) of a function $f(x)$ by a nonzero real number b results in a new function whose points are of the form (bx,y). Select a function f and use ordered pairs to show why the equation of the new function is

$$y = f\left(\frac{1}{b}x\right).$$

Assignment

1.1 **a.** **1.** Sketch a graph of the parent function $y = \cos x$.

2. Identify the domain and range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is reflected in the x-axis.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is translated horizontally 1 unit to the left.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

d. **1.** Sketch a graph of the family member that results when every y-value of the parent function is multiplied by 3.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

e. **1.** Sketch a graph of the family member that results when every x-value of the parent function is multiplied by $1/2$.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.2 **a.** **1.** Sketch a graph of the parent function $y = 1/x$ over the domain of nonzero real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is translated –3 units vertically.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is reflected in the x-axis.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.3 **a.** **1.** Sketch a graph of $y = 3^x$ over the domain of real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the *y*-values of the parent function are multiplied by 0.25.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is translated –5 units horizontally.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.4 **a.** **1.** Sketch a graph of $y = x^3$ over the domain of real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is translated –4 units horizontally.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when every *x*-value of the parent function is multiplied by 10.

2. Determine the domain and range of the resulting function.

3. Write the equation for the new function.

* * * * *

1.5. **a.** Sketch a graph of the parent function $y = \log x$.

b. Sketch a graph of the parent function translated 5 units horizontally.

c. Sketch a graph of the family member that results when the *x*-values of the parent function are multiplied by $1/3$.

1.6 **a.** Sketch a graph of the parent function $y = \sqrt{x}$.

b. Sketch the graph of the parent translated 3 units vertically.

c. Sketch a graph of the family member that results when every *y*-value of the parent function is multiplied by 0.2.

1.7 **a.** Sketch a graph of the parent function $y = |x|$.

b. Sketch the graph of the family member that results when the *x*-values of the parent function are multiplied by 0.25.

c. Sketch the graph of the family member that results when the parent function is translated 4 units horizontally.

* * * * * * * * * *

Research Project

The terms *pitch*, *frequency*, *period*, and *amplitude* are associated with both the properties of sound waves and a particular family of functions. If you have access to an oscilloscope, record the graphs of some musical notes. Prepare a brief summary of your observations for the class, including answers to the following questions:

a. What does the graph of a musical note look like?

b. How does raising the pitch of a sound affect the period of its graph?

c. How does the intensity of a sound affect the amplitude of its graph?

Activity 2

In Activity **1**, you performed transformations on a parent function to obtain new family members, then determined the equations of the resulting functions. In this activity, you use the equations of family members to determine the transformations performed on the parent.

Exploration

Table **1** shows two parent functions along with seven members of their corresponding families. Complete Parts **a** and **b** for the functions in each column.

a. Predict how the graph of each transformed function in Table **1** differs from the graph of its parent function.

Table 1: Parent functions and family members

	Parent $y = 1/x$	**Parent $y = x^2$**
1.	$y = \dfrac{1}{x-3}$	$y = (x-3)^2$
2.	$y - 2 = 1/x$	$y - 2 = x^2$
3.	$y = -(1/x)$	$y = -\left(x^2\right)$
4.	$y - 4 = -\left(\dfrac{1}{x+1}\right)$	$y - 4 = -(x+1)^2$
5.	$y = \dfrac{1}{2x}$	$y = (2x)^2$
6.	$y = \dfrac{1}{(0.5x)}$	$y = (0.5x)^2$
7.	$y + 1 = -\left(\dfrac{1}{3(x-2)}\right)$	$y + 1 = -\left(3(x-2)^2\right)$

b. Graph each transformed function in Table **1** on the same set of axes as its parent. Record your observations.

c. Compare your observations for the transformations of $y = 1/x$ to the transformations of $y = x^2$.

d. Record any generalizations that appear to apply to both families of functions.

Discussion

a.
1. Given an unknown function $y = f(x)$, how would you write the equation of the function whose graph represents $y = f(x)$ translated 3 units horizontally and –5 units vertically?
2. How would you write the equation of the function whose graph represents $y = f(x)$ reflected in the x-axis, stretched vertically by 4, and stretched horizontally by 2?

b. Consider an equation of the form $(y - d) = f\big(b(x - c)\big)$. How does each of the constants b, c, and d transform the graph of $y = f(x)$?

Assignment

2.1 Describe how the graph of $y - 5 = 3(x - 2)$ is related to the graph of $y = x$.

2.2 In Parts **a–c** below, you determine how the transformation rules developed in the discussion apply to a family of exponential functions based on a parent function of the form $y = a^x$.

a. Select a parent for a family of exponential functions and sketch its graph.

b. Determine the equation of the function that results from the following transformations of the parent: a translation 6 units vertically, a translation –3 units horizontally, and a reflection in the x-axis.

c. Verify your equation in Part **b** using a graphing utility.

2.3 For each equation below, write the equation of a possible parent function and describe the corresponding transformations from the parent.

a. $y = \sin\left[3\left(x - \frac{\pi}{2}\right)\right]$

b. $y = 2x^3 - 90$

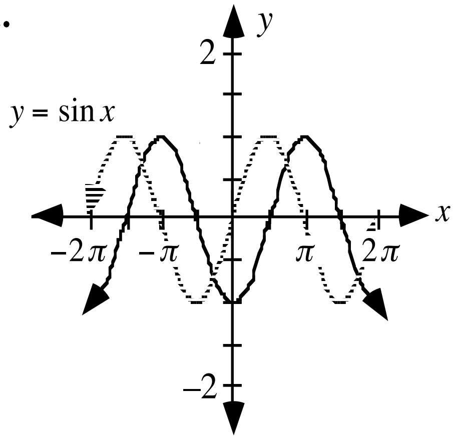
c. $y = 10^{(x-8)}$

d. $y = \dfrac{-100}{(x + 100)^2}$

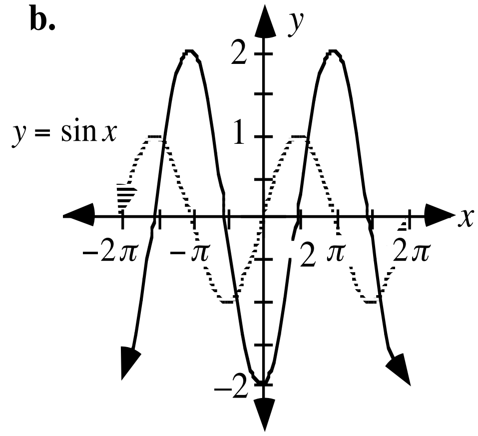
e. $y = \log(-x + 2.3) - 4.7$

2.4 On each set of axes below, the graph of the parent function $y = \sin x$ is represented by a dotted line, while the graph of the transformed function is represented by a solid line. Write an equation for the graph of each family member in terms of the parent function.

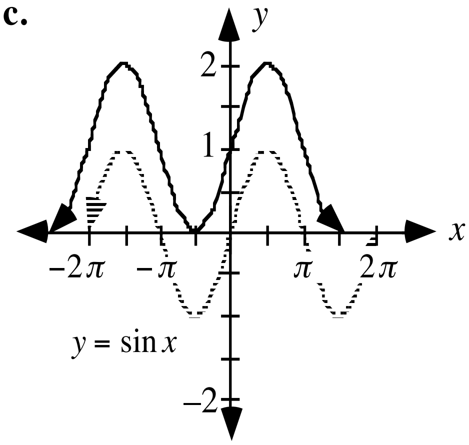
a.



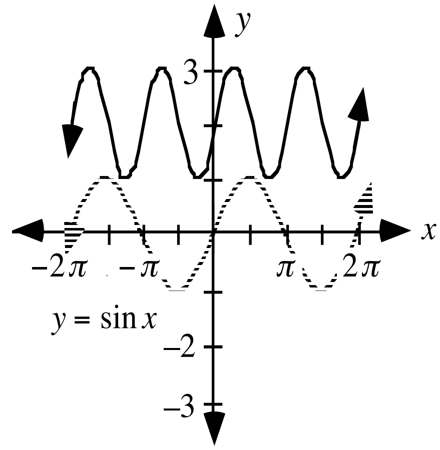
b.



c.



d.



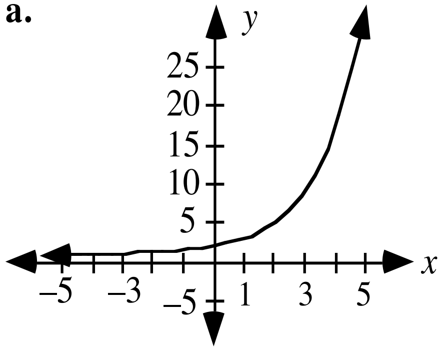
2.5 A horizontal translation of a periodic function is often referred to as a **phase shift**.

a. Are there any phase shifts in the graphs for Problem **2.4**? If so, describe the magnitude and direction of each one.

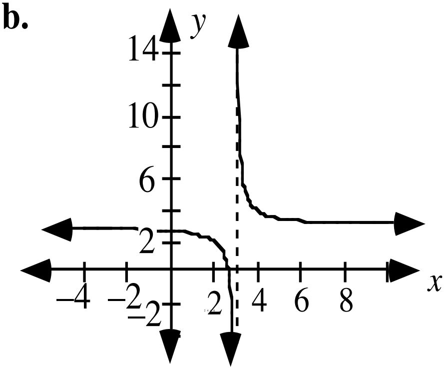
b. If the function $y = 3\cos x$ undergoes a phase shift of $-\pi/6$ radians, what is the equation for the transformed function?

2.6 Write a possible equation for the function in each of the following graphs and identify its corresponding parent.

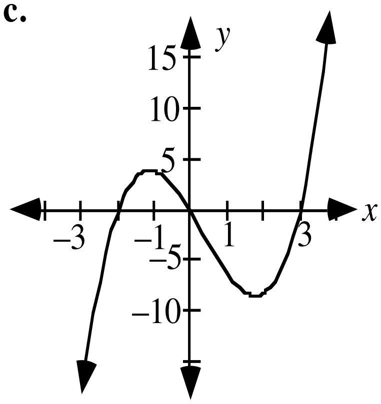
a.



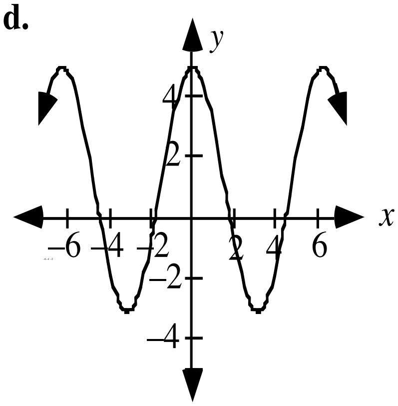
b.



c.



d.



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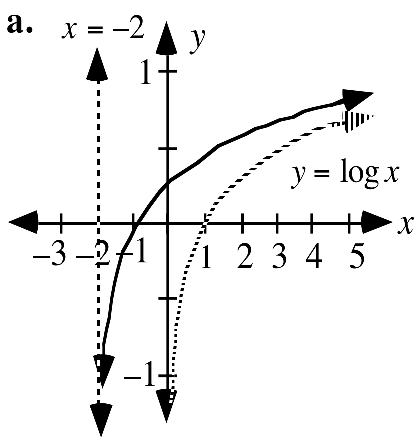
2.7 List at least five members of the family of second-degree polynomials. Describe how each one is related to the parent function.

2.8 On each set of axes below, the graph of the parent function $y = \log x$ is represented by a dotted line, while the graph of the transformed function is represented by a solid line. Write an equation for the graph of each family member in terms of the parent function.

a.



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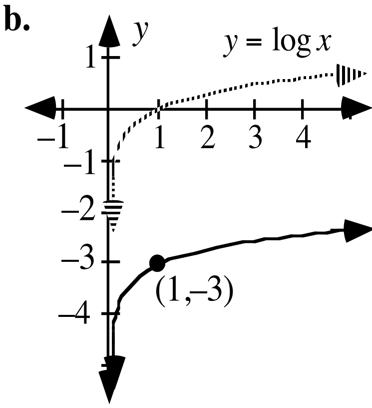
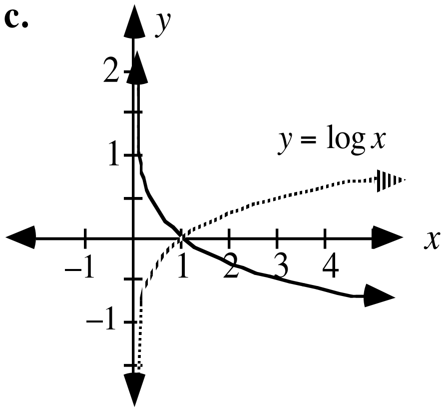
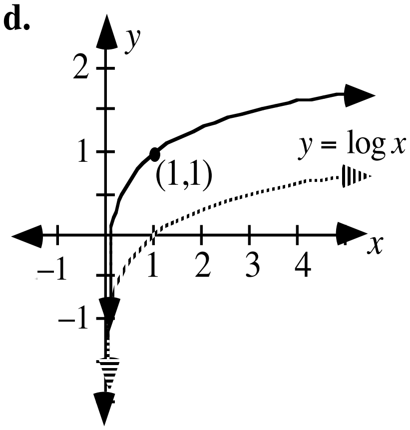


c.



d.



2.9 Describe how the constants a, b, c, and d transform the graph of each parent function below.

	Parent Function	Family Member
a.	$y = \tan x$	$(y - d) = a \tan[b(x - c)]$
b.	$y = 2^x$	$(y - d) = a \bullet 2^{(b(x-c))}$
c.	$y = \lvert x \rvert$	$(y - d) = a \bullet \lvert b(x - c) \rvert$

2.10 Write the equation of a parent function, if any, that illustrates each of the following characteristics:

a. has a vertical asymptote

b. has a horizontal asymptote

c. can be drawn without lifting a pencil

d. is periodic

e. has a discontinuity

f. contains the origin.

* * * * * * * * * *

Activity 3

Your knowledge of families of functions may prove especially helpful when modeling data sets. Selecting the best function with which to model a data set can be a difficult task. Any finite number of data points, for example, can be modeled by an infinite number of polynomial functions of different degrees. Although each of these polynomial functions may fit the points perfectly, they may not prove useful for making predictions.

A good model must do more than give a reasonable approximation of the data. It should also provide some insight into the relationships that exist in the real-world situation. For example, Figure **5** below shows two curves, $f(x)$ and $g(x)$, that fit a set of data points. One of the curves is a graph of a fifth-degree polynomial function, the other is the graph of a linear function. Which is the better model? That depends on the situation in which the data was collected.

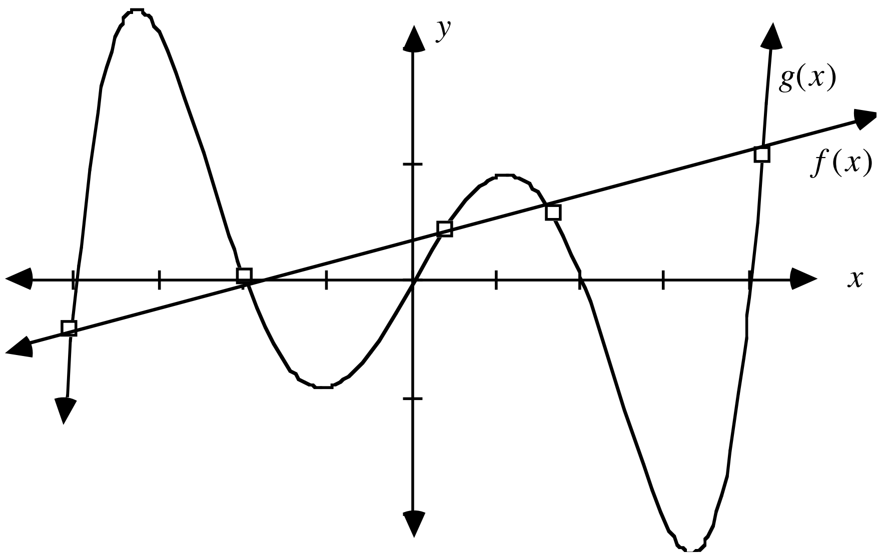


Figure 5: Two models for a set of data

Exploration

In this exploration, you investigate the relationship between light intensity and distance.

a. Obtain the following equipment from your teacher: a lamp with a 40-watt bulb, a science interface device with light sensor, and a meterstick.

b. As shown in Figure **6**, place the lamp on a flat surface. Position the meterstick 30 cm from the lamp.

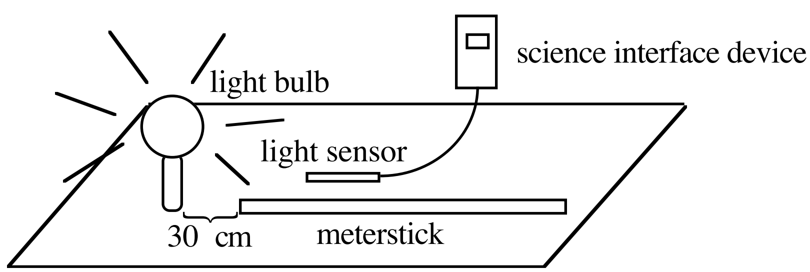


Figure 6: Lamp experiment

Darken the room. For best results, the room should contain no sources of light other than the lamp.

c. 1. Hold the sensor 30 cm from the lamp. Record both the distance from the lamp and the light intensity.

2. Move the sensor 5 cm farther from the lamp. Record both the distance and the intensity.

3. Repeat Step **2** until you have collected at least 15 data points.

d. Make a scatterplot of the data.

Discussion

a. Describe the shape of the scatterplot you created in Part **d** of the exploration.

b. What group of functions might be used to model this data? Explain your response.

c. Identify a possible parent function for a model. Defend your choice.

Assignment

3.1 The following table shows the frequencies (in beats per minute) of some musical notes. The first note in the table is concert A, the note an orchestra tunes to at the beginning of a concert.

Note	Frequency (beats/min)
A	440
B	494
C	523
D	587
E	659
F	698
G	784
A	880
B	987
C	1109
D	1245

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.2 The following data was collected during an experiment in which a ball was rolled up an inclined ramp. The ball eventually came to a stop, then rolled back down the incline. The distances in the table below were measured from a sonar range finder at the top of the ramp.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	2.1	1.4	0.9
0.2	2.1	1.6	1.0
0.4	1.9	1.9	1.1
0.6	1.5	2.2	1.6
0.8	1.2	2.4	2.0
1.0	1.0	2.6	2.2
1.2	0.9	2.8	1.3

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.3 Doctors use radioactive tracers to detect some diseases and injuries in patients. Since radioactive materials decay over time, the amount of tracer left in the patient decreases each day. The following table shows the percentage of tracer remaining at the end of each day.

Day	Percentage Remaining	Day	Percentage Remaining
0	100	5	33
1	81	6	26
2	65	7	21
3	51	8	16
4	41	9	13

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.4 The tide in the area near Boston, Massachusetts, varies by approximately 2.9 m from high tide to low tide. The tide changes from high to low and back to high approximately every 12.4 hr. In the following table, height represents distance in meters above low tide.

Time (hr)	Height (m)	Time (hr)	Height (m)
0	2.9	7	0.1
1	2.7	8	0.6
2	2.2	9	1.2
3	1.5	10	2.0
4	0.8	11	2.6
5	0.3	12	2.9
6	0.0	13	2.8

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

* * * * *

3.5 The following data was collected as a pendulum swung back and forth in front of a motion detector.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	1.393	2.4	1.856
0.4	1.145	2.8	1.838
0.8	0.851	3.2	1.549
1.2	0.682	3.6	1.308
1.6	0.859	4.0	0.841
2.0	1.328		

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.6 The table below shows some information on the eggs of 12 birds.

Type of Egg	Length (cm)	Mass (g)
hummingbird	1.3	0.5
black swift	2.5	3.5
dove	3.2	6.4
partridge	3.0	8.7
Arctic tern	4.2	18.0
grebe	4.3	19.7
Louisiana egret	4.5	27.5
mallard duck	6.2	80.0
great black-backed gull	7.6	111.0
Canada goose	8.9	197.0
condor	11.0	270.0
ostrich	17.0	1400.0

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.7 The data in the table below was collected during an experiment in which 11 mL of basic solution were added to an acidic solution, 1 mL at a time.

Basic Solution Added (mL)	pH of Solution
1	3.57
2	4.98
3	5.54
4	5.79
5	5.97
6	6.06
7	6.16
8	6.25
9	6.33
10	6.38
11	6.44

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

* * * * * * * * * *

Summary Assessment

1. Explain why it is possible to consider $y = \sin x$ as the parent function of $y = \cos x$.

2. In almost every presidential election since 1936, there has been at least three candidates on the ballot: one Democrat, one Republican, and at least one representative from a third party. The table below shows the number of votes (in thousands) that candidates from each party won in the presidential elections from 1976 to 1992.

Elections Since 1976	**Democratic Party**	**Republican Party**	**Other Major Parties**
1	40,831	39,148	910
2	35,484	43,904	6,641
3	37,577	54,455	307
4	41,809	48,886	649
5	44,909	39,104	20,034

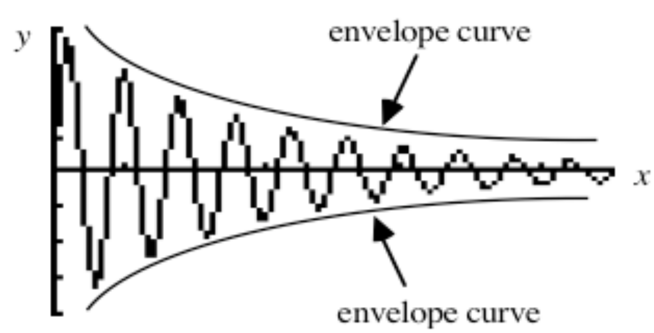
   a. Create a separate scatterplot of the data for each political party.

   b. Determine the equation of a function that could be used to model each data set. Graph each equation on the same coordinate system as the corresponding scatterplots from Part **a**.

   c. Using your equations from Part **b**, predict which party will win the next presidential election, then use the same models to predict the fate of each party in future elections.

   d. Write a paragraph describing the dangers of using your models to predict the outcomes of presidential elections.

3. Musical instruments like the piano cannot sustain notes for long periods of time. For example, the graph below represents the sound wave of a single note that was played then allowed to fade. The two curves tangent to the graph at its maximum and minimum values represent the **envelope** of the graph. What pair of equations might describe these envelope curves?



Module Summary

- A **family** of functions is a set of functions that have a common **parent**. Each family member is generated by performing one or more transformations on the parent function.
- For any point (x,y) of a function $f(x)$, adding a nonzero real number d to y results in an image point $(x,y+d)$. When this occurs, the graph is translated d units vertically. The equation of the resulting graph is $y = f(x) + d$.
- For any point (x,y) of a function $f(x)$, subtracting a nonzero real number c from x results in an image point $(x-c, y)$. When this occurs, the graph is translated c units horizontally. The equation of the resulting graph is $y = f(x-c)$.
- For any point (x,y) of a function $f(x)$, multiplying y by a nonzero real number a results in an image point (x,ay). When this occurs, the graph appears to be vertically "stretched" or "shrunk," depending on the value of a. The equation of the resulting graph is $y = a \bullet f(x)$.
- For any point (x,y) of a function $f(x)$, multiplying x by a nonzero real number b results in an image point (bx, y). When this occurs, the graph appears to be horizontally "stretched" or "shrunk," depending on the value of b. The equation of the resulting graph is

$$y = f\left(\frac{1}{b}x\right)$$

- The horizontal translation of a periodic function is often referred to as a **phase shift**.

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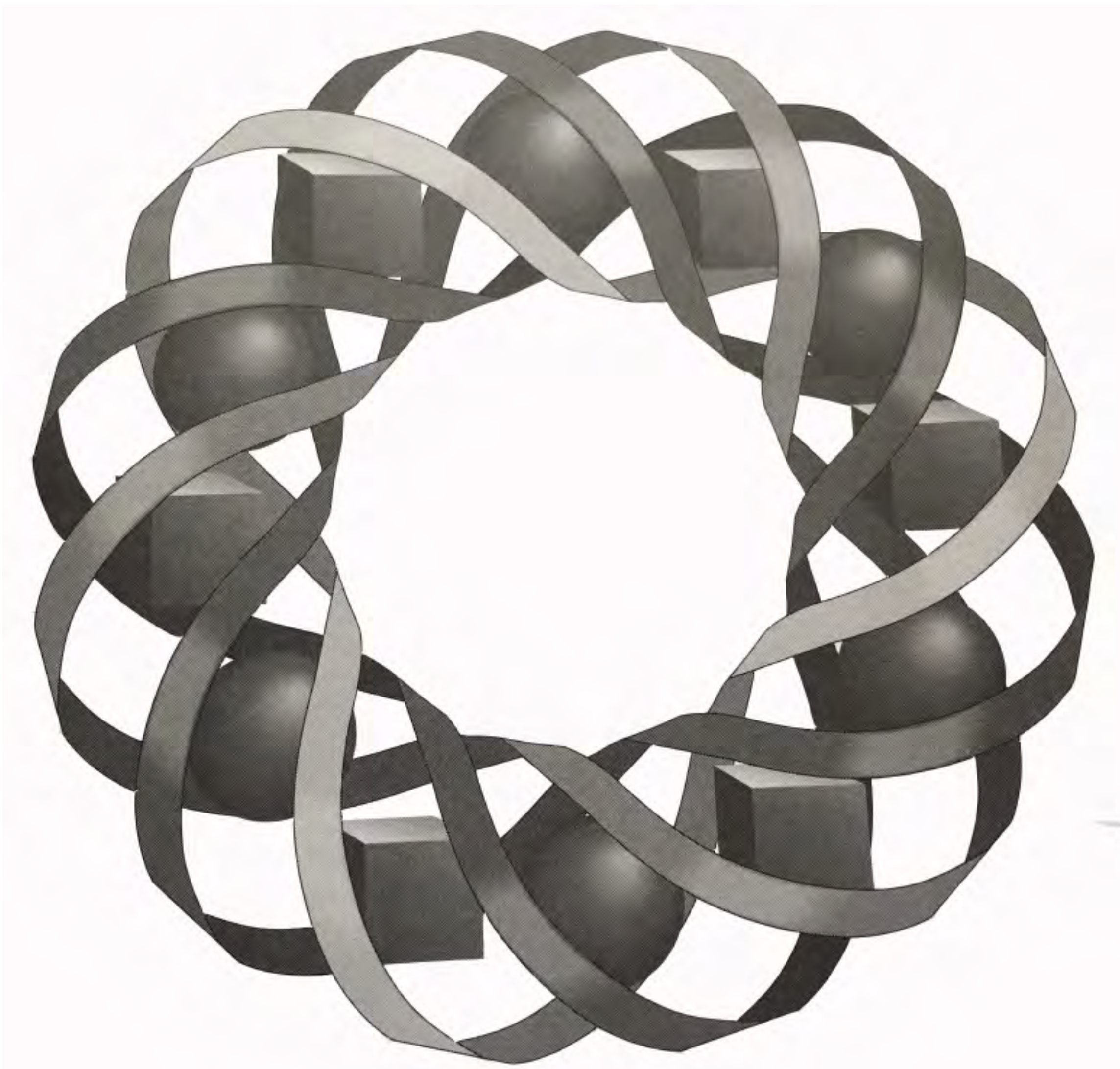
Wells, T. *The Technique of Electronic Music*. New York: Schirmer Books, 1981.

Confidence Builder



When a light-bulb company reports a mean life expectancy for its products, how confident can you be that their bulbs won't leave you in the dark?

Danny Jones • Kate Riley • John Knudson-Martin

Confidence Builder

Introduction

To obtain reliable information on populations, researchers depend on sampling techniques. The statistics generated through sampling typically are used to predict the parameters of a population. For example, political parties and government officials use the results of polls to help them gauge public opinion. Manufacturers rely on sampling to monitor the quality of their products. And economists use business statistics to study market trends.

Discussion

a. Describe the differences between a statistic and a parameter.

b. Describe some populations in your school and in your community.

c. What parameters might be of interest for each population described in Part **b**?

d.
1. Describe some different ways in which samples could be taken from the populations in Part **b**.
2. Which of these sampling methods generate simple random samples?

Activity 1

Imagine that your school board is planning to purchase some new student desks. Your class has been asked to describe the dimensions of a comfortable desk. Since these desks must accommodate a wide range of students, the class decides to investigate student heights first.

Some classmates suggest that the heights of students in the class could be used to estimate the mean height of all students in the school. Others argue that since the class is not a representative sample of the student population, the sample mean will not provide a good estimate of the population mean. How could you determine if the class mean is a reasonable approximation of the mean height of the entire student population?

Mathematics Note

The mean value for a population, or **population mean**, is denoted by the Greek letter μ (mu).

The **population standard deviation** is denoted by the Greek letter σ (sigma). It can be calculated using the following formula:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}}$$

where the population has N members represented by $x_1, x_2, \ldots, x_N$.

The mean value for a sample, or **sample mean**, is denoted by $\bar{x}$ (read "x-bar"). The **sample standard deviation**, denoted by s, can be calculated as follows:

$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n - 1}}$$

where the sample has n members represented by $x_1, x_2, \ldots, x_n$.

Notice that the denominator used to calculate the sample standard deviation is slightly different from the denominator used to calculate the population standard deviation. When calculating sample standard deviation, the denominator $n - 1$ provides a better estimate of the population standard deviation.

For example, consider a population of the digits from 0 to 9. The population mean μ is 4.5. The population standard deviation σ is approximately 2.87. For the following sample of 5 digits taken from this population—4, 6, 8, 7, 9—the sample mean $\bar{x}$ is 6.8. The sample standard deviation s is approximately 1.9.

Exploration

In this exploration, you use sampling techniques to estimate the mean height of your class.

a. Select a simple random sample of students in your class. Measure and record the height (in centimeters) of each student in the sample.

b. Calculate the mean height ($\bar{x}$) and the standard deviation (s) of the sample data.

c. Measure and record the height (in centimeters) of all students in the class population.

d. Calculate the mean height (μ) and the standard deviation (σ) of the class data.

e. As shown in Figure **1**, a relative frequency histogram consists of bars of equal width whose heights indicate the relative frequencies of measurements in the corresponding intervals.

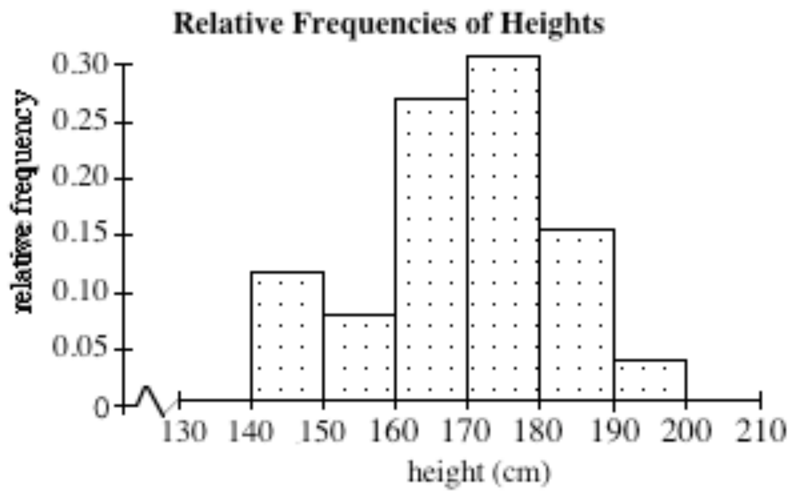


Figure 1: A relative frequency histogram

1. Create a relative frequency histogram of the class data using appropriate intervals for student heights.
2. Sketch the relative frequency polygon of the class data on the histogram from Step **1**.

Discussion

a.

1. How well did the sample mean approximate the actual mean height of the class?
2. What factors might have affected the accuracy of the sample mean as an estimate of the population mean?
3. Do you think that the heights of students in the class represent an unbiased sample of the heights of students in the entire school? Explain your response.

b. Describe the shape of the relative frequency polygon created in Part **e** of the exploration.

c. How would the shape of the relative frequency polygon change if the population consisted of professional basketball players?

Mathematics Note

A **normal distribution** is a continuous probability distribution. The graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. As in all continuous probability distributions, the total area between the *x*-axis and a normal curve is 1.

Although all normal curves have the same general shape, the width of any particular curve depends on the standard deviation (σ) of the distribution that the curve models. For example, Figure **2** shows two normal curves that have the same mean but different standard deviations.

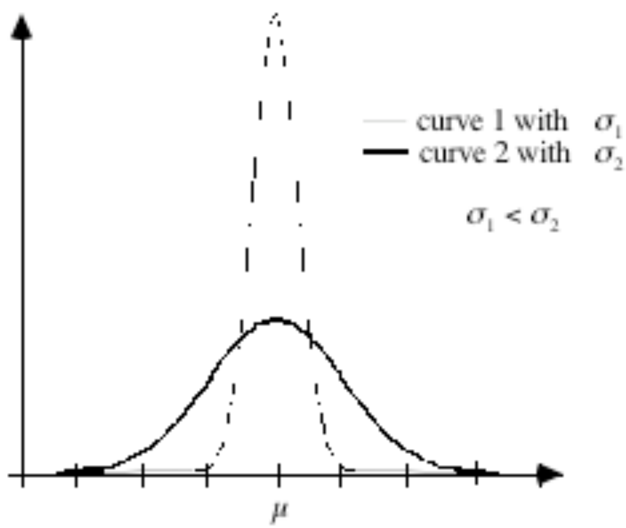


Figure 2: Two normal curves with the same mean

Since the population mean (μ) is located at the point where the curve's axis of symmetry intersects the *x*-axis, the position of the curve along the *x*-axis depends on the value of μ. Two normal curves with different means but the same standard deviation are shown in Figure **3**.

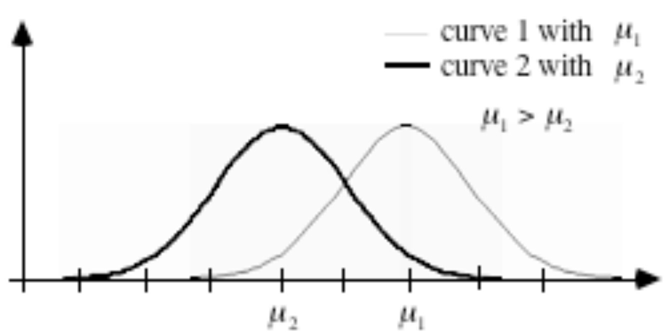


Figure 3: Two normal curves with the same standard deviation

d. **1.** What differences do you observe between the two normal curves in Figure **2**?

2. What differences do you observe between the two normal curves in Figure **3**?

e. Describe how the value of σ affects the shape of a normal curve.

f. Describe how the value of μ affects the position of a normal curve.

Assignment

1.1 **a.** For a survey on the size of local households, record the number of people living in each of five different households in your community.

b. Calculate the mean and standard deviation of your data.

c. Describe any biases your sample may contain with respect to each of the following populations:

1. your community
2. the United States
3. the world.

1.2 Write the whole numbers from 1 to 30, one at a time, on 30 identical slips of paper. Place the slips of paper in a container.

a. What is the mean of this population?

b. Select a random sample of two numbers from the container. Find the mean of the sample.

c. Repeat Part **b** for a sample of 5 numbers and a sample of 20 numbers.

d. Compare the three sample means to the population mean. Describe any trends you observe.

Mathematics Note

The **law of large numbers** states that, for very large sample sizes, there is a high probability that the sample mean is close to the population mean.

For example, suppose that you want to estimate the mean number of hours spent on homework in a population of 150 students. If you plan to sample this population, a sample of 50 students is likely to provide a better estimate of the population mean than a sample of 5 students.

1.3 **a.** Consider the whole numbers from 1 to 99 as a population. Using appropriate technology, generate three random samples from this population: one sample of 10 numbers, one of 20 numbers, and one of 80 numbers. Find the mean of each sample.

b. Repeat Part **a** at least three more times.

c. Compare the three sample means for each sample size to the population mean, $\mu = 50$. In a paragraph, describe how the results of this experiment relate to the law of large numbers.

* * * * *

1.4 The following table shows the summer earnings of a population of students.

$1872	$1341	$1792	$1650	$1422
$1413	$1900	$2143	$786	$451
$2432	$0	$243	$1381	$187
$0	$2443	$1408	$187	$0
$1228	$1119	$748	$949	$2011
$896	$1740	$0	$483	$846
$556	$780	$314	$768	$635

a. Calculate the mean and standard deviation of the data.

b. **1.** Select a random sample of five values from this population.

2. Calculate the mean and standard deviation of the sample.

c. **1.** Is your sample mean a good estimate of the population mean?

2. How could you select a sample that would provide a better estimate?

* * * * * * * * * *

Activity 2

How well do you think the mean height of your class estimates the mean height of your entire school? Your answer is likely to depend on the size of the school population. Since finding parameters for a large population is usually difficult (if not impossible), researchers use a variety of sampling techniques to obtain estimates.

In this activity, you examine how sample size can affect the distribution of sample means.

Exploration

A biologist has just received a shipment of 90 live fish for an upcoming research project. The hatchery claims that the mean length of fish in the shipment is 25 cm. The biologist, however, thinks that the fish look a little on the small side. Although it would be too much trouble to catch and measure every fish, you can help the biologist decide whether or not to believe the hatchery by sampling the population.

a.
   1. Using the template provided by your teacher, calculate the mean length and standard deviation of the fish population.
   2. Create a relative frequency polygon that represents the distribution of the data.

b. Cut the numbers representing fish lengths from the template and place them in a container. Mix the numbers thoroughly.

c. Select a simple random sample of 30 "fish" from your "pond." Record the length of one fish at a time, returning it to the pond before drawing the next. Be sure to mix the fish before each draw.

d. Calculate the mean and standard deviation of your sample.

e. Repeat Parts **c** and **d** nine more times.

f. Find the mean and standard deviation of your set of 10 sample means.

g. Create a relative frequency polygon that represents the distribution of the 10 sample means.

h. Collect the sample means of the entire class. Create a relative frequency polygon that represents the distribution of the class data.

Discussion

a. How did the mean and standard deviation of your 10 sample means in Part **f** of the exploration compare with those of your classmates?

b. How did the mean and standard deviation of the sample means calculated in Part **f** of the exploration compare to the mean and standard deviation of the original population?

c. Do you think that the mean length claimed by hatchery officials was a reasonable estimate of the population mean?

d. How did the relative frequency polygon you created in Part **g** of the exploration compare with those of your classmates?

e. Describe the relative frequency polygon created in Part **h** of the exploration using the class statistics.

Mathematics Note

The **central limit theorem** states that even if the population from which samples are taken is not normally distributed, the distribution of the means of all possible samples of the same size will be approximately normal. In other words, if you collect many samples of size *n* and create a relative frequency histogram and polygon of the sample means, the graph will tend to assume the bell shape of a normal curve. Figure **4** shows an example of such a distribution.

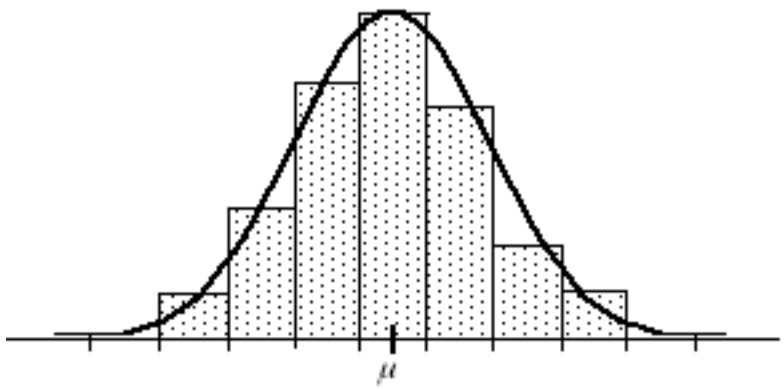


Figure 4: Normal distribution of sample means

This approximation becomes more accurate as the sample size *n* increases. Statisticians generally agree that for $n \geq 30$, the distribution of sample means can be modeled reasonably well by a normal curve. This requirement is not necessary if the population itself is normally distributed.

The mean of the distribution of all sample means equals μ, the population mean. The **standard deviation of all sample means**, denoted $\sigma_{\bar{x}}$, can be calculated using the following formula:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

where σ is the standard deviation of the population and *n* is the sample size.

For example, consider a population in which some characteristic is not normally distributed, with $\mu = 35$ and $\sigma = 5$. The standard deviation of the means for all samples of size 40 can be calculated as follows:

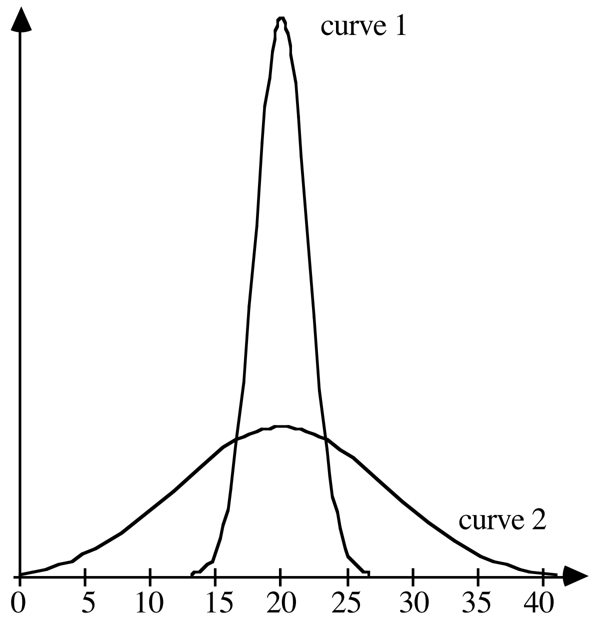
$$\sigma_{\bar{x}} = \frac{5}{\sqrt{40}} \approx 0.79$$

f. How does the sample size *n* affect the shape of the distribution of all possible sample means? Explain why this occurs.

g. When sampling a population, how can a researcher be reasonably sure that the sample mean is a good estimate of the population mean?

Assignment

2.1 The figure below shows two normal curves. One curve represents the distribution of a characteristic in a population. The other represents the distribution of sample means for that characteristic in the same population, using a sample size of 16.



a. Which curve represents the distribution of sample means? Defend your choice, including an explanation for the difference in the shapes of the two curves.

b. Use the appropriate curve to estimate the mean and standard deviation of the population.

c. Use your responses from Part **b** to estimate the mean and standard deviation of the sample means.

2.2 A family of six has the following heights (in centimeters): 145, 156, 163, 170, 174, and 188.

a. Find μ for this population.

b. Use combinations to determine the number of samples of two heights that can be taken from this population.

c. List all the possible samples of two heights from the population.

d. Find the mean of each sample in Part **c**.

e. Find the mean of five randomly chosen sample means from Part **d**.

f. Find the mean of all the sample means from Part **d**.

g. Compare the values obtained in Parts **a**, **e**, and **f** and write a summary of your findings.

2.3 **a.** Randomly generate 10 whole numbers from 1 to 9. Calculate the mean of these numbers.

b. Repeat Part **a** 19 more times, creating a population of 200 numbers from 1 to 9, and a group of 20 sample means.

c. Create a relative frequency histogram of the population of 200 numbers and describe the shape of the graph.

d. Create a relative frequency histogram of the 20 sample means and describe the shape of the graph.

e. How does the distribution of sample means compare to the distribution of the population from which the samples were taken? Use the central limit theorem to explain why the difference occurs.

2.4 A bottling plant fills bottles with soda. The volumes of the population of filled bottles are normally distributed, with a mean of 355 mL and a standard deviation of 2 mL. As part of the quality control process, a sample of four bottles is selected every hour. A technician records the mean volume of each sample. What are the mean and standard deviation of these sample means?

* * * * *

2.5 Consider a population consisting of the whole numbers from 1 to 99.

a. Determine the standard deviation of this population (σ).

b. Determine the standard deviation of the sample means ($\sigma_{\bar{x}}$) for samples of size 30 taken from this population.

c. How does the standard deviation of the population differ from the standard deviation of the sample means? Explain why you would expect this difference to occur.

2.6 The Sure Grip Tire Company manufactures motorcycle tires. The life spans of a population of its tires are normally distributed with a mean of 85,000 km and a standard deviation of 3,750 km.

a. What is the standard deviation of the sample means for samples of size 100 taken from this population?

b. How could the company decrease the size of the standard deviation of the sample means?

* * * * * * * * * *

Activity 3

According to the central limit theorem, even if a population is not normally distributed, the distribution of sample means can often be approximated reasonably well by a normal curve. This is one of the most useful facts in statistics.

Quality-control engineers, for example, frequently measure quality in terms of a product's mean life. Whenever possible, they model the results of their experiments with normal curves. When working with normal curves, they can express the proportion of the data located within a specific interval as a percentage. In Figure **5**, for example, the area under the curve that corresponds to the proportion of data in the interval $[a,b]$ is 55%. In this activity, you explore some of the properties that make normal curves so useful.

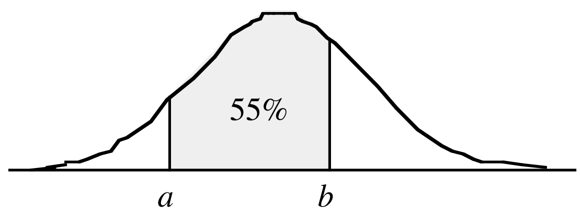


Figure 5: Percentage of area corresponding to [*a*, *b*]

Exploration

In this exploration, you use the set of whole numbers from 1 to 999 as a model population ($\mu = 500$ and $\sigma = 288$).

a. Calculate the standard deviation of the sample means ($\sigma_{\bar{x}}$) for samples of size 30 taken from this population.

b. Sketch a normal curve that models the distribution of sample means for samples of size 30.

c. Label the mean (μ) on the *x*-axis, as well as the values that are 1, 2, and 3 standard deviations from μ.

d.
1. Use technology to generate 50 random samples of size 30 from the population of whole numbers from 1 to 999.
2. Calculate the means of the 50 samples.

e. Determine the number of sample means that falls within each of the following intervals under the normal curve from Part **c**.
1. $[\mu - 1\sigma_{\bar{x}},\ \mu + 1\sigma_{\bar{x}}]$
2. $[\mu - 2\sigma_{\bar{x}},\ \mu + 2\sigma_{\bar{x}}]$
3. $[\mu - 3\sigma_{\bar{x}},\ \mu + 3\sigma_{\bar{x}}]$

f. Calculate the percentage of sample means that falls within each of the intervals described in Part **e**.

g. Collect the sample means obtained by the entire class. Determine the percentage of the class data that falls within each of the intervals described in Part **e**.

Mathematics Note

The **68–95–99.7 rule** states that approximately 68% of the total area between the normal curve and the *x*-axis lies within 1 standard deviation of the mean, 95% lies within 2 standard deviations of the mean, and 99.7% lies within 3 standard deviations of the mean. This rule is illustrated in Figure **6**.

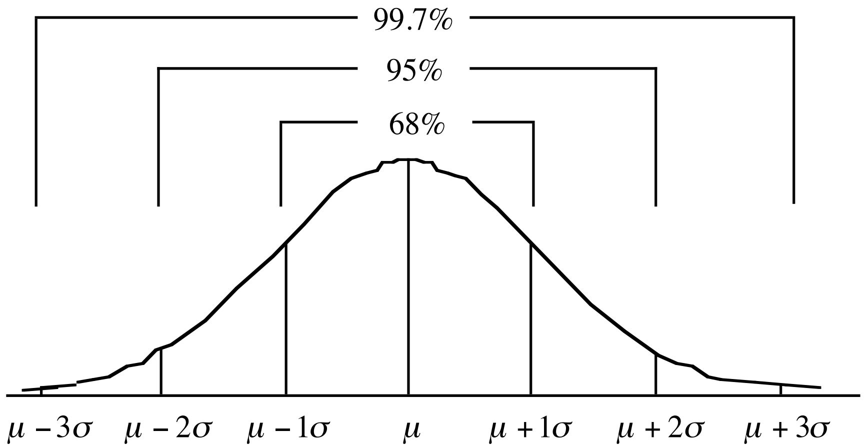


Figure 6: A normal curve and the 68–95–99.7 rule

For example, if the population mean is 100 and the standard deviation is 10, then you might expect about 68% of the sample means to lie between 90 and 110, 95% of the sample means to lie between 80 and 120, and 99.7% of the sample means to lie between 70 and 130.

Discussion

a. How did the percentages you obtained for the distribution of sample means compare to those described in the mathematics note?

b. How did the class percentages for the distribution of sample means compare to those in the mathematics note?

c. Explain why you might expect some differences between the percentages obtained in the exploration and the percentages given in the mathematics note.

d. How would you modify the exploration to obtain more accurate percentages?

Assignment

3.1 While researching automobile tires, you find that the mean life for samples of one particular brand is 60,000 km with a standard deviation of 1200 km. Assume that the life spans of tires in this population are normally distributed.

a. What percentage of the population would you expect to have life spans in the interval [58,800, 61,200]? Explain your response.

b. What are the upper and lower bounds of the interval within which you would expect the life spans of 95% of these tires to be found? Describe how you determined this interval.

3.2 Samples of a certain brand of light bulbs have a mean life of 1015 hr with a standard deviation of 75 hr.

a. What percentage of these bulbs would you expect to last longer than 1090 hr?

b. If a company purchases 1000 of these bulbs for use in its manufacturing plant, how many of them would you expect to burn out after less than 865 hr of use?

3.3 A quality control engineer who tests automobile seat belts selects a random sample of 36 men, aged 18 to 74, and measures their masses in kilograms. These masses are shown in the following table.

67.7	81.0	50.6	74.4	94.8	65.8
79.7	93.8	56.2	72.8	89.6	89.3
63.6	76.3	85.2	88.4	69.6	77.6
64.9	69.1	55.2	80.7	63.4	59.4
53.6	59.2	96.4	57.4	103.4	90.2
103.8	81.4	65.6	101.2	87.5	65.6

a. The masses of the population from which the engineer selected the sample are normally distributed with a mean of 75 kg and a standard deviation of 14.9 kg. Determine an interval that the engineer could expect to contain 68% of the data.

b. Find the actual percentage of data that lies within the interval you determined in Part **a**.

c. Explain why the percentage calculated in Part **b** might differ from 68%.

3.4 The quality control engineer described in Problem **3.3** decides to collect data from 19 more random samples of men (20 in all). The means of the samples (in kilograms) are shown in the table below.

76.0	83.1	75.8	82.1	68.4
76.1	80.6	77.4	79.6	75.4
90.0	68.9	85.4	80.5	82.0
81.2	73.4	62.3	83.2	82.5

a. Find the percentage of sample means that lie within the interval calculated in Problem **3.3a**.

b. Compare the percentage of sample means that lie within the interval to the percentage of masses from the single sample in Problem **3.3**. Explain any differences you observe.

* * * * *

3.5 A test was given to 3,000,000 people. The scores on the test were normally distributed, with a mean of 900 and a standard deviation of 212.

a. Determine the interval that contains 99.7% of the scores.

b. What percentage of the population received scores less than or equal to 1324?

c. What percentage of the population received scores less than or equal to 688?

3.6 The supervisor of the waiters and waitresses at El Burrito restaurant noticed that the amount of tips reported per shift seemed to be normally distributed with a mean of $27.35 and a standard deviation of $10.17.

a. Find the interval that contains 95% of the reported tips.

b. Find the interval that contains 99.7% of the reported tips.

c. What percentage of the staff make more than $17.18 in tips per shift?

d. From your experience with restaurants, would you have expected the amount of tips reported per shift to be normally distributed? Explain your response.

* * * * * * * * * *

Activity 4

When conducting statistical studies, researchers usually are concerned with large populations in which the mean value of a characteristic is unknown. In these situations, an estimate of the population mean is made using the statistics from a single sample. In this activity, you examine how confident researchers can be in the accuracy of their estimates.

Exploration

In this exploration, you again use the numbers from 1 to 999 as a model population ($\mu = 500$ and $\sigma = 288$).

a. Select a random sample of 30 numbers from the population and calculate $\bar{x}$.

b. Determine s, the standard deviation of this sample.

c. In most real-world situations, the mean of the population is unknown. In such cases, the sample mean ($\bar{x}$) is used to approximate the population mean (μ). In a similar manner, the sample standard deviation (s) is used to estimate the population standard deviation (σ).

1. Use the value for s from Part **b** to determine a value for the standard deviation of the sample means. In other words, estimate $\sigma_{\bar{x}}$ by substituting s for σ in the formula below, where n is the sample size:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

2. You can now use the mean of the sample ($\bar{x}$)and standard deviation of the sample means $(\sigma\ \bar{x})$to describe an interval that may contain the population mean. Create an interval that describes values that are no more than 2 standard deviations of the sample means on either side of $\bar{x}$.

d. Determine if the interval you created in Part **c** contains μ.

e. Repeat Parts **a–d** 49 more times (for a total of 50 samples). Record the number of times that the interval generated did not contain the population mean.

Discussion

a. What percentage of the time did the interval created in the exploration not contain the population mean?

b. The intervals you created in Part **c** of the exploration are known as **95% confidence intervals**. What would you expect to be true of these intervals?

c. Do your results in the exploration support the 95% figure?

d. How would you modify the process described in Part **c** of the exploration to obtain a 68% confidence interval?

e. How would you modify the sampling procedure to obtain a narrower 95% confidence interval? Explain your response.

Mathematics Note

A **confidence interval** for a parameter is an interval of numbers in which one would expect to find the value of that parameter. Every confidence interval has two aspects: an interval determined by the statistics collected from a random sample and a confidence level that gives the probability that the interval includes the parameter.

For example, a 95% confidence interval is generated by a process that results in an interval in which the probability that the parameter lies in that interval is 95%. In other words, you would expect 95% of the intervals produced by this process to contain the parameter, while 5% of the intervals would not.

The mean of all the sample means of a given sample size equals the mean of the population μ. Since sample means are normally distributed, the 68–95–99.7 rule can be applied. For example, the mean $\bar{x}$ of any one sample will fall within 2 standard deviations of μ 95% of the time. This fact also indicates a 95% probability that μ is within 2 standard deviations of $\bar{x}$. This 95% confidence interval can be represented algebraically as shown below:

$$\bar{x} - 2\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{x} + 2\left(\frac{\sigma}{\sqrt{n}}\right)$$

Since the population standard deviation usually is unknown, it is necessary to use the sample standard deviation s as an estimate of σ. In order to ensure that s properly approximates σ, the sample size n must be at least 30.

For example, imagine that a biologist selects a random sample of 100 fish from a lake. The mean length of the fish in the sample is 11 cm, with a standard deviation of 2.5 cm. To determine a 95% confidence interval, the biologist substitutes 11 for $\bar{x}$, 2.5 for s, and 100 for n as shown below:

$$11 - 2\left(\frac{2.5}{\sqrt{100}}\right) \le \mu \le 11 + 2\left(\frac{2.5}{\sqrt{100}}\right)$$

$$10.5 \le \mu \le 11.5$$

The biologist can then declare with 95% confidence that the mean length of fish in the population is in the interval [10.5, 11.5].

Assignment

4.1 **a.** Imagine that you have selected two random samples of the same size from the same population. Using the statistics from the two samples, you then determine two 95% confidence intervals for the population mean. Would you expect the two confidence intervals to be the same? Explain your response.

b. If you took 20 samples of the same size from the same population and determined 20 corresponding 95% confidence intervals, how many of them would you expect to contain the population mean? Explain your response.

4.2 **a.** For a given sample, would a 99.7% confidence interval be larger or smaller than a 95% confidence interval? Explain your response.

b. Write an algebraic representation of a 99.7% confidence interval given the sample mean ($\bar{x}$), sample standard deviation (s), and sample size (n).

4.3 A fish food manufacturer is developing a product to increase first-year growth in trout. After a year on this experimental diet, a random sample of 100 trout revealed a mean gain in mass of 84 g with a standard deviation of 14 g.

a. Use s to approximate the standard deviation of all possible sample means for $n = 100$.

b. Determine a 95% confidence interval for the mean gain in mass of trout fed the experimental diet.

c. Describe the meaning of the confidence interval in Part **b**.

d. How might the standard deviation of sample means be affected if the sample size were increased to 400 trout?

e. Why might a smaller standard deviation of sample means help the company market its fish diet?

4.4 A car manufacturer claims that its new model consumes fuel at a rate of 14 km/L. To verify this claim, the quality control engineers at a competing company selected a sample of 16 cars. Their tests yielded a mean of 13.5 km/L and a standard deviation of 1.6 km/L.

a. Create a 68% confidence interval for the rate of fuel consumption and determine if it contains the figure claimed by the manufacturer.

b. Create a 95% confidence interval for the rate of fuel consumption and determine if it contains the figure claimed by the manufacturer.

c. Do you believe the manufacturer's claim? Write a paragraph explaining how you reached your conclusion.

4.5 An advertising agency is investigating the number of hours that the residents of a particular city spend watching television each day. As part of their study, they surveyed a random sample of city households. The results of the survey are shown in the frequency table below.

Number of Hours Watching Television Per Day	Frequency
1	4
2	5
3	10
4	6
5	4
6	2
7	1
8	1

a. Construct a 95% confidence interval for the mean number of hours that city households spend watching television per day.

b. What does your response to Part **a** indicate about television viewing?

4.6 A 95% confidence interval for the mean life (in hours) of a particular brand of batteries is $410 \leq \mu \leq 450$.

a. Determine a 99.7% confidence interval for the mean life of these batteries.

b. What does your response to Part **a** indicate about battery life?

4.7 What factors should investigators consider when deciding whether to use a 68%, a 95%, or a 99.7% confidence interval?

* * * * *

4.8 A soft-drink company is monitoring the performance of its bottling equipment. According to the label, each bottle should contain 340 mL of soda. A quality-control specialist selects a random sample of 100 bottles each day for 5 days and measures their volumes. The table below displays the values for $\bar{x}$ and s for each day.

Day	$\bar{x}$ (mL)	s (mL)
1	340.8	2.6
2	340.2	2.1
3	340.9	1.7
4	340.4	2.3
5	340.1	0.9

a. Using the data for day 1, construct a 95% confidence interval for the mean volume of soda per bottle.

b. Using the data for day 2, construct a 95% confidence interval for the mean volume of soda per bottle.

c. **1.** Compare the two confidence intervals from Parts **a** and **b**.

2. How many different 95% confidence intervals would you get if you calculated one for each day of available data?

d. Is there any guarantee that any of the 95% confidence intervals obtained from the data will contain the actual mean volume per bottle? Explain your response.

e. Do you think that the company's labels accurately describe the milliliters of soda in a bottle? Write a paragraph to explain your conclusion.

* * * * * * * * * *

Research Project

Select a population and choose one of its characteristics to study. For example, if you select the students at your school as a population, you might wish to examine the mean number of hours spent studying per week.

Develop a sampling method that will allow you to determine a reasonably good estimate of the mean of this characteristic. After conducting your study, write a report that includes the following:

- descriptions of the population you surveyed, the characteristic you examined, and your sampling method
- the data you collected as well as the statistics your data generated, including a confidence interval
- a statement summarizing the conclusions you drew from your analysis
- any suggestions or recommendations that you believe your study supports.

Summary Assessment

Imagine that you are a journalist for the local newspaper. Your editor has requested that you write an article that uses statistics to describe one characteristic of a population. Your article must include two parts.

The first part of your article should describe the population, the characteristic you measured, and the sampling method used. It also should include the sample data you obtained.

To write this portion of your article, complete the following steps.

- Create a hypothetical population with some measurable characteristic.
- Design a method to sample this population.
- Create a simulation that models both your population and your sampling method.
- Use your simulation to generate some sample data.

The second part of your article should report on the results of your study. Include an estimate of the mean value of the characteristic for the population, and support your estimate with a discussion of confidence intervals and the normal distribution of sample means. To illustrate the mathematics involved in your analysis, you also should include the values for $\overline{x}$ and $\sigma_{\overline{x}}$ generated from your data, as well as a 95% confidence interval for μ .

Your article should conclude with any recommendations or suggestions that you feel your study supports.

Module Summary

- The mean value for a population, or **population mean**, is denoted by the Greek letter μ (mu).
- The **population standard deviation** is denoted by the Greek letter σ (sigma). It can be calculated using the following formula:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}}$$

  where the population has N members represented by $x_1,\ x_2,\ \ldots,\ x_N$.

- The mean value for a sample, or **sample mean**, is denoted by $\overline{x}$ (read "x-bar").
- The **sample standard deviation**, denoted by s, can be calculated as follows:

$$s = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n-1}}$$

  where the sample has n members represented by $x_1,\ x_2,\ \ldots,\ x_n$.

- A **normal distribution** is a continuous probability distribution. The graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. As in all continuous probability distributions, the total area between the x-axis and a normal curve is 1.
- Although all normal curves have the same general shape, the width of any particular curve depends on the standard deviation (σ) of the distribution that the curve models. Since the population mean (μ) is located at the point where the curve's axis of symmetry intersects the x-axis, the position of the curve along the x-axis depends on the value of μ.
- The **law of large numbers** states that, for very large sample sizes, there is a greater probability that the sample mean is close to the population mean.
- The **central limit theorem** states that, even if the population from which samples are taken is not normally distributed, the distribution of the means of all possible samples of the same size will be approximately normal.

  This approximation becomes more accurate as the sample size n increases. For $n \geq 30$, the distribution of sample means can be modeled reasonably well by a normal curve. This requirement is not necessary if the population from which samples are taken is normally distributed.

- The mean of the distribution of all sample means for a given sample size equals μ, the population mean.

- The **standard deviation of all sample means**, denoted by $\sigma_{\bar{x}}$, can be calculated using the following formula:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

  where σ is the standard deviation of the population and n is the sample size.

- The **68–95–99.7 rule** states that approximately 68% of the total area between the normal curve and the x-axis lies within 1 standard deviation of the mean, 95% lies within 2 standard deviations of the mean, and 99.7% lies within 3 standard deviations of the mean.

- A **confidence interval** for a parameter is an interval of numbers in which one would expect to find the value of that parameter. Every confidence interval has two aspects: an interval determined by the statistics collected from a random sample and a confidence level that gives the probability that the interval includes the parameter.

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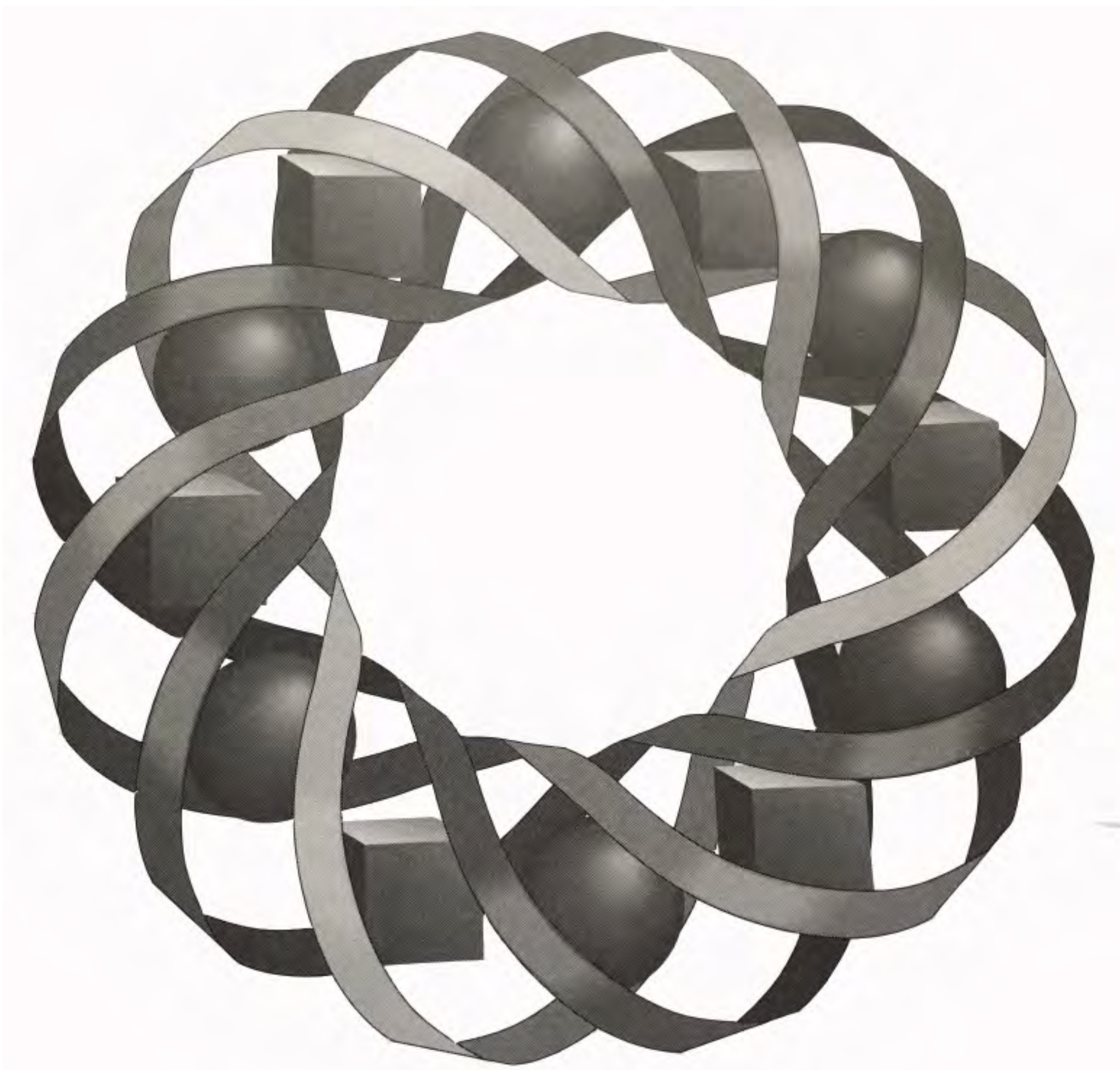
Transmitting Through Conics



What does watching television have to do with conic sections? In this module, you'll see how conics influence your favorite telecasts—from signal to satellite dish.

Anne Merrifield • Pete Stabio

Transmitting Through Conics

Introduction

Conic sections are geometric figures that play important roles in satellite, radio, and microwave communications. In this module, you investigate how conic sections can be described both geometrically and algebraically.

Exploration

In this exploration, you use a pencil to visualize a cone.

a. Hold a pencil at its midpoint between your thumb and finger, as illustrated in Figure **1**. While keeping the midpoint of the pencil stationary, move the pencil so that its tip draws a circle.

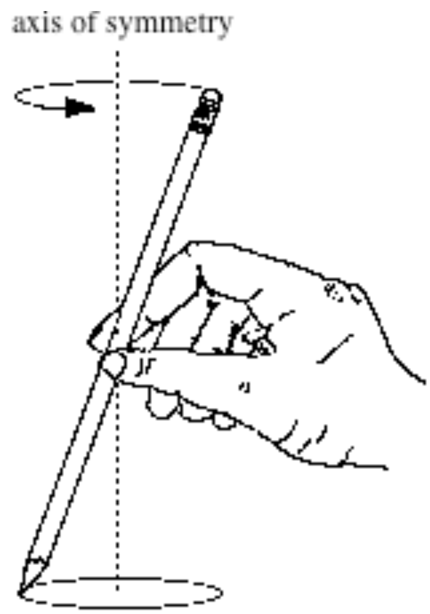


Figure 1: Pencil drawing a circle

b. Note the positions of the pencil as its tip draws a circle. Record your observations.

Mathematics Note

The set of all points that satisfy a given geometric condition is a **locus** (plural **loci**). For example, a circle is the locus of all points in a plane that are a given distance, the radius, from a fixed point, the center.

An equation defining a locus is satisfied by the coordinates of the points belonging to the locus and by no other points. In a coordinate plane, for example, the locus of the equation $y = 2x + 3$ is the set of all points on a line with a slope of 2 and a y-intercept of 3.

Discussion

a. Describe the locus of points traced by the tip of the pencil. How is this locus related to the one traced by the other end of the pencil?

b. As the pencil's tip traces a circle, describe the locus of the pencil's midpoint. This is the **apex** of the cone.

c. The pencil in the exploration represents a line segment. Describe the locus of points generated by a line moving in the same way as this pencil.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a cone. In a right circular cone, the conic section formed depends on the slope of the intersecting plane. The intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**, as shown in Figure **2** below.

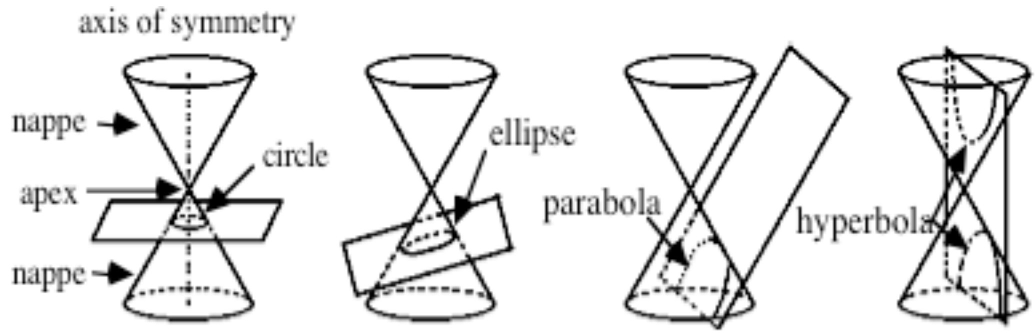


Figure 2: Conic sections

To visualize the four conic sections, imagine a plane whose slope gradually changes as it slices a cone. When the plane is perpendicular to the cone's axis of symmetry and intersects the cone in more than one point, the intersection is a circle. As the slope of the plane gradually changes, the intersection is an ellipse. When the plane is parallel to a line generating the cone, a parabola is formed. When the plane intersects both nappes, a hyperbola is formed.

d. Describe some familiar items that contain objects shaped liked conic sections.

e. How would you describe the shapes of the conic sections to someone who had never seen them before?

f. When the intersection of a plane and a cone contains the cone's apex, geometric figures other than a circle, an ellipse, a parabola, or a hyperbola are formed. These other intersections are **degenerate conic sections**.

There are three degenerate conic sections. Describe the shape of each one and the location of the plane relative to the cone.

Activity 1

In this activity, you investigate the locus of points that forms a circle.

Exploration

In this exploration, you use the Pythagorean theorem to develop an equation that describes a circle.

a. Select a point at an intersection of grid lines as the center of the circle.

b. Select a distance for the radius, r. Draw the locus of points that lie this distance from the center of the circle.

c. As you may recall from previous modules, the distance d between two points (x_1, y_1) and (x_2, y_2) can be found using a formula derived from the Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Use the distance formula to determine an equation for the circle you drew in Part **b**.

d. Repeat Parts **a–c** four more times, using a different radius for each circle and placing the center of each circle in a different quadrant.

Discussion

a. Consider a circle with center at (h, k) and a radius of r. Given an arbitrary point (x, y) on the circle, how could you use the distance formula to write an equation for the circle?

b. To write an equation without a radical, you must isolate the radical on one side of the equation, then square both sides of the equation. For example,

$$\sqrt{\left(x^2 + y^2\right)} = 4$$

$$\left(\sqrt{\left(x^2 + y^2\right)}\right)^2 = 4^2$$

$$x^2 + y^2 = 16$$

Square both sides of the equation you wrote in Part **a** of the discussion.

Mathematics Note

The **standard form** of the equation of a circle with center at (h, k) and radius r is:

$$(x-h)^2+(y-k)^2=r^2$$

For example, the standard form of the equation for a circle with center at (–2,4) and a radius of 5 is $(x+2)^2+(y-4)^2=25$.

c. How does the standard form of the equation of a circle compare to the equation you determined in Part **a** of the discussion?

d. How could you determine whether or not any point in the coordinate plane is a point on a given circle?

e. If $x^2=16$, is it always true that $\sqrt{x^2}=4$? Justify your response.

f. Some graphing utilities graph only functions. Since a circle does not represent a function, its graph may have to be produced by graphing two functions that represent halves of the circle.

1. To find these two functions for the circle described by $x^2+y^2=16$, for example, the equation can be solved for y. Identify these two functions.
2. Determine the appropriate domain and range for each function in Step **1**.
3. Do you believe that a combined graph of the two functions in Step **1** includes all the points of the circle defined by $x^2+y^2=16$? Explain your response.

Assignment

1.1 Write the equation in standard form of each of the following:

a. a circle with center at the origin and a radius of 3

b. a circle with center at (–1.3,8.9) and a radius of $\sqrt{21}$

c. a circle with center at (a, b) and a radius of $\sqrt{c}$.

1.2 Radio signals may be thought of as concentric circles (circles with the same center) emitted from a transmitter. Write the equations of three concentric circles.

1.3 Identify the centers and radii of the circles described by the following equations:

a. $(x-30)^2+(y-120)^2=289$

b. $(x+21)^2+(y-73)^2=141$

c. $x^2+y^2=121$

1.4 **a.** For each circle described by an equation in Problem **1.3**, list the coordinates of:

1. a point on the circle

2. a point inside the circle

3. a point outside the circle.

b. When the coordinates of a point outside a circle are substituted into its equation in standard form, what must be true of the result?

* * * * *

1.5 The center of a circle is located at (–2,–8). The coordinates of one point on the circle are (2,6). Find the equation of the circle in standard form.

1.6 Zhang listens to radio station KIZY, 105.3 on the FM dial. Zhang's home is located 32 km north and 75 km east of the station's transmitter, on the edge of KIZY's maximum broadcast range.

a. Determine the distance traveled by the signal when it is received by Zhang's home radio and make a sketch of the station's listening area.

b. If the radio station has coordinates (0,0), find the equation that represents the locus of KIZY's maximum broadcast range.

c. Determine the approximate coordinates of four locations—other than Zhang's home—that lie on the locus of KIZY's maximum broadcast range.

1.7 A new radio station, KZME, is building a transmitter 23 km east and 57 km north of Zhang's house.

a. Determine the location of KZME relative to KIZY.

b. Zhang's home also happens to be on the edge of KZME's maximum broadcast range. Determine the distance traveled by the signal when it is received by Zhang's home radio.

c. Determine an equation that represents the locus of KZME's maximum broadcast range.

d. Find the approximate coordinates of two locations other than Zhang's home that lie on the locus of KZME's maximum broadcast range.

e. Make a sketch of the listening areas for both KZME and KIZY.

f. Estimate the coordinates of the points where the two maximum broadcast ranges intersect.

g. Describe the location of Zhang's home relative to the intersection points of the two maximum broadcast ranges.

h. What does the region formed by the intersection of the two listening areas represent?

* * * * * * * * * *

Activity 2

In 1609, German astronomer Johannes Kepler (1571–1630) hypothesized that the planets in our solar system orbit the sun in paths that are not circular. His theory provided the best explanation for years of observations and revolutionized the science of astronomy. In this activity, you examine the characteristics of planetary orbits.

Exploration 1

In this exploration, you construct various loci using a piece of string and a pencil. You then examine how the length of string is related to a particular locus.

a.
1. Obtain a piece of string at least 15 cm long. Tie a knot at each end of the string.

2. Tape a sheet of paper to a sheet of cardboard. Near the center of the paper, stick a tack through both knots and into the cardboard.

3. Place the tip of a pencil inside the loop formed by the string. Use the pencil to pull the string taut, then place the pencil's tip on the paper, as shown in Figure **3**. Keeping the string taut, use the pencil to trace a locus of points.

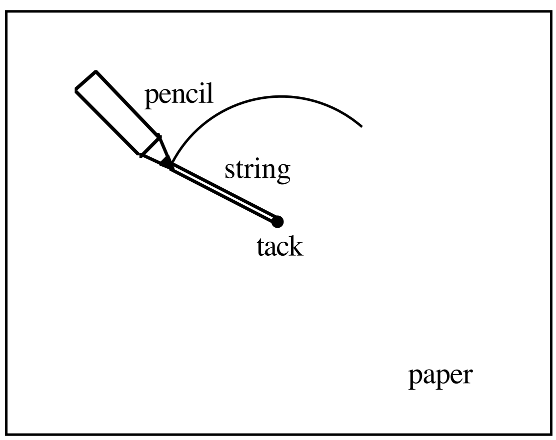


Figure 3: Construction of a locus of points using one tack

4. Describe the figure defined by the locus of points. Observe how the length of the string determines the size of the figure.

b.
1. Remove the tack and the paper from the cardboard. Tape another sheet of paper to the cardboard. Stick one tack into one of the knots in the string and another tack into the second knot. Stick the two tacks into the cardboard so that the distance between them is less than the length of the string.

2. Use the pencil to pull the string taut and place the pencil's tip on the paper, as shown in Figure **4**. Keeping the string taut, use the pencil to trace a locus of points until the tacks interfere with the construction of the locus. When this occurs, lift the pencil and string to the other side of the tacks and continue creating the locus of points. The locus is complete when the pencil begins retracing its path.

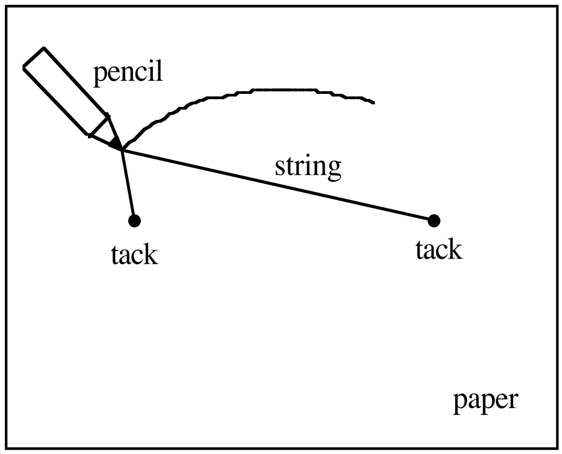


Figure 4: Construction of a locus using two tacks

c. The locus of points created in Part **b** is an **ellipse**. The location of each tack is a **focus** (plural **foci**) of the ellipse. Move the pencil to a location that is equidistant from both foci.

When the pencil is in this location, what shape is formed by the string and the segment joining the foci?

d.
1. Remove the tacks and label the foci F_1 and F_2.
2. The midpoint of the segment joining the foci is the **center** of an ellipse. To find the center of the ellipse drawn in Part **b**, construct the perpendicular bisector of $\overline{F_1F_2}$. Label the center of the ellipse O, as shown in Figure **5**.

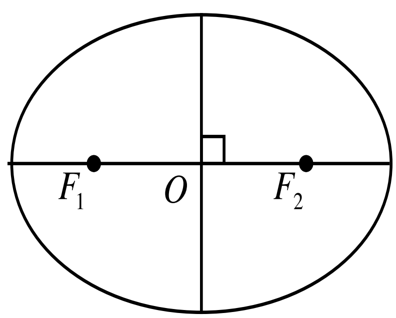


Figure 5: An ellipse, its center, and its foci

3. Compare the length of the segment that contains the foci and whose endpoints are on the ellipse to the distance between the knots in the string.

e. **1.** Remove your drawing of an ellipse from the cardboard and tape down another sheet of paper. Position the two tacks so that the distance between them is different than the distance in Part **b**.

2. Repeat Parts **b–d**. Describe the changes in the ellipse produced by moving the tacks.

Discussion 1

a. **1.** Describe the locus of points created using one tack in Part **a** of Exploration **1**.

2. Describe the lines of symmetry for this locus of points.

b. How is the distance between the knots in the string related to the size of the locus of points created using one tack?

Mathematics Note

An **ellipse** is a locus of points in a plane such that the sum of the distances from two fixed points, the foci, is a constant.

For example, Figure **6** shows an ellipse with its center at the origin of a two-dimensional coordinate system. Point P is on the ellipse because $d_1 + d_2 = 2a$, where $2a$ is a constant.

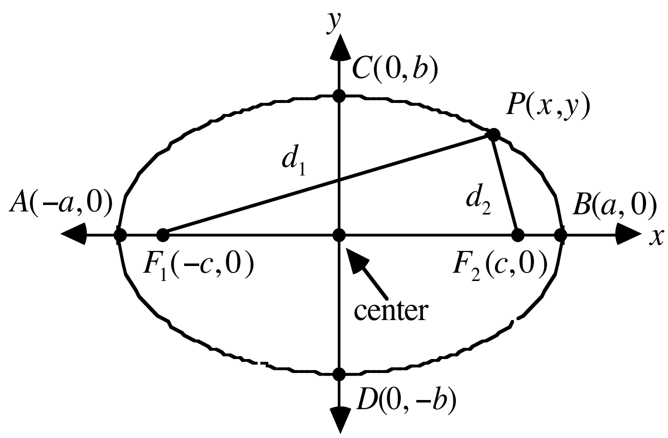


Figure 6: An ellipse and its foci

In Figure **6**, the points $F_1\,(c,0)$ and $F_2\,(-c\,,0)$ are the **foci** of the ellipse. Points A, C, B, and D are the **vertices** of the ellipse. The **major axis** of the ellipse is $\overline{AB}$; its length is $2a$. The **minor axis** is $\overline{CD}$, which has a length of $2b$. The major axis is the longer of the two and always contains the foci. The intersection of the major and minor axes is the **center** of the ellipse.

c. How does the locus of points created in Part **b** of Exploration **1** satisfy the definition of an ellipse given in the mathematics note?

d. Describe the symmetries found in an ellipse.

e. **1.** Describe the distance between the knots in the string in Part **b** of Exploration **1** in terms of a in Figure **6**.

2. Assuming that the distance between the knots in the string remains unchanged, how does the distance between the tacks affect the shape of the ellipse created?

f. In Part **c** of Exploration **1**, you placed the pencil at the location indicated by point P in Figure **7**. At this point, the distance from P to each focus is the same. What equation describes OP in terms of PF_1 and OF_1?

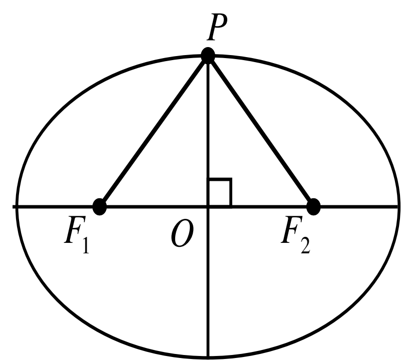


Figure 7: A point *P* on an ellipse

g. If P is any point on the ellipse, how does the distance between the knots in the string compare to $PF_1 + PF_2$?

h. As mentioned in the mathematics note, the sum of the distances from the foci to a point P is the constant $2a$, the length of the major axis. Using this fact, along with the distance formula, determine an equation for the ellipse in Figure **6**.

Exploration 2

In Exploration **1**, you created circles and ellipses using a piece of string, tacks, and a pencil. In one sense, it can be argued that an ellipse is formed when the segment between two tacks at the center of a circle is "stretched." In this exploration, you examine how this "stretching" transformation can be used to find another form of the equation for an ellipse.

a. Write an equation for a circle with its center at the origin.

b. In the Level 4 module "It's All in the Family," you observed that a function can be stretched horizontally by m if each point in the function is transformed from (x,y) to (mx,y), where $m \neq 0$. The equation which results in this transformation can be found by replacing x with x/m in the original equation.

Similarly, a function can be stretched vertically by n if each point in the function is transformed from (x,y) to (x,ny), $n \neq 0$. The equation which results in this transformation can be found by replacing y with y/n in the original equation.

Write the equation that results when the equation for the circle in Part **a** is stretched horizontally by 3 and stretched vertically by 2.

c. **1.** Rewrite the equation from Part **b** by multiplying both sides of the equation by $1/r^2$, where r represents the radius of the circle.

2. Simplify this equation and write it in the form below:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

d. **1.** Create a graph of the equation from Part **c**.

2. Identify the points where the graph intersects the x- and y-axes.

e. Repeat Parts **b–d** when the equation for the circle in Part **a** is stretched horizontally by 2 and stretched vertically by 3.

Discussion 2

a. Compare the ellipses you created in Parts **d** and **e** of Exploration **2**.

b. How do the intersections of each ellipse and the x- and y-axes relate to its equation in the form below?

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

c. How could you use an equation in the form above to identify the lengths of the major and minor axes of the ellipse?

d. How could you use such an equation to identify the foci of the ellipse?

Mathematics Note

The **standard form** of the equation of an ellipse with center at the origin and foci on the x-axis is written as follows:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $b^2=a^2-c^2$ and c is the distance from the center to each focus.

The equation in standard form can be used to quickly sketch the graph of an ellipse. As shown in Figure **6**, the x-intercepts of the ellipse are $(a,0)$ and $(-a,0)$ and the length of the major axis is $2a$. The y-intercepts of the ellipse are $(b,0)$ and $(-b,0)$ and the length of the minor axis is $2b$.

Assignment

2.1 **a.** Sketch a graph of the ellipse described by the equation below:

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

b. Find the foci of the ellipse in Part **a**.

c. Write an equation for this ellipse in the form based on the distance formula (described in Part **h** of Discussion **1**).

2.2 The following equation describes an ellipse with center at the origin:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

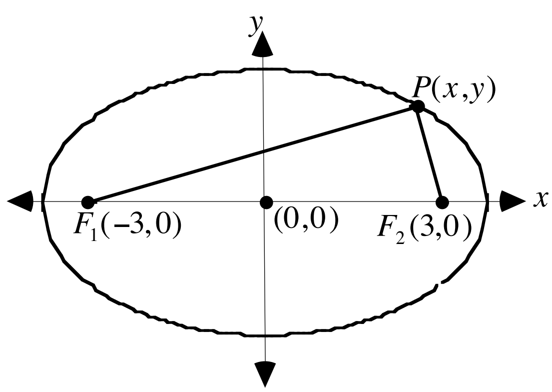
a. Does this equation represent a function?

b. Describe any adjustments you would have to make in order to graph this equation on a graphing utility.

c. The vertices of this ellipse occur at the x- and y-intercepts. Determine the coordinates of the vertices.

d. If P is any point on the ellipse and F_1 and F_2 are the foci, what is the value of $PF_1 + PF_2$?

2.3 The foci of the ellipse in the diagram below are located at (–3,0) and (3,0). Its locus of points is described by the sum $PF_1 + PF_2 = 16$.



a. Use the distance formula to express PF_1 and PF_2 in terms of x and y.

b. Use your response to Part **a** to write an equation for the ellipse based on the distance formula.

c. Determine the coordinates of the vertices that lie on the y-axis.

d. Determine the coordinates of the vertices that lie on the x-axis.

e. Determine the equation of the ellipse in standard form.

Mathematics Note

When an equation contains two radicals, the following process may be used to eliminate them:

- Given: $\sqrt{x} + \sqrt{y} = k$
- Isolate one of the radicals. $\sqrt{x} = k - \sqrt{y}$
- Square both sides and simplify. $\left(\sqrt{x}\right)^2 = \left(k - \sqrt{y}\right)^2$

$$x = k^2 - 2k\sqrt{y} + \left(\sqrt{y}\right)^2$$

$$x = k^2 - 2k\sqrt{y} + y$$

- Isolate the next radical. $\dfrac{x - k^2 - y}{-2k} = \sqrt{y}$
- Square both sides and simplify. $\left(\dfrac{x - k^2 - y}{-2k}\right)^2 = \left(\sqrt{y}\right)^2$

$$\left(\frac{x - k^2 - y}{-2k}\right)^2 = y$$

If the equation contains more than two radicals, the process of isolating one radical at a time is continued until all radicals are removed.

For example, to remove the radicals from the equation $\sqrt{x} + \sqrt{y} = 10$, you could use the following steps:

$$\sqrt{x} + \sqrt{y} = 10$$

$$\sqrt{x} = 10 - \sqrt{y}$$

$$\left(\sqrt{x}\right)^2 = \left(10 - \sqrt{y}\right)^2$$

$$x = 100 - 20\sqrt{y} + y$$

$$\frac{x - 100 - y}{-20} = \sqrt{y}$$

$$\left(\frac{x - 100 - y}{-20}\right)^2 = \left(\sqrt{y}\right)^2$$

$$\left(\frac{x - 100 - y}{-20}\right)^2 = y$$

2.4 **a**. Demonstrate that the following equation of an ellipse,

$$\sqrt{(x+2)^2+(y-0)^2}+\sqrt{(x-2)^2+(y-0)^2}=10$$

is equivalent to its equation in standard form:

$$\frac{x^2}{25}+\frac{y^2}{21}=1$$

b. Verify that the equation involving radicals defines the same set of points as the equation in standard form by solving each one for y.

2.5 Consider the ellipse described by the equation below:

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

a. Write an equation that describes the image of this ellipse translated 8 units horizontally and 6 units vertically. List the coordinates of its center.

b. Using the equation of the translated ellipse from Part **a**:

1. determine the coordinates of the endpoints of the major axis

2. determine the coordinates of the endpoints of the minor axis.

c. What are the coordinates of the foci of the translated ellipse?

d. Write the standard form of the equation of an ellipse with center at (h,k) and foci on the line $y=k$.

2.6 Earth's orbit is an ellipse with the sun located at one focus. The sun is $2.99 \bullet 10^6$ km from the center of the ellipse. The length of the major axis is $2.99 \bullet 10^8$ km.

a. As the earth travels its elliptical path, what is the shortest distance between the earth and the sun?

b. What is the greatest distance between the earth and the sun?

* * * * *

2.7 Demonstrate that the following equation of an ellipse,

$$\sqrt{(x+4)^2+(y-0)^2}+\sqrt{(x-4)^2+(y-0)^2}=34$$

is equivalent to its equation in standard form:

$$\frac{x^2}{289}+\frac{y^2}{273}=1$$

2.8 A communications satellite has been placed in an elliptical orbit around Earth. The satellite's orbital path is directly above the equator. One focus is at Earth's center, and each focus is 410 km from the center of the ellipse. The length of the major axis is 13,960 km.

a. Write an equation that describes the satellite's orbit.

b. How does the satellite's orbit compare to a circular orbit?

c. Earth's radius at the equator is about 6400 km. At its lowest point, how close is the satellite to Earth's surface?

d. At its highest point, how far is the satellite from Earth's surface?

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Activity 3

As you discovered in the previous activity, an ellipse is the locus of points in a plane for which the sum of the distances from two foci is a constant. In this activity, you use technology to continue your investigations of the ellipse, then explore the geometry of another conic section.

Exploration 1

a. Construct a circle and label its center O. Construct P_1 on the circle.

b. Construct P in the interior of the circle, but not at the center.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct $\overleftrightarrow{OP_1}$.

f. Mark the intersection of the perpendicular bisector from Part **d** and $\overleftrightarrow{OP_1}$. Label this point X. Your construction should now resemble the one shown in Figure **8**.

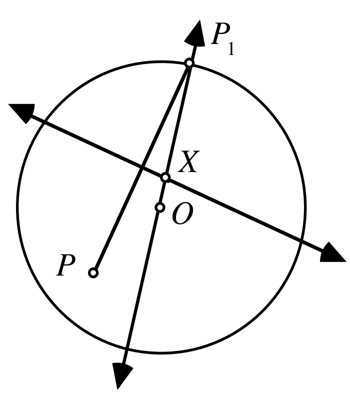


Figure 8: Completed construction

g. Trace the path of X as P_1 is moved around the circle. Note which conic is formed by the path.

h. Move P to a different location inside the circle and repeat Part **g**. Experiment with several other locations of P, including the center of the circle. **Note:** Save this construction for use in Exploration **2**.

Discussion 1

a. Describe how the location of point P affects the shape of the figure generated by the construction in Exploration **1**.

b. The figure generated when P is located inside the circle appears to be an ellipse, but appearance alone is not enough to prove that it is one. For this to be true, the figure must satisfy the definition of an ellipse.

In Figure **9**, P is any point within the circle that is not the center. The point X is a point on the figure.

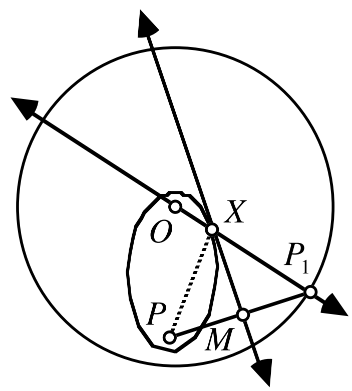


Figure 9: Construction from Exploration 1

1. Which points appear to be the foci of the figure?
2. If the figure generated is an ellipse, what sum must remain constant?
3. Since $\overline{MX}$ is the perpendicular bisector of $\overline{PP_1}$, what is the relationship between P_1X and PX?
4. Why is the length of $\overline{OP_1}$ always the same, no matter where on the circle P_1 is located?
5. Why is the length of $\overline{OP_1}$ equal to $OX + XP$?
6. How does the fact that $OX + XP = OP_1$ prove that the generated figure is indeed an ellipse?

Exploration 2

In the previous exploration, you created a construction that generated two conic sections: an ellipse and a circle. In this exploration, you use the same construction to generate another conic section, the hyperbola.

a. Using the construction from Exploration **1**, locate *P* outside the circle. Trace the path of point *X* as P_1 moves around the circle.

b. Move *P* to several different locations outside the circle and at least one point on the circle. Repeat Part **a** for each location. Note the differences in the figures generated.

Mathematics Note

A **hyperbola** is the locus of points in a plane for which the positive difference of the distances from two designated foci is a constant. In Figure **10**, for example, points F_1 and F_2 represent the foci. The hyperbola formed is the set of all points *H* in a plane for which the difference between HF_1 and HF_2 is a constant.

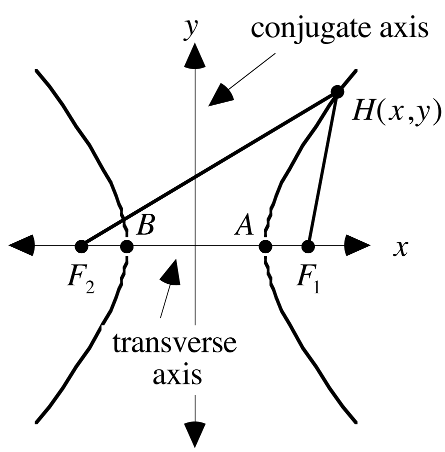


Figure 10: A hyperbola

The midpoint of $\overline{F_1F_2}$ is the **center** of the hyperbola. The **vertices** of the hyperbola occur at the intersections of the **branches** and $\overline{F_1F_2}$. In the hyperbola in Figure **10**, the vertices occur at points *A* and *B*.

The line segment joining the vertices is the **transverse axis**. The perpendicular bisector of the transverse axis lies on the **conjugate axis**.

Discussion 2

a. Describe how the location of point *P* affects the shape of the figure generated by the construction in Exploration **2**.

b. The figure generated when *P* is located outside the circle appears to be a hyperbola, but as with the ellipse, appearance alone is not enough to prove this conjecture. In Figure **11**, *P* is any point outside the circle. The point *X* is a point on the figure.

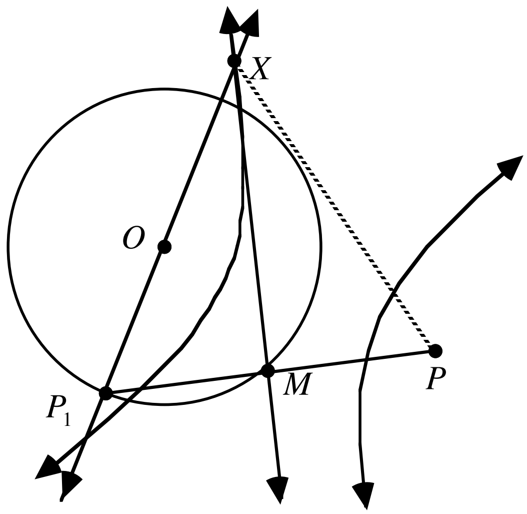


Figure 11: Construction from Exploration 2

1. Which points appear to be the foci of the figure?
2. As P_1 moves along the circle, how is the length of $\overline{OP_1}$ affected? Explain why this occurs.
3. Since $\overline{MX}$ is the perpendicular bisector of $\overline{PP_1}$, what is the relationship between P_1X and PX?
4. Why is the length of $\overline{PX}$ equal to $OX + OP_1$?
5. Why does $PX - OX = OP_1$?
6. How does the fact that $PX - OX = OP_1$ prove that the generated figure is indeed a hyperbola?

c. Describe the symmetries that exist within a hyperbola.

d.
1. The hyperbola shown in Figure **10** has its center at the origin. If the coordinates of F_1 are $(c, 0)$, what are the coordinates of F_2?
2. If the coordinates of vertex *A* are $(a, 0)$, what is the difference between AF_1 and AF_2?
3. Is the difference between BF_1 and BF_2 the same as the difference between AF_1 and AF_2? Why or why not?
4. What can you conclude about the constant difference for the hyperbola?

Mathematics Note

The **standard form** of the equation of a hyperbola with center at the origin and foci on the *x*-axis is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = c^2 - a^2$, c is the distance from the center to each focus, and a is the distance from the center to each vertex.

For example, suppose that the coordinates of F_1 in Figure **10** are $(-5, 0)$, the coordinates of F_2 are $(5, 0)$, and the constant difference $HF_1 - HF_2 = 6$ for any point *H* on the hyperbola. It follows that $a = 3$, $c = 5$, and $b = 4$. The equation, in standard form, of this hyperbola would be:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

e. **1.** If a graphing utility graphs only functions, it will not graph the equation of a hyperbola in standard form. Explain why this occurs.

2. How could you create a graph of a hyperbola on such a graphing utility?

Assignment

3.1 **a.** The positive difference of the distances from any point $H(x, y)$ on a hyperbola to each focus is a constant, $2a$. Use this fact, along with the distance formula, to write an equation for the hyperbola with center at the origin and foci at (4,0) and (–4,0), with $a = 3$.

b. Use appropriate technology to graph the hyperbola in Part **a**. What considerations, if any, must be addressed in order to create the graph?

3.2 **a.** How does the equation of a hyperbola compare to the equation of an ellipse?

b. Use appropriate technology to demonstrate that the following equation for a hyperbola,

$$\sqrt{(x-4)^2 + (y-0)^2} - \sqrt{(x+4)^2 + (y-0)^2} = 6$$

is equivalent to the equation in standard form below:

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

3.3 Consider the hyperbola described by the equation below:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

a. Identify the vertices of the hyperbola.

b. **1.** Use a symbolic manipulator to solve the equation of the hyperbola for y.

2. Graph the equations from Step **1**.

3. As $|x|$ increases, what does the value of $\sqrt{(x^2 - 4)}$ approach?

4. Rewrite the equations from Step **1** for large $|x|$. What kind of function is described by these equations?

5. Do the expressions that you wrote in Step **4** describe the actual values of y for the hyperbola?

c. In the Level 4 module "Big Business," an **asymptote** to a curve is described as a line such that the distance from a point P on the curve to the line approaches 0 as the distance from point P to the origin increases without bound.

Are the lines $y = (3/2)x$ and $y = -(3/2)x$ asymptotes of the hyperbola? Explain your response.

d. Describe how the denominators of the equation of the hyperbola can be used to find the equations of the asymptotes.

e. Identify the asymptotes for a hyperbola described by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3.4 Consider the hyperbola described by the following equation:

$$\frac{x^2}{9} - \frac{y^2}{144} = 1$$

a. Find the vertices and foci for the hyperbola.

b. Determine the equations of the asymptotes.

c. Sketch a graph of the equation including the asymptotes.

d. Determine two points on the curve where the x-coordinate equals 6.

3.5 Consider a hyperbola with the equation below:

$$\frac{x^2}{25} - \frac{y^2}{4} = 1$$

a. Write the equation of the image obtained by a horizontal translation of –4 units and a vertical translation of 3 units.

b. 1. Determine the coordinates of the center of the translated hyperbola.

2. Determine the coordinates of the foci.

3. Determine the equations of the asymptotes for the translated hyperbola.

3.6 Miguel is designing a stained glass window 1.5 m high and 1 m wide. He plans to incorporate the distinct curve of a hyperbola in the window. In order to accomplish this, Miguel must decide how the hyperbola should appear, then determine an equation that can be used to make a template to cut the glass.

a. Design a stained glass window that includes a hyperbola.

b. Determine an equation that describes the hyperbola.

c. Write a paragraph that explains how you determined the equation for the hyperbola. Be sure to mention the constant difference and include the coordinates of the foci and vertices.

* * * * *

3.7 Graph the following hyperbola:

$$\frac{(x+1)^2}{4} - \frac{(y+3)^2}{1} = 1$$

On your graph, list the equations of the asymptotes and the coordinates of the foci, center, and vertices.

3.8 Demonstrate that the following equation for a hyperbola,

$$\sqrt{(x-5)^2 + (y-0)^2} - \sqrt{(x+5)^2 + (y-0)^2} = 8$$

is equivalent to the equation below:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

* * * * * * * * * *

Research Project

In this module, you have encountered equations for ellipses and hyperbolas, with center at the origin and foci located on the x-axis, in two very different forms: the standard form and a form based on the distance formula. For this research project, create algebraic proofs that show that these forms are equivalent.

a. For an ellipse, prove that $\sqrt{(x-c)^2+(y-0)^2}+\sqrt{(x+c)^2+(y-0)^2}=2a$ and the equation below are equivalent:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

b. For a hyperbola, prove that $\sqrt{(x-c)^2+(y-0)^2}-\sqrt{(x+c)^2+(y-0)^2}=2a$ and the equation below are equivalent:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Activity 4

As you observed in the previous activity, circles, ellipses, and hyperbolas can all be generated in a similar manner. The remaining conic section, however, has different restrictions than the others. In this activity, you examine the geometric properties of a **parabola**.

Mathematics Note

A **parabola** is the set of all points in a plane equidistant from a line and a point not on the line. The line is the **directrix** of the parabola. The point is the **focus** of the parabola. In Figure **12**, for example, the directrix of the parabola is line l and the focus is point F. The distances from any point P on the parabola to F or l are equal.

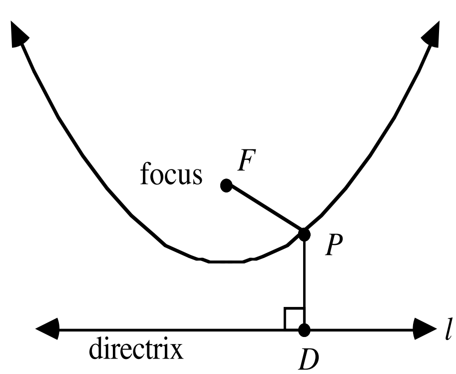


Figure 12: A parabola

Exploration

In this exploration, you create a construction for generating a parabola and determine an equation that describes its locus of points.

a. Using a geometry utility, construct a horizontal line to represent the directrix of a parabola. Construct point D on the line.

b. Construct a point F not on the line. This point represents the focus of the parabola.

c. Construct $\overline{FD}$.

d. Construct the perpendicular bisector of $\overline{FD}$.

e. Construct a line perpendicular to the horizontal line through D.

f. Construct a point at the intersection of the line from Part **e** and the perpendicular bisector of $\overline{FD}$. Label this intersection P. This point represents a point on a parabola.

Your construction should now resemble the one shown in Figure **13**.

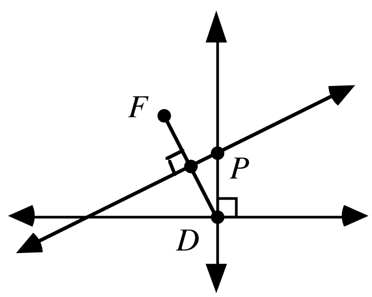


Figure 13: Completed construction

g. Trace the path of P as D is moved along the horizontal line. As you trace the path of P, record its coordinates for at least 10 different locations.

h.
1. Graph the coordinates of each point you recorded in Part **g** in a scatterplot.
2. As with the other conic sections, the equation of a parabola contains a second-degree equation. Determine an equation that models the scatterplot.
3. Graph the equation found in Step **2** on the same coordinate system as the scatterplot.

i. Move point F to at least two other locations and repeat Parts **g** and **h**. Locate point F below the line through D at least once.

Discussion

a. How could you prove that the construction in the exploration produces a parabola?

b. Describe any symmetries you observe in the parabola in Figure **14** below.

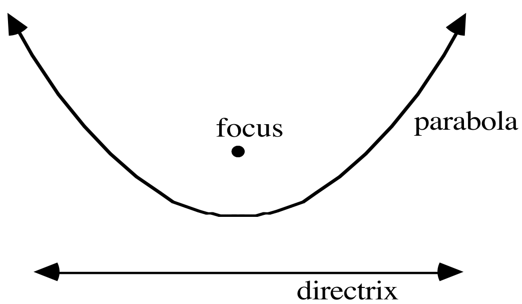


Figure 14: A parabola, its focus, and its directrix

c. The **vertex** of a parabola occurs at the point where the axis of symmetry intersects the parabola.

Describe how you could use the construction in the exploration to find the vertex of the parabola.

d. How does changing the distance from the focus to the directrix change the shape of the resulting parabola?

e.

1. How did locating the focus below the directrix affect the resulting parabola?
2. How did it affect the equation that defines the parabola?

f. How would you change the construction in the exploration to draw a parabola that opened to the left? to the right?

g. In the Level 4 module "It's All in the Family," you explored several transformations of the function $y = x^2$. Suppose that the parabola in Figure **14** is a graph of $y = x^2$.

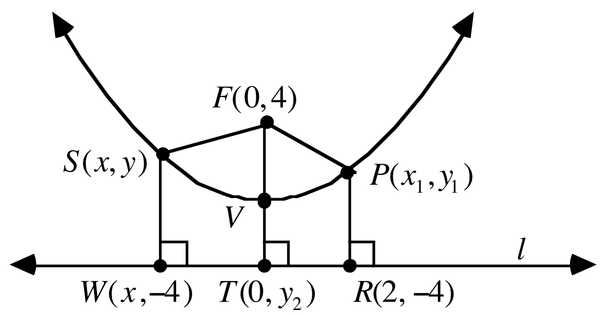
1. Describe where the x- and y-axes are located.
2. Describe the graph of $y = -x^2$.
3. Describe the graph of $y = x^2 + 3$.
4. Describe the graph of $y = (x - 2)^2$.

h. The shapes of the transformed graphs in Part **g** are identical to the shape of the parent. How would you modify the equation $y = x^2$ to change the shape of the parabola?

i. Describe how the graph of $y = x^2$ is transformed in the graph of $y = 3(x - 4)^2 + 5$.

Assignment

4.1 Use the diagram of the parabola below to complete Parts **a–e**.



a. Find y_2.

b. What are the coordinates of the vertex?

c. Find x_1.

d. Use the distance formula to describe the relationship between *PF* and *PR*.

e. Use the relationship you described in Part **d** to determine y_1.

4.2 Use the parabola shown in Problem **4.1** to complete Parts **a–d**.

a. Why are the *x*-coordinates of points *S* and *W* equal?

b. Use the distance formula to show that $FS = SW$.

c. Use the relationship in Part **b** to determine the equation of the parabola in the form $y = ax^2$. This is the **standard form** of the equation for a parabola with vertex at the origin.

d. Verify your equation using the coordinates of *P* from Problem **4.1**.

Mathematics Note

The **standard form** of the equation for a parabola with a horizontal directrix and vertex $V(h,k)$ is:

$$y = a(x - h)^2 + k$$

When a is positive, the parabola opens upward. When a is negative, the parabola opens downward.

The distance p from the vertex to the focus is the same as the distance from the vertex to the directrix. As shown in Figure **15**, the coordinates of the focus are $(h, k + p)$ and the equation of the directrix is $y = k - p$.

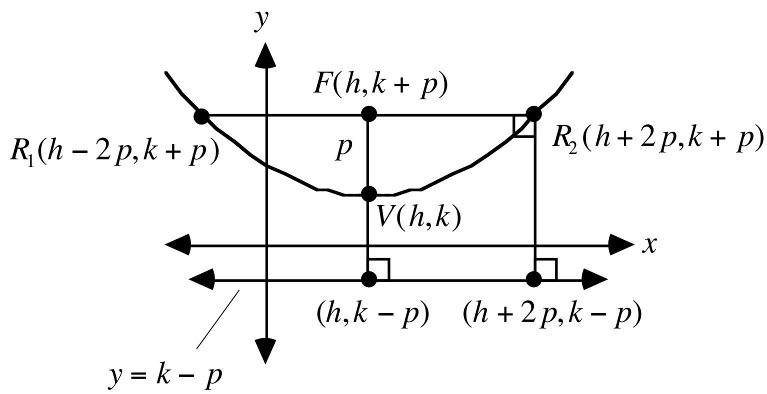


Figure 15: A parabola with horizontal directrix

4.3 **a.** What transformations of the parent function $y = x^2$ are described by $y - k = (x - h)^2$?

b. What additional transformation of the parent function is described by $y - k = a(x - h)^2$?

c. How is R_1R_2 in Figure **15** related to a in $y - k = a(x - h)^2$?

d. Write an equation that relates a to R_1R_2.

e. Since $R_1R_2 = 4p$, write an equation that relates a to $4p$.

Mathematics Note

Given the equation of a parabola in standard form, $y = a(x - h)^2 + k$, then

$$|a| = \frac{1}{4p}$$

where p is the distance from the focus to the vertex or the vertex to the directrix.

For example, Figure **16** shows the graph of the equation

$$y - 4 = -\frac{1}{8}(x - 8)^2$$

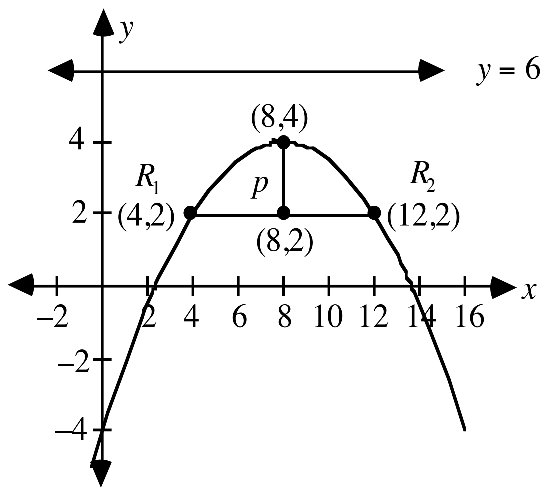


Figure 16: The graph of $y - 4 = -\frac{1}{8}(x - 8)^2$

For this parabola, $R_1R_2 = 8$, $p = 2$, and

$$|a| = \frac{1}{4 \bullet 2} = \frac{1}{8}$$

4.4 Consider the parabola with focus at (2,6) and directrix described by the equation $y = 4$.

a. Plot the focus and vertex and draw the directrix.

b. Determine the equation of the parabola.

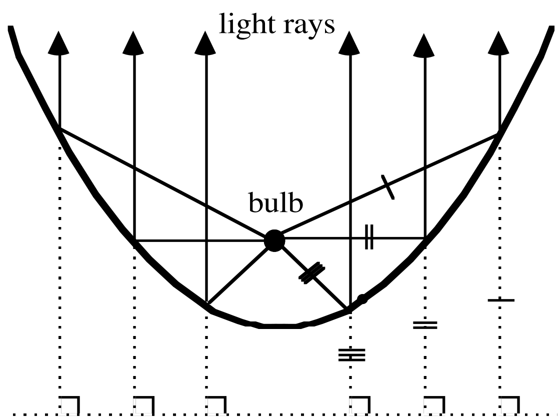
c. By inspection, determine two points on the parabola (other than the vertex) which have coordinates that are integers.

d. Verify that the points you identified in Part **c** are on the parabola by substituting their coordinates in the equation from Part **b**.

* * * * *

4.5 Determine the equation, in standard form, of the parabola with focus at (–1,2) and directrix $y = 5$.

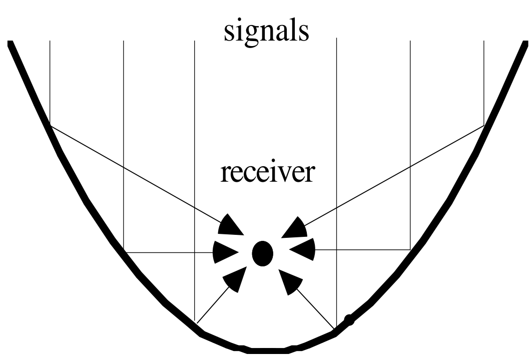
4.6 A paraboloid is a three-dimensional figure whose cross section is a parabola. The reflectors used in searchlights and satellite dishes, for example, are paraboloids. In a searchlight, the bulb is placed at the focus of the paraboloid. As shown in the diagram below, all light rays emitted from the bulb are reflected parallel to the axis of symmetry of the parabola that generated the shape of the reflector.



a. Consider a parabolic reflector formed by revolving the portion of the parabola $20y = x^2$ between $x = -20$ and $x = 20$ about the y-axis. Determine the coordinates of the location where the bulb should be placed.

b. If each unit on the coordinate system represents 1 cm, what is the maximum depth of the reflector?

4.7 Broadcast signals traveling into the opening of a parabolic reflector parallel to its line of symmetry are reflected through the focus of the paraboloid. This is an important property for collecting and receiving satellite television broadcasts.



a. Imagine that you have been asked to design a television satellite dish at least 180 cm in diameter at its opening, but no more than 60 cm deep. Determine an equation for a parabola that can be used to generate a suitable parabolic reflector.

b. If each unit on the coordinate system represents 1 cm on the satellite dish, where should the receiver be located?

4.8 Consider the parabola with focus at (2,6) and directrix $x = 0$.

a. Determine the equation of the parabola.

b. Graph the focus, vertex, and directrix on a sheet of graph paper.

c. By inspection, determine two points on the parabola (other than the vertex) which have coordinates that are integers.

d. Using the coordinates of the points identified in Part **c**, verify the equation you determined in Part **a**.

* * * * * * * * * *

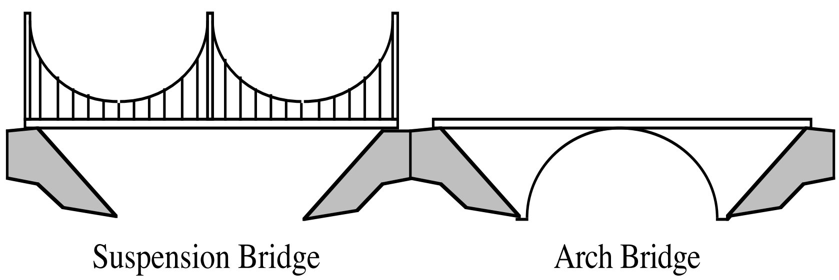
Summary Assessment

1. Imagine that you have just observed a comet that you cannot find in any star chart or atlas. So that others may observe the comet and verify your discovery, you must describe its orbit. If the comet is indeed a new discovery, it will be named after you.

From Kepler's laws of planetary motion, you know that the comet's orbit is an ellipse with the sun at one focus. According to your observations, the comet's closest approach to the sun is $9.00 \bullet 10^6$ km, while its farthest distance from sun is $1.79 \bullet 10^9$ km.

Using an appropriate coordinate system, write an equation that describes the comet's elliptical orbit. On a graph of the equation, label all important points and measurements.

2. Many famous bridges around the world contain conic sections in their design. Others contain curves that closely resemble conic sections. For example, the diagram below shows a suspension bridge and an arch bridge.



a. Find pictures of a suspension bridge and an arch bridge. Identify the conic sections that most closely approximate the curves found in the designs. Justify your selections.

b. Determine the equations of the conic sections identified in Part **a**.

Module Summary

- The set of all points that satisfies one or more given conditions is a **locus** (plural **loci**).
- A **conic section** can be formed by the intersection of a plane with a cone. In a right circular cone, the conic section formed depends on the slope of intersecting plane. The intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.

  It is also possible for the intersection of a plane and a cone to form a point, a line, or two intersecting lines. These intersections are called **degenerate conic sections**.
- A **circle** is the locus of points in a plane that are a given distance, the **radius**, from a fixed point, the **center**.
- The **standard form** of the equation of a circle with center at (h,k) and radius r is:

$$(x-h)^2+(y-k)^2=r^2$$

- An **ellipse** is a locus of points in the plane such that the sum of the distances from two fixed points, the **foci**, is a constant.

  An ellipse is symmetric with respect to the lines containing its two axes. The **major axis** is the longer of the two and always contains the foci; the **minor axis** is the shorter of the two. The intersection of the major and minor axes is the **center** of the ellipse. The intersections of the ellipse and the major and minor axes are the **vertices** of the ellipse.
- The **standard form** of the equation of an ellipse with center at (h,k) is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

  where $b^2=a^2-c^2$, $2a$ is the length of the major axis, and c is the distance from the center to each focus.
- A **hyperbola** is the locus of points in a plane for which the positive difference of the distances from two designated foci is constant.

  The midpoint of the segment joining the foci is the **center** of the hyperbola. The intersections of the **branches** and this segment are the **vertices** of the hyperbola. The line segment joining the vertices is the **transverse axis**. The perpendicular bisector of the transverse axis lies on the **conjugate axis**.

- The **standard form** of the equation of a hyperbola with center at (h,k) is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

  where $b^2=c^2-a^2$, $2a$ is the distance between the vertices, and c is the distance from the center to each focus.

- The **asymptotes** for a hyperbola with equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

  are $y=(b/a)x$ and $y=-(b/a)x$.

- A **parabola** is a locus of points in a plane for which each point is equidistant from a fixed line, the **directrix**, and a fixed point, the **focus**, not on the line. The **vertex** of a parabola is the midpoint of the line segment from the focus perpendicular to the directrix.

- The **standard form** of the equation of a parabola with a horizontal directrix and vertex $V(h,k)$ is:

$$y=a(x-h)^2+k$$

  The coefficient a in the equation depends on the distance from the directrix to the focus. It can be shown algebraically that:

$$|a|=\frac{1}{4p}$$

  where p is the distance from the focus to the vertex or the vertex to the directrix.

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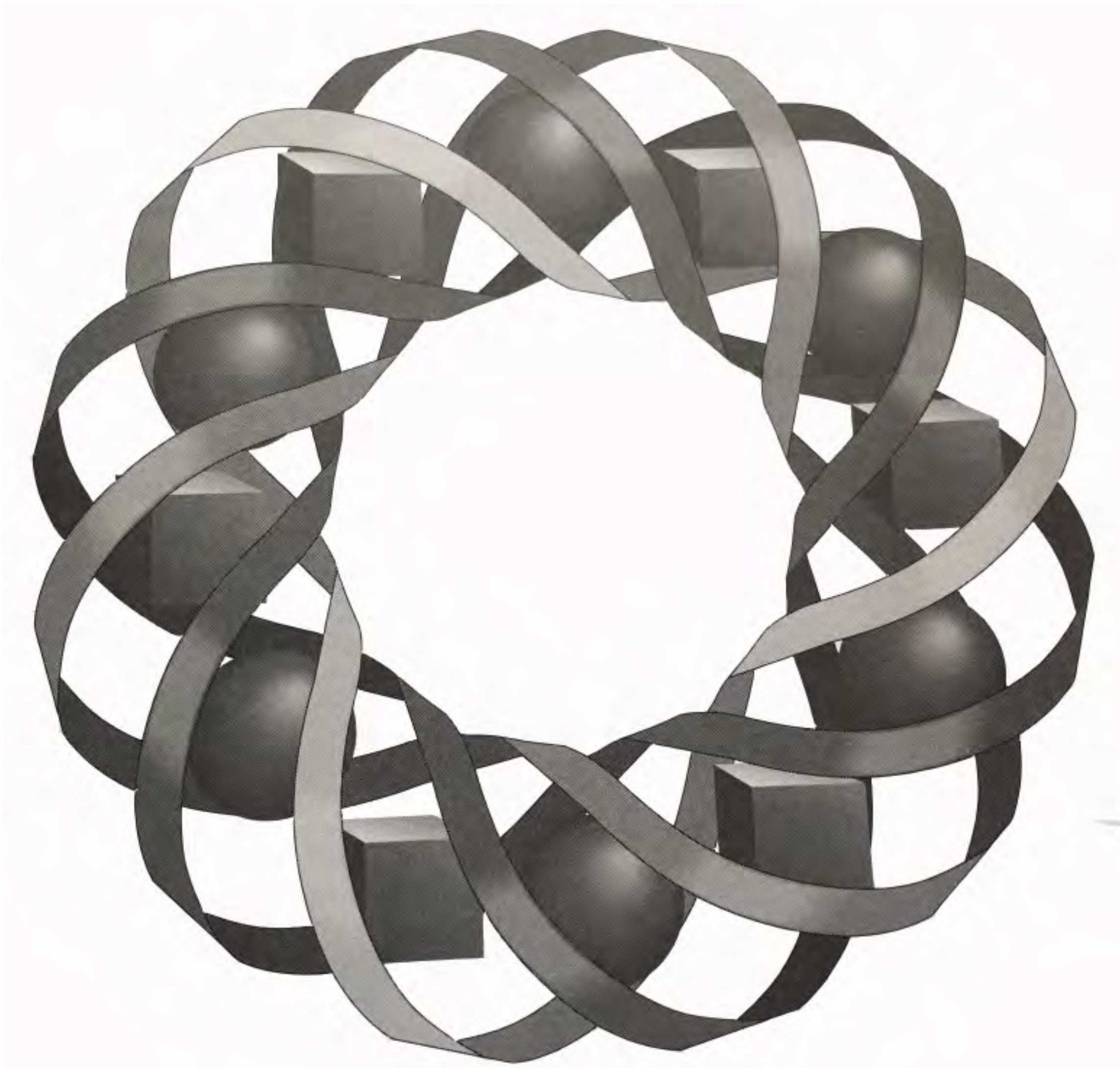
Controlling the Sky with Parametrics



Air traffic controllers must know the precise locations of all nearby planes at any given time. In this module, you discover how mathematics can help model these situations.

Sherry Horyna • Ed Sisolak • Dan West

Controlling the Sky with Parametrics

Introduction

An air traffic controller stares in surprise at the radar screen. Two airplanes are flying toward each other on a collision course! In order to avoid catastrophe, the controller must alter the flight path of one of the planes.

In this module, you explore the mathematics needed to help the controller handle this situation.

Activity 1

Minute Aviation sponsors an air show in which a large jet and a small plane fly by the control tower at the same time. In this activity, you use **parametric equations** to simulate the paths of these flights on a graphing utility.

Exploration 1

The small plane flies at a speed of 55 m/sec, while the large jet flies at a speed of 90 m/sec. They fly past the tower at the same moment and at the same altitude, on parallel courses from west to east. The small plane is on a course closer to the tower. The controller watches the flight paths on a radar screen, as shown in Figure **1**.

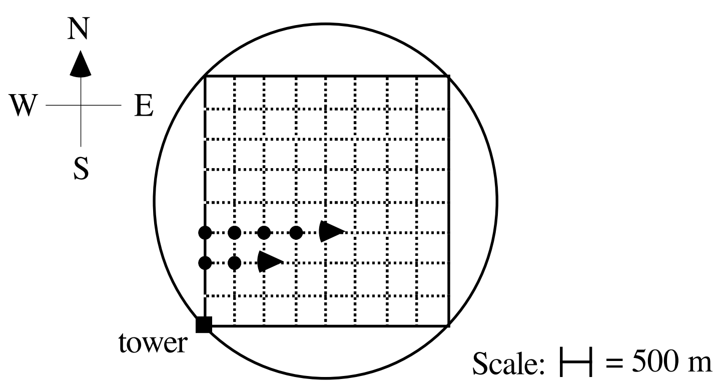


Figure 1: Large jet and small plane flying by control tower

Imagine that the control tower shown in Figure **1** is located at the origin of a rectangular coordinate system. Each side of the squares on the grid in Figure **1** represents a distance of 500 m.

a. Determine how far each airplane is from the tower when it first enters the radar screen.

b. Write a linear equation in slope-intercept form $(y = mx + b)$ to model the path of each plane on the grid.

c. **1.** Determine the coordinates of each plane 1 sec after it passes the tower.

2. Determine the coordinates of each plane 2 sec after it passes the tower.

3. Determine the number of seconds each plane remains on the radar screen.

d. **1.** Identify the corresponding domain and range for the segment of the line that models each path as it appears on the radar screen.

2. Use a graphing utility to graph both equations simultaneously over the appropriate intervals.

3. On the graph, indicate the location of each airplane 20 sec and 30 sec after passing the tower.

Discussion 1

a. If you traced values on either line segment graphed in Exploration **1**, what type of information would you get from the ordered pairs?

b. Federal regulations for air shows control a plane's speed, altitude, and distance from the crowd. The equations and graphs in Exploration **1** model the flight paths of two airplanes. If the controller used these equations and graphs to monitor the flights, what information would be missing?

c. What difficulties arose when you attempted to identify the planes' positions in Part **d** of Exploration **1**?

Exploration 2

In Exploration **1**, you wrote equations that described the path of each airplane as it flew by the tower. The small plane traveled at a speed of 55 m/sec, while the large jet traveled at a speed of 90 m/sec. In this exploration, you examine a mathematical method for visualizing this difference in speeds.

a. Copy and complete Table **1** for the small airplane. The values in the column with the heading "Time" represent the number of seconds after the small airplane passes the control tower.

Table 1: Position of small airplane at different times

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0		
1		
2		
3		
⋮		
10		
⋮		
t		

b. Copy and complete Table **2** for the large jet. The values in the column with the heading "Time" represent the number of seconds after the large jet passes the control tower.

Table 2: Position of large jet at different times

Time (sec)	***x*-coordinate (m)**	***y*-coordinate (m)**
0		
1		
2		
3		
$\vdots$		
10		
$\vdots$		
t		

Mathematics Note

Parametric equations allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. On an xy-plane, for example, both x and y can be expressed as functions of a third variable t:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

When parametric equations are used to model linear relationships, they can be written in the general form:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

where (a,c) are the coordinates of the graph when $t = 0$. The coefficients b and d represent the rates of change in the x- and y-components, respectively.

For example, the graph of the parametric equations below has coordinates (3,–4) when $t = 0$. The horizontal and vertical rates of change are –2 and 5 units, respectively, for every unit change in t.

$$\begin{cases} x = 3 - 2t \\ y = -4 + 5t \end{cases}$$

c. Write the parametric equations that describe the x- and y-coordinates of the small airplane at any given time t.

d. Write the parametric equations that describe the x- and y-coordinates of the large jet at any given time t.

e.
1. Set your graphing utility to graph parametric equations simultaneously. Using the same intervals for x and y as in Exploration **1** and the interval $[0, 73]$ for the parameter t, graph both pairs of equations from Parts **c** and **d**.
2. Experiment with different increments for t. Record your observations.

Discussion 2

a. Which graphs do you think give a more accurate depiction of the location of the two airplanes when they fly by the tower: the graphs in Exploration **1** or the graphs in Exploration **2**? Explain your response.

b.
1. How does changing the increment for the parameter affect the graph?
2. Given that the parameter represents time, what would you expect to occur as the size of the increment decreases?

c. Does tracing values on either line segment allow you to determine an airplane's position at a particular time? Explain your response.

d. What would you need to change to model the motion of the airplanes before they pass the control tower?

e. In the parametric equations below, which variables are represented in the domain and which variables are represented in the range?

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

f.
1. Suppose that the jet and the airplane are flying toward each other, with the jet entering the radar screen from the right, along the same path and at the same speed as in the explorations. How would you modify your parametric equations to describe this situation?
2. How could you determine when the two planes would pass each other?

Assignment

1.1 **a.** Use a graphing utility and the parametric equations from Exploration **2** to model the motion of the small plane and the large jet for 8 sec after they pass the tower.

b. **1.** Use your graph to estimate the time required (to the nearest 0.1 sec) for the small airplane to travel 100 m past the tower.

2. Use algebra to determine the time required (to the nearest 0.1 sec) for the small airplane to travel 100 m.

c. Repeat Part **b** for the large jet.

d. After 8 sec, what is the distance between the two airplanes?

1.2 **a.** The parametric equations below model the motion of two planes as they fly across the radar screen.

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases} \quad \begin{cases} x = 4000 - 90t \\ y = 1500 + 0t \end{cases}$$

b. Use the trace feature to estimate when the planes will pass each other.

c. Use algebra to determine when the planes will pass each other.

1.3 **a.** Write parametric equations to model the motion of a plane that enters the radar screen 1000 m east of the tower flying due north at 55 m/sec.

b. Write parametric equations to model the motion of a plane that leaves the radar screen 1500 m east of the tower flying due south at 90 m/sec.

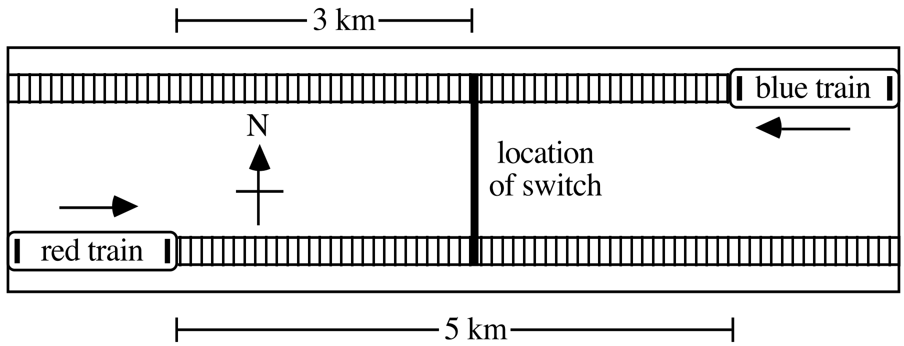
1.4 The small plane and large jet described in Activity **1** also plan to stage a near collision in the air show. The planes will fly on perpendicular paths at the same altitude. The small airplane will fly east at 55 m/sec along a path 1000 m north of the control tower. The large jet will fly due north at 90 m/sec along a path somewhere to the east of the tower.

a. Write parametric equations to describe the paths that will produce an actual collision.

b. The pilots have decided that at the moment the small plane reaches the point where their paths intersect, the large jet should be 25 m away. Modify the parametric equations from Part **a** to model this situation.

* * * * *

1.5 Shown in the diagram below, two subway trains are traveling toward a switch, where trains can change from one set of tracks to another. The red train is 3 km from the switching point, traveling east at 60 km/hr. At the same time, the blue train is 2 km from the switch, traveling west at 55 km/hr.

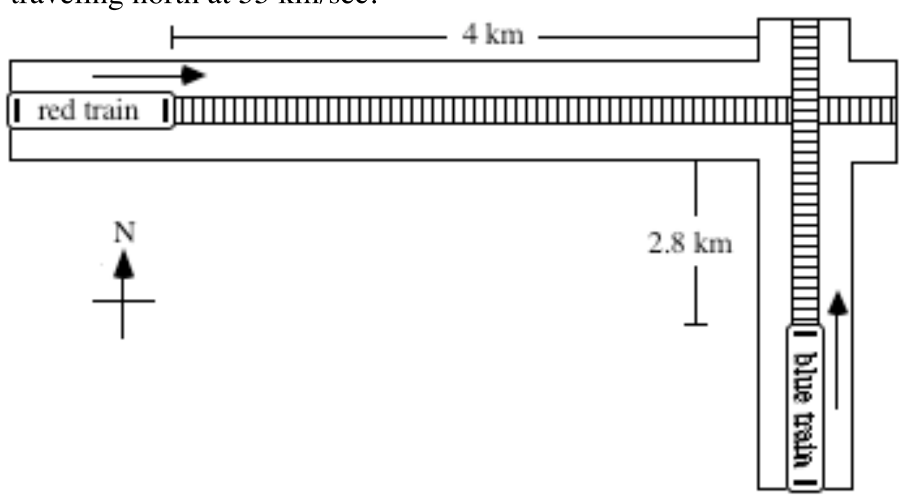


a. Write sets of parametric equations to model this situation.

b. Judging from a graph of the equations in Part **a**, will it be safe to switch the blue train to the south tracks if both trains continue at their present speeds? Justify your response.

c. If it is safe to switch the blue train, determine the red train's location when the blue train reaches the switch. If it is not safe, adjust the speed of one train so that the switch can occur safely and demonstrate algebraically that your adjustment will result in a safe switch of the blue train to the south track.

1.6 As shown in the following diagram, two trains are traveling toward an intersection. The red train is 4 km from the intersection and traveling east at 60 km/hr. The blue train is 2.8 km from the intersection and traveling north at 55 km/sec.



a. Model this situation parametrically.

b. Will the two trains collide? Justify your response algebraically.

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Activity 2

In the real world of air travel, flight paths seldom can be modeled as strictly horizontal or vertical line segments on a coordinate system. In this activity, you explore how to represent other types of linear motion in parametric form.

Exploration

Imagine that a Boeing 747 enters an air traffic controller's radar screen from the south. As shown in Figure **2**, the 747 appears at a point 2000 m east of the tower. At the same moment, a Piper Cub enters the screen at a point 500 m north of the tower.

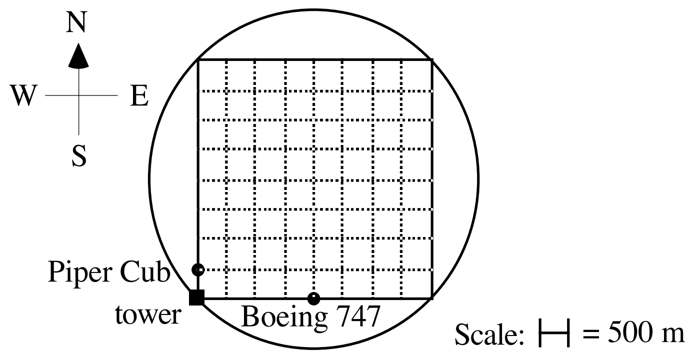


Figure 2: Initial positions of two planes on a radar screen

The radar reports the same altitude for both planes. After 1 sec, the 747 is 150 m west and 100 m north of its initial position. The Piper Cub, meanwhile, is now at a point 60 m east and 40 m north of its initial position. If both planes continue on these courses at constant speeds, will they collide?

a. Represent the grid on the radar screen in Figure **2** as a coordinate plane with the tower located at the origin.

1. Determine the coordinates for the location of each airplane as it enters the radar screen.
2. Find the coordinates for the location of each airplane after 1 sec.
3. Determine equations for the lines that represent the planes' flight paths.

b. The direction and speed of the Boeing 747 can be represented by a velocity vector.

1. Determine the horizontal component ($\mathbf{v}_x$) of the 747's velocity.
2. Determine the vertical component ($\mathbf{v}_y$) of the 747's velocity.
3. Determine the ratio $\mathbf{v}_y / \mathbf{v}_x$.
4. Compare the ratio $\mathbf{v}_y / \mathbf{v}_x$ to the slope of the line that represents the 747's flight path.

c. Create tables with headings like those in Tables **3** and **4**. Complete each table with the appropriate distances. The values in the columns with the heading "Time" represent the number of seconds after the air traffic controller began monitoring the planes.

Table 3: The Boeing 747's position

Time (sec)	x-coordinate (m)	y-coordinate (m)
0		
1		
2		
3		
⋮		
7		
⋮		
t		

Table 4: The Piper Cub's position

Time (sec)	x-coordinate (m)	y-coordinate (m)
0		
1		
2		
3		
⋮		
7		
⋮		
t		

d. Write parametric equations to model the path of each airplane.

e. Graph the equations simultaneously on a graphing utility.

f.
1. Graph the two linear equations from Part **a** on a coordinate grid with the same scales as in Part **e**.
2. Compare these graphs to the parametric graphs in Part **e**.

g. Determine the coordinates of the point where the paths of the two airplanes intersect.

h. Suppose that the air traffic controller did not begin monitoring the planes until the position of the 747 was (1700,200) and the position of the Piper Cub was (120, 580). Repeat Parts **c–f** for this situation.

Discussion

a. Describe what each term represents in the parametric equations that model the path of the Piper Cub.

b.
  1. How did changing the positions of the planes at $t = 0$ affect the parametric equations you used to model their flight paths?

  2. How did this affect the corresponding graphs?

c. Describe how you could determine the slope of a line from its parametric equations.

d. Why would you expect there to be infinitely many parametric equations that can model a single line?

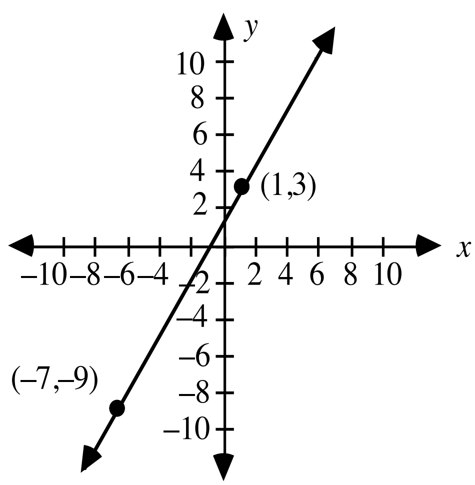
e.
  1. Will the two airplanes in the exploration collide? Explain your response.

  2. Which graphs better support your response: the ones from Part **e** of the exploration or the ones from Part **f**?

f. How could you determine the distance between the planes when the first plane reaches the intersection of the flight paths?

Assignment

2.1 The graph below shows the line that passes through (–7,–9) and (1,3).



a. Write a set of parametric equations that represents an object moving along the line segment from (–7,–9) to (1,3).

b. Write a second set of parametric equations that represents an object moving along a different segment of the line.

c. Write a third set of parametric equations that represents an object moving along the segment in Part **a** at a different velocity.

d. How many different sets of parametric equations could be used to describe a segment of this line?

2.2 **a.** Determine the distance between the planes in the exploration when the first plane reaches the intersection of their flight paths.

b. Determine the distance between the planes in the exploration when the second plane reaches the intersection of their flight paths.

2.3 The path of an airplane on a radar screen is represented by the parametric equations below, where x and y represent distances in meters and t represents time in seconds.

$$\begin{cases} x = 3500 - 30t \\ y = 1000 + 60t \end{cases}$$

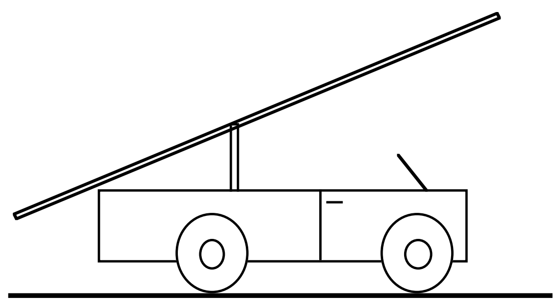
a. Graph the flight of the airplane for the interval from 0 sec to 50 sec using an appropriate increment.

b. At what location was the airplane first detected on the screen?

c. What is the slope of the line segment depicted by the parametric equations?

d. Determine the equation of the line that contains this segment in slope-intercept form.

2.4 To load baggage onto a plane, airline employees use a mechanized ramp. When the ramp is elevated as shown in the diagram below, the higher end is 4 m above the ground while the lower end is 1 m above the ground. The length of the ramp is 8.5 m.



a. Using the point on the ground directly below the lower end of the ramp as the origin of a coordinate system, determine a set of coordinates to represent:

1. the lower end of the ramp
2. the higher end of the ramp

b. What is the slope of the ramp?

c. Determine a set of parametric equations that could be used to model the path of a suitcase moving up the ramp.

d. If the suitcase falls off the ramp three-fourths of the way to the top, how far will it fall to the ground? Explain your response.

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2.5 The path of a moving object is described by the parametric equations below, where t represents time in seconds:

$$\begin{cases} x = 4 - 3t \\ y = -5 + 2t \end{cases}$$

a. Graph the path using the interval [0, 3] for the domain.

b. What is the position when $t = 0$?

c. What is the slope of the line depicted by the parametric equations?

d. Determine the equation of the line in slope-intercept form.

e. What is the distance between the object's locations at $t = 0$ and $t = 2.5$?

2.6 The motion of object A is defined by the following parametric equations, where x and y represent distances in meters and t represents time in seconds:

$$\begin{cases} x = -6 + 2t \\ y = -7 + 3t \end{cases}$$

The motion of object B is defined by the parametric equations below:

$$\begin{cases} x = 9 - 2t \\ y = -2 + 2t \end{cases}$$

a. Graph the parametric equations for both objects simultaneously.

b. At what point do the paths of the objects intersect?

c. Which object reaches the point of intersection first?

d. After the first object reaches the point of intersection, how long does it take the second object to reach that same point?

e. Modify the set of parametric equations for object B so that the two objects reach the point of intersection from Part **b** at the same time.

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Activity 3

The day-to-day operations of an airport feature many kinds of motion, including some that cannot be modeled with linear equations. An airplane waiting its turn to land, for example, might travel in a circle. The path of a helicopter blade also is circular, as is the motion of an anemometer, a device used to measure wind speed. In this activity, you use parametric equations to model circular paths.

Exploration

Airports use many different methods for delivering baggage to arriving passengers. In one popular system, luggage rotates around a carousel until claimed by the owners. Imagine that a lone suitcase is revolving on a circular luggage carousel, as shown in Figure **3**. Assume that the carousel has a radius of 1 unit and that the center of the carousel is located at the origin of a rectangular coordinate system.

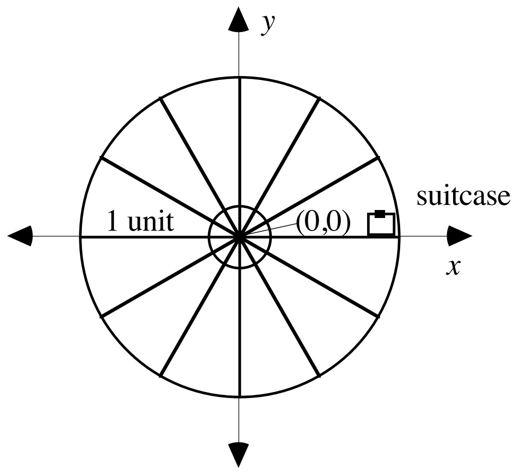

Figure 3: Suitcase on a luggage carousel

a. Represent the coordinates of any ordered pair (x, y) on the unit circle in terms of a central angle of the circle. Use radians to measure the angle.

b. Create a table with headings like those in Table **5** below. Use your results from Part **a** to complete the table for one revolution of the suitcase counterclockwise from the positive *x*-axis.

Table 5: Positions on a unit circle

Angle Measure (radians)	*x*-coordinate	*y*-coordinate
0		
$\pi/6$		
$\pi/3$		
$\pi/2$		
$2\pi/3$		
$5\pi/6$		
π		
$7\pi/6$		
$4\pi/3$		
$3\pi/2$		
$5\pi/3$		
$11\pi/6$		
2π		

c. Convert the relationships you wrote in Part **a** to parametric equations of the form below:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

d. Graph the parametric equations in Part **c**. Verify that the coordinates of the points on the graph agree with the values in Table **5**.

Mathematics Note

A circle with radius r and center at (h,k) can be represented by parametric equations in the following form:

$$\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases}$$

For example, the parametric equations below represent a circle with a radius of 3 units and center at the point (–2,5).

$$\begin{cases} x = -2 + 3\cos t \\ y = 5 + 3\sin t \end{cases}$$

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e. Write parametric equations to describe a luggage carousel with a radius of 2 units and center at the origin of a two-dimensional coordinate system.

Graph these equations simultaneously with the parametric equations from Part **c**.

f. Many larger airports place two or more carousels side by side in the baggage claim area. To model this situation, write parametric equations for a luggage carousel with a radius of 1 unit and center at (2,3).

Graph these equations simultaneously with the parametric equations from Part **c**.

Discussion

a. In Activities **1** and **2**, the parameter t represented time. What does the parameter t represent in Parts **c–f** of the exploration?

b. How do the coefficients of $\cos t$ and $\sin t$ affect the graphs in Part **e** of the exploration?

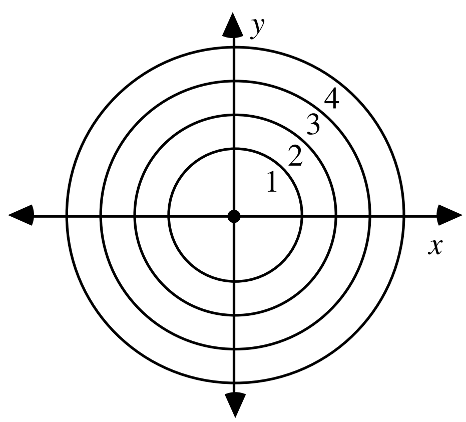
c. In Activities **1** and **2**, the constant terms represented the position of an object at $t = 0$. What do the constant terms represent in Part **f** of the exploration?

d. What are some advantages to graphing circles parametrically?

e. In the Level 4 module "Can It," you showed that $(\sin t)^2 + (\cos t)^2 = 1$ for any real-number value of t. Describe how to use this knowledge to write the standard form of the equation for a circle given its parametric equations. **Note:** The standard form of the equation for a circle with center at (h,k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$.

Assignment

3.1 A radar transmitter sweeps the surrounding area in a circular motion. The diagram below shows four concentric circles with a radar transmitter at the center. The innermost circle (1) has a radius of 40 km. The radius of each successive circle increases by 20 km. Assume that the transmitter is located at the origin of a two-dimensional coordinate system.

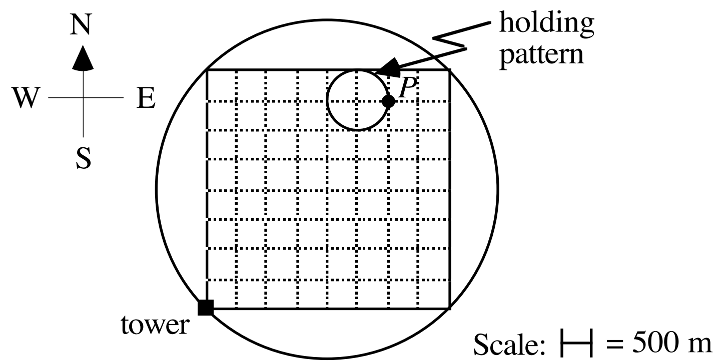


a. Write parametric equations for the circle farthest from the radar transmitter (circle 4).

b. What is the parameter for the equations you wrote in Part **a**?

c. What are the dependent variables in the equations you wrote in Part **a**? What are the independent variables?

3.2 While waiting for permission to land, a plane flies in a circular holding pattern as shown in the diagram below. The path has a diameter of 1 km and the plane completes one revolution every 5 min. The center of the circle is 2.5 km east and 3.5 km north of the tower.



a. Assume that the tower is located at the origin of a rectangular coordinate system and that point P is the initial location of the plane. Use parametric equations to model the airplane's flight path.

b. Determine the coordinates that describe the plane's location after it has traveled $1/5$ of a revolution counterclockwise from its initial position.

c. If the airplane began circling counterclockwise from point P, find the location of the plane after 9 min.

d. How far will the airplane have traveled after 9 min?

3.3 **a.** Graph the parametric equations below using appropriate technology:

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$$

b. What was the first point plotted on the circle in Part **a**? In which direction was the circle plotted?

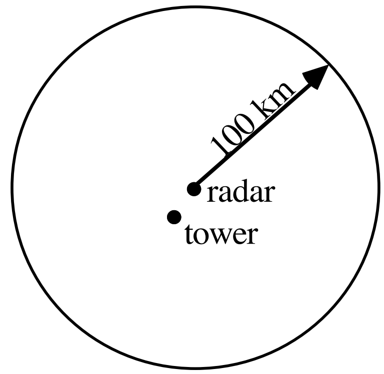
c. Change the parametric equations so that the circle will be plotted in the clockwise direction. (The starting point may be different than the starting point in Part **a**.)

d. Change the parametric equations so that the plotting starts on the positive x-axis and proceeds in a clockwise direction.

e. Change the parametric equations so that the plotting starts on the positive y-axis and proceeds in a counterclockwise direction.

* * * * *

3.4 As shown in the diagram below, the radar transmitter for an airport is located 1.5 km east and 2 km north of the control tower. The radar has a range of 100 km.



a. How far (in km) is the radar transmitter from the control tower?

b. Assuming that the tower is located at the origin of a rectangular coordinate system, use parametric equations to model the outer perimeter of the radar.

c. **1.** The radar detects an airplane 75 km west and 10 km south of the tower. Write coordinates for the location of the plane.

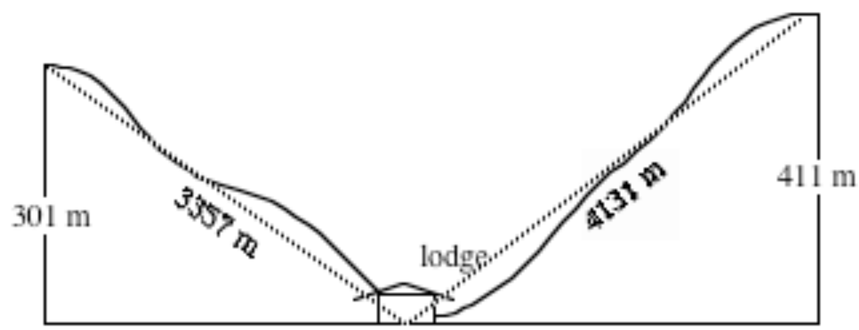
2. Find the distance from the plane to the radar transmitter.

3. Write parametric equations to model a circle with its center at the radar transmitter and a radius equal to the distance in Step **2**.

* * * * * * * * * *

Summary Assessment

1. A ski lodge is located at the junction of two ski runs. As shown in the diagram below, the ski run to the left of the lodge is 3357 m long and has a vertical drop of 301 m. The run to the right of the lodge is 4131 m long and has a vertical drop of 411 m.



Jolene is at the summit of the run to the left, while Michael is at the summit of the run to the right. The two friends plan to meet at the lodge for lunch. Both skiers start toward the lodge at the same moment, and both travel downhill at a rate of 0.8 m of elevation every second.

Assume that the lodge is located at the origin of a rectangular coordinate system and that the skiers' paths are linear.

a. Write the ordered pairs that represent Jolene's and Michael's initial positions.

b. **1.** Determine the horizontal distance that Jolene covers in 1 sec.

2. Determine the horizontal distance that Michael covers in 1 sec.

c. Write parametric equations that model the each skier's trip down the mountain.

d. Using appropriate intervals for the domain and range, graph the parametric equations from Part **c** on a graphing utility.

e. Determine which skier arrives at the lodge first, and find the time required to descend.

f. What is the difference in the arrival times of the two skiers?

g. At the time the first skier arrives at the lodge, how far (to the nearest meter) is the other skier from the lodge?

2. Anton and Julia board the Ferris wheel at the local carnival. The diameter of the wheel is 16 m and the bottom of the wheel is 3 m above the ground. When the ride operator finishes loading the passengers, Anton and Julia's chair is at the highest point of the wheel. At that moment, the Ferris wheel starts to turn counterclockwise at a constant rate of 4 revolutions per minute. A complete ride lasts 4.75 min.

a. Use parametric equations to model the path of Anton and Julia's chair. (Remember that Anton and Julia start their ride at the top of the wheel.)

b. Determine how far Anton and Julia will travel after 30 sec.

c. Determine how far Anton and Julia will travel after one complete ride.

d. How far above the ground will the two friends be after 10 sec? after 20 sec? after 30 sec? (Hint: In this case, the parameter t represents radian measure.)

Module Summary

- **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. On an *xy*-plane, for example, both *x* and *y* can be expressed as functions of a third variable *t*:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

- When parametric equations are used to model linear relationships, they can be written in the general form:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

  where (a, c) are the coordinates of the graph when $t = 0$. The coefficients *b* and *d* represent the constant rates of change in the *x*- and *y*-components, respectively, in terms of *t*.

- A circle with radius *r* and center at (h, k) can be represented by parametric equations in the following form:

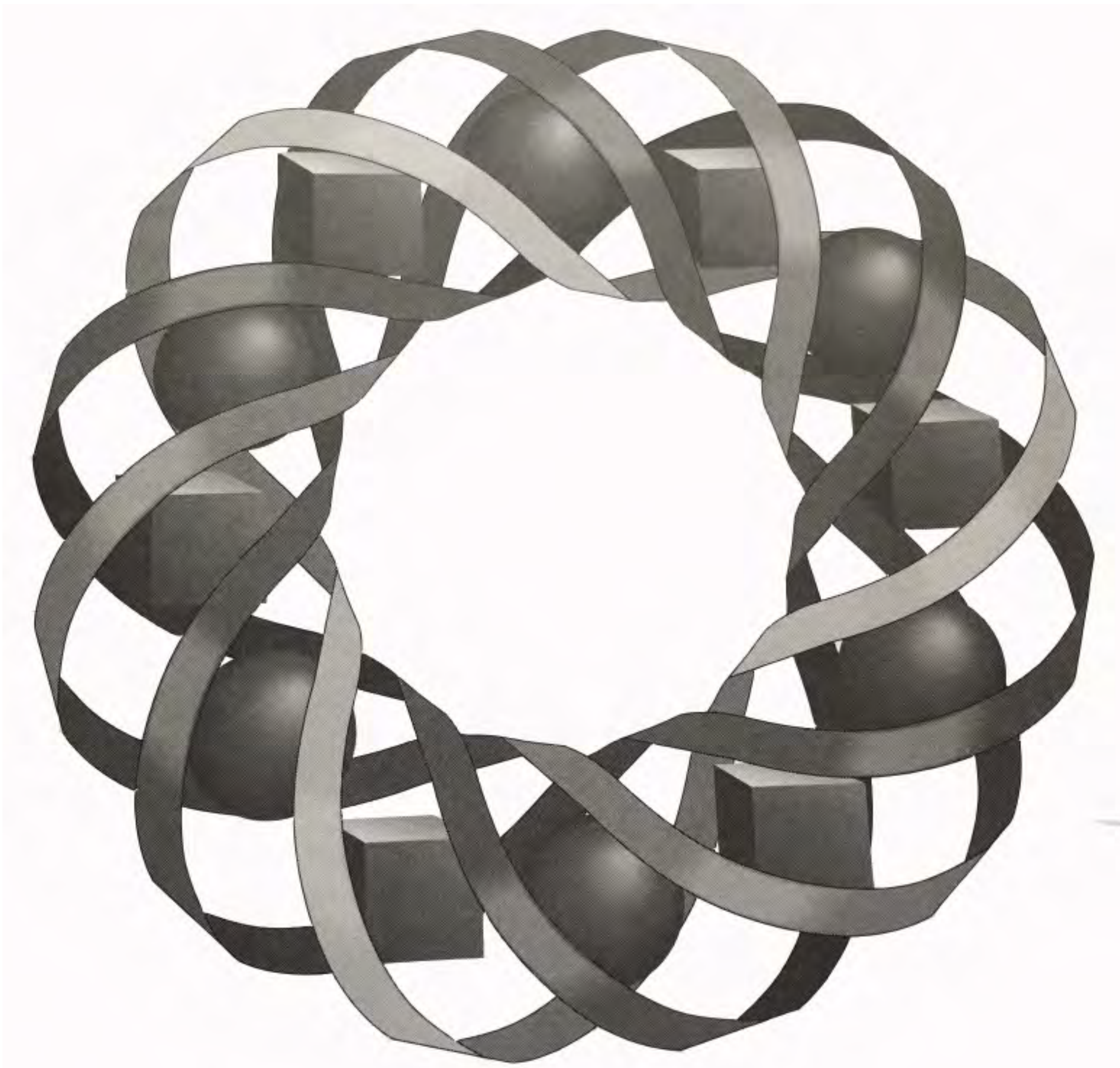
$$\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases}$$

Selected References

Cieply, J. F. "Parametric Equations: Push 'Em Back, Push 'Em Back, Way Back!" *The Mathematics Teacher* 86 (September 1993): 470–74.

Mathematical Association of America (MAA). *Teaching Mathematics with Calculators: A National Workshop*. Providence, RI: MAA, 1994.

Having a Ball



News flash! The sum of the measures of a triangle's interior angles is not always 180˚. In this module, you discover how—and why—this can occur.

Monty Brekke • Janet Kuchenbrod • Tim Skinner

Having a Ball

Introduction

Imagine yourself on board an airplane flying from Los Angeles to Tokyo, as illustrated in Figure **1**. What route should the pilot choose in order to travel the shortest distance between these two cities?

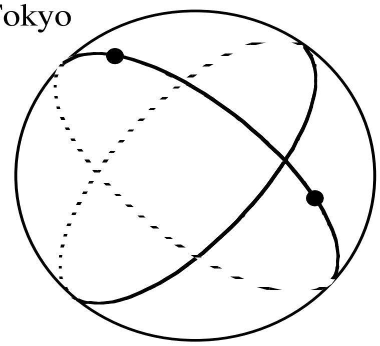


Figure 1: Flight path from Los Angeles to Tokyo

On a flat surface, the shortest distance between two points is along the line segment that connects the points. The earth, however, is shaped roughly like a sphere. A **sphere** is a set of points in space at a given distance from a fixed point. The fixed point is the **center**, the given distance is the **radius**. What is the shortest distance between two points on a sphere? How do other geometric properties on a sphere compare to the geometric properties of a flat surface, or a plane?

History Note

Although recognized for centuries, geometry on a flat surface was formalized by Greek mathematicians around 300 B.C. It is commonly known as **Euclidean geometry**, in honor of the Greek geometer Euclid. The flat surface of Euclidean geometry is the **Euclidean plane**.

Euclid's plane geometry was based on five **postulates**, statements assumed to be true without proof. The fifth postulate was the subject of controversy for over 20 centuries. According to this postulate, "if a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles."

The fifth postulate is illustrated in Figure **2**. If $m\angle 1 + m\angle 2 < 2 \bullet 90^\circ$, lines *l* and *m* will eventually meet on the same side of the vertical line *t* as $\angle 1$ and $\angle 2$.

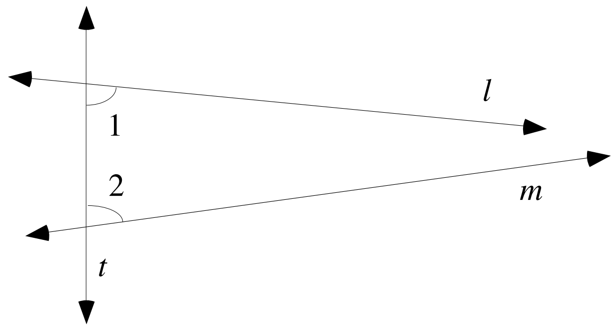


Figure 2: Geometrical representation of Euclid's fifth postulate

By modifying Euclid's fifth postulate and the concept of parallelism, mathematicians have developed other geometries, referred to as **non-Euclidean geometries**. One example is **spherical geometry**, the study of geometric figures constructed on the surface of a sphere. In spherical geometry, Euclid's fifth postulate is not satisfied.

Activity 1

In this activity, you determine whether or not each of the Euclidean concepts listed below also is true in spherical geometry.

- Two points determine a unique line.
- The length of the segment determined by two points is the shortest distance between the points.
- Lines cannot be measured because they are of infinite length.
- For any three points on a line, exactly one point is between the other two.
- Two lines either intersect or are parallel.

Exploration 1

In this exploration, you investigate the properties of lines on a sphere.

a. Obtain a sphere. Cut two pieces of string that are slightly longer than the circumference of the sphere.

b.
1. Mark two points on the sphere that are not endpoints of a diameter of the sphere.
2. Determine whether it is possible to wrap a string around the sphere so that it passes through the two points marked in Step **1** and forms a circle with circumference equal to the circumference of the sphere.

3. If it is possible to complete Step **2**, determine whether it is possible to wrap a second string as described in Step **2** so that it does not coincide with the first string.

c. Repeat Part **b** for two additional pairs of points. Record your observations.

d. Locate the endpoints of a diameter of the sphere. Repeat Part **b** using these two points and record your observations.

Discussion 1

a. If the sphere were cut along the paths of the strings in Exploration **1**, what characteristics would be shared by all the cross sections?

Mathematics Note

A **great circle** of a sphere is a set of points determined by the intersection of the sphere and a Euclidean plane that contains the center of the sphere. Any two points on the surface of the sphere and contained in such an intersection determine a great circle. In spherical geometry, a **line** is defined as a great circle.

In Figure **3**, for example, points *A* and *B* lie on line *l*, a great circle of the sphere, and are not endpoints of a diameter of the sphere. These two points divide line *l* into **major** and **minor arcs**. The minor arc is the shorter of the two. Minor arcs typically are named by two letters, such as *AB*, while major arcs are named with three letters, such as *ADB*. If *A* and *B* are endpoints of a diameter of the sphere, then the two arcs are equal in length, and neither is a major nor minor arc.

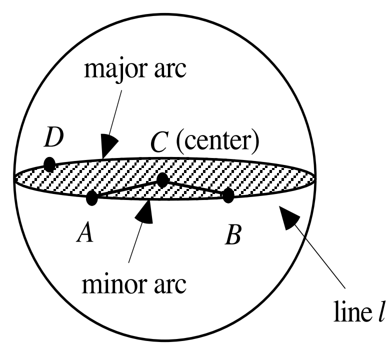


Figure 3: A line on a sphere

On a sphere, the **distance** between two points that are not endpoints of a diameter is the length of the minor arc of the great circle determined by those two points. The distance between two points that are endpoints of a diameter is half the circumference of the sphere. In Figure **3**, for example, the distance between points *A* and *B* is equal to the length of arc *AB*.

b. Is it possible for more than one great circle to pass through two given points on a sphere?

c.
1. In spherical geometry, what is the least number of points needed to determine a unique line?
2. Will this number of points always determine a unique line?

d. Locations on Earth are often described using lines of longitude and latitude. How do lines of longitude and latitude compare to great circles?

e. Recall that the measure of an arc is the measure of its central angle and that the length of an arc can be found by multiplying the circumference of the circle by the fractional part of the circle represented by the arc.
1. In Figure **3**, what fractional part of the circle is represented by arc AB?
2. Given the length of arc AB, how could you determine $m\angle ACB$?

Mathematics Note

An **angle** on a sphere is defined by two minor arcs from different great circles that have a common endpoint. The intersection of the arcs is the vertex of the angle.

The measure of an angle on a sphere is defined by the planes that contain the great circles forming the angle. The tangents to the great circles at their point of intersection and in the planes of the circles intersect to form a planar angle. The measure of the angle on the sphere is the same as the measure of this planar angle. As in Euclidean geometry, angle measure is greater than or equal to 0° and less than 180°.

As shown in Figure **4**, for example, minor arc AP and minor arc BP form $\angle APB$. The measure of $\angle APB$ is the same as the measure of $\angle A'PB'$.

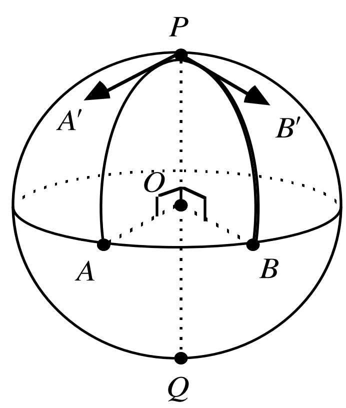


Figure 4: An angle on a sphere

f. In Figure **4**, points P and Q are endpoints of a diameter and the great circle containing A and B is perpendicular to $\overline{PQ}$.

Recall that in a plane, a line tangent to a circle is perpendicular to the radius that contains the point of tangency. Similarly, a plane tangent to a sphere is perpendicular to the radius containing the point of tangency.

Given these facts, what is the relationship between the plane tangent to the sphere at point P and the great circle that contains points A and B? Explain your response.

g.
1. Considering your response to Part **f** above, what is the relationship between $\angle A'PB'$ and $\angle AOB$?
2. What is the relationship between $\angle AOB$ and $\angle APB$?

h. If $m\angle APB = 90^\circ$, what is the length of arc AB in Figure **4**? Explain your response.

i. On a sphere, the endpoints of a diameter are equidistant from the great circle perpendicular to the diameter. How does the fact that central angles of equal measure intercept arcs of equal length prove this statement?

j. Describe how to determine the measure of $\angle ABC$ in Figure **5** below.

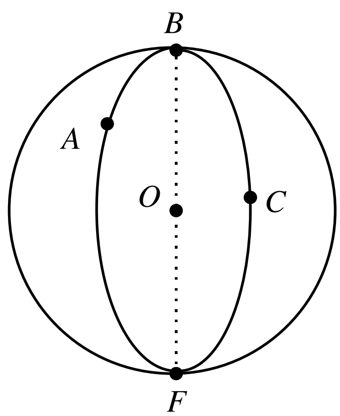


Figure 5: Angle *ABC* on a sphere

Exploration 2

Earth's north and south poles have a special relationship with the equator. In the following exploration, you examine similar relationships among points and lines on a sphere.

a.
1. Using a straightedge and a sheet of paper, determine the number of points of intersection possible for two distinct lines in a Euclidean plane.
2. Using rubber bands and a sphere, determine the corresponding number of points on the surface of a sphere.

b.
1. Determine the number of perpendicular lines that can be drawn from a point not on a line to that line in a Euclidean plane.
2. Determine the corresponding number on the surface of a sphere.

c. Construct two lines perpendicular to the same line in a plane. Repeat this construction on a sphere. Record any observations you make.

d.
1. Determine the number of points that are equidistant from all the points on a line in the Euclidean plane.
2. Determine the number of points that are equidistant from all the points on a line on a sphere.

Mathematics Note

A point is a **polar point** or **pole** of a given line on a sphere if is not on the given line and all lines perpendicular to the given line pass through the point. Every line on a sphere has two polar points.

In Figure **6**, for example, J and K are polar points of line l since every line perpendicular to line l passes through points J and K.

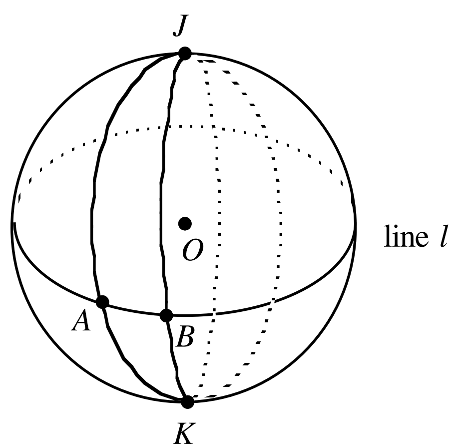


Figure 6: Sphere with line *l* and polar points *J* and *K*

Discussion 2

a. Does the term *pole* as defined in the mathematics note above also apply to Earth's north and south poles?

b. In Part **b** of Exploration **2**, you determined the number of perpendicular lines on a sphere that can be drawn from a point not on a line to that line. How would your response change if you do not consider polar points?

c. In Figure **6**, what is the relationship between the lengths of arc AJ and arc AK? Justify your response.

d. Describe how you could use perpendicular lines to locate the polar points of any line on a sphere.

e. Can any point on a sphere be a polar point? Explain your response.

f. Describe how you could find the measure of an angle on a sphere using an arc of the great circle for which the vertex of the angle is a polar point.

g. Is it possible for two lines on a sphere to be parallel? Explain your response.

Assignment

1.1 **a**. Are all of the circles that can be drawn on the surface of a sphere great circles? Explain your response.

b. What geometric figures can be formed by the intersection of a Euclidean plane and a sphere?

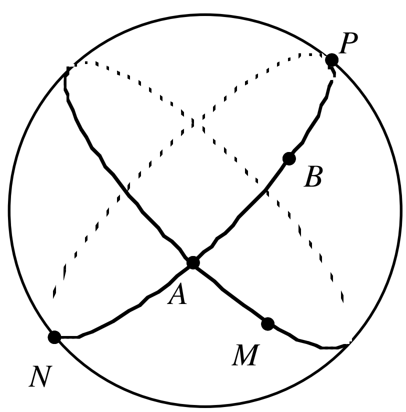
1.2 In Euclidean geometry, lines have infinite length. Is this also true of lines in spherical geometry? Explain your response.

1.3 The cities of Libreville, Gabon, and Quito, Ecuador, are very close to the equator. Libreville is located at approximately 0° N latitude and 9.5° E longitude. Quito is located at approximately 0° S latitude and 78.5° W longitude.

a. What is the measure of the angle formed by the great circles passing through the poles and each of the two cities? Explain your response.

b. Given that the diameter of Earth is approximately 12,756 km, what is the distance between Libreville and Quito?

1.4 As shown in the diagram below, *P* and *N* are poles of line *AM*.



a. Determine the number of lines that can be drawn perpendicular to line *AM* through each of the points listed below. Justify your responses.

1. point *A*
2. point *B*
3. point *P*

b. Summarize your findings in Part **a**.

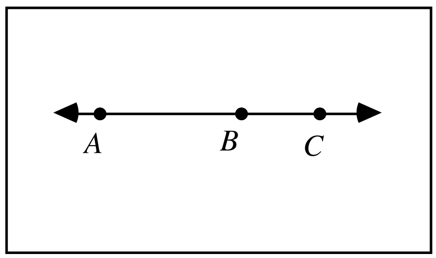
1.5 **a.** As mentioned in Problem **1.3**, Earth's diameter is approximately 12,756 km. Given this fact, what is the greatest possible distance between two cities on Earth?

b. City A is located on the same great circle as cities B and C, and is equidistant from both B and C. If you know the locations of B and C, how many locations are possible for A? Describe these locations.

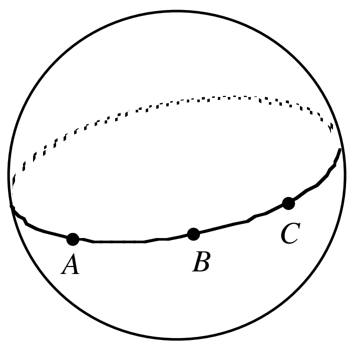
c. If city A is not located on the same great circle as cities B and C and is equidistant from both cities, how many locations are possible for A?

d. Describe how your response to Part **c** compares to the set of points in a Euclidean plane that are equidistant from the endpoints of a line segment.

1.6 On a line in a Euclidean plane, point B lies between points A and C if and only if $AB + BC = AC$.



As shown in the diagram below (not drawn to scale), points A, B, and C lie on the same great circle.



Using the Euclidean definition of "betweenness," when is B between A and C on a sphere?

* * * * *

1.7 **a.** On Earth, which line of latitude is perpendicular to every line of longitude? Explain your response.

b. What is the distance from the north or south pole to any point on the equator?

1.8 Compare lines, collinear points, parallelism, and perpendicularity in Euclidean geometry with the same four concepts in spherical geometry.

* * * * * * * * * *

Research Project

Write a report on Euclid's contributions to the study of plane geometry, including an explanation of the significance of the work recorded in his *Elements*. Your report also should include descriptions of the five geometric postulates (or axioms) upon which Euclidean geometry is based, along with a geometrical illustration of each one.

Activity 2

In the previous activity, you discovered that lines in spherical geometry are very different from lines in Euclidean geometry. In this activity, you investigate triangles and other polygons on the surface of a sphere.

Exploration 1

In this exploration, you investigate the sum of the measures of the interior angles of a triangle on the surface of a sphere.

Mathematics Note

A **triangle** on a sphere is the union of three noncollinear points together with the minor arcs of the great circles defined by these points. For example, Figure **7** shows ΔABC on a sphere.

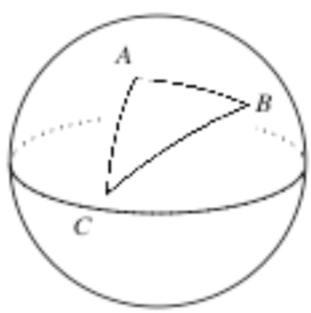


Figure 7: ΔABC on a sphere

For the purposes of this exploration, a triangle is any simple closed figure with three vertices connected by minor arcs of great circles.

a. Obtain a sphere and determine its circumference. Record this measurement to the nearest 0.1 cm.

b. Create three thin strips of paper: one 20 cm long, one 24 cm long, and one 27 cm long. Draw a line down the length of each strip, as shown in Figure **8**.

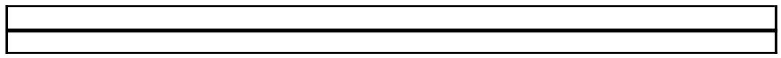


Figure 8: Paper strip with center line

c. Cut the 20-cm strip into three pieces that will form a triangle. Use the pieces to create a triangle *ABC* with a perimeter of 20 cm on the sphere.

Make sure that the center lines on the strips meet at each vertex, then tape the vertices securely.

d. Place three rubber bands around the sphere so that each rubber band lies along a side of the triangle and traces a great circle on the surface of the sphere.

e. As you discovered in Activity **1**, the measure of an angle on a sphere equals the measure of the minor arc of the great circle for which the vertex of the angle is a polar point.

1. Consider each vertex of the triangle as a pole of the sphere. Locate and mark the intersection of the great circles containing the sides of each angle (the rubber bands) with the great circle for which each vertex is a polar point.
2. Use the length of the minor arc determined by the points of intersection in Step **1** to determine the measure of each interior angle of the triangle. Record these measures in a table with headings like those in Table **1** below.

Table 1: Angle measures and sums

Perimeter of Triangle (cm)	$m\angle ACB$	$m\angle ABC$	$m\angle BAC$	**Sum of Angle Measures**
20				
24				
27				

f. Repeat Parts **c–e** using the 24-cm and 27-cm strips of paper.

g. Compare the results you recorded in Table **1** with those of your classmates.

Discussion 1

a. How do triangles constructed on the surface of a sphere differ from triangles constructed in a Euclidean plane?

b. How does the perimeter of a triangle constructed on the surface of a sphere appear to influence the sum of the measures of its interior angles?

c. How does the circumference of a sphere appear to influence the sum of the measures of a triangle's interior angles?

d. Is it possible to construct a triangle with two right angles on a sphere? Explain your response.

e. Is it possible to construct a triangle with three obtuse angles on a sphere? Explain your response.

f. Do you think that two-sided polygons can be constructed on a sphere? Explain your response.

Mathematics Note

In general, a polygon with *n* sides on the surface of a sphere is the union of *n* points that are the vertices of the polygon, with no three vertices collinear, together with the minor arcs of great circles defined by consecutive vertices. For example, Figure **9** shows a quadrilateral *ABCD* constructed on the surface of a sphere.

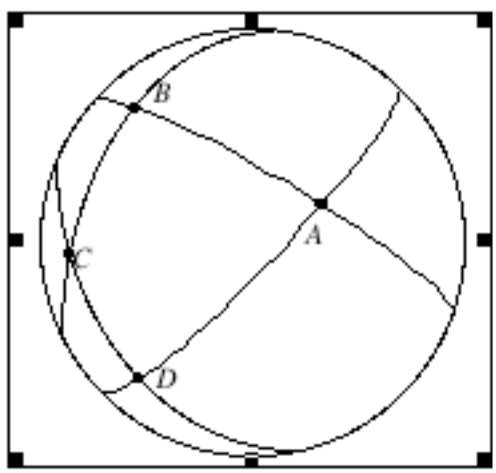


Figure 9: Quadrilateral *ABCD* on a sphere

Exploration 2

In this exploration, you examine the properties of quadrilaterals on a sphere.

a. Using paper strips and rubber bands, construct a quadrilateral on a sphere with a base of 6 cm and two sides of 3 cm perpendicular to the base.

b. Record the length of the side opposite the base and the measures of all interior angles.

c. Repeat Parts **a** and **b** for a quadrilateral with a base of 12 cm and perpendicular sides of 6 cm.

d. Construct a quadrilateral on the sphere with four sides of varying lengths. Record the measures of the interior angles.

e. For each quadrilateral constructed in Parts **a–d**, find the sum of the measures of the interior angles.

Discussion 2

a.
1. When two of the angles in a quadrilateral on a sphere are right angles, what are the measures of the other two angles?
2. If, as described in Part **a** of Exploration **2**, the two sides perpendicular to the base of a quadrilateral are congruent, the quadrilateral is a **Saccheri quadrilateral**. What can you determine about the non-right angles of a Saccheri quadrilateral?

b. If you constructed a quadrilateral with a base of 6 cm and two sides of 3 cm perpendicular to the base in a Euclidean plane,
1. what would be the length of the side opposite the base?
2. what would be the measures of the angles opposite the base?

c. In Parts **b** and **c** of Exploration **2**, how does the length of the base compare to the length of its opposite side? Do you think this relationship will always hold true on a sphere?

d. Do you think that rectangles can be constructed on the surface of a sphere? Explain your response.

e. In a Euclidean plane, the sum of the measures of a quadrilateral's interior angles is 360˚. Based on your results in Exploration **2**, suggest a general rule for the sum of the measures of a quadrilateral's interior angles on a sphere.

Assignment

2.1 **a.** What is the upper bound for the measure of an individual angle in a triangle on a sphere?

b. What is the upper bound for the sum of the measures of the angles of a triangle on a sphere?

2.2 In Euclidean geometry, the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Do you think this is also true in spherical geometry? Explain your response.

2.3 On the surface of a sphere, is there an upper bound for the sum of the lengths of a triangle's sides? Explain your response.

2.4 In Euclidean geometry, the sum of the measures of a triangle's interior angles is 180°. A quadrilateral can be divided into two triangles. Therefore, the sum of the measures of a quadrilateral's interior angles is $2 \bullet 180^\circ$ or 360°.

a. Use a similar argument to support your general rule for the sum of the measures of a triangle's interior angles on a sphere.

b. What do you think is the upper bound for the sum of the measures of a quadrilateral's interior angles on a sphere? Explain your reasoning.

2.5 In spherical geometry, what is the lower bound for the sum of the measures of a hexagon's interior angles? What is the upper bound for this sum? Explain your responses.

2.6 **a.** Determine a formula for finding the lower bound for the sum of the measures of the interior angles of an n-sided polygon on a sphere.

b. Determine a formula for finding the upper bound for the sum of the measures of the interior angles of an n-sided polygon on a sphere.

2.7 Describe the Euclidean quadrilaterals which cannot be constructed on the surface of a sphere and explain why they cannot exist.

* * * * *

2.8 **a.** In spherical geometry, a triangle may have three right angles. Determine the dimensions of a triangle with three right angles on Earth's surface. (Earth's diameter is approximately 12,756 km.)

b. What portion of Earth's surface is represented by the area of the triangle?

2.9 Compare triangles and quadrilaterals in Euclidean geometry to triangles and quadrilaterals in spherical geometry.

* * * * * * * * * *

Activity 3

In Euclidean geometry, two polygons are **similar** if there is a one-to-one correspondence between their vertices so that corresponding sides are proportional and corresponding angles are congruent. In this activity, you compare similarity in a plane with similarity on a sphere.

Discussion

a. Is it possible for two triangles in a plane to be similar if the lengths of their sides are different? Explain your response.

b. On a sphere, what is the relationship between a triangle's perimeter and the sum of the measures of its interior angles?

c. Can two triangles on a sphere be similar if the lengths of their sides are different? Explain your response.

d. When is it possible for two triangles on a sphere to be similar? Explain your response.

Assignment

3.1 **a.** Are all right triangles in a plane similar? Explain your response.

b. Are all squares in a plane similar? Explain your response.

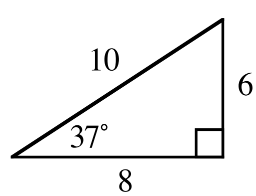
c. Are all rectangles in a plane similar? Explain your response.

3.2 **a.** Construct two rectangles that are similar but not congruent in a plane.

b. Is it possible to construct two quadrilaterals that are similar but not congruent on a sphere? Explain your reasoning.

3.3 Use similarity to describe the patterns on the surface of a soccer ball.

3.4 **a.** Consider a triangle in a plane similar to the one in the diagram below. Given that its longest side measures 12 cm, determine the lengths of the remaining sides and the measures of the triangle's three interior angles.



b. If the triangles in Part **a** were on a sphere, what would you expect to be true about the measures of their interior angles?

* * * * *

3.5 A baseball diamond is a square that measures 90 feet between bases. The pitcher's mound is 60 feet 6 inches from home plate.

a. An artist wants to create a scale drawing of a baseball diamond on a baseball card. The card measures 6 cm by 9 cm. Describe how the artist can use similarity to create the scale drawing, including some possible dimensions (in centimeters) for the drawing.

b. Why would the process you described in Part **a** not produce an accurate scale drawing on a baseball?

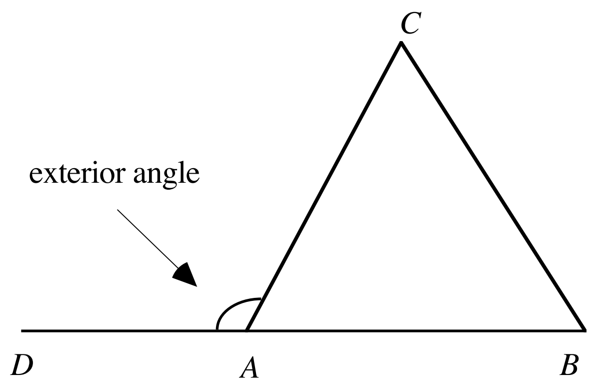
3.6 **a.** In a Euclidean plane, how many triangles can be drawn that have angle measures of 45°, 55°, and 80°, but different perimeters? Explain your response.

b. On a sphere, how many triangles can be drawn that have angle measures of 50°, 65°, and 95°, but different perimeters? Explain your response.

* * * * * * * * * *

Summary Assessment

1. **a.** In Euclidean geometry, determine the relationship between the measure of an exterior angle of a triangle and the sum of the measures of two non-adjacent interior angles, as shown in the diagram below.



b. Does this same relationship hold true in spherical geometry? Justify your response.

2. The locations of points on the surface of a sphere can be described in a variety of ways.

a. Explain how degrees of latitude and degrees of longitude are determined for a point on Earth's surface.

b. Using longitude and latitude, describe a triangle on the Earth's surface for which the sum of the measures of the interior angles is 270°.

c. Does a line of latitude satisfy the definition of a line in spherical geometry? Explain your response.

3. **a.** Use your knowledge of circles in a plane to define a circle on a sphere. Draw an example of a circle that fits your definition.

b. Can a circle on a sphere also be a line? Explain your response.

Module Summary

- A **postulate** is a statement that is assumed to be true without proof.
- A **sphere** is a set of points in space at a given distance from a fixed point. The fixed point is the **center**, the given distance is the **radius**.
- **Spherical geometry** is the study of geometric figures constructed on the surface of a sphere. In spherical geometry, Euclid's fifth postulate is not satisfied.
- A **great circle** of a sphere is a set of points determined by the intersection of the sphere and a Euclidean plane that contains the center of the sphere. Any two points on the surface of the sphere and contained in the intersection determine a great circle.
- In spherical geometry, a **line** is defined as a great circle.
- On a sphere, the **distance** between two points that are not endpoints of a diameter is the length of the minor arc of the great circle determined by those two points. The distance between two points that are endpoints of a diameter is half the circumference of the sphere.
- An **angle** on a sphere is defined by two minor arcs from different great circles that have a common endpoint. The intersection of the arcs is the vertex of the angle.
- The measure of an angle on a sphere is defined by the planes that contain the great circles forming the angle. The tangents to the great circles at their point of intersection and in the planes of the circles intersect to form a planar angle. The measure of the angle on the sphere is the same as the measure of this planar angle. It also equals the measure of the minor arc of the great circle for which the vertex of the angle is a polar point. As in Euclidean geometry, angle measure is greater than or equal to 0° and less than 180°.
- Point J is a **polar point** of the line l on a sphere if J is not on line l, and all lines perpendicular to l pass through J. Every line on a sphere has two polar points or **poles**.
- A **triangle** on a sphere is the union of three noncollinear points together with the minor arcs of the great circles defined by these points.
- In general, a polygon with n sides on the surface of a sphere is the union of n points that are the vertices of the polygon, with no three vertices collinear, together with the minor arcs of great circles defined by consecutive vertices.

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