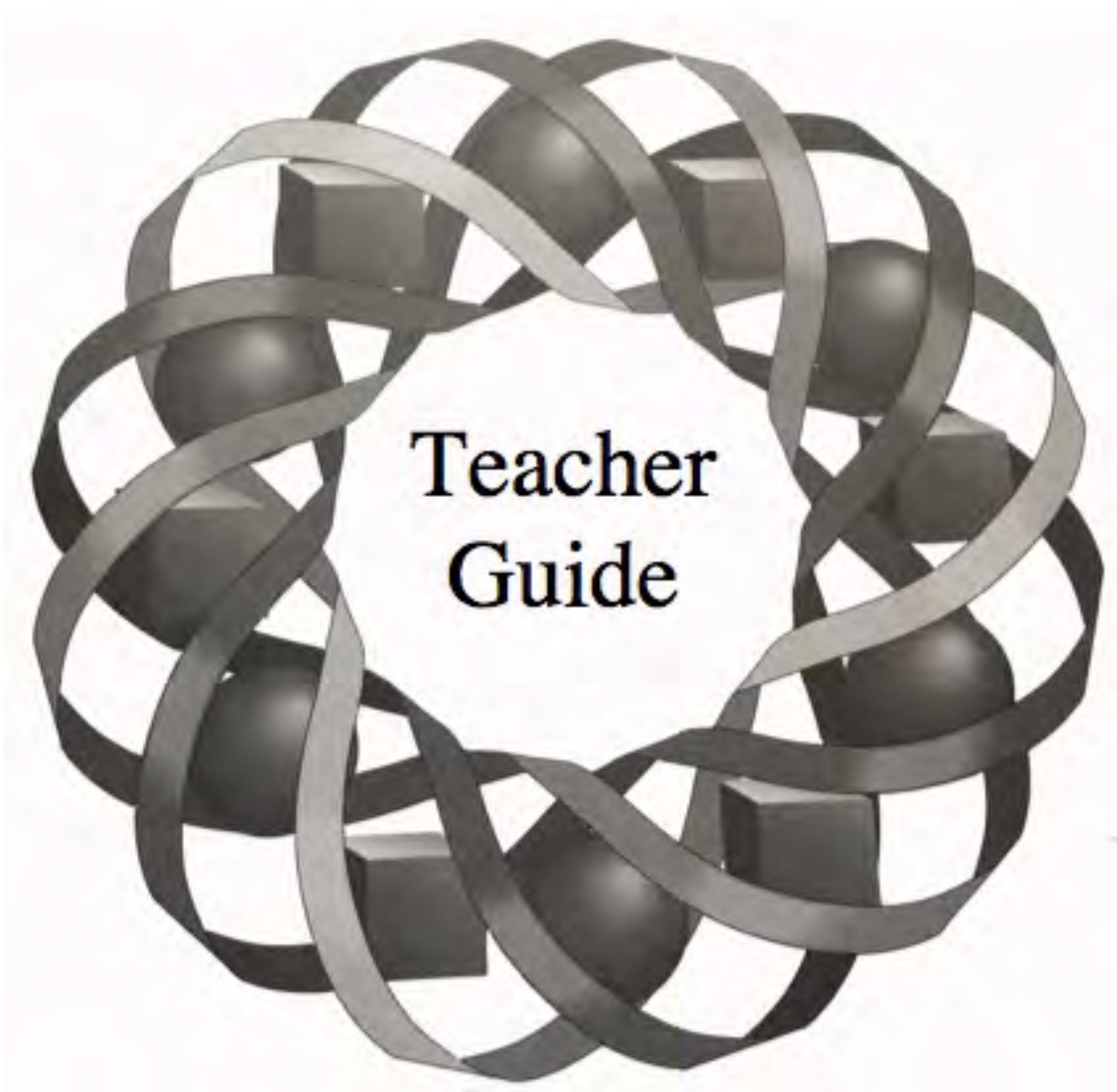
SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 4 Volumes 1-3

Teacher Guide
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About *Integrated Mathematics: A Modeling Approach Using Technology*

The Need for Change

In recent years, many voices have called for the reform of mathematics education in the United States. Teachers, scholars, and administrators alike have pointed out the symptoms of a flawed system. From the ninth grade onwards, for example, about half of the students in this country's mathematical pipeline are lost each year (National Research Council, 1990, p. 36). Attempts to identify the root causes of this decline have targeted not only the methods used to instruct and assess our students, but the nature of the mathematics they learn and the manner in which they are expected to learn. In its *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics addressed the problem in these terms:

> When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics. (p. 64)

Some Methods for Change

Among the major objectives of the *Integrated Mathematics* curriculum are:

- offering a 9–12 mathematics curriculum using an integrated inter-disciplinary approach for *all* students.
- incorporating the use of technology as a learning tool in all facets and at all levels of mathematics.
- offering a *Standards*-based curriculum for teaching, learning, and assessing mathematics.

The *Integrated Mathematics* Curriculum

An integrated mathematics program "consists of topics chosen from a wide variety of mathematical fields. . . [It] emphasizes the relationships among topics within mathematics as well as between mathematics and other disciplines" (Beal, et al., 1992; Lott, 1991). In order to create innovative, integrated, and accessible materials, *Integrated Mathematics: A Modeling Approach Using Technology* was written, revised, and reviewed by secondary teachers of mathematics and science. It is a complete, *Standards*-based mathematics program designed to replace all currently offered secondary mathematics courses, with the possible exception of advanced placement classes, and builds on middle-school reform curricula.

The *Integrated Mathematics* curriculum is grouped into six levels. All students should take at least the first two levels. In the third and fourth years, *Integrated Mathematics* offers a choice of courses to students and their parents, depending on interests and goals. A flow chart of the curriculum appears in Figure 1.

Each year-long level contains 14–16 modules. Some must be presented in sequence, while others may be studied in any order. Modules are further divided into several activities, typically including an exploration, a discussion, a set of homework assignments, and a research project.

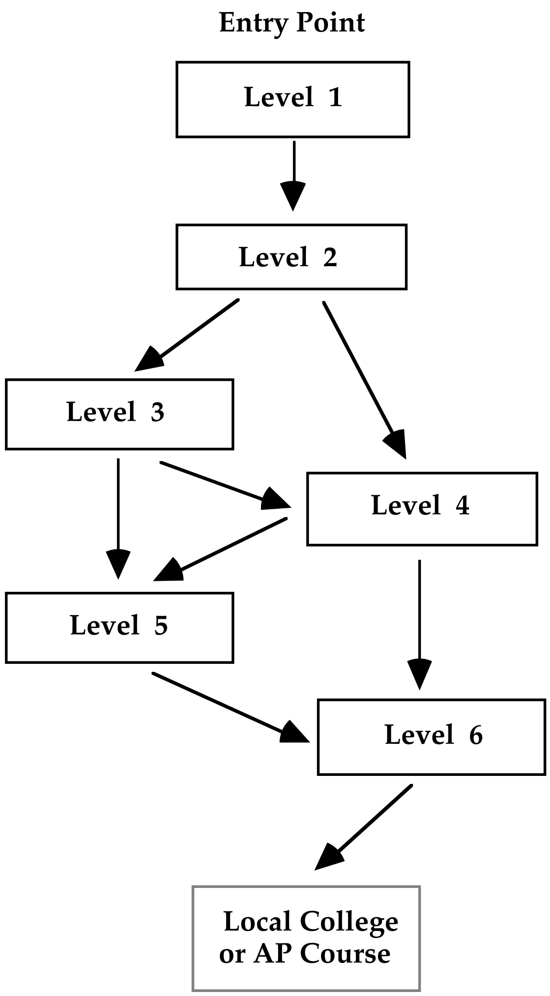

Figure 1: ***Integrated Mathematics*** **course sequence**

Assessment materials—including alternative assessments that emphasize writing and logical argument—are an integral part of the curriculum. Suggested assessment items for use with a standard rubric are identified in all teacher editions.

Level 1: a first-year course for ninth graders (or possibly eighth graders)

Level 1 concentrates on the knowledge and understanding that students need to become mathematically literate citizens, while providing the necessary foundation for those who wish to pursue careers involving mathematics and science. Each module in Level 1, as in all levels of the curriculum, presents the relevant mathematics in an applied context. These contexts include the properties of reflected light, population growth, and the manufacture of cardboard containers. Mathematical content includes data collection, presentation, and interpretation; linear, exponential, and step functions; and three-dimensional geometry, including surface area and volume.

Level 2: a second-year course for either ninth or tenth graders

Level 2 continues to build on the mathematics that students need to become mathematically literate citizens. While retaining an emphasis on the presentation and interpretation of data, Level 2 introduces trigonometric ratios and matrices, while also encouraging the development of algebraic skills. Contexts include pyramid construction, small business inventory, genetics, and the allotment of seats in the U.S. House of Representatives.

Levels 3 and 4: options for students in the third year

Both levels build on the mathematics content in Level 2 and provide opportunities for students to expand their mathematical understanding. Most students planning careers in math and science will choose Level 4. While Level 3 also may be suitable for some of these students, it offers a slightly different mixture of context and content.

Contexts in Level 4 include launching a new business, historic rainfall patterns, the pH scale, topology, and scheduling. The mathematical content includes rational, logarithmic, and circular functions, proof, and combinatorics.

In Level 3, contexts include nutrition, surveying, and quality control. Mathematical topics include linear programming, curve-fitting, polynomial functions, and sampling.

Levels 5 and 6: options for students in the fourth year

Level 6 materials continue the presentation of mathematics through applied contexts while embracing a broader mathematical perspective. For example, Level 6 modules explore operations on functions, instantaneous rates of change, complex numbers, and parametric equations.

Level 5 focuses more specifically on applications from business and the social sciences, including hypothesis testing, Markov chains, and game theory.

More About Level 4

"Colorful Scheduling" introduces coloring theory in the contexts of map making and scheduling events. "Nearly Normal" and "Confidence Builder" continue to explore statistics, sampling, and normal distributions. "Can It" stresses trigonometric concepts via data modeling. Other modules with a geometric theme include "Motion Pixel Productions," "Transmitting Through Conics," and "Having A Ball," which introduces non-Euclidean geometry.

"Drafting and Polynomials," "Big Business," "It's All in the Family," and "Controlling the Sky with Parametrics" focus on functions and equations. In the remaining modules, students investigate a variety of mathematical topics, including vectors, proof, limits, and logarithms.

The Teacher Edition

To facilitate use of the curriculum, the teacher edition contains these features:

Overview /Objectives/Prerequisites

Each module begins with a brief overview of its contents. This overview is followed by a list of teaching objectives and a list of prerequisite skills and knowledge.

Time Line/Materials & Technology Required

A time line provides a rough estimate of the classroom periods required to complete each module. The materials required for the entire module are listed by activity. The technology required to complete the module appears in a similar list.

Assignments/Assessment Items/Flashbacks

Assignment problems appear at the end of each activity. These problems are separated into two sections by a series of asterisks. The problems in the first section cover all the essential elements in the activity. The second section provides optional problems for extra practice or additional homework.

Specific assignment problems recommended for assessment are preceded by a single asterisk in the teacher edition. Each module also contains a Summary Assessment in the student edition and a Module Assessment in the teacher edition, for use at the teacher's discretion. In general, Summary Assessments offer more open-ended questions, while Module Assessments take a more traditional approach. To review prerequisite skills, each module includes brief problem sets called "Flashbacks." Like the Module Assessment, they are designed for use at the teacher's discretion.

Technology in the Classroom

The *Integrated Mathematics* curriculum takes full advantage of the appropriate use of technology. In fact, the goals of the curriculum are impossible to achieve without it. Students must have ready access to the functionality of a graphing utility, a spreadsheet, a geometry utility, a statistics program, a symbolic manipulator, and a word processor. In addition, students should have access to a science interface device that allows for electronic data collection from classroom experiments, as well as a telephone modem.

In the student edition, references to technology provide as much flexibility as possible to the teacher. In the teacher edition, sample responses refer to specific pieces of technology, where applicable.

Professional Development

A program of professional development is recommended for all teachers planning to use the curriculum. The *Integrated Mathematics* curriculum encourages the use of cooperative learning, considers mathematical topics in a different order than in a traditional curriculum, and teaches some mathematical topics not previously encountered at the high-school level.

In addition to incorporating a wide range of context areas, *Integrated Mathematics* invites the use of a variety of instructional formats involving heterogeneous classes. Teachers should learn to use alternative assessments, to integrate writing and communication into the mathematics curriculum, and to help students incorporate technology in their own investigations of mathematical ideas.

Approximately 30 classroom teachers and 5 university professors are available to present inservice workshops for interested school districts. Please contact Kendall Hunt Publishing Company for more information.

Student Performance

During the development of *Integrated Mathematics*, researchers conducted an annual assessment of student performances in pilot schools. Each year, two basic measures—the PSAT and a selection of open-ended tasks—were administered to two groups: students in classes using *Integrated Mathematics* and students in classes using other materials. Students using *Integrated Mathematics* materials typically had access to technology for all class work. During administration of the PSAT, however, no technology was made available to either group. Student scores on the mathematics portion of this test indicated no significant difference in performance.

During the open-ended, end-of-year test, technology was made available to both groups. Analysis of student solutions to these tasks showed that students using *Integrated Mathematics* were more likely to provide justification for their solutions and made more and better use of graphs, charts, and diagrams. They also demonstrated a greater variety of problem-solving strategies and were more willing to attempt difficult problems.

References

Beal, J., D. Dolan, J. Lott, and J. Smith. *Integrated Mathematics: Definitions, Issues, and Implications; Report and Executive Summary*. ERIC Clearinghouse for Science, Mathematics, and Environmental Education. The Ohio State University, Columbus, OH: ED 347071, January 1990, 115 pp.

Lott, J., and A. Reeves. "The Integrated Mathematics Project," *Mathematics Teacher* 84 (April 1991): 334–35.

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National Research Council. *A Challenge of Numbers: People in the Mathematical Sciences*. Washington, DC: National Academy Press, 1990.

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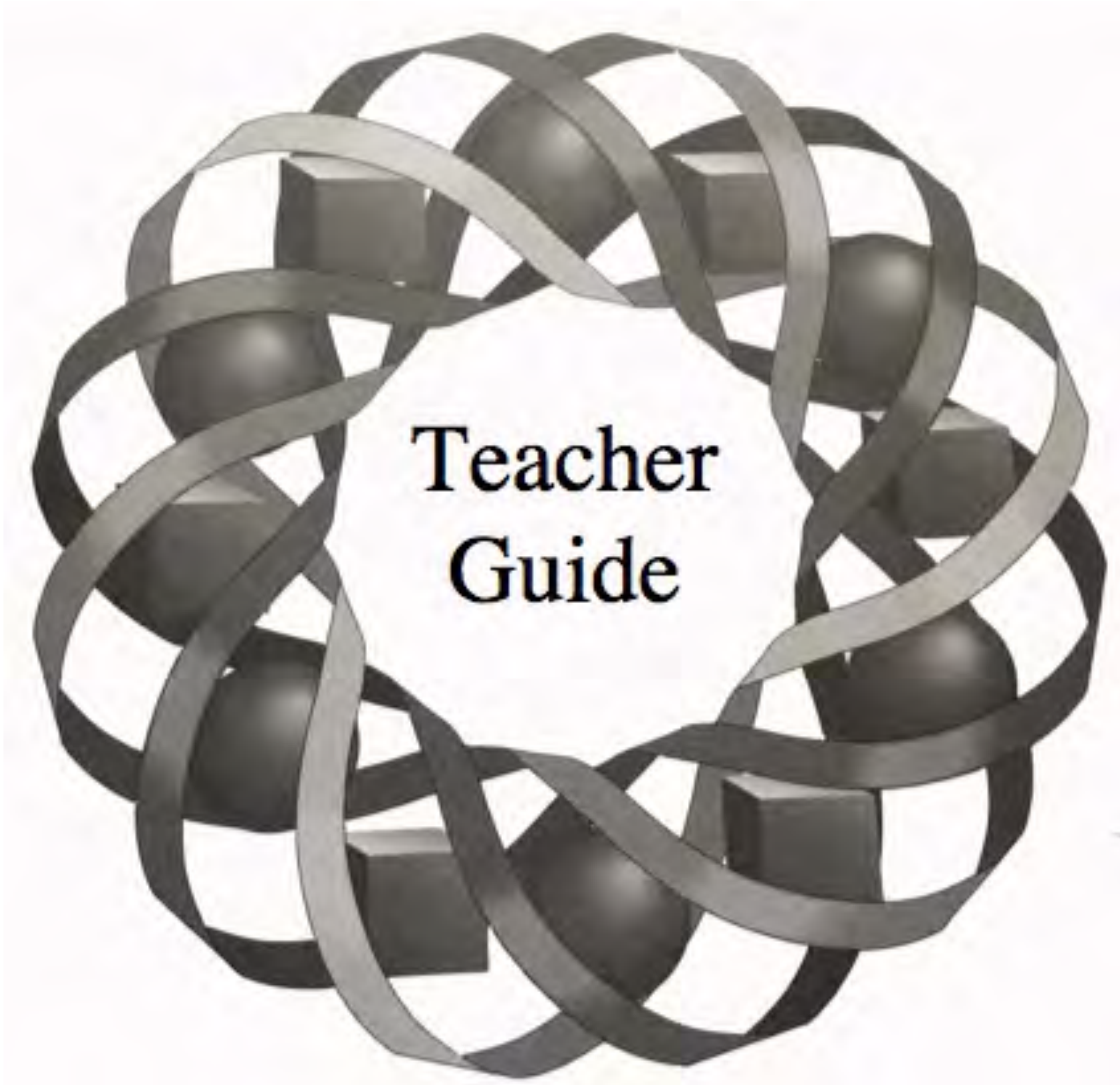
Colorful Scheduling



Have you outgrown coloring? Mathematicians haven't. Mathematicians use coloring theory to solve problems in scheduling, organization, and cartography.

Masha Albrecht • Darlene Pugh

Teacher Edition

Colorful Scheduling

Overview

This module integrates three branches of mathematics: coloring theory, topology, and graph theory. Although these topics have not traditionally been linked, new advances in molecular biology, physics, chemistry, and mathematics have led to their combination in many applications. **Note:** Students were introduced to graph theory in the Level 1 module "Going in Circuits."

Coloring theory is a branch of graph theory originally developed to study map-coloring problems. The proof of the four-color theorem in 1976 was a major accomplishment in mathematics. Coloring theory has many applications, including the analysis of scheduling and organizational problems.

Objectives

In this module, students will:

- determine the chromatic number of a map
- investigate the four-color theorem for maps drawn on flat surfaces and spheres
- create graphs of maps
- solve scheduling problems using graphs and coloring theory
- identify topologically equivalent graphs
- identify planar graphs
- determine the relationship between chromatic number and the number of vertices of a complete planar graph.

Prerequisites

For this module, students should know:

- the definition of a simple, closed curve
- the definition of a one-to-one correspondence

Time Line

Activity	Intro.	1	2	3	4	Summary Assessment	Total
Days	1	2	2	2	3	1	11

Materials Required

Materials	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
colored pencils	X	X	X			
map of western United States	X					
unlined paper		X				
string		X				
straightedge		X				
map of Australia		X				
map of South America		X	X			
map of local region			X			
map of continental United States				X		
non-permanent markers					X	
balloons					X	
string					X	

Technology

Software	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
geometry utility				X	X	

Colorful Scheduling

Introduction
(page 3)

Students make an initial exploration of map coloring using a map of the continental United States west of the Mississippi.

Materials List

- map of the western United States (one per student; a blackline master appears at the end of the teacher edition for this module)
- colored pencils (one set of at least five different colors per group)

Exploration
(page 3)

Answers will vary. Some students may observe that the least number of colors required to color this map is 4. This notion is investigated in detail in Activities **1–3**.

Discussion
(page 3)

a. Sample response: The number of different colors required is 4.

b. Sample response: Yes, the process used to color a map of the western United States with four colors could be continued for the entire country.

c. Answers will vary. Students explore the chromatic number for maps drawn on spherical surfaces in Activity **4**. With some restrictions, the least number of colors required is 4.

(page 4)

Activity 1

Note: You may wish to display a selection of road maps, atlases, and globes. Although four colors are sufficient for any of these maps, many publishing companies choose to use more.

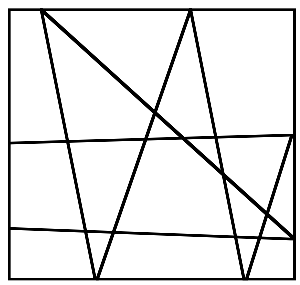
Materials List

- map of South America (one per student; a blackline master appears at the end of the teacher edition for this module)
- map of Australia (one per student; a blackline master appears at the end of the teacher edition for this module)
- colored pencils (one set of at least five different colors per group)
- unlined paper
- straightedge
- string (for Problem **1.7**)

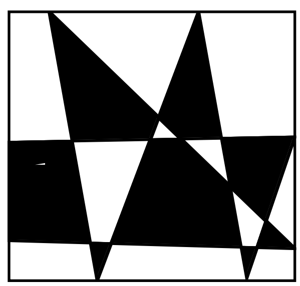
Exploration

(page 4)

a. Sample map:

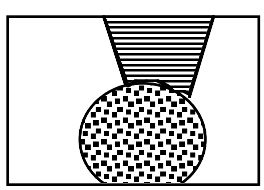


b. This map requires two colors. Sample response:



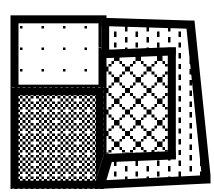
c. Student maps will vary. Sample responses are shown below.

1.



Chromatic number of 3

2.



Chromatic number of 4

d. Answers will vary. **Note:** Students encounter the four-color theorem in Activity **4**.

Discussion

(page 5)

a.

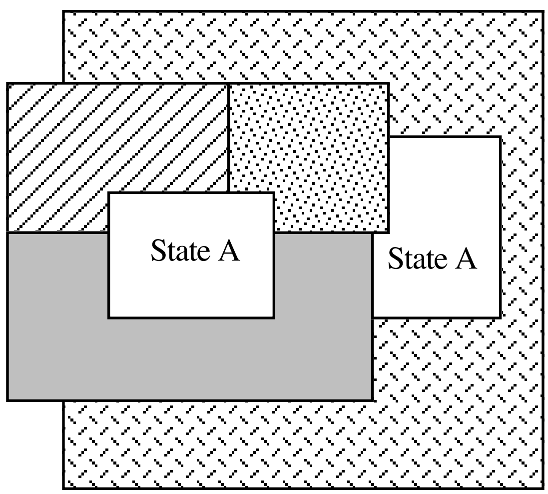
1. All straight-line maps should have a chromatic number of 2.

2. The chromatic number remains the same, regardless of the number of lines. **Note:** This can be proved using mathematical induction. You may wish to encourage students to think about the problem with this in mind. Sample jusification: If there is only one line, label the regions on the two sides of the lines with different colors.

If an additional line is added to the drawing, label all sections of the map on one side of the additional line with the color currently in use. To complete the coloring, change every region on the other side of the additional line to the opposite color.

This same scheme applies for a map colored with two colors and consisting of n lines. When the nth $+ 1$ line is added, retain the color already in use for all regions on one side of the new line and reverse the colors on the other side of the new line.

b. Answers will vary. Some students may agree that four colors are sufficient to color any map of the earth because none of the maps encountered so far have required more than four. Others may feel that they do not have enough information at this point to agree. **Note:** The four-color theorem requires that all parts of the same region be contained in one continuous border. If this stipulation is not met, a map that requires five colors can be created, as shown below.



Teacher Note

To complete Problem **1.2**, students will require maps of South America and Australia. Blackline masters appear at the end of the teacher edition for this module. Each student will need a length of string to complete Problem **1.7**.

Assignment (page 5)

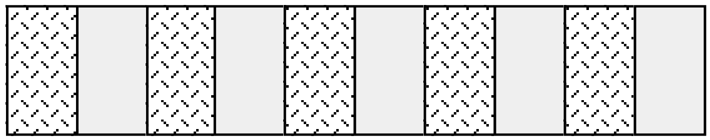
1.1 Sample response: A checkerboard has a chromatic number of 2 because it is an example of a straight-line map created on a flat surface.

***1.2** **a.** The chromatic number of a map of South America is 4. Sample response:

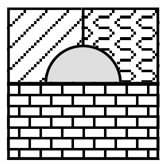
b. The chromatic number of a map of Australia is 3. Sample response:

1.3 **a.** If the chromatic number of a map is 2, then the map must have a minimum of two regions, by the definition of chromatic number. (If a map had only one region, then it would only need one color.)

b. There is no maximum number of regions for a map with a chromatic number of 2. In the following map, for example, the alternating pattern of regions could continue indefinitely since there is no limit to the number of vertical lines that can be drawn to represent boundaries.



1.4 **a.** The minimum number of regions for a map with a chromatic number of 4 is 4. Sample map:



b. There is no maximum number of regions for a map with a chromatic number of 4.

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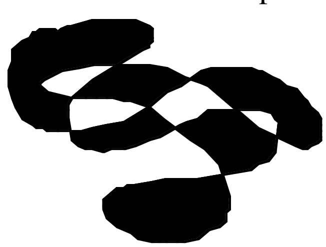
1.5 On some maps, each region is bounded by a simple closed curve. However, if nations or states that consist of several islands (such as Hawaii) are considered single regions, then this is not true. Although the boundaries of each of Hawaii's islands could be considered a simple closed curve, the boundary of the state as a whole is not.

Note: For the remainder of this module, students will only consider maps in which the boundaries of regions are simple closed curves.

1.6 **a.** Sample response: There would be no reason to use more than 14 colors for a map with 14 regions. With 14 colors, each region could have its own color.

b. As noted in Part **a** of the discussion in this activity, the minimum number of colors required for any straight-line map is 2.

1.7 The chromatic number for such a map is 2. Sample response:



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Activity 2

In this activity, students investigate a second method for determining a map's chromatic number.

Materials List

- map of South America (one per student; a blackline master appears at the end of the teacher edition for this module)
- map of local area (one per student)
- colored pencils (one set of at least five different colors per group)

Exploration

(page 7)

a. Sample drawing:

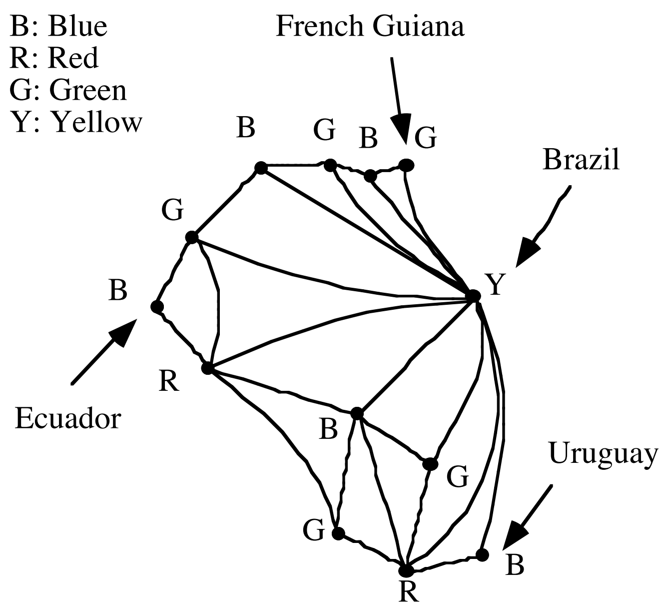
b. **1.** Students should discover that although their graphs may differ slightly in appearance, they are essentially the same.

2. Two graphs are equivalent if there is a one-to-one correspondence between their vertices and their edges so that corresponding edges connect corresponding vertices.

c. The chromatic number of the graph of South America is 4. One possible method for coloring the graph uses a greedy algorithm, as follows: Start with a color and assign it to every vertex for which it is available. Repeat this process for a second color, a third, and so on, until all vertices are colored. (Students may recall greedy algorithms from the Level 1 module "Going in Circuits.")

Note: For very large graphs, this algorithm may not determine the minimum number of colors. In fact, even the most efficient computer algorithm for coloring large graphs may not give the minimum number of colors.

Sample graph:



Discussion

(page 8)

a. The vertex that represents Brazil has the greatest degree (10). The vertices represent French Guiana, Ecuador, and Uruguay, respectively. Each have degree 2. (See sample graph given in Part **c** of the exploration.)

b. If students drew their graphs correctly, all the graphs should be equivalent. Sample response: Yes, the graphs are equivalent because they are graphs of the same map. A one-to-one correspondence exists between vertices (countries) and edges (shared boundaries), and countries sharing boundaries correspond to vertices sharing edges.

c. Sample response: When depicting maps using graphs, it does not make sense to use loops at vertices. A loop would signify that a region shares a border with itself.

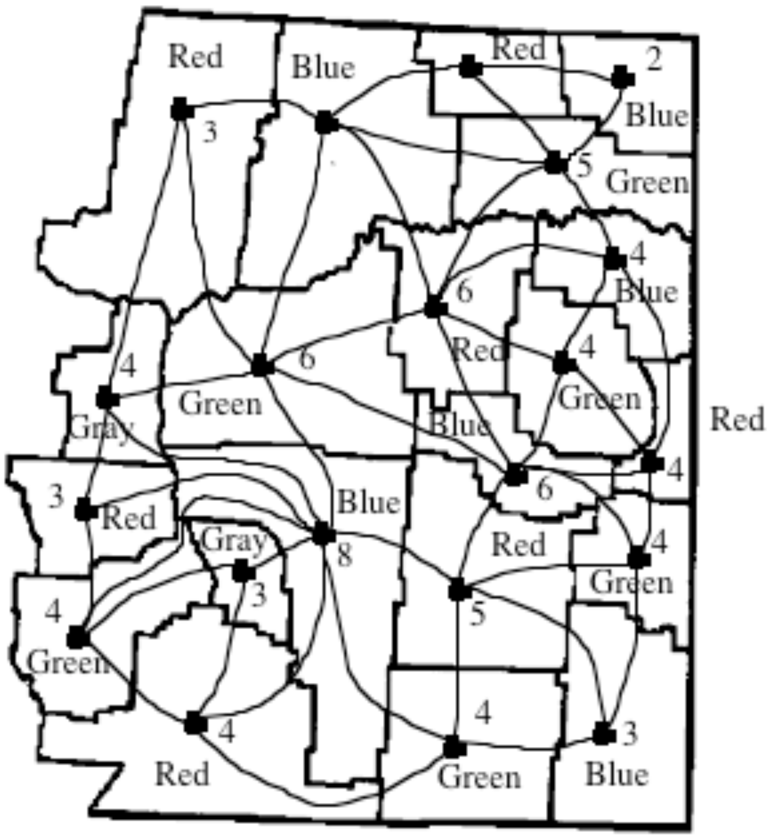
Teacher Note

Problem **2.1** instructs students to obtain a map of their home state or local area. As an alternative, you may wish to use the map of counties in eastern Montana which appears at the end of this teacher edition.

Assignment

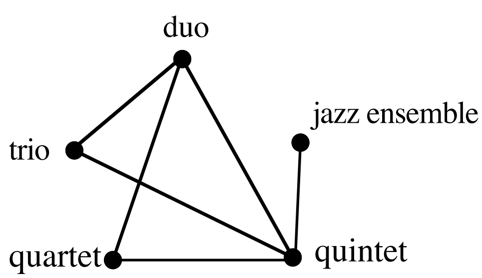
(page 8)

2.1 **a–b.** The following sample response is based on the map of counties in eastern Montana. The number at each vertex represents its degree.

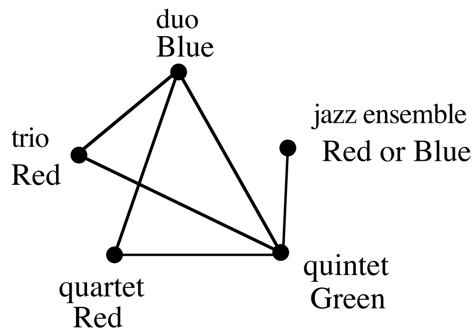


c. The chromatic number of this map is 4.

***2.2** **a**. Sample graph:



b. The chromatic number of the graph is 3. Because the duo, trio, and quintet share instruments, at least three colors are required. In the sample response below, the vertex representing the jazz ensemble could be colored red or blue, since it is connected only to a green vertex.

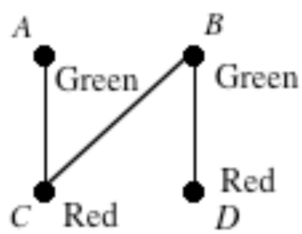


c. Sample response: In the graph in Part **b**, vertices connected by an edge represent groups that cannot practice at the same time without a conflict. At least three colors are required for this graph. Because no two vertices connected by an edge share the same color, the groups can practice without conflict using three different practice times. If only two colors (indicating two practice times) were used, there would be at least one conflict. Therefore the minimum number of practice times is 3.

d. Sample schedule:

Time	**Group**
3:00–3:30 (Green)	quintet
3:30–4:00 (Blue)	duo, jazz ensemble
4:00–4:30 (Red)	trio, quartet

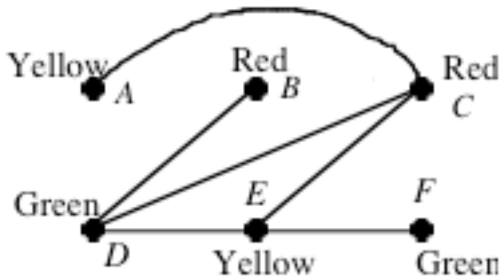
2.3 **a.** Sample graph (with colors indicated):



b. Sample response: Two settings are needed to prevent collisions since the chromatic number of the graph is 2.

c. Sample response: Cars can move without colliding in the directions represented by vertices that have the same color. This means that cars can move in directions *A* and *B* at the same time, and in directions *C* and *D* at the same time.

***2.4** **a.** Sample graph (with colors indicated):



b. Sample response: Three traffic-light settings are needed because the chromatic number is 3. The graph shows that cars can move in directions *A* and *E* on the same green light, in directions *C* and *B* on another green light, and in directions *D* and *F* on yet another green light. To prevent collisions, cars should not move in directions *A* and *C* on the same light; in directions *B* and *D* on the same light; in directions *C*, *B*, *D*, and *E* on the same light; or in directions *E* and *F* on the same light.

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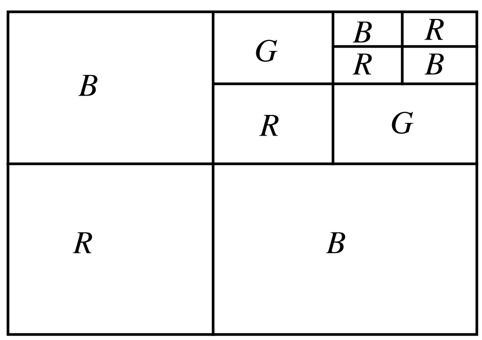
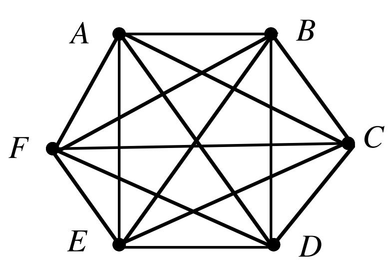
2.5 In the following sample response, *R* represents red, *G* represents green, and *B* represents blue.



2.6 Sample response: Let the six people be represented by vertices connected by edges, as shown below. If two people know each other, you can color the edge red; if they don't, you can color the edge blue.



Consider vertex *A*. This vertex, like every other vertex in the graph, has degree 5. Since each person either knows or does not know each other person, at least three of the five edges at *A* must be the same color. This means that *A* either knows three people—say *B*, *C*, and *D*—or there are three people A does not know.

Consider the case when A knows B, C, and D. This means there are three edges at A that are red: $\{A,B\}$, $\{A,C\}$, and $\{A,D\}$. If $\{B,C\}$ is red, then A, B, and C know each other.

If $\{B,C\}$ is blue, consider $\{B,D\}$. If $\{B,D\}$ is red, then A, B, and D know each other.

If $\{B,D\}$ is blue, consider $\{C,D\}$. If $\{C,D\}$ is red, then A, C, and D know each other.

If $\{C,D\}$ is blue, then $\{B,C\}$, $\{B,D\}$, and $\{C,D\}$ are all blue. This means that B, C, and D are complete strangers.

A similar argument can be used when A does not know B, C, or D by interchanging red and blue in the paragraphs above.

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(page 10)

Activity 3

Students investigate some connections between topology and coloring theory.

Materials List

- map of the United States (one per student; a blackline master appears at the end of the teacher edition for this module)

Technology

- geometry utility

Discussion 1

(page 11)

a. Sample response: The graph in Figure **5a** is not topologically equivalent to the one in Figure **6b** because there is not a one-to-one correspondence between all vertices and edges. For instance, the graph in Figure **6b** has more edges than the one in Figure **5a**. Some vertices in Figure **6b** have a degree of 3, while all the vertices in Figure **5a** have a degree of 2.

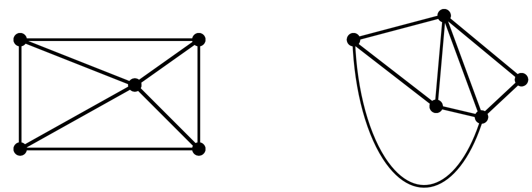
b. Sample response: To make the graph in Figure **5c** equivalent to the one in Figure **6a**, an edge could be drawn between W'' and Y'' and between X'' and Z'', and the edge between W'' and X'' erased.

Note: To make Figure **6a** equivalent to Figure **5c**, an edge could be drawn between A' and B', and the edges between A' and C' and B' and D' erased.

Exploration (page 11)

This exploration allows students to grapple with the concept of topologically equivalent graphs. Using a drawing utility, they distort graphs by stretching, twisting, shrinking, and pulling. **Note:** It may be difficult to create curved edges on some drawing utilities.

a. **1–2.** Students should manipulate their graphs until there are no overlapping edges. Any graph with five vertices and eight edges drawn according to the directions will be topologically equivalent to one of the following planar graphs:



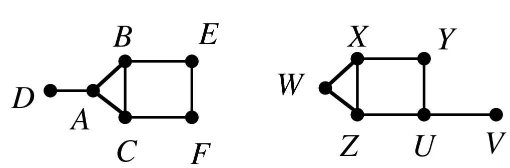
3–5. Students should find that the original and the distorted copy are topologically equivalent. The number of vertices and edges stay the same, and the vertices remain connected in the same manner.

b. Students construct four topologically equivalent graphs, modify them, then challenge other students to determine which of the four graphs are topologically equivalent. **Note:** There are several possible ways to organize the rest of this exploration. For example, students can design their graphs as a group, then challenge another group; or individuals within a group may challenge other group members.

Discussion 2 (page 12)

a. Some students may move the vertices and edges of one graph to show that it is planar, then use a similar process to test the other graphs. Other students may try to show a one-to-one correspondence between the vertices and edges of different graphs.

Note: Some students may consider the degree of each vertex, and compare degrees from graph to graph. Considering only the degrees of vertices is not sufficient to determine the topological equivalency of graphs. The figure below, for example, shows two graphs, each with six vertices and seven edges.



Graph 1 Graph 2

Each graph has three vertices of degree 3, two vertices of degree 2, and one vertex of degree 1. However, they are not topologically equivalent because the vertices are not connected in the same way.

For example, vertices *A*, *B*, and *C* in Graph 1 each have degree 3. For the two graphs to be topologically equivalent, vertex *A* in Graph 1 must be paired with either *X*, *Z*, or *U* in Graph 2. Because *A* is connected to *D* (with degree 1), *A* must be paired with *U* and *D* paired with *V*. Now *A* is connected with *B* and *C* in Graph 1, each of degree 3. *U* is connected to *Y* and *Z* in Graph 2, each of degree 2. Therefore, the graphs are not topologically equivalent.

b. Sample response: Adding or deleting edges means that the vertices will no longer be connected in the same way.

c. Sample response: Topological equivalency is hard to determine when graphs have many vertices with high degrees.

d. Graphs of the same map represent the same information. Therefore, there must be a one-to-one correspondence between vertices and edges. In both graphs, corresponding vertices represent the same countries and corresponding edges represent the same shared borders.

e. Sample response: The graph of any flat map must be planar. After marking the vertices in each country, the edges can be drawn so that they cross the shared borders. This guarantees that there are no overlapping edges. **Note:** This requires that no portion of one country be found within the borders of another country.

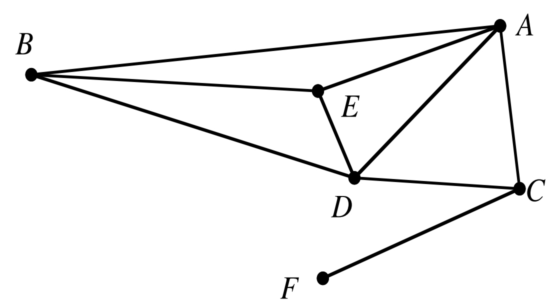
Teacher Note

Each student will require a map of the United States to complete Problems **3.3** and **3.5**. They also may request maps of South America and Australia. Blackline masters appear at the end of the teacher edition for this module.

Assignment

(page 12)

***3.1** Answers will vary. Check for a one-to-one correspondence between the vertices and edges of the two graphs. A topologically equivalent graph must have six vertices with edges connecting the vertices that correspond to *A* and *B*, *A* and *C*, *A* and *D*, *A* and *E*, *B* and *D*, *B* and *E*, *C* and *D*, *C* and *F*, and *D* and *E*. Sample graph:

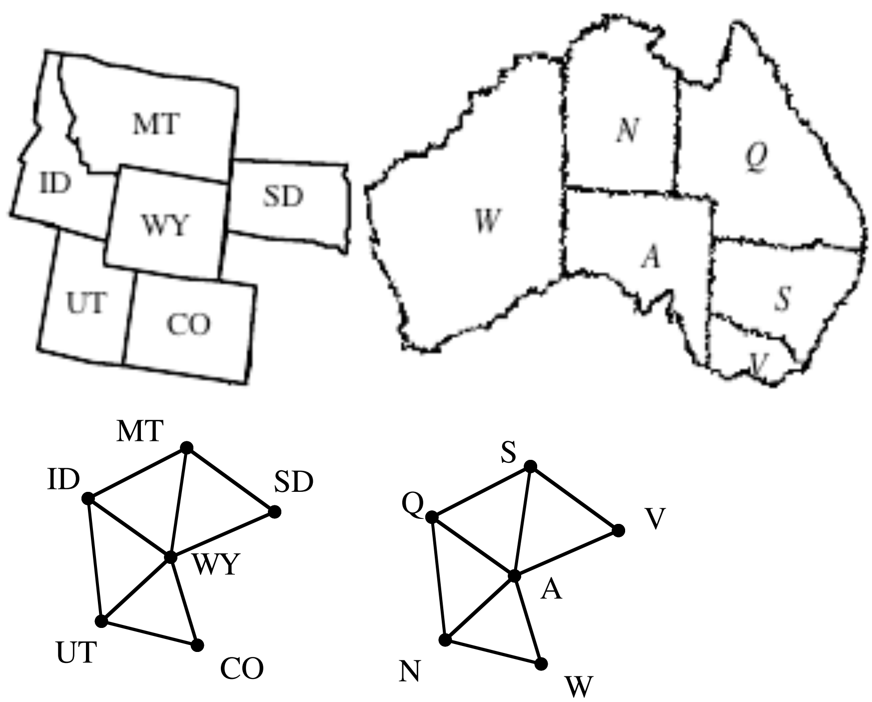


***3.2** Sample response: Even though the two graphs have the same number of vertices of each degree, they may not have a one-to-one correspondence between their vertices and edges. For example, the two graphs below both have the given characteristics, but they are not topologically equivalent because their vertices are not connected in the same way.

(See response to Part **a** of the previous discussion.)

3.3 **a.** The graph of any flat map must be planar. After marking the vertices in each country, the edges can be drawn so that they cross the shared borders. For this reason, the graphs of maps of South America, Australia, and the United States are all planar.

b. Answers will vary. Sample response: These graphs are topologically equivalent because there is a one-to-one correspondence among the vertices and corresponding edges connect corresponding vertices.



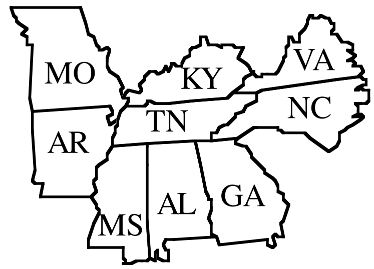
c. Sample response: No. Brazil borders 10 other countries. No state in the United States borders 10 other states.

* * * * *

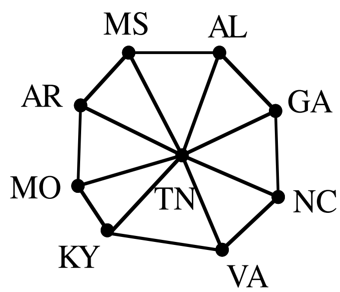
3.4 Answers will vary, depending on student choices. Sample response based on map given in Problem **3.3b**: A possible one-to-one correspondence is between the names of the states and the letters designating the vertices of the graph, as shown below.

Idaho × ID	Montana × MT	Wyoming × WY
Utah × UT	Colorado × CO	South Dakota × SD

3.5 **a.** Sample response:



b. The following sample response corresponds with the states identified in Part **a**.



3.6 Sample response: Because Alaska and Hawaii do not share a border with any of the other states, a graph of the map has two vertices that are not connected to any of the other vertices. In a graph of a map of South America, the degree of every vertex is at least 2.

* * * * * * * * * *

(page 13)

Activity 4

In this activity, students investigate complete graphs, the chromatic number of maps drawn on spherical surfaces, and the four-color theorem. **Note:** Kenneth Appel and Wolfgang Haken published a proof of the four-color theorem in 1976. The theorem was first believed to have been proven by A. B. Kempe in 1879. In 1890, however, P. J. Heawood showed a fallacy in Kempe's proof. In the same paper, Heawood suggested an equation for the number of colors required by maps on more complex surfaces. (See research project at the end of this activity.)

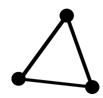
Materials List

- balloons (one per group)
- non-permanent markers (one set of at least five different colors per group)
- string approximately 12 inches long (5 pieces per group)

Exploration 1

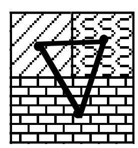
(page 13)

a. **1.** Sample graph:



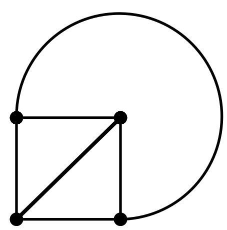
2–3. Because there are no overlapping edges (except at the vertices), any complete graph with three vertices is planar.

4. Student maps will vary. Sample response:



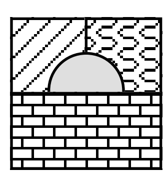
5. The chromatic number of both the map and the graph is 3.

b. **1.** Sample complete graph with four vertices:



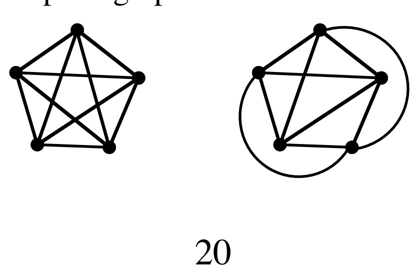
2–3. Because a complete graph with four vertices is topologically equivalent to a graph drawn on a plane with no overlapping edges, it is always planar.

4. Sample map:



5. The chromatic number of both the map and the graph is 4.

c. **1.** Sample complete graphs with five vertices:

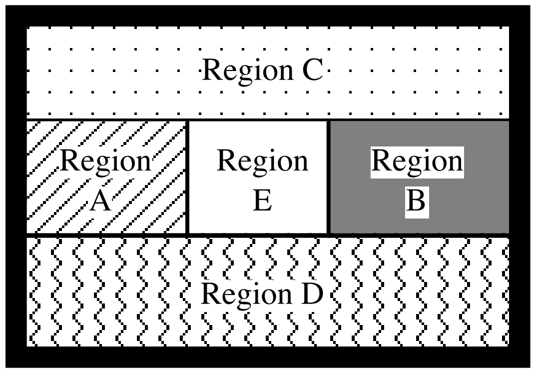


2–3. Though mathematicians have not proven that complete graphs with five vertices are not planar, such graphs cannot be drawn in a plane without overlapping edges.

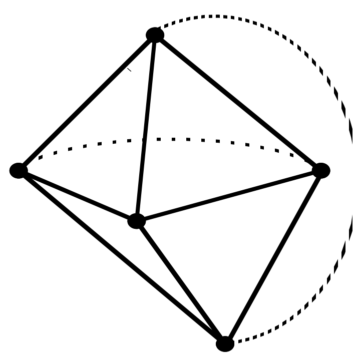
4. No, the complete graph is not planar.

5. The chromatic number for the graph is 5. There must be a different color for each vertex because each pair is connected. There is no planar map.

d. Because it takes a minimum of five colors to color this map, its chromatic number is 5.



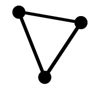
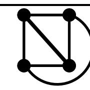
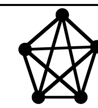
e. Answers may vary. One edge must extend from the vertex that represents region C (at the top of the screen) to the vertex that represents region D (at the bottom of the screen). Another edge must connect the vertices that represent regions A and B. In the following sample, dotted lines indicate the edges that wrap around.



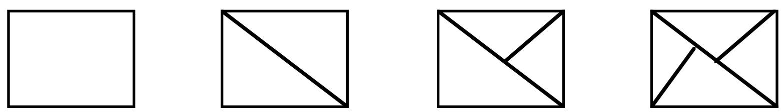
f. Since the term *planar* may be confused as meaning "on a plane," the more general expression "can be drawn without overlapping edges" is used in Table **1**. Graphs drawn on surfaces other than flat ones are still considered planar when they are topologically equivalent to graphs drawn on a flat surface without overlapping edges.

Because it is not possible to draw the corresponding five-region map for a complete graph with five vertices and a chromatic number of 5 on a flat surface, students should begin to believe that a chromatic number of 4 is sufficient for any map drawn on a flat surface.

A sample table is shown below.

No. of Vertices	1	2	3	4	5
Sample Graph					
Chromatic Number	1	2	3	4	5
Possible Surfaces	any surface	any surface	any surface	any surface	computer screen

g. Each sample map below is labeled with the number of vertices required to draw its corresponding planar graph. It is not possible to draw a flat map with five regions that has a complete planar graph.



1 region, 1 vertex — 2 regions, 2 vertices — 3 regions, 3 vertices — 4 regions, 4 vertices

Discussion 1

(page 15)

a. Sample response: It was not possible to draw the corresponding map for a complete graph with five vertices on a flat surface.

b. Sample response: Every region borders every other region.

c. If a map can be drawn on a flat surface, then so can its graph. Therefore, the graph of any flat map is always planar.

d. Any complete graph is always planar if it has less than five vertices.

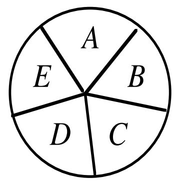
e. Sample response: A complete graph with five vertices is not planar, and thus cannot be modeled on a flat surface. It can be modeled on a computer screen because the computer can permit "wrap-around" borders.

f. The number of vertices in a complete graph and the chromatic number of the corresponding map are equal. Since each vertex is connected to every other vertex, each vertex needs its own color.

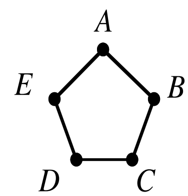
g. After working through Exploration **1**, students may realize that complete graphs with more than four vertices cannot be planar. (This is part of the proof of the four-color theorem. If one could construct a complete planar graph with five vertices, then some flat maps would require five colors. See the following teacher note.)

Teacher Note

When discussing the proof of the four-color theorem, it is critical to note that the following statement is false: "If a map has chromatic number equal to *n*, then the graph has a complete subgraph with *n* vertices." For example, the "pie map" in the diagram below has a chromatic number of 3, yet its graph does not have a complete subgraph with three vertices. (Students investigate this counterexample in Problem **4.4**.)



Pie Map



Graph of Pie Map

As a result, even though mathematicians were able to show that no planar graph with five vertices was complete—and thus required five colors—they still had to consider all possible incomplete planar graphs with five or more vertices. Considering all these possibilities required the help of computers.

Since 1976, many other theorems have been proved or verified using computers. However, because the four-color theorem is so simple—and its proof so long, complex, and computerized—some controversy remains.

Exploration 2 (page 15)

a. Students will probably predict a chromatic number of 4 or more.

b. The chromatic number for a complete graph with four vertices drawn on a sphere is 4.

c. It is not possible to draw a complete graph with five vertices and no overlapping edges on a spherical surface.

Discussion 2 (page 16)

a. Sample responses may include the surface of a football, a cube, or a banana.

b. Sample response: It does not seem possible to draw a complete graph with five vertices and no overlapping edges on a balloon. Every attempt resulted in some edges that intersected.

c. A chromatic number of 4 is sufficient for any map drawn on a flat surface or a spherical surface.

d. All graphs of maps drawn on a sphere are planar. The surface of a sphere can be drastically changed if a single point is removed from it. The chromatic number of the map on the sphere doesn't change by removing this single point. However, the resulting surface—a punctured sphere—can be stretched to form a flat surface. A punctured sphere is topologically equivalent to a plane, and has the same chromatic number as the plane. Hence a sphere, though not topologically equivalent to a plane, has a chromatic number of 4.

e. Sample response: Because it is not possible to draw a complete graph with five vertices and no intersecting edges on a sphere, it does not model the computer game screen.

f. A chromatic number of 4 is sufficient for any map drawn on a surface that is topologically equivalent to a sphere. This is because objects that are topologically equivalent have the same chromatic number.

Assignment

(page 16)

***4.1** **a.** The graph is not planar because it is not topologically equivalent to a graph without overlapping edges.

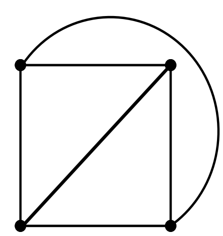
b. The vertices *A*, *B*, *C*, and *D*, along with the edges that connect them, represent one example of a complete subgraph with four vertices.

c. **1.** The vertices *A*, *B*, *C*, *D*, and *E*, along with the edges that connect them, represent a complete subgraph with five vertices.

2. Sample response: No. The edges cannot be drawn without intersections.

3. In Exploration **1**, students discovered that a complete graph with five vertices can never be planar. If a subgraph is not planar, then the original graph cannot be planar.

4.2 **a.** Answers will vary. Sample graph:

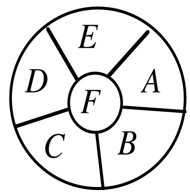


b. The degree of each vertex is 3.

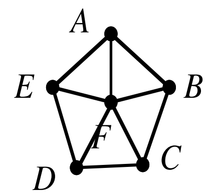
c. The degree of each vertex in a complete graph with n vertices is $n - 1$.

***4.3** Because both objects are topologically equivalent to a sphere, the chromatic number for both of them is 4. Therefore, the maximum number of vertices for any complete graph with no overlapping edges drawn on these surfaces is also 4.

4.4 Sample response:



Pie Map



Graph of Pie Map

* * * * *

4.5 Except for triangles and quadrilaterals, polygons and their diagonals cannot be represented by planar graphs. Graphs of such polygons cannot be drawn without overlapping edges.

4.6 Sample response: A planar graph is one that can be drawn without overlapping edges. A planar figure is any geometric figure that can be drawn in a plane. Sample response: a hexagon with all its diagonals is a planar figure, but it could not represent a planar graph. **Note:** Any planar graph is a planar figure.

* * * * * * * * * * *

Research Project (page 17)

Although four colors are sufficient for any map on a plane or sphere, a map on a doughnut-shaped surface—or torus—can require more colors. Since maps can wrap around and through the hole of a torus, mapped regions can share more boundaries. A chromatic number of 7 is sufficient for any map drawn on a torus. This is true in part because the maximum number of vertices for any complete graph without intersecting edges on a torus is 7. Any complete graph with seven vertices requires seven colors, since each vertex is connected to every other vertex.

In 1890, P. J. Heawood hypothesized that the following equation would give the chromatic number H_g for maps drawn on a surface with g holes.

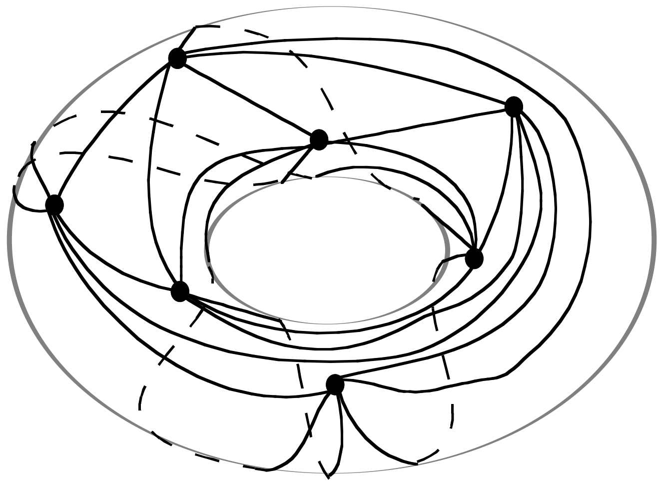
$$H_g = \left[\frac{1}{2}\left(7 + \sqrt{1 + 48g}\right)\right]$$

(The brackets in the equation indicate the greatest integer function.) Since a torus has one hole, Heawood's equation predicts that a chromatic number of 7 is sufficient for any map drawn on a torus:

$$H_g = \left[\frac{1}{2}\left(7 + \sqrt{1 + 48(1)}\right)\right] = 7$$

In 1952, G. A. Dirac proved that this equation worked for some surfaces, but the case of the torus was not proven until 1954 by Gerhard Ringel.

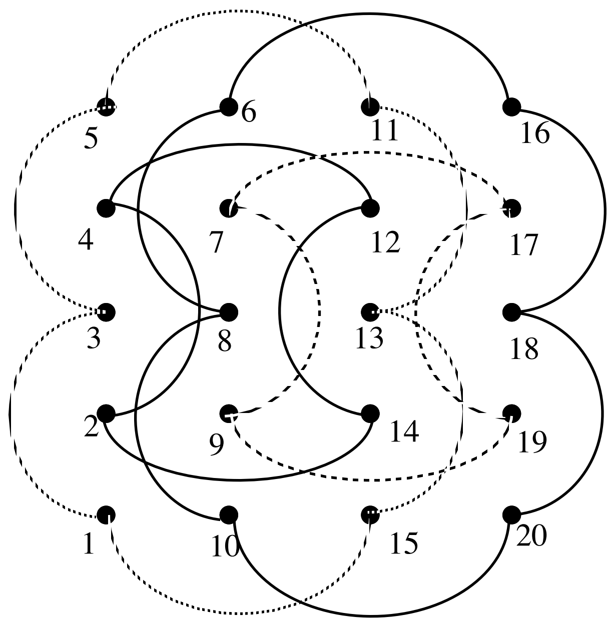
The diagram below shows a complete graph with seven vertices drawn on a torus:



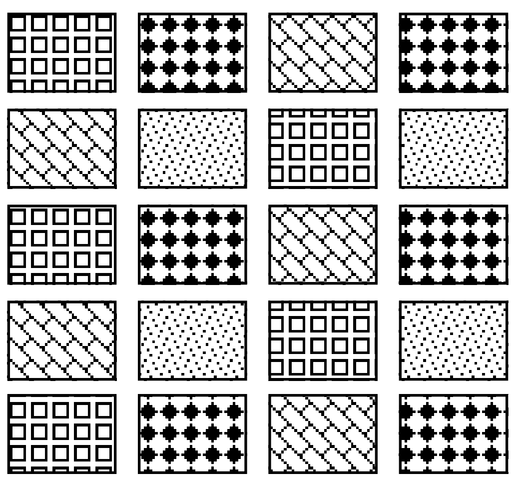
Answers to Summary Assessment (page 18)

1. **a.** Sample response: The teacher needs four different versions of the test. A chromatic number of 4 is sufficient for any map drawn on a flat surface, and the chromatic number corresponds to the number of different tests needed.

b. Sample response: Each vertex in the graph below represents a desk. Edges connect vertices that can use the same version of the test.



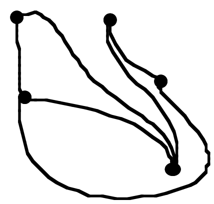
The corresponding arrangement of the four versions of the test can be seen in the following four-pattern map:



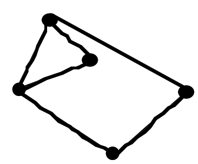
Module Assessment

1. Which of the following graphs are topologically equivalent? Justify your responses.

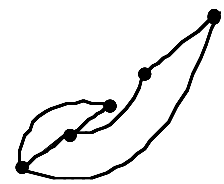
a.



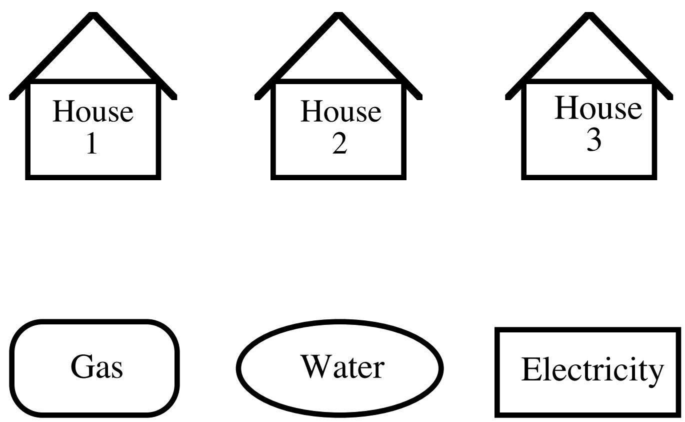
b.



c.



2. The diagram below shows the locations of three houses and their sources of gas, water, and electricity. Each house must be directly connected to each utility.

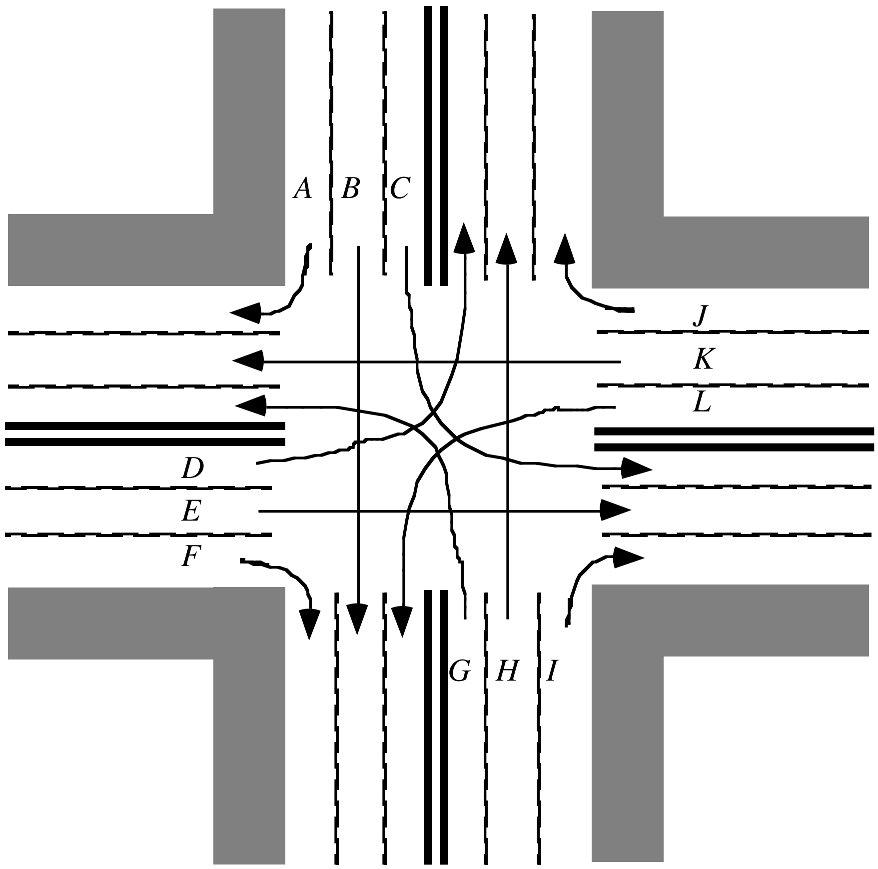


a. Draw a graph that illustrates these connections.

b. Is your graph planar? Explain your response.

c. Is your graph complete? Explain your response.

3. The following diagram represents a busy traffic intersection. To minimize the risk of collisions, how many traffic-light settings should this intersection have? Justify your response with a graph.



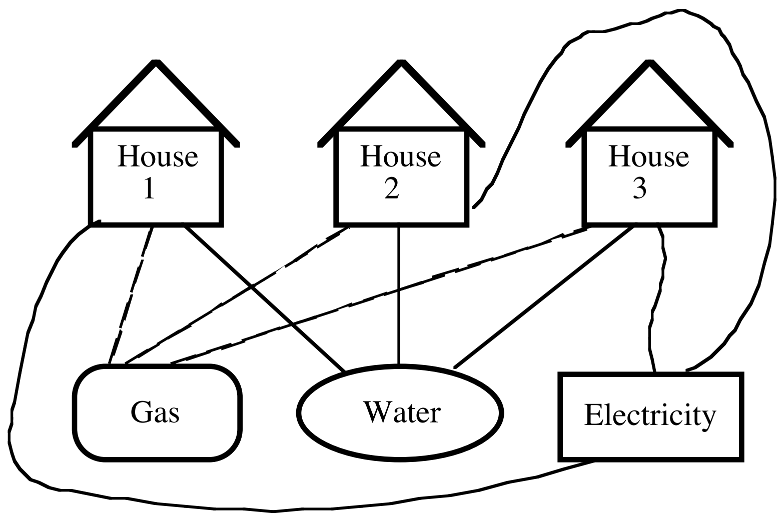
4. The table below shows the class lists for eight classes at Exponent High School. (Exponent High is a very small school.) Use coloring theory to design a testing schedule that has the fewest test periods—with no student scheduled for two tests at the same time. Describe the process you used to determine your response.

Class A	**Class B**	**Class C**	**Class D**	**Class E**	**Class F**	**Class G**	**Class H**
Akihide	Ebdul	Giselle	Ingmar	Charlotte	Leota	Dena	Akihide
Dena	Giselle	Ingmar	Jim	Jim	Milo	Nadia	Lia
Xiao	Wyatt	Tristan	Kami	Leota	Nam	Stormie	Nadia
Zhen	Ying	Xiao	Singrid	Nadia	Shen	Wyatt	Thad

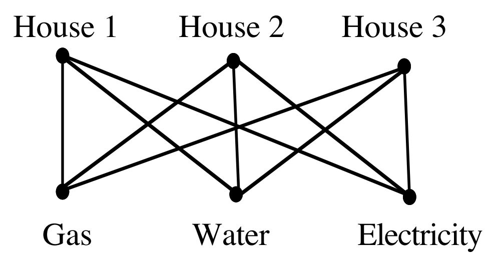
Answers to Module Assessment

1. All three graphs are topologically equivalent because there is a one-to-one correspondence between their vertices and between their edges and corresponding vertices are connected by corresponding edges.

2. **a.** Sample map with graph:



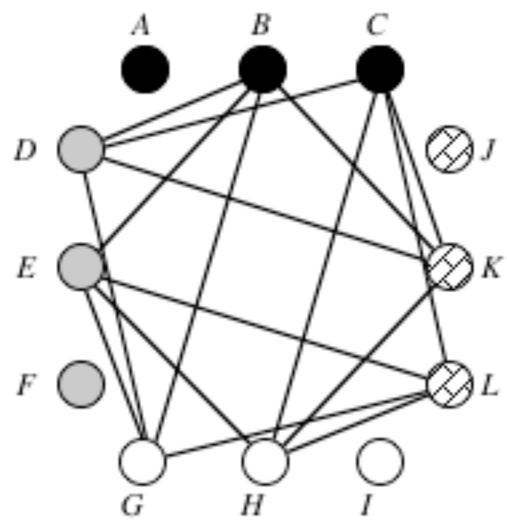
Sample graph using vertices instead of figures:



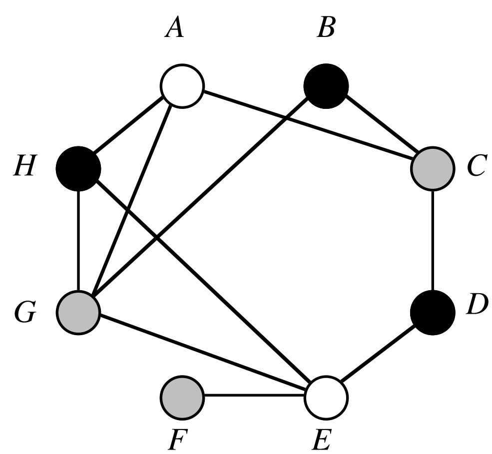
b. Sample response: The graph is not planar because it is not possible to draw the edges without overlapping.

c. The graph is not complete because each vertex is not connected to every other vertex. In this situation, the utilities are not connected to each other and neither are the houses.

3. The minimum number of traffic-light settings required at this intersection is 4. Student graphs will vary. Sample graph:



4. The fewest number of class periods required is 3. A graph of the class information is shown below. Each vertex represents a class, and edges connect the classes that share at least one student. The chromatic number of the graph corresponds to the number of test periods required to avoid conflicts.



The following table shows one possible test schedule with three periods:

Period	**Class**
1	A, E
2	C, F, G
3	B, D, H

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Flashbacks

Activity 1

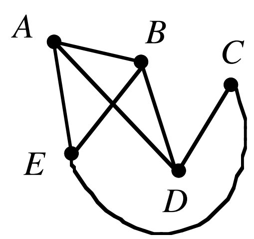
1.1 Draw an example of a simple curve and explain why it is simple.

1.2 Draw an example of a closed curve and explain why it is closed.

1.3 Draw an example of a curve that is neither simple nor closed.

Activity 2

2.1 Consider the graph shown below.



a. How many vertices are there in this graph? Explain your response.

b. How many edges are there in this graph?

Activity 3

3.1 Define a one-to-one correspondence between two sets.

3.2 Describe a one-to-one correspondence between sets A and B below.

$$A = \{a,\ b,\ c\}$$
$$B = \{1,\ 2,\ 3\}$$

3.3 Describe an example of a linear function that does not represent a one-to-one correspondence.

Activity 4

4.1 Consider the vertices of each polygon below. How many segments are required to connect every vertex to every other vertex?

a. a triangle

b. a quadrilateral

c. a pentagon

d. a hexagon

e. a 30-gon

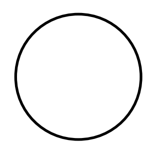
4.2 How many diagonals does a cube have?

Answers to Flashbacks

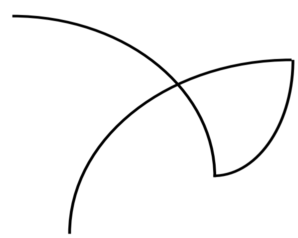
Activity 1

1.1 Sample response: A simple curve does not cross itself, although it may start and stop at the same point. It may be thought of as a curve that can be drawn without lifting a pencil. An example is shown below:

1.2 Sample response: A closed curve must start and stop at the same point. A circle is one example. It may or may not cross itself.



1.3 Sample response:



Activity 2

2.1

a. Sample response: This graph has five vertices: *A*, *B*, *C*, *D*, and *E*. The edge connecting vertices *A* and *D* does not appear to intersect the edge connecting vertices *B* and *E*, since it is not named.

b. This graph has seven edges.

Activity 3

3.1 A one-to-one correspondence exists between two sets when there is a function defined between the two sets such that each element of the second set, or range, is paired with exactly one element of the first set, or domain, and each element of the range is paired with some element of the domain.

3.2 One possible one-to-one correspondence is shown below.

$$\begin{array}{rl} A = & \{a, b, c\} \\ & \;\;\downarrow\;\; \downarrow\;\; \downarrow \\ B = & \{1, 2, 3\} \end{array}$$

3.3 Any linear function that represents a horizontal line is not a one-to-one correspondence.

Activity 4

4. **a.** 3

b. 6

c. 10

d. 15

e. 435

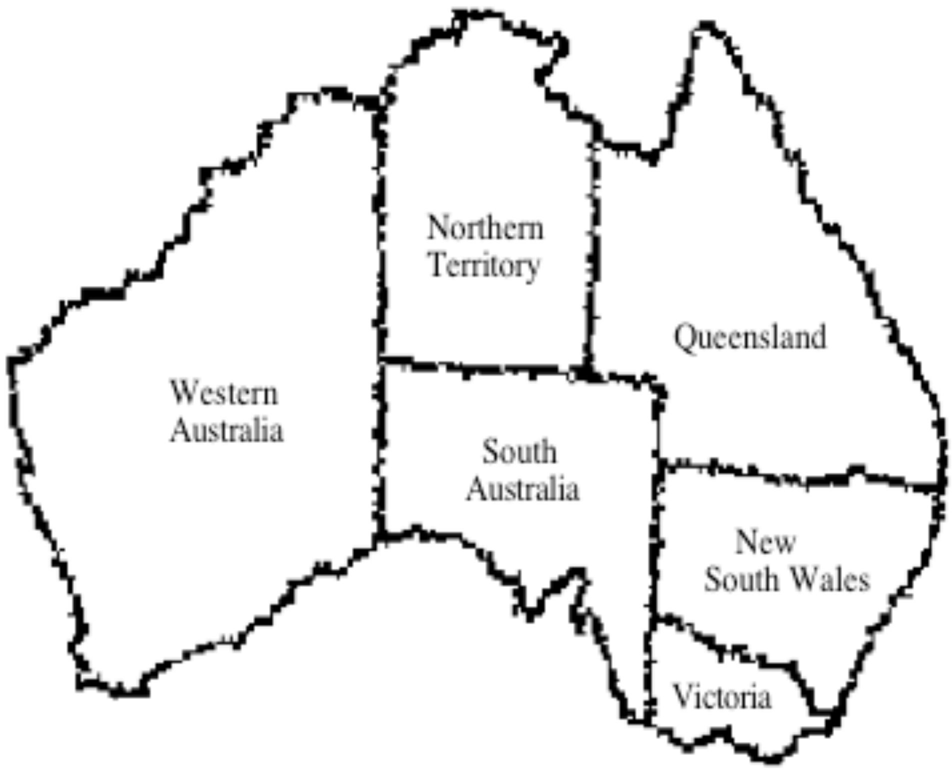
The formula for the number of segments, including sides and diagonals, in an *n*-gon is:

$$\frac{n(n-1)}{2}$$

4.2 A cube has 4 diagonals.

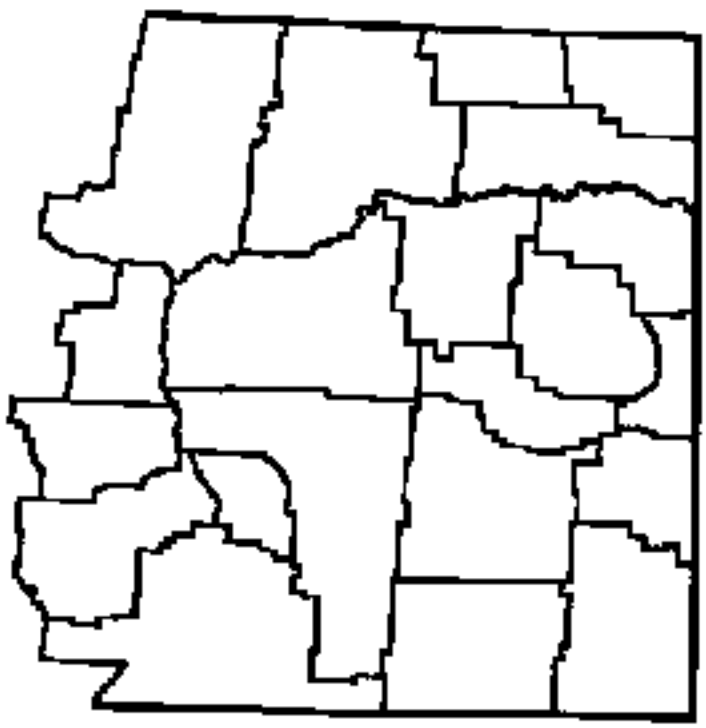
Map of Western United States

Map of Australia



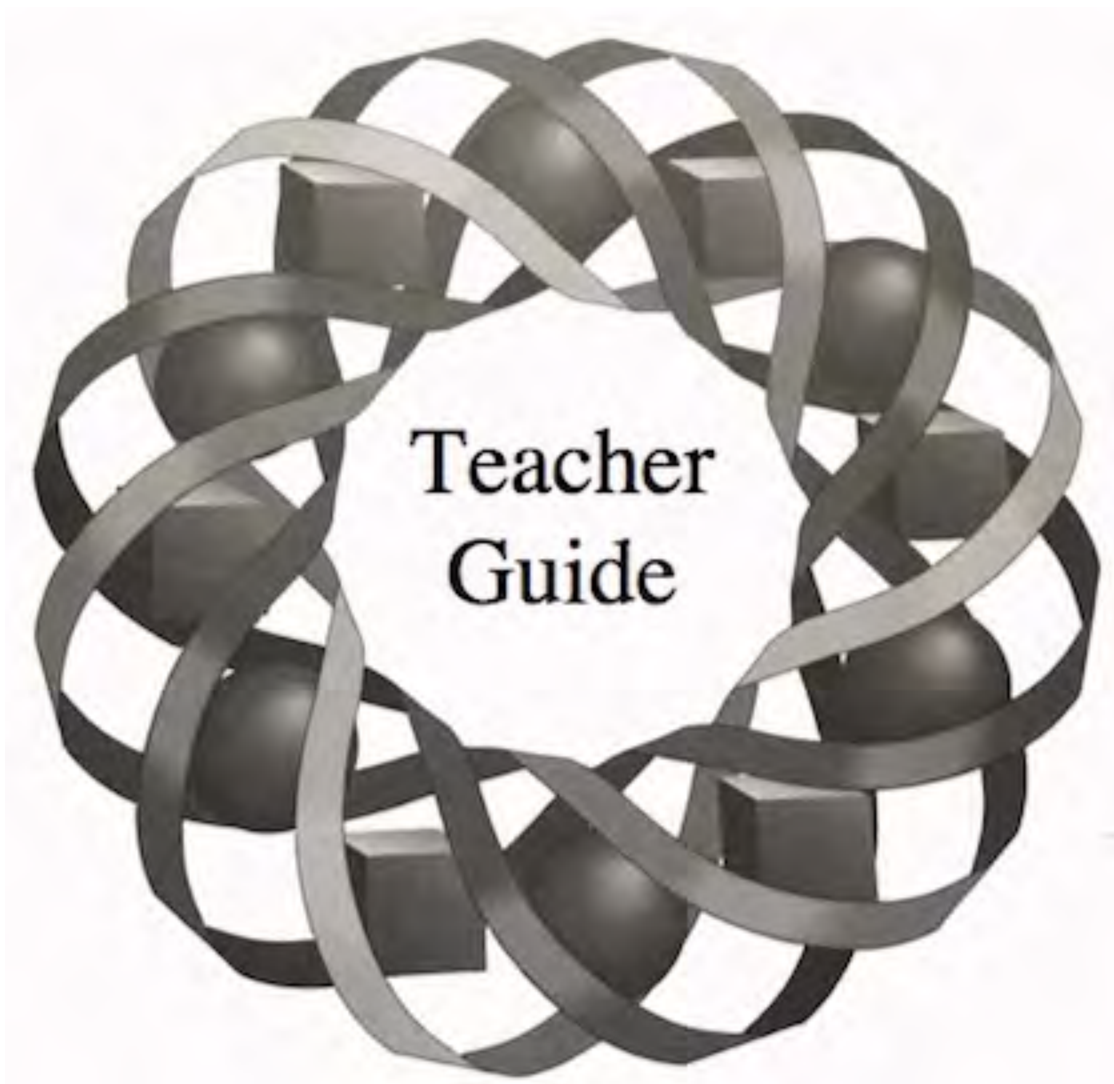
Map of South America

Map of Eastern Montana Counties



Map of Continental United States

Can It!



How can you model such diverse phenomena as the height of tides in the ocean, the flow of electricity in a circuit, and the hours of daylight in a year? In this module, you explore functions that can model events which occur over and over again.

Margaret Plouvier • Todd Robins • Teri Willard

Teacher Edition

Can It!

Overview

This module develops the circular functions using trigonometry on the unit circle. Students are introduced to radian measure and examine the relationship between radians and degrees. They investigate the graphs of circular functions, identify period and amplitude from graphs and equations, and use periodic functions to model real-world events.

Objectives

In this module, students will:

- identify circular functions by the shapes of their graphs
- identify the amplitude and period of circular functions
- write the equations of sine or cosine curves from graphs
- use sine or cosine functions to model real-world data
- determine transformations of the graphs of circular functions
- identify the inverse functions for sine, cosine, and tangent
- use inverse trigonometric functions to solve trigonometric equations.

Prerequisites

For this module, students should know:

- the definitions of the trigonometric ratios sine, cosine, and tangent
- how to calculate the circumference of a circle
- the definition of a central angle
- the definition of a function
- how to identify reflections, rotations, and translations.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	3	2	1	10

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
large can	X				
graph paper	X				
scissors	X	X			
ruler	X				
protractor	X				
strips of paper or ribbon	X	X			
glue		X			
freezer paper		X			

Technology

Software	Activity				
	1	2	3	4	Summary Assessment
spreadsheet	X			X	
graphing utility	X	X	X	X	
geometry utility		X			
symbolic manipulator				X	

Can It!

Introduction
(page 23)

After students are introduced to circular functions in Activities **1–3**, they will use them to model several types of cyclic events, including uniform circular motion, ocean tides, and alternating current.

(page 23)

Activity 1

Students are introduced to unit circles, radian measure for angles, and the relationships among angles measured in radians, real numbers, arc lengths, and points on a unit circle.

Materials List

- ruler (one per group)
- protractor (one per group)
- large empty cans of varying diameter (one per group)
- narrow strips of paper or ribbon (one long strip per group)
- scissors (one pair per group)
- graph paper (two sheets per group)

Exploration
(page 23)

This exploration introduces students to radian measure using a can-wrapping activity. The relationship between a radian and the radius of a unit circle is emphasized as a precursor to the exploration of trigonometric functions.

a–b. Using the radius of a can to define 1 unit, students create a unit circle and a number line.

c. Students identify and mark the locations of various real numbers from the number line as the line is wrapped on the unit circle. **Note:** Values may differ from those expected due to the difference in the radius of the can and its rim

d. Students approximate the ordered pair that corresponds to each labeled point on the unit circle. In this mapping process, students should realize that a real number is paired with an ordered pair of real numbers. For example, $\pi/2$ is paired with the ordered pair (0,1). See table given in Part **f** below.

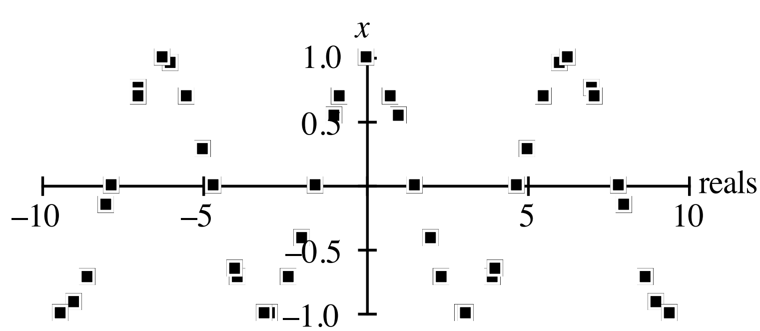
e. Students construct a central angle that measures 1 radian.

f. Students repeat Parts **c** and **d** using the negative values on the number line. Sample table:

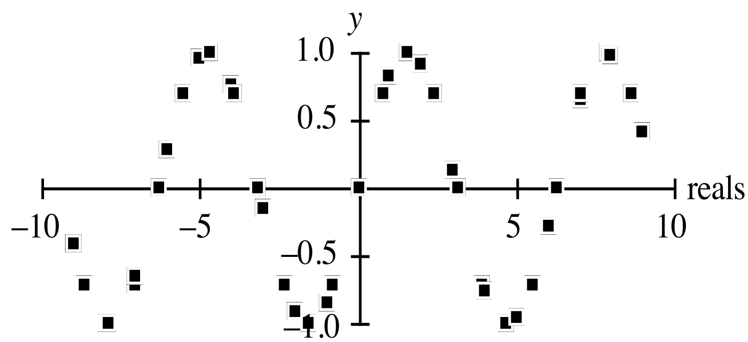
Real No.	***x***	***y***	**Real No.**	***x***	***y***
0	1.0	0.0	0	1.0	0.0
$\pi/4$	0.7	0.7	$-\pi/4$	0.7	–0.7
1	0.5	0.8	–1	0.5	–0.8
$\pi/2$	0.0	1.0	$-\pi/2$	0.0	–1.0
2	–0.4	0.9	–2	–0.4	–0.9
$3\pi/4$	–0.7	0.7	$-3\pi/4$	–0.7	0.7
3	–1.0	0.1	–3	–1.0	–0.1
π	–1.0	0.0	$-\pi$	–1.0	0.0
$5\pi/4$	–0.7	–0.7	$-5\pi/4$	–0.7	0.7
4	–0.7	–0.8	–4	–0.7	0.8
$3\pi/2$	0.0	–1.0	$-3\pi/2$	0.0	1.0
5	0.3	–1.0	–5	0.3	1.0
$7\pi/4$	0.7	–0.7	$-7\pi/4$	0.7	0.7
6	1.0	–0.3	–6	1.0	0.2
2π	1.0	0.0	-2π	1.0	0.0
7	0.8	0.7	–7	0.8	–0.7
$9\pi/4$	0.7	0.7	$-9\pi/4$	0.7	–0.7
$5\pi/2$	0.0	1.0	$-5\pi/2$	0.0	–1.0
8	–0.1	1.0	–8	–0.1	–1.0
$11\pi/4$	–0.7	0.7	$-11\pi/4$	–0.7	–0.7
9	–0.9	0.4	–9	–0.9	–0.4
3π	–1.0	0.0	-3π	–1.0	0.0

g. Sample scatterplots:

***x*-coordinates vs. real numbers**



y-coordinates vs. real numbers



Discussion

(page 25)

a. Sample response: If the number line were extended indefinitely, an infinite number of points on the number line would be paired with each point on the circle. This is because the number line continues to repeatedly wrap around the unit circle.

b.
1. The domain of the wrapping function is the set of real numbers.
2. The range is the set of ordered pairs that describe the location of the points on the unit circle. **Note:** Later in this module, students discover that these ordered pairs can be expressed in the form $(\cos\theta, \sin\theta)$, where θ is the measure of a central angle in a unit circle.

c. Sample response: The values measured on the vertical axes repeat over and over in the scatterplots. This indicates that the graphs are cyclic.

d.
1. The greatest value is 1 and the least value is –1.
2. The greatest value is 1 and the least value is –1.

e. The ratio is equal to 1.

f.
1. 2π radians
2. The circumference of any circle is $2\pi r$, where r is the radius of the circle. The number of radians in the circumference is the ratio of the arc length to the radius or $2\pi r/r$. The result is always 2π radians regardless of the size of the circle.

g. Since $360° = 2\pi$ radians, the following proportion shows the relationship between radian measure and degree measure:

$$\frac{2\pi}{360} = \frac{\text{radians}}{\text{degrees}}$$

Assignment (page 26)

1.1 **a.** Sample response: The point (0,1) corresponds to $\pi/2$ when wrapping counterclockwise from the *x*-axis.

b. **1.** When measured counterclockwise from the point (1,0), the central angle measures 90°.

2. When measured counterclockwise from the point (1,0), the central angle measures $\pi/2$ radians.

1.2 Answers will vary. Sample response: Since there are 2π radians in one complete revolution around the circle, the real number $2\pi + 3$ is also paired with this point.

1.3 Answers will vary. Sample responses are given below.

a. 2π radians, -2π radians

b. π radians, $-\pi$ radians

c. $3\pi/2$ radians, $-\pi/2$ radians

d. $8\pi/3$ radians, $-4\pi/3$ radians

1.4 **a.** They have both rotated through 6π radians.

b. The child 1 m from the center has traveled a distance of $6\pi \approx 19$ m. The child 1.5 m from the center has traveled $6\pi(1.5) = 9\pi \approx 28$ m.

***1.5** **a.** 2π sec

b. π sec

1.6 **a.** Using dimensional analysis:

$$\frac{2\pi(1.496 \bullet 10^8 \text{ km})}{1 \text{ year}} \bullet \frac{1 \text{ year}}{365 \text{ days}} \bullet \frac{1 \text{ day}}{24 \text{ hr}} \approx \frac{107{,}300 \text{ km}}{1 \text{ hr}}$$

b. Since there are 91 days between April 1 and July 1, the distance traveled is:

$$2\pi \bullet (1.496 \bullet 10^8) \bullet \frac{91}{365} \approx 2.343 \bullet 10^8 \text{ km}$$

* * * * *

1.7 5π radians/sec

1.8 **a.** $(5\pi/2) \bullet 10 = 25\pi \approx 78.54$ m

b. $A = \left(\pi r^2/8\right) = 100\pi/8 = 12.5\pi \approx 39.27 \text{ m}^2$

1.9 Answers will vary, depending on the placement of the coordinate system. In the sample responses below, the hour representing 3 is at the point (1,0).

a. Sample response:

Hour	Coordinates
3	(1,0)
6	(0,–1)
9	(–1,0)
12	(0,1)

b. Sample response:

Hour	Coordinates
11	(–0.5,0.87)
7	(–0.5,–0.87)
5	(0.5,–0.87)
1	(0.5,0.87)

c. (–1,0)

* * * * * * * * * *

(page 28)

Activity 2

Students create the graphs for the sine, cosine, and tangent functions from the unit circles they created in Activity **1**.

Materials List

- freezer paper (one long sheet per group)
- long, unmarked strips of paper or ribbon (several strips per group)
- scissors (one pair per group)
- glue (one bottle per group)

Technology

- graphing utility
- geometry utility (optional)

Teacher Note

In the following exploration, students generate graphs for the sine, cosine, and tangent functions. Following this exploration—or as a replacement activity—you may wish to demonstrate the relationship between trigonometric functions and the unit circle using technology.

A demonstration can be created using parametric equations on a graphing utility. Use $x = \cos t$ and $y = \sin t$ to generate the unit circle and $x = t$ and $y = \sin t$ to generate the sine curve. Graph them simultaneously. The coordinates can be compared by using the trace feature and toggling between graphs. Care should be taken to set the domain and range so that the unit circle does not appear distorted. The same process may be used for cosine and tangent.

Discussion 1

(page 28)

a.

1. Since AC is 1, AB represents $\sin\angle ACB$.
2. Since AC is 1, BC represents $\cos\angle ACB$.

b.

1. The value of $\sin(t)$ is positive in the first and second quadrants since the *y*-coordinate of a point on the unit circle is positive in those quadrants. It is negative in the third and fourth quadrants since the *y*-coordinate is negative in those quadrants.
2. The value of $\cos(t)$ is positive in the first and fourth quadrants since the *x*-coordinate of a point on the unit circle is positive in those quadrants. It is negative in the second and third quadrants since the *x*-coordinate is negative in those quadrants.

c.

1. A tangent is perpendicular to the radius of a circle at the point of tangency. The positive *x*-axis from 0 to 1 is a radius of the unit circle, so the tangent is perpendicular to the *x*-axis and $m\angle EDC = 90^\circ$.
2. Since both triangles are right triangles and share a common angle, $\angle C$, the measures of the third angles are also equal. By the Angle-Angle-Angle Property, the two triangles are similar.
3. By the properties of similar triangles,
$$\frac{DE}{CD} = \frac{AB}{BC}$$
Since $CD = 1$, $DE = AB/BC$.

d. From right-triangle trigonometry,
$$\tan\angle ACB = \frac{AB}{BC}$$
Since $DE = AB/BC$, $\tan\angle ACB = DE$.

e. Sample response: From Figure **5**, when $m\angle ACB$ is an odd multiple of $\pi/2$, the ray *CA* and tangent line passing through *D* are parallel. Since ray *CA* and the line do not intersect, *DE* does not exist.

f. The value of $\tan(t)$ is positive in the first and third quadrants since both the *x*- and *y*-coordinates have the same sign in those quadrants. It is negative in the second and fourth quadrants since the *x*- and *y*-coordinates have different signs in those quadrants.

Exploration (page 30)

a. At least one group should graph each trigonometric function.

b. Students create a coordinate system on which to graph the selected function.

c–e. Students use their graphs of a unit circle from Activity **1** to create paper strips whose lengths represent the values of the selected function at each labeled point on the unit circle. **Note:** The use of colored ribbons may produce distinctive graphs suitable for display.

f. To approximate the shape of the graph of the function, students sketch a smooth curve over the paper-strip graph.

g. Students graph the three circular functions on a graphing utility and note their characteristics.

Discussion 2 (page 32)

a. Students share their graphs with the class.

b. Sample response: The first coordinate for a sine graph represents a real number *r*, while the second coordinate represents $\sin r$.

The first coordinate for a cosine graph represents a real number *r*, while the second coordinate represents $\cos r$.

The first coordinate for a tangent graph represents a real number *r*, while the second coordinate represents $\tan r$.

c. Students should observe that corresponding graphs and scatterplots are similar in shape and characteristics.

d.
1. The interval for the sine and cosine is 2π. The interval for the tangent is π.
2. Sample response: The intervals for the graphs of the sine and cosine equal the circumference of the unit circle. The interval for the tangent is equal to half the circumference of the unit circle.

e. **1.** The domain for the sine and cosine functions is the real numbers. The domain for the tangent is all real numbers except for $n\pi/2$, where n is an odd integer.

The range for the sine and cosine functions is the interval $[-1,1]$. The range for the tangent is the real numbers.

2. Since the x- and y-coordinates of the points on the unit circle are in the interval $[-1,1]$, the range values for the sine and cosine functions are also in that interval. The range of the tangent function is determined by the ratio of the y-coordinate to the x-coordinate. Since all real numbers can be expressed by these ratios, the range is the interval $(-\infty,\infty)$.

f. For both the sine and cosine functions, the amplitude is 1 and the period is 2π. Since the tangent has neither an absolute maximum nor absolute minimum, its amplitude does not exist. Its period is π.

g. Sample response: When the denominator of a fraction approaches 0, the value of the fraction increases without bound. When x is close to $\pi/2$, or any odd multiple of $\pi/2$, the values of $\cos x$ are near 0 and the value of the tangent increases without bound.

Assignment

(page 33)

2.1 Sample response: The coordinates of A are $(r\cos\angle DOC, r\sin\angle DOC)$. Since $\Delta DOC \sim \Delta AOB$,

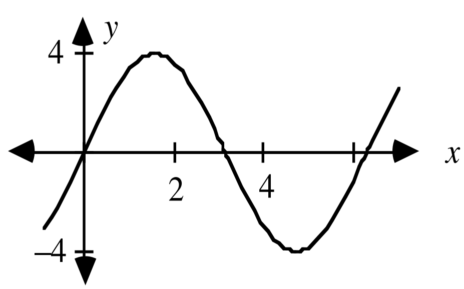
$$\frac{OC}{1} = \frac{OB}{r} \text{ and } \frac{CD}{1} = \frac{AB}{r}$$

This means that $r \bullet OC = OB$ and $r \bullet CD = AB$. Therefore, OB (the x-coordinate of A) equals $r\cos\angle DOC$ and AB (the y-coordinate of A) equals $r\sin\angle DOC$.

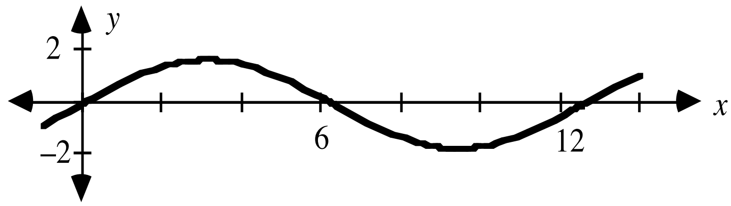
***2.2** **a.** The period is 2π.

b. The amplitude is 2.

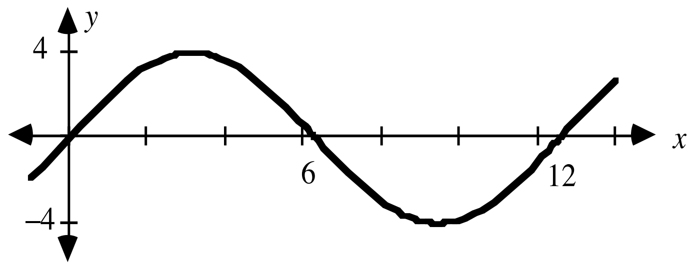
c. Sample graph:



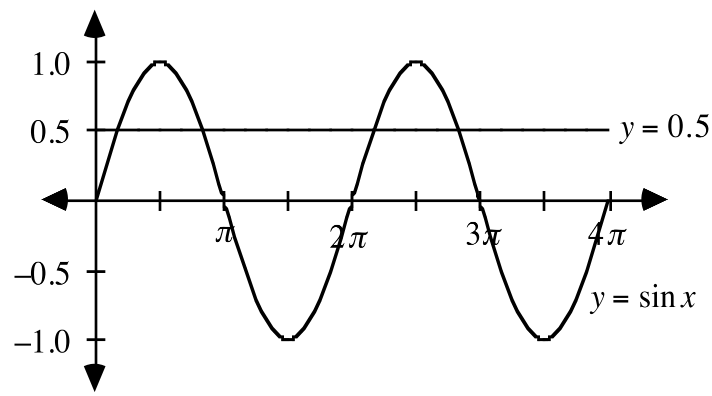
d. Sample graph:



e. Sample graph:



2.3 **a.** Sample graph:



b. Sample response: There are four solutions: $\pi/6$, $5\pi/6$, $13\pi/6$, and $17\pi/6$. These occur wherever the two graphs intersect.

c. Sample response: There would be four more solutions. Since the graph of $y = \sin x$ repeats every 2π, there are two more intersections for every interval of 2π.

2.4 **a.** Students demonstrate that each identity is true by substituting several values for θ.

b. **1.** Sample response: Since *P* and *Q* represent points with the same *y*-coordinate, they have the same value for the sine function. The angle formed by the positive *x*-axis and $\overline{OQ}$ is $\pi - \theta$. Therefore, $\sin\theta = \sin(\pi - \theta)$.

2. Sample response: Since *P* and *Q* represent points with the same *x*-coordinate, they have the same value for the cosine function.

The angle formed by the positive *x*-axis and $\overline{OQ}$ is $-\theta$. Therefore, $\cos\theta = \cos(-\theta)$.

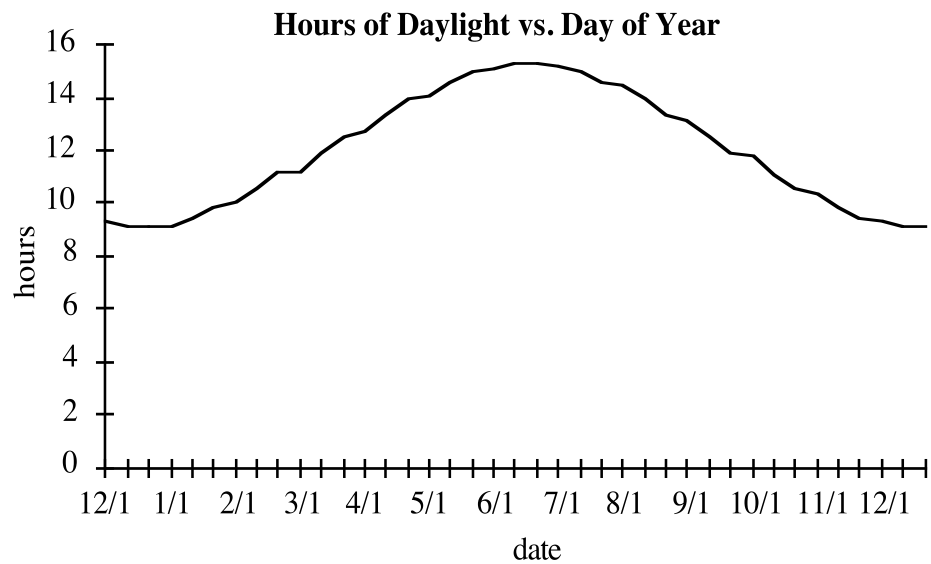
3. Sample response: The coordinates of point Q are additive inverses of the coordinates of point P. The coordinates of P are $(\cos\theta, \sin\theta)$. The coordinates of point Q are $(-\cos\theta, -\sin\theta)$. By the definition of the tangent function, $\tan\theta = \sin\theta/\cos\theta$ and

$$\tan(\theta + \pi) = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

Therefore, $\tan\theta = \tan(\pi + \theta)$.

c. Sample response: Once the first real number is found, the identity can be used to find the second. For example, if $\sin(\pi/6) = 0.5$, then the sine of $\pi - \pi/6 = 5\pi/6$ has the same value.

2.5 **a.** Sample graph:



b. Sample response: Yes. The shape of the sine curve is included in this graph, but it has been moved up and has a different period and amplitude.

c. The amplitude of the curve is the difference between the maximum and minimum, divided by 2:

$$\frac{15.27 - 9.2}{2} \approx 3$$

d. Since the pattern in the hours of daylight repeats annually, the period is 365 days.

* * * * *

2.6 **a.** Students substitute several values for t into each equation to demonstrate that each appears to be an identity.

b. Since ΔABC is a right triangle, $AB^2 + BC^2 = AC^2$ by the Pythagorean theorem. As a radius of a unit circle, $AC^2 = 1$. Using right-triangle trigonometry, $AB = \sin\angle ACB$ and $BC = \cos\angle ACB$. By substitution, $(\sin\angle ACB)^2 + (\cos\angle ACB)^2 = 1$.

c. Since ΔABC and ΔDEC are right triangles with $\angle ACB \cong \angle CDE$ and each hypotenuse a radii of the unit circle centered at the origin, it follows that $\angle BAC \cong \angle DCE$. Since all their corresponding angles are congruent, the triangles are similar. The ratio of proportion is 1 (since the hypotenuse of each triangle is a radius), so $\Delta ABC \cong \Delta CED$.

Because the triangles are congruent, $\overline{CE} \cong \overline{AB}$. In ΔDCE, $m\angle DCE = (\pi/2) - m\angle CDE$, since the sum of the measures of the interior angles of a triangle is equal to π radians. This means, by substitution, $m\angle DCE = (\pi/2) - m\angle ACB$.

Also in ΔDCE, $CE = \cos\angle DCE = \cos\left((\pi/2) - m\angle ACB\right)$, while in ΔABC, $AB = \sin\angle ACB$. By substitution, $\sin\angle ACB = \cos\left((\pi/2) - m\angle ACB\right)$. **Note:** A similar argument can be used for the identity $\cos t = \sin\left((\pi/2) - t\right)$.

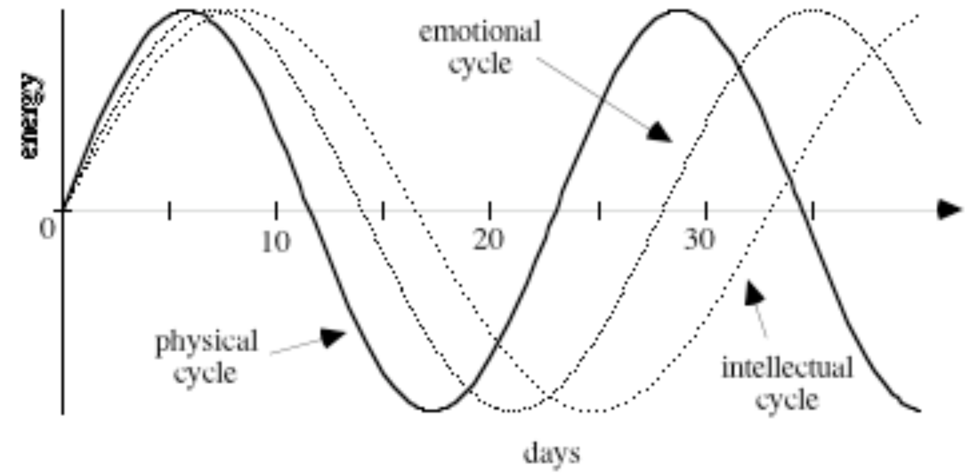
2.7 **a.** 12π radians

b. (5,0)

c. (3.5,0)

d. The coordinates for Moonlight's location are: $(5\cos 13\pi/4, 5\sin 13\pi/4) \approx (-3.54, -3.54)$. The coordinates for Red Ribbon's location are: $(3.5\cos 13\pi/4, 3.5\sin 13\pi/4) \approx (-2.47, -2.47)$.

2.8 **a.** Sample graph:



b. The period for the physical cycle is 23 days, for the emotional cycle 28 days, and for the intellectual cycle 33 days.

c. Sample response: The graphs of all three cycles are the same shape as the sine curve but with different periods and different amplitudes.

d. Critical days occur when the curves cross the horizontal axis. Students should label these points on their graphs.

e. Answer will vary. Students should calculate their age in days, then divide this figure by the period of each cycle. A student who is exactly 16 years old, for example, has lived 5840 days. The physical cycle has gone through approximately 253.9 cycles. Since $0.9 \bullet 23$ is about 21 days, the physical cycle is on the negative side and increasing daily. The emotional cycle has gone through approximately 208.6 cycles. Since $0.6 \bullet 28$ is about 17 days, the emotional cycle is on the negative side and decreasing daily. The intellectual cycle has gone through about 177.0 cycles. This means the intellectual cycle is at the baseline and increasing daily.

f. The three cycles coincide initially at birth and again about 58 yr later. The latter date is calculated by finding the least common multiple of 23, 28, and 33, then converting this number of days into years.

* * * * * * * * * *

(page 39)

Activity 3

The general form of a sine function is $y = a(\sin b(x + c)) + d$, where a, b, c, and d are real-number parameters. In this activity, students investigate the transformations resulting from different values of the four parameters. The general sine and cosine functions are then used to model real-life phenomena.

Materials List

- none

Technology

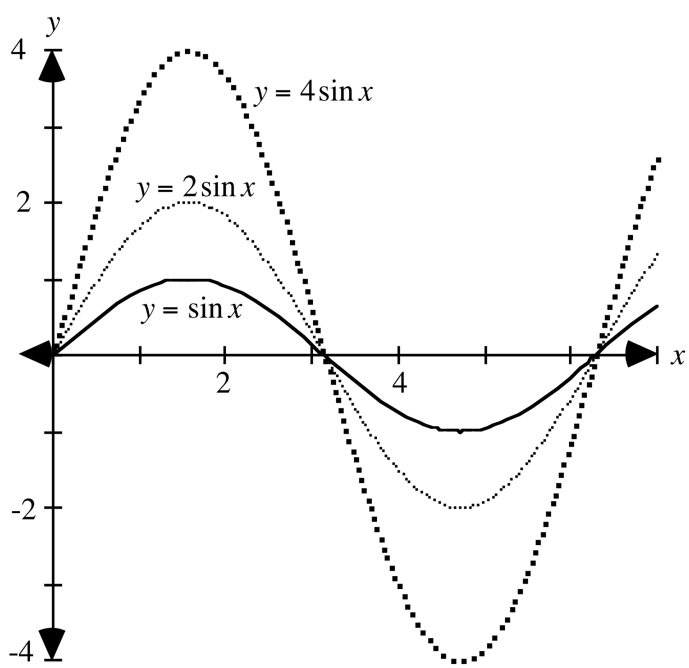
- graphing utility

Exploration

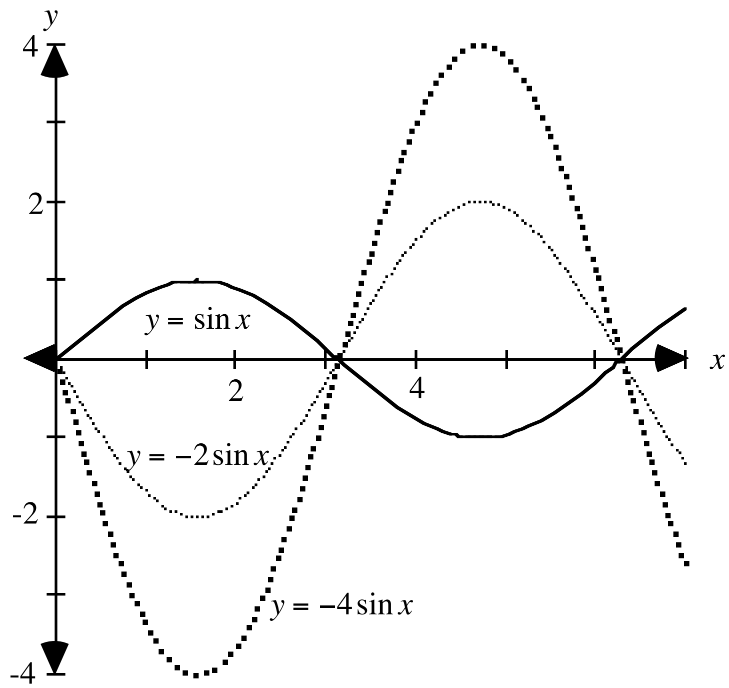
(page 39)

Students use a graphing utility to examine the general equation of the sine curve.

a. **1.** Sample graph:

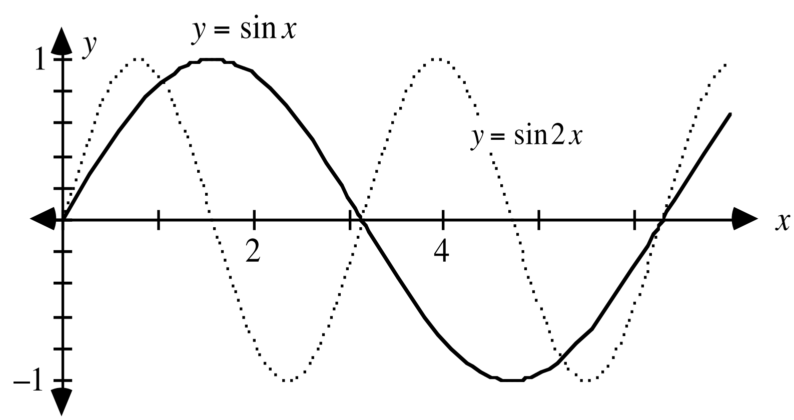


2. Sample graph:



3–4. The value of *a* determines the maximum and minimum values of the function and therefore the amplitude of the curve. (Some students may use the terms *vertical shrink* or *stretch* to describe this effect.) If *a* is a negative number, the curve is reflected in the *x*-axis.

b. **1.** Sample graph using $b = 2$:

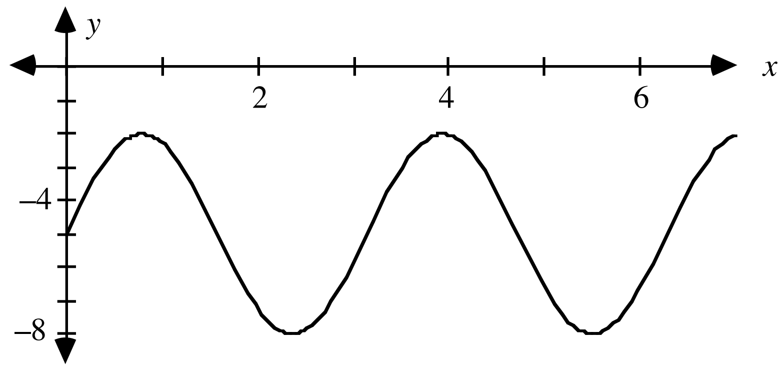


2–4. The value of *b* determines the period of the curve. A negative value for *b* reflects the graph in the *y*-axis. Some students may use the terms *horizontal stretch* or *shrink* to describe this effect.

c. The parameter *c* causes a horizontal translation of the graph of –*c* units.

d. The parameter *d* causes a vertical translation of the graph of *d* units.

e. Sample graph:



Discussion
(page 40)

a. Given an equation of the form $y = a(\sin b(x + c)) + d$, the amplitude is $|a|$ and the period is $2\pi/b$. If a is negative, the graph of $y = \sin x$ is reflected in the x-axis. If b is negative, the graph is reflected in the y-axis. The parameter c translates the graph horizontally by $-c$ units, while d translates the graph vertically d units.

b. Sample response: The two functions have the same amplitude and period. They also have the same shape. One graph can be obtained from the other by a horizontal translation.

Assignment
(page 41)

3.1 **a.** $y = \sin x$

b. $y = \sin 2x$

c. $y = 3\sin 2x$

d. $y = -3\sin 2x$

e. $y = -3\sin(2(x - \pi/4))$

f. $y = -3\sin(2(x - \pi/4)) + 1$

3.2 $A = 20\sin(120\pi \bullet t) \approx 20\sin(380t)$

***3.3** $h = 1.45\cos((\pi/6.2)t) + 1.45$

3.4 **a.** **1.** $y = 4\cos((\pi/6.2)t) + 4$

2. 4.203 – 2.238 = 1.965 m

b. **1.** $y = 4\sin((\pi/6.2)t) + 4$

2. 8 – 7.464 = 0.536 m

3.5 **a.** The period is about 365 days; the amplitude is approximately 23°.

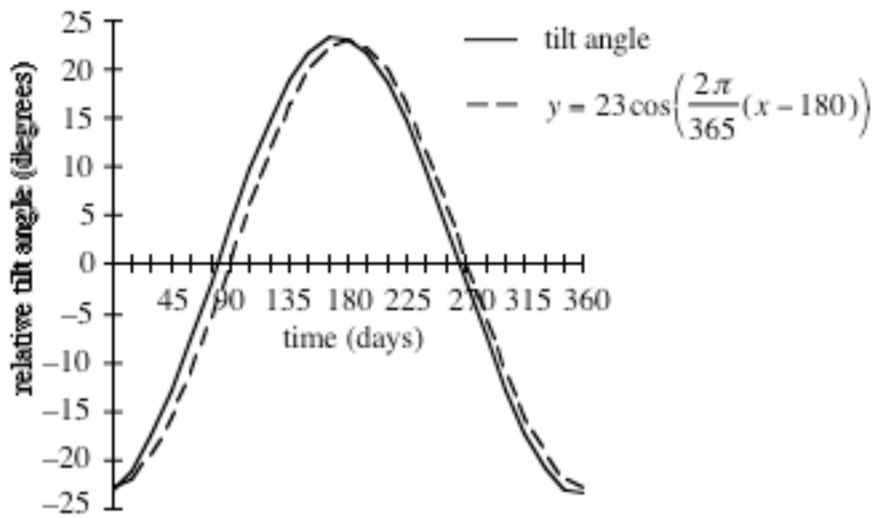
b. Either the sine or the cosine could be used to model this graph. In both cases, however, the function would have to be modified.

c. Sample response: The amplitude needs to be increased to 23 and the period needs to be adjusted to 365.

d. The following equation also models the data well:

$$y = -23\cos\left(\frac{2\pi}{365}x\right)$$

Note: The sample graph below shows both the "Relative Tilt Angle" curve from the student edition, and a graph of the model equation.



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3.6 **a.** If t represents the number of minutes the Ferris wheel has been rotating, the cosine function that models the vertical motion of the chair is:

$$h = 38\cos\left(\frac{\pi}{0.6}(t-0.6)\right) + 38$$

The corresponding sine function is:

$$h = 38\sin\left(\frac{\pi}{0.6}(t-0.6) + \frac{\pi}{2}\right) + 38$$

$$= 38\sin\left(-\frac{\pi}{0.6}(t+0.3)\right) + 38$$

b. Since the diameter of the wheel is 76 m, the amplitude is 38. This number is substituted for a in the general form of the sine or cosine equation. The period (1.2 min) is substituted for p in the expression $2\pi/p$ to determine the value of b ($\pi/0.6$).

c. Using the cosine function, translations of 38 units vertically and 0.6 units horizontally are needed. Using sine, the horizontal translation required is 0.3 units.

d. If t represents the number of minutes the wheel has been rotating, the cosine function that models the motion of the chair is:

$$h = 38\cos(\pi(t-1)) + 38$$

The corresponding sine function is:

$$\begin{aligned} h &= 38\sin(\pi(t-1) + \pi/2) + 38 \\ &= 38\sin(\pi(t-0.5)) + 38 \end{aligned}$$

3.7 **a.** Answers will vary: The following sample responses are based on a period of 30 min or 0.5 hr and a temperature range from 20–22° C.

b. Students may select either the sine or cosine function. The following sample response assumes that the temperature is 22° C at $t = 0$:

$$y = 2\sin\left(\frac{2\pi}{1/2}x\right) = 2\sin(4\pi x)$$

c. Students may realize that the time required to heat a room is generally shorter than the time it takes to cool. The length of each complete cycle depends on the outside temperature.

* * * * * * * * * *

(page 44)

Activity 4

In this activity, students are introduced to the concept of inverse functions.

Materials List

- none

Technology

- spreadsheet
- graphing utility
- symbolic manipulator

Teacher Note

In the following exploration, students use the spreadsheet from the exploration in Activity **1** to create a scatterplot of the inverse relationship between the real numbers and the sine and cosine. Because the scatterplot for the inverse tangent is difficult to interpret, it has been omitted from the exploration. In Part **b** of Discussion **1**, students are asked to predict how the scatterplot of the inverse tangent relationship might look. You may wish to sketch this graph so students can compare it to their predictions.

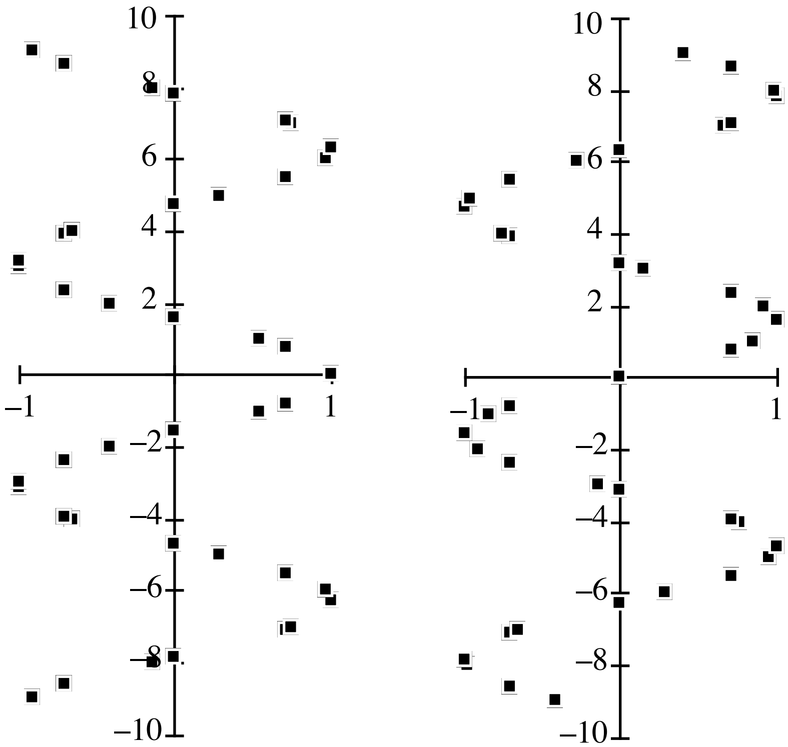
Exploration 1

(page 45)

In this exploration, students graph and examine scatterplots of inverse sine and cosine relationships. **Note:** These scatterplots are easier to interpret when the intervals and scales on both axes are approximately equal.

a. Students update the column headings of the spreadsheet created in Activity **1**.

b. Sample graphs: Inverse of the cosine function and inverse of the sine function, respectively



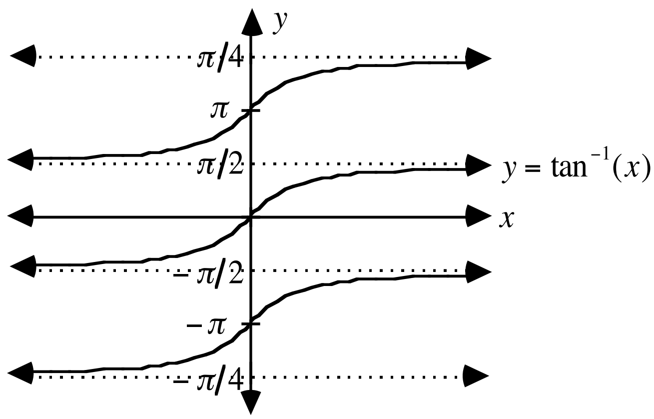
c. Neither inverse relation is a function. In Exploration **2**, students restrict the domains of the sine and cosine so that their inverses are also functions.

Discussion 1

(page 46)

a. Sample response: The inverse cosine relationship looks like a graph of the cosine function that has been rotated about the origin $\pi/2$ radians in a clockwise direction. The inverse sine relationship looks like a graph of the sine function that has been reflected in the *x*-axis, then rotated about the origin $\pi/2$ radians in a clockwise direction.

b. **1.** Sample response: The inverse tangent relationship looks like a graph of the tangent function that has been reflected in the x-axis, then rotated about the origin $\pi/2$ radians in a clockwise direction.



2. The inverse tangent relationship is not a function. There is more than one y-value for each x-value in the domain.

c. Sample response: No. In each of these inverse relationships, there is more than one y-value for each x-value in the domain.

d. The lines $d = 2$ and $d = -2$ represent positions 2 cm above and 2 cm below the equilibrium point. The graph of $d = -3\cos t$ represents the object's distance above or below the equilibrium point over time. The x-values where the lines intersect the curve indicate the times when the object is either 2 cm above or 2 cm below its equilibrium point.

Teacher Note

In the following exploration, students determine a domain over which the inverses of the sine, cosine, and tangent represent functions. To ensure that the class examines all three inverses, you may wish to assign each group a different function.

Exploration 2

(page 46)

a. Students identify a portion of the scatterplot or spreadsheet from Exploration **1** for which the inverse is a function.

b. Students graph the inverse function on a graphing utility.

c. The domain of the sine function over which the inverse sine function is defined is the interval $[-\pi/2, \pi/2]$. The corresponding range is the interval [–1,1].

The domain of the cosine function over which the inverse cosine function is defined is the interval $[0, \pi]$. The corresponding range is the interval [–1,1].

The domain of the tangent function over which the inverse tangent function is defined is the interval $(-\pi/2, \pi/2)$. The corresponding range is the interval $(-\infty, \infty)$.

d. Students experiment with values inside and outside of the restricted domain of their selected functions. They should observe that for any value selected within the respective domains, $\sin^{-1}(\sin x) = x$, $\cos^{-1}(\cos x) = x$, and $\tan^{-1}(\tan x) = x$. For values outside the domain, these relationship may not hold true.

e. **1.** Most calculators will report only one solution: $\cos^{-1}(2/3) \approx 0.841$. Some symbolic manipulators may report all possible solutions.

2. Students should solve as follows:

$$2 = -3\cos t$$
$$2/-3 = \cos t$$
$$\cos^{-1}(-2/3) = \cos^{-1}(\cos t)$$
$$\cos^{-1}(-2/3) = t$$

Most calculators will report only one solution: $\cos^{-1}(-2/3) \approx 2.3$. Some symbolic manipulators may report all possible solutions.

Discussion 2

(page 47)

a. Sample response: I selected a continuous set of domain values from the function such that it contained every possible range value exactly once.

b. Answers will vary. Some students may have selected the same restricted domain used by the technology. Others may identify a different set of values for the domain.

c. **1.** The following relationships are true within the restricted domain of the respective functions: $\sin^{-1}(\sin x) = x$, $\cos^{-1}(\cos x) = x$, and $\tan^{-1}(\tan x) = x$.

2. Students should observe that the above relationships may not exist outside the restricted domain of the function.

3. The inverse function is only defined over a restricted domain for the original function. Within that domain, $f^{-1}(f(x)) = x$.

d. Sample response: The solutions determined in Part **e** of Exploration **2** included only one value. This occurs because the inverse function is only defined over a restricted domain. By using the graph in Discussion **1**, several other solutions could be identified.

e. Sample response: The identities allow you to find a second value that is also a solution. Applying the period to each of the two solutions will allow you to find all the other solutions. **Note:** Some symbolic manipulators may report all possible solutions.

f. $3x + 1$

Assignment

(page 48)

4.1 **a.** **1.** Answers will vary. Sample response: Two approximate values for x are 3.99 and –0.85.

$$-\sin(x) = 0.75$$
$$\sin(x) = -0.75$$
$$\sin^{-1}(\sin(x)) = \sin^{-1}(-0.75)$$
$$x \approx -0.85$$
$$\pi - x \approx 3.99$$

2. Answers will vary. Sample response: Two approximate values for x are 1.27 and 4.41.

$$2\tan(x) = 6.5$$
$$\tan(x) = 3.25$$
$$\tan^{-1}(\tan(x)) = \tan^{-1}(3.25)$$
$$x \approx 1.27$$
$$\pi + x \approx 4.41$$

3. Answers will vary. Sample response: Two approximate values for x are 0.47 and –0.47.

$$\cos(3x) = 0.15$$
$$\cos^{-1}(\cos(3x)) = \cos^{-1}(0.15)$$

The two equations below result since $\cos x = \cos(-x)$.

$$3x \approx 1.42 \qquad 3x \approx -1.42$$
$$x \approx 0.47 \qquad x \approx -0.47$$

4. Answers will vary. Sample response: Two approximate values for x are 0.10 and 1.48.

$$4\sin(2x) + 1 = 1.75$$
$$4\sin(2x) = 0.75$$
$$\sin(2x) \approx 0.19$$
$$\sin^{-1}(\sin(2x)) \approx \sin^{-1}(0.19)$$

The two equations below result since $\sin x = \sin(\pi - x)$.

$$2x \approx 0.19 \qquad 2x \approx \pi - 0.19$$
$$x \approx 0.10 \qquad x \approx 1.48$$

5. Answers will vary. Sample response: Two approximate values for x are 1.64 and 0.36.

$$-2\cos(3(x-1)) - 5 = -4.32$$
$$-2\cos(3(x-1)) = 0.68$$
$$\cos(3(x-1)) = -0.34$$
$$\cos^{-1}(\cos(3(x-1))) = \cos^{-1}(-0.34)$$

The two equations below result since $\cos x = \cos(-x)$.

$$3(x-1) = 1.92 \qquad 3(x-1) = -1.92$$
$$x-1 = 0.64 \qquad x-1 = -0.64$$
$$x \approx 1.64 \qquad x \approx 0.36$$

b. Students use a symbolic manipulator to check their Part **a** results.

4.2 **a.** **1.** To determine the mean of 12 hr, students may substitute 0 for d and solve for h.

2. The following solutions describe the numbers of days after March 21 on which the most and least daylight hours occur, respectively:

$$15 = 12 + 3\sin\left(\frac{2\pi}{365}d\right) \qquad 9 = 12 + 3\sin\left(\frac{2\pi}{365}d\right)$$
$$d \approx 91 \qquad d \approx 274$$

From these solutions, the shortest day occurs on December 20, while the longest day occurs on June 20.

b. Students should realize that the daylight hours increase immediately after March 21. Therefore, the interval over which there is at least 12 hr of daylight begins on March 21 and ends at the next date that the number of daylight hours equals 12. This date can be found by the following:

$$12 = 12 + 3\sin\left(\frac{2\pi}{365}d\right)$$
$$d \approx 183$$

The interval over which there is at least 12 hr of daylight begins on March 21 and ends on September 20.

c. The model is:

$$h = 12 - 3\sin\left(\frac{2\pi}{365}d\right)$$

d. Students should realize that the daylight hours in the southern hemisphere decrease immediately after March 21. Therefore, the interval over which there is at least 12 hr of daylight ends on March 21. It begins at the next date when the number of daylight hours equals 12. This date can be found as follows:

$$12 = 12 - 3\sin\left(\frac{2\pi}{365}d\right)$$

$$d \approx 183$$

The interval over which there is at least 12 hr of daylight begins on September 20 and ends on March 21.

4.3 **a.** The water depth remains above 4.4 m for approximately 4.9 hr after 12:00 noon, or until approximately 4:54 P.M. This can be found as follows:

$$4.4 = \cos(0.51t) + 5.2$$

$$t \approx 4.9$$

b. Using the identity $\cos x = \cos(-x)$, the next time the water is at 4.4 m can be found as follows:

$$4.4 = \cos(0.51t) + 5.2$$

$$-0.8 = \cos(0.51t)$$

$$-0.8 = \cos(-0.51t)$$

$$t \approx -4.9$$

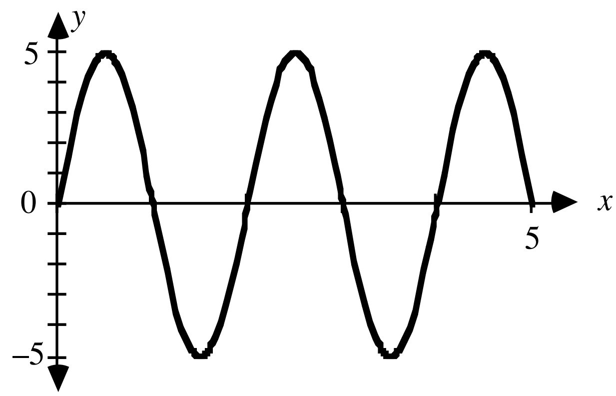
The period of the function is $2\pi/(0.51) \approx 12.3$. The earliest that the ship can leave the port is $-4.9 + 12.3 = 7.4$ hr after 12:00 noon, or about 7:24 P.M.

c. Sample response: The water level is below 4.4 m from 4.9 hr to 7.4 hr after 12:00 noon. Since the period is 12.3 hr, the tide will be below 4.4 m from 17.2 hr to 19.7 hr after 12:00 noon, and again from 29.5 hr to 32 hr after 12:00 noon.

If the ship must be back at sea 26 hr after entering the harbor, it must leave no more than 30.9 hr after 12:00 noon. To beat the falling tide, however, the ship must leave no later than 29.5 hr after 12:00 noon. This is about 5:30 P.M. on the following day.

4.4 **a.** Sample function: $y = 5\sin \pi x$.

b. Sample response: The swing momentarily stops at its highest point before it starts moving in the opposite direction. Therefore, the swing is at the highest point when the speed is 0 m/sec. The swing's horizontal velocity is 0 at each point where the graph intersects the x-axis.



c. The swing will be traveling 2 m/sec at any time t such that $t \approx 2(n + 0.07)$ or $t \approx 2(n + 0.43)$, where n is any whole number.

* * * * *

4.5 **a.** Sample response: Since the vertical shift is 3.2 units above the x-axis and the amplitude is 1.1 units, the maximum and minimum of the function are at 3.2 + 1.1 and 3.2 – 1.1, respectively. The farthest the pendulum gets from the motion detector is 4.3 cm. The closest it gets is 2.1 cm.

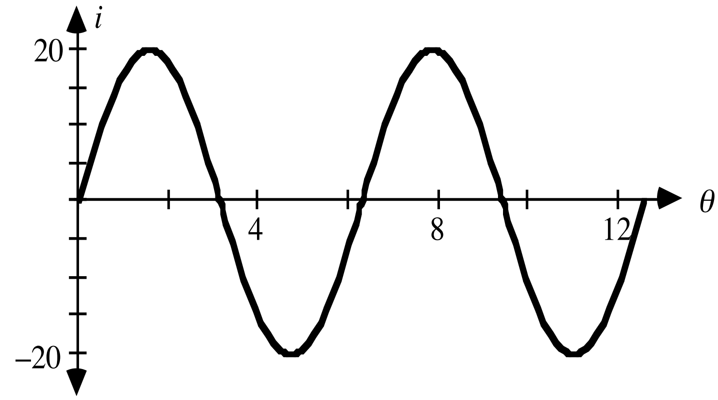
b. Sample response: Setting the function equal to 4.3 and solving for t determines when the pendulum is at its farthest point from the detector. It is at its farthest point after 0.08 sec. The period of the function is 0.3 sec. Therefore, the pendulum will be at its greatest distance from the detector at $0.08 + 0.3n$, where n is an integer.

c. The pendulum will be hanging straight down when d is equal to the mean distance from the detector. At this point, $d = 3.2$. The pendulum will pass this point twice during each period, so it will be hanging straight down once every 0.157 sec.

The first time it will be hanging straight down occurs at $t = 0.007$ (half the time required to reach the farthest point for the first time). The pendulum will return to this position again at $t = 0.007 + 0.157n$, where n is a whole number.

4.6 **a.** $i = 20\sin\theta$

b. Sample graph:



c. Sample response: The effective value of the current is $0.707(20) = 14.14$. Solving the following equation for θ gives one possible value:

$$14.14 = 20\sin\theta$$
$$\frac{14.14}{20} = \sin\theta$$
$$\sin^{-1}\left(\frac{14.14}{20}\right) = \sin^{-1}(\sin\theta)$$
$$0.785 \approx \theta$$

Using the identity $\sin\theta = \sin(\pi - \theta)$, another possible value is approximately 2.356. The other possible values for θ can be determined by adding 2π to each of these solutions.

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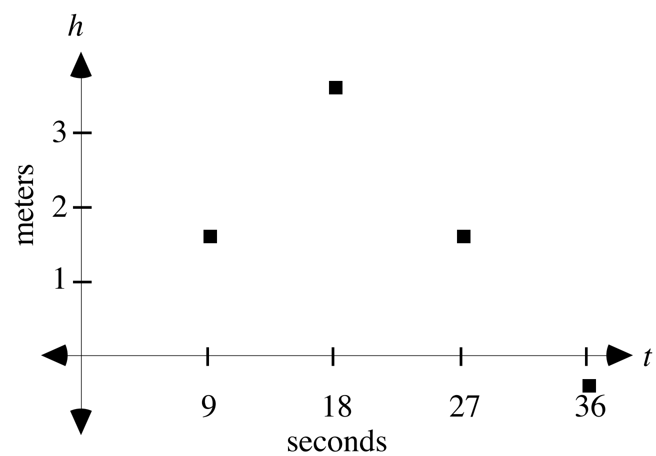
Answers to Summary Assessment (page 51)

1. Some students may wish to create a table of values, then use a graphing utility to produce a scatterplot. The following sample response uses the paddle which is in the water at $t = 0$ and assumes the origin is at the center of the wheel.

Time (sec)	Distance (m)
0	–0.35
9	1.65
18	3.65
27	1.65
36	–0.35

Sample graph:



a. The amplitude is 2 m.

b. The period is 36 sec.

c. Sample response: Since $a = -2$, $b = 2\pi/36$, $c = 0$, and $d = -0.35 - (-2) = 1.65$, the equation of the cosine function is:

$$h = -2\cos\left(\frac{\pi}{18}t\right) + 1.65$$

d. Sample response:

$$h = -2\sin\left(\frac{\pi}{18}(t+9)\right) + 1.65$$

2. **a.** $2\pi \bullet (6/36) = \pi/3$

b. Since it takes 36 sec to rotate through 2π radians, it follows that:

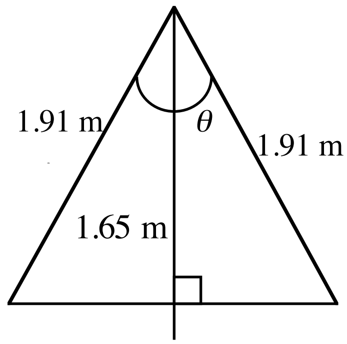
$$\left(\frac{15\pi/4}{2\pi}\right) \bullet 36 = 67.5 \text{ sec}$$

c. Solving the following equation for *v*:

$$\frac{2\pi}{36} = \frac{v}{60}$$

$v = 10\pi/3$ radians/min.

3. As shown in the figure below, the distance between the paddles is $2d$.



Students may solve for θ using $\cos^{-1}\theta = \frac{1.65}{1.91}$. Using this equation, $\theta \approx 0.5279$ radians. The number of paddles can be found by dividing 2π by 0.5279. This results in approximately 12 paddles on the wheel.

Module Assessment

1. As you move around a unit circle from quadrant to quadrant, how do the signs for the values of the tangent function change?

2. The table below gives the average monthly temperature (in degrees Celsius) for a city in the Rocky Mountain region for January 1990 through December 1991. Use the table to answer the following questions.

Month	Average Temp.	Month	Average Temp.
1	–1.94	13	–4.72
2	–2.06	14	3.28
3	1.72	15	1.50
4	7.94	16	5.33
5	10.22	17	10.06
6	15.83	18	15.00
7	20.17	19	20.56
8	19.44	20	21.50
9	17.28	21	15.17
10	8.39	22	6.94
11	3.33	23	–1.28
12	–9.22	24	–1.44

a. Graph this data using a graphing utility.

b. What are the advantages and disadvantages of modeling this data using a circular function?

c. What is the amplitude of a curve that would approximate the data?

d. What is the period of a function that would approximate the data?

e. How would you alter the equation $y = \sin x$ so that its graph would approximate the graph in Part **a**?

f. Predict the average monthly temperature in this city for each of the next five months.

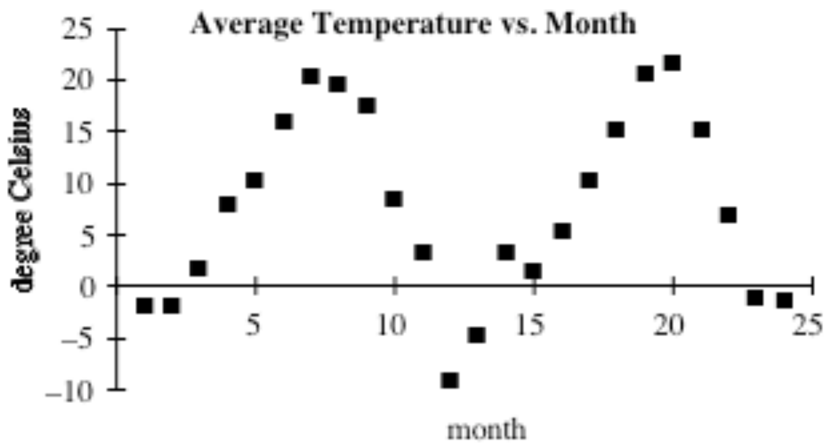
3. Describe the differences in the graphs of $y = \tan x$ and $y = 3\tan(0.5x)$.

Answers to Module Assessment

1. Student responses should reflect the contents of the table below.

Interval	Sign of Tangent
$0 < x < \pi/2$	+
$\pi/2 < x < \pi$	–
$\pi < x < 3\pi/2$	+
$3\pi/2 < x < 2\pi$	–

2. **a.** Sample graph:



b. Sample response: Since the data have the general pattern of a sine function and seasons do repeat, a sine function with a period of 1 year might give an adequate picture of the trends in temperature for these 2 years. It would not, however, predict extremes in temperature which occur over the years.

c. Based on the data from the table, the amplitude of a curve would be approximately:

$$\frac{21.50-(-9.22)}{2}=15.36$$

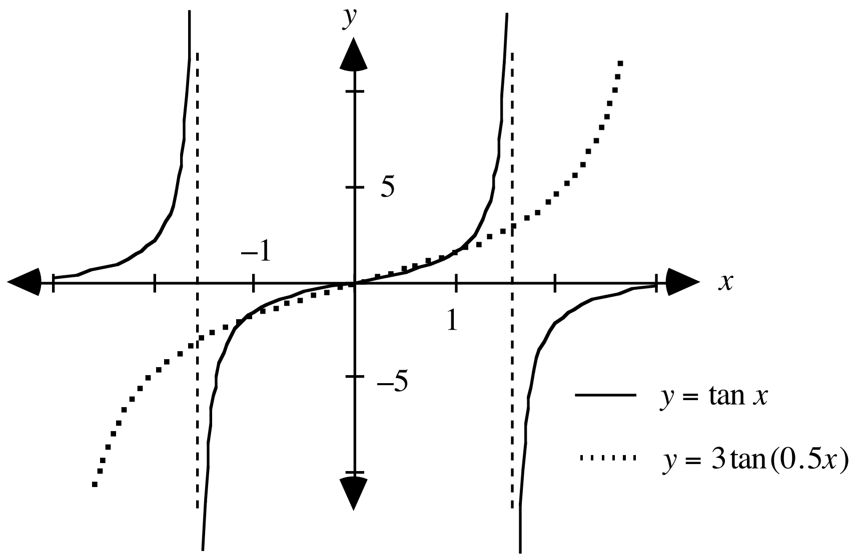
d. Since the pattern in average temperatures appears to repeat annually, the period is 12 months.

e. Sample response: In order to use the sine function, you need to change the amplitude to 15.36, change the period to 12 months, shift the curve down approximately 6 degrees, and shift the curve to the left about 4 months. The following equation is one possible model:

$$y = 15.36\sin\left(\frac{\pi}{6}(x-4)\right)+6$$

f. Sample response: It is very difficult to get accurate predictions from the graph. It looks like the temperature will drop for the next month, then increase. Based on the model, the next five average temperatures for the city would be $-9.4, -7.3, -1.7, 6$, and 13.7.

3. Sample response: The period of the graph of $y = 3\tan(0.5x)$ is twice that of $y = \tan x$. The absolute values of $y = \tan x$ increase more quickly than those for $y = 3\tan(0.5x)$. This is indicated by the steepness of the graph of $y = \tan x$.



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Flashbacks

Activity 1

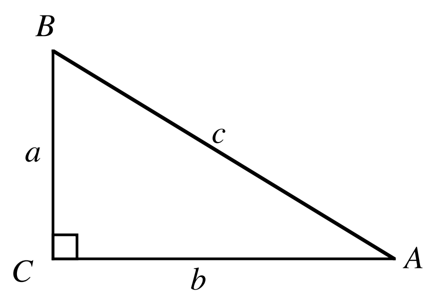
1.1 What is the relationship between the circumference of a circle and its radius?

1.2 Describe one method of locating the center of a circle.

1.3 Draw a central angle of approximately 60° in a circle. Label the arc intercepted by the angle.

Activity 2

2.1 **a.** Write expressions for the sine, cosine, and tangent for each of the acute angles in ΔABC below.



b. Describe $\tan \angle A$ in terms of $\cos \angle A$ and $\sin \angle A$.

Activity 3

3.1 Graph the equation $y = x^2$.

3.2 Graph the equation $y = x^2 + 4$. Describe how adding 4 affects the graph of $y = x^2$.

3.3 Graph the equation $y = (x - 9)^2 + 4$. Describe how subtracting 9 from x affects the graph of $y = x^2 + 4$.

3.4 Graph the equation $y = -2(x - 9)^2 + 4$. Describe how multiplying $(x - 9)$ by –2 affects the graph of $y = (x - 9)^2 + 4$.

Activity 4

4.1 Solve each of the following equations for x:

a. $3x + 4 = 12$

b. $7 = 5 - 8(x - 1)$

4.2 Identify the inverse for each of the following operations:

a. addition

b. division

c. finding the square root

d. raising to the third power

4.3 A 2×3 matrix is multiplied on the right by a 2×2 matrix. This product is then multiplied by the inverse of the 2×2 matrix. What is the final result?

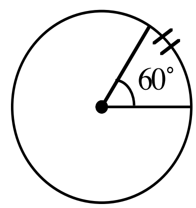
Answers to Flashbacks

Activity 1

1.1 The circumference of a circle is $2\pi r$.

1.2 Sample response: Construct two or more chords for the circle and determine the perpendicular bisector for each chord. The intersection of the perpendicular bisectors is the center of the circle.

1.3 Sample sketch:



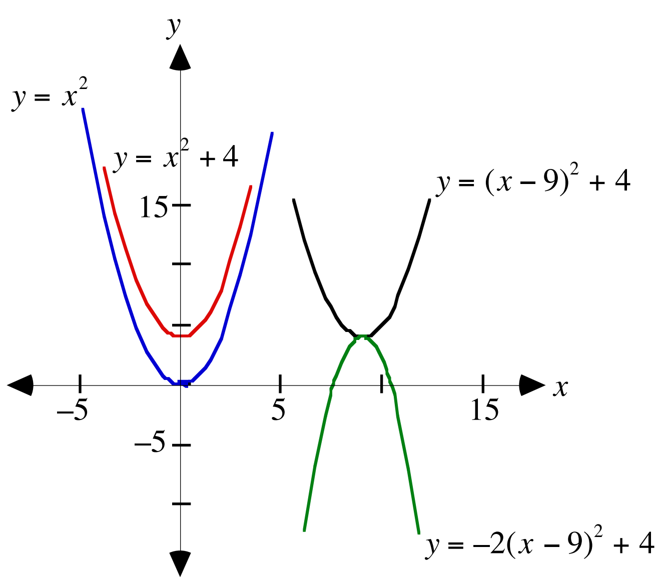
Activity 2

2.1 **a.** $\sin\angle A = a/c$; $\cos\angle A = b/c$; $\tan\angle A = a/b$; $\sin\angle B = b/c$; $\cos\angle B = a/c$; $\tan\angle B = b/a$

b. $\tan\angle A = \sin\angle A/\cos\angle A$

Activity 3

3.1 Sample graphs for Flashbacks **3.1–3.4** are shown below:



3.2 Sample response: The 4 produces a vertical translation of 4 units in the graph of $y = x^2$.

3.3 Sample response: The –9 produces a horizontal translation of 9 units in the graph of $y = x^2 + 4$.

3.4 Sample response: The negative sign of –2 produces a reflection in the x-axis of $y = (x - 9)^2 + 4$. The magnitude of –2 causes the values of y to increase more rapidly so the shape of the graph becomes narrower.

Activity 4

4.1 **a.** $x = 8/3$

b. $x = 3/4$

4.2 **a.** subtraction

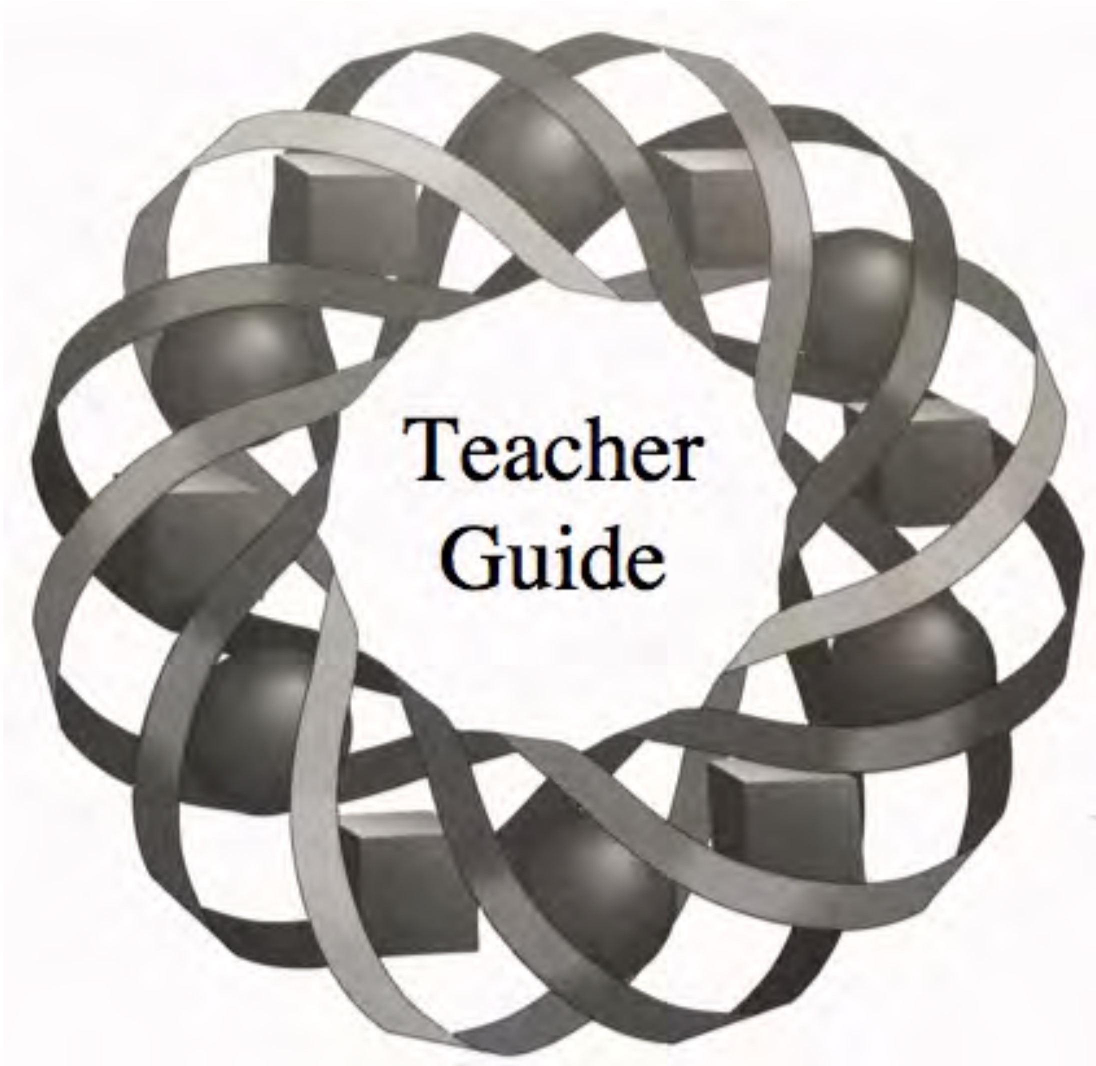
b. multiplication

c. raising to the second power (squaring)

d. taking the cube root

4.3 The result is the original 2×3 matrix.

Motion Pixel Productions



Thanks to computer animation, you can now practice landing an airplane, synthesizing dangerous chemicals, even performing medical surgery. And all without fear of mistakes or injury. In this module, you use matrices to explore the applications of transformational geometry in computer graphics.

Gary Bauer • Franklin Lund • Margaret Plouvier

Teacher Edition

Motion Pixel Productions

Overview

Many of the computer graphics seen in cartoons and television commercials are done by transforming shapes using matrices. In this module, students explore how to transform two-dimensional shapes through matrix multiplication.

Objectives

In this module, students will:

- represent reflections, rotations, translations, and dilations of objects in the plane using matrices
- represent points in homogeneous matrix form
- find the image of a figure under a composite transformation
- express a glide reflection as a composition of a reflection and a translation
- express transformations and compositions of transformations as single 3×3 matrices.

Prerequisites

For this module, students should know:

- how to determine the dimensions of a matrix
- matrix notation
- how to multiply and add matrices
- how to determine angle measures using inverse trigonometric functions
- how to identify a transformation given the preimage and image
- how to perform transformations using technology.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	3	2	3	2	1	11

Materials Required

Materials	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
centimeter graph paper	X	X	X	X	X
straightedge	X	X	X	X	X

Technology

Software	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
geometry utility			X	X	
symbolic manipulator	X	X			

Motion Pixel Productions

Introduction
(page 57)

Computer animation has many applications. To accomplish these animations, figures may be represented by matrices and manipulated through transformations.

Discussion
(page 57)

a. Sample responses may include video games, movies, and cartoons.

b. Students may describe a recent cartoon, television commercial, or printed advertisement.

c. Sample responses may include graphic arts, architecture, engineering, advertising, and movie production.

(page 57)

Activity 1

Students review matrix notation, matrix operations, and transformation matrices.

Materials List

- centimeter graph paper (several sheets per student)
- straightedge (one per student)
- acetate film (one sheet per student)

Technology

- symbolic manipulator (optional)

Discussion 1
(page 58)

a. The three vertices may be represented in either a 2×3 or a 3×2 matrix. **Note:** The conventional method of using matrices to perform transformations involves multiplying the preimage matrix on the left by the transformation matrix. To comply with this convention, the dimensions of the preimage matrix must be 2×3. This convention will be followed throughout this module.

b. The dimensions of the matrices that represent the image and preimage are the same.

c. A preimage represented in a 2×3 matrix requires multiplication by a 2×2 transformation matrix on the left.

d. Sample response: A dilation alters the size of a figure equally in all directions. A translation moves a figure a given distance in a given direction by changing the *x*- and *y*-values by some constant values. A rotation rotates a figure about a center through a prescribed angle. A reflection flips the original figure over a given line of reflection.

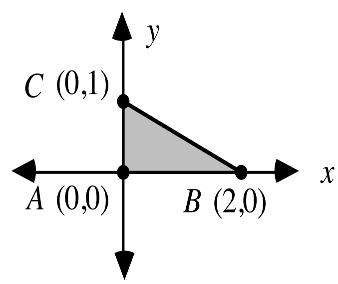
e. For dilations, translations, and rotations, the orientation of the image is the same as that of the preimage. A reflection, however, changes the orientation of the preimage.

f. Since this transformation maps a figure onto itself, it is referred to as an identity transformation.

Exploration 1 (page 59)

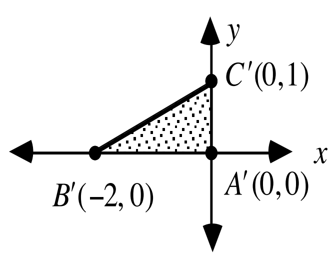
Students use a reference triangle with vertices at $A(0,0)$, $B(2,0)$, and $C(0,1)$ to develop appropriate matrices for specific transformations.

a. Sample graph:



b. Selections will vary. The sample responses given in Parts **c–g** below use a reflection in the *y*-axis.

c. Sample graph using a reflection in the *y*-axis:



d. Sample matrix:

$$\begin{matrix} & A' & B' & C' \\ x & \left[\begin{matrix} 0 \\ 0 \end{matrix}\right. & \begin{matrix} -2 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] \\ y & & & \end{matrix}$$

e. The matrix expression for the reference triangle and its image after a reflection in the y-axis is shown below:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

f. The values of a, b, c, and d for the transformation matrix can be found as shown below:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0a+0c & 2a+0c & 0a+c \\ 0b+0d & 2b+0d & 0b+d \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\begin{aligned} 2a+0c &= -2 \\ 2b+0d &= 0 \\ 0a+c &= 0 \\ \textit{and}\ \ 0b+d &= 1 \end{aligned}$$

Therefore $a=-1, b=0, c=0$, and $\text{d}=1$ resulting in the transformation matrix below.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

g. Students should use matrix multiplication to verify their transformation matrix as shown below:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

h. Students repeat Parts **c–g** using a different reflection from Part **b**. The resulting transformation matrices for r_x, r_l, and r_m are listed below.

$$\text{r}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{r}_l = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{r}_m = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Discussion 2

(page 61)

a. Sample response: The sheet of acetate represents the plane that contains ΔABC. When the image $\Delta A'B'C'$ is repositioned, the entire plane (sheet of acetate) is repositioned in the same manner.

b. Sample response: When ΔABC is reflected in the y-axis, the images of the vertices are $A'(0,0)$, $B'(-2,0)$, and $C'(0,1)$, respectively. Matrix multiplication resulted in the following equation:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a(0)+c(0) & a(2)+c(0) & a(0)+c(1) \\ b(0)+d(0) & b(2)+d(0) & b(0)+d(1) \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the values of a, b, c, and d are -1, 0, 0, and 1, respectively.

c. Sample response: Graph ΔABC and its image under the desired transformation. Then substitute the image matrix for the product matrix in the following equation and solve for a, b, c, and d.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$$

d. Sample response: Use ΔABC from the exploration to determine the transformation matrix necessary to reflect it in the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since this is also the transformation matrix for any object in the plane, the matrix equation for the given figure is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & 5 \end{bmatrix}$$

Teacher Note

In Exploration **2**, students determine transformation matrices that result in rotations about the origin for various angle measures. You may wish to assign particular angle measures to each student.

Exploration 2 (page 61)

a–c. Using the process described in Exploration **1**, students determine the transformation matrix necessary to create the appropriate image matrix. Their transformation matrices should be of the form below, where θ is the angle of rotation:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

d. Students repeat Parts **a–c** using angle measures of their choice.

Discussion 3 (page 61)

a. Sample response: The element in the first row, first column was equal to the cosine of the angle of rotation, as was the element in the second row, second column. The element in the first row, second column was equal to the additive inverse of the sine of the angle of rotation and the element in the second row, first column was equal to the sine of the angle of rotation.

b. Sample response: No. Both degrees and radians measure the amount of rotation so the general form of the matrix would be the same.

c. Sample response: Using the preimage and image matrices, it is possible to determine the transformation matrix using the method outlined in Exploration **1**. Since the element in the first row, first column of the transformation matrix represents the cosine of the angle of rotation, the angle of rotation can be found using the inverse cosine function. The inverse sine function could also be used by considering the element in the second row, first column.

Assignment (page 62)

1.1 **a.** The rotation matrix is:

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

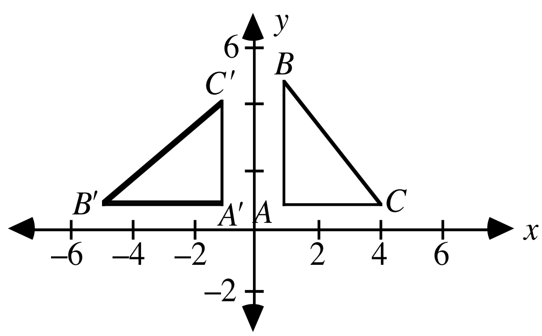
b. Sample response:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 & 1 & 4 \\ 1 & 5 & 1 \end{bmatrix}$$

c. The vertices of the image can be expressed in the matrix below:

$$\begin{bmatrix} -1 & -5 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

d. Sample graph:



***1.2** Students should be encouraged to use the reference triangle with vertices at $A(0,0)$, $B(2,0)$, and $C(0,1)$ to determine these matrices.

a. $r_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b. $R_{O,270^\circ} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

c. $R_{O,-270^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

d. $r_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

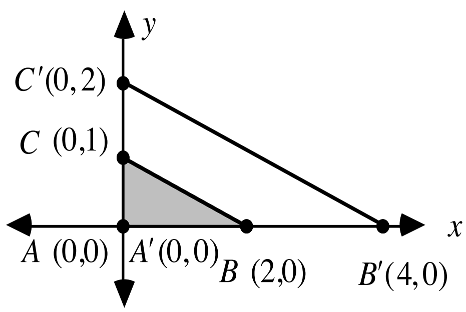
e. $r_l = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

f. $R_{O,26^\circ} = \begin{bmatrix} \cos 26^\circ & -\sin 26^\circ \\ \sin 26^\circ & \cos 26^\circ \end{bmatrix} \approx \begin{bmatrix} 0.899 & -0.438 \\ 0.438 & 0.899 \end{bmatrix}$

g. $R_{O,-78^\circ} = \begin{bmatrix} \cos(-78^\circ) & -\sin(-78^\circ) \\ \sin(-78^\circ) & \cos(-78^\circ) \end{bmatrix} \approx \begin{bmatrix} 0.208 & 0.978 \\ -0.978 & 0.208 \end{bmatrix}$

***1.3** Students use rotations and reflections to transform a simple figure, then create a flip-card animation by taping the images to index cards.

1.4 **a.** The following sample graph uses a reference triangle with vertices at $A(0,0)$, $B(2,0)$, and $C(0,1)$.



b. The 2×2 transformation matrix for a dilation by a scale factor of 2, with center at the origin, is:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

1.5 **a.** Answers will vary. Students should mention a dilation with a scale factor greater than 1. Some students also may include translations or rotations.

b. Students should mention a dilation with a scale factor between 0 and 1. Some students also may include translations or rotations.

* * * * *

1.6 **a.** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \bullet \begin{bmatrix} 1.5 & 1.9 & 2.5 & 2.9 \\ 2 & 3.1 & 2 & 3.1 \end{bmatrix} = \begin{bmatrix} 4.5 & 5.7 & 7.5 & 8.7 \\ 6 & 9.3 & 6 & 9.3 \end{bmatrix}$

c. The area of the book in the enlarged photograph is 9 times that of the original.

1.7 The coordinates listed in the problem may be used as reference points in the preimage. A dilation with a scale factor of 15 and center at the origin results in the following coordinates for the image: (–45,0), (–22.5,105), (0,90), (22.5, 105), (45,0), and (0,45).

$$\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix} \bullet \begin{bmatrix} -3 & -1.5 & 0 & 1.5 & 3 & 0 \\ 0 & 7 & 6 & 7 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -45 & -22.5 & 0 & 22.5 & 45 & 0 \\ 0 & 105 & 90 & 105 & 0 & 45 \end{bmatrix}$$

* * * * * * * * * *

(page 64)

Activity 2

In this activity, students discover that it is possible to perform all transformations in the plane using matrix multiplication by a 3×3 transformation matrix.

Materials List

- centimeter graph paper (several sheets per student)
- straightedge (one per student)

Technology

- symbolic manipulator (optional)

Teacher Note

You may wish to allow students to use a symbolic manipulator to perform the matrix multiplications in this activity.

Exploration

(page 64)

To generate a translation using multiplication of 3×3 matrices, students express points in homogeneous form. This concept is then extended to other transformations.

a. Since the dimensions of the preimage matrix are 2×1, the dimensions of the transformation matrix must be 2×2.

b. In general, a 2×2 matrix multiplied with the matrix of a point (x,y) gives:

$$\begin{bmatrix} q & r \\ s & t \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} qx + ry \\ sx + ty \end{bmatrix}$$

If the product matrix above is equal to the image matrix of the translation

$$\begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

then q must equal 1 and ry must equal a. The only way that ry can equal a is if y is a constant—which it is not. A similar argument applies using the second row of both matrices. Therefore, there is no 2×2 matrix that can be multiplied by the matrix of a point (x,y) to produce a translation.

c. Since the dimensions of the preimage matrix are 3×1, the dimensions of the transformation matrix must be 3×3.

d. The transformation matrix may be found by solving the following matrix equation:

$$\begin{bmatrix} e & f & g \\ h & i & j \\ k & l & m \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ex + fy + g \\ hx + iy + j \\ kx + ly + m \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix}$$

This implies that $ex + fy + g = x + a$, which is true when $e = 1$, $f = 0$, and $g = a$. Similarly, $hx + iy + j = y + b$ when $h = 0$, $i = 1$, and $j = b$, while $kx + ly + m = 1$ when $k = 0$, $l = 0$, and $m = 1$. Therefore, the 3×3 matrix that results in the translation of the point (x,y) expressed in homogeneous form is:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

e. Students should determine that the 3×3 matrix necessary to reflect a point in the x-axis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

f. Students should determine that the 3×3 matrix necessary to dilate a point by a scale of factor of n, with center at the origin, is:

$$\begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Discussion

(page 65)

a. The 3×3 translation matrix is of the form below, where (a,b) is the translation vector with its endpoint at the origin:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

b. This portion of the matrix is the 2×2 identity matrix. It does not affect the *x*- and *y*-values and preserves orientation.

c. Sample response: The 3×3 transformation matrix that results in a reflection in the *x*-axis is the 2×2 matrix that results in the same transformation with a third row and third column added. The elements of the third row are 0, 0, and 1, respectively. The elements of the third column are also 0, 0, and 1.

d. The 3×3 transformation matrix that results in a dilation of scale factor *n* with center at the origin is the 2×2 matrix that results in the same transformation with a third row and third column added. The elements of the third row are 0, 0, and 1, respectively. The elements of the third column are also 0, 0, and 1.

e. Sample response: For the other transformations that could be represented by a 2×2 matrix, the 3×3 matrix is formed by adding a third row of elements 0, 0, and 1 and a third column of the elements 0, 0, and 1.

If this pattern continues, the following 3×3 matrix would represent a rotation of θ degrees with center at the origin:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

f. The following 3×3 matrix represents the identity transformation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

g. One advantage in representing transformations as 3×3 matrices is that all transformations, include translations, can be completed using matrix multiplication.

Assignment

(page 66)

2.1 **a.** The following matrix represents the vertices of the triangle in homogeneous form:

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

b. The following matrix represents a dilation by a scale factor of 4 with center at the origin:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. Sample response: The two matrices have the same form. In this case, $n = 4$.

2.2 Students verify that the transformation matrix identified in Part **e** of the previous discussion produces the appropriate image for a rotation of 30° with center at the origin.

2.3 **a.** The following matrix represents a dilation by a scale factor of k with center at the origin:

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The following matrix represents a reflection in the line $y = x$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

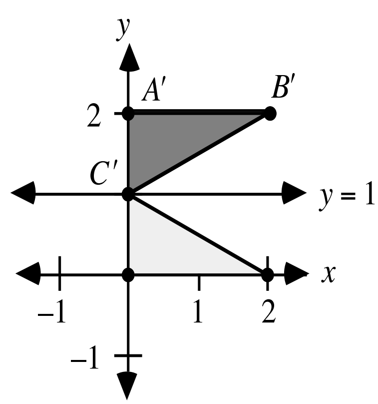
2.4 **a.** The transformation matrix for a reflection in the y-axis is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The transformation matrix for a reflection in the line $y = -x$ is:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.5 **a.** Sample graph:



b. The transformation matrix that creates this reflection is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: Students should be able to find this matrix using the process outlined in the exploration. In the next activity, students explore composition of transformations. Problem **3.7** describes how to determine such a matrix without graphing the preimage and image.

* * * * *

2.6 **a.** The transformation matrix that results in a reflection in the x-axis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The transformation matrix that results in a translation of 2 units vertically is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

c. **1.** Students multiply the matrix from Part **b** on the left of the matrix from Part **a**, as shown below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

The product matrix is the same as the transformation matrix found in Problem **2.5b**.

2. Sample response: The same final image is obtained by reflecting a figure in the line $y = 1$ as by reflecting the figure in the x-axis, followed by a vertical translation of 2 units.

2.7 **a.** The homogenous form of the ordered triple (x, y, z) is:

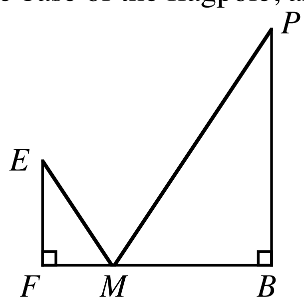
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

b. The matrix that translates the point (x, y, z) to $(x + a, y + b, z + c)$ is:

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* * * * *

2.8 **a.** In the sample sketch below, E represents the eye of the student, F represents the feet of the student, M represents the mirror, B represents the base of the flagpole, and P represents the top of the flagpole.



b. Sample response: ΔEFM is transformed by a reflection in a line perpendicular to $\overline{FB}$ at M. It is then dilated by an unknown scale factor with the center of dilation at M.

c. **1.** The coordinates of the preimage right triangle are (0,0), (–2,0), and (–2,1.7). The image found by a reflection in the y-axis is represented by the matrix equation below:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 1.7 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1.7 \\ 1 & 1 & 1 \end{bmatrix}$$

2. The equation that results in the dilation by a scale factor of 3 of the image from Step **1** is:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1.7 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 \\ 0 & 0 & 5.1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus, the ordered pair that represents the top of the flagpole is (6,5.1), and the height of the flagpole is 5.1 units.

3. The product of the two transformation matrices is:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. The product of the preimage matrix and the single transformation matrix in Step **3**, shown below, results in the same image as the final image from Step **2**.

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 1.7 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 \\ 0 & 0 & 5.1 \\ 1 & 1 & 1 \end{bmatrix}$$

* * * * * * * * * *

(page 68)

Activity 3

In this activity, students examine composite transformations and their corresponding matrices. Composite transformation techniques increase efficiency in computer animation by reducing the number of mathematical calculations required.

Materials List

- centimeter graph paper (several sheets per student)
- straightedge (one per student)

Technology

- geometry utility

Exploration

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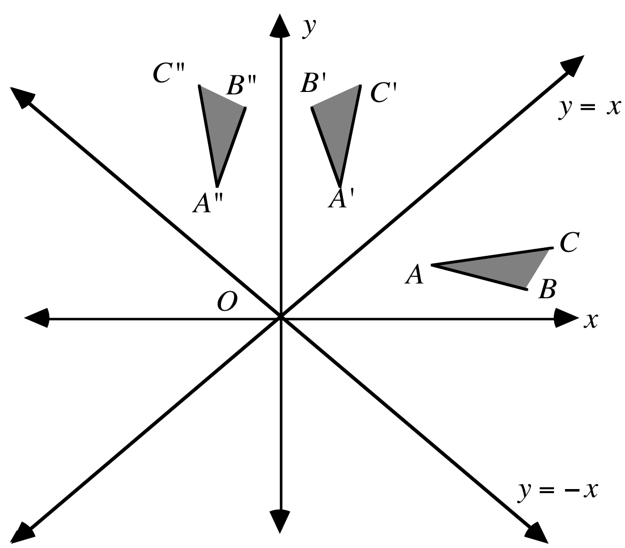
Students perform two reflections in intersecting lines and discover that this composition can be expressed as a single transformation. **Note:** To help distinguish among the images of multiple reflections, students may wish to use a different color for the interior of each image.

a. **1–5.** Sample graph:



6. The preimage and the final image have the same orientation.

7. The measures of $\angle AOA''$, $\angle BOB''$, and $\angle COC''$ are all 90°.

8. A rotation of 90° with center at the origin will also map ΔABC onto $\Delta A''B''C''$.

b. **1.** The transformation matrices for the two reflections, where l is the line $y = x$ and y is the y-axis, are shown below.

$$r_l = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. The transformation matrix for a rotation of 90° with center at the origin is:

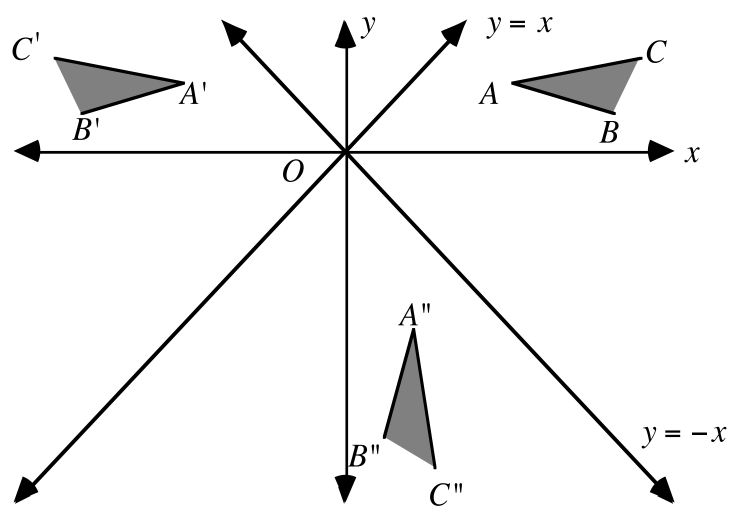
$$R_{O,90^\circ} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. The corresponding matrix equation is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. The two matrices are the same.

c. **1–4.** Answers will vary. The following sample graph shows a reflection in the y-axis followed by a reflection in the line $y = x$.



5. For the sample response given above, the measures of $\angle AOA''$, $\angle BOB''$, and $\angle COC''$ are all 90°.

6. Sample response: A rotation of 270° or –90° with center at the origin will also map the preimage onto the final image. **Note:** As some geometry utilities may measure the absolute value of the angle, you may wish to remind students that the angle of rotation is negative for a clockwise rotation.

d. The following responses correspond with the sample response given in Part **c**.

1. The transformation matrices for a reflection in the *y*-axis and a reflection in the line $y = x$ are shown below.

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_l = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. The transformation matrix for a rotation of 270° with center at the origin is:

$$R_{O,270^\circ} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. The corresponding matrix equation is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. The two matrices are the same.

e. **1.** The composition of two dilations with the same center is a dilation whose scale factor equals the product of the two scale factors of the original dilations.

2. The composition of two rotations with the same center is a rotation whose angle measure is the sum of the measures of the angles of the original rotations.

3. The composition of two translations is a translation whose resultant vector is the sum of the vectors of the two original translations.

Discussion

(page 70)

a. Changing the order of reflections changes the order of multiplication for the transformation matrices. As shown in the following example, this may result in a different product matrix and a different image since matrix multiplication is not commutative.

$$r_y \circ r_l = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r_l \circ r_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b.

1. Every pair of reflections in intersecting lines can be represented as a single rotation. The composition of any number of rotations can also be represented as a single rotation. Therefore, the image of any even number of reflections in intersecting lines can be expressed as a single rotation.

2. This would not be true for an odd number of reflections, since the preimage and image do not have the same orientation.

c.

1. The composition of two reflections in the same line results in an image that is exactly the preimage. This can also be represented as the identity transformation.

2. The result of a composition of two reflections in parallel lines is a translation.

d.

1. The composition of two translations is a translation whose resultant vector is the sum of the vectors of the two original translations.

2. If the two translations are represented by the matrices below,

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

then their composition results in one of the following transformation matrices:

$$\begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & c+a \\ 0 & 1 & d+b \\ 0 & 0 & 1 \end{bmatrix}$$

Since addition of real numbers is commutative, the order does not matter.

e. **1.** The composition of two dilations with the same center is a dilation whose scale factor equals the product of the two scale factors used in the original dilations.

2. If the two dilations are represented by the matrices below,

$$\begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then their composition results in one of the following transformation matrices:

$$\begin{bmatrix} n \bullet k & 0 & 0 \\ 0 & n \bullet k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} k \bullet n & 0 & 0 \\ 0 & k \bullet n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since multiplication of real numbers is commutative, the order does not matter.

f. **1.** The composition of two rotations with the same center is a rotation whose angle measure is the sum of the measures of the angles used in the original rotations.

2. The order of rotations does not affect the final image. Students should use an example to support their response.

g. It is more efficient to use compositions, since several transformations can be represented in a single matrix.

Assignment

(page 71)

3.1 **a.** Answers may vary. Sample response: A reflection in the x-axis followed by a reflection in the line $y = -x$ maps ΔCDE onto $\Delta C''D''E''$. Using the coordinates, $C(a,c)$, $D(a,b)$, and $E(d,b)$, the matrix equation is:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} a & a & d \\ c & b & b \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c & b & b \\ -a & -a & -d \\ 1 & 1 & 1 \end{bmatrix}$$

Note: Students may use numeric coordinates.

b. Sample response: A rotation of –90° with center at the origin maps ΔCDE onto $\Delta C''D''E''$. Using the coordinates, $C(a,c)$, $D(a,b)$, and $E(d,b)$, the matrix equation is:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} a & a & d \\ c & b & b \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c & b & b \\ -a & -a & -d \\ 1 & 1 & 1 \end{bmatrix}$$

c. In the matrix equation below, *m* represents the line $y = -x$, *x* represents the *x*-axis, and *O* represents the origin:

$$\mathrm{r}_m \circ \mathrm{r}_x = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{R}_{O,-90^\circ}$$

***3.2** In the following table, I represents the identity transformation.

B₂ (columns), B₁ (rows)

$\circ$	$\mathbf{r}_x$	$\mathbf{r}_l$	$\mathbf{R}_{0,90^\circ}$	$\mathbf{R}_{0,180^\circ}$	$\mathbf{R}_{0,270^\circ}$
$\mathbf{r}_x$	I	$\mathrm{R}_{0,90^\circ}$	r_l	r_y	r_m
$\mathbf{r}_l$	$\mathrm{R}_{0,270^\circ}$	I	r_y	r_m	r_x
$\mathbf{R}_{0,90^\circ}$	r_m	r_x	$\mathrm{R}_{0,180^\circ}$	$\mathrm{R}_{0,270^\circ}$	I
$\mathbf{R}_{0,180^\circ}$	r_y	r_m	$\mathrm{R}_{0,270^\circ}$	I	$\mathrm{R}_{0,90^\circ}$
$\mathbf{R}_{0,270^\circ}$	r_l	r_y	I	$\mathrm{R}_{0,90^\circ}$	$\mathrm{R}_{0,180^\circ}$

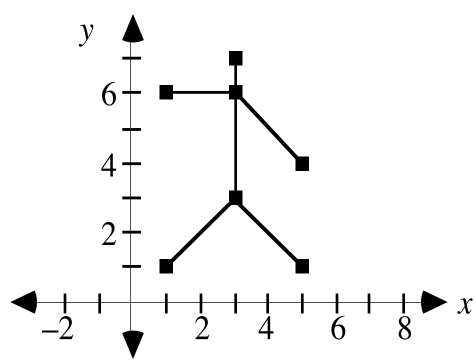
3.3 a–b. In both Parts **a** and **b**, the product is:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

c. Sample response: The answers are the same. Matrix multiplication is associative.

d. Sample response: If *y* represents the *y*-axis and *l* represents the line with equation $y = x$, the composition may be denoted as $\mathrm{r}_y \circ \mathrm{r}_l$.

***3.4 a–b.** Sample graph with vertices (3,7), (3,6), (1,6), (5,4), (3,3), (1,1), and (5,1):



c. This composition is a translation which maps (0,0) to (5,–1).

d. $T_{A,B} \circ T_{C,D} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

e. Using the sample ordered pairs from Part **a**, the image matrix is:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 3 & 3 & 1 & 5 & 3 & 1 & 5 \\ 7 & 6 & 6 & 4 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 6 & 10 & 8 & 6 & 10 \\ 6 & 5 & 5 & 3 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3.5 Using the sample ordered pairs from Problem **3.4a**, the image matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} \cos 285^\circ & -\sin 285^\circ & 0 \\ \sin 285^\circ & \cos 285^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 3 & 3 & 1 & 5 & 3 & 1 & 5 \\ 7 & 6 & 6 & 4 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 7.54 & 6.57 & 6.05 & 5.16 & 3.67 & 1.22 & 2.26 \\ 1.09 & 1.34 & -0.59 & 3.79 & 2.12 & 0.74 & 4.57 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3.6 **a.** Answers will vary. Sample response: The vertices $A(3,1)$, $B(6,1)$, and $C(6,2)$ of the preimage ΔABC may be mapped to $\Delta A'''B'''C'''$ using the following transformations: $D_{O,2} \circ R_{O,-60^\circ} \circ r_x$. The matrix equation that gives the coordinates of the image is:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) & 0 \\ \sin(-60^\circ) & \cos(-60^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 3 & 6 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.27 & 4.27 & 2.54 \\ -6.20 & -11.39 & -12.39 \\ 1 & 1 & 1 \end{bmatrix}$$

b. Sample response:

$$D_{0,2} \circ R_{O,-60^\circ} \circ r_x \approx \begin{bmatrix} 1 & 1.732 & 0 \\ 1.732 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.7 **a.** The angle formed by the line $y = 2x$ and the x-axis is $\tan^{-1}(2/1) \approx 63^\circ$.

b. The matrix required to rotate the line $y = 2x$ so that it coincides with the x-axis is:

$$\begin{bmatrix} \cos(-63^\circ) & -\sin(-63^\circ) & 0 \\ \sin(-63^\circ) & \cos(-63^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. The matrix that produces a reflection in the x-axis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d. The matrix that will return the line $y = 2x$ to its original position must rotate the plane by the additive inverse of the angle of rotation used in Part **a**:

$$\begin{bmatrix} \cos(63^\circ) & -\sin(63^\circ) & 0 \\ \sin(63^\circ) & \cos(63^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e. The matrix for the composition that will produce the same result as the three transformations described in Parts **b–d** is found by multiplying the three transformation matrices as follows:

$$\begin{bmatrix} \cos(63^\circ) & -\sin(63^\circ) & 0 \\ \sin(63^\circ) & \cos(63^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} \cos(-63^\circ) & -\sin(-63^\circ) & 0 \\ \sin(-63^\circ) & \cos(-63^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.59 & 0.81 & 0 \\ 0.81 & 0.59 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The image is the product of the composition matrix and the preimage matrix:

$$\begin{bmatrix} -0.59 & 0.81 & 0 \\ 0.81 & 0.59 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1.18 & 0.81 \\ 0 & 1.62 & 0.59 \\ 1 & 1 & 1 \end{bmatrix}$$

f. Sample response: The coordinates of the vertices for the reflection of ΔABC in the line $y = 2x$ found by graphing are approximately (0,0), (–1.2,1.6), and (0.8,0.6). These are approximately equal to the coordinates found in Part **e**.

3.8 **a.** The composite matrix is:

$$T_{(-2,3)} \circ R_{0,45^\circ} \circ T_{(2,-3)} \approx \begin{bmatrix} 0.71 & -0.71 & 1.54 \\ 0.71 & 0.71 & 2.29 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The image under the composition in Part **a** is:

$$\begin{bmatrix} 1.54 & 2.95 & 0.83 \\ 2.29 & 3.71 & 3.00 \\ 1 & 1 & 1 \end{bmatrix}$$

c. Sample response: The coordinates of the vertices for the rotation of ΔABC with a center at (–2,3) found by graphing are approximately (1.6,2.3), (2.9,3.8), and (0.9,3.0). These are approximately equal to the coordinates found in Part **b**.

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3.9 **a.** The composite matrix is:

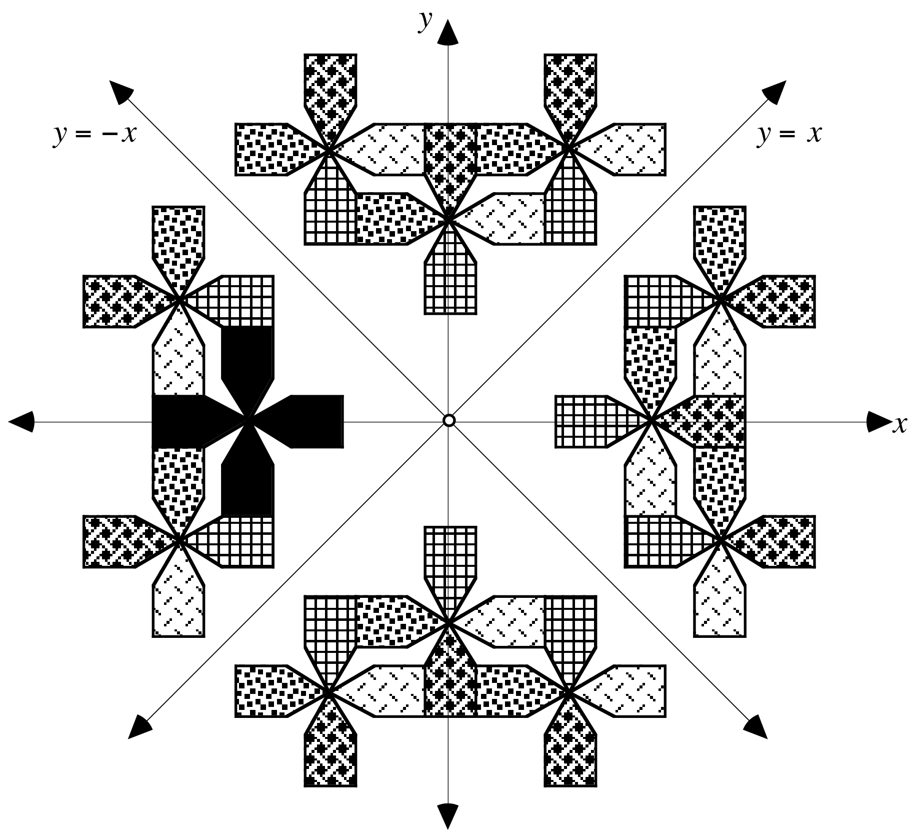
$$T_{(0,5)} \circ R_{0,-71.6^\circ} \circ r_x \circ R_{0,71.6} \circ T_{(0\text{-}5)} \approx \begin{bmatrix} -0.80 & -0.60 & 2.99 \\ -0.60 & 0.80 & 0.99 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The image under the composition in Part **a** is:

$$\begin{bmatrix} 2.99 & 1.39 & 2.39 \\ 0.99 & -0.20 & 1.80 \\ 1 & 1 & 1 \end{bmatrix}$$

c. Sample response: The coordinates of the vertices for the rotation of ΔABC with a center at (–2,3) found by graphing are approximately (3,1), (1.4,–0.3), and (2.4,1.9). These are approximately equal to the coordinates found in Part **b**.

3.10 Answers will vary. Sample response: I designed the black figure in the following design using an equilateral triangle sharing the side of a square. I rotated it 90˚ about the vertex of the triangle not on the side of the square and repeated this process two more times. This gave me a basic 4-leafed “flower.” I shaded each of the 4 leaves with a different pattern. I translated the flower twice by vectors to create the 3-flower design. I reflected the result in the *y*-axis. I then rotated the result 90˚ about the origin, and finally reflected that result in the *x*-axis.



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Activity 4

This activity introduces students to the glide reflection, a composite transformation consisting of a reflection and a translation parallel to the line of reflection.

Materials List

- centimeter graph paper (several sheets per student)
- straightedge (one per student)

Technology

- geometry utility

Exploration 1

(page 75)

a. Students create ΔABC using a graphing utility and perform a glide reflection using the translation followed by the reflection.

b. Students repeat the glide reflection using a reflection followed by a translation. They then compare their results with the image in Part **a**.

Discussion 1 (page 75)

a. Sample response: No. The resulting image is the same, regardless of the order in which the translation and reflection are performed.

b. Sample response: It is the opposite direction. This is because the reflection changes the orientation once and the translation does not change it.

c.
1. The two lines must be parallel.
2. The two parallel lines are perpendicular to the original line of reflection.

Teacher Note

In the following exploration, students express transformations as the composition of reflections. To save time, you may wish to assign one transformation in Part **b** to each group, then omit Part **d**.

Exploration 2 (page 76)

a. Students use a geometry utility to create a quadrilateral to be transformed. Its vertices should be labeled so that corresponding vertices can be easily identified later.

b. Students perform a selected transformation on their quadrilateral.

c. Students determine the number of reflections that must be composed to result in the same image as the selected transformation.

d. Students repeat the exploration for another transformation in Part **b**.

e. Students determine the number of reflections necessary to produce any congruent image of their quadrilateral.

Discussion 2 (page 77)

a. All four transformations listed in Part **b** of Exploration **2**—reflections, rotations, translations, and glide reflections—can be expressed as a composition of reflections.

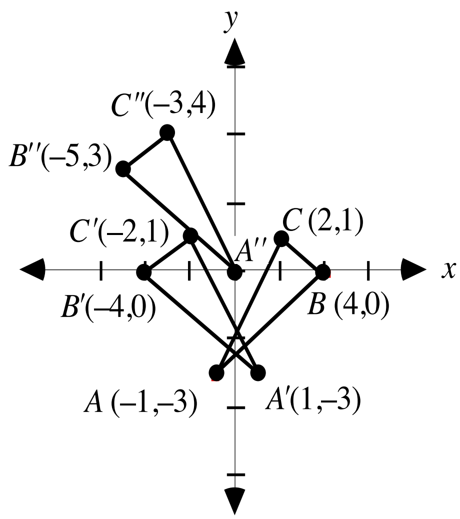
b. Sample response: Yes, the transformation could be expressed as a composition of three reflections.

c. Any transformation whose image is congruent to the preimage can be expressed as a composition of reflections in no more than three lines.

d. Sample response: No. Dilations cannot be expressed as a composition of reflections. This is because reflections result in congruent images and the images under dilations are similar.

e. By only using reflections to perform transformations, a programmer can write a single procedure that can be used repeatedly. Otherwise, a different procedure must be written for each type of transformation.

4.1 **a–c.** Sample graph:



d. $T_{P,P'} \circ r_y$

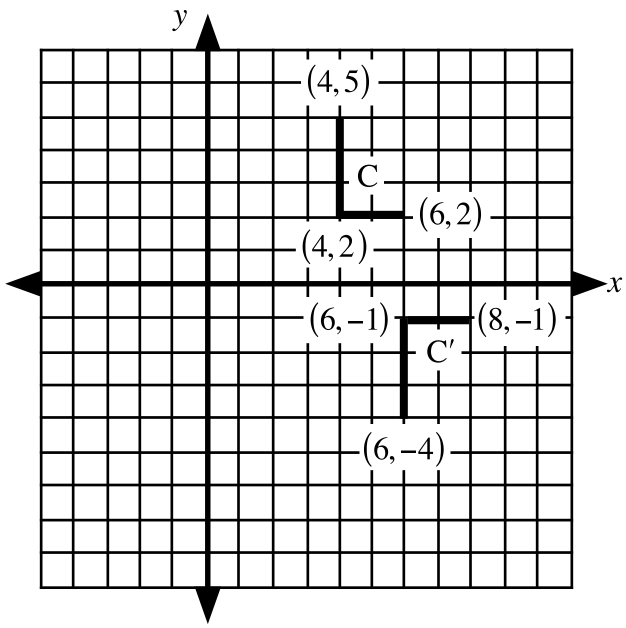
e. Sample response:

$$T_{P,P'} \circ r_y \circ \Delta ABC = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 4 & 2 \\ -3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

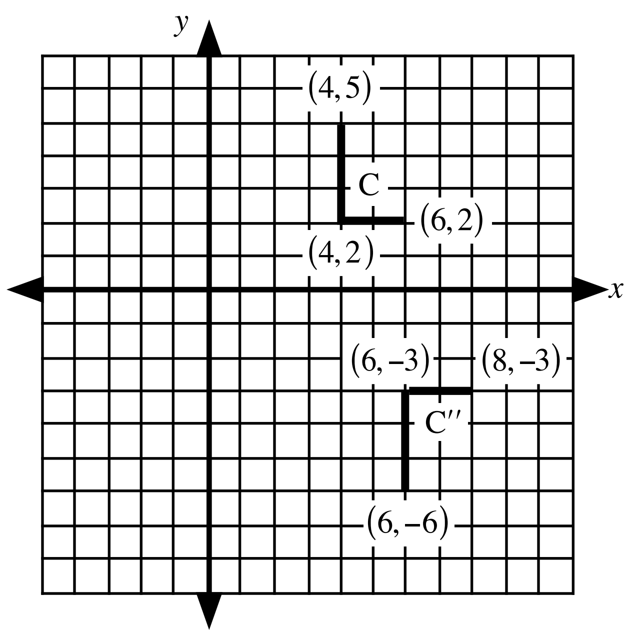
$$= \begin{bmatrix} 0 & -5 & -3 \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

The resulting matrix represents the coordinates of the vertices for $\Delta A''B''C''$.

***4.2** **a.** **1–3.** Sample graph:



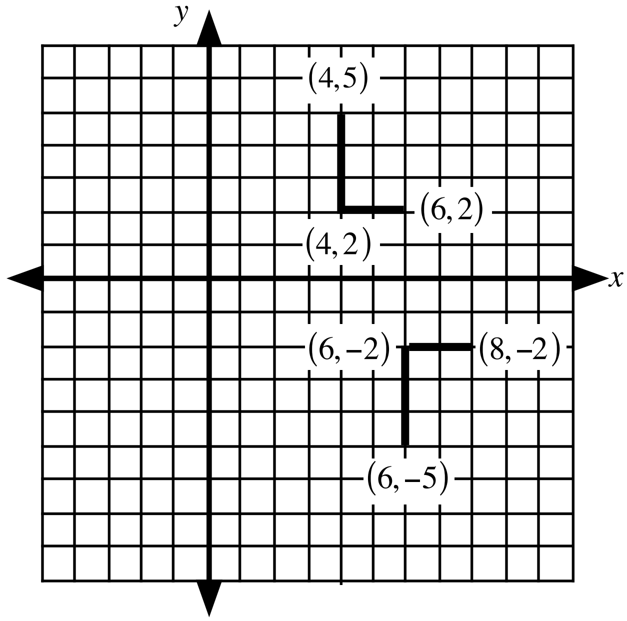
4. Sample graph:



5. The order in which the transformations are performed makes a difference in the positions of the final images, as illustrated in the graphs of C' and C'' above.

b. Since the translation vector (2 units horizontally and 1 unit vertically) is not parallel to the line of reflection (the *x*-axis), this is not a glide reflection.

***4.3** **a.** Sample graph:



b. The graph is the same as in Part **a**.

c. Sample response: No. This composition is a glide reflection, because the translation is parallel to the line of reflection. The order of transformations in a glide reflection does not affect the image.

d. Sample responses using the figure from Problem **4.2**:

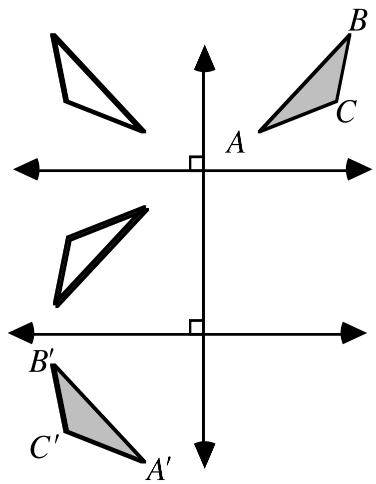
$$T_{S,S'} \bullet r_x \bullet \mathbf{L} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 4 & 4 & 6 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 8 \\ -5 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$r_x \bullet T_{S,S'} \bullet \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 4 & 4 & 6 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 8 \\ -5 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4 **a.** In the following sample sketch, the shaded triangle, ΔABC, is the preimage and $\Delta A'B'C'$ is the image under a glide reflection. The unshaded triangles are intermediate images when the glide reflection is shown as the composition of three reflections.



b. Sample response: Two of the lines are parallel to each other and are perpendicular to a third line. **Note:** Students may find a set of three reflection lines that do not satisfy these conditions. However, it is possible to prove that any glide reflection can be expressed as a composition of three reflections: two in parallel lines and one in a line perpendicular to the parallel lines.

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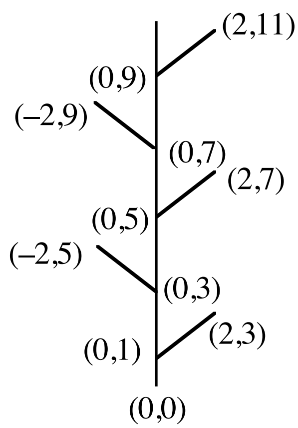
4.5 Sample response: To design a figure that resembles the tree twig, start by drawing a vertical line segment for the stem and label it as the *y*-axis of a coordinate plane. Next draw a segment that represents one of the branches and assign coordinates to the endpoints of the segment. List the ordered pairs in matrix form. Then multiply the matrix representing the segment by the appropriate transformation matrices. Using (0,1) and (2,3) as the endpoints of the original segment under the composition $T_{S,S'} \circ r_y$ where *y* is the *y*-axis, point *S* has coordinates (0,0) and S' has coordinates (0,2):



The second segment on the tree has coordinates (–2,5) and (0,3), as found by the following matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 5 & 3 \\ 1 & 1 \end{bmatrix}$$

The third segment on the tree has coordinates (2,7) and (0,5):

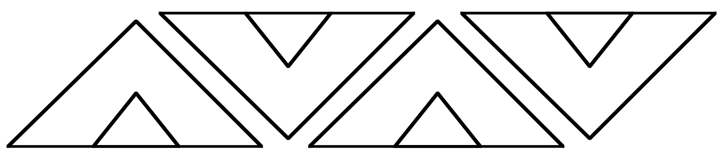
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -2 & 0 \\ 5 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 7 & 5 \\ 1 & 1 \end{bmatrix}$$

Similarly the fourth segment on the tree has coordinates (0,7) and (–2,9); while the fifth segment has coordinates (0,9) and (2,11).

4.6 Sample response: The following border was designed using the matrix **A** to represent the original figure of a triangle within a triangle.

$$\mathbf{A} = \begin{bmatrix} -5 & 0 & 5 & 2 & 0 & -2 \\ 0 & 5 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The triangles were then transformed by $T_{A,A'} \circ r_x$, where x represents the x-axis, point A has coordinates (0,0) and A' has coordinates (6,5).



The matrix equation representing this glide reflection is shown below:

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -5 & 0 & 5 & 2 & 0 & -2 \\ 0 & 5 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 11 & 8 & 6 & 4 \\ 5 & 0 & 5 & 5 & 3 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The other triangles were found in a similar manner.

* * * * * * * * * *

Answers to Summary Assessment (page 80)

1–2. Students should include at least one example of each of the following transformations: reflections, rotations, dilations, translations, and glide reflections.

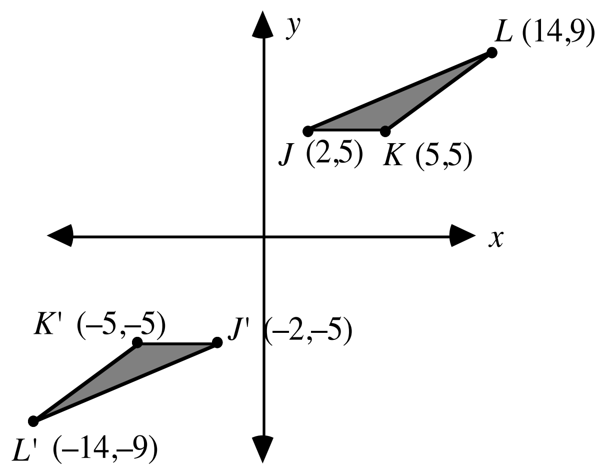
3. **Note:** If students use complicated preimages, then you may wish to allow them to indicate each image using only a few selected points.

4. The product of this transformation matrix and the preimage matrix should represent the matrix of the final image in Problem **3**.

Module Assessment

In each of the following problems, one or more transformations have been performed on ΔJKL to obtain its image $\Delta J'K'L'$. All reflections have been done in the lines $y = x$, $y = -x$, the x-axis, or the y-axis. All rotations and dilations have been performed with the origin as the center.

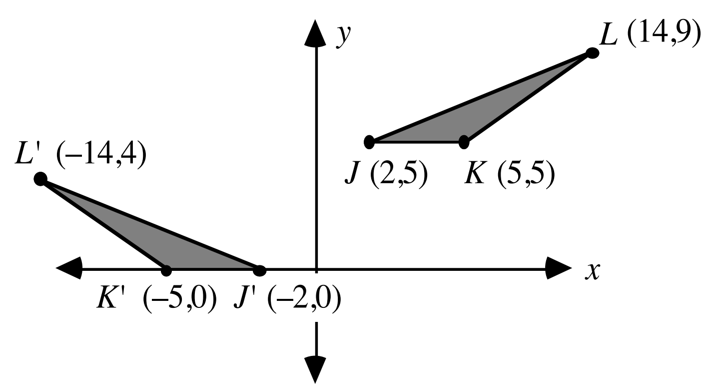
1. Use the following diagram to answer Parts **a** and **b**.



a. Use appropriate notation to indicate the transformation or composition of transformations performed on ΔJKL to obtain its image $\Delta J'K'L'$.

b. Write a matrix equation using 3×3 transformation matrices that shows the relationship between the matrix for the preimage, the transformation matrices, and the matrix for the image.

2. Use the following diagram to answer Parts **a–c**.

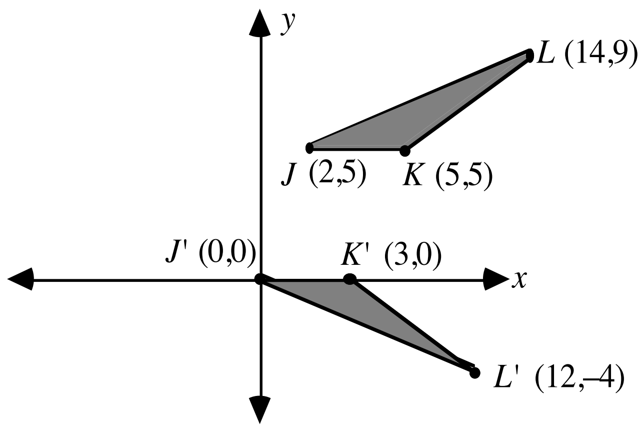


a. Use appropriate notation to indicate the transformation or composition of transformations performed on ΔJKL to obtain its image $\Delta J'K'L'$.

b. Determine a 3×3 matrix that accomplishes this transformation or composition of transformations.

c. Write a matrix equation using a 3×3 transformation matrix that shows the relationship between the matrix for the preimage, the transformation matrix, and the matrix for the image.

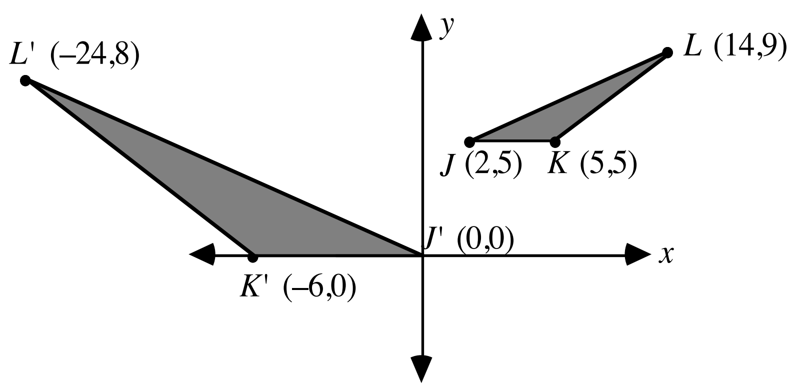
3. Use the following diagram to answer Parts **a** and **b**.



a. Use appropriate notation to indicate a composition of transformations that would generate the image $\Delta J'K'L'$ from ΔJKL.

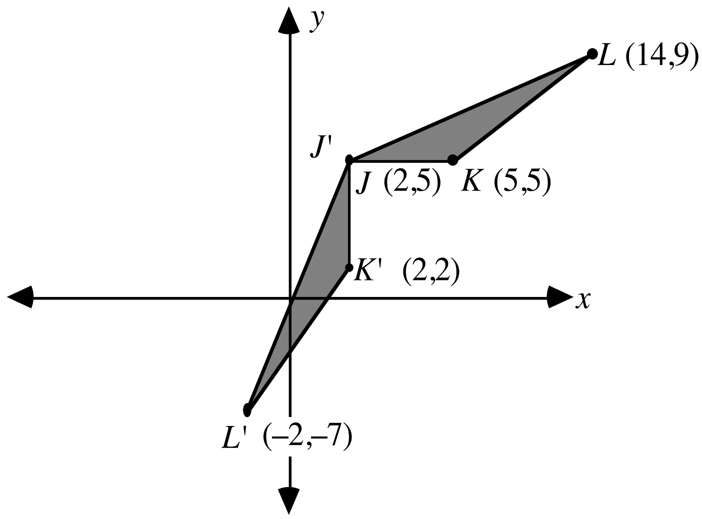
b. Determine a 3×3 transformation matrix that accomplishes this composition.

4. Write a matrix equation that generates the matrix for the image $\Delta J'K'L'$ in the following diagram.



5. When a series of transformations is performed on the preimage ΔJKL from Problem **4**, the image that results has the vertices J' (–2,9), K'(–2,6), and L' (–6,–3). Write a matrix equation containing a 3×3 transformation matrix that will result in this image.

6. Write a matrix equation that generates the matrix for the image $\Delta J'K'L'$ in the following diagram.



Answers to Module Assessment

1. **a.** Answers may vary. The image may be the result of a rotation of 180° about the origin, denoted $R_{O,180^\circ}$, or a reflection in the *x*-axis, followed by a reflection in the *y*-axis, denoted $r_y \circ r_x$.

b. Matrix equation for $R_{O,180^\circ}$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -14 \\ -5 & -5 & -9 \\ 1 & 1 & 1 \end{bmatrix}$$

Matrix equation for $r_y \circ r_x$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -14 \\ -5 & -5 & -9 \\ 1 & 1 & 1 \end{bmatrix}$$

2. **a.** Answers may vary. Sample response: The image is the result of the composition $r_y \circ T_{J,A}$, where *A* is the point (2,0) and *y* is the *y*-axis.

b. The 3×3 transformation matrix which corresponds with the sample response in Part **a** is:

$$r_y \circ T_{J,A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

c. The matrix equation below shows the relationship between the preimage matrix, the transformation matrix, and the image matrix:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -14 \\ 0 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

3. **a.** Answers may vary. Sample response: The image is the result of the composition $r_x \circ T_{J,J'}$, where *x* is the *x*-axis.

b. The 3×3 transformation matrix that represents $r_x \circ T_{J,J'}$ is:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Answers may vary. Sample response: The image is the result of the composition $D_{O,2} \circ r_y \circ T_{J,J'}$, where *y* is the *y*-axis and *O* is the origin. The transformation matrix is:

$$D_{O,2} \circ r_y \circ T_{J,J'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 4 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix equation that generates the matrix for the image is:

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -6 & -24 \\ 0 & 0 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

5. Students may find it helpful to graph the preimage and final image on the same coordinate system. One composition that results in the correct image is $T_{O,J'} \circ r_y \circ R_{O,-90^\circ} \circ T_{J,O}$, where *O* is the origin and *y* is the *y*-axis. The transformation matrix is:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix equation that generates the matrix for the image is:

$$\begin{bmatrix} 0 & -1 & 3 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -6 \\ 9 & 6 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

6. Answers may vary. Sample response: The image is the result of the composition $T_{A,A'} \circ r_l$, where *l* is the line $y = -x$, *A* is the point (0,0), and A' is the point (7,7). The transformation matrix is:

$$T_{A,A'} \circ r_l = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 7 \\ -1 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix equation that generates the matrix for the image is:

$$\begin{bmatrix} 0 & -1 & 7 \\ -1 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 5 & 14 \\ 5 & 5 & 9 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 2 & -7 \\ 1 & 1 & 1 \end{bmatrix}$$

Selected References

Fleischer, D. "Computer Animation: The Art of Hyper-Reality." *Science Digest* (February 1982): 46–53.

North Carolina School of Science and Mathematics, Department of Mathematics and Computer Science. *Matrices*. Reston, VA: National Council of Teachers of Mathematics, 1988.

Wedge, J. C. "Role of the Animator in the Generation of Three-Dimensional Computer-Generated Animation." ERIC, 1985. ED 282512.

Flashbacks

Activity 1

1.1 Solve each of the following systems of equations for *x* and *y*.

a. $\begin{cases} x + 2y = -2 \\ -3x + 4y = 0 \end{cases}$

b. $\begin{cases} -x + 2y = 2 \\ 3x + 4y = -12 \end{cases}$

1.2 Consider the triangle whose vertices are represented in the matrix below.

$$\begin{bmatrix} 2 & 3 & -3 \\ 0 & -2 & 5 \end{bmatrix}$$

Identify the transformation that results when this preimage matrix is multiplied by each of the following matrices:

a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.3 Use the matrices below to determine whether or not matrix multiplication is commutative.

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \bullet \begin{bmatrix} -5 & 7 \\ 1 & -4 \end{bmatrix}$$

Activity 2

2.1 Find the product of the matrices below:

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

2.2 **a.** Given an $m \times n$ matrix, what must the dimensions of a second matrix be for multiplication of the two matrices to be possible?

b. What are the dimensions of the product?

2.3 **a.** What 2×2 matrix results in r_x where x represents the x-axis?

b. What 2×2 matrix results in $\mathrm{R}_{O,60^\circ}$ where O is the origin?

Activity 3

3.1 Multiply the following matrices:

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 3 & 2 \\ -2 & -3 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 3 & -1 \\ -2 & -3 & 1 \\ -1 & 2 & 3 \end{bmatrix} \bullet \begin{bmatrix} -1 & -1 & 2 \\ 2 & -2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

3.2 Solve for each variable in the matrix expression below:

$$\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 9 \\ 0 & 2 & -16 \\ 1 & 1 & 1 \end{bmatrix}$$

3.3 Multiply the 3×3 transformation matrix that results in a reflection in the x-axis on the left by the 3×3 transformation matrix that results in a rotation of 60° about the origin.

Activity 4

4.1 Sketch the line of reflection that maps point A below to its image A' and describe how you identified this line.

4.2 Find a 3×3 transformation matrix that represents a reflection in the y-axis, followed by a clockwise rotation of 72°, followed by a dilation with center at the origin and a scale factor of 4.

4.3 Multiply the following matrices and explain the result.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} a & p & x \\ b & q & y \\ c & r & z \end{bmatrix}$$

Answers to Flashbacks

Activity 1

1.1 **a.** $x = -4/5$; $y = -3/5$

b. $x = -16/5$; $y = -3/5$

1.2 **a.** a reflection in the x-axis

b. a reflection in the line $y = x$

c. a dilation by a factor of 2 with the center at the origin

d. Sample response: There is no transformation. The matrix is the identity matrix.

1.3 As illustrated below, the products of the two matrices are different when the order is changed.

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \bullet \begin{bmatrix} -5 & 7 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 18 \\ -17 & 16 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 7 \\ 1 & -4 \end{bmatrix} \bullet \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ -14 & -13 \end{bmatrix}$$

This indicates that matrix multiplication is not commutative.

Activity 2

2.1 Some students may wish to use technology to find this product:

$$\begin{bmatrix} 8 & 17 & 26 \\ 5 & 14 & 23 \\ 5 & 14 & 23 \end{bmatrix}$$

2.2 **a.** To multiply on the right, the second matrix must have the dimensions $n \times p$. To multiply on the left, its dimensions must be $p \times m$.

b. The product of multiplication on the left has dimensions $m \times p$. The product of multiplication on the right has dimensions $p \times n$.

2.3 **a.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b. $\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$

Activity 3

3.1 The product is:

$$\begin{bmatrix} -1 & 21 & -13 \\ -6 & -6 & 2 \\ 7 & -15 & 11 \end{bmatrix}$$

3.2 Students may solve the expression using a symbolic manipulator, or set up the following system of equations and solve algebraically:

$$0a + 0b = 0 \quad a + 2c = 7 \quad -3a + 4c = 9$$
$$0b + 0d = 0 \quad b + 2d = 2 \quad -3b + 4d = -16$$

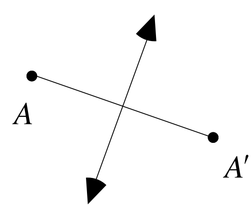
Solving the system yields $a = 1$, $b = 4$, $c = 3$, and $d = -1$.

3.3 The resulting matrix is:

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Activity 4

4.1 The line of reflection is the perpendicular bisector of $\overline{AA'}$.



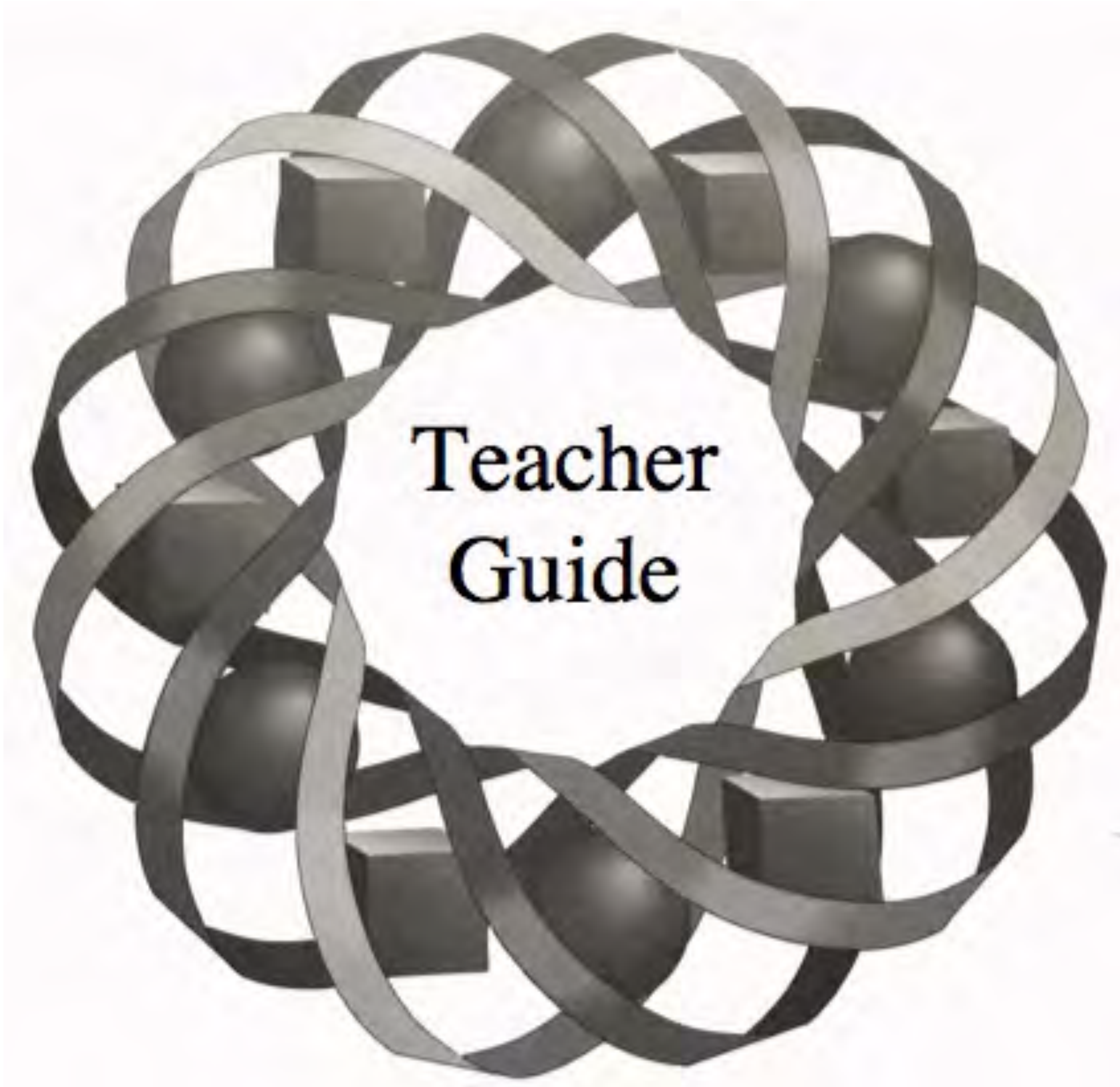
4.2 The resulting matrix is approximately:

$$\begin{bmatrix} -1.24 & -3.80 & 0 \\ -3.80 & 1.24 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.3 The first three matrices are multiplicative identity matrices. Since multiplication by the multiplicative identity matrix maps the preimage onto itself, the solution is:

$$\begin{bmatrix} a & p & x \\ b & q & y \\ c & r & z \end{bmatrix}$$

Drafting and Polynomials



Many computer-generated images are defined by polynomial curves. In this module, you use polynomial functions and their corresponding graphs to study some elements of graphic design.

Masha Albrecht • Tom Teegarden

Teacher Edition

Drafting and Polynomials

Overview

Students investigate polynomial functions and their corresponding graphs through the context of computer-assisted graphic design.

Objectives

In this module, students will:

- recognize and identify the graphs of polynomial functions
- use the distributive property to expand polynomials
- recognize the relationships among the zeros, degree, and factors of a polynomial function
- use quadratic and other functions and their graphs as mathematical models

Prerequisites

For this module, students should know:

- the distributive property of multiplication over addition
- the definition and properties of a function
- function notation
- interval notation
- the definitions of the domain and range of a function.

Time Line

Activity	**1**	**2**	**3**	**4**	**Summary Assessment**	**Total**
Days	2	3	2	2	2	11

Materials Required

Materials	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
straightedge	X				
graph paper	X	X	X		
thin, straight pasta or string-licorice candy		X	X		

Technology

Software	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
graphing utility	X	X	X	X	X
symbolic manipulator	X				

Drafting and Polynomials

Introduction

(page 85)

The introduction describes situations in which polynomial curves are used in graphic design. In the remainder of the module, students explore a simplified form of the process a computer graphics program might use to characterize a section of a curve. Such programs may employ a variety of strategies for coordinatizing the screen. For example, one program coordinatizes the screen pixel by pixel with the origin in the lower left-hand corner.

(page 86)

Activity 1

In this activity, students experiment with one method of fitting a line to a set of points and expressing it in factored form. (This method can be generalized for any polynomial function with all real roots.)

Materials list

- straightedge
- graph paper

Technology

- graphing utility
- symbolic manipulator

Teacher Note

When using a graphing utility, students should plot points so that both a scatterplot and a function can be illustrated simultaneously.

Exploration

(page 86)

a–b. Students should plot two points not on the same vertical line. (If both points are on the same vertical line, the line does not represent a function from x to y.) Sample response: (7,3) and (2,–6).

c. In the case of two points, the "smoothest, simplest" curve is a line. **Note:** Most graphics programs consider a curve "smooth" when it is continuous or can be drawn without lifting a pencil from the paper.

d. Answers will vary. The equation of the line that contains (7,3) and (2,–6) is:

$$y = \frac{9}{5}x - \frac{48}{5}$$

e. Student graphs should intersect the *x*-axis in only one point. The *x*-intercept of the line that contains the points (7,3) and (2,–6) is approximately 5.3.

f. Students express the equation from Part **d** in the form $y = m(x + b/m)$.

1. Sample response:

$$y = \frac{9}{5}\left(x - \frac{16}{3}\right)$$

2. The zero of the function is equal to $-b/m$. For the sample equation given above, the zero is $16/3$ or approximately 5.3.

Discussion

(page 87)

a. Sample response: Yes, there is exactly one value of *y* that corresponds to every value of *x*.

b. The degree of a linear function is 1.

c. The graph of function *g* intersects the *x*-axis at $x = 7$.

d. Students should suggest equations of the form $y = a(x - 12)$.

e. **1.** Sample response: Changing the value of a changes the slope of the line.

2. Sample response: Using the distributive property, $y = a(x - c)$ is equivalent to $y = ax - ac$. Interpreting this form as the slope-intercept form of a line means the value of a represents the slope of the line and the quantity $-ac$ is the *y*-intercept.

f. Sample response: An equation of the form $y = a(x - c)$ contains the zero *c*, as well as the slope a.

Assignment

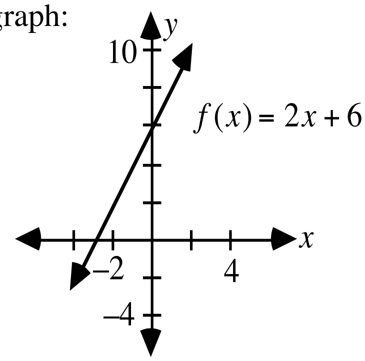
(page 89)

1.1 **a.** Using a method similar to that described in the exploration, the equation of the line in factored form is $y = 2(x - 0)$.

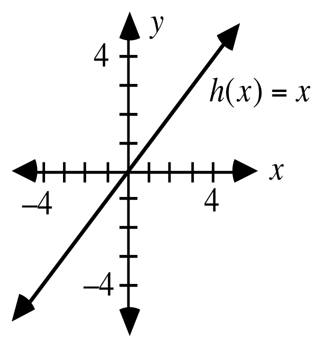
b. The root of the equation $y = 2(x - 0)$ is 0.

1.2 **a.** Since each function except $h(x)$ is written as a product of factors in the form $(x - c_n)$, students should predict that the zeros equal the values of *c*.

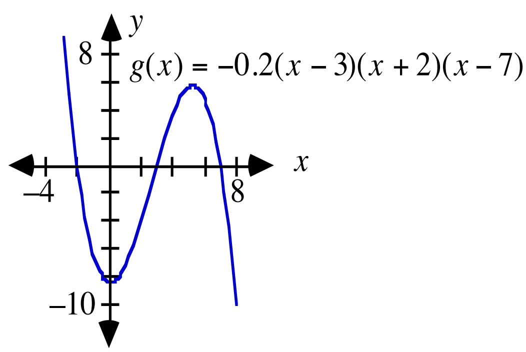
b–c. **1.** The function $f(x) = 2(x + 3)$ has a zero of –3 since $f(-3) = 0$. Sample graph:



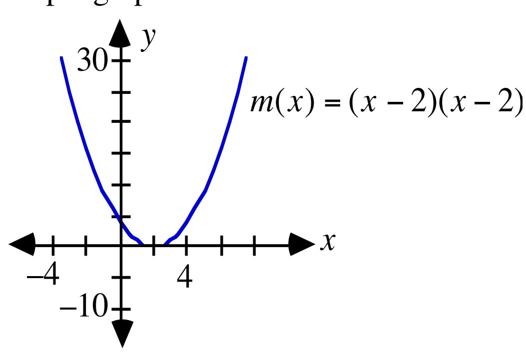
2. The function $h(x) = x$ has a zero of 0 since $h(0) = 0$. Sample graph:



3. The function $g(x) = -0.2(x - 3)(x + 2)(x - 7)$ has zeros of 3, –2, and 7 since $g(3)$, $g(-2)$, and $g(7)$ all equal 0. Sample graph:



4. The function $m(x) = (x - 2)(x - 2)$ has a zero of 2 since $m(2) = 0$. Sample graph:

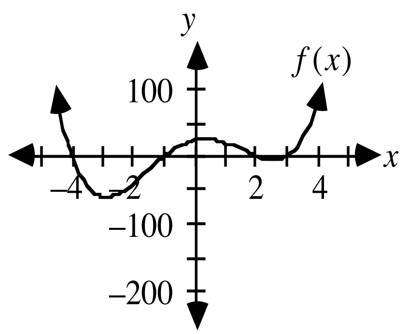


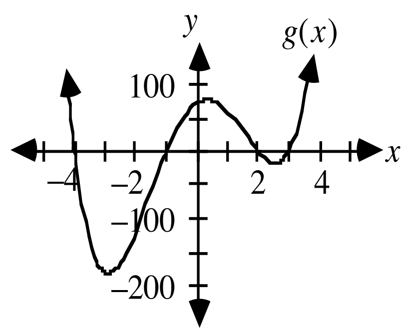
***1.3** Answers will vary. The following sample responses use the roots 3, 2, –1, and –4.

a. One polynomial function with roots of 3, 2, –1, and –4 is $f(x) = (x - 3)(x - 2)(x + 1)(x + 4)$.

b. Another polynomial function with the same roots is $g(x) = 3(x - 3)(x - 2)(x + 1)(x + 4)$.

c. The two graphs have the same *x*-intercepts and the same basic shape, but the coefficient of 3 makes the graph of $g(x)$ vary more from the *x*-axis in the interval $-4 < x < 3$.



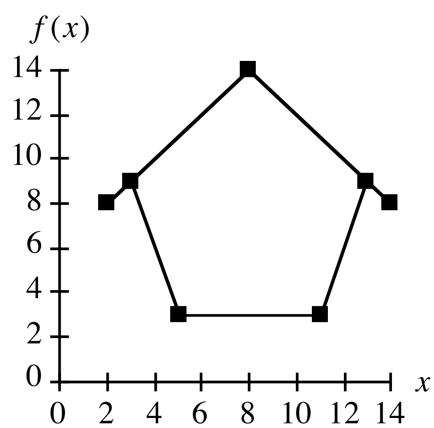


1.4 **a.** Using the distributive property, $3(x + 2) = 3(x) + 3(2) = 3x + 6$.

b. Using the distributive property:

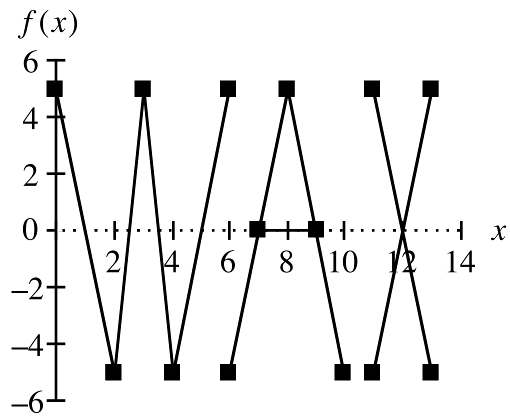
$$\begin{aligned}(x + 7)(x - 4) &= x(x - 4) + 7(x - 4) \\ &= x^2 - 4x + 7x - 28 \\ &= x^2 + 3x - 28\end{aligned}$$

1.5 Sample response: The roof can be formed by $f(x) = -x + 22$ for *x* in [8, 14] and $f(x) = x + 6$ for *x* in [2, 8]. The sides of the birdhouse can be formed by $f(x) = -3(x - 6)$ for *x* in [3, 5] and $f(x) = 3(x - 10)$ for *x* in [11, 13]. The floor can be formed by $f(x) = 3$ for *x* in [5, 11]. A graph of the design is shown below:



***1.6** Sample response: The functions that correspond to each letter are shown in the following table.

Letter	**Function**	**Domain**
W	$f(x) = -5(x-1)$	[0, 2]
W	$f(x) = 10(x-2.5)$	[2, 3]
W	$f(x) = -10(x-3.5)$	[3, 4]
W	$f(x) = 5(x-3.5)$	[4, 6]
A	$f(x) = -5(x-9)$	[8, 10]
A	$f(x) = 5(x-7)$	[6, 8]
A	$f(x) = 0$	[7, 9]
X	$f(x) = -5(x-12)$	[11, 13]
X	$f(x) = 5(x-12)$	[11, 13]



* * * * *

1.7 **a.** **1.** $y = 2x + 6$; degree 1

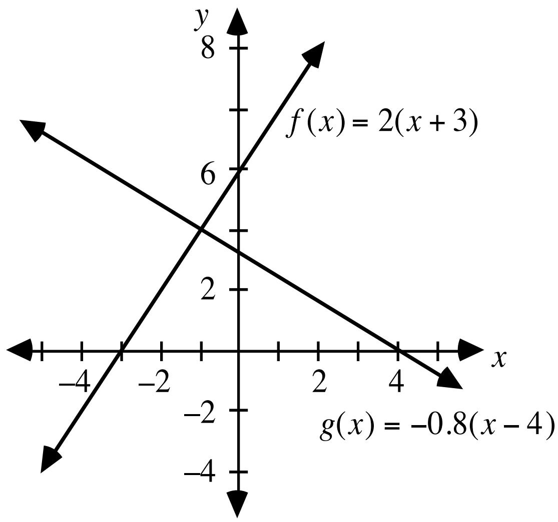
2. $y = -0.8x + 3.2$; degree 1

3. $y = x^2 - 4x + 4$; degree 2

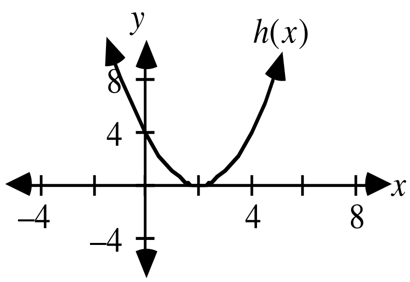
4. $y = 0.2x^3 - 1.6x^2 + 0.2x + 8.4$; degree 3

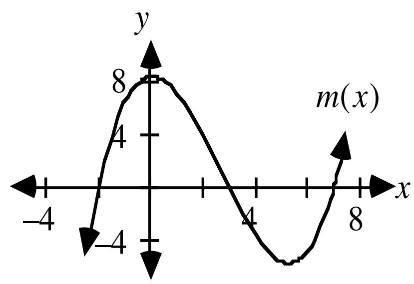
b. Students verify the products in Part **a** using a symbolic manipulator.

c. **1–2.** Sample graphs of $f(x) = 2x + 6$ and $g(x) = 0.8x + 3.2$ are shown below. In each case, the zero is the value of the constant c in the factor $(x - c)$.



3–4. Sample graphs of $m(x) = 0.2x^3 - 1.6x^2 + 0.2x + 8.4$ and $h(x) = x^2 - 4x + 4$ are shown below. In each case, the zeros are the values of the constants c in the factors $(x - c)$.





1.8 **a.** The polynomial function in w that represents the area of the sidewalk is $f(w) = (20 + 2w)(30 + 2w) - (20)(30)$.

b. Using the distributive property, the function can be simplified as follows:

$$\begin{aligned} f(w) &= (20 + 2w)(30 + 2w) - (20)(30) \\ &= 600 + 100w + 4w^2 - 600 \\ &= 4w^2 + 100w \end{aligned}$$

* * * * * * * * * *

(page 91)

Activity 2

Students experiment with connecting three noncollinear points using smooth curves.

Materials List

- graph paper
- straight thin pasta, soaked for about 1 hr (less time in hot water) or string licorice candy

Technology

- graphing utility

Teacher Note

In the exploration, students use soaked pasta as a model spline. String, pipe cleaners, or wire may be substituted for the pasta. However, pasta models curves extremely well and also sticks to the paper.

In Part **c**, each curve should represent a function and intersect the *x*-axis at two points. This ensures that all roots are real roots. In Parts **g** and **h**, you may wish to point out to students that it may not be possible to precisely model a curve formed with a spline using a single, simple function—even though the spline curve and the function have the same *x*-intercepts.

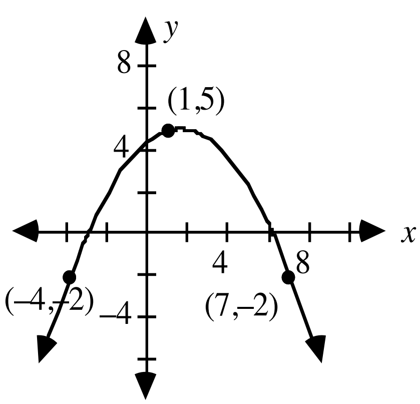
Exploration

(page 92)

a–b. Since three noncollinear points can always be connected with a parabola, students will be modeling the graphs of second-degree polynomials.

c. The curve formed by the spline should intersect the *x*-axis in two points.

d. The following sample graph uses the points (–4,–2), (1,5), and (7,–2):



e. Students approximate the two zeros of the polynomial. For the sample response given in Part **d**, the zeros are approximately –3.2 and 6.2.

f–h. The factors which correspond to the sample response given in Part **d** are $(x+3.2)$ and $(x-6.2)$. One function which approximates the three points is $f(x) = -0.23(x+3.2)(x-6.2)$.

Discussion

(page 93)

a. Sample response: A line is appropriate as a smooth curve when joining any two points, or any three or more collinear points.

b. Students should obtain second-degree polynomials. Although it is possible to fit higher-order polynomials to three points, the curves will contain unnecessary "bends."

c. Sample response: Yes. If the points are collinear, then a first-degree polynomial (a line) would model them.

d. Answers will vary. Students could subsitute one of the points to solve for the coefficient *a* in a polynomial of the form $y = a(x-c_1)(x-c_2)$.

e.
1. If $a \neq 0$ in a function of the form $f(x) = a(x-c_1)(x-c_2)$, changing the value of a changes the function without changing its zeros.
2. Changing $|a|$ stretches the graph vertically. If the sign of a is changed, the graph of the function is reflected in the *x*-axis.

f. Since projectiles follow parabolic trajectories, students may describe the paths of a thrown ball or a model rocket (after its engines burn out). Some brands of downhill skis are designed using parabolic curves, as are certain types of fishing rods. Cross sections of many satellite dish antennas and headlight reflectors also are parabolas. **Note:** Some students may suggest the curves of telephone or power lines as possible examples. Although this shape resembles a parabola, it is in fact a catenary—which is not defined by a polynomial function.

Assignment

(page 94)

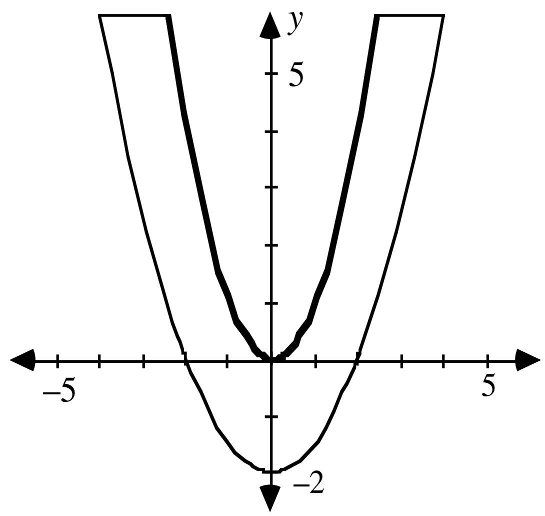
2.1

a. Students may use the trace feature on a graphing utility to find the coordinates of the vertex: (5,25).

b. Sample response: Because the vertex lies on the parabola's axis of symmetry, the *x*-coordinate of the vertex is the mean of the values of the two zeros. This value can be substituted into the equation to find the *y*-coordinate of the vertex.

c. The method described in Part **b** should work for any parabolic function with two real zeros.

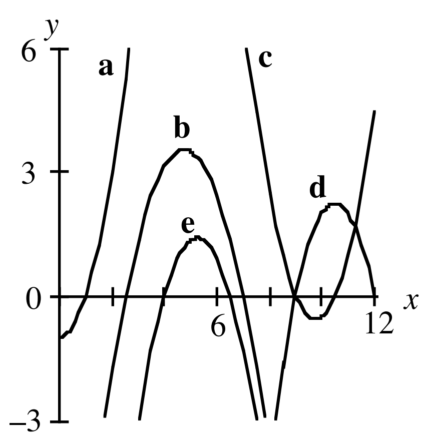
2.2 Answers will vary. The sample response below uses the following functions: $y = x^2$ for x in $\left[-\sqrt{6}, \sqrt{6}\right]$, $y = 0.5(x+2)(x-2)$ for x in $\left[-4, 4\right]$, and $y = 6$ for x in $\left[-4, -\sqrt{6}\right]$ and $\left[\sqrt{6}, 4\right]$.



Note: The shapes students create with parabolas may look more like the letter V than the letter U. They will have an opportunity to create another version in Activity **4**. The outline of the sample letter above contains only four distinct curves. The outline of the letter U shown in the student edition requires at least 12 equations to describe.

2.3 Sample response: The functions that correspond to each curve are shown in the following table.

Function	Domain
a. $f(x) = (x-1)(x+1)$	[0, 2.6]
b. $f(x) = -0.7(x-2.5)(x-7)$	[1.6, 7.8]
c. $f(x) = (x-9)(x-10.5)$	[7.2, 12]
d. $f(x) = -(x-9)(x-12)$	[8.2, 12]
e. $f(x) = -0.9(x-4)(x-6.5)$	[3, 7.5]



***2.4** **a.** One polynomial function *f* in *x* that represents the possible area of the rectangular design is $f(x) = (55 - 2x)(40 - 2x)$.

b. Sample response: The value of $f(x)$ represents the area of the rectangular design in square centimeters; the value of *x* represents the width of the border in centimeters.

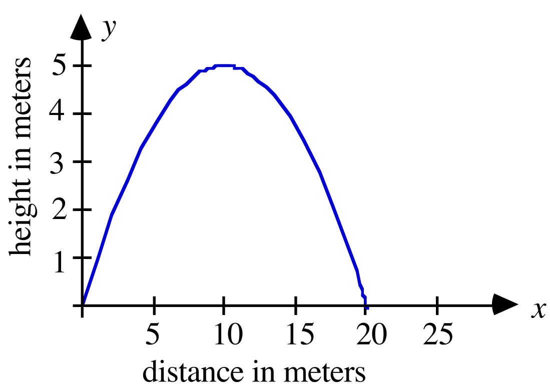
c. The domain of the function is approximated by the interval [0, 6.7]; its range is approximated by the interval [0, 1100].

* * * * *

2.5 **a.** Answers may vary. If students place the origin at the bottom of the grid, directly under the first image at the left, some points might be: (0,13), (3,12), (5.8,10), (10,3.8), (11.6,0).

b. Answers may vary. Using the sample points given in Part **a**, one possible model is $y = -0.1(x + 11.6)(x - 11.6)$.

2.6 **a.** Sample graph:



b. The zeros are 0 and 20; the coordinates of the vertex are (10,5).

c. Answers may vary. Using trial and error, one equation that fits reasonably well is $y = -0.05x(x - 20)$.

d. Sample response: The point from which the ball was kicked is assumed to be the origin. The variable *y* is the height of the ball above the ground (the vertical distance); *x* is the horizontal distance the ball has traveled.

* * * * * * * * * *

(page 96)

Activity 3

Students explore connecting several points with smooth curves and examine the graphs of higher-degree polynomials.

Materials List

- graph paper
- straight thin pasta, soaked for about 1 hr (less time in hot water) or string licorice candy

Technology

- graphing utility

Teacher Note

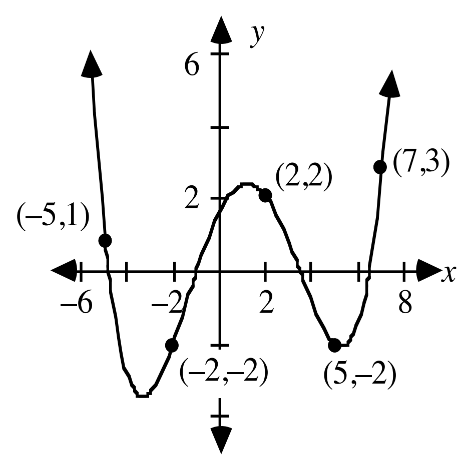
The following exploration focuses on the relationship between a function's factors and its roots—not on fitting curves to data points. As in the previous activity, you may wish to point out that it may not be possible to model a curve formed with a spline precisely using a single, simple function—even though the spline curve and the function have the same *x*-intercepts.

Exploration

(page 97)

Students use smooth curves to connect five noncollinear points and model polynomials of degrees higher than 2.

a–d. Sample response based on the points (–5,1), (–2,–2), (2,2), (5,–2), and (7,3):



e. Students approximate the zeros of the polynomial. For the sample response above, the zeros are approximately $x = -4.8$, $x = -1$, $x = 3.5$ and $x = 6.5$.

f. Using the sample response above, the factors of the polynomial are $(x+4.8)$, $(x+1)$, $(x-3.5)$, and $(x-6.5)$.

g. Using the factors from Part **f**, one function that approximates the graph is $f(x)=0.1(x+4.8)(x+1)(x-3.5)(x-6.5)$.

h. Students should modify their graphs by adjusting the value of the coefficient *a*.

Discussion

(page 97)

a. Students may describe their curves in terms of the number of "bends" and the number of *x*-intercepts. For the sample response given in the exploration, there are three bends and four *x*-intercepts.

b. The polynomial should be of degree 4 or higher.

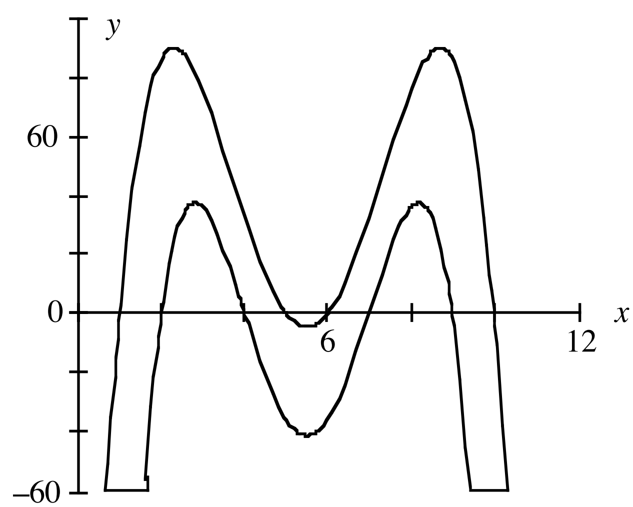
c. Sample response: Yes, a polynomial with multiple roots of the same value can have fewer *x*-intercepts but the same degree. It is also possible to change the number of zeros in a polynomial by translating it vertically, without affecting its degree.

d. Sample responses may include an S-shaped curve on a highway, the cross section of a ship's hull, the "scalloped" contours of some furniture, or the tracks of a roller coaster.

Assignment

(page 98)

3.1 **a.** Answers will vary. The following three functions could be used to create the outline of the letter M:
$f(x)=-1.5(x-2)(x-4)(x-7)(x-9)$ for *x* in [1.6,9.4];
$f(x)=-0.9(x-1)(x-5)(x-6)(x-10)$ for *x* in [0.7,10.3]; and
$f(x)=-57.5$ for *x* in [0.7,1.6] and [9.4,10.3].

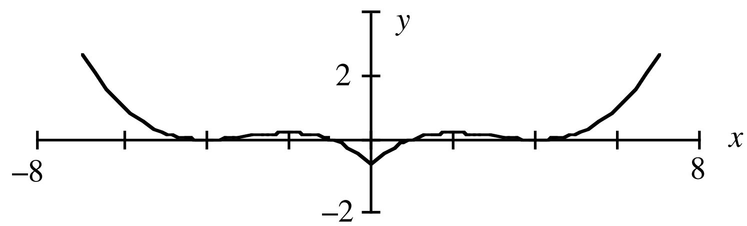


b. Student responses should include a list of functions along with their appropriate domains.

3.2 **a.** Sample response: $f(x) = 0.05(x - 0.5)(x - 4)^2$ for x in [0,7]

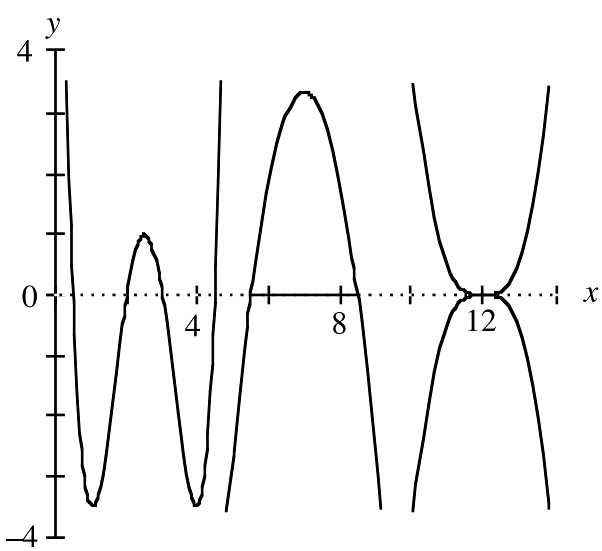
b. Sample response: $f(x) = -0.05(x + 0.5)(x + 4)^2$ for x in [–7,0]

c. Sample response: The right side and left side of the design are reflections of each other in the y-axis. The roots and leading coefficients (a) of the corresponding functions are additive inverses.



3.3 Sample response: The functions that correspond to each letter are shown in the following table, along with the intervals that approximate their domains and ranges.

Letter	Function	Domain	Range
W	$f(x) = (x - 0.5)(x - 2)(x - 3)(x - 4.5)$	[0.32, 4.69]	[–3.52, 3.50]
A	$f(x) = -1.5(x - 5.5)(x - 8.5)$	[4.86, 9.14]	[–3.49, 3.38]
A	$f(x) = 0$	[5.5, 8.5]	[0, 0]
X	$f(x) = 0.3(x - 12)^3$	[9.73, 14.27]	[–3.51, 3.51]
X	$f(x) = -0.3(x - 12)^3$	[9.73, 14.27]	[–3.51, 3.51]



***3.4** Answers will vary. Student responses should include a sketch of the design, a list of the functions used and their zeros, as well as a description of the domain and range used to view the picture.

* * * * *

3.5 **a.** After one year, the account balance was $250(1+0.055)^1 = \$263.75$.

b. The account balance after these five years can be found as follows:

$$\begin{aligned} f(x) &= 250(1.055)^5 + 200(1.055)^4 + 325(1.055)^3 + \\ &\quad 450(1.055)^2 + 400(1.055)^1 + 675(1.055)^0 \\ &= \$2553.99 \end{aligned}$$

c. If $x = 1 + r/n$, a polynomial function in x that models the account balance on Christine's 18th birthday is:

$$f(x) = 250x^5 + 200x^4 + 325x^3 + 450x^2 + 400x^1 + 675$$

d. If $x = 1 + r/n$, a polynomial function in x that models the account balance on Christine's 30th birthday is:

$$f(x) = 250x^{17} + 200x^{16} + 325x^{15} + 450x^{14} + 400x^{13} + 675x^{12}$$

3.6 The length of the box is $(96 - 2x)$. The width of the box is $(54 - 2x)$. The height is x.

a. **1.** The polynomial function that models the surface area of the box is:

$$\begin{aligned} A(x) &= (96)(54) - 4(x^2) \\ &= -4(x^2 - 1296) \\ &= -4(x - 36)(x + 36) \end{aligned}$$

2. The domain of the function is all real numbers, $(-\infty, \infty)$. In the context of this problem, however, only the interval $(0, 27)$ should be considered, since a box cannot be formed for any x-value outside this interval.

3. The zeros of this function are $x = 36$ and $x = -36$. Both are outside the interval of values that should be considered in this context. Therefore, the zeros of this function have no meaning in this setting.

b. **1.** The polynomial function in x that models the volume of the box is:

$$\begin{aligned} V(x) &= x(96 - 2x)(54 - 2x) \\ &= 4x^3 - 300x^2 + 5184x \\ &= 4x(x - 48)(x - 27) \end{aligned}$$

2. The domain of the function is all real numbers, $(-\infty, \infty)$. In the context of this problem, however, only the interval $(0, 27)$ should be considered, since a box cannot be formed for any x-value outside this interval.

3. The zeros of the function are 0, 27, and 48. Since all of these values are outside the interval to be considered in this problem, they have no meaning in this setting.

c. Students may trace a graph of the function to determine the approximate maximum volume of about 26,000 cm^3. This occurs when x is approximately 11 cm.

d. The surface area of the open-topped box with the maximum volume is approximately 4700 cm^2, since $A(11) = 5184 - 4(11)^2 = 4700$.

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(page 100)

Activity 4

In this activity, students explore the graphs of polynomials of higher degrees. They determine the domain and range of the corresponding polynomial functions and investigate the end behavior of such functions.

Materials List

- none

Technology

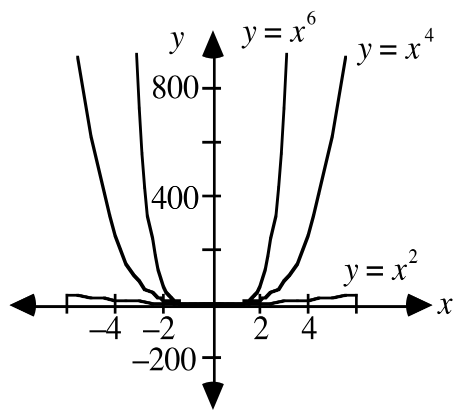
- graphing utility

Exploration 1

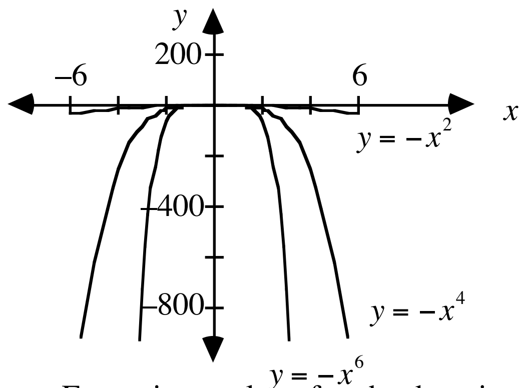
(page 100)

Students compare graphs of polynomials of the form $y = ax^n$.

a. Students should observe that these graphs have the same general shape. The domain of all functions of the form $y = x^n$, when n is an even, positive integer, is the set of real numbers. The corresponding range is the set of non-negative real numbers.

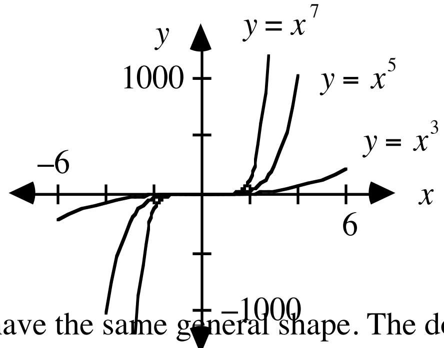


b. Students should again observe that the graphs have the same general shape. The domain of all functions of the form $y = -x^n$, when n is an even, positive integers is the set of real numbers. The corresponding range is the set of nonpositive real numbers.

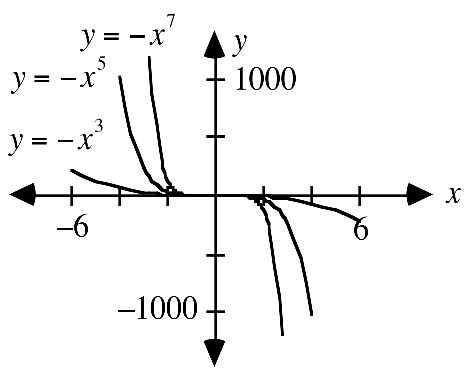


c. Sample response: For a given value of n, the domains of $f(x) = x^n$ and $f(x) = -x^n$ are the same; the respective values in the ranges are additive inverses.

d. The graphs have the same general shape. The domain of all functions of the form $y = x^n$, when n is an odd integer greater than 1, is the set of real numbers. The corresponding range is also the set of real numbers.



e. The graphs have the same general shape. The domain of all functions of the form $y = -x^n$, when n is an odd integer greater than 1, is the set of real numbers. The corresponding range is also the set of real numbers.



f. Sample response: The domains are equal and the ranges are the same.

g. Sample response: When n is even, the graph is U-shaped like a parabola and the domain is the set of real numbers, or the interval $(-\infty,\infty)$. If a is positive, the range is the set of non-negative real numbers, or the interval $[0,\infty)$. If a is negative, the range is the set of nonpositive real numbers, or the interval $(-\infty,0]$.

When n is odd and a is positive, the graphs rise from left to right and are nearly horizontal in the vicinity of the origin. The domain and range of each function is the set of real numbers, or the interval $(-\infty,\infty)$.

When n is odd and a is negative, the graphs fall from left to right and are nearly horizontal in the vicinity of the origin. The domain and range of each function is the set of real numbers, or the interval $(-\infty,\infty)$.

Discussion 1

(page 101)

a.

1. Sample response: If n is even, $|a|$ affects the shape of the graph—the greater the $|a|$, the narrower the curve. If n is odd, $|a|$ affects how steep the curve is—the greater the $|a|$ is, the steeper the curve becomes and the less flat it is in the vicinity of the origin. In either case, if a is negative, the curve is reflected in the x-axis.
2. For each value of n, the domain is always the set of real numbers, or the interval $(-\infty,\infty)$. If n is even, the range is either the set of non-negative real numbers or the set of nonpositive real numbers. If n is odd, the range is always the set of real numbers.

b.

1. The y-values decrease from a positive value at $x=-5$ to a value of 0 at $x=0$. The y-values then increase as x increases in value from 0 to 5.
2. The y-values increase across the domain from a negative value at $x=-5$, to 0 at $x=0$, to positive values when $x>0$.

c. For functions of the form $y=x^n$, where n is even, the y-values increase without bound as $|x|$ increases. For functions of the form $y=-x^n$, where n is even, the y-values decrease without bound as $|x|$ increases.

For functions of the form $y=x^n$, where n is odd, the y-values approach $-\infty$ as x decreases without bound. As x increases without bound, the y-values approach $+\infty$. For functions of the form $y=-x^n$, where n is odd, the y-values approach $+\infty$ as x decreases without bound. As x increases without bound, the y-values approach $-\infty$.

d. Sample response: It is not possible. Either the subset of positive real numbers or negative real numbers will be missing because of the effects of raising a value to an even power.

e. Sample response: The range for a polynomial function of odd degree will always be the set of real numbers because of its end behavior. As the absolute values of x increase, one end of the graph increases without bound, while the other end decreases without bound.

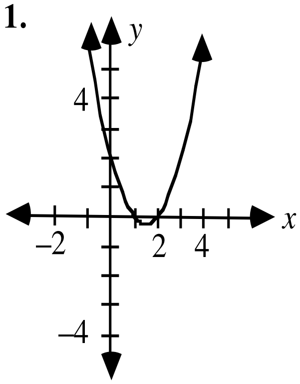
Exploration 2

(page 101)

In this exploration, students discover that some polynomial functions cannot be expressed as the product of first-degree factors.

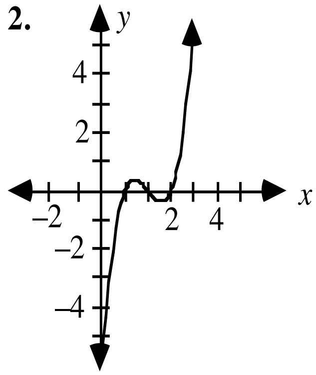
a. Students create graphs of four polynomials and approximate their roots. Sample graphs of each function are given below.

1.



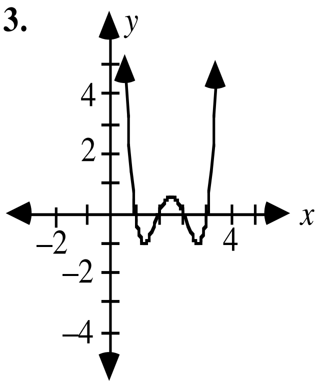
real roots: $x = 1$, $x = 2$

2.



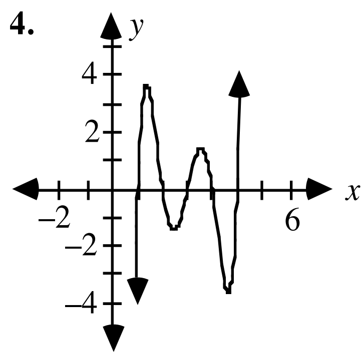
real roots: $x = 1$, 2, and 3

3.



real roots: $x = 1$, 2, 3, and 4

4.



real roots: $x = 1$, 2, 3, 4, and 5

b. These polynomial functions can be written in factored form since all of their roots are real numbers.

1. $y = (x-1)(x-2)$

2. $y = (x-1)(x-2)(x-3)$

3. $y = (x-1)(x-2)(x-3)(x-4)$

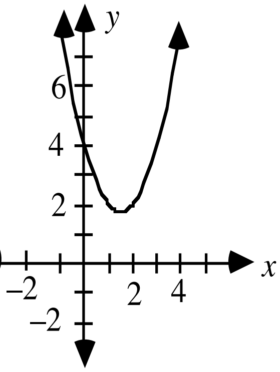
4. $y = (x-1)(x-2)(x-3)(x-4)(x-5)$

c. Adding a constant to a function of the form $f(x) = (x-c_1)(x-c_2)\cdots(x-c_n)$ translates the graph vertically.

d–e. Students should discover that a graph of a polynomial function of degree n that does not intersect the x-axis n times cannot be expressed in factored form using only first-degree factors. **Note:** Students will be introduced to imaginary roots in later modules. However, you may choose to discuss them with your students at this time.

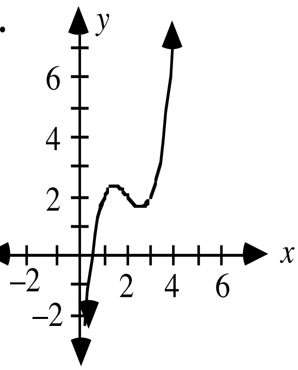
The following sample graphs show vertical translations of the original functions in Part **a** by 2 units, along with the approximate real roots, if any.

1.



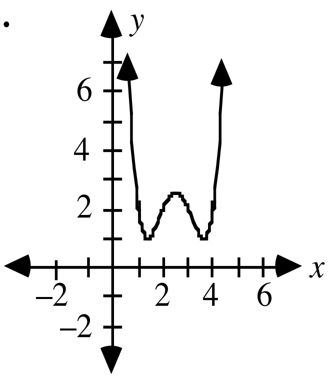
real roots: none

2.



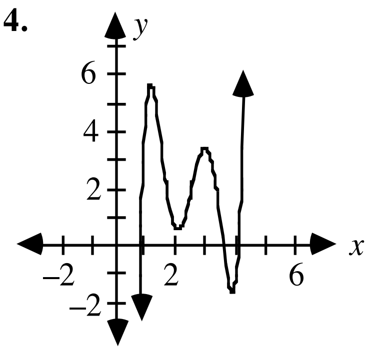
real roots: $x \approx 0.5$

3.



real roots: none

4.



real roots: $x \approx 0.9,\ 4.3,$ and 4.9

Discussion 2

(page 102)

a. If the linear function $f(x) = mx + b$, has a non-zero value of m, translating the function vertically will change the value of the root but not the number of roots. If $m = 0$, the function has no zeros unless $b = 0$.

b. Sample response: No. Translating a function only affects the function's position, not its degree.

c. Sample response: No. The translated functions cannot be expressed as a product of first-degree factors. The value of c in factors of the form $(x - c)$ are determined by the points where the graph crosses the x-axis. Although they have the same degree, the translated graphs cross the x-axis fewer times than the original functions.

d. **1.** The number of real roots of a polynomial function in factored form is equal to its degree.

2. Sample response: No. The maximum number of real roots is equal to the degree of the polynomial function. The actual number of real roots can vary.

e. Although the degree of a polynomial can sometimes be determined by the shape of its graph, this is not always true. In most cases, the graph can be used to identify only the least possible degree of the polynomial.

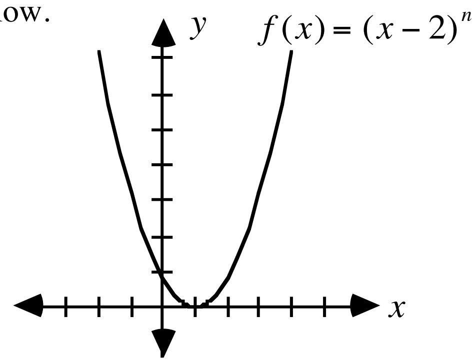
f. **1.** Sample response: Based on the number of times the function must cross the x-axis, the function has 5 real roots. This would indicate a minimum degree of 5.

2. Sample response: No. Polynomial functions of a higher degree could model the set of points, as could pieces of several functions of lower degrees.

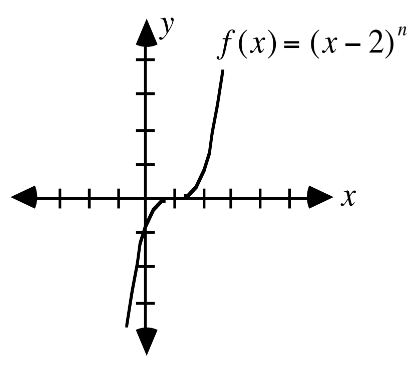
Assignment

(page 103)

4.1 Sample response: When $n = 1$, $f(x)$ is a line that intercepts the x-axis at 2 and the y-axis at –2. When n is even, the graph is tangent to the x-axis at $x = 2$ and the range is the set of non-negative real numbers. The general shape of the graph of $f(x) = (x-2)^n$, when n is even, is illustrated below.



When n is odd and $n \geq 3$, the graph intersects the x-axis at $x = 2$ and the range is the set of real numbers. The general shape of the graph of $f(x) = (x-2)^n$, when n is odd and $n \geq 3$, is illustrated below.



4.2 **a.** Answers will vary. The function should have two factors of the form $(x - c)$ that are raised to even powers. Sample response: $f(x) = (x - 1)^2(x - 4)^2$.

b. Answers will vary. The function should have two factors of the form $(x - c)$ with exactly one of them raised to an odd power. Sample response: $f(x) = (x - 1)^3(x - 4)^2$.

4.3 **a.** Answers will vary. Students should find a polynomial function of even degree with exactly two distinct real roots. This can be done by finding a polynomial in factored form with an even number of distinct real roots greater than 2 and translating it vertically until the graph crosses the *x*-axis exactly two times. Sample response:

$$\begin{aligned} f(x) &= (x + 4)(x + 2)(x - 2)(x - 1) + 6 \\ &= x^4 + 3x^3 - 8x^2 - 12x + 22 \end{aligned}$$

b. Answers will vary. Students should find a polynomial function of odd degree with exactly three distinct real roots. This can be done by finding a polynomial in factored form with an odd number of distinct real roots greater than 3 and translating it vertically until the graph crosses the *x*-axis exactly three times. Sample response:

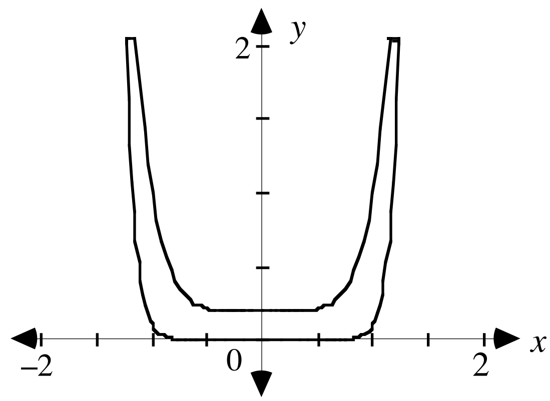
$$\begin{aligned} f(x) &= (x + 4)(x + 2)(x - 2)(x - 1)(x - 1) + 4 \\ &= x^5 + 2x^4 - 11x^3 - 4x^2 + 28x - 12 \end{aligned}$$

4.4 **a.** Sample response: Any function of the form $f(x) = -(x - b)^n$, where $-5 < b < 5$ and *n* is even, will satisfy these conditions.

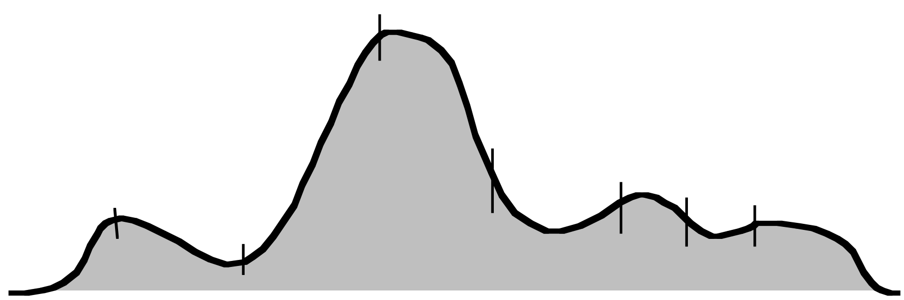
b. Any function of the form $f(x) = -(x - b)^n$, where *n* is odd, will satisfy these conditions.

4.5 Because some higher-degree polynomials are better models for the shapes needed to define the boundaries of the letter U, this version should look better than the one made with parabolas in Problem **2.4**. The following sample response uses even-degree polynomials of a high degree to create a flattened outline at the bottom of the letter. The appropriate domains for these curves can be found using the trace feature on a graphing utility.

Sample response: The letter U in the following diagram was defined by the functions $y = 0.8x^6 + 0.2$ for x in [–1.15, 1.15], $y = 0.1x^{14}$ for x in [–1.24, 1.24], and $y = 2.05$ for x in [1.15, 1.24] and [–1.24, 1.15].



***4.6** **a**. Answers may vary. Some students may indicate the "peaks" and "valleys" of the curve, since these regions contain no more than two turning points. The sample sketch below shows the points where a drawing program spliced the cubic curves together.



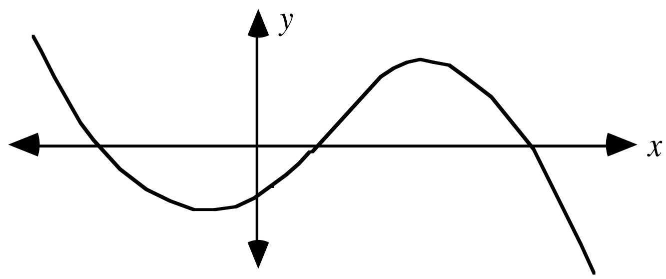
b. Sample response: A collection of cubics is easier to manipulate than a single higher-degree polynomial. It is also easier to determine several equations, each modeling some of the points, than it is to find a single equation that models all of the points.

* * * * *

4.7 **a.** By examining the graph, students should observe that the *y*-values increase as *x* increases in the intervals (–3.56, 0.40) and $(3.18, \infty)$.

b. By examining the graph, students should observe that the *y*-values decrease as *x* increases in the intervals: $(-\infty, -3.56)$ and (0.40, 3.18).

4.8 It is possible to argue that the complete graph is a polynomial of any degree greater than or equal to 2. Sample response:



a. The sample graph shown appears to have degree 3.

b. There are three zeros: two positive and one negative.

c. Sample response: The end behavior approaches positive infinity for small values of x. It approaches negative infinity for large values of x.

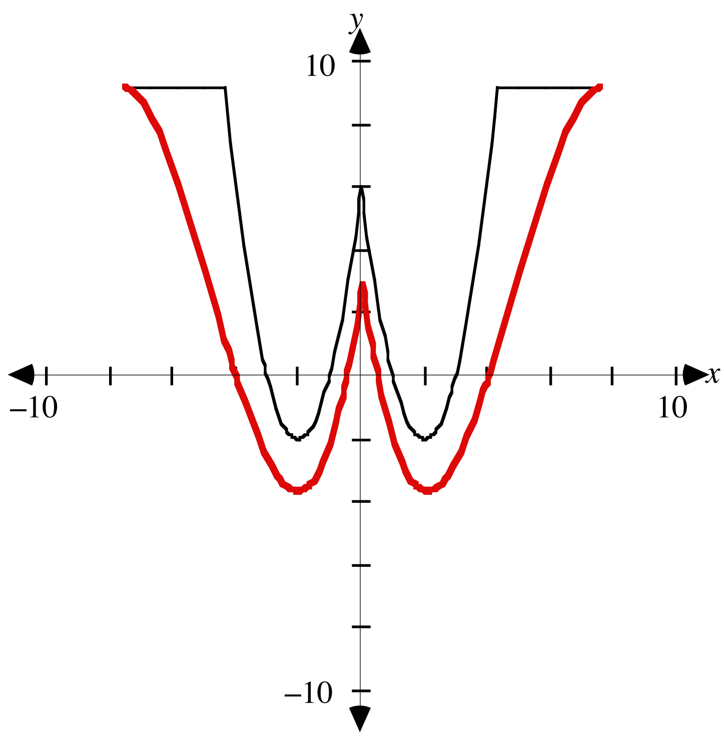
* * * * * * * * * *

Answers to Summary Assessment

(page 105)

Answers will vary. Students may use two fourth-degree equations to define the boundaries of the letter W, since some fourth-degree polynomials have three "turning points"—for example, $x^2(x+3)^2$ and $0.25(x-1)^2(x+4)^2-7$. Others may use linear equations. The sample below was created by experimenting with a graphing utility. It is composed of a quadratic (the inside edge), a cubic (the outside edge), horizontal lines, and the reflections of all these polynomials in the *y*-axis. Appropriate values for the domains were found by tracing curves to find the points of intersection. The letter is defined by the following functions:

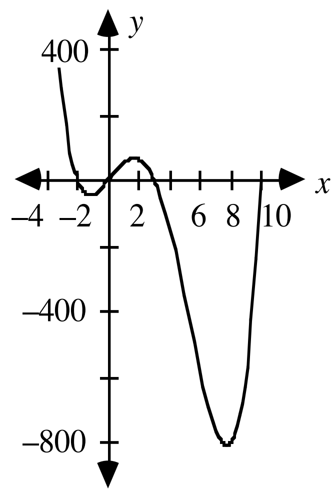
- $y = 2(x+1)(x+3)$ defined over the domain $[-4.36, 0)$
- $y = 2(x-1)(x-3)$ defined over the domain $[0, 4.36)$
- $y = 0.15(x+0.5)(x+4)(x+10)$ defined over the domain $[-7.55, 0]$
- $y = -0.15(x-0.5)(x-4)(x-10)$ defined over the domain $(0, 7.55]$
- $y = 9.15$ defined over the domain $[-7.55, -4.36)$ and $[4.36, 7.55]$



Module Assessment

1. Find two polynomials of two different degrees that could model a spline curve containing the points (–4,1), (1,–1), and (5,5).

2. Use the following graph to complete Parts **a–d** below.



a. Estimate the zeros of a function that models the graph.

b. Write a polynomial function that models the graph.

c. List the domain and range for the function in Part **b**.

d. Explain whether or not the function in Part **b** is a "good" model of the graph.

3. Describe the graph of $g(x) = -x^4 + 7x^3 - 5x^2 - 31x + 30$ using the concepts and vocabulary included in this module.

Answers to Module Assessment

1. Answers may vary. Sample response: The following three equations of second, third, and fourth degree all model the points: $y = 0.2(x+3)(x-2)$, $y = 0.025(x+6)(x+2.5)(x-2.5)$, and $y = -0.004(x+6)(x+3)(x-2.5)(x-10)$.

2.

a. The zeros of the function are –2, 0, 3, and 10.

b. Sample response: $f(x) = x(x+2)(x-3)(x-10)$.

c. The domain for the sample function is the set of real numbers. The range for the sample function is $f(x) \geq -807.6$.

d. Sample response: The function is probably not an exact model of the graph. The zeros are approximations and the coordinates of other points on the graph can only be estimated. It is not possible to verify that the graph of the model function fits the given graph exactly. Therefore, being a "good" model requires interpretation.

3. Sample response: The function is a fourth-degree polynomial with four zeros: –2, 1, 3, and 5. The domain is the set of real numbers. The range is the set of real numbers in the interval $(-\infty, 48.13)$. The graph of the function has increasing *y*-values in the interval $(-\infty, -0.96)$, decreasing *y*-values in the interval $(-0.96, 1.96)$, increasing *y*-values in the interval $(1.96, 4.22)$, and decreasing *y*-values in the interval $(4.22, \infty)$. The function's end behavior can be described as follows: As $|x|$ increases without bound, the function values eventually decrease.

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Brodlie, K. W. *Mathematical Methods in Computer Graphics and Design*. London: Academic Press, 1980.

Gardan, Y. *Mathematics and CAD. Volume 1: Numerical Methods for CAD*. Cambridge, MA: The MIT Press, 1986.

Gilfillan, S. C. *Inventing the Ship*. Chicago: Follett Publishing Co., 1935.

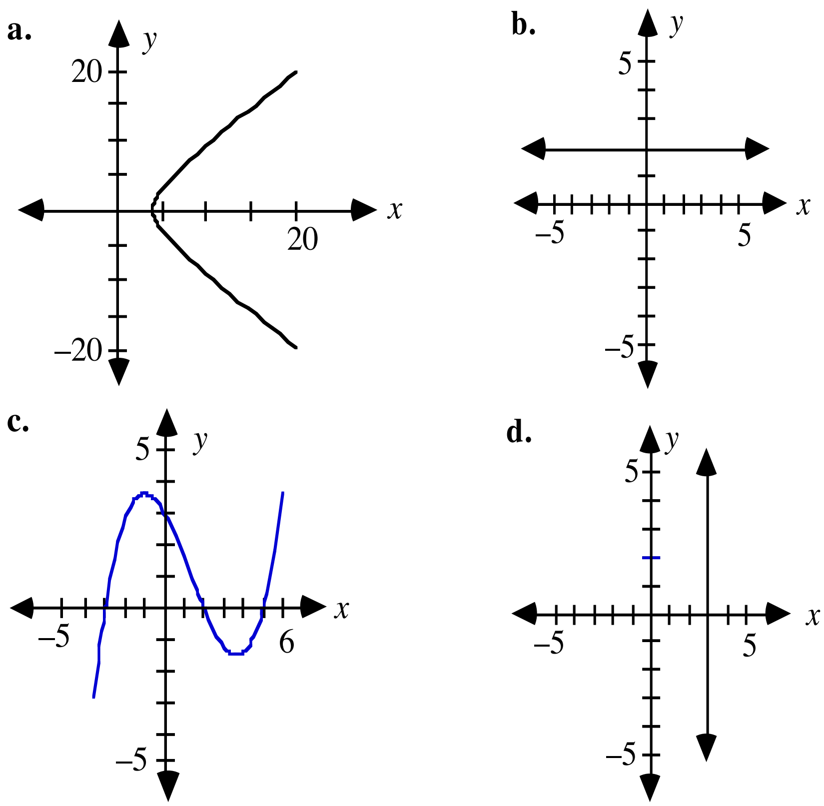
Lewell, J. *A–Z Guide to Computer Graphics*. New York: McGraw-Hill, 1985.

Pokorny, C. K. and C. F. Gerald. *Computer Graphics: The Principles Behind the Art and Science*. Irvine, CA: Franklin, Beedle & Associates, 1989.

Flashbacks

Activity 1

1.1 Which of the following are graphs of functions? Explain your responses.



1.2 Use the distributive property to simplify each of the following expressions:

a. $5(80 + 4)$

b. $4(10 + 2) + 3(40 - 3)$

c. $3(2x - 3)$

1.3 List the domain and range for each of the following functions:

a. $f(x) = 3x - 4$

b. $h(x) = x^2$

c. $j(x) = \sqrt{x}$

d. $g(x) = 5/x$

Activity 2

2.1 Given the two functions in Parts **a** and **b** below, use the sum of the squares of the residuals to determine which function provides a better approximation of the following data set. Justify your response.

x	y
0.5	1.6
–1.0	5.0
Ï–2.5	0.0
1.0	–3.0
0.1	3.2

a. $y = -2x^2 - 4x + 3$

b. $y = -2x^2 - 4x + 3$

2.2 Identify the zeros of each of the following polynomial functions:

a. $h(x) = 4(x - 3)(x + 5)(x - 2)$

b. $g(x) = (x - 4.9)(x + 4.9)$

2.3 Use the distributive property to expand each function in Flashback **2.2**.

Activity 3

3.1 Identify the zeros and list the domain and range of each function below.

a. $f(x) = (x - 3)^2$

b. $g(x) = (x - 2)^2 (x + 3)$

3.2 Evaluate each function below for the given value of x.

a. $f(x) = (x - 3)^2 (x + 1)$; $x = 4.7$

b. $g(x) = (x - 2)(x + 1)(x - 3)(x + 4)$; $x = 3.8$

3.3 Identify the degree of the function $h(x) = 4(x - 3)^2 (x + 5)$.

Activity 4

4.1 Rewrite each of the following using the distributive property.

a. $h(x) = (x - 1)(x - 3)$

b. $f(x) = (2x + 1)(x - 1)$

c. $k(x) = (x - 7)(x - 1)(x + 2)$

4.2 Without graphing, determine the zeros, the domain, and the range of each of the following functions:

a. $g(x) = 2(x + 3)^2$

b. $f(x) = -0.5(x - 2)^3$

Answers to Flashbacks

1.1 Graphs **b** and **c** are graphs of functions because each x-value is paired with only one y-value. Graphs **a** and **d** are not functions.

1.2
- **a.** $5 \bullet 80 + 5 \bullet 4 = 400 + 20 = 420$
- **b.** $(4 \bullet 10 + 4 \bullet 2) + (3 \bullet 40 - 3 \bullet 3) = (40 + 8) + (120 + (-9)) = 159$
- **c.** $3(2x) - 3 \bullet 3 = 6x + (-9) = 6x - 9$

1.3
- **a.** The domain is the set of real numbers, $(-\infty,\infty)$; the range is the set of real numbers, $(-\infty,\infty)$.
- **b.** The domain is the set of real numbers, $(-\infty,\infty)$; the range is the set of non-negative real numbers, $[0,\infty)$.
- **c.** The domain is the set of non-negative real numbers, $[0,\infty)$; the range is the set of non-negative real numbers, $[0,\infty)$.
- **d.** The domain is the set of real numbers except 0, $(-\infty,0)$ and $(0,\infty)$. The range is all real numbers, $(-\infty,\infty)$

Activity 2

2.1 The equation in Part **b**, $y = -2x^2 - 4x + 3$, is a better model because the sum of the squares of the residuals (292.4) is less than that for the equation in Part **a** (343.2).

2.2
- **a.** The zeros of the function $h(x)$ are 3, –5, and 2.
- **b.** The zeros of the function $g(x)$ are 4.9 and –4.9.

2.3
- **a.** $h(x) = 4x^3 - 76x = 120$
- **b.** $g(x) = x^2 - 24.01$

Activity 3

3.1
- **a.** The function $f(x)$ has a double root of 3. The domain is the set of reals, $(-\infty,\infty)$, and the range is the set of non-negative reals, $[0,\infty)$.
- **b.** The function $g(x)$ has a root of –3 and a double root of 2. The domain is the set of reals, $(-\infty,\infty)$, and the range is the set of reals, $(-\infty,\infty)$.

3.2
- **a.** $f(4.7) = (4.7 - 3)^2(4.7 + 1) = 16.473$
- **b.** $g(3.8) = (3.8 - 2)(3.8 + 1)(3.8 - 3)(3.8 + 4) = 53.9136$

3.3 The function $h(x) = 4(x - 3)^2(x + 5)$ has degree 3.

Activity 4

4.1 **a.** $h(x) = (x - 1)(x - 3) = x^2 - 4x + 3$

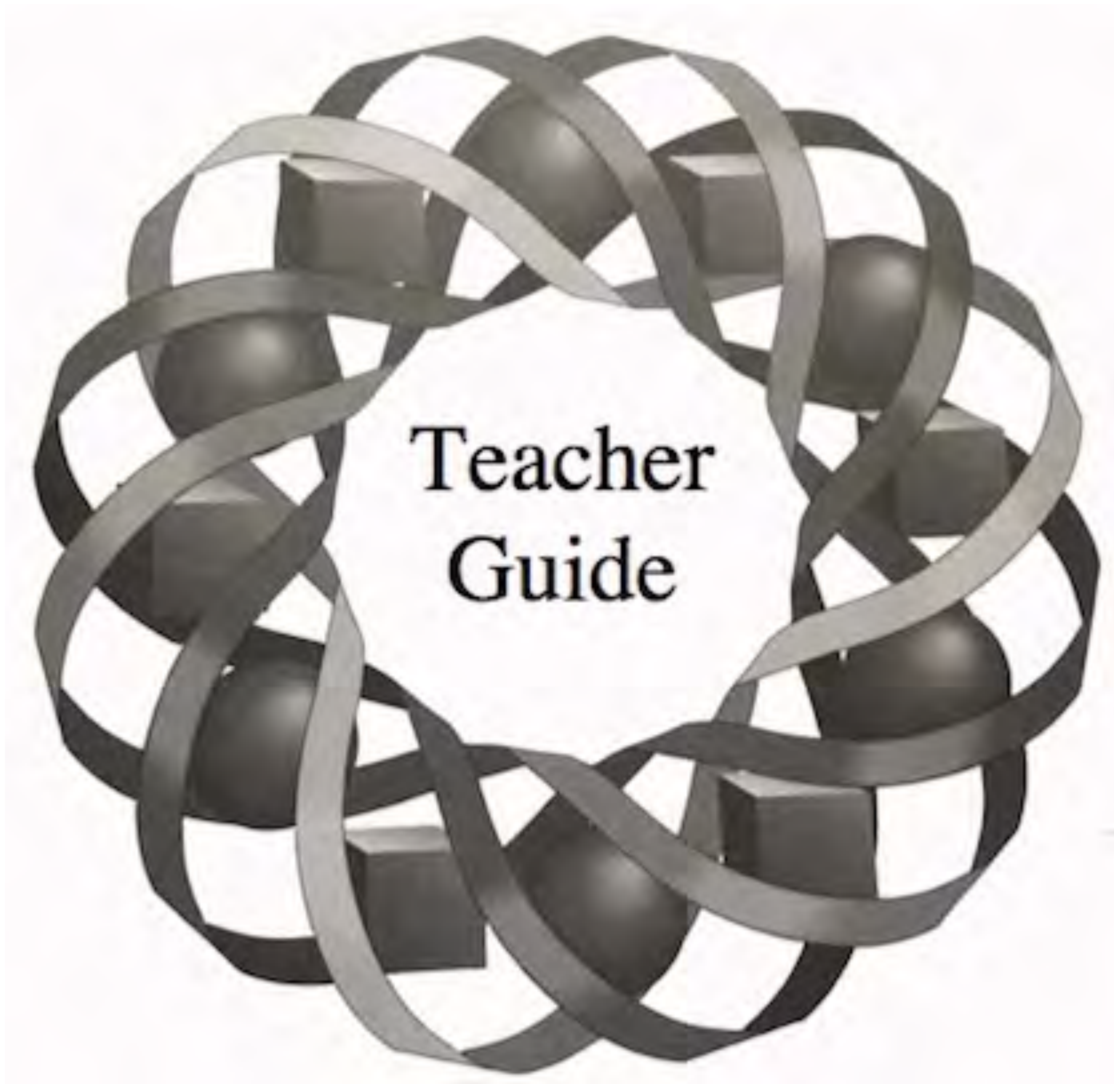
b. $f(x) = (2x + 1)(x - 1) = 2x^2 - x - 1$

c. $k(x) = (x - 7)(x - 1)(x + 2) = x^3 - 6x^2 - 9x + 14$

4.2 **a.** The function $g(x) = -2(x + 3)^2$ has a zero of –3. Its domain is the set of real numbers, $(-\infty, \infty)$; its range is the set of nonpositive real numbers, $(-\infty, 0]$.

b. The function $f(x) = -0.5(x - 2)^3$ has a zero at 2. Both its domain and range are the set of real numbers, $(-\infty, \infty)$.

Log Jam



What do earthquakes, noise levels, and upset stomachs have in common? Data about these phenomena can be difficult to interpret when graphed on a linear scale. In this module, you use logarithms to investigate another type of scale.

Dick Sander • Paul Swenson • Dan West

Teacher Edition

Log Jam

Overview

In this module, students investigate the use of logarithms in chemistry (pH), geology (the Richter scale), and mathematics. They examine the properties of exponents and develop the properties of logarithms.

Objectives

In this module, students will:

- interpret real-number exponents
- explore the properties of exponents
- use logarithmic scales to represent data that covers a wide range of values
- explore properties of logarithms
- use base-10 logarithms to solve equations.

Prerequisites

For this module, students should know:

- how to find the equation of a line
- how to interpret negative exponents
- how to use scientific notation
- how to determine the inverse of a relation
- how to model exponential growth.

Time Line

Activity	Intro.	1	2	3	Summary Assessment	Total
Days	1	2	3	2	1	9

Materials Required

Materials	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
carbonated soft drink	X	X			
milk	X				
vinegar	X				
orange juice	X				
litmus paper	X				
baking soda	X	X			
250-mL beaker	X	X			
100-mL beaker		X			
eye droppers		X			
graph paper			X		
semilog graph paper			X		

Technology

Software	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
graphing utility		X	X	X	X
spreadsheet		X	X	X	X
pH probe	X	X			
scientific interface device	X	X			

Log Jam

Introduction
(page 111)

Students are introduced to the pH scale and the chemical concepts of acid and base. **Note:** Although it is possible to prepare a solution with a pH less than 0 or greater than 14, only the more common values are discussed in this module.

Materials List

- carbonated soft drink
- milk
- baking soda
- vinegar
- orange juice
- 250-mL beakers
- litmus paper (optional)

Technology

- pH probe
- scientific interface device

Exploration
(page 112)

Students test the pH of several familiar solutions. Although a pH probe and scientific interface device (such as Texas Instruments' CBL) are preferable, the exploration may be completed using litmus paper.

a. Student predictions may vary.

b.
1. acidic
2. neutral
3. basic
4. acidic
5. acidic

Discussion

(page 112)

a. **1.** Carbonated soft drinks, vinegar, and orange juice are acidic.

2. Sample responses may include lemon juice, tomato juice, or urine.

b. **1.** A solution of baking soda and water is basic.

2. Many household cleansers, such as ammonia, are weak bases. Human blood is slightly basic (pH 7.3–7.5).

c. Sample response: The resulting mixture would tend to be neutralized, though mixing the two solutions could be dangerous as a result of chemical reactions.

(page 112)

Activity 1

In this activity, students conduct an experiment using a weak acid (carbonated soft drink) and a weak base (baking soda). They monitor pH levels and graph the results. Students also investigate the properties of exponents revealed by multiplying and dividing numbers with like bases raised to a power.

Note: Students are not formally introduced to logarithms until Activity **2**, where they discover that the pH of a solution is equal to the negative logarithm of the hydrogen ion concentration.

Materials List

- carbonated soft drink (one glass per group)
- baking soda (two teaspoons per group)
- 250-mL beaker (one per group)
- 100-mL beaker (one per group)
- eye droppers (one per group)
- water

Technology

- graphing utility
- spreadsheet
- pH probe (one per group)
- scientific interface device

Teacher Note

In the exploration, students neutralize a weak acid with a weak base. In a neutralization reaction, hydrogen ions react with hydroxide ions to form water. You may wish to consult with a chemistry teacher prior to this experiment.

To ensure consistent results, each pH probe should be calibrated according to the manufacturer's instructions before beginning the exploration. If the number of pH probes is limited, the exploration may be conducted as a demonstration, with all students recording the pH values generated.

Exploration

(page 112)

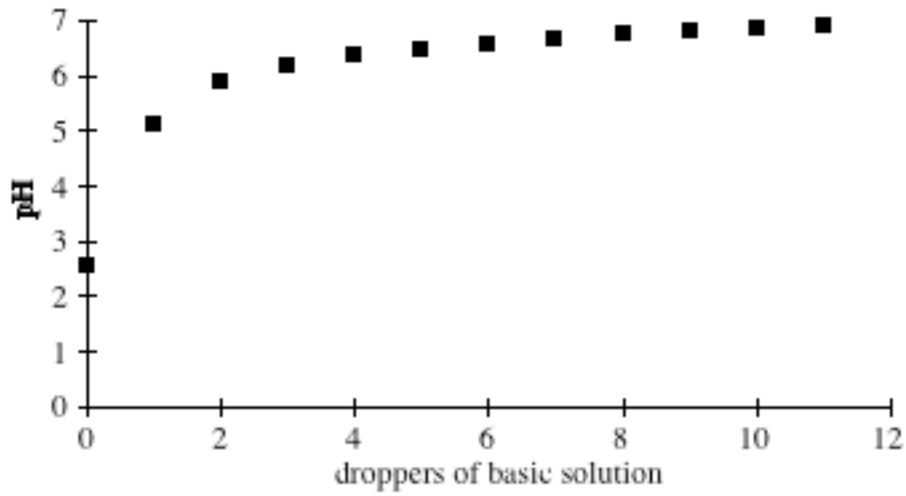
a. To prevent foaming, the carbonated soft drink should be nearly flat. Opening the soft-drink container the night before the exploration will speed this step.

b–f. Sample data:

Droppers of Baking Soda Solution	pH	Change in pH
0	2.568	
1	5.104	2.536
2	5.905	0.801
3	6.194	0.289
4	6.350	0.156
5	6.483	0.133
6	6.572	0.089
7	6.661	0.089
8	6.750	0.089
9	6.817	0.067
10	6.862	0.045
11	6.906	0.044

g. It should take about 10 more droppers of baking soda solution to change the pH an amount equivalent to the change produced by the second dropper. Student results may vary from 3 to 15 units. In the sample data, the change in pH produced by adding the second dropper is 0.801. Adding 6 more droppers of baking soda solution resulted in a change of 0.845.

h. Sample scatterplot:



Teacher Note

Students without some prior knowledge of chemistry may have difficulty with the definitions of ions and moles. An **ion** is an electrically charged atom or molecule. When dissolved in water, most acids form two ions: a negatively charged anion and a positively charged hydrogen ion (H^+). The equation below is for acetic acid, the weak acid in vinegar:

$$CH_3CO_2H \rightarrow CH_3CO_2^- + H^+$$

In reality, hydrogen ions in solution bond with molecules of water to form hydronium ions, H_3O^+. For the purposes of this module, however, pH is discussed strictly in terms of hydrogen ion concentration.

At room temperature, the concentration of hydrogen ions in water is $1 \bullet 10^{-7}$ moles/L, as is the concentration of hydroxide ions. Because these concentrations are equal, water is neutral. The product of the concentrations of hydrogen ions and hydroxide ions in water and dilute solutions is a constant: $1 \bullet 10^{-14}$.

Discussion

(page 113)

a. Sample response: The pH values increase as the baking soda solution is added, becoming closer and closer to 7.

b. Sample response: The change in pH per added dropper of baking soda solution decreases as the number of droppers increases.

c. Some possible reasons for this variation include inconsistencies in measurement, differences in the calibration of the pH probes, and differences in the sensitivity of the probes.

d. **1.** Each spike represents the addition of 1 unit of baking soda solution.

2. Student graphs should be similar in overall shape.

3. Sample response: The change in pH from 3.57 to 4.98 is 1.41. Judging from the results of the exploration, we would expect the change in pH to reach 1.41 units after about 10 more droppers. (After the addition of 9 more droppers, the pH changes from 4.98 to 6.44, a difference of 1.46.)

e. Sample response: The pH is the additive inverse of the exponent for the concentration of hydrogen ions in moles per liter, when the concentration is expressed in scientific notation.

f. Sample response: Water has a hydrogen ion concentration of $1 \bullet 10^{-7}$ moles/L, while a base with a pH of 13 has a hydrogen ion concentration of $1 \bullet 10^{-13}$ moles/L. Thus, water has a greater concentration because $1 \bullet 10^{-7}$ is greater than $1 \bullet 10^{-13}$.

g. **1.** Sample response: Using the commutative and associative properties of multiplication, regroup the two numbers, multiply the decimal numbers, then multiply the powers of 10.

$$\left(1 \bullet 10^{-7}\right)\left(6.02 \bullet 10^{23}\right) =$$
$$1 \bullet \left(10^{-7} \bullet 6.02\right) \bullet 10^{23} =$$
$$1 \bullet \left(6.02 \bullet 10^{-7}\right) \bullet 10^{23} =$$
$$(1 \bullet 6.02)\left(10^{-7} \bullet 10^{23}\right) = 6.02 \bullet 10^{16}$$

2. Using the definition of multiplication of fractions:

$$\frac{\left(5.4 \bullet 10^{5}\right)}{\left(2.5 \bullet 10^{-2}\right)} = \left(\frac{5.4}{2.5}\right) \bullet \left(\frac{10^{5}}{10^{-2}}\right) \approx 2.2 \bullet 10^{7}$$

h. Sample response: A change of 10^4 occurs. The ratio of the two numbers is $10^{-3}/10^{-7}$ which equals 10^4.

i. Sample response: The negative exponent –2.5 indicates that the expression could be rewritten as $1/10^{2.5}$. The 2.5 means that $10^{2.5}$ could be rewritten as $10^{5/2}$ or $\sqrt[2]{10^5}$. Hence, $1/10^{2.5}$ could be rewritten as $1/\sqrt[2]{10^5}$.

Using a different interpretation, $10^{-2.5}$ also could be rewritten as:

$$10^{-2+(-0.5)} =$$
$$10^{-2} \bullet 10^{-0.5} =$$
$$\frac{1}{10^2} \bullet \frac{1}{10^{0.5}} = \frac{1}{100\sqrt{10}}$$

j. **1.** Because $10^{597/100}$ has an exponent that contains a fraction that cannot be reduced, but can be rewritten as $10^{5.97}$, the two expressions are equivalent. Using the properties of exponents:

$$10^{5.97} = 10^5 \bullet 10^{0.9} \bullet 10^{0.07} = 10^5 \bullet 10^{9/10} \bullet 10^{7/100} = 10^{597/100}$$

2. With a repeating decimal as an exponent, it is difficult to determine a value for the expression $10^{0.\overline{3}}$.

3. The expressions are equal. **Note:** The response to this question (and to Part **k** below), depends on the continuity of the function $y = 10^x$.

k. The only way to evaluate such an expression is to approximate it by using a decimal expansion of π.

Assignment

(page 115)

1.1 **a.** $1 \bullet 10^{-4}$ moles/L

b. $1 \bullet 10^{-11}$ moles/L

c. 12

***1.2** **a.** **1.** The following sample response shows the use of dimensional analysis to cancel units:

$$(1\ \cancel{L})\left(\frac{1 \bullet 10^{-3}\ \cancel{\text{moles}}}{1\ \cancel{L}}\right)\left(\frac{6.02 \bullet 10^{23}\ \text{ions}}{1\ \cancel{\text{mole}}}\right) =$$

$$1\left(1 \bullet 10^{-3}\right)\left(6.02 \bullet 10^{23}\right)\ \text{ions} = 6.02 \bullet 10^{20}\ \text{ions}$$

2. $6.02 \bullet 10^{15}$ ions

3. $6.02 \bullet 10^{(23-n)}$ ions

b. pH=n

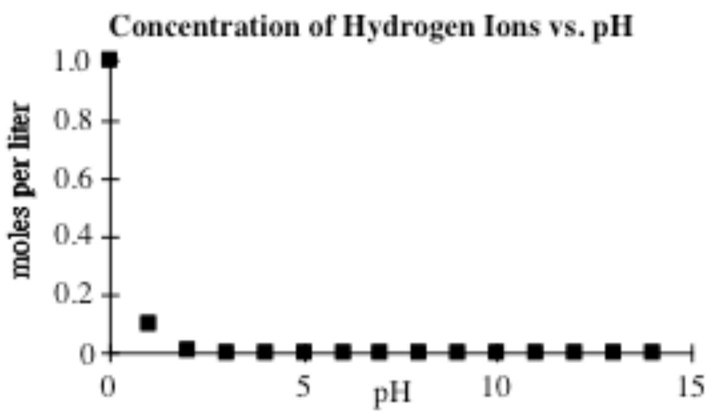
1.3 **a.** The ratio of the concentration of hydrogen ions in solutions with pH 3 and pH 4 is $10:1$; of pH 4 and pH 8 is $10,000:1$ or $10^4:1$; and of pH 3 and pH 8 is $100,000:1$ or $10^5:1$.

b. For every 1 unit of decrease in pH, the hydrogen ion concentration increases by a factor of 10.

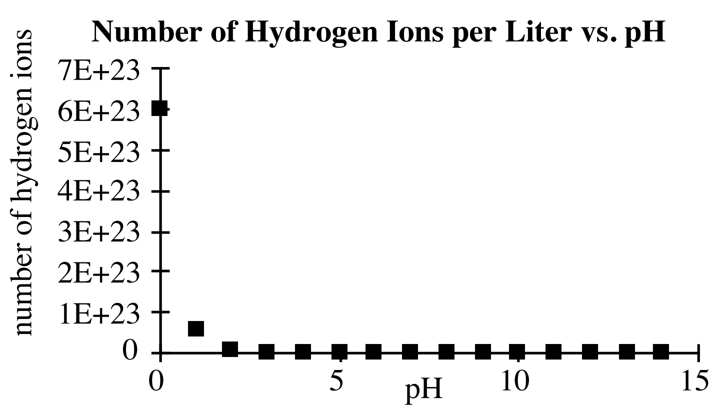
***1.4** **a.** Sample table:

pH	Concentration of Hydrogen Ions (Moles/L)	No. of Hydrogen Ions per Liter
0	1	6.02E+23
1	0.1	6.02E+22
2	0.01	6.02E+21
3	0.001	6.02E+20
4	0.0001	6.02E+19
5	0.00001	6.02E+18
6	0.000001	6.02E+17
7	0.0000001	6.02E+16
8	0.00000001	6.02E+15
9	1E–09	6.02E+14
10	1E–10	6.02E+13
11	1E–11	6.02E+12
12	1E–12	6.02E+11
13	1E–13	6.02E+10
14	1E–14	6.02E+9

b. Sample scatterplot:



c. Sample scatterplot:



d. The two graphs have similar shapes.

e. Sample response: Because the range of values on the vertical axis is so large, the graphs are of little use for estimating the number of ions or moles per liter for a pH greater than 3 unless you can magnify that portion of the graph.

* * * * *

1.5 The ratio of the intensities of an earthquake of magnitude 7 on the Richter scale and a quake of magnitude 4 is:

$$\frac{10^7}{10^4} = \frac{10^{7-4}}{1} = \frac{10^3}{1} = \frac{1000}{1}$$

1.6 **a.** The 1906 San Francisco quake was $10^{8.3-6.9} = 10^{1.4} \approx 25$ times as strong as the 1989 quake.

b. Using the property of multiplying two numbers with the same base: $10^{1.4} = 10^1 \bullet 10^{0.4} \approx 10 \bullet 2.5 \approx 25$

c. **1.** $10^{8.3} = 10^{83/10} = \sqrt[10]{10^{83}}$

2. $10^{8.3} = 10^8 \bullet 10^{0.3}$

3. Answers will vary depending on the type of technology available. Using a TI-92 calculator, for example, entering "10^8.3" yields $1.99526 \bullet 10^8$, while entering "$10 \wedge (83 \div 10)$" yields $100,000,000 \bullet 10^{3/10}$.

* * * * * * * * * *

(page 117)

Activity 2

Students explore the effects of logarithmic scales on the graphs from Activity **1**. They discover that logarithms are powers of numbers.

Materials List

- graph paper (optional)
- semilog graph paper (optional)

Technology

- graphing utility
- spreadsheet

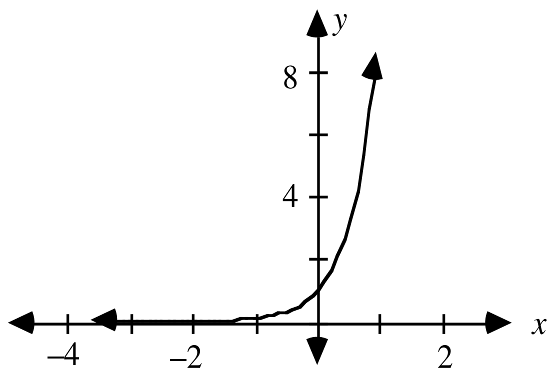
Teacher Note

On most spreadsheets, students can create scatterplots using different logarithmic scales on either axis. To give students a better feel for the effects of a logarithmic scale, however, you may wish to require them to create some of the graphs in this activity by hand, using either regular graph paper or semilog graph paper.

Exploration 1

(page 117)

a. The graph of $y = 10^x$ is asymptotic to the x-axis.

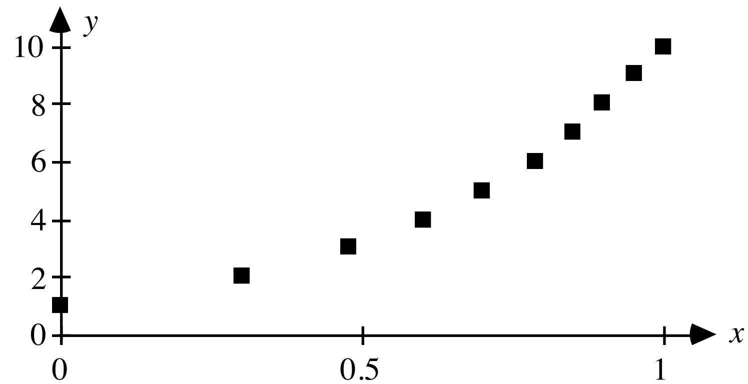


b. 1, 10, 100, 1000

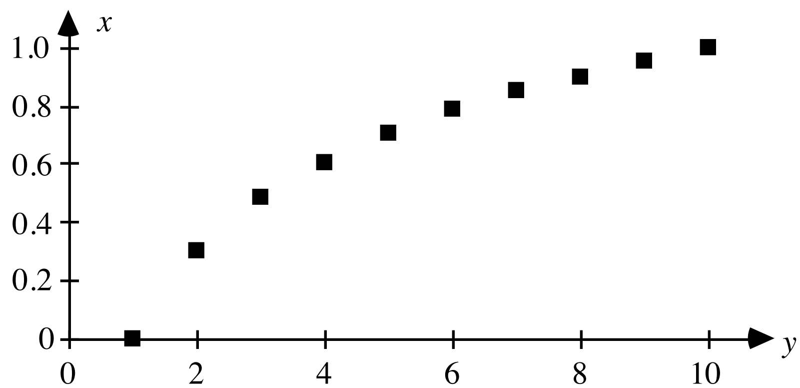
c. Sample table:

x	***y***
0	1
0.30	2
0.48	3
0.60	4
0.70	5
0.79	6
0.85	7
0.90	8
0.95	9
1	10

d. Sample graph:



e. Sample graph:



Discussion 1

(page 118)

a. Sample response: Since the x-values and y-values were interchanged, the graphs are inverses of each other.

b. There is no value of x such that $10^x = 0$. Therefore, 0 is not in the domain of the inverse of $y = 10^x$.

c.
1. The sequence is arithmetic with a common difference of 1.
2. No. Since the x-values do not form an arithmetic sequence, they cannot be used to form a linear scale.

d.
1. Sample response: The only number that can be written as a power of 1 is 1.
2. Sample response: The base b cannot be negative because the value of a negative number raised to some exponents does not exist. For example, $(-2)^{0.5}$ does not exist.

e. Sample response: No positive number raised to any exponent yields a negative result.

f. **1.** $\log_5 25 = 2$

2. $\log_3 27 = 3$

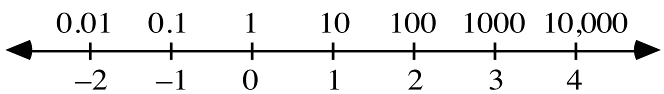
3. $\log_3 81 = 4$

Exploration 2

(page 119)

Students graph concentration of hydrogen ions vs. pH using a logarithmic scale on the vertical axis. They also examine how changing the base of a logarithmic scale affects the graph.

a. Students should label each power of 10 with its common log, as shown below:

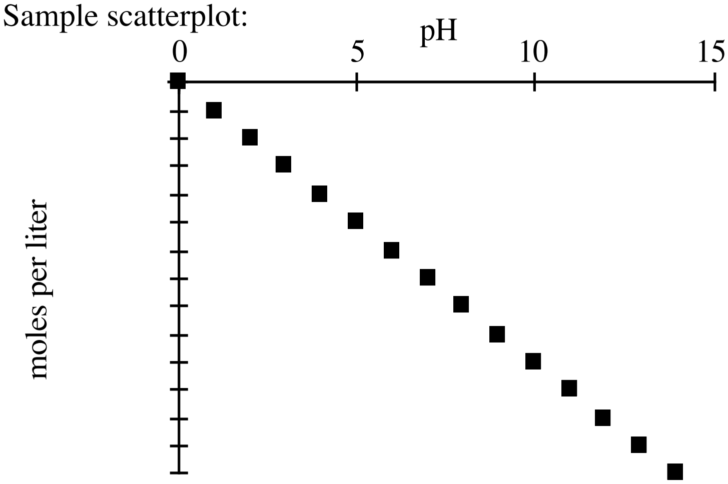


b. Answers may vary. Sample response: Changing the *y*-axis to a logarithmic scale would stretch the data along the *y*-axis, making it easier to interpret.

c. **1.** Sample table:

pH	Concentration of Hydrogen Ions (Moles per Liter)	log(Concentration of Hydrogen Ions)
0	1	0
1	0.1	–1
2	0.01	–2
3	0.001	–3
4	0.0001	–4
5	0.00001	–5
6	0.000001	–6
7	0.0000001	–7
8	0.00000001	–8
9	$1 \bullet 10^{-9}$	–9
10	$1 \bullet 10^{-10}$	–10
11	$1 \bullet 10^{-11}$	–11
12	$1 \bullet 10^{-12}$	–12
13	$1 \bullet 10^{-13}$	–13
14	$1 \bullet 10^{-14}$	–14

2–3. Sample scatterplot:



Discussion 2 (page 120)

a. Sample response: The graph in Figure **5** is curved, while the graph of the same data on a semilog coordinate system looks like a straight line.

b. Sample response: The ratio of the concentrations is almost impossible to visualize using the graph in Figure **5**. Using the graph of the same data on a semilog coordinate system, you can see that a pH of 5 corresponds with a hydrogen ion concentration of $1 \bullet 10^{-5}$ moles/L, while a pH of 10 corresponds with a concentration of $1 \bullet 10^{-10}$ moles/L.

c. Sample response: It is easier to interpret the pH data on the semilog coordinate system.

d. The equation $3 = \log(x+2)$ can be written as $10^3 = x + 2$. This can be solved for x as follows:

$$\begin{aligned} 10^3 &= x + 2 \\ 10^3 - 2 &= x \\ 1000 - 2 &= x \\ 998 &= x \end{aligned}$$

Assignment (page 121)

2.1 Sample response: The number system most commonly used is based on units of 10. When multiplying a decimal number by 10^x, where x is an integer, you can simply move the decimal x places to the right if x is positive or to the left if x is negative.

2.2 **a.** $\log_6(1296) = 4$

b. $\log_5(78{,}125) = 7$

c. $\log_2(1024) = 10$

2.3 **a.** –2

b. 4

c. –4

***2.4** The correct choice is **c**, $\text{pH} = -\log[\text{H}^+]$. Response **a**, $\text{pH} = \log\left(-\left[\text{H}^+\right]\right)$, is not possible since 10 raised to any power is always positive. Response **b** is incorrect since pH value is positive and $\log[\text{H}^+]$ is negative.

2.5 **a.** The pH of lemon juice is 2.3.

b. The pH of vinegar is 2.8.

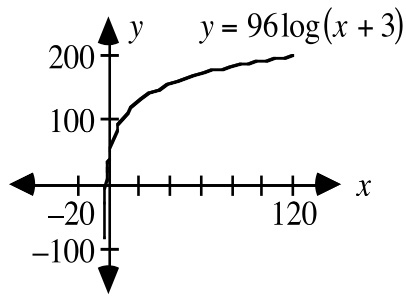
c. The pH of carbonated water is 3.0.

d. The pH of milk is 6.6.

e. The pH of blood is 7.4.

* * * * *

2.6 **a.** As x increases, y also increases, but at a continually diminishing rate. Sample graph:



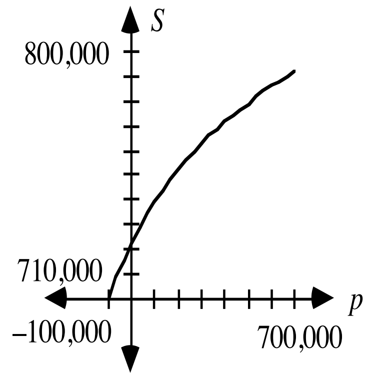
b. According to this model, the steel production in 1980 was about 138.9 million tons. In 1995, the steel production was about 156.8 million tons. Students may trace the graph to determine these values, or solve as follows:

$$\begin{aligned} y &= 96\log(25+3) \\ &= 96\log 28 \\ &\approx 138.9 \end{aligned} \qquad \begin{aligned} y &= 96\log(40+3) \\ &= 96\log 43 \\ &\approx 156.8 \end{aligned}$$

c. Using this model, the country will produce 200 million tons of steel 118.2 years after 1955, or in the year 2073. Students may trace the graph to determine this value or solve as follows:

$$\begin{aligned} 200 &= 96\log(x+3) \\ 200/96 &= \log(x+3) \\ 2.08\overline{3} &= \log(x+3) \\ 10^{2.08\overline{3}} &= x+3 \\ 121.2 &\approx x+3 \\ 118.2 &\approx x \end{aligned}$$

2.7 **a.** As p increases, S also increases, but at a continually diminishing rate. Sample graph:



b. The company's current sales are about $722,980. Students may trace the graph to determine this value or solve as follows:

$$\begin{aligned} S &= 132,000 \log(0 + 300,000) \\ &\approx 132,000(5.477) \approx 722,980 \end{aligned}$$

c. If the company doubles its advertising budget, its total sales increase to about $762,716. Students may trace the graph to determine this value or solve as follows:

$$\begin{aligned} S &= 132,000 \log(300,000 + 300,000) \\ &= 132,000 \log(600,000) \approx 762,716 \end{aligned}$$

d. Sample response: The company should not double its advertising budget. They would spend an additional $300,000 in advertising to gain only an additional $39,736 in total sales. This would be a disastrous fiscal move.

* * * * * * * * * *

(page 122)

Activity 3

This activity focuses on the properties of logarithms and their use in solving equations.

Materials List

- none

Technology

- spreadsheet
- graphing utility

Exploration

(page 122)

Students use a spreadsheet to discover four basic properties of logarithms: the product, quotient, power, and inverse properties.

a–b. Sample table:

x	y	$\log x$	$\log y$	$\log(x \bullet y)$
1	25	0.000	1.398	1.398
2	24	0.301	1.380	1.681
3	23	0.477	1.362	1.839
⋮	⋮	⋮	⋮	⋮
24	2	1.380	0.301	1.681
25	1	1.398	0.000	1.398

c. Students should discover that $\log(x \bullet y) = \log x + \log y$.

d–e. Sample spreadsheet:

x	y	$\log x$	$\log y$	$\log(x/y)$	$\log x^y$	$\log 10^x$	$10^{\log x}$
1	25	0.000	1.398	–1.398	0.000	1	1
2	24	0.301	1.380	–1.079	7.225	2	2
3	23	0.477	1.362	–0.885	10.974	3	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
23	3	1.362	0.477	0.885	4.085	23	23
24	2	1.380	0.301	1.079	2.760	24	24
25	1	1.398	0.000	1.398	1.398	25	25

Students should discover the following equalities:

- $\log(x/y) = \log x - \log y$
- $\log x^y = y \log x$
- $\log 10^x = x$
- $10^{\log x} = x$

f. Students recreate the spreadsheet using a base other than 10. On Microsoft Excel, for example, this may be done using the function "log(number,base)." The values in the following sample table were determined using logarithms of base 2. The relationships found are the same as those found in Parts **c–e**.

x	y	$\log_2 x$	$\log_2 y$	$\log_2(x \bullet y)$	$\log_2(x/y)$	$\log_2 x^y$	$2^{\log x}$	$\log 2^x$
1	25	0.000	4.644	4.644	–4.644	0.000	1	1
2	24	1.000	4.585	5.585	–3.585	24.000	2	2
3	23	1.585	4.524	6.109	–2.939	36.454	3	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
23	3	4.524	1.585	6.109	2.939	13.571	23	23
24	2	4.585	1.000	5.585	3.585	9.170	24	24
25	1	4.644	0.000	4.644	4.644	4.644	25	25

Discussion

(page 123)

a. As students experiment with other bases, they should find that the relationships found for common logarithms remain true for $b > 0$ and $b \neq 1$.

b. Sample response: When multiplying two numbers with the same bases raised to powers, you add the exponents. Since logarithms are exponents to same bases, the logarithm of a product is the sum of the logarithms of the numbers.

When dividing two numbers with the same bases raised to powers, you subtract the exponents. Since logarithms are exponents to same bases, the logarithm of a quotient is the difference of the logarithms of the two numbers.

When raising a power to a power, the exponents are multiplied. Since $\log_b x$ is an exponent, the expression $(\log_b x)^y$ is a power raised to a power. Therefore, the powers are multiplied resulting in $y \bullet \log_b x$. **Note:** You may wish to demonstrate how this property can be derived directly from the properties of exponents, as shown below.

Let $y \bullet \log_b x = z$. This implies that

$$\log_b x = z/y$$

$$x = b^{z/y}$$

Let $\log_b x^y = q$. This implies that $x^y = b^q$. Substituting for x yields the following:

$$\left(b^{z/y}\right)^y = b^q$$

$$b^z = b^q$$

$$z = q$$

Therefore, $\log_b x^y = y \log_b x$.

c. The equation $\log_b x = \log_b y$ is true exactly when $x = y$.

Assignment

(page 124)

3.1 **a.** $x \approx 3.0$

b. $x \approx 4.3$

c. $x \approx 5.6$

3.2 Sample response:

$$[H^+] = 1/10^{pH}$$
$$[H^+] = 10^{-pH}$$
$$\log[H^+] = \log 10^{-pH}$$
$$\log[H^+] = -pH$$
$$-\log[H^+] = pH$$

***3.3** **a.** Predictions will vary.

b. Sample response: $c = 4^d$, where c represents the number of cells and d represents days.

c. Students may use a spreadsheet, graphing utility, or symbolic manipulator to approximate their answers. Using this model, it should take approximately 8.3 days to reach the desired population.

d. Solving algebraically:

$$100{,}000 = 4^d$$
$$\log 100{,}000 = \log 4^d$$
$$5 = d \log 4$$
$$5/\log 4 = d$$
$$8.3 \approx d$$

e. Some symbolic manipulators will solve using natural logs (ln) with base e, as shown below. The solution is the same using any base.

$$\ln 100{,}000/\ln 4 \approx 8.3$$

3.4 Since the change in magnitude on the Richter scale is the logarithm of the change in intensity, the magnitude of the Sylmar earthquake can be calculated as follows:

$$1 \bullet 10^{6.9} = 4 \bullet 10^x$$
$$10^{6.9}/4 = 10^x$$
$$\log\left(10^{6.9}/4\right) = \log 10^x$$
$$\log 10^{6.9} - \log 4 = x$$
$$6.9 - 0.6 = x$$
$$6.3 \approx x$$

3.5 **a.** **1.** Applying the logarithmic property of multiplication, $\log(2n) = \log 2 + \log n$. The magnitude of the quake on the Richter scale is therefore increased by log 2, or about 0.301.

2. Sample response: No, any initial intensity can be substituted for *n*. The increase remains log 2.

b. Using the logarithmic property of multiplication, $\log(3n) = \log 3 + \log n$. The magnitude of the quake on the Richter scale is therefore increased by log 3, or approximately 0.477.

* * * * *

3.6 **a.** Estimates will vary.

b. It will take a little over 25 years for the principal and interest to reach $450. Some students may enter the formula $100(1+(0.06/12))^{12x}$ in a spreadsheet and vary the values for *x* until they find a result close to $450. Others may enter the equations $100(1+(0.06/12))^{12x}$ and $y = 450$ in a graphing utility and identify the point of intersection. Still others may use a symbolic manipulator to solve for *x*.

c. **1.** Substitute appropriate quantities into the formula.

2. Simplify quantity in the parentheses.

3. Divide both sides of the equation by 100.

4. Simplify the left side of the equation.

5. Take the logarithm of both sides of the equation.

6. Apply power property of logarithms to right side of equation.

7. Divide both sides of the equation by log(1.005).

8. Simplify the left side of the equation.

9. Divide both sides of the equation by 12.

10. Simplify left side of the equation.

d. The power property of logarithms was used to proceed from Step **5** to Step **6**.

e. In Step **4**, *t* is in the exponent. Therefore, logarithms are needed to solve for the value of *t*. In Step **5**, the $\log(1.005)^{12t}$ can be simplified using the power property of logarithms. This provides an avenue for which *t* may be isolated and determined.

* * * * * * * * * *

Research Project

(page 126)

Student projects will vary. Large earthquakes have recently been reported in California, Japan, China, and the Philippines. The Internet may provide a convenient source of information on recent quakes.

Answers to Summary Assessment (page 127)

1. **a.** Students should graph the data on a semilog scale, then re-express it to find the relationship $0.1d - 4.7 = \log P$.

b. Solving algebraically:

$$\begin{aligned} d &= 10 \bullet \log(P/P_r) \\ 0.1 \bullet d &= \log P - \log P_r \\ 0.1 \bullet d &= \log P - \log 0.00002 \\ 0.1 \bullet d &\approx \log P + 4.7 \\ 0.1 \bullet d - 4.7 &\approx \log P \end{aligned}$$

c. Student responses may take any of the following forms:

$$P \approx 10^{(0.1 \bullet d - 4.7)} \text{ or } P \approx \frac{(10^{0.1})^d}{10^{4.7}} \text{ or } P \approx (0.00002)(1.259^d)$$

2. **a.** Using the formula from Problem **1c**, the pressure exerted on the eardrum by normal conversation is approximately 5.04 N/m^2. The pressure exerted on the eardrum at a rock concert is approximately $1.27 \bullet 10^7 \text{ N/m}^2$. This is over 2.5 million times the pressure exerted by normal conversation.

b. Solving the formula $P \approx 10^{(0.1 \bullet d - 4.7)}$ for d results in $d \approx 10(\log P + 4.7)$. A pressure of $1 \bullet 10^3 \text{ N/m}^2$ corresponds to a decibel level of approximately 77. This sound would be somewhat louder than a normal speaking voice, but not painful.

3. **a.** From the table, the pressure created by heavy traffic is approximately $20,000 \text{ N/m}^2$. This corresponds to a depth of about 2 m.

b. Sample response: If a person dove to a depth of 100 m, the pressure on the eardrum would be $100 \bullet 10,000 = 1,000,000 = 1 \bullet 10^6 \text{ N/m}^2$. From the table, this pressure corresponds to a decibel level of about 110, or about as loud as a power mower.

Module Assessment

1. A game farm currently has a population of 200 deer. Without any harvest, this population will increase by 25% each year. This situation can be modeled by the equation $P = P_0 \bullet (1 + r)^t$, where P is the total population, P_0 is initial population, r is the annual growth rate, and t represents time in years. Using this model, how many years will it take for the deer population to exceed 1000?

2. Solve for x in each of the following equations:

a. $9^x = 76$

b. $\log x = 5.3$

c. $10^x = 10,000,000$

3. Suppose that your stomach feels good at a pH of 2.5. On the day before a big event, you become extremely nervous. This causes the amount of acid in your stomach to increase, lowering the pH to 2.0. Since this feels uncomfortable, you take one tablet of antacid, raising the pH to 3.0. An antacid company claims that its product "neutralizes" stomach acid. A neutral solution has a pH of 7. How many antacid tablets would you have to take to raise the pH to 7? Is the company's claim realistic? Explain your response.

Answers to Module Assessment

1. $t \approx 8$ years

2. **a.** $x = \log_9 76$ or ≈ 1.97

b. $x = 10^{5.3} \approx 1.995 \bullet 10^5$

c. $x = 7$

3. Sample response: If *neutralizes* means that the pH in the stomach is raised to 7, the company's claim is unrealistic. To raise the pH from 3 to 7 would require 10^{7-3} or 10^4 times as much antacid as the amount required to raise the pH from 2 to 3. That means a consumer would have to take 10,000 tablets.

Selected References

Fessenden, R. J., and J. S. Fessenden. *Chemical Principles for the Life Sciences*. Boston, MA: Allyn and Bacon, 1979.

Malnic, E., and C. Decker. “Massive Quake Hits Bay Area.” *The Los Angeles Times* 18 Oct. 1989: A1.

Stahler, A. N. *The Earth Sciences*. New York: Harper and Row, 1971.

Woram, J. M. *The Recording Studio Handbook*. Plainview, NY: Sagamore, 1977.

Flashbacks

Activity 1

1.1 Write each of the following expressions as a fraction in the form a/b where a and b are integers.

a. 10^{-5}

b. 10^{3}

c. 2^{-3}

d. 3^{4}

1.2 Write each of the expressions in Flashback **1.1** as a decimal.

1.3 Write each of the following expressions as a power of 10.

a. $1000 \bullet 100$

b. $10{,}000{,}000/100$

Activity 2

2.1 The equation $y = 10^{x}$ is an example of an exponential equation. Give another example of an exponential equation and describe its parts using the mathematical terms *base* and *exponent*.

2.2 Find the value of x in the equation $1{,}000{,}000 = 10^{x}$.

2.3 Identify the power of 10 equal to each of the following:

a. 1

b. 0

c. –1

Activity 3

3.1 Use the properties of exponents to simplify the following expressions:

a. $5^3 \bullet 5^4$

b. $x^3 \bullet x^{-4}$

c. $(7x^5)^0(3x^4y)^3$

d. $6^7/6^2$

e. $\dfrac{c^{-3}}{c^{-5} \bullet c^2}$

3.2 Since $7 = 7^1$, it follows that there must be a value for *x* such that $(7^5)^x = 7^1$. What is the value for *x*?

Answers to Flashbacks

Activity 1

1.1 **a.** $10^{-5} = 1/100,000$

b. $10^{3} = 1000/1$

c. $2^{-3} = 1/8$

d. $3^{4} = 81/1$

1.2 **a.** 0.00001

b. 1000

c. 0.125

d. 81

1.3 **a.** $1000 \bullet 100 = 100,000 = 10^{5}$

b. $10,000,000/100 = 100,000 = 10^{5}$

Activity 2

2.1 Sample response: Another example of an exponential equation is $y = 2^{x}$. The 2 represents the base, x represents the exponent, and y represents the value of 2^{x}.

2.2 $x = 6$

2.3 **a.** 10^{0}

b. Sample response: There is no power when $10^{x} = 0$. However, as the value of x decreases without bound, 10^{x} approaches 0.

c. There is no power of 10 that results in a negative value.

Activity 3

3.1 **a.** $5^{3} \bullet 5^{4} = 5^{7}$

b. $x^{3} \bullet x^{-4} = x^{-1}$

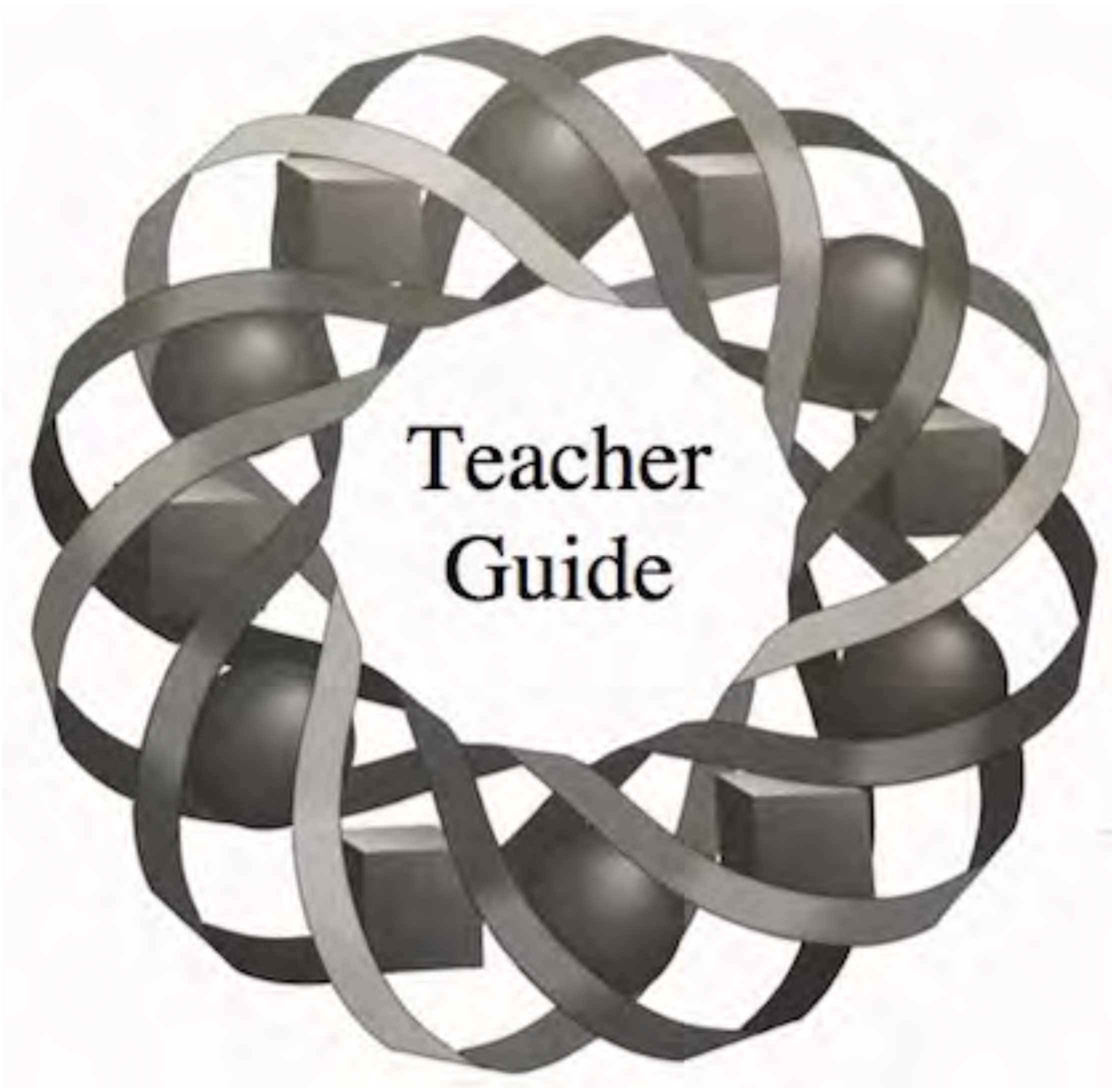
c. $\left(7x^{5}\right)^{0}\left(3x^{4}y\right)^{3} = 7^{0} \bullet x^{0} \bullet 3^{3} \bullet x^{12} \bullet y^{3} = 27x^{12}y^{3}$

d. $6^{7}/6^{2} = 6^{5}$

e. $\dfrac{c^{-3}}{c^{-5} \bullet c^{2}} = \dfrac{c^{-3}}{c^{-3}} = 1$

3.2 $x = 1/5$ or 0.2

More or Less?



What do car engines and compact discs have in common? When manufacturing either product, quality control engineers use inequalities to represent tolerance and precision intervals.

Kyle Boyce • Ruth Brocklebank • Mark Lutz
Pete Stabio • Tom Teegarden

Teacher Edition

More or Less?

Overview

In this module, students explore inequalities in various forms: linear, absolute value, and polynomial.

Objectives

In this module, students will:

- interpret and solve linear, absolute-value, and polynomial inequalities
- create a graphical representation leading to the concept of limits
- use mapping diagrams to represent mathematical relationships.

Prerequisites

For this module, students should know:

- interval notation
- function notation
- definitions of inequalities
- how to express an interval as an inequality.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	2	1	9

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
compass	X				
scissors	X				
rulers	X				
template A	X				
string	X				
graph paper		X		X	

Teacher Note

A blackline master for template A appears at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity				
	1	2	3	4	Summary Assessment
graphing utility			X	X	
geometry utility		X		X	
symbolic manipulator	X			X	

More or Less?

Introduction

(page 133)

This module uses tolerance and precision intervals in quality control to explore inequalities.

Discussion

(page 133)

a. Consumers want to purchase the best possible product for the price. Quality control helps assure that consumers are satisfied. Product quality also can be crucial to consumer safety, as in seat belts or prescription medicines, for example.

b. Quality control is important to manufacturers because it results in customer satisfaction and repeat buyers. Good quality control can also help companies avoid lawsuits triggered by faulty products.

c. Answers will vary. Sample responses: BMW automobiles, Sony electronics, and General Electric appliances.

d. Sample response: Given a choice between two comparably priced items, consumers will purchase the better one. This encourages manufacturers to produce high-quality items at a competitive price.

e. Differences in raw materials and imperfections in the manufacturing process (due to machine error) make it impossible for manufacturers to consistently produce identical items.

f. Defective items may be returned to the place of purchase or to the manufacturer. If neither of these options produces a satisfactory result, there are many state and federal organizations devoted to protecting consumers, including the Consumer Protection Agency, the Better Business Bureau, and Consumer Affairs.

(page 134)

Activity 1

Students develop a tolerance interval for the circumference of simulated compact discs and investigate its relationship to the interval containing the discs' radii.

Materials List

- compass (one per student)
- string (one 40-cm length per student)
- scissors (one per student)
- ruler (one per student)
- template A (one per student)

Teacher Note

A blackline master for template A appears at the end of the teacher edition for this module.

Technology

- symbolic manipulator

Exploration

(page 134)

Students should observe that a tolerance interval involving a linear function produces a predictable precision interval. The sample responses in Parts **b–d** are based on the data given in Part **a**.

a. **1.** Students create five paper circles that represent compact discs with a circumference of 37 cm.

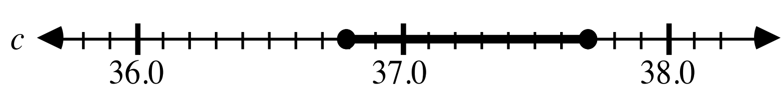
2. Students should find that their five discs are not exactly the same size.

3. The appropriate number of significant digits should be determined by the calibration of the measuring instrument. The following sample data is reported to the nearest 0.1 cm:

36.8	37.4	37.1	37.7	37.4

b. **1.** The bounds of the tolerance interval that contains the sample data are 37.7 and 36.8.

2. Graph of sample data:

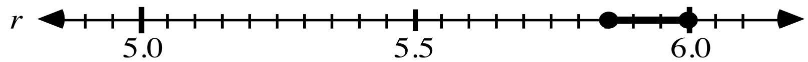


3. The tolerance interval is $36.8 \le c \le 37.7$, where c represents the circumference of the discs. Using set notation, this inequality may be represented as follows: $c \in [36.8, 37.7]$.

c. 1. The radius r for the desired circumference is $37/2\pi \approx 5.89$ cm.

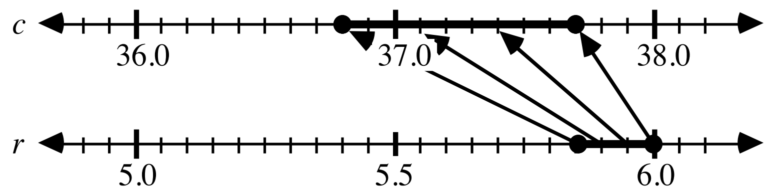
2. The bounds of the precision interval from the sample data are 5.86 and 6.00. The inequality representing the interval is $5.86 \le r \le 6.00$.

3. Graph of sample data:



4. The precision interval may be represented as $r \in [5.86, 6.00]$, where r represents the radius of the discs.

d. Sample mapping diagram:



e–f. Students gather the class data, determine a tolerance interval for the circumference, then determine the corresponding precision interval for the radius. They represent these intervals as inequalities and create graphs of the intervals on template A.

Discussion

(page 136)

a. Sample response: The five discs were not exactly the same because of the lack of precision in the tools and in the cutting process.

b. The function that relates radius to circumference is $c = f(r) = 2\pi r$.

c. Using the class data, all students should describe the same tolerance and precision intervals (except for possible rounding differences). Using the sample data given in the exploration, the tolerance interval for the circumference, in centimeters, is [36.8, 37.7]. The corresponding precision interval for the radius in centimeters is [5.86, 6.00].

d. Answers will vary, depending on class data.

e. Answers will vary. Sample response: If $r \in [5.86, 6.00]$, then $c \in [36.8, 37.7]$, and vice versa.

f. Because the function $c = f(r) = 2\pi r$ is continuous and strictly increasing, students may divide the minimum and maximum values of the circumference by 2π to determine the corresponding minimum and maximum values of the radius. If c is an element of $[a,b]$, then

$$a \le c \le b$$
$$a \le 2\pi r \le b$$
$$\frac{a}{2\pi} \le r \le \frac{b}{2\pi}$$

Thus, r is an element of $\left[a/2\pi,\ b/2\pi\right]$.

g. **1.** $[0,\infty)$

2. $(-\infty,0)$

Assignment

(page 136)

1.1 **a.** Sample response:

$$113.4 < 3x$$
$$113.4/3 < 3x/3$$
$$37.8 < x$$

Students should substitute values for x greater than 37.8 into the original inequality and verify that they are correct solutions.

b. Sample response:

$$3x < 114.6$$
$$3x/3 < 114.6/3$$
$$x < 38.2$$

Students should substitute values for x less than 38.2 into the original inequality and verify that they are correct solutions.

c. A symbolic manipulator produces the same results.

d. $(37.8,\infty)$ and $(-\infty,38.2)$

e. The solution is the intersection of the two intervals in Part **d**. This may be written as the inequality $37.8 < x < 38.2$. Using interval notation, x is an element of $(37.8, 38.2)$.

1.2 **a.** $36.94 \le 2\pi r \le 37.06$ or $5.88 \le r \le 5.90$

b. $[5.88, 5.90]$

c. Sample response: Decreasing the tolerance interval creates a narrower precision interval. This means that the machines must be more accurate in order to produce acceptable discs.

1.3 **a.** $237 \le 6x + 30 \le 243$ or $34.5 \le x \le 35.5$

b. $x \in [34.5, 35.5]$

c. Sample response: I subtracted 30 from each term in the inequality, then divided each term by 6.

1.4 **a.** **1.** Students may respond with either $5 > 3$ or $3 < 5$. The following sample responses are based on the inequality $3 < 5$.

2. $15 < 25$

3. The inequality is true.

b. **2.** $-15 < -25$

3. This inequality is false. By changing the sense of the inequality, it can be made into the true statement $-15 > -25$.

c. When multiplying an inequality by a positive number, the inequality sign remains the same. When multiplying an inequality by a negative number, the inequality sign (the sense of the inequality) must be reversed. By multiplying an inequality by 0, one achieves an equality.

d. Students should demonstrate that their generalization holds true for at least four other examples.

e. Sample response: If $b = a + c$ and $n < 0$, then $nb = n(a + c)$. This implies that $nb = na + nc$. Since nc is negative, $nb < na$ or $na > nb$.

1.5 **a.** Sample response: Division may be performed by multiplying by the reciprocal of the divisor. Thus, the results for multiplication and division are the same. Division by 0 is impossible.

b. Students should demonstrate that the generalization holds true for at least four examples.

***1.6** **a.** $30 < 630 - 6x < 36$

b. $99 < x < 100$; (99, 100)

* * * * *

1.7 Sample response: This situation can be described by the inequality $43 + 0.5w > 70$, where w represents number of weeks. Solving the inequality yields $w > 54$. The snow leopard will require at least 54 weeks to reach a healthy weight of 70 kg.

1.8 **a.** $-50 \le x \le 50$

b. $l_0 + \left(1.05 \bullet 10^{-5}(-50)\right)l_0 \le l \le l_0 + \left(1.05 \bullet 10^{-5}(50)\right)l_0$

c. $299.8425 \le l \le 300.1575$

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Activity 2

In this activity, students use a graphical model to explore the relationship between tolerance and precision intervals.

Materials List

- graph paper (one sheet per student)

Technology

- geometry utility

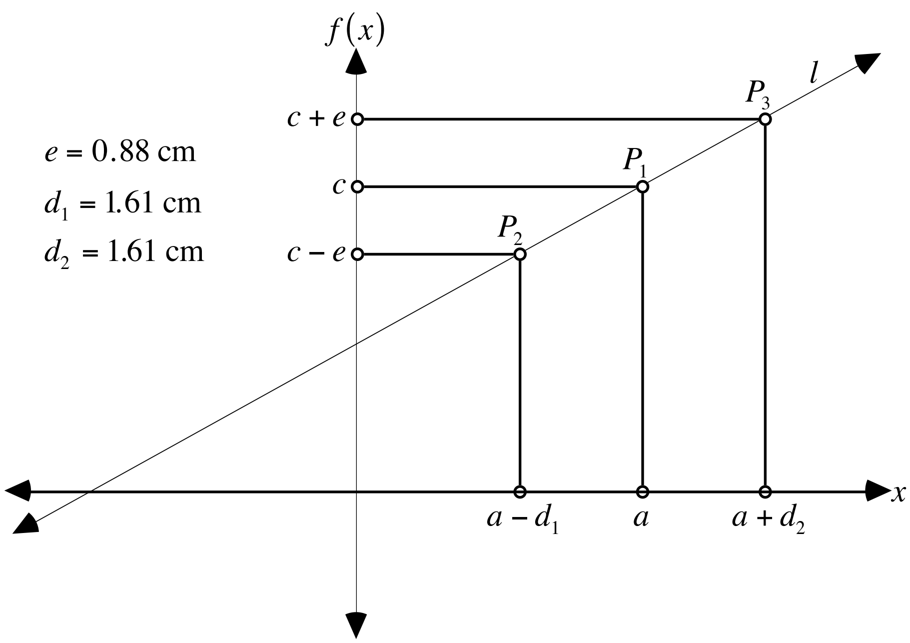
Teacher Note

To save class time, you may wish to produce the sketch described in Part **a** of the exploration yourself, then allow students to copy it. The construction should be created so that when the point with coordinates $(0, c - e)$ is moved, c and a remain fixed, while the intervals $(c - e, c + e)$ and $(a - d_1, a + d_2)$ move (along with the corresponding perpendicular segments to the axes). This shows a relation between e and the values of d_1 and d_2.

Exploration

(page 139)

a–b. The construction should resemble the one shown below:



c. Sample data:

e	d_1	d_2
0.494	0.904	0.904
0.600	1.098	1.098
0.776	1.421	1.421
0.882	1.614	1.614
0.988	1.808	1.808
1.164	2.131	2.131
1.270	2.325	2.325
1.446	2.647	2.647
1.552	2.841	2.841
1.799	3.293	3.293

d. **1.** $c - e < f(x) < c + e$

2. $a - d_1 < x < a + d_2$

Teacher Note

After discussing the mathematics note, you may wish to emphasize that $\lim_{x \to a} f(x)$ is not always equal to $f(a)$.

Discussion

(page 140)

a. In the exploration in Activity **1**, the desired circumference of the compact disc, 37 cm, could be represented by c, while the corresponding radius, 5.9 cm, could be represented by a.

b. As e increases, both d_1 and d_2 increase. As e decreases, both d_1 and d_2 decrease.

c. The larger the tolerance interval, the larger the corresponding precision interval. The smaller the tolerance interval, the smaller the corresponding precision interval.

d. Since the precision interval $(a - d_1, a + d_2)$ corresponds with the tolerance interval $(c - e, c + e)$, it follows that if $P(x_1, 0)$ is a point in the interval $(a - d_1, a + d_2)$, then $f(x_1)$ must be in the interval $(c - e, c + e)$.

e. In this case, because l is a strictly increasing line, one can determine the x-values that correspond to the $f(x)$-values that are the endpoints of the tolerance interval. These resulting x-values are the endpoints of the precision interval. **Note:** Although this process works on oblique lines, you may wish to caution students that it does not work in general.

f. Sample responses may include the gap setting of a spark plug, the size of a shoe or other article of clothing, or the mass of a box of cereal.

Assignment

(page 142)

2.1 **a.** $x = 4.5$

b. Sample response:

$$11.8 < 2x + 3 < 12.2$$
$$8.8 < 2x < 9.2$$
$$4.4 < x < 4.6$$

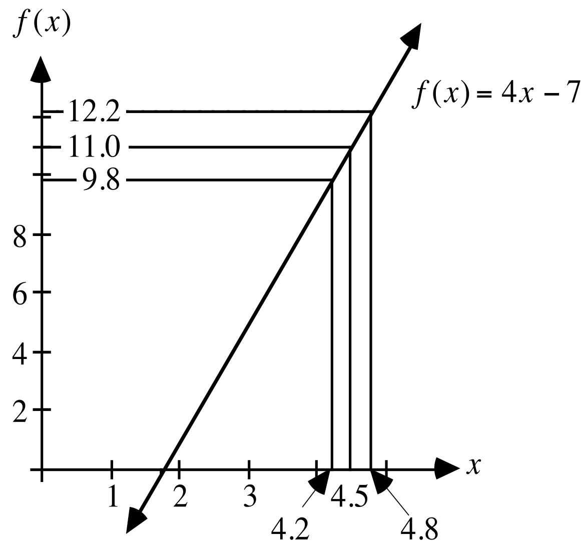
c. $(12 - 0.2,\ 12 + 0.2)$

d. $4.45 < x < 4.55$ or $(4.45, 4.55)$

***2.2** **a.** Sample response: The limit is 11.

b. $(4.2, 4.8)$

c. Sample sketch:



2.3 **a.** Sample response:

$$13 - e < 3x + 1 < 13 + e$$
$$12 - e < 3x < 12 + e$$
$$\frac{12 - e}{3} < x < \frac{12 + e}{3}$$
$$4 - \frac{e}{3} < x < 4 + \frac{e}{3}$$

b. Since $a = 4$ and $d = e/3$, the interval is:

$$\left(4 - \frac{e}{3}, 4 + \frac{e}{3}\right)$$

2.4 **a.** $f(x) = 56x + 2$

b. $47.9 - e \le 56x + 2 \le 47.9 + e$

c. $0.82 - \frac{e}{56} \le x \le 0.82 + \frac{e}{56}$

d. $a \approx 0.82$

2.5 **a.** $f(x) = 8x + 12$

b. $(292, 300)$

c. $(35,36)$

* * * * *

2.6 If t is the time spent in the air, then t must satisfy the inequality $3.25 < t + 0.5 < 3.5$. Since $t = d/r$ where d is the distance traveled and r is the average speed, the inequality becomes $3.25 < (540/r) + 0.5 < 3.5$. Solving for r results in the inequality $180 < r < 196.4$, where r is the speed in kilometers per hour. Thus, r is in the interval $(180,196.4)$.

An alternative method of solution begins with $r = d/t$. Using the endpoints of the time interval to determine the smallest and largest possible rates results in the following inequality:

$$\frac{2(270)}{3.5 - 0.5} < r < \frac{2(270)}{3.25 - 0.5}$$

Simplifying this expression results in the interval $(180,196.4)$.

2.7 Sample response: For x^3 to fall in the interval (7.9, 8.1), x must fall in the interval below:

$$\left(\sqrt[3]{7.9},\ \sqrt[3]{8.1}\right)$$

or approximately (1.992, 2.008). Therefore x must be within 0.008 of 2.

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(page 143)

Activity 3

In this activity, the absolute value function is introduced as a means of finding the distance between two numbers. Absolute values are then used to represent tolerance and precision intervals.

Materials List

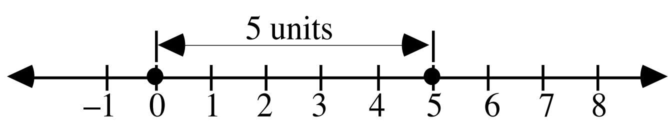
- none

Technology

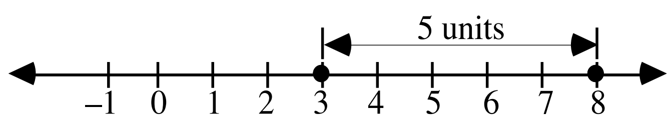
- graphing utility

Teacher Note

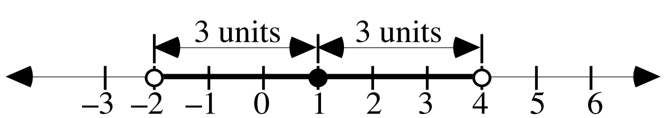
The absolute value of $x - a$, where x and a are real numbers, represents the distance between x and a. For example, the expression $|5 - 0| = |5|$ can be represented on a number line as shown below.



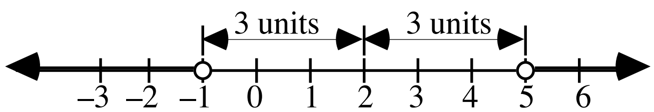
Similarly, the expression $|3 - 8| = |-5|$ can be represented as follows:



The inequality $|x - a| < b$, where b is a positive number, is satisfied by any real number x whose distance from a is less than b units. These real numbers constitute the interval $(a - b, a + b)$. For example, the inequality $|x - 1| < 3$ is shown on the number line below. The corresponding interval is $(1 - 3, 1 + 3) = (-2, 4)$. Another way to write this is $x \in (-2, 4)$.



The inequality $|x - a| > b$, where b is a positive number, is satisfied by any real number x whose distance from a is more than b units. These real numbers constitute the intervals $(-\infty, a - b)$ or $(a + b, \infty)$. For example, the inequality $|x - 2| > 3$ is shown on the following number line.

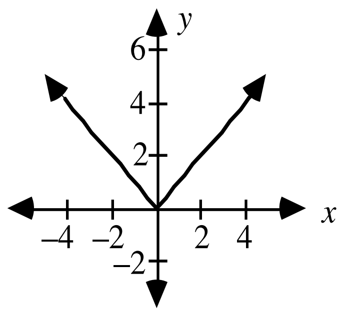


The corresponding intervals that satisfy the inequality are $(-\infty, 2 - 3) = (-\infty, -1)$ or $(2 + 3, \infty) = (5, \infty)$. Note that in this case, solutions in either interval satisfy the original inequality. Another way to write this is $x \in (-\infty, -1) \cup (5, \infty)$.

Exploration

(page 143)

a. Sample graph:



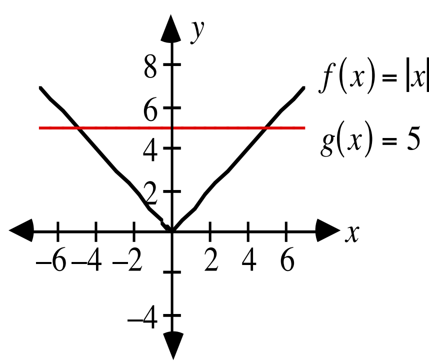
b. Sample response: The value of the *y*-coordinate has the same magnitude as the value of the *x*-coordinate, and is non-negative.

c. **1.** $x = 5$ and $x = -5$

2. $x = 0$

3. There are no solutions because absolute values are never negative.

d. **1.** The solution set to the inequality $|x| < 5$ is the interval $(-5, 5)$. Sample graph:

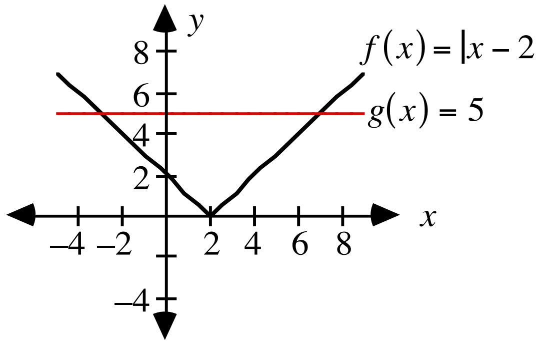


2. The solution set to the inequality $|x| > 5$ is the solution to the inequalities: $x < -5$ or $x > 5$.

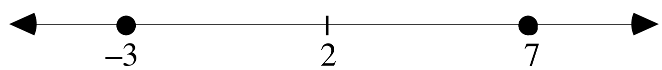
3. Using interval notation, x is an element of $(-\infty,-5)$ or $(5,\infty)$. This can be expressed as $(-\infty,-5)\cup(5,\infty)$.

e.

1. The solution set to the equation $|x-2|=5$ is $x=-3$ and $x=7$. Sample graph:



2. Sample graph:

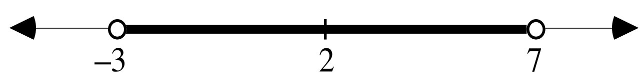


3. Sample response: Each solution is 5 units from 2.
4. $x\in\{3,7\}$

f. In each case, the solutions will be b units from a, or $x\in\{a-b,a+b\}$.

g.

1. The solution to the inequality $|x-2|<5$ is $-3<x<7$.
2. Sample graph:



3. Sample response: The solutions are all the numbers that are less than 5 units from 2.
4. $x\in(-3,7)$

h. In each case, the solutions will be all the numbers that are within b units of a; or $x\in(a-b,a+b)$.

Discussion

(page 144)

a. Sample response: The graph is a V-shaped figure with the vertex of the V at the origin and the sides of the V in the first and second quadrants. The graph is made of two diverging rays from the origin. The line that contains the ray in the second quadrant has slope –1; the line that contains the ray in the first quadrant has slope 1.

b. The domain is the set of real numbers and the range is the set of non-negative real numbers.

c. The absolute value of x represents the distance between x and 0.

d. These expressions represent the distance between x and 2 and x and –2, respectively.

e. Sample response: The interval is symmetrical about 7 with its endpoints 2 units from 7. Two equivalent expressions are $|x - 7| < 2$ and $|2x - 14| < 4$. There are many others.

f. **1.** $x \in (a - b, a + b)$

2. $x \in (-\infty, a - b) \cup (a + b, \infty)$

Assignment (page 145)

3.1 **a.** The interval consists of the numbers within 4 units of 11:

b. The graph consists of the numbers more than 4 units from 7:

c. The interval consists of the numbers within 10 units of –15:

3.2 Answers may vary. Some possible responses are shown below.

a. $|x - 1| < 6$

b. $|x + 5.5| < 1.5$

c. $|x - 26| > 2$

d. $|x - 4.5| < 1.5$

e. $|x - 1| > 13$

3.3 $x \in (352, 356)$

3.4 **a.** $|V - 1020| < 3$ or $|6x - 1020| < 3$

b. $|x - 170| < 0.5$

***3.5** **a.** $|M - 224| < 3$

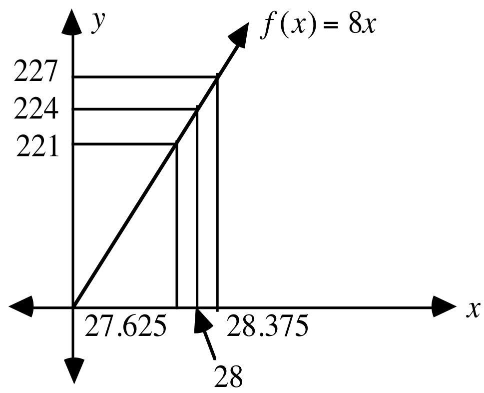
b. $f(x) = 8x$

c. $221 < 8x < 227$; $27.625 < x < 28.375$

d. $a = 28$; $d = 0.375$

e. $|x - 28| < 0.375$

f. Sample graph:



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3.6 **a.** $440 < 4x + 6 < 464$

b. $(108.5, 114.5)$

3.7 $(-\infty, -13) \cup (86, \infty)$

* * * * * * * * * *

(page 146)

Activity 4

In this activity, students explore nonlinear inequalities.

Materials List

- graph paper

Technology

- geometry utility
- graphing utility
- symbolic manipulator

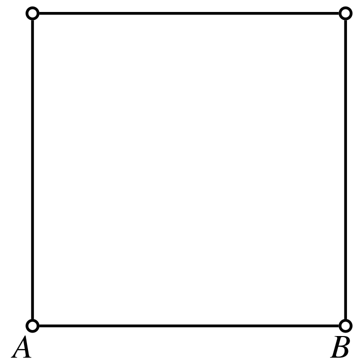
Exploration (page 146)

a–c. Sample response:

Distance $(A \text{ to } B) = 4.0$ cm

Perimeter (Polygon 1) $= 16.0$ cm

Area (Polygon 1) $= 16.0 \text{ cm}^2$

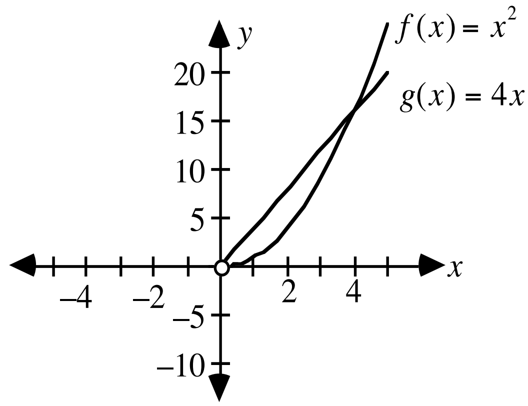


d.

1. The function $f(x) = x^2$ represents the area of a square, where x represents the length of a side and $x > 0$.

2. The function $g(x) = 4x$ represents the perimeter of a square, where x represents the length of a side and $x > 0$.

3. Sample graph:



4. $0 < x^2 < 4x$, which implies $0 < x < 4$ since $x > 0$

5. From inspection of the graphs, the interval in which the value of the area function is less than the value of the perimeter function is $(0, 4)$.

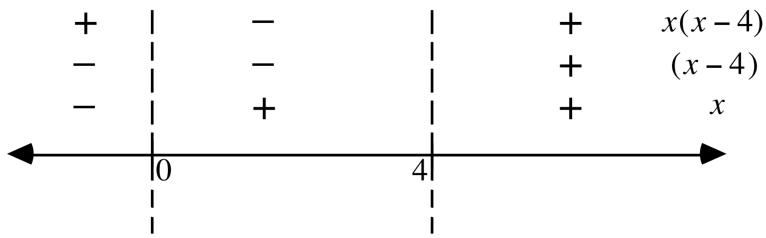
e.

1. The resulting inequality is $x^2 - 4x < 0$ or $4x - x^2 > 0$. It can be written in factored form as $x(x-4) < 0$.

2. The factors x and $(x-4)$ have zero values at $x = 0$ and $x = 4$, respectively.

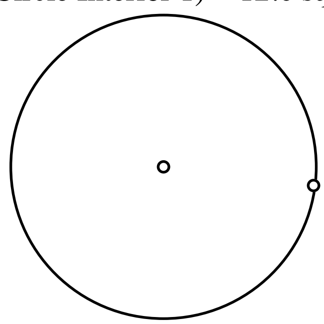
3. These values are the endpoints of the interval $0 < x < 4$, which is the interval determined from the graphs of the functions in Part **d**.

4. **Note:** You may wish to demonstrate the use of a number line, as shown below, to complete this step. The product of the factors is negative only in the interval $(0, 4)$.

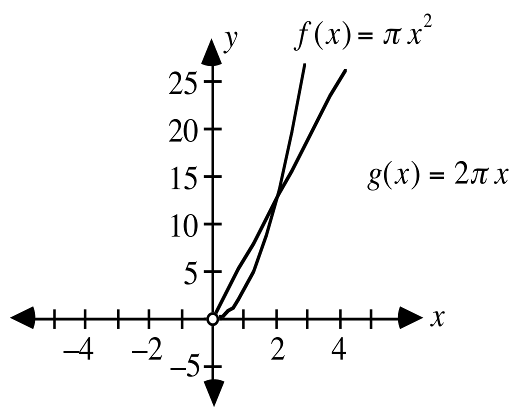


f. Sample response:

Radius (Circle Interior 1) = 2.0 cm
Circumference (Circle Interior 1) = 12.6 cm
Area (Circle Interior 1) = 12.6 square cm



The function $f(x) = \pi x^2$ represents the area of the circle, where x represents the radius and $x > 0$. The function $g(x) = 2\pi x$ represents the circumference of the circle, where x represents the radius and $x > 0$.
Sample graph of $f(x)$ and $g(x)$:

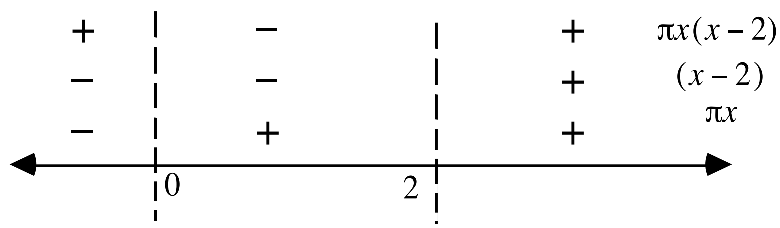


The values of x for which the magnitude of the circle's area is less than the magnitude of its circumference is $0 < \pi x^2 < 2\pi x$.

From inspection of the graphs, the interval in which the value of $f(x)$ is less than the value of $g(x)$ is $0 < x < 2$.

The difference between $f(x)$ and $g(x)$ can be expressed as $\pi x^2 - 2\pi x < 0$. This can be written in factored form as $\pi x(x-2) < 0$. The factors (πx) and $(x-2)$ have zero values at $x = 0$ and $x = 2$, respectively.

The signs of the expressions πx, $(x-2)$, and $\pi x(x-2)$ are shown in the following diagram:



Discussion

(page 147)

a. **1.** The graph is a straight line, represented by a first-degree equation.

2. The graph is a parabola, represented by a quadratic (second-degree) equation.

b. **1.** Sample response: When a portion of the parabola representing the area function is above the line representing the perimeter function, the magnitude of the area is greater than the magnitude of the perimeter.

2. Sample response: When a portion of the parabola representing the area function is below the line representing the perimeter function, the magnitude of the area is less than the magnitude of the perimeter.

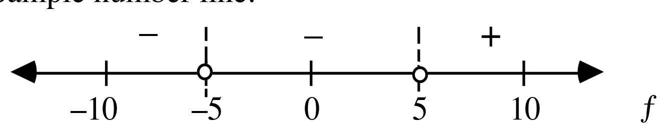
c. **1.** Sample response: Graph the functions $f(x) = x^n$ and $g(x) = k$. The intervals of the domain in which the graph of *f* is above the graph of *g* represent the inequality $x^n > k$.

2. Sample response: Graph the functions $f(x) = x^n$ and $g(x) = k$. The intervals of the domain in which the graph of *f* is below the graph of *g* represent the inequality $x^n < k$.

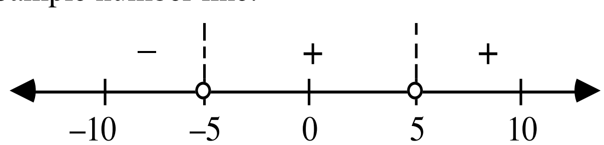
Assignment

(page 148)

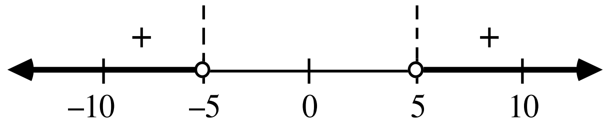
4.1 **a.** Sample number line:



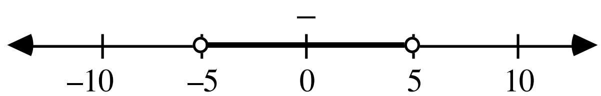
b. Sample number line:



c. The inequality $(x-5)(x+5)>0$ is true when $x<-5$ or $x>5$. Sample number line:

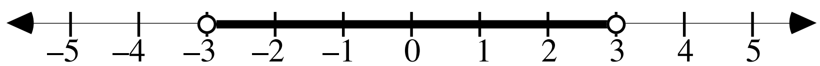


d. The inequality $(x-5)(x+5)<0$ is true when $-5<x<5$. Sample number line:



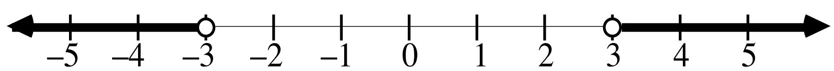
e. The solution for $x^2<25$ is $-5<x<5$ or $x\in(-5,5)$. This is true because $x^2<25$ is equivalent to $x^2-25<0$, which is equivalent to $(x-5)(x+5)<0$. Similarly, $x^2>25$ has the same solution as $(x-5)(x+5)>0$, or $x\in(-\infty,-5)\cup(5,\infty)$.

4.2 **a.** The solution set is the inequality $-3<x<3$ or $x\in(-3,3)$. Sample graph:

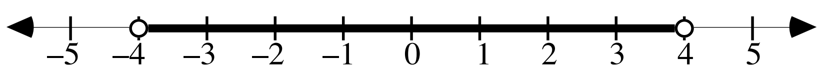


b. $|x|<3$

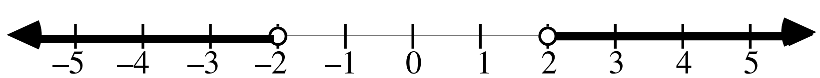
c. The solution set is the intervals $x<-3$ or $x>3$ or $x\in(-\infty,-3)\cup(3,\infty)$. Sample graph:



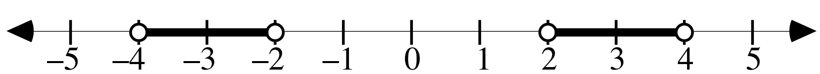
***4.3** **a.** The solution set is the inequality $-4<x<4$ or $x\in(-4,4)$. Sample graph:



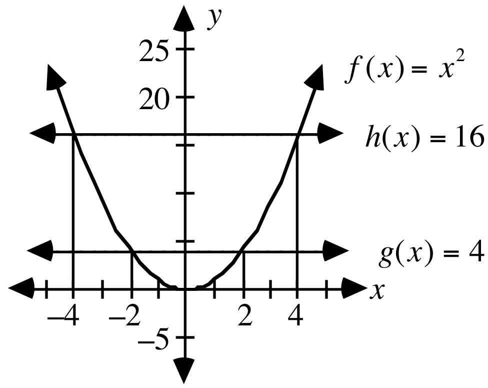
b. The solution set is the intervals $x<-2$ or $x>2$, or $x\in(-\infty,-2)\cup(2,\infty)$. Sample graph:



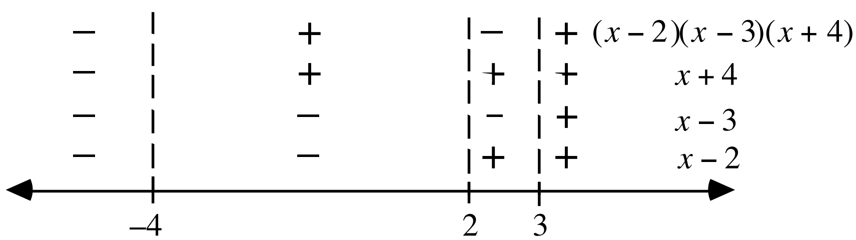
c. The solution set to the inequality $4<x^2<16$ must satisfy both conditions in Parts **a** and **b**. Thus, the solution set is $-4<x<-2$ or $2<x<4$, or $x\in(-4,-2)\cup(2,4)$. Sample graph:



d. Sample graph:



4.4 The inequality is true when $x \in (-\infty, -4) \cup (2, 3)$. Students may use a number line (as described in Part **e** of the exploration) to determine this solution.



4.5 **a.** A function that represents the volume is $V(x) = 4\pi x^2$, where x is the radius of the base and $x > 0$. A function that represents the surface area is $A(x) = \pi x^2 + 8\pi x$, where $x > 0$. Solving the inequality $4\pi x^2 > \pi x^2 + 8\pi x$ results in $x \in (8/3, \infty)$.

b. The volume of the jar for which the magnitude of the volume is approximately equal to the magnitude of the surface area occurs when $x = 8/3$. At that point, the volume is approximately 89.4 cm^3.

4.6 **a.** The desired length for a side is approximately 7.05 cm.

b. Since the tolerance is $\pm 5\%$ of 1 L or 0.05 L, the tolerance interval is (0.95, 1.05) in liters or (950, 1050) in cubic centimeters.

c. **1.** Sample response: An acceptable interval for the side length, in centimeters, is found by solving $950 < 20.1x^2 < 1050$. The interval is (6.87, 7.23).

2. The largest acceptable error from the desired length is 0.18 cm.

***4.7** **a.** Since $V = lwh$, $f(h) = (1.618h)3h = 4.854h^2$ when $h > 0$.

b. The height of the box should be about 4.97 cm; the length of the box should be about 8.04 cm.

c. The tolerance interval for the volume in cubic centimeters is (118.8, 121.2) or $118.8 < f(h) < 121.2$.

d. The interval for acceptable heights, in centimeters, is approximately (4.95, 5.00). The interval for acceptable lengths is approximately (8.00, 8.08).

e. The interval for acceptable heights, in centimeters, would be approximately (4.96, 4.98). The interval for acceptable lengths would be approximately (8.02, 8.06).

f. Sample response: Smaller tolerances for volume require smaller variations in the length and height. When the tolerance interval is decreased, so is the precision interval.

* * * * *

4.8 **a.** Since $V = \pi r^2 h$, $f(r) = \pi r^2 (3r) = 3\pi r^3$.

b. The desired radius is about 31.7 cm.

c. **1.** The interval for volume in liters is (297, 303). The interval for volume in cubic centimeters is (297,000, 303,000).

2. The interval for the radius in centimeters is about (31.6, 31.8).

3. The interval for the height in centimeters is approximately (94.8, 95.4).

4.9 Solving the inequality $900 < 3000 - 4.9t^2 < 1500$ gives the safe interval for the skydiver. The skydiver should open her parachute between 17.5 sec and 20.7 sec after the jump begins. The desired interval for t is $17.5 < t < 20.7$.

4.10 **a.** The minimum volume for the vat is approximately 39.901 m^3. This is found by letting the initial volume be 40 and the temperature change be -36˚.

b. [39.901, 40]

c. [370.4, 370.7]

* * * * * * * * * *

Answers to Summary Assessment (page 151)

Sample response: The 1% precision of the 8-cm radius for the base of the container results in an interval for the radius of (7.92, 8.08). The 2.5% tolerance of the volume results in an interval for the volume in cubic centimeters of (975, 1025). The equation $V = Bh$ can be used to determine the possible heights by using the endpoints of the radius interval in the base area formula $B = \pi r^2$, and by using the corresponding endpoints of the volume interval for V.

The interval for the area of the circular base in square centimeters is approximately $\left(7.92^2\pi, 8.08^2\pi\right)$, or about (197.06, 205.10). There are two height intervals that result: h_1 uses the smallest base with the volume interval; h_2 uses the largest base with the volume interval.

The h_1 interval in centimeters is $(975/197.06, 1025/197.06)$, or about (4.95, 5.20).

The h_2 interval in centimeters is $(975/205.10, 1025/205.10)$, or about (4.75, 5.00).

The height intervals must be checked to determine if they would produce volumes in the desired interval using $V = Bh$. To find the smallest acceptable height, the volume using the smallest base must be acceptable: $4.75 \bullet 197.06 \approx 936$ (reject); $4.95 \bullet 197.06 \approx 975$ (accept). A height of 4.95 cm is the smallest height that would result in a volume within the interval using the smallest base area.

Similarly, the height values must be checked to determine the largest height using the largest base area that will result in an acceptable volume: $5.20 \bullet 205.10 \approx 1066$ (reject); $5.00 \bullet 205.10 \approx 1026$ (slightly outside interval, may be due to rounding); $4.99 \bullet 205.10 \approx 1023$ (accept). A height of 4.99 cm is the largest height that would result in an acceptable volume using any acceptable base.

Therefore, the interval of acceptable heights, in centimeters, is approximately [4.95, 4.99]. This represents a desired height of 4.97 cm with a precision of 0.4%. This level of accuracy may not be possible, so the recommendation may be to reevaluate the acceptable tolerance or to discuss the tolerance of the base.

Module Assessment

1. Beth plans to spend between \$5500 and \$7000 on a car. She has already saved \$4000, and currently earns \$5.00 per hour.

a. Determine a function that describes the money Beth currently earns in terms of hours worked.

b. Express the additional amount of money Beth needs to meet the estimated costs for the car as an inequality.

c. Express the interval of hours Beth must work to earn this money as an inequality.

d. Graph the function from Part **a**, and label your response to Part **c** on the graph.

2. Solve each of the following inequalities and graph the solutions on a number line.

a. $3x < 12$

b. $4x + 2 \geq 18$

c. $7 - 2x < -13$

d. $12 < 2x - 8 \leq 28$

e. $|x - 4| < 9$

f. $|2x - 10| \geq 4$

3. **a.** Graph the functions $f(x) = 0.5x^2$ and $g(x) = 2x$ on a graphing utility.

b. By inspecting the graphs, determine the interval for which $f(x) < g(x)$. Express your answer as an inequality.

4. Solve the conjunction of inequalities $9 < x^2 \leq 25$ and sketch a graph of the solution set.

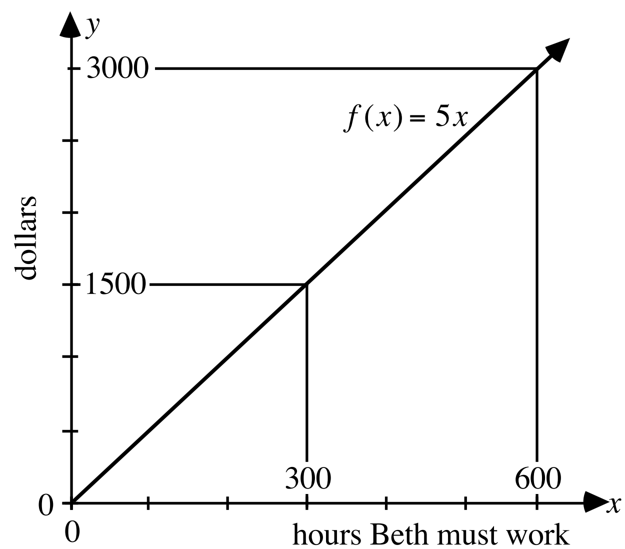
Answers to Module Assessment

1. **a.** $f(x) = 5x$ where $x \geq 0$

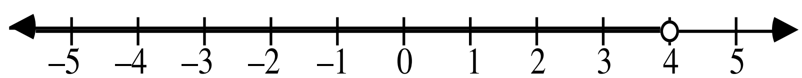
b. $5500 - 4000 < f(x) < 7000 - 4000$

c. $300 \leq x \leq 600$

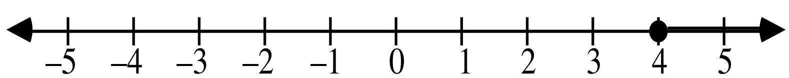
d. Sample graph:



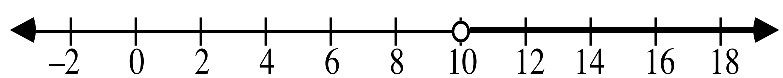
2. **a.** Sample graph:



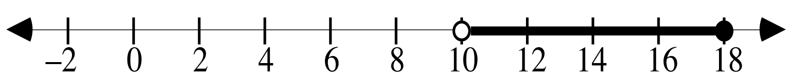
b. Sample graph:



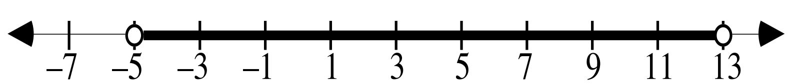
c. Sample graph:



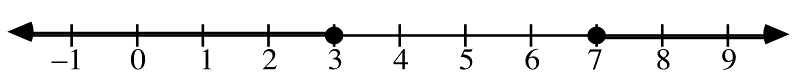
d. Sample graph:



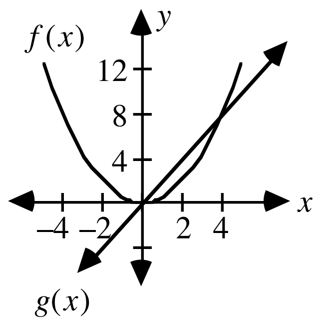
e. Sample graph:



f. Sample graph:

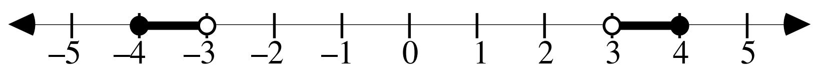


3. **a** Sample graph:



b. The graph of $f(x)$ is below the graph of $g(x)$ for the interval $(0, 4)$.

4. The solution set to the inequality $9 < x^2 \leq 25$ is $[-4, -3) \cup (3, 4]$.
Sample graph:



Selected References

Buchanan, O. L., Jr. *Limits: A Transition to Calculus*. Boston: Houghton Mifflin Co., 1970.

Juran, J. M., F. M. Gryna, Jr., and R. S. Bingham, Jr., eds. *Quality Control Handbook*. New York: McGraw-Hill, 1974.

Stroyan, K. D. *Introduction to the Theory of Infinitesimals*. New York: Academic Press Inc., 1976.

Flashbacks

Activity 1

1.1 Graph the solution to each of the following on a number line.

a. $x = 3$

b. $x > -2$

c. $x \neq 0$

d. $1 \geq x$

e. $-4 < x$

1.2 Write each of the following inequalities using interval notation:

a. $1 \leq x \leq 3$

b. $1 < x < 3$

1.3 Write each of the following intervals using inequalities:

a. $[2, 3]$

b. $(2, 3)$

Activity 2

2.1 Solve the following inequalities:

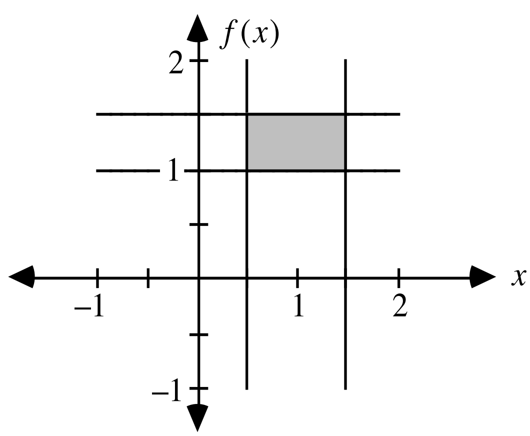
a. $3x > 111$

b. $2x - 9 < 5$

c. $-4 < 2x - 4 < 4$

d. $2 < -x + 4 < 5$

2.2 Consider the shaded region in the following figure:



a. Describe the shaded region using interval notation.

b. Describe the shaded region using inequalities.

Activity 3

3.1 Solve each of the following inequalities. Express each result using interval notation and graph the solution set on a number line.

a. $0.5x \le -2$

b. $-5x - 4 < 6$

c. $-5 < 3x - 2 < 7$

d. $3 \le -2x + 1 < 5$

3.2 Determine the precision interval for a desired tolerance interval of $14.7 < 2x - 2 < 15.3$.

3.3 Describe all values of x that satisfy each of the following:

a. $|x| = 2$

b. $|x + 2| = 4$

Activity 4

4.1 **a.** Use absolute value to write an inequality for a point x that is at least 3 units from the point 5.

b. Use absolute value to write an inequality for a point x that is less than or equal to 3 units from the point –2.

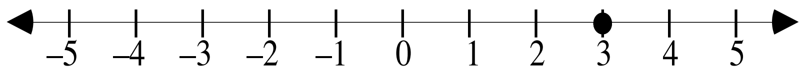
4.2 **a.** What is the area of a square with a perimeter of 24 cm?

b. What is the volume of a rectangular prism with a height of 10 cm and a square base like the one described in Part **a**?

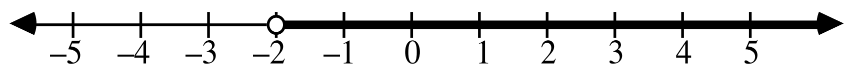
Answers to Flashbacks

Activity 1

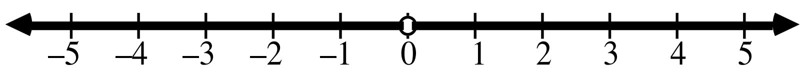
1.1 **a.** Sample graph:



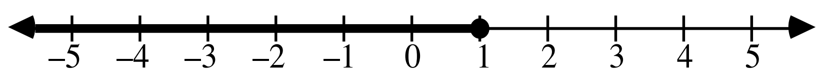
b. Sample graph:



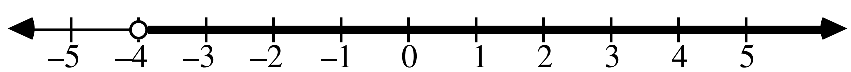
c. Sample graph:



d. Sample graph:



e. Sample graph:



1.2 **a.** $[1, 3]$

b. $(1, 3)$

1.3 **a.** $2 \le x \le 3$

b. $2 < x < 3$

Activity 2

2.1 **a.** $x > 37$

b. $x < 7$

c. $0 < x < 4$

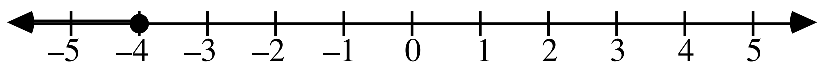
d. $-1 < x < 2$

2.2 **a.** Sample response: The value of x is in the interval $[0.5, 1.5]$; $f(x)$ is in the interval $[1, 1.5]$.

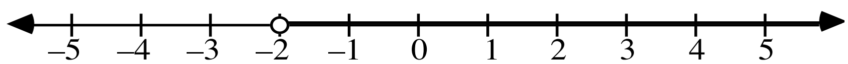
b. $0.5 \le x \le 1.5$ and $1 \le f(x) \le 1.5$

Activity 3

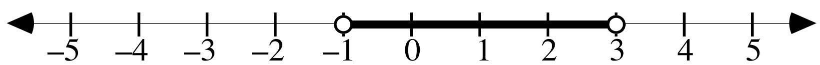
3.1 **a.** $x \le -4$ or $(-\infty, -4]$



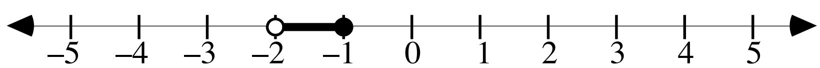
b. $x > -2$ or $(-2, \infty)$



c. $-1 < x < 3$ or $(-1, 3)$



d. $-2 < x \le -1$ or $(-2, -1]$



3.2 $8.35 < x < 8.65$

3.3 **a.** $x = 2$ or -2

b. $x = 2$ or -6

Activity 4

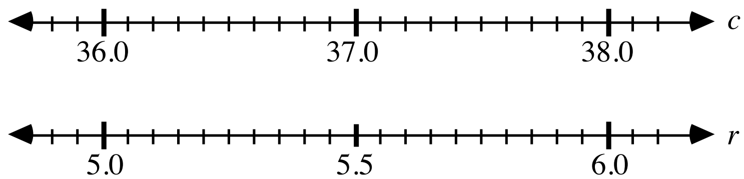
4.1 **a.** $|x - 5| > 3$

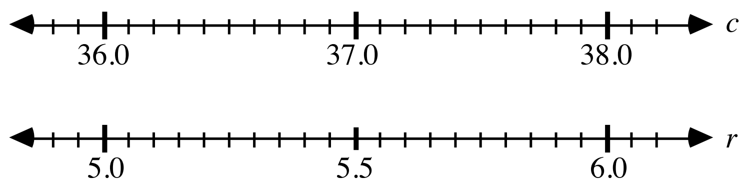
b. $|x + 2| \le 3$

4.2 **a.** Since each side measures 6 cm, the area is $36\ \text{cm}^2$.

b. The volume is $360\ \text{cm}^3$.

Template A





Nearly Normal

Would you recognize a normal curve if you saw one? In this module, you'll explore the mathematics of normal while examining the historic patterns of rainfall in a Northwest town.

Lee Brown • Roger Patterson • Karen Umbaugh

Teacher Edition

Nearly Normal

Overview

Students use precipitation data to examine the normal distribution by creating relative frequency polygons. They also examine the differences between discrete and continuous probability distributions.

Objectives

In this module, students will:

- organize data using relative frequency tables and graphs
- use mean and standard deviation to describe data sets
- explore the following types of probability distributions: discrete, binomial, continuous, uniform, and normal.

Prerequisites

For this module, students should know:

- how to construct histograms
- the basic properties of probability
- how to determine mean, median, and standard deviation
- the difference between experimental and theoretical probability
- how to determine expected value.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	3	2	11

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
graph paper	X	X	X	X	X
coins		X			
containers		X			

Technology

Software	Activity				
	1	2	3	4	Summary Assessment
statistics package	X	X	X	X	X
spreadsheet	X	X	X	X	X
graphing utility	X	X	X	X	X

Nearly Normal

Introduction

(page 157)

You may wish to provide reading material about climate-controlled communities, such as the BioSphere II experiment. In this experiment, eight crew members lived for two years in a self-contained environment.

(page 157)

Activity 1

Students use precipitation data for an imaginary city to create a relative frequency table, a relative frequency histogram, and a relative frequency polygon.

To begin the activity, you may wish to present a current National Oceanographic and Atmospheric Administration (NOAA) summary of precipitation in your town or region, or discuss weather data from the local newspaper or airport.

Materials List

- graph paper (optional)

Technology

- spreadsheet
- statistics package (optional)
- graphing utility (optional)

Teacher Note

The population standard deviation, denoted by σ, can be found using the following formula, where the population has *N* members represented by $x_1, x_2, \ldots, x_N$ and μ is the population mean:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}}$$

The sample standard deviation, denoted by *s*, is found using the formula below, where the sample has *n* members represented by $x_1, x_2, \ldots, x_n$ and $\bar{x}$ is the sample mean:

$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n - 1}}$$

Note that the denominator used to calculate sample standard deviation is slightly different than the one used to calculate population standard deviation. There are theoretical reasons for dividing by $n - 1$ rather than n; these have to do with degrees of freedom. Division by $n - 1$ gives unbiased estimates of variance.

In this module, all data sets should be treated as populations. Students will investigate the differences between s and σ in later modules.

Exploration

(page 160)

Note: To save time, you may wish to enter the data in Table **1** in a spreadsheet, then allow students to copy or download the file.

a. The mean of the data in Table **1** is 13.46 cm. The median is 13.41 cm. The standard deviation is 3.4 cm.

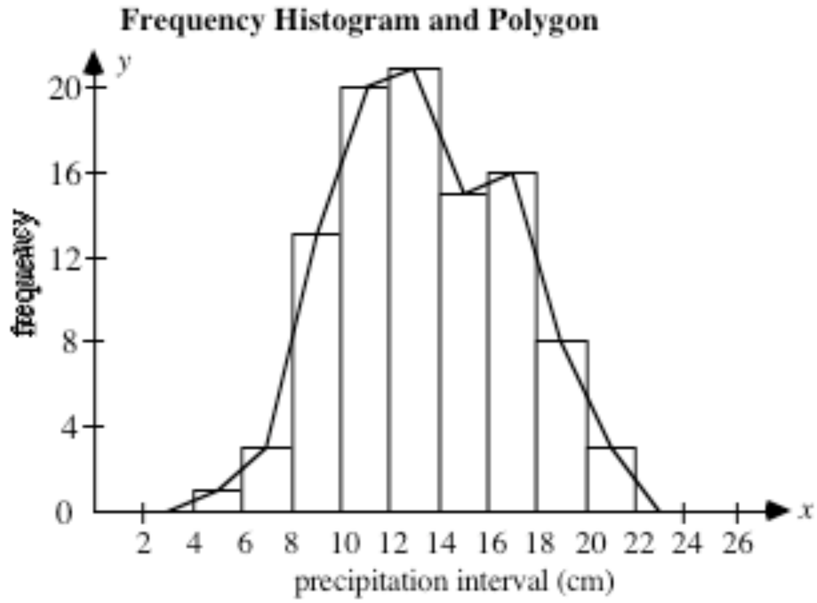
b. To simplify counting the frequencies in each interval, students may use a spreadsheet to sort the data.

1. Sample relative frequency table:

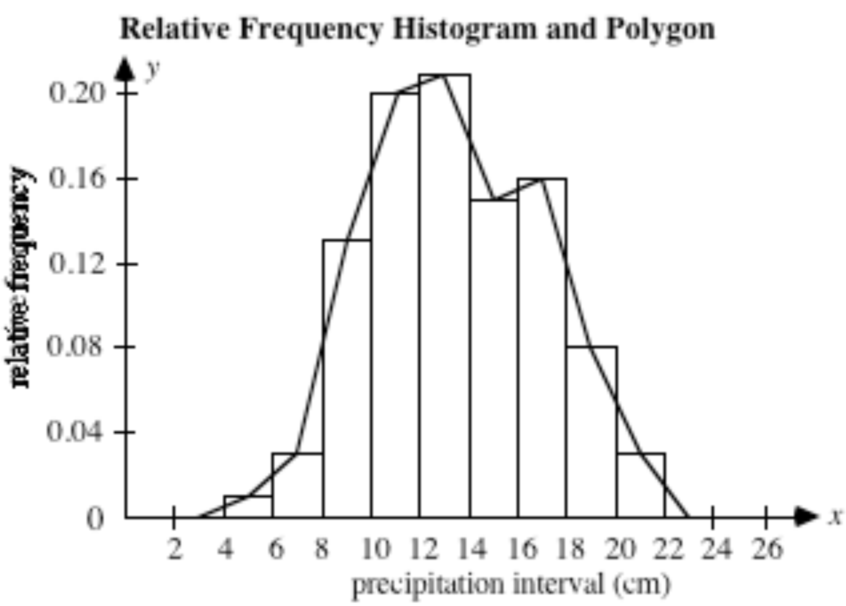
Precipitation (cm)	Frequency	Relative Frequency
$4 \leq x < 6$	1	$1/100 = 0.01$
$6 \leq x < 8$	3	$3/100 = 0.03$
$8 \leq x < 10$	13	$13/100 = 0.13$
$10 \leq x < 12$	20	$20/100 = 0.20$
$12 \leq x < 14$	21	$21/100 = 0.21$
$14 \leq x < 16$	15	$15/100 = 0.15$
$16 \leq x < 18$	16	$16/100 = 0.16$
$18 \leq x < 20$	8	$8/100 = 0.08$
$20 \leq x < 22$	3	$3/100 = 0.03$

2. The sum of the frequencies is 100. The sum of the relative frequencies is 1.

c. **Note:** Since the base of a relative frequency polygon is the *x*-axis, students should include an interval with a frequency of 0 on each side of their data sets. Sample frequency histogram and frequency polygon:



d. The histogram and polygon from Part **d** will have the same shape as those in Part **c** as long as the *x*-scales are the same and the *y*-scales are similar. For example, the distance from 0 to 4 in the sample graph in Part **c** is the same as the distance from 0 to 0.04 in the sample graph in Part **d**.



e.

1. On each of the graphs, the mean is located at 13.46 cm.
2. The points 1 standard deviation below the mean and 1 standard deviation above the mean are 10.06 and 16.86, respectively.

3. The points 2 standard deviations below the mean and 2 standard deviations above the mean are 6.66 and 20.26, respectively.

Discussion

(page 160)

a.
1. The interval that has the highest frequency is $12 \le x < 14$.
2. Sample response: The interval with the highest frequency will not always contain either the mean or the median. It is possible that the distribution of rainfall could cause the mean to be located elsewhere. For example, if the intervals with the greatest and least rainfall both had high frequencies, the mean might fall in an interval with a low frequency.

b. When reporting levels of rainfall above or below normal, weather forecasters are usually referring to the mean rainfall for a given period.

c. Sample response: Both the frequency polygon and the relative frequency polygon have the same basic shape if the *x*-scales are the same. Both polygons have a peak near the center.

d.
1. If the rainfall amounts in the 19th century were similar to those in the 20th century, the general shape of the frequency polygon for the combined data would be similar. It would display similar measures of central tendency and spread, but would be taller and appear more smooth.
2. The shape of the relative frequency polygon should remain almost the same, since the ratios for the combined data should resemble those of the 20th century.

e.
1. Since 21 of the 100 observations fall in this interval, the probability is 0.21 or 21%.
2. The relative frequency of an interval equals its probability of occurring in the 20th century.

f.
1. 67%
2. 96%

g. Answers may vary. Sample response: Since relative frequency indicates the probability of a particular interval of rain, the polygon could be used to determine the most probable interval of rainfall. This interval could then be used to determine the May precipitation in New Bernoulli.

Assignment

(page 161)

1.1
- **a.** 0.20 or 20%
- **b.** 0.27 or 27%
- **c.** 0.04 or 4%

***1.2** Sample response: The relative frequencies are based on the ratio of the frequencies to the total number of observations. Hence, the sum of the relative frequencies must be 1. It is also true that the total of the probabilities of all possible outcomes of an experiment must equal 1. The intervals represent all possible outcomes.

1.3 The table below shows the annual amounts of rainfall in centimeters for Missoula, Montana, for the years 1942–1991. The sample responses for Problems **1.3** and **1.4** are based on this data.

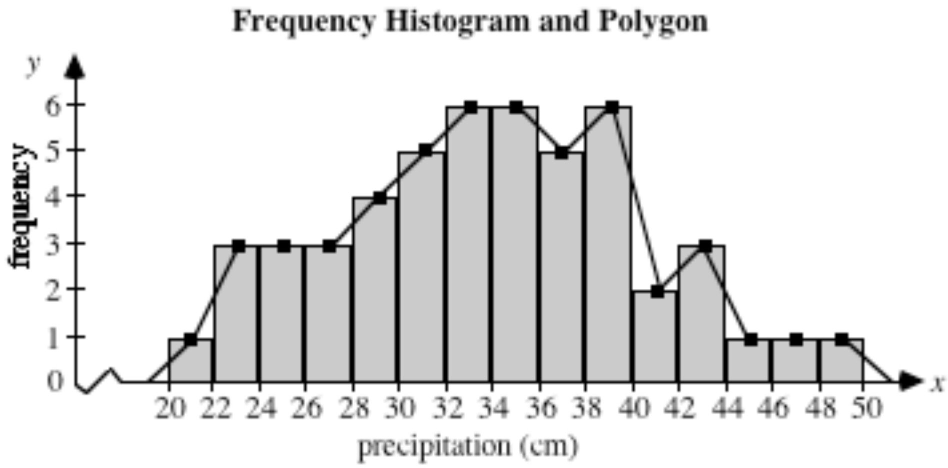
Year	**(cm)**	**Year**	**(cm)**	**Year**	**(cm)**	**Year**	**(cm)**	**Year**	**(cm)**
1991	33.0	1981	44.1	1971	33.3	1961	35.8	1951	38.2
1990	34.3	1980	49.1	1970	38.3	1960	25.1	1950	36.0
1989	34.4	1979	26.3	1969	33.5	1959	41.4	1949	22.7
1988	28.1	1978	29.9	1968	32.6	1958	43.1	1948	37.8
1987	25.7	1977	33.0	1967	31.5	1957	31.0	1947	37.3
1986	42.1	1976	23.0	1966	28.2	1956	38.5	1946	31.6
1985	31.8	1975	46.9	1965	36.1	1955	40.2	1945	24.8
1984	33.8	1974	27.3	1964	38.7	1954	34.5	1944	26.9
1983	42.4	1973	22.9	1963	37.8	1953	28.9	1943	34.6
1982	39.1	1972	34.8	1962	30.5	1952	21.9	1942	38.1

a. The mean is approximately 33.8 cm, the median is 34.1 cm, and the standard deviation is approximately 6.4 cm.

b–c. The following table uses intervals 2 cm wide.

Precipitation (cm)	**Frequency**	**Relative Frequency**
$20 \le x < 22$	1	0.02
$22 \le x < 24$	3	0.06
$24 \le x < 26$	3	0.06
$26 \le x < 28$	3	0.06
$28 \le x < 30$	4	0.08
$30 \le x < 32$	5	0.10
$32 \le x < 34$	6	0.12
$34 \le x < 36$	6	0.12
$36 \le x < 38$	5	0.10
$38 \le x < 40$	6	0.12
$40 \le x < 42$	2	0.04
$42 \le x < 44$	3	0.06
$44 \le x < 46$	1	0.02
$46 \le x < 48$	1	0.02
$48 \le x < 50$	1	0.02

d–e. The frequency histogram and frequency polygon below are based on the intervals given in Parts **b** and **c**.



1.4 Answers will vary, depending on the data students use. The following responses are based on the sample data given in Problem **1.3**.

a. The mean of the sample data is 33.8 cm. The points 1 standard deviation below and above the mean are 27.4 cm and 40.2 cm, respectively. The points 2 standard deviations below and above the mean are 21.0 cm and 46.6 cm, respectively.

b. **1.** Based on the intervals from Problem **1.3b**, 32 of the values, or 64%, are within 1 standard deviation of the mean.

2. Based on the intervals from Problem **1.3b**, 48 of the values, or 96%, are within 2 standard deviations of the mean.

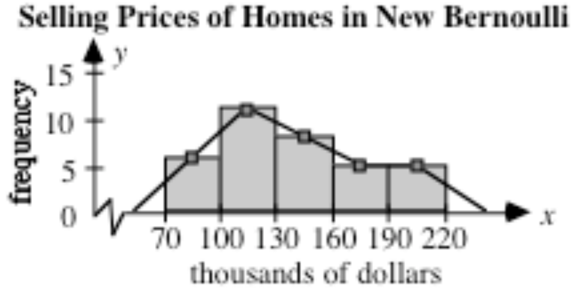
* * * * *

1.5 **a.** The mean is approximately $138,217, the median is $131,400, and the standard deviation is approximately $38,047.

b–c. The following table is based on an interval width of $30,000.

Selling Price	Frequency	Relative Frequency
$70,000–$99,999	6	0.17
$100,000–$129,999	11	0.31
$130,000–$159,999	8	0.23
$160,000–$189,999	5	0.14
$190,000–$219,999	5	0.14

d–e. The frequency histogram and frequency polygon below are based on the intervals given in Parts **b** and **c**.



1.6 **a.** The mean of the data is $138,217.

b. **1.** Based on the intervals given in Problem **1.5b**, 24 of the prices, or 68.6%, are within the interval [$100170, $176264]. These values are within 1 standard deviation of the mean.

2. Based on the intervals given in Problem **1.5b**, 34 of the prices, or 97.1%, are within the interval [$62123, $214311]. These values are within 2 standard deviations of the mean.

c. The sum of the relative frequencies of the intervals with values of $130,000 or greater is 0.51. The probability, therefore, is 51%.

* * * * * * * * * *

(page 163)

Activity 2

In this activity, students are introduced to probability distributions and binomial experiments through coin tossing. They then use a simulation to model the precipitation intervals in Table **1**.

Materials List

- coins (12 per group)
- container for coins (one per group)
- graph paper (optional)

Technology

- spreadsheet
- graphing utility
- statistics package (optional)

Exploration 1

(page 164)

Students investigate the theoretical probabilities of obtaining a certain number of heads when tossing *n* coins. Some students may wish to use tree diagrams to determine these probabilities.

a. Sample table:

No. of Heads	Frequency	Probability
0	1	0.25
1	2	0.50
2	1	0.25
Total	4	1.00

b. Sample table:

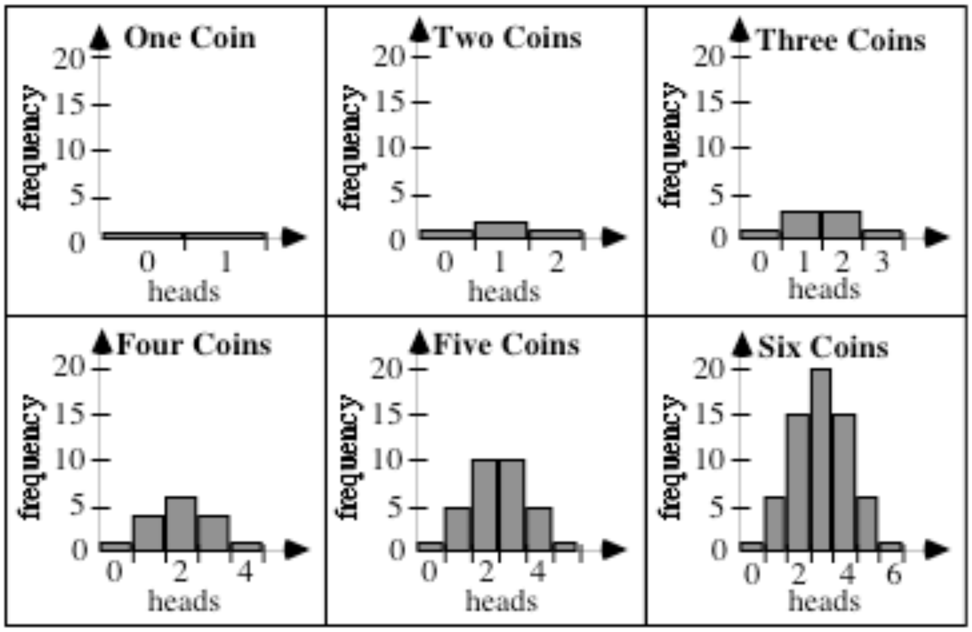
No. of Heads	Frequency	Probability
0	1	0.0625
1	4	0.25
2	6	0.375
3	4	0.25
4	1	0.0625
Total	16	1.00

c. There are 2^n possible outcomes.

d. The resulting pattern is Pascal's triangle. Sample table:

No. of Coins	No. of Heads						
	0	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

e. The following histograms show the frequencies of the number of heads when tossing 1, 2, 3, 4, 5, and 6 coins. Students should observe that, with the exception of the histogram for tossing one coin, all have taller bars in the middle with bars on either side that gradually decrease in size.



f. Students may use appropriate technology or formulas derived in previous modules to find mean and standard deviation.

Note: Some technology may calculate only sample standard deviation (division by $n-1$). In this situation, however, the population standard deviation (division by n) should be used since the outcomes represent the entire population.

No. of Coins	Mean	Standard Deviation
1	0.5	0.5
2	1	0.707
3	1.5	0.866
4	2	1
5	2.5	1.118
6	3	1.225

Discussion 1 (page 165)

a. Answers may vary. The pattern developed is Pascal's triangle. Sample response: The first number and last number in each row is 1. The remaining numbers can be found by adding the number directly above the cell being calculated to the number to the left of the number above the cell being calculated.

b. Students should observe that, with the exception of the histogram for tossing one coin, all have taller bars in the middle with bars on either side that gradually decrease in size.

c. Students should observe some similarity in the shapes of the two histograms. They both peak near the center and gradually drop off on either side.

d. Sample response: Yes, as long as the probability of obtaining a head on each coin is the same.

e. Sample response: Yes, the coin tossing fits all the criteria for a binomial experiment. There are 3 repetitions of the same action. The result of any one toss does not influence the result of any other. Each toss has only two possible outcomes: a head or a tail. And the probability of obtaining a head on each toss is 0.5.

f.

1. $n = 6$
2. $p(\text{H}) = 0.5$
3. $\mu = 6 \bullet 0.5 = 3$
4. The two values are the same.

g.

1. Sample response: The sum of the probabilities for all possible outcomes of an experiment is 1. In the case of a binomial trial, there are only two possible outcomes. If the probability of a success is p, the probability of a failure must be $(1 - p)$.
2. $\sigma = \sqrt{6(0.5)(0.5)} = \sqrt{1.5} \approx 1.225$
3. The two values are the same.

h. Sample response: An experimental probability distribution may never be the same as the theoretical probability distribution. As the number of tosses in the simulation increases, however, the closer the two distributions are likely to be.

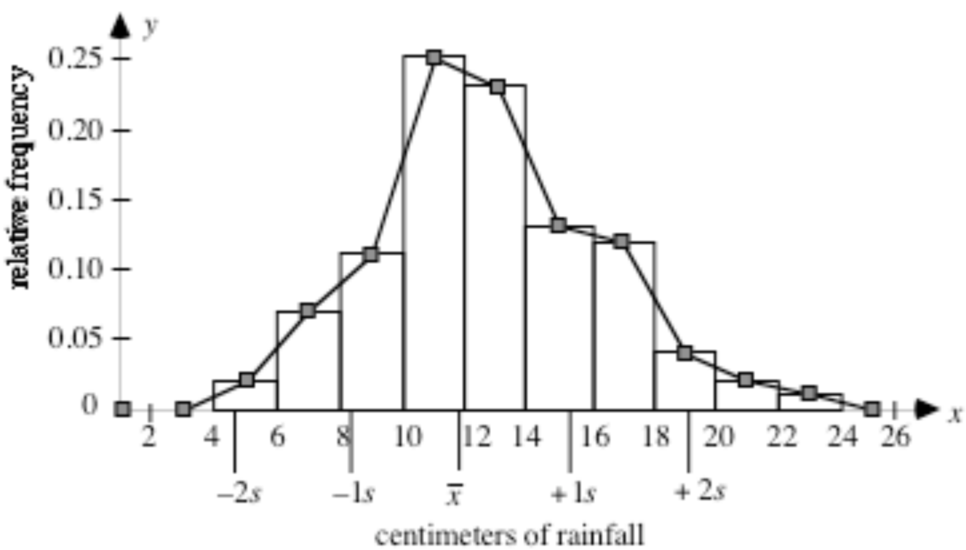
Exploration 2 (page 167)

a–b. Students simulate the pattern of precipitation in New Bernoulli by tossing 12 coins. The corresponding amount of rainfall is determined by counting the number of heads, then using Table **7**. Each group should generate 100 trials to represent 100 years of observations. Sample data:

No. of Heads	Precipitation (cm)	Frequency	Relative Frequency
0	[0, 2)	0	0.00
1	[2, 4)	0	0.00
2	[4, 6)	2	0.02
3	[6, 8)	7	0.07
4	[8, 10)	11	0.11
5	[10, 12)	25	0.25
6	[12, 14)	23	0.23
7	[14, 16)	13	0.13
8	[16, 18)	12	0.12
9	[18, 20)	4	0.04
10	[20, 22)	2	0.02
11	[22, 24)	1	0.01
12	[24, 26)	0	0.00

c. Answers will vary. For the sample data given in the table above, the mean is 5.86 heads, the median is 6, and the standard deviation is 1.78.

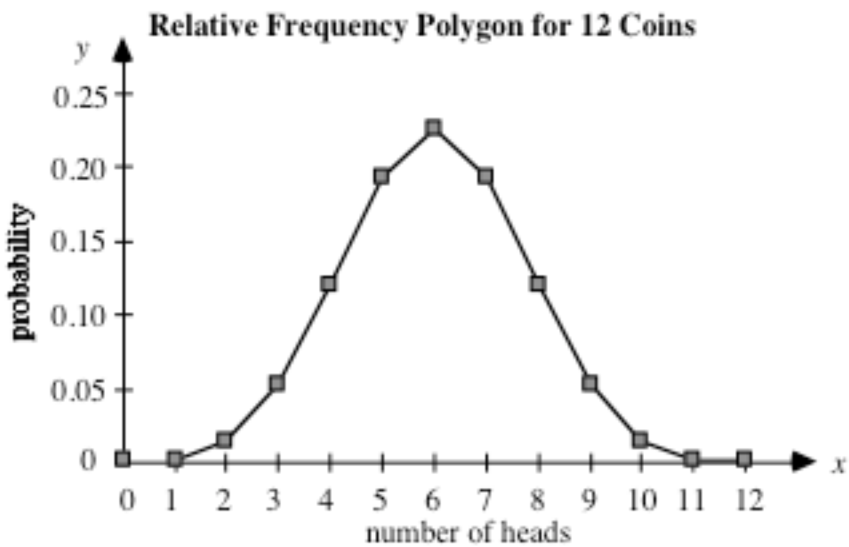
d–e. Sample graph:



f. The table below shows the theoretical probability distribution when tossing 12 coins.

No. of Heads	Frequency	Probability
0	1	0.0002
1	12	0.0029
2	66	0.0161
3	220	0.0537
4	495	0.1208
5	792	0.1934
6	924	0.2256
7	792	0.1934
8	495	0.1208
9	220	0.0537
10	66	0.0161
11	12	0.0029
12	1	0.0002

Sample graph:



g. In this case, $\mu = 12(0.5) = 6$ and $\sigma = \sqrt{12(0.5)(0.5)} \approx 1.732$. These are close to the mean and standard deviation of the sample data given in Part **c**.

Discussion 2 (page 169)

a. Sample response: The shapes of the relative frequency polygons are similar. The greatest frequencies are close to the mean value and the heights of the polygons on either side of the peak tend to decrease at about the same rate on either side.

b. Sample response: The probability of obtaining each number with the random number generator would be the same: $1/12$. The probability of obtaining each number of heads when tossing 12 coins varies with the number of heads. For example, the probability of obtaining 1 head is $3/1024$, while the probability of obtaining 6 heads is $231/1024$.

c. Answers will vary. For the sample data given in Exploration **2**, an interval of [10 cm, 14 cm) seems typical.

d. Sample response: Yes, because the shape of the relative frequency histogram for the simulation is very close to the shape of the relative frequency histogram for the historic rainfall. **Note:** Students should be reminded that this does not guarantee that the simulation will produce the same pattern of rainfall as in Table **1**. In fact, it could produce some very different patterns.

Assignment

(page 169)

2.1 **a.** Answers will vary. For the sample data given in Exploration **2**, 61% are within 1 standard deviation of the mean and 83% are within 2 standard deviations of the mean.

b. Answers will vary. For the sample data given in Exploration **2**, the percentages are slightly lower than those for the historic rainfall data.

2.2 **a.** Answers will vary. The following responses are based on the sample data given in Exploration **2**.

1. 0.25 or 25%
2. 0.19 or 19%
3. 0.09 or 9%

b. **1.** Answers will vary. For the sample data given in Exploration **2**, these values are noticeably different.

2. Sample response: In Problem **1.2**, the percentages were calculated using observed rainfall amounts. In Problem **2.2a**, the percentages were based on data gathered using a simulation. Although the simulation is likely to produce similar results, it does not guarantee exactly the same pattern.

***2.3** **a.** Since the relative frequency is equal to the probability, the probabilities are 0.12, 0.42, 0.28, and 0.18, respectively.

b. Since the relative frequency equals the frequency divided by the total number of observations, the frequency can be found by multiplying the relative frequency by 50. The frequencies are 6, 21, 14, and 9, respectively.

c. The mean can be found by multiplying each number of heads by its corresponding probability and adding the results:

$$0(0.12) + 1(0.42) + 2(0.28) + 3(0.18) = 1.52$$

The median is 1 head, since more than half the trials resulted in 0 or 1 heads and less than half the trials resulted in 0 heads.

d. Sample response: The relative frequencies should approach the theoretical probabilities of 0.125, 0.375, 0.375, and 0.125 for 0 heads, 1 head, 2 heads, and 3 heads, respectively. This is because the more times the experiment is repeated, the closer the relative frequencies should get to the theoretical probabilities.

e. The expected value, in number of heads, can be calculated as follows:

$$0.125 \bullet 0 + 0.375 \bullet 1 + 0.375 \bullet 2 + 0.125 \bullet 3 = 1.5$$

f. **1.** $\mu = np = 3(0.5) = 1.5$

2. The expected value and the mean are equal.

* * * * *

2.4 **a.** Some students may model this situation on a spreadsheet using 120 fours, 253 fives, 217 sixes, and 410 sevens for a total of 1000 entries. Others may generate random numbers from 1 to 1000 where 1–120, 121–373, 374–690, and 691–1000 represent a 4-game, 5-game, 6-game, and 7-game series, respectively.

Sample data:

No. of Games	Frequency
4	5
5	11
6	19
7	15

b. Answers will vary. The mean of the sample data is 5.88; the standard deviation is 0.95.

c. The mean of the historic data is approximately 5.92 games; the standard deviation is approximately 1.07 games.

* * * * * * * * * *

(page 171)

Activity 3

In this activity, students examine the differences between discrete and continuous probability distributions. They also explore uniform probability distributions.

Materials List

- graph paper (optional)

Technology

- graphing utility
- statistics package (optional)
- spreadsheet (optional)

Discussion 1

(page 171)

a. Sample response: The sum of the relative frequencies is 1. Whether they are thought of as the percentages of all the outcomes of an experiment or as the probabilities of all the outcomes of an experiment, the total must be 1.

b. Sample response: The relative frequency for each equally likely interval could be found by dividing 1 by the number of intervals.

c. Sample response: There are an infinite number of rainfall values within each interval.

d. Sample response: The width of the entire interval is 10. Since the area under the distribution equals 1, the height must be 0.1.

e. The height of a uniform probability distribution over the interval $[a,b]$ is:

$$\frac{1}{b-a}$$

Teacher Note

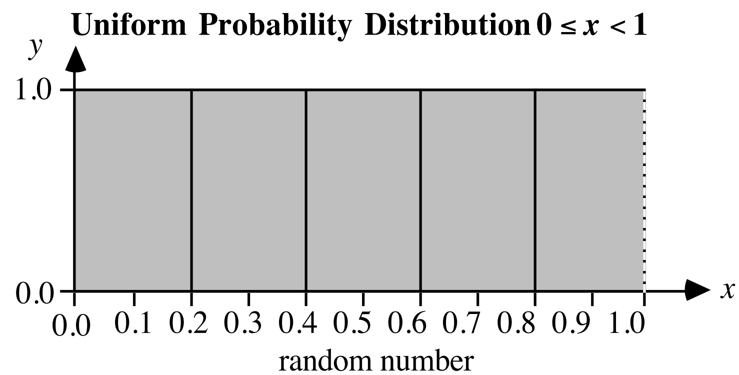
Because the random number generator on many graphing calculators does not generate all real numbers in an interval, it does not produce a continuous distribution. However, the data generated can be modeled by a continuous distribution.

Using the TI-92, for example, entering "Rand" then repeatedly pressing "Enter" generates a set of random numbers between 0 and 1. Students should seed the generator first using "RandSeed."

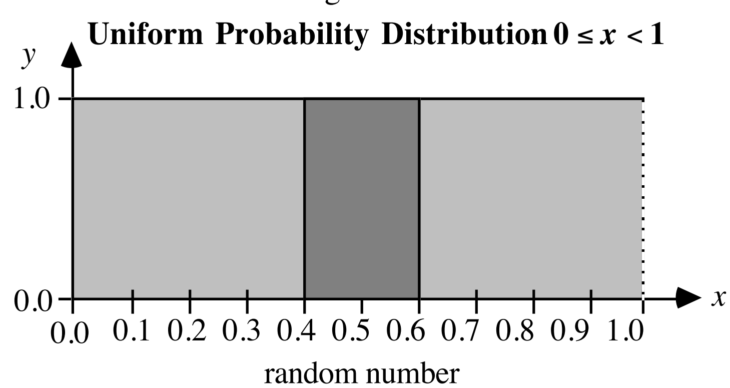
Exploration (page 172)

a. You may wish to point out that students are using a continuous distribution to model the discrete distribution produced by a random number generator.

b. The following sample graph uses five intervals of width 0.2.



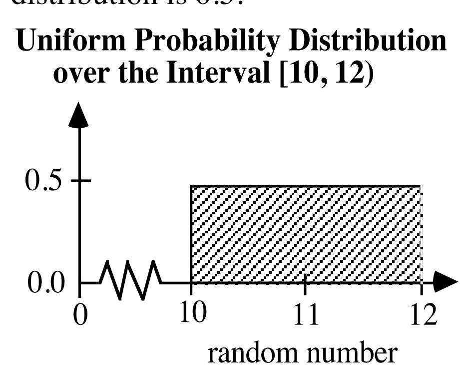
c. In the following sample graph, the interval between 0.4 and 0.6 is shaded. The area of this region is 0.2.



d. Answers will vary. About $1/5$ or approximately 40 of the 200 numbers should fall in the interval between 0.4 and 0.6.

e. The percentage and the area calculated in Part **c** should be about the same. Using the interval between 0.4 and 0.6, the percentage should be approximately 20%.

f. Since the width of the interval is 2, the height of the uniform probability distribution is 0.5.



Discussion 2 (page 173)

a. Sample response: No. The experiment is not binomial because there are more than two possible outcomes for each trial.

b. The total area in a uniform probability distribution is 1.

c. Sample response: Determine the width of the selected interval by subtracting as follows, $0.5 - 0.4 = 0.1$. This represents $1/10$ of the entire interval between 0 and 1. Since the total area of the uniform probability distribution is 1, the area of the selected region is 0.1. Therefore, the probability that the number will fall in the interval $[0.4, 0.5)$ is 0.1.

d.
1. 0.05
2. 0.01
3. 0.001
4. 0.0001

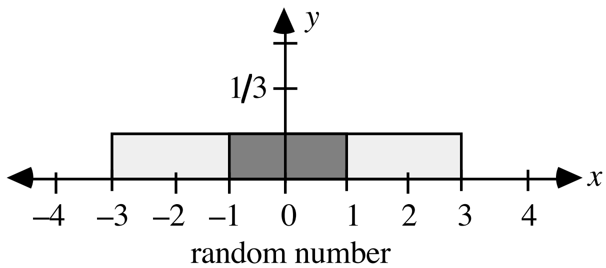
e.
1. Sample response: If the pattern continues, the probability of obtaining 0.4 would be 0.
2. Sample response: The probability is 0. This is because the probability is equal to the area under the curve bounded by the endpoints of the interval. If a specific value is used to define the interval, the interval would be represented by a segment. Since a segment has no area, the probability would be 0.

f.
1. Sample response: The distribution appears to include all the possible real-number values within an interval. This indicates that the distribution is continuous.
2. Sample response: The distribution is not uniform. The areas of equal intervals that are closer to the center of the distribution are greater than those farther away. Since the areas, and therefore the probabilities, are not equal, this cannot be a uniform distribution.

Assignment (page 174)

3.1 Sample response: Since the probability of obtaining any specific value is 0, the inclusion of the endpoints would not change the probability that a value will fall in the interval.

3.2 **a–b.** The sample graph below shows the uniform probability distribution between –3 and 3 with the shaded interval (–1, 1). Since the width of the entire interval is 6, the height of the distribution must be $1/6$. The probability of obtaining a number in the interval (–1,1) is $1/3$.



c. **1.** Answers will vary. Using a TI-92 calculator, for example, the command $6 * \text{rand}() - 3$ generates random numbers between –3 and 3.

2–4. The percentage should be close to 33%, the probability determined in Part **b**.

3.3 Sample response: No. Intervals of equal width do not have an equal probability of being selected, therefore, the distribution is not uniform.

***3.4** **a.** Sample response: Since the curve is a continuous probability distribution, the area under the curve must be 1. This can be supported by counting the squares under the curve. There are approximately 50 squares, each measuring 0.02 units^2.

b. Sample response: The mean is 10 cm. Approximately half of the area under the curve is on either side of this value.

c. The area bounded by $x = 6$, $x = 14$, the curve, and the x-axis is about 0.95.

d. **1.** The probability that the precipitation will be within 1 standard deviation of the mean is about 0.68.

2. The probability that the precipitation will be within 2 standard deviations of the mean is about 0.95.

* * * * *

3.5 **a.** Sample response: The polygon and its interior could represent a continuous probability distribution because they include all possible real numbers in the interval $\left[1/3, 3\right]$ and the area of the trapezoid is:

$$\frac{1}{2}h\left(b_1 + b_2\right) = \frac{1}{2}\left(\frac{8}{3}\right)\left(\frac{1}{4} + \frac{1}{2}\right) = 1$$

b. Sample response: The polygon and its interior cannot represent a uniform distribution because the probabilities (areas) over intervals of equal widths are not equal. For example, even though the widths of the intervals $[1/3, 2/3)$ and $[8/3, 3)$ are identical, the area inside the polygon in the interval $[1/3, 2/3)$ is smaller than the area inside the polygon in the interval $[8/3, 3)$.

c. **1.** Since there is no interval for the number 2, the probability is 0.

2. Students should find the area of the trapezoid bounded by the lines $-3x + 32y = 7$, $x = 1/3$, $x = 2$, and the x-axis:

$$\frac{1}{2}h(b_1 + b_2) = \left(\frac{1}{2}\right)\left(\frac{5}{3}\right)\left(\frac{1}{4} + \frac{13}{32}\right) = \frac{35}{64}$$

3. Students should find the area of the trapezoid bounded by the lines $-3x + 32y = 7$, $x = 2$, $x = 3$, and the x-axis:

$$\frac{1}{2}h(b_1 + b_2) = \left(\frac{1}{2}\right)(1)\left(\frac{13}{32} + \frac{1}{2}\right) = \frac{29}{64}$$

Note: Although the sum of the probabilities from Steps **2** and **3** is 1, the events are *not* complementary. This is because the outcome $x = 2$ is not considered in this sum. However, the probability from Step **3** is equal to 1 minus the probability from Step **2** plus the probability from Step **1**.

3.6 **a.** **1.** $P(0.6 < x < 0.8) = 0.2$

2. $P(0.69 < x < 0.71) = 0.02$

3. $P(0.69999 < x < 0.70001) = 0.00002$

b. The trend suggests that $P(x = 0.7) = 0$.

* * * * * * * * * *

(page 177)

Activity 4

This activity introduces normal distributions. Students use normal distributions to model various coin-tossing simulations in an attempt to find one that approximates the pattern of rainfall in Table **1**.

Materials List

- graph paper (optional)

Technology

- graphing utility
- spreadsheet
- statistics package (optional)

Teacher Note

To complete the following exploration, students will require technology that can compute the probability distribution for the number of heads when *n* coins are flipped, calculate its mean and standard deviation, graph the distribution as a scatterplot, then overlay the graph of a normal curve.

For example, the following program has been written for the TI-92 calculator and may be modified for other utilities. This program asks students to enter the number of coins to be tossed in a simulation and the probability of obtaining a head for one coin. It then reports the mean and standard deviation. Upon pressing the "ENTER" key, the program plots the normal curve over the points that represent the vertices of the relative frequency polygon.

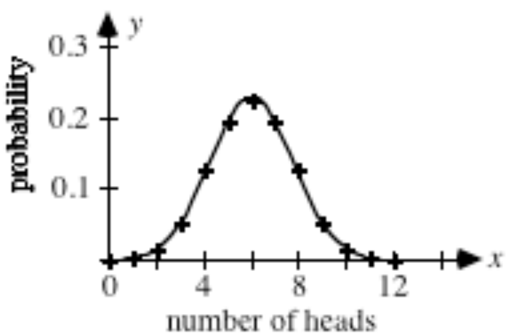
```
pascal()
Prgm
: FnOff
: ClrDraw
: ClrHome
: DelVar L1,L2
: DISP "NUMBER OF COINS"
: Input N
: DISP "PROB. OF HEADS"
: Input P
: 0→I
: Lbl a
: I+1→I
: I–1→ L1 [I]
: nCr(N,I–1) * P^(I–1) * (1–P)^(N–I+1)→L2 [I]
: If I < N+1
: Goto a
: 0→ Xmin
: (N+2)→ Xmax
: 1→Xscl
: 0→ Ymin
: 0.5→ Ymax
: 0.1→Yscl
: N*P→M
: √(N * P * (1 – P)) → S
: Disp "MEAN:"
```

```
: Disp M
: Disp "ST. DEV.:"
: Disp S
: Disp "PRESS ENTER FOR GRAPH"
: Pause
: 1 / (√(2πS²)) * e^(−0.5(X − M)² / S²) → Y1(x)
: NewPlot 1,1,L1,L2,,,4
: DispG
: Scatter
: EndPrgm
```

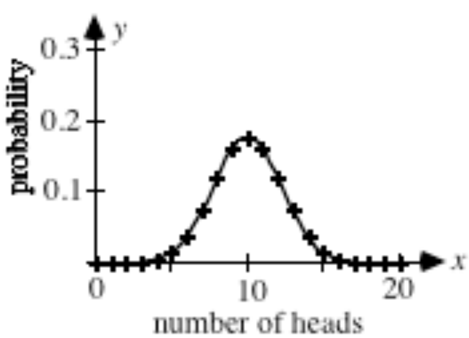
Exploration (page 178)

Students use technology to investigate normal distributions.

a. The mean and standard deviation of the distribution are 6 and approximately 1.73, respectively. Sample graph:



b. The mean and standard deviation of the distribution are 10 and approximately 2.24, respectively. Sample graph:



c. Responses will vary. The general shape of the graph should resemble those illustrated in Parts **a** and **b**.

Discussion

(page 178)

a. Sample response: It appears to be a good model because it has the same general shape as the frequency polygon for the historic rainfall data. It is also a continuous distribution, which allows the determination of any rainfall amount.

b. Sample response: As the number of coins increases, both the mean and the standard deviation of the distribution increase. This occurs because the mean equals $(n \bullet p)$ and the standard deviation is $\sqrt{n(p)(1-p)}$.

c. Sample response: As the number of coins increases, the normal curve becomes flatter and more spread out. This must happen if the area under the curve is to remain 1.

d. Answers may vary. Sample response: The coin-tossing simulation is a binomial experiment. From the exploration, it appears that the more coins that are tossed, the closer the relative frequency polygon is to a normal curve.

e.

1. Using the 68–95–99.7 rule, approximately 95% of the data lies in the interval (3.6, 8.4).
2. Since approximately 95% of the data lies in the interval (3.6, 8.4), this means that 5% of the data lies outside this interval. Therefore, the probability that an *x*-value is greater than 8.4 is:

$$\frac{1}{2} \bullet 0.05 = 0.025$$

3. Approximately 68% of the data lies in the interval (4.8, 7.2). The percentage of the data less than 4.8 is:

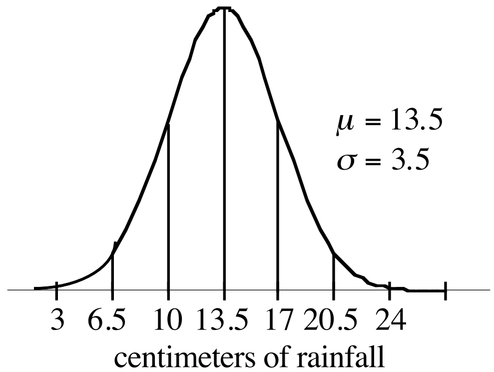
$$\frac{1}{2} \bullet 0.32 = 0.16$$

Therefore, the probability that an *x*-value is less than 7.2 is $0.68 + 0.16 = 0.84$.

Assignment

(page 179)

4.1 **a.** Sample sketch:



b. **1.** Using the 68–95–99.7 rule, the probability of less than 3 cm of rainfall is half of 0.003, since it lies over 3 standard deviations from the mean. The probability is 0.0015 or 0.15%.

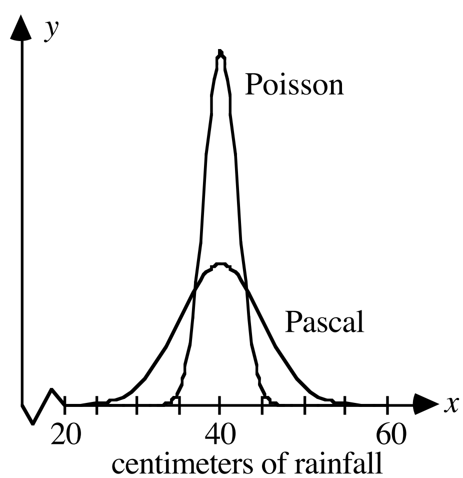
2. The probability of more than 20.5 cm of rainfall is half of 0.05 since it lies over 2 standard deviations from the mean. The probability is 0.025 or 2.5%.

3. The probability of between 10 cm and 17 cm of rainfall is 0.68 or 68% since the interval lies within 1 standard deviation of the mean.

c. Answers will vary. Students should mention the probabilities of drought or flood calculated in Part **b**.

***4.2** The area under the curve outside of 2 standard deviations is 0.05. The probability of less than 100 cm of rain is $(1/2)0.05 = 0.025 = 2.5\%$.

***4.3** **a.** Sample sketch:



b. Sample response: The mayor of Poisson is correct. Her city has annual precipitation between 38 cm and 42 cm 68% of the time. Pascal has annual precipitation between 35 cm and 45 cm 68% of the time. Some of that 68% would fall outside the interval from 38 cm to 42 cm.

c. Sample response: The curve for Poisson is steeper and taller than the one for Pascal. The height and spread must change so that the area under the curve remains equal to 1.

d. Sample response: The amount of precipitation in Poisson will be very consistent over the years. Almost 100% of the time, the residents will see between 34 cm and 46 cm of precipitation. The residents of Pascal will see more variation in their rainfall. They can expect between 25 cm and 55 cm of precipitation.

* * * * *

4.4 **a.** **1.** $\mu = 8$; $\sigma \approx 2.19$

2. $\mu = 16$; $\sigma \approx 3.10$

3. $\mu = 24$; $\sigma \approx 3.79$

b. Sample response: Yes. These experiments represent binomial experiments where *n* is the number of coins, the probability of a success is 0.4, and the probability of a failure is $(1 - 0.4) = 0.6$.

c. When the probability of obtaining a head is 0.2, the means are 1, 2, and 6, respectively. The standard deviations are approximately 1.79, 2.53, and 3.10, respectively.

d. Students should observe that no matter what the probability of obtaining a head, or how many coins are tossed, the mean is *np* and the standard deviation is $\sqrt{np(1-p)}$.

4.5 **a.** The table below shows the frequency distribution for rolling two dice and recording the sum.

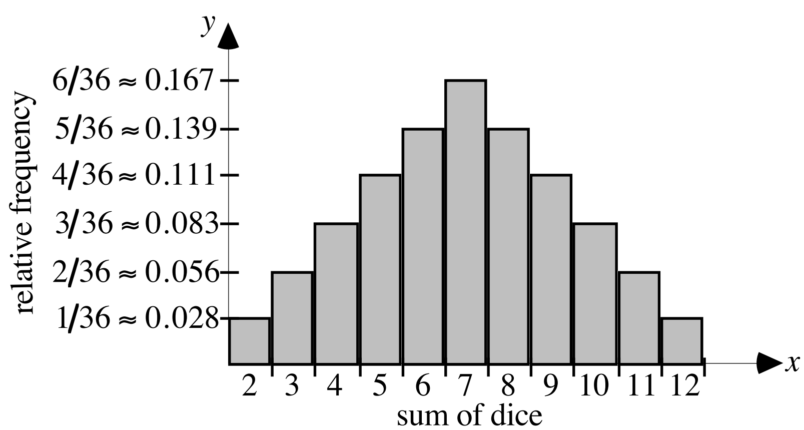
Sum of Dice	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

b. The mean is 7, the median is 7, and the standard deviation is approximately 2.42.

c. The table below shows the theoretical probability distribution for rolling two dice.

Sum of Dice	Probability
2	$1/36 \approx 0.028$
3	$2/36 \approx 0.056$
4	$3/36 \approx 0.083$
5	$4/36 \approx 0.111$
6	$5/36 \approx 0.139$
7	$6/36 \approx 0.167$
8	$5/36 \approx 0.139$
9	$4/36 \approx 0.111$
10	$3/36 \approx 0.083$
11	$2/36 \approx 0.056$
12	$1/36 \approx 0.028$

d. Sample graph:



e. **1.** 67%

2. 95%

3. 100%

f. Sample response: Yes. Although the data is not continuous, it could be modeled by a normal distribution. The shape of the frequency histogram resembles that of a normal curve and the data appears to fit the 68–95–99.7 rule that describes a normal distribution.

g. On any single turn, the probability of rolling a 10 is $3/36 \approx 0.083$. The probability that this occurs on three consecutive turns is $(3/36)^3 = 27/46,656 = 1/1728 \approx 0.0006$.

* * * * * * * * * *

Research Project (page 181)

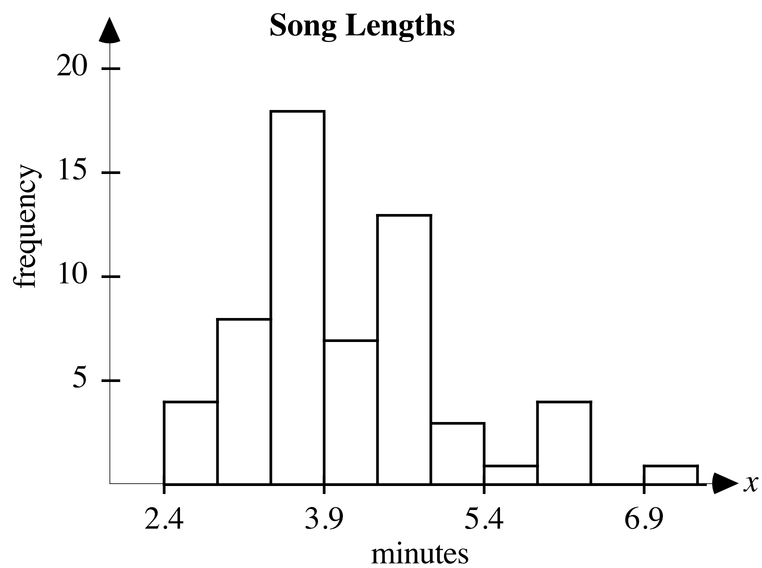
The Bernoulli distribution is named after Jacques Bernoulli (1654–1705). Although Jacques and his brother Jean were perhaps the most famous members of the family, at least 10 others also achieved prominence in mathematics and science. Bernoulli's principle of hydrodynamics, for example, is named in honor of Jean's son Daniel (1700–1782).

Answers to Summary Assessment

(page 182)

Responses will vary, depending on the selection of songs. To obtain a varied sample, students should survey a number of different artists and take no more than two song lengths from any individual performer. Time should be converted to minutes and fractions of minutes.

The histogram below displays sample data for 60 songs. The song lengths are closely grouped around the mean but skewed to the left. (Students should understand that "approximately normal" allows for some variation from a perfectly smooth curve.) The mean is 4.13 min and the standard deviation is 0.97 min. Using this data, the radio station would be able to play about 10 songs in 45 min.



Module Assessment

1. The table below shows the mean daily temperatures for a weather station during a three-month period.

Temp. (˚C)	**Frequency**	**Temp. (˚C)**	**Frequency**
$[17,18)$	0	$[24,25)$	19
$[18,19)$	3	$[25,26)$	18
$[19,20)$	0	$[26,27)$	14
$[20,21)$	2	$[27,28)$	8
$[21,22)$	7	$[28,29)$	5
$[22,23)$	5	$[29,30)$	2
$[23,24)$	6	$[30,31)$	1

a. Make a relative frequency table of this data.

b. Use your table to draw a relative frequency histogram.

2. The heights of a population of young women are normally distributed with a mean of 168 cm and a standard deviation of 6.5 cm.

a. Identify the intervals centered at the mean that will include 68%, 95%, and 99.7% of the heights of the young women.

b. Sketch a graph of a normal distribution with a mean of 168 cm and a standard deviation of 6.5 cm. On the x-axis of your graph, mark the point where the mean occurs and the points 1, 2, and 3 standard deviations from the mean.

c. Leigh is 181 cm tall. What is the approximate percentage of young women in the population who are taller than Leigh?

3. **a.** Consider a random number generator that produces numbers in the interval [2, 7] so that any number is equally likely to occur. Draw a graph of the probability distribution for all the possible outcomes.

b. What are the mean and median of this probability distribution? Describe how you determined your responses.

c. What is the probability that a number x from this random number generator will fall in each of the following intervals?

1. $2 \le x \le 3$

2. $0 \le x \le 1$

3. $x > 4$

4. $x = 5$

4. The table below contains the names of three professional baseball players, their personal best batting averages for a season, the decade in which they achieved those averages, the mean batting average for all players in that decade, and the standard deviation for all players in that decade.

Name	Batting Average	Decade	Mean	Standard Deviation
Ty Cobb	0.420	1910–19	0.266	0.0371
Ted Williams	0.406	1940-49	0.267	0.0326
George Brett	0.390	1970-79	0.261	0.0317

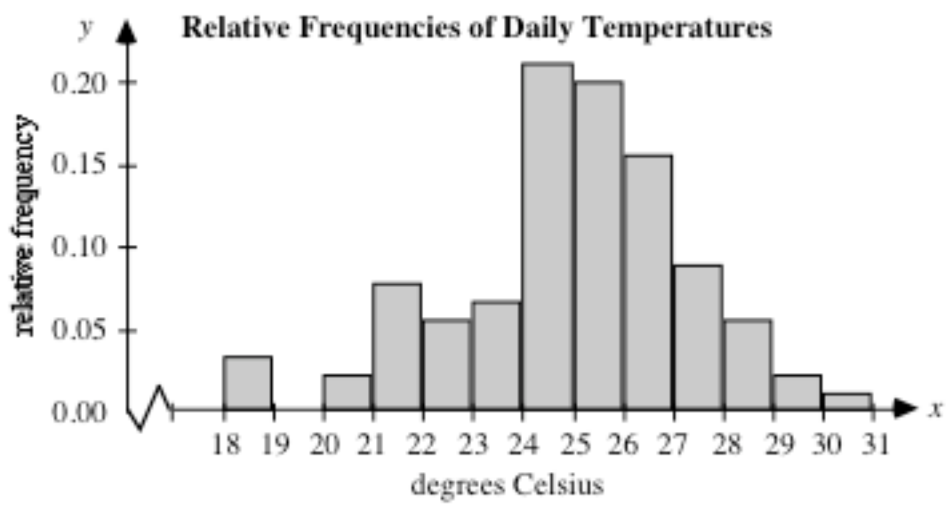
Compare the batting averages of the three players. In relation to the other players in the decade, which player appears to have achieved the greatest accomplishment? Justify your response.

Answers to Module Assessment

1. **a.** Sample table:

Temperature (Celsius)	Frequency	Relative Frequency
$[17,18)$	0	0.00
$[18,19)$	3	0.03
$[19,20)$	0	0.00
$[20,21)$	2	0.02
$[21,22)$	7	0.08
$[22,23)$	5	0.06
$[23,24)$	6	0.07
$[24,25)$	19	0.21
$[25,26)$	18	0.20
$[26,27)$	14	0.16
$[27,28)$	8	0.09
$[28,29)$	5	0.06
$[29,30)$	2	0.02
$[30,31)$	1	0.01
Total	90	1.00

b. Sample histogram:

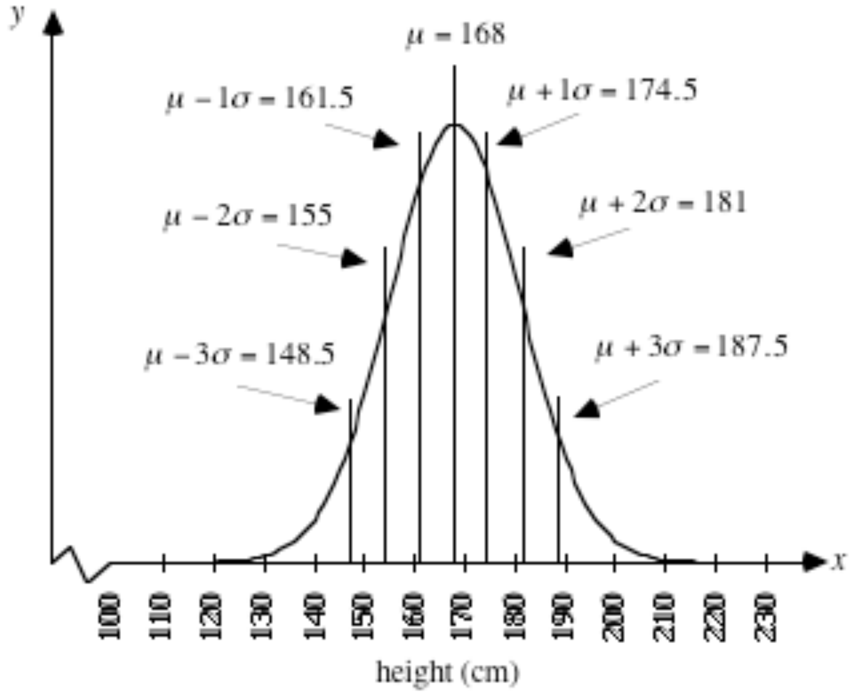


2. **a.** 68%: $161.5 \le h < 174.5$

95%: $155 \le h < 181$

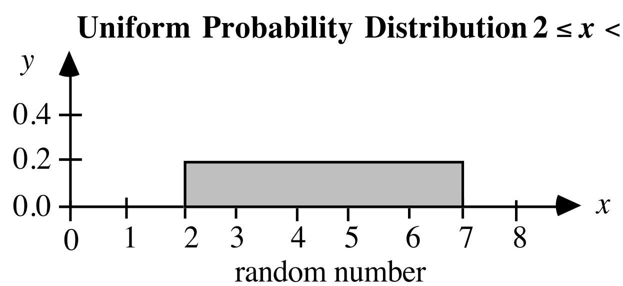
99.7%: $148.5 \le h < 187.5$

b. Sample graph:



c. $(1/2)5\%$ or 2.5%

3. **a.** The graph is a rectangle of height 0.2 over the interval from 2 to 7.



b. Sample response: In a continuous probability distribution, the mean and median are in the middle of the graph. Both the mean and the median are 4.5.

c. **1.** $p(2 \le x \le 3) = 1/5$ or 0.2

2. $p(0 \le x \le 1) = 0$

3. $p(x > 4) = 3/5$ or 0.6

4. $p(x = 5) = 0$

4. Answers may vary, but students should recognize that Ted Williams' personal best is more standard deviations from the mean (4.26) of his decade than either of the other players' best averages.

Selected References

Freedman, D., R. Pisani, and R. Purves. *Statistics*. New York: W. W. Norton & Co., 1980.

Moore, D. S., and G. P. McCabe. *Introduction to the Practice of Statistics*. New York: W. H. Freeman and Co., 1989.

National Climatic Data Center. *Climatological Data Annual Summary Montana*. Vols. 43–94. Asheville, NC: National Oceanic and Atmospheric Administration, 1992.

For information on weather summaries for your area, write the National Climatic Data Center, National Oceanic and Atmospheric Administration, Federal Building, 37 Battery Park Avenue, Asheville, NC 28801-2733.

Flashbacks

Activity 1

1.1 Write the following real-number intervals as inequalities in terms of x.

a. $[-6,10]$

b. $(-6,10]$

c. $[-6,10)$

d. $(-6,10)$

1.2 Use the following table to answer Parts **a** and **b**.

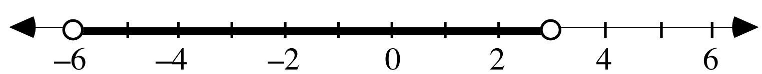
5.5	12.1	2.7	16.4	3.5
4.0	5.1	3.4	8.5	2.7

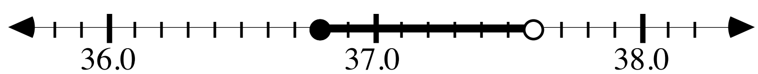
a. Determine the mean, median, and mode of this data.

b. Determine the standard deviation of the data.

c. Consider a random sample of four data items from the table above. How do you think the mean, median, and mode of the sample will compare with the values you calculated in Part **a**?

1.3 Describe each of the following graphs using interval notation.

a.


b.


Activity 2

2.1 A music store questioned 60 customers about their purchases of compact discs (CDs) in the past month. The frequency table shown below summarizes the results of this survey.

No. of CDs Purchased	Frequency
1	25
2	15
3	8
4	10
5	1
6	1
Total	60

a. Determine the experimental probabilities of a customer purchasing 1, 2, 3, 4, 5, or 6 compact discs in a month.

b. Create a histogram to represent the information in Part **a**.

2.2 The images on the screen of a graphing calculator are made up of pixels. On most screens, each pixel is either on or off. Consider two pixels located side by side on this type of screen. Assume that the probability of any pixel being on is the same as its probability of being off, and that each pixel is independent of the others.

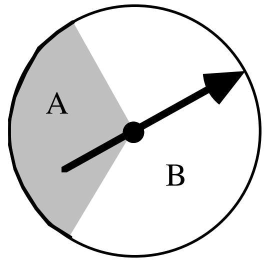
a. List all the possible outcomes for the two pixels.

b. Determine the theoretical probability that exactly one of the two pixels is on.

2.3 The daily earnings of a lawn-mowing business depend on the weather. On days when it rains, the business loses $45. On days when it does not rain, the business makes $155. What is the expected value for the business if the probability of rain on any one day is 0.11?

Activity 3

3.1 The diagram below shows a spinner like those used in many games.



The arrow for the spinner may land in either region A or region B. Determine the probability of the arrow landing in region B if the central angle that forms region A measures 120˚.

3.2 Find the length of a rectangle that has a width of 6 cm and an area of 3 cm^2 .

Activity 4

4.1 **a.** Create a scatterplot of relative frequency versus *x*-value to represent the data in the table below.

***x*-value**	**Relative Frequency**
1	0.06
2	0.08
3	0.09
4	0.10
5	0.11
6	0.12
7	0.11
8	0.10
9	0.09
10	0.08
11	0.06

b. Use your graph to estimate the mean of the data.

c. Verify your estimate by calculating the mean of the data.

d. Calculate the standard deviation of the data.

Answers to Flashbacks

Activity 1

1.1 **a.** $-6 \le x \le 10$

b. $-6 < x \le 10$

c. $-6 \le x < 10$

d. $-6 < x < 10$

1.2 **a.** The mean is 6.39, the median is 4.55, and the mode is 2.7.

b. The standard deviation is approximately 4.36.

c. The mean, median, and mode are likely to be different, since the standard deviation is relatively large and the sample size is small.

1.3 **a.** $(-6, 3)$

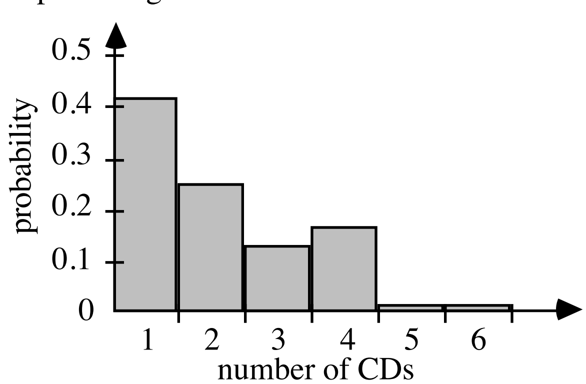
b. $[36.8, 37.6)$

Activity 2

2.1 **a.** The experimental probabilities are shown in the table below.

No. of CDs Purchased	Frequency	Experimental Probability
1	25	$25/60 \approx 0.42$
2	15	$15/60 = 0.25$
3	8	$8/60 \approx 0.13$
4	10	$10/60 \approx 0.17$
5	1	$1/60 \approx 0.02$
6	1	$1/60 \approx 0.02$
Total	60	1

b. Sample histogram:



2.2 **a.** The possible outcomes for the two pixels are OO, OF, FO, and FF, where O represents on and F represents off.

b. From the following table of theoretical probabilities, the probability that exactly one pixel is on is 0.5.

No. of Pixels On	Frequency	Probability
0	1	0.25
1	2	0.50
2	1	0.25
Total	4	1.00

2.3 $-45 \bullet 0.11 + 155 \bullet 0.89 = \$133$ per day

Activity 3

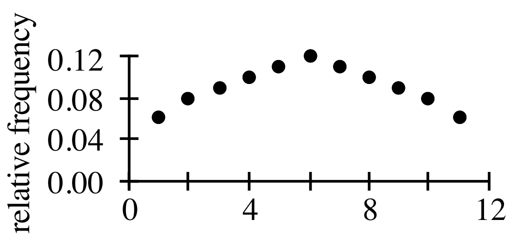
3.1 Sample response:

$$\frac{360 - 120}{360} = \frac{2}{3}$$

3.2 Since $A = l \bullet w$, $l = 0.5 \text{ cm}$.

Activity 4

4.1 **a.** Sample scatterplot:



b. The mean appears to occur at $x = 6$.

c. The mean is 6.

d. The standard deviation is approximately 2.86 if we assume there are 100 data values.

Big Business

How can mathematics help launch a new business? In this module, you'll explore how rational functions and nonlinear inequalities can offer direction to a fledgling cosmetics company.

Sherry Horyna • Jeff Hostetter • Peter Rasmussen

Teacher Edition
Big Business

Overview

This module uses the creation of a new product—from research to packaging to financing—to introduce students to rational functions. Students simulate the pH balance of a home permanent and explore constraints on container sizes.

Objectives

In this module, students will:

- graph and interpret rational functions, including domains, discontinuities, and asymptotes
- evaluate functions around discontinuities
- graph nonlinear inequalities and systems of relations.

Prerequisites

For this module, students should know how to:

- recognize polynomials by degree
- write polynomials as products of first-degree factors
- graph inequalities
- solve systems of equations.

Time Line

Activity	**Intro.**	**1**	**2**	**3**	**4**	**5**	**Summary Assessment**	**Total**
Days	1	2	3	2	2	1	1	12

Materials Required

Materials	Activity						
	Intro.	**1**	**2**	**3**	**4**	**5**	**Summary Assessment**
household ammonia	X						
distilled water	X						
1-L beakers	X						
100-mL graduated cylinders	X						
1-mL pipettes	X						
graph paper					X	X	
straightedge					X	X	
scissors					X		
tape					X		

Technology

Software	Activity						
	Intro.	**1**	**2**	**3**	**4**	**5**	**Summary Assessment**
graphing utility	X	X	X	X	X	X	X
spreadsheet	X	X	X	X			
science interface device	X						
pH probe	X						
symbolic manipulator		X	X	X			X

Big Business

Introduction

(page 189)

Students use a formula for concentration by mass to generate a rational function.

Materials List

- distilled water
- pH probe
- clear household ammonia (6.80% by weight)
- 1-L beakers (two per group)
- 100-mL graduated cylinders (one per group)
- 1-mL pipette (one per group)

Technology

- spreadsheet
- graphing utility
- science interface device

Teacher Note

Although a relatively weak base, ammonium hydroxide is a powerful household cleanser and should be handled with care. The density and concentration given in the student edition (0.993 g/mL and 6.80% by weight) were obtained by calling the consumer information number on the label of a bottle of Parson's™ Clear Ammonia. These figures may vary for other brands. If the density and concentration are different from those given in Table **1**, students should substitute the appropriate values.

Students should recall their experience with pH from the Level 4 module "Log Jam." In water, ammonia (NH_3) forms a solution of ammonium ions (NH_4^+) and hydroxide (OH^-) ions by the formula $NH_3 + H_2O \Leftrightarrow NH_4^+ + OH^-$.

To ensure consistent results, each pH probe should be calibrated according to the manufacturer's instructions before beginning the exploration.

Exploration

(page 189)

Students explore the relationship between the pH of the solution and the mass of distilled water added as well as the relationship between the percent concentration of ammonium hydroxide and the mass of distilled water added.

a. The pH of the ammonia solution should be about 11.0.

b. If 1 mL of 6.80% ammonium hydroxide solution has a mass of 0.993 g, then:

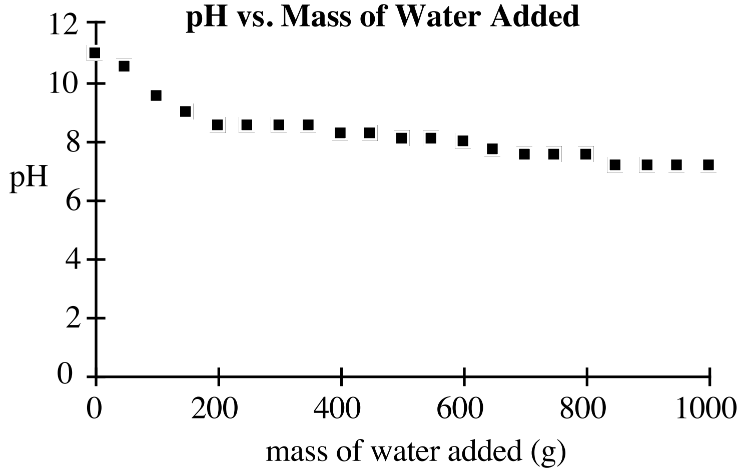
$$6.80 = \frac{p}{0.993} \bullet 100$$
$$p \approx 0.0675 \text{ g}$$

c. **Note:** To save time, you may wish to premeasure 1 mL of the ammonia solution for each group.

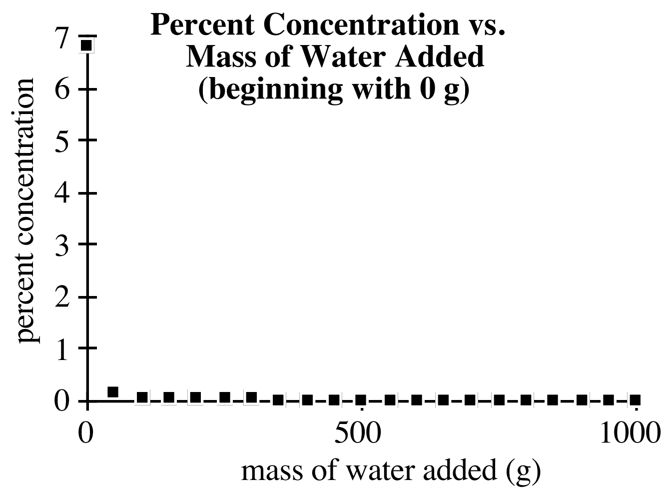
d–e. Students may wish to use a spreadsheet to record their data and create graphs. They should obtain a final pH close to 7 (neutral). Sample data:

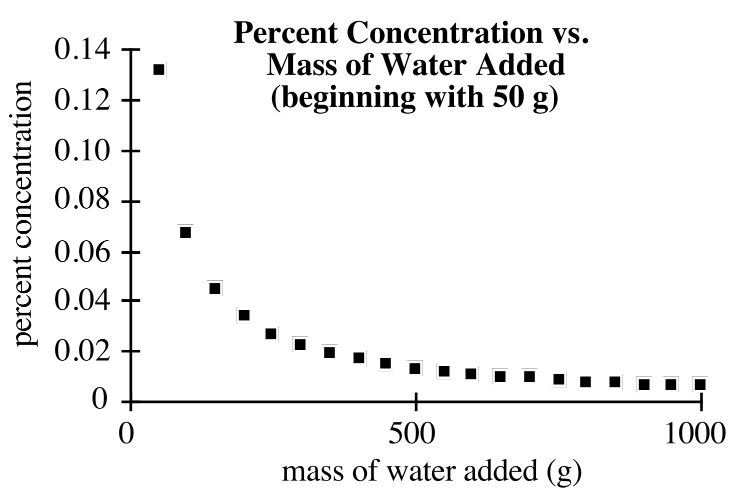
Mass of Distilled Water (g)	Total Mass of Solution (g)	pH	Concentration of Ammonium Hydroxide (%)
0	0.993	11.0	6.8
50	50.993	10.5	0.132
100	100.993	9.5	0.067
150	150.993	9.0	0.045
200	200.993	8.5	0.034
250	250.993	8.5	0.027
300	300.993	8.5	0.022
350	350.993	8.5	0.019
400	400.993	8.2	0.017
450	450.993	8.2	0.015
500	500.993	8.1	0.013
550	550.993	8.1	0.012
600	600.993	8.0	0.011
650	650.993	7.7	0.010
700	700.993	7.5	0.010
750	750.993	7.5	0.009
800	800.993	7.5	0.008
850	850.993	7.2	0.008
900	900.993	7.2	0.007
950	950.993	7.2	0.007
1000	1000.993	7.2	0.007

f. Sample graphs:



Note: The two graphs below illustrate the same information using different scales on the axis that represents percent concentration.



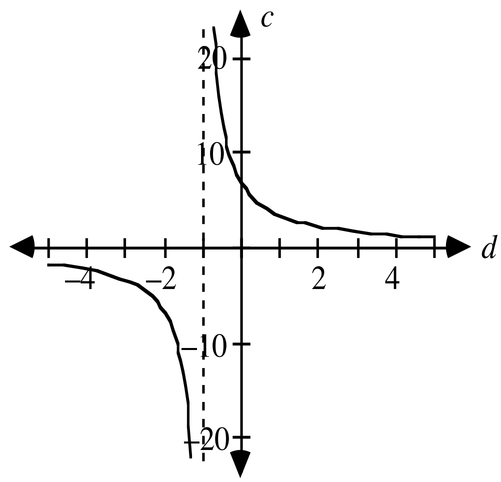


g. **1.** In the following sample function, c represents percent concentration and d represents the mass, in grams, of the distilled water added.

$$c = \frac{0.0675}{d + 0.993} \bullet 100$$

2. Sample response: The domain for this the setting is the set of real numbers in the interval [0, 1000].

h. **1.** Sample graph:



2. The domain is the set of all real numbers except –0.993. The function is undefined at this value.

Discussion

(page 191)

a. Answers will vary. It should take approximately 600–700 mL of water to obtain a pH less than 8.

b. Answers will vary. The pH of the solution continues to approach 7 as more water is added. (Many alkaline perms use a solution of hydrogen peroxide to neutralize the base.)

c. Sample response: It took about 650 mL of distilled water to reduce the percent concentration to 0.01%. This can be found by solving the function in Part **g** of the exploration for d when the function is the set equal to 0.01.

d. Students should observe that neither graph is linear and that both have the same general shape.

e. Sample response: The portion of the graph from Part **h** in which x is positive is very similar to the graphs in Part **f**. However, the graph of the equation in Part **h** also has a negative portion. [There is also a vertical asymptote where $x = -0.993$.]

f. The domain of the function is all real numbers except –0.993. The domain for the setting in the exploration is the interval [0, 1000].

g. Sample response: When $x = -0.993$, the function is not defined. In other words, –0.993 is not in the domain. Both sides of the graph approach the line with the equation $x = -0.993$.

(page 191)

Activity 1

In this activity, students are introduced to a formal definition of rational functions. They identify discontinuities in functions and graph rational functions with holes.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator
- spreadsheet (optional)

Teacher Note

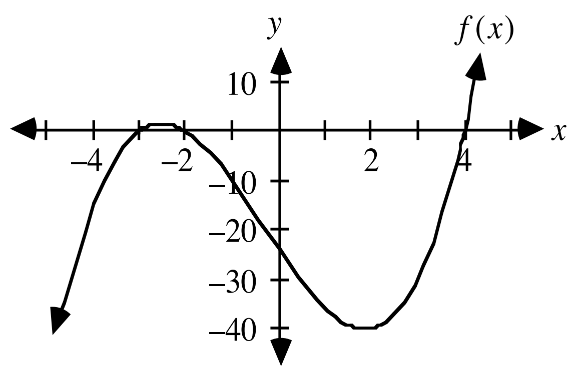
Discontinuities may not be readily apparent on the graphs generated by some graphing utilities. Students may need to experiment with intervals for the domain and range in order to view holes.

Exploration 1

(page 192)

In this exploration, students examine rational functions in which all of the factors of the denominator are also factors of the numerator. They also use a symbolic manipulator to examine the result of the division indicated by the rational function.

a. **1.** Sample graph:

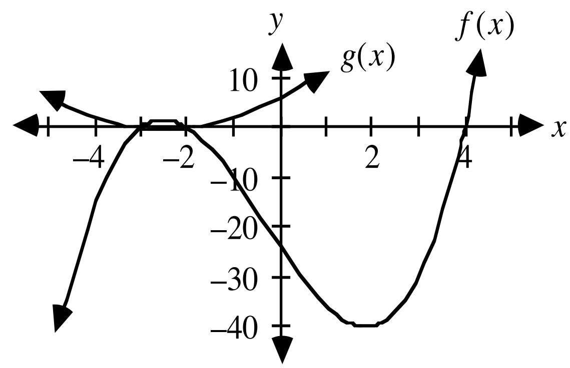


2. The roots of $f(x)$ are –3, –2, and 4.

3. The domain of $f(x)$ is the set of real numbers.

4. $f(x) = (x+2)(x+3)(x-4)$

b. 1. Sample graph:



2. The roots of $g(x)$ are –3 and –2.

3. The domain of $g(x)$ is the set of real numbers.

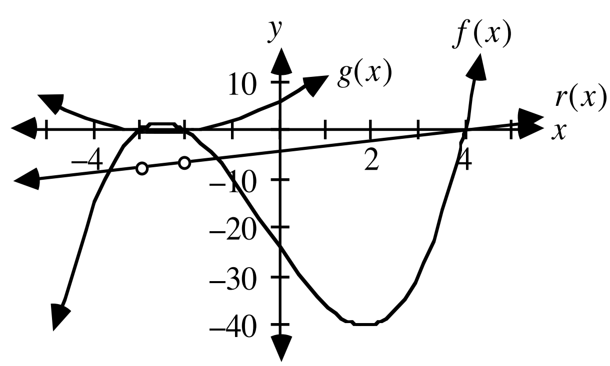
4. $g(x) = (x+2)(x+3)$

c. 1. The rational function is:

$$r(x) = \frac{x^3 + x^2 - 14x - 24}{x^2 + 5x + 6}$$
$$= \frac{(x+2)(x+3)(x-4)}{(x+2)(x+3)}$$

2. The domain of $r(x)$ is all real numbers except –2 and –3.

3. Depending on the technology used, students may or may not observe the holes in the graph of $r(x)$ at $x = -2$ and $x = -3$. Sample graph:



d. Students may use a variety of methods to approximate the equation of the line. Sample response: $y = x - 4$.

e. Sample response:

$$\begin{aligned} t(x) &= \frac{(x+2)(x+3)(x-4)}{(x+2)(x+3)} \\ &= \frac{(x+2)}{(x+2)} \bullet \frac{(x+3)}{(x+3)} \bullet (x-4) \\ &= 1 \bullet 1 \bullet (x-4) \\ &= x-4, \text{ as long as } x \neq -2 \text{ and } -3 \end{aligned}$$

Note: The domain for $t(x)$ must be the same as the domain of $r(x)$ for the two functions to be equivalent.

f. Sample response: When $f(x)$ is divided by $g(x)$, the result is $x-4$. This expression is included in the equation of the line found in Part **d**. Although $t(x)$ also equals $x-4$, its domain does not include $x=-2$ and $x=-3$. **Note:** Students discuss the equivalence of these functions in Part **e** of Discussion **1**.

Discussion 1

(page 193)

a. The graphs of $f(x)$ and $g(x)$ intersect at the points with coordinates (–2,0) and (–3,0). This occurs because –2 and –3 are roots of both functions.

b. The function $r(x)$ intersects the x-axis at the point with coordinates (4,0). It also intersects $f(x)$ at this point. This occurs because the only factor of $f(x)$ that is not contained in $g(x)$ is $(x-4)$. The graphs of $r(x)$ and $g(x)$ do not intersect.

c. If the graphs of two polynomials intersect each other at the x-axis, they share common factors of the form $(x-c_n)$, where c_n is the x-coordinate of each point of intersection.

d. Sample response: The pairs of common factors from the numerator and denominator can be rewritten as 1s. The product of the remaining factors is a polynomial expression. The graph of the polynomial function defined by this expression is similar to the graph of the rational function.

e.

1. No. The domain of the function $r(x)$ does not include –2 and –3. The domain of the line is all real numbers.

2. Sample response: Yes. Their domains are the same and they are equal for all values in the domain.

3. Sample response: The result of the division is equivalent to $r(x)$ as long as the domain is restricted to exclude –2 and –3.

f. Sample response: A symbolic manipulator could be used to factor both the numerator and denominator of the rational function so that common factors could be rewritten as 1s. The product of the remaining factors results in a polynomial expression.

A symbolic manipulator could also be used to divide the numerator of the rational expression by the denominator. The result is a polynomial expression.

The restrictions on the domain must be determined by identifying the values of x for which the function is undefined.

Exploration 2

(page 194)

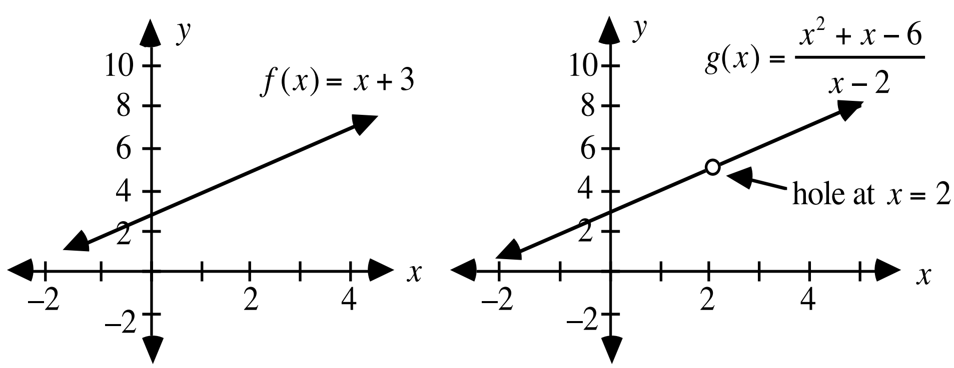
In this exploration, students compare linear functions and related rational functions. The exploration leads to a discussion of discontinuities and holes in graphs.

a. The domain for $f(x)$ is the set of all real numbers. The domain for $g(x)$ is the set of all real numbers except 2.

b. Students may use a symbolic manipulator to perform this division. You may wish to note the fact that there is no remainder.

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

c. Students should note that the graphs are similar except for a hole in $g(x)$ at $x = 2$. Sample graphs:



d. Sample table:

x	$f(x)$	$g(x)$
1.5	4.5	4.5
1.6	4.6	4.6
1.7	4.7	4.7
1.8	4.8	4.8
1.9	4.9	4.9
2.0	5.0	undefined
2.1	5.1	5.1
2.2	5.2	5.2
2.3	5.3	5.3
2.4	5.4	5.4
2.5	5.5	5.5

e. The two functions are not equivalent because their domains are not the same.

Discussion 2

(page 194)

a.
1. The function $f(x)$ is a polynomial function. While $g(x)$ is not, its numerator and denominator are both polynomial functions.
2. The functions $f(x)$ and $g(x)$ are both rational functions. **Note:** All polynomial functions are rational functions. Students encounter this question in Problem **1.5**.

b. **Note:** The two equivalent functions must have the same domain. The following sample responses were obtained by multiplying the numerator and denominator of the original function by the same constant.

1. Sample response:

$$h(x) = \frac{2x + 6}{2}$$

2. Sample response:

$$t(x) = \frac{4x^2 + 4x - 24}{4x - 8}$$

c.
1. Sample response: No, the function is not continuous because it is not defined at $x = 2$.
2. Sample response: The factors of the numerator are $(x + 1)$, $(x - 2)$, and $(x - 3)$. The factor of the denominator is $(x - 2)$. The factor $(x - 2)$ in the numerator can be paired with the $(x - 2)$ in the denominator to create a 1. The product of the remaining factors is the polynomial expression $x^2 - 2x - 3$. Therefore, the graphs of $h(x)$ and $p(x)$ would be similar.

3. Sample response: The graph of $h(x)$ has a hole at $x = 2$ that is not present in the graph of $p(x)$. This is because $h(x)$ is not defined at $x = 2$.

4. Answers may vary. Sample response:

$$r(x) = \frac{x^3 + x^2 - 9x - 9}{x + 3}$$

Assignment

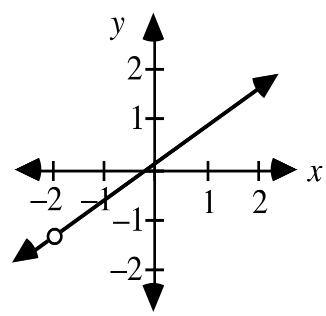
(page 196)

1.1 **a.** The domain is the set of all real numbers except 7. A hole appears at $x = 7$.

b. The domain is the set of all real numbers except –4 and –1. Holes appear at $x = -4$ and $x = -1$.

c. The domain is the set of all real numbers. No discontinuities exist.

1.2 Answers will vary. Sample response:

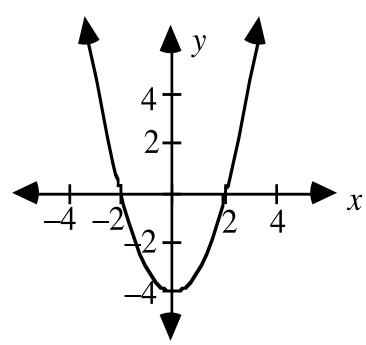


1.3 **a.** Students may use a symbolic manipulator to complete this division.

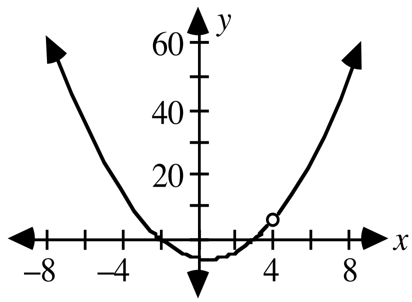
$$\frac{x^3 + 5x^2 + 3x - 9}{x + 3} = x^2 + 2x - 3$$

b. These two functions are not equivalent because their domains are not the same. The domain for $f(x)$ is the set of all reals. The domain for $g(x)$ is the set of all reals except –3.

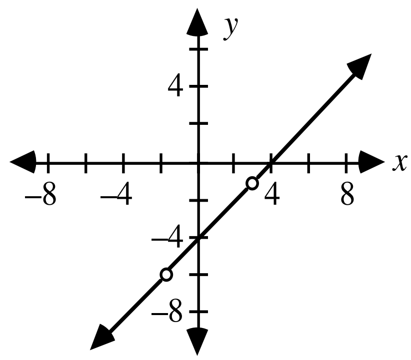
***1.4** **a.** The domain is the set of all real numbers. There are no discontinuities.



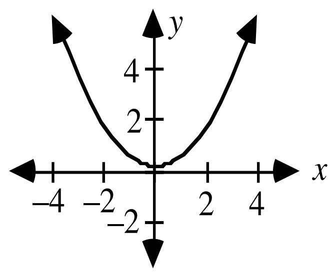
b. The domain is the set of all real numbers except 4. A hole occurs at $x = 4$.



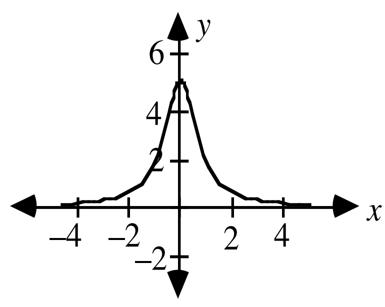
c. The domain is the set of all real numbers except 3 and –2. Holes occur at $x = 3$ and $x = -2$.



d. The domain is the set of all real numbers. There are no discontinuities.



e. The domain is the set of all real numbers. There are no discontinuities.

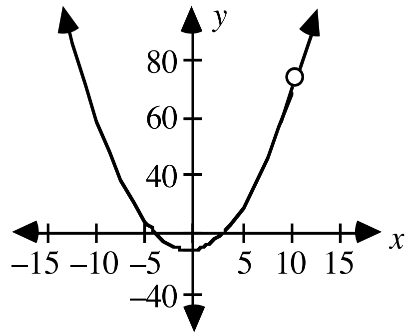


1.5 A constant function, such as $f(x) = 1$, is a polynomial. Therefore any polynomial $p(x)$ can be written in rational form as:

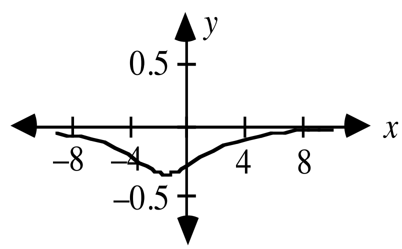
$$\frac{p(x)}{f(x)}$$

* * * * *

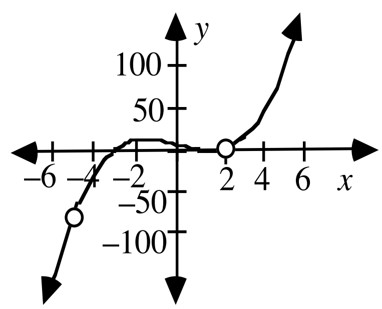
1.6 **a.** The domain is the set of all real numbers except 11. A hole occurs at $x = 11$.



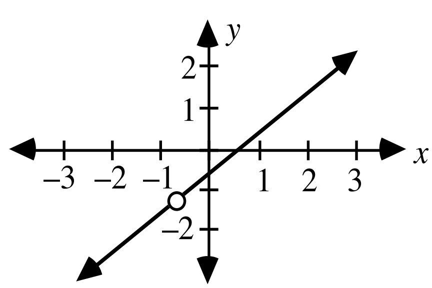
b. The domain is the set of all real numbers. There are no discontinuities.



c. The domain is the set of all real numbers except –5 and 2. Holes occur at $x = -5$ and $x = 2$.



d. The domain is the set of all real numbers except –0.75. A hole occurs at $x = -0.75$.



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(page 198)

Activity 2

Students continue to explore rational functions and their graphs. Vertical and horizontal asymptotes are introduced.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator
- spreadsheet (optional)

Teacher Note

In Parts **d** and **e** of Exploration **1**, students are expected to use a symbolic manipulator to rewrite a rational expression in the form $Q(x) + R(x)/d(x)$. They may wish to refer to an instruction manual to determine how to represent a rational expression in this form.

Exploration 1

(page 198)

a. Students verify that the final expression given in the example is equivalent to the original rational expression.

b–c. Students should rewrite this expression as shown below, then verify their results as described in Part **a**.

$$\begin{aligned}\frac{6x-5}{x+4} &= \frac{6(x+4)-24-5}{x+4} \\ &= \frac{6(x+4)}{x+4} - \frac{29}{x+4} \\ &= 6 - \frac{29}{x+4}\end{aligned}$$

d. Students use a symbolic manipulator to repeat Part **b**.

e. Students select a rational function in which the degree of the numerator is greater than or equal to the degree of the denominator. They then use a symbolic manipulator to determine an equivalent mixed expression.

Discussion 1

(page 199)

a. Sample response: All of the factors in the denominator can be paired with the same factors in the numerator. These pairs are equivalent to 1. The product of the remaining factors in the numerator and the 1s is a polynomial with no remainder.

b.
1. Sample response: When the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, the value of $Q(x)$ is 0. The numerator cannot be expressed as a product that contains the factors in the denominator.

2. Sample response: When the degree of the polynomial in the numerator is equal to the degree of the polynomial in the denominator, the value of $Q(x)$ is a real number. The numerator can be expressed as a product of a real number and the factors of the denominator.

c. Answers will vary, depending on the technology used. Using a TI-92 calculator, for example, students should type "propFrac," then the rational expression enclosed by parentheses. Pressing the enter key results in the rewritten expression.

Teacher Note

On some graphing utilities, especially graphing calculators, vertical asymptotes may appear to be part of the graph of the function itself. In such cases, you may wish to select another graphing tool or graph the functions using a "dot mode" found on some graphing calculators.

Exploration 2

(page 199)

Students examine the values of a rational function $f(x)$ near its vertical asymptote and when $|x|$ is large. They also investigate the relationship between the result of the division of the function's numerator by the denominator and the equations of the function's asymptotes.

a. $d(x) = 0$ when $x = 1$

b. The domain of $f(x)$ is the set of all real numbers except 1.

c. $$\frac{3x+2}{x-1} = 3 + \frac{5}{x-1}$$

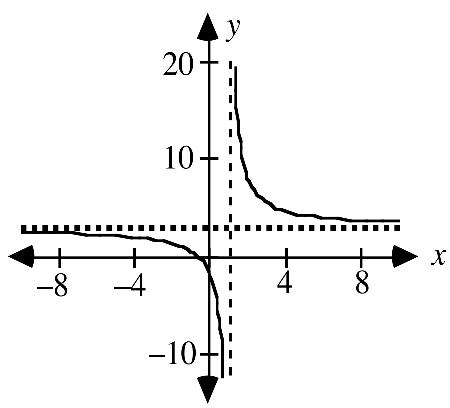
d. Sample table:

x	$f(x)$
0.9	–47
0.95	–97
0.99	–497
0.999	–4997
1.001	5003
1.01	503
1.05	103
1.1	53

e. Sample table:

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–10,000	2.9995	3	–0.0005
–1000	2.995	3	–0.005
–100	2.9505	3	–0.05
100	3.0505	3	0.0505
1000	3.005	3	0.005
10,000	3.0005	3	0.0005

f. A vertical asymptote occurs at $x = 1$. There is also a horizontal asymptote with the equation $y = 3$. Sample graph:



g. Sample response: The function is undefined at $x = 1$. As x approaches 1, the value of $f(x)$ gets very large or very small. As x gets close to 1, the graph approaches the vertical line with the equation $x = 1$.

h. Sample response: As $|x|$ gets very large, the value of $R(x)/d(x)$ approaches 0 and can be ignored. The value of $f(x)$ approaches $Q(x)$.

i. As with the function examined in Parts **a–f**, the value of $R(x)/d(x)$ approaches 0 as $|x|$ gets very large and can be ignored. The value of $f(x)$ approaches $Q(x)$.

Discussion 2

(page 201)

a. A discontinuity occurs at $x = 1$.

b.

1. As the value of x gets closer to 1 from the left, the values of $f(x)$ approach $-\infty$. The graph approaches the line $x = 1$.
2. As the values of x get closer to 1 from the right, the values of $f(x)$ approach $+\infty$. The graph approaches the line $x = 1$.

c. As $|x|$ gets very large, the values of $f(x)$ get close to 3. The graph approaches the horizontal line $y = 3$.

d. Sample response: As $|x|$ gets very large, the expression $5/(x-1)$ approaches 0 and the value of the function gets close to 3.

e. This function has a vertical asymptote with the equation $x = 1$ and a horizontal asymptote with the equation $y = 3$.

f. Sample response: No. A discontinuity only occurs if a function is undefined at some point, or if there is a break or a hole in the graph for some value in the domain. None of these things happen with a horizontal asymptote.

g.

1. Sample response:

$$f(x) = 2 + \frac{3}{x-4}$$

Note: This graph also has a horizontal asymptote at $y = 2$.

2. Sample response:

$$g(x) = 2 + \frac{0}{x-4}$$

3. The discontinuities in both sample functions occur at $x = 4$.

h. Rational expressions in which the numerator shares all the factors of the denominator have discontinuities represented by holes in the graph. These can be expressed in the form

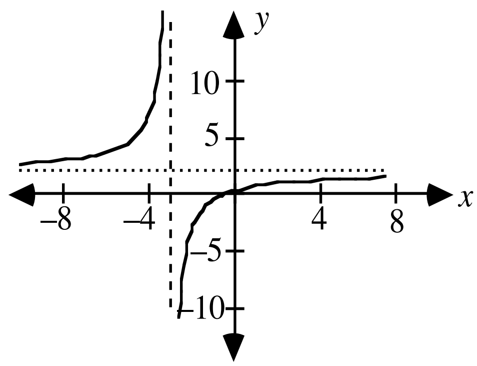
$$Q(x) + \frac{R(x)}{d(x)}$$

where $R(x) = 0$.

Rational expressions which can be expressed in the form above, where $R(x)$ is a real number other than 0, have discontinuities represented by vertical asymptotes at the value where the function is undefined. They also have horizontal asymptotes at $y = Q(x)$, if $Q(x)$ is a constant.

Assignment (page 202)

2.1 Answers will vary. Sample graph:



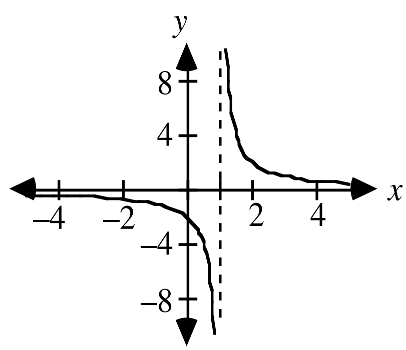
2.2 **a.** $d(x) = 0$ when $x = 1$

b. The domain of $f(x)$ is the set of all real numbers except 1.

c. **Note:** You may wish to point out that even though the quotient is 0, this division is still useful. The quotient provides information about the graph's asymptotes (in this case, the *x*-axis).

$$\frac{2}{x-1} = 0 + \frac{2}{x-1}$$

d. **1.** A horizontal asymptote occurs at $y = 0$. A vertical asymptote occurs at $x = 1$. Sample graph:

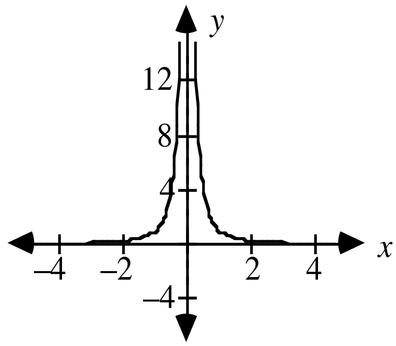


2. Sample response: By dividing, $f(x)$ may be rewritten as

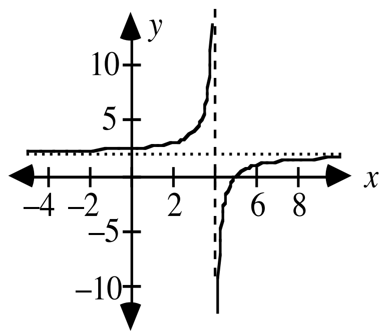
$$f(x) = 0 + \frac{2}{x-1}$$

The equation of the horizontal asymptote is related to the quotient of this division. The equation of the vertical asymptote is related to the value of *x* that makes the denominator 0.

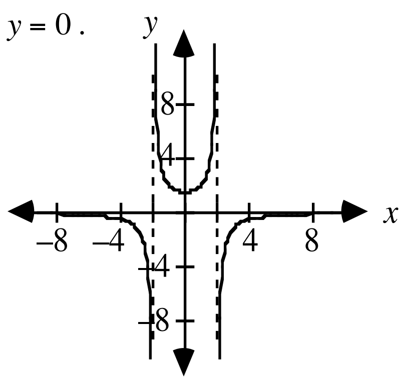
***2.3** **a.** The domain is the set of all real numbers except 0. A vertical asymptote occurs at $x = 0$. A horizontal asymptote occurs at $y = 0$.



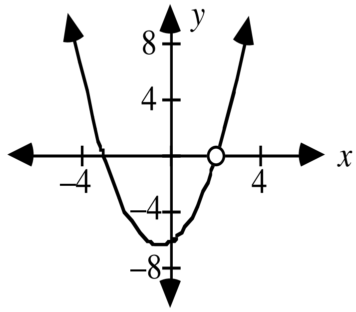
b. The domain is the set of all real numbers except 4. A vertical asymptote occurs at $x = 4$. A horizontal asymptote occurs at $y = 2$.



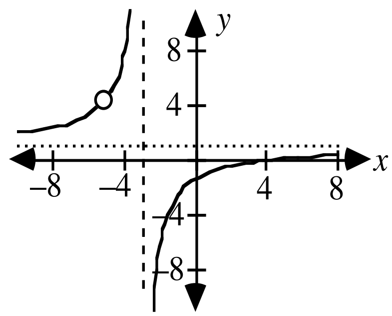
c. The domain is the set of all real numbers except –2 and 2. Vertical asymptotes occur at $x = 2$ and $x = -2$. A horizontal asymptote occurs at $y = 0$.



d. The domain is the set of all real numbers except 2. A hole occurs at $x = 2$.



e. The domain is the set of all real numbers except –5 and –3. A hole occurs at $x = -5$. A vertical asymptote occurs at $x = -3$. A horizontal asymptote occurs at $y = 1$.



Note: This function can be factored as follows:

$$f(x) = \frac{x^3 + 4x^2 - 17x - 60}{x^3 + 11x^2 + 39x + 45} = \frac{(x-4)(x+3)(x+5)}{(x+3)^2(x+5)}$$

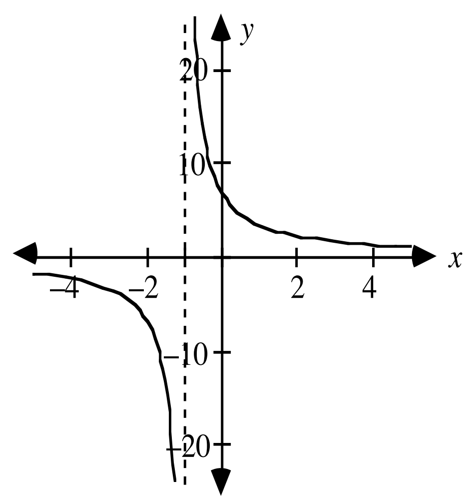
On some symbolic manipulators, however, the division yields:

$$f(x) = \frac{x^3 + 4x^2 - 17x - 60}{x^3 + 11x^2 + 39x + 45} = 1 + \frac{-7}{x+3}$$

In this case, some students may overlook the hole at $x = -5$.

2.4 **a.** The domain is the set of all real numbers except –0.993. **Note:** Some students may argue that a negative mass of distilled water is not realistic and therefore the domain should be all positive real numbers.

b. Sample graph:



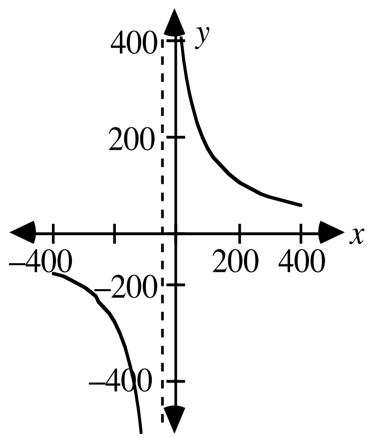
c. A vertical asymptote occurs at $x = -0.993$. A horizontal asymptote occurs at $y = 0$.

2.5 **a.** $C(5) = \$490.91$, $C(1000) = \$25.71$, and $C(10{,}000) = \$2.69$

b. Sample response: There are always some fixed costs for setting up the manufacturing equipment. These costs are the same whether 5000 or 5,000,000 perms are produced. The more perms the company makes, the lower the cost per perm.

c. The domain is the set of all real numbers except –50. **Note:** Some students may argue that it is unrealistic to make a negative number of perms or a fractional number of perms, therefore the domain should be the positive integers.

d. A vertical asymptote occurs at $x = -50$. A horizontal asymptote occurs at $y = 0$. Sample graph:

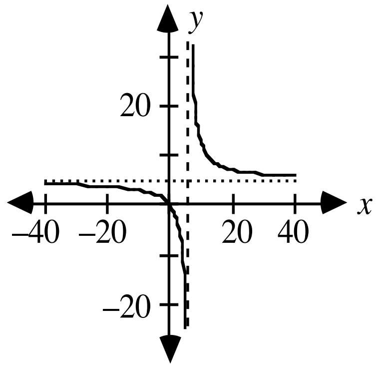


e. The startup costs are \$540. This is found by substituting $x = 0$ into the cost function.

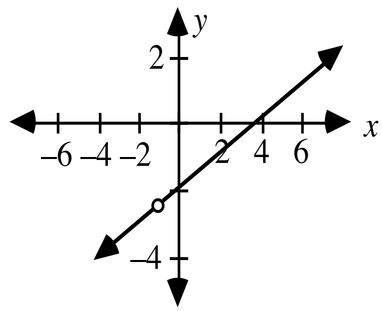
f. 11,015 units

* * * * *

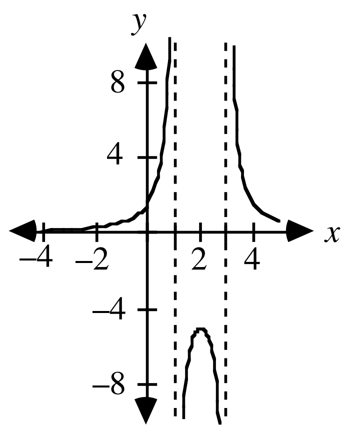
2.6 **a.** The domain is the set of all real numbers except 6. A vertical asymptote occurs at $x = 6$. A horizontal asymptote occurs at $y = 5$.



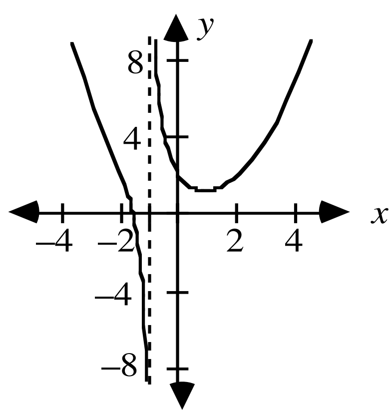
b. The domain is the set of all real numbers except –0.5. A hole occurs at $x = -0.5$.



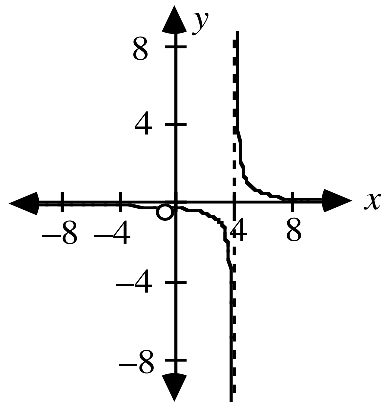
c. The domain is the set of all real numbers except 1 and 3. Vertical asymptotes occur at $x = 1$ and $x = 3$. A horizontal asymptote occurs at $y = 0$.



d. The domain is the set of all real numbers except –1. A vertical asymptote occurs at $x = -1$.



e. The domain is the set of all real numbers except 4 and –1. A hole occurs at $x = -1$. A vertical asymptote occurs at $x = 4$. A horizontal asymptote occurs at $y = 0$.



2.7 **a.** The domain of the following function is the set of all real numbers except –10.

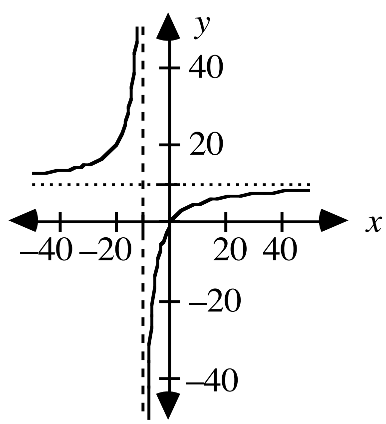
$$R(x) = \frac{10x}{10 + x}$$

Note: Some students may argue that a negative resistance is not realistic and therefore the domain should be all positive real numbers.

b. The graph has a vertical asymptote at $x = -10$ and a horizontal asymptote at $y = 10$.

$$R(x) = 10 - \frac{100}{x + 10}$$

c. Sample graph:



d. $r_2 = 4.3$ ohms

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(page 204)

Activity 3

In this activity, students explore the graphs of rational functions with oblique asymptotes.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator
- spreadsheet (optional)

Exploration

(page 204)

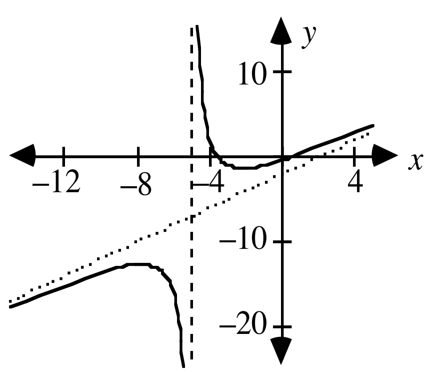
a. The domain of $f(x)$ is the set of all real numbers except –5.

b. $$\frac{x^2+3x-2}{x+5}=(x-2)+\frac{8}{x+5}$$

c. Sample table:

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–1000	–1002	–1002	–0.008
–500	–502	–502	–0.02
–100	–102.1	–102	–0.084
–50	–52.2	–52	–0.18
50	48.15	48	0.145
100	98.08	98	0.076
500	498	498	0.016
1000	998	998	0.008

d. A vertical asymptote occurs at $x=-5$. Sample graph:



e. Sample response: As $|x|$ becomes large, the function $Q(x) = x - 2$ has values very close to $f(x)$. It is therefore a possible oblique asymptote for the graph of $f(x)$.

Discussion

(page 205)

a. The graph has a vertical asymptote at $x = -5$ and an oblique asymptote at $y = x - 2$.

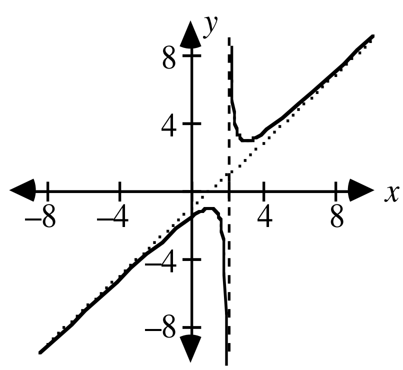
b. As $|x|$ gets large, $R(x)/d(x)$ approaches 0.

c. Sample response: As $|x|$ gets large, the values of $f(x)$ and $Q(x)$ get closer to being equal.

Assignment

(page 206)

3.1 Answers will vary. Sample graph:



3.2 Answers will vary. Students should identify features of the graph of each function by writing it in the following form:

$$f(x) = Q(x) + \frac{R(x)}{d(x)}$$

a. Sample response:

$$f(x) = \frac{x^2 + 3x + 2}{x + 1} = x + 2 + \frac{0}{x + 1}$$

This graph looks like the graph of $y = Q(x) = x + 2$ with a hole at $x = -1$, where the function is undefined.

b. Sample response:

$$f(x) = \frac{3x + 2}{x + 1} = 3 - \frac{1}{x + 1}$$

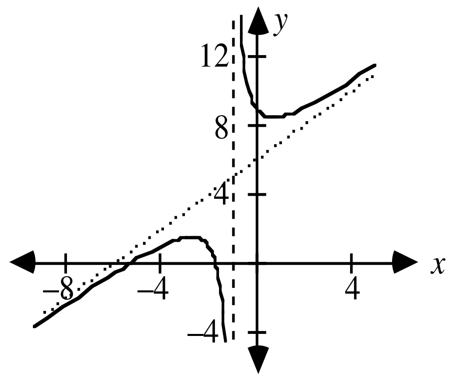
The graph of this function has a horizontal asymptote at $y = Q(x) = 3$. A vertical asymptote occurs at $x = -1$, where the function is undefined.

c. Sample response:

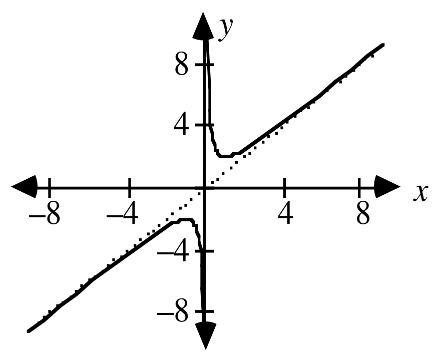
$$f(x)=\frac{x^2-4x+2}{x+1}=x-5+\frac{7}{x+1}$$

The graph of this function has an oblique asymptote at $y=Q(x)=x-5$. A vertical asymptote occurs at $x=-1$, where the function is undefined.

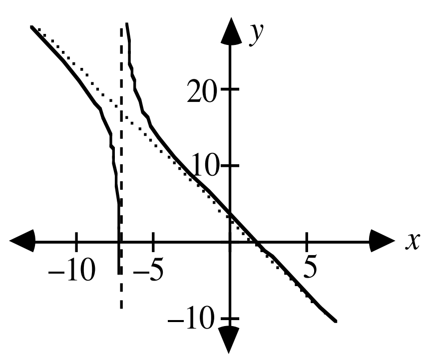
***3.3** **a.** The domain is the set of all real numbers except –1. A vertical asymptote occurs at $x=-1$. An oblique asymptote occurs at $y=x+6$.



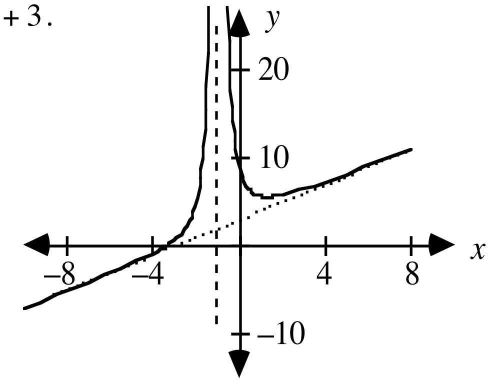
b. The domain is the set of all real numbers except 0. A vertical asymptote occurs at $x=0$. An oblique asymptote occurs at $y=x$.



c. The domain is the set of all real numbers except –7. A vertical asymptote occurs at $x=-7$. An oblique asymptote occurs at $y=-2x+3$.

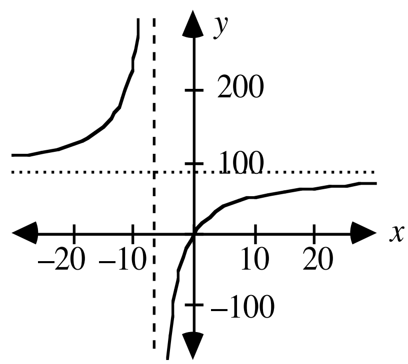


d. The domain is the set of all real numbers except –1. A vertical asymptote occurs at $x = -1$. An oblique asymptote occurs at $y = x + 3$.



***3.4** **Note:** Some students may argue that a negative dosage is not realistic and therefore the domain should be all positive real numbers.

a. A vertical asymptote occurs at $x = -6.5$. A horizontal asymptote occurs at $y = 85$. Sample graph:



b. The dosage for a 2-year-old child is 20 mg.

c. The dosage for a 15-year-old is approximately 59.3 mg.

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3.5 **a.** Sample response:

$$f(x) = \frac{x^2 - 3x - 10}{x - 5} = x + 2 + \frac{0}{x - 5}$$

The graph of this function looks like the graph of $y = Q(x) = x + 2$ with a hole at $x = 5$, where the function is undefined.

b. Sample response:

$$f(x) = \frac{3x - 16}{x - 5} = 3 - \frac{1}{x - 5}$$

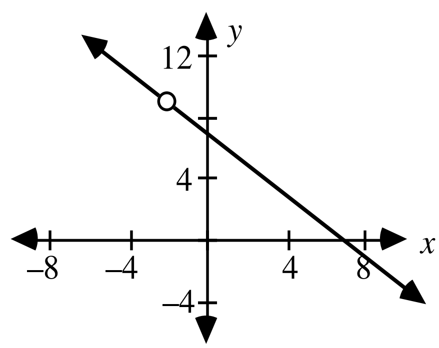
The graph of this function has a horizontal asymptote at $y = Q(x) = 3$. A vertical asymptote occurs at $x = 5$, where the function is undefined.

c. Sample response:

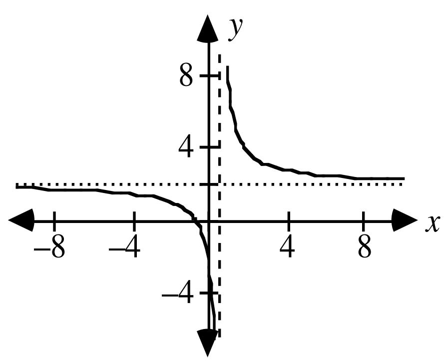
$$f(x) = \frac{x^2 - 10x + 32}{x - 5} = x - 5 + \frac{7}{x - 5}$$

The graph of this function has an oblique asymptote at $y = Q(x) = x - 5$. A vertical asymptote occurs at $x = 5$, where the function is undefined.

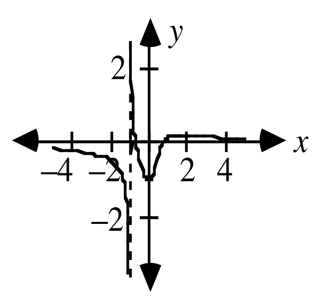
3.6 **a.** The domain is the set of all real numbers except –2. A hole occurs at $x = -2$.



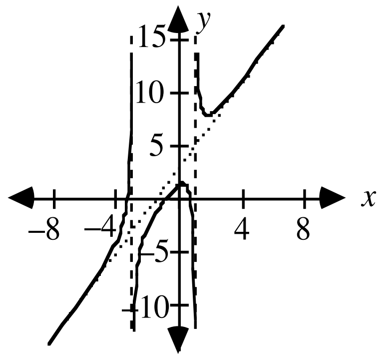
b. The domain is the set of all real numbers except 0.5. A vertical asymptote occurs at $x = 0.5$. A horizontal asymptote occurs at $y = 2$.



c. The domain is the set of all real numbers except –1. A vertical asymptote occurs at $x = -1$. A horizontal asymptote occurs at $y = 0$.



d. The domain is the set of all real numbers except –3 and 1. Vertical asymptotes occur at $x = -3$ and $x = 1$. An oblique asymptote occurs at $y = 2x + 3$.



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(page 208)

Activity 4

In this activity, students examine restrictions on the domains of rational functions. They graph systems of equations, solve for their intersections, and review inequalities.

Materials List

- straightedge
- graph paper
- scissors
- tape

Technology

- graphing utility

Exploration

(page 208)

Students explore restrictions on the domain of a function by modeling the heights of cylindrical bottles of various radii.

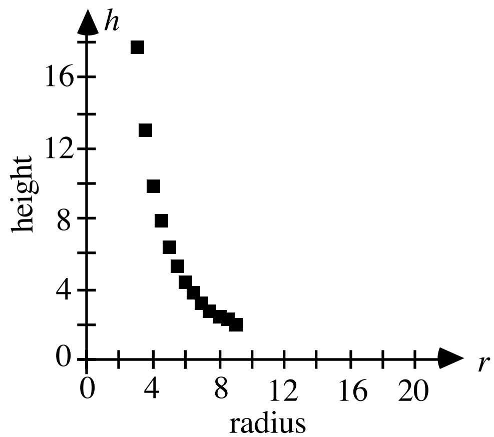
a. $h = 500/\pi r^2$

b. Students may find the cylinders easier to construct if they leave a small tab to overlap when taping. **Note:** You may wish to use the values of *r* shown in the sample table given in Part **c**.

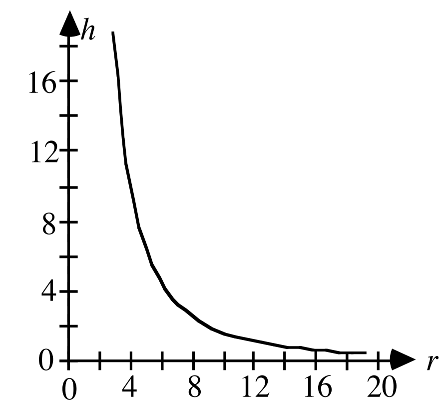
c. The values in the following sample table are rounded to the nearest 0.1 cm.

Radius	Height	Radius	Height
2.0	39.8	6.0	4.4
2.5	25.5	6.5	3.8
3.0	17.7	7.0	3.2
3.5	13.0	7.5	2.8
4.0	9.9	8.0	2.5
4.5	7.9	8.5	2.2
5.0	6.4	9.0	2.0
5.5	5.3		

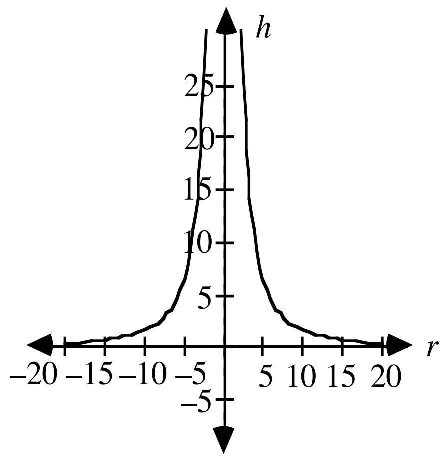
Sample graph:



d. Sample graph:



e. The domain for this graph is the set of nonzero real numbers.



Discussion

(page 208)

a. Sample response: The cylinders with large radii would take up too much room on store shelves. The cylinders with large heights would be too easy to knock over.

b. Solving for r in terms of h, yields the following equation:

$$r = \frac{\sqrt{500}}{\sqrt{\pi h}} = \frac{500^{1/2}}{(\pi h)^{1/2}}$$

A rational function is defined as the quotient of two polynomial functions. The expressions $500^{1/2}$ and $(\pi h)^{1/2}$ are not polynomials because their exponents are not whole numbers. Therefore, the equation is not a rational function.

c. Sample response: A reasonable domain for the function is $1 \leq r \leq 12.6$, where r represents radius in centimeters. A cylinder with a radius less than 1 cm would be very difficult to manufacture. A radius of 12.6 cm results in a height of 1 cm. Anything shorter also would be very difficult to make.

d. Sample response: The graph in Part **d** does not have a negative portion because it is not realistic to have a negative radius.

e.
1. The following function is undefined when $r = 0$:

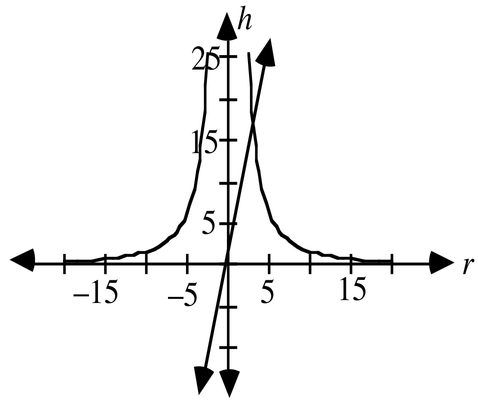
$$h(r) = \frac{500}{\pi r^2}$$

2. The domain is the set of all real numbers except 0.
3. A vertical asymptote occurs at $r = 0$, where the function is undefined. (The graph also has a horizontal asymptote at $h(r) = 0$, although this does not represent a discontinuity.)

Assignment (page 209)

4.1 **a.** Sample response: The equation for the relationship is $h = 2\pi r$. Since the radius of a cylinder cannot be negative, the domain for this function is the positive real numbers.

b. Sample graph:



c. The intersection of $h = 2\pi r$ and $h = 500/\pi r^2$ occurs at the point (2.94,18.45).

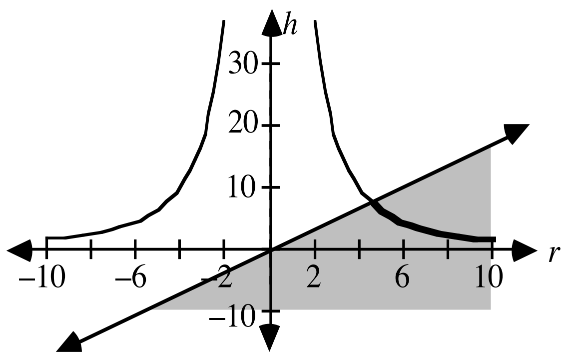
d. Sample response: The point of intersection is the solution to the system of equations. It also means that a cylinder with a radius of 2.94 cm and a height of 18.45 cm satisfies all the constraints in this situation.

e. Sample response: If the two functions did not intersect, then it would be impossible to construct a 500-mL bottle whose height equals its circumference.

f. The bottle will have a radius of 2.94 cm and a height of 18.45 cm.

4.2 **a.** $h \leq 2d$ or $h \leq 4r$

b. Since negative values for *h* and *r* are not reasonable in this context, some students may graph the first quadrant only. Sample graph:



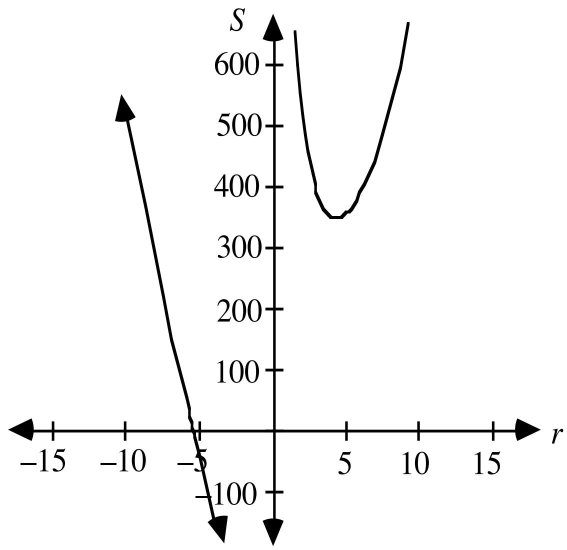
c. The solution of the system is the portion of $h = 500/\pi r^2$ in the shaded region of the graph shown in Part **b**.

d. Sample response: The point of intersection of the two graphs is (3.41,13.66). This means that the radius of the bottle must be greater than or equal to 3.41 cm. The corresponding height will be less than or equal to 13.66 cm. The height and the radius must satisfy the equation $h = 500/\pi r^2$.

***4.3** **a.** $S = 2\pi r^2 + 2\pi rh$

b. $S = 2\pi r^2 + 2\pi r\left(500/\pi r^2\right)$

c. Since negative values for *r* are not reasonable in this context, some students may show a graph of the first quadrant only.



d. The minimum surface area is approximately 349 cm^2.

e. The bottle's radius is about 4.26 cm. Its height is about 8.77 cm.

f. Sample response: Yes, the height is less than twice the diameter.

g. Sample response: This bottle fits all of the criteria given in the assignment and is therefore a reasonable choice. When I made a model of the cylinder, however, I felt that the container could be taller with a smaller radius. It would be more attractive and take up less room on the shelf.

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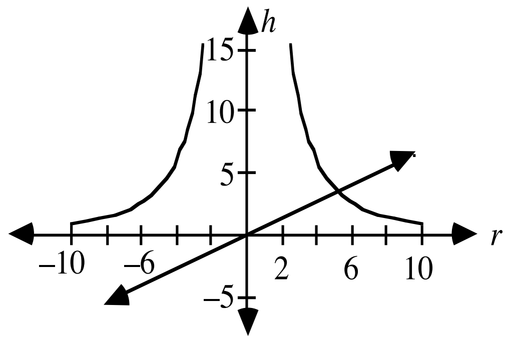
4.4 **a.** $h = 300/\pi r^2$

b. $h = \dfrac{2}{3}r$

c. Sample response: An appropriate domain for this system of equations is the set of all real numbers, *r*, such that $r > 0$. It is unrealistic to have a negative radius or a radius of 0.

d. Since negative values for r are not reasonable in this context, some students may show a graph of the first quadrant only.



e. The bowl's radius is about 5.2 cm. Its height is about 3.5 cm.

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(page 210)

Activity 5

Materials List

- graph paper (optional)
- straightedge (optional)

Technology

- graphing utility

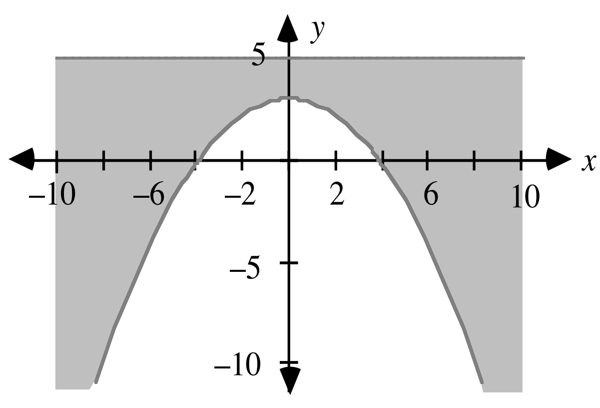
Teacher Note

Some graphing utilities may not have a shade function. In this case, students may redraw the boundary curves on graph paper, then shade the solution region by hand.

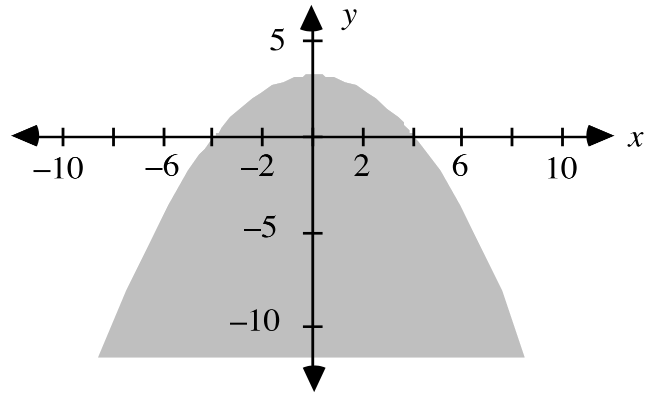
Exploration

(page 211)

a. The boundary is part of the solution set. Sample graph:

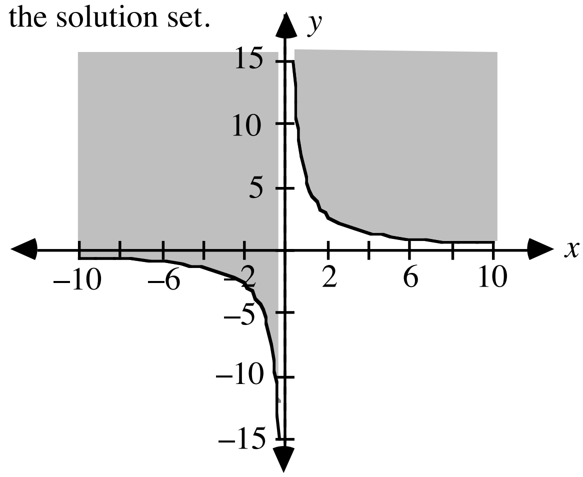


b. The boundary is not part of the solution set. Sample graph:

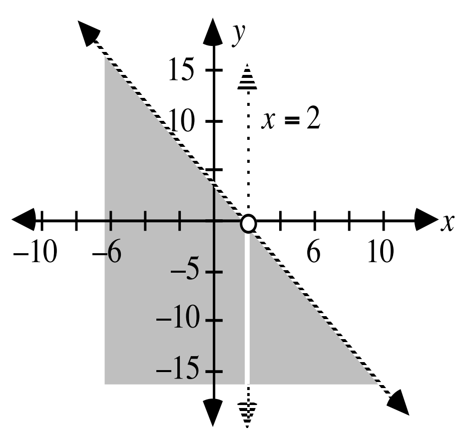


c. Sample response: The shading is on opposite sides of the curve for the two inequalities. The curve is included in the solution set for $y \le -0.2x^2 + 3$; it is not included for $y > -0.2x^2 + 3$. Together, the two solution regions encompass the whole plane.

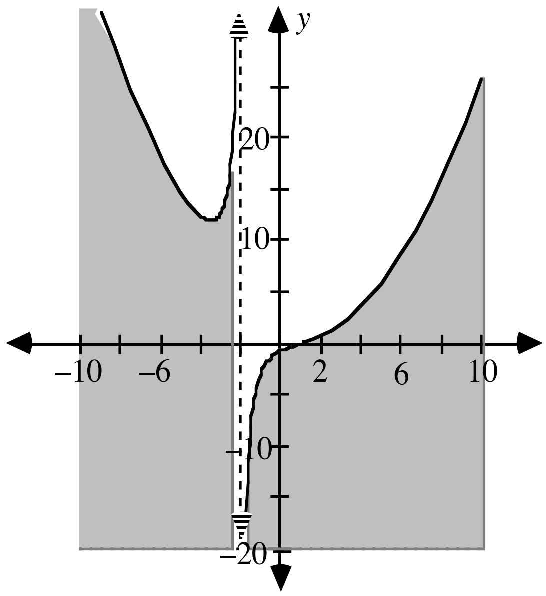
d. **1.** The graph has a vertical asymptote with equation $x = 0$ and a horizontal asymptote with equation $y = 0$. The curve is included in the solution set.



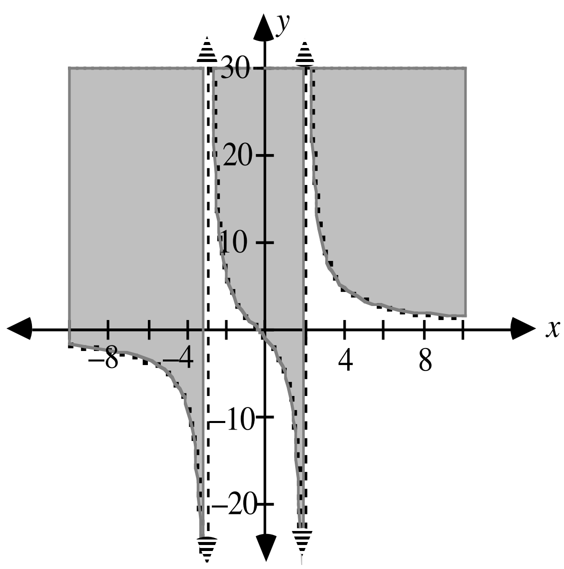
2. The graph has a hole at $x = 2$. The curve is not included in the solution set.



3. The graph has a vertical asymptote with equation $x = -2$. The curve is included in the solution set.



4. The graph has vertical asymptotes with equations $x = 2$ and $x = -3$, and a horizontal asymptote with equation $y = 0$. The curve is not included in the solution set.



Discussion

(page 212)

a. Methods will vary, depending on the graphing utility used. Sample response: We graphed the boundary curve on a graphing utility, then printed a copy of the graph. After determining which side of the graph represented the solution region, we shaded the appropriate area by hand.

b. Sample response: For $y \le -0.2x^2 + 3$, the solution set includes the curve and the region below it. The solution set for $y > -0.2x^2 + 3$ is the region above the curve. Together, the two solution regions encompass the whole plane.

c. Sample response: There is no solution to a rational inequality on a line marking a hole because the rational function is not defined for that value.

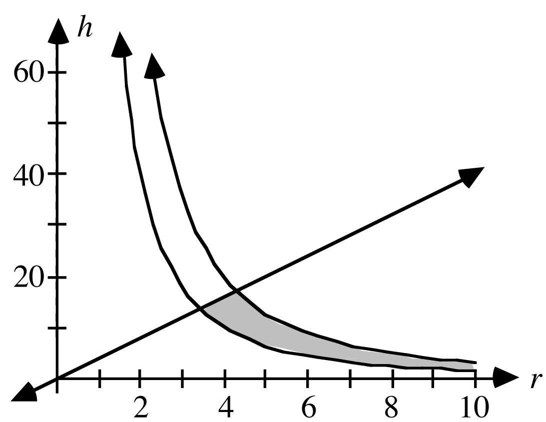
Assignment

(page 212)

5.1 **a.** Sample response:

$$\begin{cases} h \ge 500/\pi r^2 \\ h \le 1000/\pi r^2 \\ h \le 4r \end{cases}$$

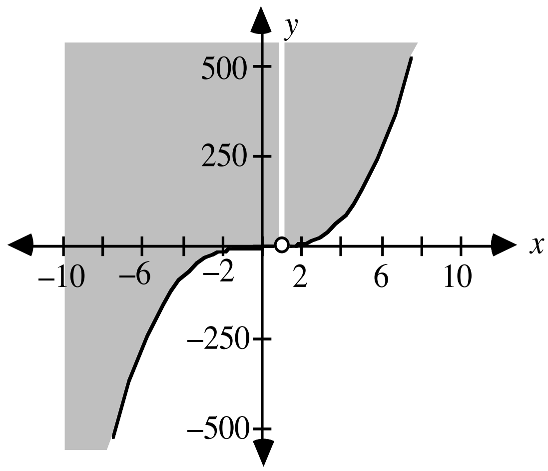
b–c. The solution to the system is the shaded region bounded by the three functions that define the inequalities in Part **a**. Sample graph:



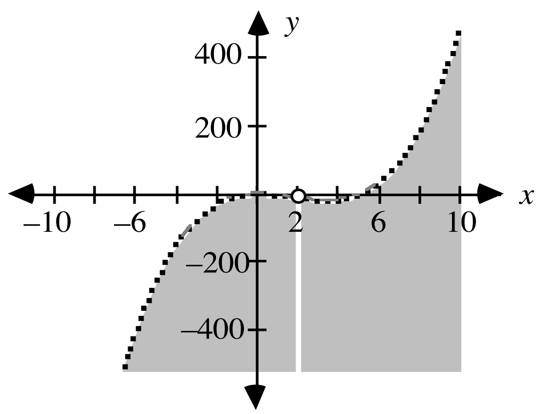
d. Sample response: All the points in the solution region represent possible dimensions for the radius and height of the bottle.

e. Sample response: One bottle that satisfies the constraints has a radius of about 5 cm and a height of about 6.37 cm.

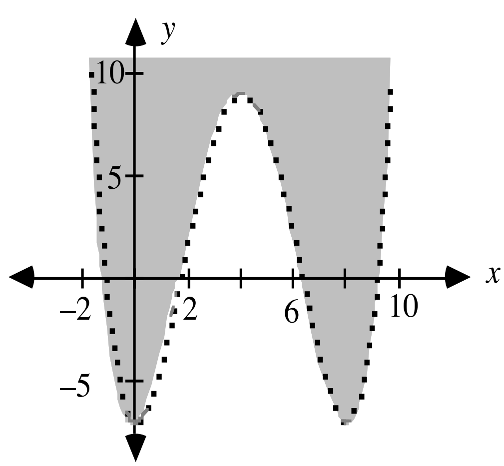
5.2 **a.** The graph has a hole at $x = 1$. The curve is included in the solution set. Sample graph:



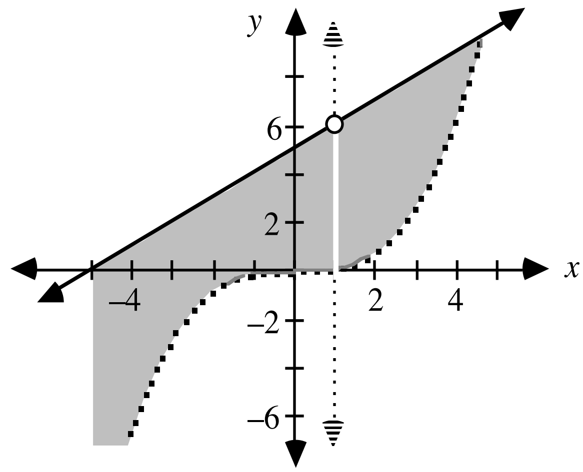
b. The graph has a hole at $x = 2$. The curve is not included in the solution set. Sample graph:



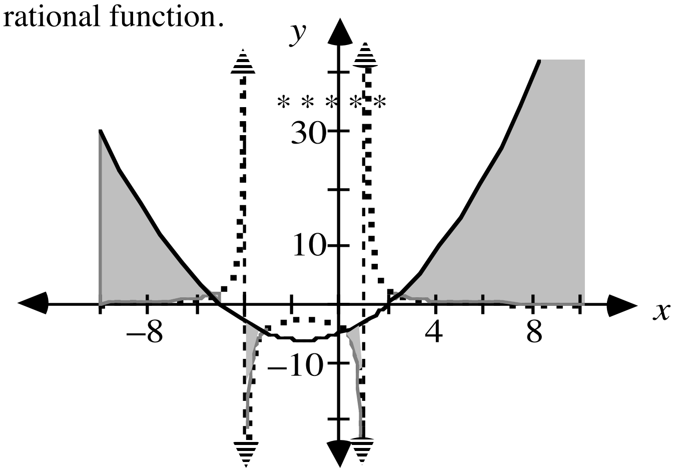
c. The graph has no discontinuities. The curve is not included in the solution set. Sample graph:



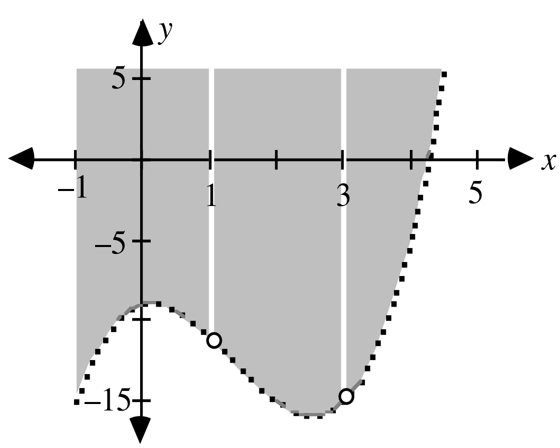
***5.3** **a.** Sample response: A hole occurs at $x = 1$. The solution includes the upper curve but not the lower one.



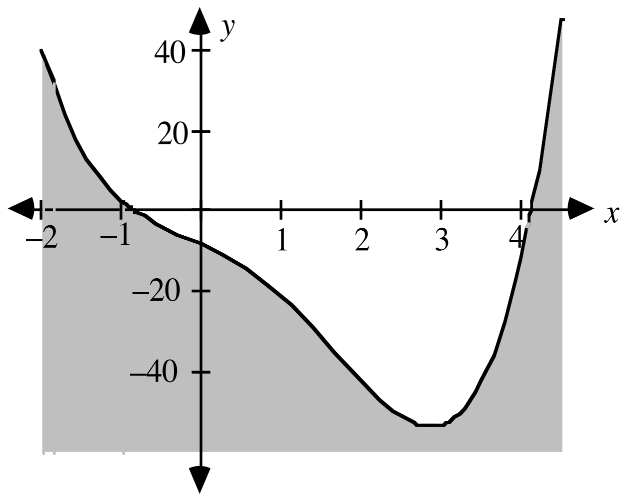
b. Sample response: There are vertical asymptotes at $x = 1$ and $x = -4$. The solution set includes part of the curve defined by the second-degree polynomial but none of the curve defined by the rational function.



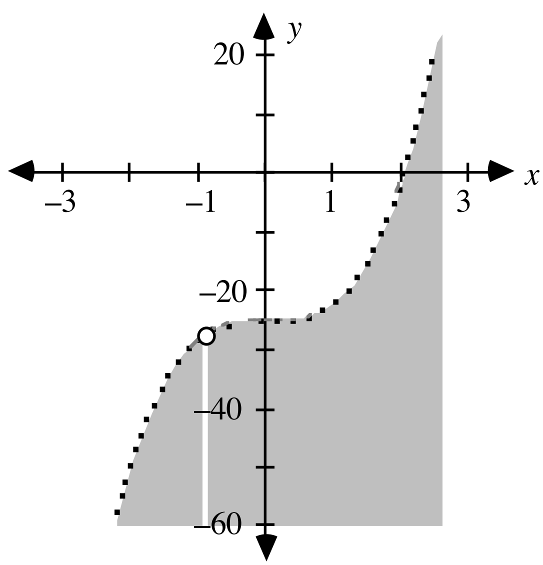
5.4 **a.** The graph has holes at $x = 1$ and $x = 3$. The curve is not included in the solution set.



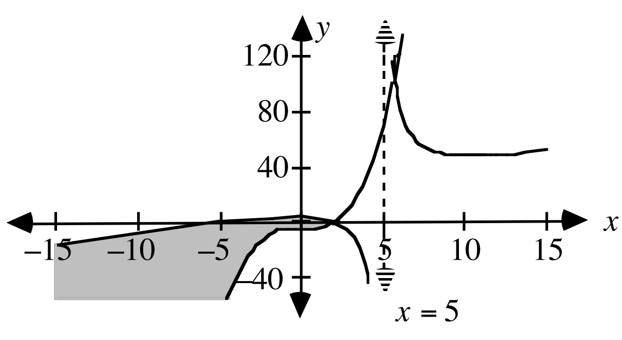
b. There are no discontinuities. The curve is part of the solution set.



c. The graph has a hole at $x = -1$. The curve is not part of the solution set.



5.5 There is a vertical asymptote at $x = 5$.



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Answers to Summary Assessment (page 214)

Note: This summary assessment is intended for small groups.

1. The domain of the following function is the set of all real numbers except 0. **Note:** Some students may suggest that, in this setting, an appropriate domain is the set of positive real numbers.

$$P = \frac{65000r(1+r)^{240}}{(1+r)^{240}-1}$$

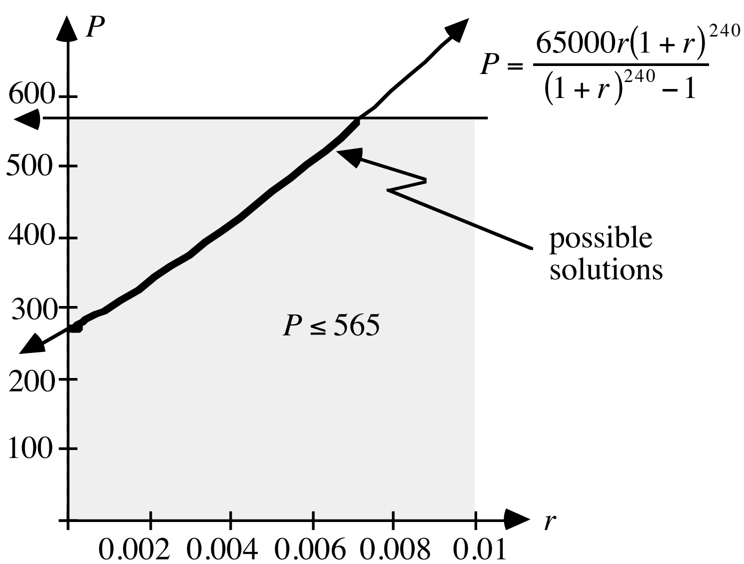
2. 2%

3. $(0, 0.02]$

4. Sample response: There is a discontinuity when $r = 0$. Although an interest rate of 0 in the real world is possible, it is very unlikely. In any case, the problem situation is unaffected by the discontinuity of the function since the value of the payment at $r = 0$ can be calculated simply by dividing the amount of the loan by the number of payments:

$$\frac{\$65{,}000}{240 \text{ payments}} = \$270.8\overline{3}$$

5. Given the constraints in this context, the highest monthly rate is 0.71%. The corresponding interval, therefore, is $(0, 0.0071]$. This solution may be found using a number of methods. Some students may graph the function in Problem **1**, then use the trace feature on a graphing utility. Others may graph both the original function and $P = 565$, then find the intersection of the two functions.



6. The corresponding annual percentage rate is approximately 8.52%.

Module Assessment

1. After a year in business, your company's marketing department has compiled some data on the sales of the home permanent. During the summer months, for example, sales surged upward. Based on this data, they have created a model to predict weekly sales during the period from May through August for the coming year. In this function, $s(w)$ is the number of bottles sold and w is the week number, starting with 1:

$$s(w) = \frac{-312.5w^3 + 8125w^2 - 50,000w}{w - 10} + 500$$

a. Identify the domain of $s(w)$.

b. Graph the function above over an appropriate domain. (Assume that there are four weeks in each of the summer months.)

c. The function that models weekly sales has a discontinuity. Where is this discontinuity located and what is its effect on the graph?

d. Does the discontinuity indicate a problem in sales for that week? Explain your response.

e. Despite what the model predicts, the company has never sold fewer than 1000 bottles in a week. Discuss the limitations of the model.

f. When does the model predict that sales will peak during the summer?

g. How many bottles can the company expect to sell during this week?

2. Determine the domain of each function below. Sketch a graph of each function and label any discontinuities.

a. $l(x) = \dfrac{(2x + 1)(-1.5x - 3)}{2x + 1}$

b. $g(x) = \dfrac{0.2x^3}{x - 1}$

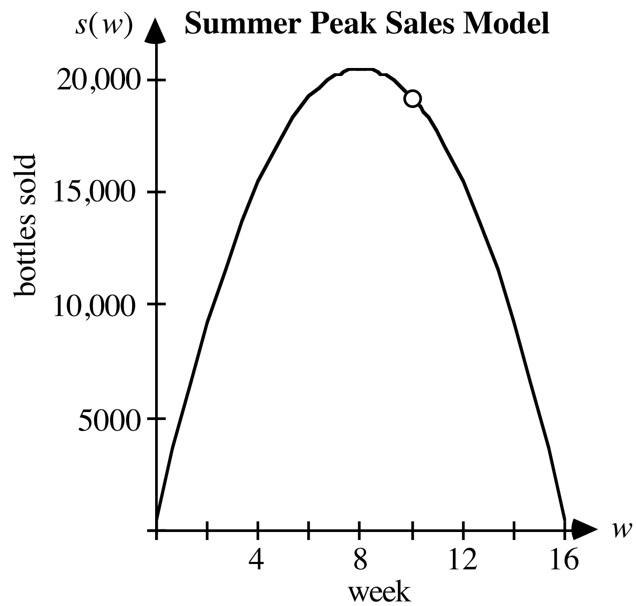
3. Graph the following system of inequalities. Shade the solution region and label any discontinuities that exist.

$$\begin{cases} y < \dfrac{x + 3}{x - 1} \\ y \geq -4 \end{cases}$$

Answers to Module Assessment

1. **a.** The domain is the set of all real numbers except 10.

b. Sample graph:



c. Sample response: A hole occurs at $w = 10$. This means that the graph does not indicate what will happen on week 10.

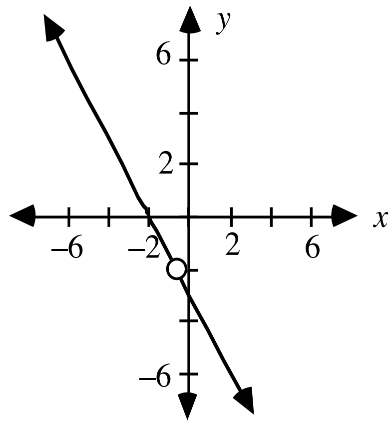
d. Sample response: No. The model can still be used to predict sales during that week by estimating from the curve.

e. Sample response: This model is only useful for making predictions over a limited number of weeks. After week 16, for example, it predicts sales less than 0.

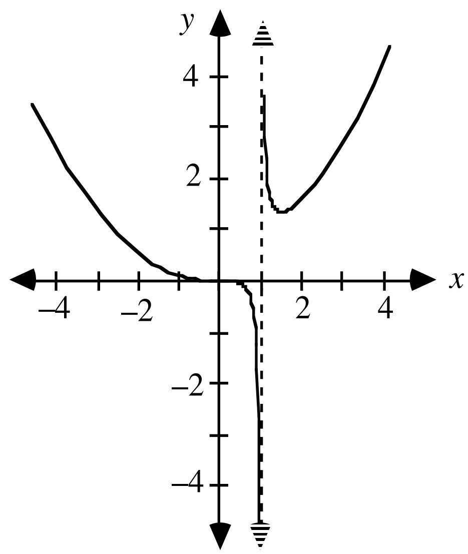
f. According to the model, sales should peak during week 8 (or the fourth week of June).

g. During this week, the company should sell about 20,500 bottles.

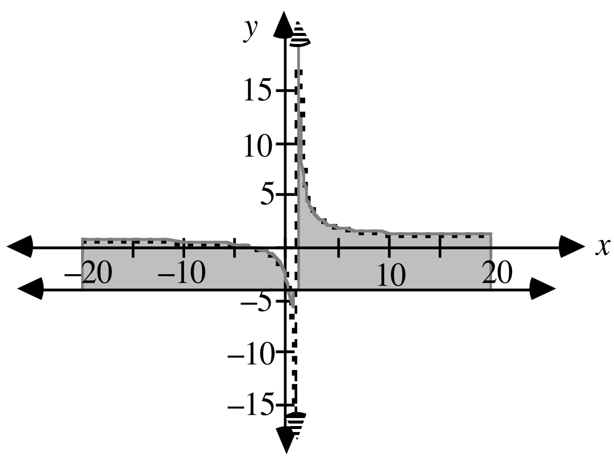
2. **a.** The domain is the set of real numbers except –0.5. A hole occurs in the graph when $x = -0.5$. Sample graph:



b. The domain is the set of real numbers except 1. A vertical asymptote occurs at $x = 1$. Sample graph:



3. The line is included in the solution set; the curve is not. The equation of the vertical asymptote is $x = 1$.



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Flashbacks

Activity 1

1.1 Consider the function $f(x) = 2x^2 + x - 11$.

a. Determine the domain of $f(x)$.

b. Find $f(5)$.

1.2 Solve the following equation for x: $x^2 - 3 = 0$.

1.3 Identify the degree of each of the polynomials on the left hand side of the equations below.

a. $(x+2)(x-3)(x+1) = 0$

b. $x^2 - 5x + 6 = 0$

c. $x^4 + x^3 - 7x^2 - x + 6 = 0$

1.4. Determine the roots of each polynomial in Flashback **1.3** and write it as the product of first-degree factors in the form $(x - c)$, if possible.

Activity 2

2.1 Solve the following equation for x:

$$3x^3 + 2x^2 - 19x + 6 = 0$$

2.2 Consider the function $f(x) = 3x^2 - 2x + 17$.

a. Determine the domain of $f(x)$.

b. Find $f(10,000)$.

2.3 For each of the following, describe what happens to the numerator, the denominator, and the values of $f(x)$ as $|x|$ becomes large.

a. $f(x) = 1/x$

b. $f(x) = \dfrac{x+1}{x+2}$

c. $f(x) = x/2$

Activity 3

3.1 Describe the discontinuity in the graph of each function below as a hole or as an asymptote.

a. $f(x) = \dfrac{2}{x-3}$

b. $f(x) = \dfrac{3x+3}{x+1}$

c. $f(x) = \dfrac{(x+2)(x+4)(x-3)}{(x+2)(x-3)}$

d. $f(x) = \dfrac{5(x-2)}{(x+3)(x-2)}$

3.2 Identify the vertical and horizontal asymptotes, if any, for each of the following functions.

a. $f(x) = \dfrac{5x+27}{x+5}$

b. $f(x) = \dfrac{6x-11}{x-2}$

c. $f(x) = \dfrac{-6x+11}{x-2}$

d. $f(x) = \dfrac{2x^2+2x-5}{x^2+x-6}$

Activity 4

4.1 **a.** Solve the following equation for G.

$$F = \frac{Gm_1m_2}{r^2}$$

b. Solve the following equation for l.

$$P = 2\pi\sqrt{l/g}$$

4.2 Identify the domain of each of the following functions:

a. $V = \dfrac{4}{3}\pi r^3$ (the formula for the volume of a sphere)

b. $y = 3x^2 + 11x + 6$

c. $y = \dfrac{2x^2 - 11x + 15}{x-3}$

Activity 5

5.1 Graph the following system of equations and find its point(s) of intersection.

$$\begin{cases} y = x + 12 \\ y = 3x + 9 \end{cases}$$

5.2 Graph the following system of inequalities and identify the solution set.

$$\begin{cases} y \geq 3x^2 + 12x - 9 \\ y < 2x - 3 \end{cases}$$

Answers to Flashbacks

Activity 1

1.1 **a.** The domain is the set of real numbers.

b. $f(5) = 44$

1.2 $x = \sqrt{3}$, $x = -\sqrt{3}$

1.3 **a.** degree 3

b. degree 2

c. degree 4

1.4 **a.** This polynomial is written as the product of first-degree factors. Its roots are –2, 3, and 1.

b. This polynomial can be written as $(x - 6)(x + 1)$. Its roots are 6 and –1.

c. This polynomial can be written as $(x + 1)(x - 1)(x + 3)(x - 2)$. Its roots are –1, 1, –3, and 2.

Activity 2

2.1 $x = 2$, $x = 1/3$, $x = -3$

2.2 **a.** The domain is the set of real numbers.

b. $f(10,000) = 3,000,199,810,006$.

2.3 **a.** As $|x|$ becomes large, the numerator is 1, the absolute value of the denominator increases, and $f(x)$ approaches 0.

b. As $|x|$ becomes large, the absolute values of the numerator and denominator increase and $f(x)$ approaches 1.

c. As x increases without bound, the denominator is 2, the numerator increases and $f(x)$ increases without bound. As x decreases without bound, the denominator is 2, the numerator decreases without bound, and $f(x)$ decreases without bound.

Activity 3

3.1

a. asymptote

b. hole

c. 2holes

d. asymptote and a hole

3.2

a. A horizontal asymptote occurs at $f(x) = 5$; a vertical asymptote occurs at $x = -5$.

b. A horizontal asymptote occurs at $f(x) = 6$; a vertical asymptote occurs at $x = 2$.

c. A horizontal asymptote occurs at $f(x) = -6$; a vertical asymptote occurs at $x = 2$.

d. A horizontal asymptote occurs at $f(x) = 2$; vertical asymptotes occur at $x = -3$ and $x = 2$.

Activity 4

4.1

a. $G = \dfrac{Fr^2}{m_1 m_2}$

b. $l = \dfrac{gP^2}{4\pi^2}$

4.2

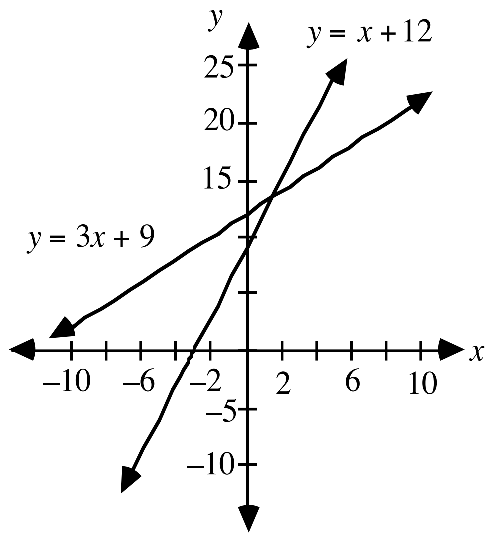
a. The domain is the set of positive real numbers.

b. The domain is the set of all real numbers.

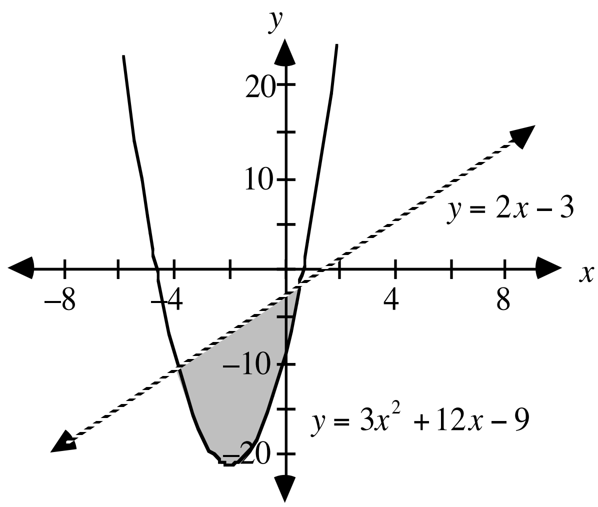
c. The domain is the set of positive real numbers except 3.

Activity 5

5.1 The point of intersection is (1.5,13.5).



5.2 The points of intersection are approximately (0.52,–2.0) and (–3.85,–10.7).



Believe It or Not

How can you convince yourself—and others—that a statement is true or false? In this module, you'll explore some useful tools for developing a reasoned argument.

Terri Dahl • Satinee Lightbourne • Steve A. Yockim

Teacher Edition

Believe It or Not

Overview

This module introduces students to formal reasoning in the context of stories, advertisements, and mathematical situations. The activities emphasize if-then statements, truth values, and proofs. Students begin with situations in which informal proof is sufficient and progress to cases requiring more formal proof.

Objectives

In this module, students will:

- identify the hypothesis and conclusion of a conditional statement
- identify the truth value of a conditional
- use Venn diagrams to represent conditionals
- negate conditionals
- find the converse, inverse, and contrapositive of a conditional
- identify logically equivalent conditionals
- write proofs using a chain of if-then statements
- explore proof by exhaustion
- find counterexamples
- use deductive reasoning
- write direct proofs
- develop indirect proofs.

Prerequisites

For this module, students should know:

- how to write an if-then statement
- how to use Venn diagrams to represent mathematical situations
- how to use the transitive property of equality.

Time Line

Activity	1	2	3	4	5	Summary Assessment	Total
Days	2	2	2	2	2	1	11

Materials Required

	Activity					
	1	**2**	**3**	**4**	**5**	**Summary Assessment**
template A	X					
template B					X	
scissors					X	

Teacher Note

The use of template A is optional. Blackline masters of both templates appear at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity					
	1	**2**	**3**	**4**	**5**	**Summary Assessment**
geometry utility				X		

Believe It or Not

Introduction

(page 221)

This module examines logic, conditionals, and proof. Students explore these ideas both mathematically and in everyday settings.

(page 221)

Activity 1

In this activity, students are introduced to conditional statements written in if-then form. They also use Venn diagrams to illustrate conditionals.

Materials List

- template A (optional; a blackline master appears at the end of the teacher edition for this module)

Exploration

(page 221)

In the following exploration, each student or group selects a verdict in a fictional murder case, then develops an argument to present to the class.

a. Students read "The Murder of Sam Barone" and select a possible verdict. **Note:** Template A shows each of the six possible verdicts: "Hank is guilty," "Hank is innocent," "Bill is guilty," "Bill is innocent," "Paula is guilty," and "Paula is innocent." You may wish to make a copy of this template, cut it into separate slips of paper, and ask students to draw slips randomly.

b. Students may organize the evidence in various ways, depending on the verdict they select.

c. Encourage students to look for flaws in their reasoning prior to the discussion. To make their arguments more clear, each student or group should try to reword at least a few sentences as if-then statements.

Discussion

(page 223)

a. As audience and jury, the class should try to identify any flaws in the logic of each argument.

b. Students who find an argument unbelievable should explain any identified flaws.

c. The exploration assumes that all statements made are true. In this setting, it is not possible for a statement to be both true and false. (This is part of the mathematical definition of a statement.)

d. Since there is only one guilty individual in the story, some students may have used invalid assumptions to provide supporting arguments for their verdicts.

e. Students who selected false verdicts may attempt to rephrase some statements from the story. Depending on how a statement is rephrased, its meaning may or may not change.

f.
1. Sample response: The hypothesis describes the conditions in the situation.
2. Sample response: The conclusion describes the possible results when the conditions from the hypothesis are met.

g.
1. If an animal is a dog, then it is a German shepherd.
2. Sample response: The statement is not true because some dogs are not German shepherds. **Note:** Some students may argue that the statement can be true or false, depending on the dog selected. You may use this opportunity to point out the need for a consistent method to determine the truth value of a conditional. Students use truth tables to determine the truth value of a conditional later in this module. They will find that the only time a conditional is false is when the hypothesis is true and the conclusion is false. Otherwise, the statement is true.

Assignment

(page 224)

1.1

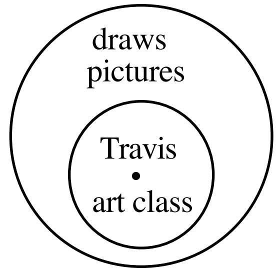
a. The animal is a stegosaurus and a dinosaur.

b. The animal is a dinosaur but is not a stegosaurus.

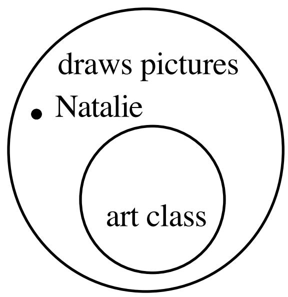
c. The animal is neither a stegosaurus nor a dinosaur.

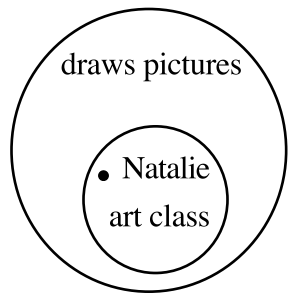
d. Answers may vary. Sample response: If an animal is a stegosaurus, then the animal is a dinosaur.

1.2 **a.** Travis' name should be placed in the "art class" circle because that is the most restrictive assumption about Travis. Sample diagram:

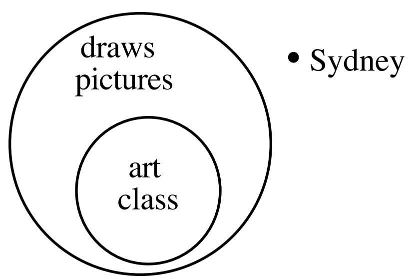


b. Natalie's name can be placed in either of the two locations shown below, since both represent people who draw pictures. There is not enough information to determine if she is or is not in the art class. Sample diagrams:

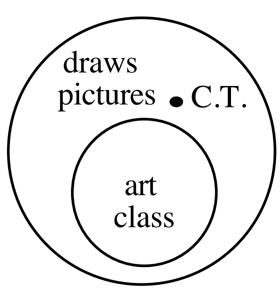


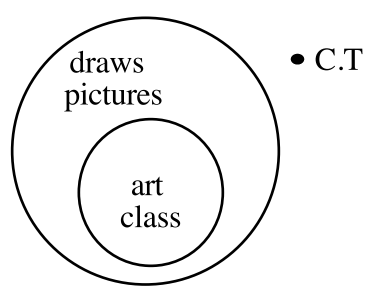


c. Since Sydney does not draw pictures, the only possible location for Sydney's name is outside the larger circle. Sample diagram:



d. C.T.'s name should be placed outside the smaller circle, either inside the "draws pictures" circle but outside the "art class" circle or outside the large circle. There is no information to indicate whether or not C.T. draws pictures. Sample diagrams:





1.3 **a.** If an animal is a whale, then (it is a mammal).

b. If a figure is a triangle, then (it is a polygon).

c. If Hank is guilty, then (he had to be at the scene of the crime).

d. If Bill was snowbound at Aspen, then (he was not in Hillside).

***1.4** Answers will vary. Sample response:

a. If a person owns the sword, then (that person knows a great deal about Egyptian history).

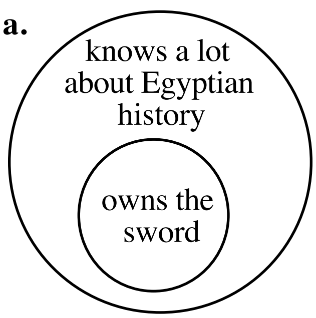
b. If the sword was purchased on January 4, 1997, then (the owner was in my shop in Hillside on that day).

c. If Hank was in Paris on January 4, 1997, then (it would have taken him over 24 hours to reach Hillside).

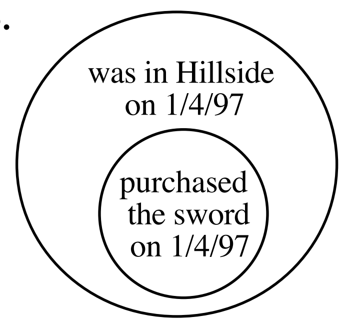
d. If Bill was in Aspen on January 2, 1997, then (he was there for the big snowstorm).

***1.5** The following diagrams correspond with the sample responses given in Problem **1.4** above.

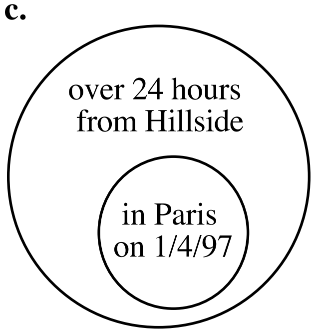
a.



b.



c.

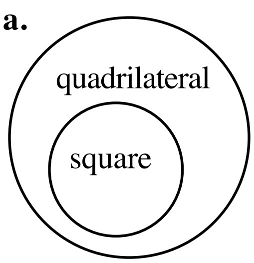


d.

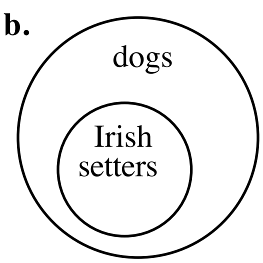
* * * * *

1.6 Sample diagrams:

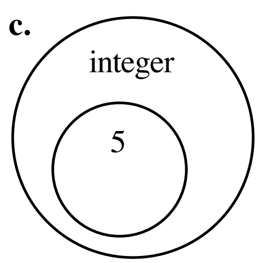
a.



b.



c.



1.7 **a.** If a triangle is isosceles, then it has two congruent sides.

b. If $x = 3$, then $x^2 = 9$.

c. If a person lives in Hawaii, then that person lives in the United States.

* * * * * * * * * *

(page 225)

Activity 2

Students use truth tables to determine the truth value of a conditional. They also explore the effects of negation on truth values.

Materials List

- none

Teacher Note

The following exploration is designed to give students an intuitive feel for the truth values of a conditional. The truth values for any conditional are illustrated in the table below.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Exploration

(page 226)

Students examine the truth value of a conditional by treating it like a promise. If the promise is kept, or there is no valid reason to complain, then the conditional is true.

a. "If you finish this exploration, then I promise that you have permission to go to the dance."

b. Students should complete the table as shown below.

	You finished the exploration	**You have permission to go to the dance**	**Is the promise kept? (truth value)**
1.	yes	yes	yes
2.	yes	no	no
3.	no	yes	yes
4.	no	no	yes

c. The promise is broken only when the student finishes the exploration and does not have permission to go to the dance.

d. **Table 2: Truth table for $p \rightarrow q$**

	Hypothesis p	Conclusion q	Conditional $p \rightarrow q$
1.	true	true	true
2.	true	false	false
3.	false	true	true
4.	false	false	true

e. Sample response: The conditional is false only when the hypothesis is true and the conclusion is false.

Discussion (page 226)

a. Sample response: If you did not finish the exploration and you have permission to go to the dance, then you have no valid reason to complain about the result. Thus, the conditional is considered true.

b. From Table **1**, a conditional is true when the promise is kept. From Table **2**, a conditional is true when both the hypothesis and the conclusion are true, or when the hypothesis is false.

c. A conditional is false only when the hypothesis is true and the conclusion is false. In all other cases, the conditional is true.

d.
1. You do not finish the exploration.
2. You do not have permission to go to the dance.

e. The double negation of a statement p has the same truth value as the original statement p.

f. Sample response: Since the statement implies that it is cloudy whenever it is rainy, all rainy days would be expected to be cloudy.

g. Sample response: Since the statement implies that all squares are rectangles, the implied quantifier could be *all* or *every*.

Assignment (page 227)

2.1 **a.** In this statement, a universal quantifier such as *all* is implied. Sample response: All mammals are animals.

b. In this statement, an existential quantifier such as *one* is implied. Sample response: There exists one person with red hair or at least one person has red hair.

2.2 **a.** Answers will vary. Sample response: If $3 = 5$, then $x + 5 = x + 7$.

b. This is not possible. If the hypothesis of a conditional is false, then the conditional is true.

2.3 **a.** Answers will vary. Sample response: If a triangle is equilateral, then the triangle is isosceles.

b. Answers will vary. Sample response: If a triangle is equilateral, then the triangle is scalene.

***2.4** There is only one case in which negating both the hypothesis and the conclusion makes a true conditional false: when the hypothesis of the original conditional is false and the conclusion of the original conditional is true. In the table below, this case is indicated with an asterisk (*).

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim p \to \sim q$
true	true	true	false	false	true
true	false	false	false	true	true
*false	true	true	true	false	false
false	false	true	true	true	true

2.5 Sample response: Always. In a false conditional, the hypothesis is true and the conclusion is false. Negating the hypothesis creates a conditional with a false hypothesis. Any conditional with a false hypothesis is true.

* * * * *

2.6 **a.** The quantifier is *every*. In this statement, it refers to all rhombuses.

b. The implied quantifier is *at least one*. There may be more solutions.

2.7 **a.** **1.** $\sim p$

2. $\sim q$

3. $p \to q$

4. $\sim q \to \sim p$

5. $q \to p$

b. Answers will vary, depending on which conditional students select. the following sample response shows the truth table for the conditional $\sim q \to \sim p$.

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$
true	true	false	false	true
true	false	false	true	false
false	true	true	false	true
false	false	true	true	true

* * * * * * * * * *

(page 229)

Activity 3

This activity uses an excerpt from Lewis Carroll's *Alice in Wonderland* to introduce the converse, inverse, and contrapositive. **Note:** Lewis Carroll was the pen name of Oxford mathematician Charles L. Dodgson (1832–1898).

Materials List

- none

Exploration

(page 229)

Students write the converse, inverse, and contrapositive of a conditional statement, along with the corresponding truth tables.

a. If it is raining, then it is cloudy.

b. Sample truth table:

Raining (p)	**Cloudy (q)**	**Statement ($p \rightarrow q$)**
true	true	true
true	false	false
false	true	true
false	false	true

c. **1.** If it is cloudy, then it is raining.

2. Sample truth table:

Cloudy (q)	**Raining (p)**	**Statement ($q \rightarrow p$)**
true	true	true
false	true	true
true	false	false
false	false	true

d. **1.** If it is not raining, then it is not cloudy.

2. Sample truth table:

Not Raining ($\sim p$)	**Not Cloudy ($\sim q$)**	**Statement ($\sim p \rightarrow \sim q$)**
false	false	true
false	true	true
true	false	false
true	true	true

e. **1.** If it is not cloudy, then it is not raining.

2. Sample truth table:

Not Cloudy ($\sim q$)	**Not Raining** ($\sim p$)	**Statement** ($\sim q \rightarrow \sim p$)
false	false	true
true	false	false
false	true	true
true	true	true

f. Students should observe that the original statement and the contrapositive have the same truth values, and that the inverse and the converse have the same truth values.

Discussion

(page 230)

a. Sample response: The truth values for the original statement and its contrapositive are the same. The truth values for the converse and inverse are not the same as the original statement, but are the same as each other.

b. Sample response: The contrapositive of the inverse is the converse. The contrapositive of a statement is formed when the hypothesis and conclusion are interchanged and negated. When this is done to the inverse of a statement, the result is the converse of the statement.

c. The original statement and its contrapositive are logically equivalent since their truth table values are identical. The converse and inverse of the original statement are logically equivalent since their truth table values are identical.

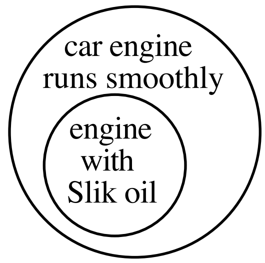
d. A conditional and its contrapositive are always logically equivalent, as are its converse and inverse. Since a definition and its converse are logically equivalent, a definition is also logically equivalent to its inverse. Therefore, a conditional statement formed from a definition, along with its converse, inverse, and contrapositive, are all logically equivalent.

Assignment (page 231)

3.1 The following sample response is based on the slogan, "Slik oil makes your car engine run smoothly."

a. If you use Slik oil in your car engine, then (your engine will run smoothly).

b. Sample Venn diagram:



c. If your car engine runs smoothly, then you use Slik oil.

d. If you do not use Slik oil in your car engine, then your engine does not run smoothly.

e. If your car engine does not run smoothly, then you do not use Slik oil.

f. Student preferences may vary, but their responses should reflect the fact that both forms convey the same meaning. In the sample response given above, the conditional is probably the better slogan.

3.2 **a.** If a triangle is equilateral, then it is isosceles.

b. The truth tables for this conditional are the same as those given in the previous exploration.

c. **1.** The conditional and the converse are not logically equivalent.

2. The conditional and the contrapositive are logically equivalent.

3. The converse and the contrapositive are not logically equivalent.

4. The converse and the inverse are logically equivalent.

5. The conditional and the inverse are not logically equivalent.

6. The inverse and the contrapositive are not logically equivalent.

d. Sample response: The conditional and its contrapositive are logically equivalent. The converse and inverse of the same conditional are also logically equivalent.

***3.3** **a.** If you live in New York, then you live in the United States.

b. If a figure is not a polygon, then it is not a triangle.

3.4 **a.** A number is a googol if and only if it consists of the numeral 1 followed by one hundred 0s.

b. A triangle is regular if and only if it is equilateral and equiangular.

* * * * *

3.5 **a.** **1.** If a number is π, then it is irrational. (true)

2. If a number is irrational, then it is π. (false)

3. If a number is not π, then it is rational. (false)

4. If a number is rational, then it is not π. (true)

b. The conditional and its contrapositive have the same truth value. The converse and the inverse have the same truth value.

3.6 **a.** It is not possible to write this statement as a definition because the converse is not true.

b. An animal is a canine if and only if it is a dog.

c. $a = b$ if and only if $b = a$

* * * * * * * * * *

(page 232)

Activity 4

Students investigate proof by exhaustion and proof by counterexample.

Materials List

- none

Technology

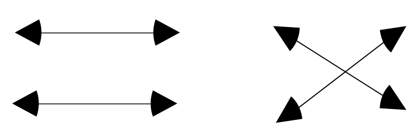
- geometry utility

Exploration

(page 233)

Students explore the relationships among angles formed by intersecting lines using four steps: drawing a picture, making a hypothesis, supporting the hypothesis by investigating many examples, and writing a generalization.

a. Two distinct lines may be parallel or may intersect in a single point. Sample diagram:



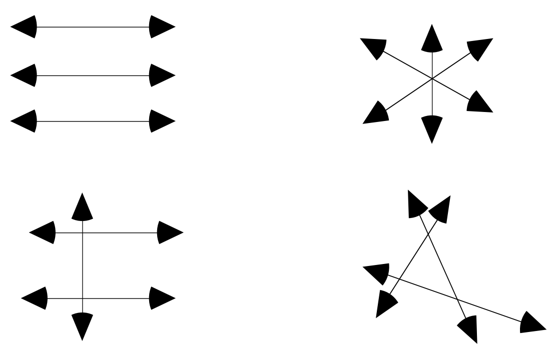
b. Some students may conjecture that the measures of vertical angles formed by two intersecting lines are equal. Others may predict that adjacent angles formed by two intersecting lines are supplementary.

c. Students use appropriate technology to test their conjectures. Sample response: When we measured the vertical angles formed by the two intersecting lines, they had equal measures. The result was the same when we changed the angles.

d. Sample responses: If two lines intersect, then the measures of the vertical angles are equal.

If two lines intersect, then adjacent angles are supplementary.

e. Three distinct lines in a plane may be parallel, may intersect in a single point, may form two parallel lines intersected by a transversal, or may intersect to form a triangle. Sample diagram:



Sample conjectures, in if-then form:

- If two parallel lines are cut by a transversal, then the measures of the pairs of corresponding angles, alternate interior angles, and alternate exterior angles, respectively, are equal.
- If two parallel lines are cut by a transversal, then the measures of the pairs of same-side interior angles (and same-side exterior angles) are supplementary.
- If three lines intersect in a single point, then each pair of vertical angles formed is congruent.

Discussion (page 233)

a. You may wish to facilitate this discussion by compiling a list of student conjectures on the board.

b. Sample response: Sketches allow you to visualize the relationships among the angles formed by intersecting lines. Drawing the diagrams on technology helps you test a conjecture by observing angle measurements while moving the lines.

c. Sample response: They are helpful, but not completely convincing. A diagram may not show all possible cases.

d. Sample response: No, using technology does not prove conjectures. It makes observing a number of different cases easier, but it still doesn't allow you to test all possible pairs of intersecting lines.

Assignment

(page 234)

4.1 **a.** Students may or may not be convinced that the remaining marbles in the box are green.

b. Sample conjecture: All the marbles in the box are green.

c. Sample response: This could be proven by examining all the marbles in the box. If they are all green, the conjecture is true.

4.2 There are an infinite number of possibilities to examine. Therefore, the relationships among vertical angles in the exploration cannot be proved by exhaustion.

***4.3** **a.** Depending on student assumptions, many arguments may be valid. Sample response: Assuming it takes 1 sec to count each number, it would take 11.6 days to count to 1,000,000. Since this is more than 1 week, the statement is true.

b. Sample response: Yes. A computer could be programmed to count from 1 to 1,000,000 at the same rate that I could count. The starting and ending time could be recorded, verifying the statement by exhaustion.

c. Sample response: No. It would take too much time. It can be proven much more quickly by using a series of logical arguments as in Part **a**.

4.4 To prove by exhaustion that the additive identity on a clock is 12, students must examine all 12 values of n. Sample proof:

$$12 + 1 = 1 + 12 = 1$$
$$12 + 2 = 2 + 12 = 2$$
$$12 + 3 = 3 + 12 = 3$$
$$\vdots$$
$$12 + 12 = 12 + 12 = 12$$

Thus, the additive identity for clock addition is 12.

4.5 Sample response: Drawing one marble that is not green would disprove the statement.

***4.6** **a.** Sample response: Your method is not correct, because there is an infinite number of quadrilaterals with four right angles and you only examined 10.

b. A rectangle is a quadrilateral with four right angles, but it is not a square.

c. If a quadrilateral has four right angles, then it is a rectangle.

d. Sample response: No. Examining a limited number of cases is not a proof. However, such observations may suggest how a proof might be pursued.

4.7 **a.** The shopkeeper said, "I could never tell those two brothers apart, but I know this: if it ain't Bill that owns the sword, then it's Hank. And if it ain't Hank then it's Bill. I know I sold it to one of them."

b. Bill said, "Sure I hated Sam, but it doesn't mean I killed him."

c. Bill said, "I was in Aspen, Colorado, skiing on January 2, 1997. Because of a big snowstorm there, nobody in Aspen could leave town for at least three days before or after January 2."

* * * * *

4.8 Sample response: There are 24 possible arrangements of the four books, where R represents red, Y represents yellow, G represents green, and B represents blue: RYGB, RYBG, RGYB, RGBY, RBYG, RBGY, YGBR, YGRB, YBGR, YBRG, YRGB, YRBG, GBRY, GBYR, GRBY, GRYB, GYBR, GYRB, BRYG, BRGY, BYRG, BYGR, BGRY, BGYR. By inspection, the green and yellow books are adjacent in 12 of the 24 arrangements, or 50%.

4.9 **a.** Sample response: $3 + 2 = 5$, but 5 is not even.

b. Sample response: Two adjacent right angles have the same measure, but the angles are not vertical.

c. Sample response: A sprinkler is spraying water on the sidewalk, but it isn't raining.

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(page 235)

Activity 5

This activity introduces two methods of proof that use deductive reasoning: direct proof and indirect proof. Students explore direct proof using a series of if-then statements. They examine both methods using proofs in paragraph form.

Materials List

- template B (one per student or group; a blackline master appears at the end of the teacher edition for this module)
- scissors (one pair per student or group)

Exploration (page 236)

a–b. Students arrange the six sentences on template B to create a logical sequence of if-then statements which show that Bill is innocent.

1. If Bill was in Aspen on January 2, 1997, then he was snowbound.
2. If Bill was snowbound, then he couldn't leave Aspen until January 5.
3. If Bill couldn't leave Aspen until January 5, then he couldn't be in Hillside on January 4.
4. If Bill couldn't be in Hillside on January 4, then he couldn't buy the sword.
5. If Bill didn't buy the sword, then he didn't own the murder weapon.
6. If Bill didn't own the murder weapon, then he didn't commit the murder.

Discussion (page 237)

a. Answers will vary. Sample response: We arranged the sentences so that the conclusion of one led directly to the hypothesis of the next.

b. Transitivity provides a way to present a series of valid arguments, each of which proceeds logically from the preceding one.

c. Sample response:

- If $2x - 3 = 5x + 6$, then $2x - 5x = 6 + 3$.
- If $2x - 5x = 6 + 3$, then $-3x = 9$.
- If $-3x = 9$, then $x = -3$.

Therefore, it can be concluded that if $2x - 3 = 5x + 6$, then $x = -3$.

d. To prove the conditional directly, the hypothesis, "n^2 is an odd integer," would have to be used to produce the conclusion "*n* is an odd integer." If n^2 is odd, then it must equal $2a + 1$, where *a* is some integer. Taking the square root of both sides of this equality, $n = \sqrt{2a + 1}$. It is not possible to determine whether $\sqrt{2a + 1}$ is even or odd.

e. The first step would be to assume that the given statement is false. If this conditional is false, that implies that "*n* is an even integer" is true, while "n^2 is an even integer" is false. If "n^2 is an even integer" is false, then n^2 is an odd integer.

Assignment (page 238)

5.1 **a.** Sample response: If Bill was in Aspen on January 2, 1997, then Bill was snowbound until January 5, 1997.

If Bill was snowbound in Aspen until January 5, 1997, then he was not in Hillside on January 4, 1997.

If Bill was not in Hillside on January 4, 1997, then Bill did not buy the sword from the shopkeeper.

If Bill did not buy sword from the shopkeeper, then Hank bought the sword from the shopkeeper.

If Hank bought the sword from the shopkeeper, then Hank owned the sword.

If Hank owned the sword, then Hank owned the murder weapon.

b. Sample response: If Hank owned the murder weapon, then Hank murdered Sam.

If Hank murdered Sam, then Hank was guilty.

***5.2** **a.** If $5(x-2) = -20$, then $5x - 10 = -20$ (by the distributive property).

If $5x - 10 = -20$, then $5x = -10$ (by the addition property of equality).

If $5x = -10$, then $x = -2$ (by the multiplicative property of equality).

Thus, the solution to $5(x-2) = -20$ is $x = -2$.

b. If 2π radians = 360°, then π radians = 180° (by the multiplicative property of equality).

If π radians = 180°, then $\pi/3$ radians = $180°/3$ (by the multiplicative property of equality).

If $\pi/3$ radians = $180°/3$, then $\pi/3$ radians = 60° (by simplification).

5.3 The contrapositive is "If $x = 2$, then $x^2 = 4$." Sample proof: If $x = 2$, then $x^2 = 2^2$ (by squaring both sides of the equation).

If $x^2 = 2^2$, then $x^2 = 4$ (by simplification).

5.4 Sample response: The Pythagorean theorem states that if a triangle is a right triangle, then the sum of the squares of the lengths of the legs equals the square of the hypotenuse. In the diagram, the area of the larger square is $(a+b)^2$, by the definition of the area of a square. The area of the smaller square is c^2 (also by the definition of the area of a square) and the area of each triangle is $\frac{1}{2}ab$, by the definition of the area of a triangle.

Because the area of a region can be expressed as the sum of the areas of its non-overlapping parts, the area of the larger square is also $\frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$. Thus, $(a+b)^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$. By combining like terms, $(a+b)^2 = 2ab + c^2$. By multiplication of polynomials, $a^2 + 2ab + b^2 = 2ab + c^2$. By subtracting $2ab$ from both sides of the equation, $a^2 + b^2 = c^2$. Since *a* and *b* are lengths of the legs of the right triangle and *c* is the hypotenuse, the sum of the squares of the lengths of the legs of a right triangle equals the square of the hypotenuse.

***5.5** **a.** Assume that the following statement is false: "If lines *l*, *m*, and *n* intersect in three different points, then it is not possible for both *l* and *m* to be perpendicular to *n*." Hence, "lines *l*, *m*, and *n* intersect in three different points" is true and "it is not possible for both *l* and *m* to be perpendicular to *n*" is false. This means that both *l* and *m* are perpendicular to *n*. If both *l* and *m* are perpendicular to *n*, then *l* and *m* are parallel. If *l* and *m* are parallel, then they do not intersect. This contradicts the given hypothesis (lines *l*, *m*, and *n* intersect in three different points). Therefore, the assumption is false and the original statement is true.

b. Assume that the following statement is false: "A triangle cannot have three interior angles each with a measure of 70˚." Hence, each of the three interior angles of a triangle has a measure of 70˚. The sum of the measures of the angles of the triangle would then be 210˚. This contradicts the theorem that the sum of the measures of the interior angles of any triangle is 180˚. Therefore, the assumption is false and the original statement is true.

5.6 Assume that the following statement is false: "If Bill did not own the murder weapon, then he was innocent." This implies that "Bill did not own the murder weapon" is true and "Bill was innocent" is false. If Bill was not innocent, then he committed the crime. If Bill committed the crime, then he owned the murder weapon. This contradicts the fact that Bill did not own the murder weapon, so the assumption is false. If the assumption is false, then its opposite must be true. Therefore, if Bill did not own the murder weapon, then he was innocent.

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5.7 **a.** If Sam owed Hank money, then Hank hated Sam.

If Hank hated Sam, then Hank had a motive.

b. If a triangle is equilateral, then all of its sides are congruent (by the definition of an equilateral triangle).

If all of a triangle's sides are congruent, then it has two sides that are congruent.

If a triangle has two sides congruent, then it is isosceles (by the definition of an isosceles triangle).

5.8 If $\angle 1$ and $\angle 2$ are supplementary angles and $\angle 1$ and $\angle 3$ are supplementary angles, then using the definition of supplementary angles, $m\angle 1 + m\angle 2 = 180°$ and $m\angle 1 + m\angle 3 = 180°$. By the transitive property of equality, $m\angle 1 + m\angle 2 = m\angle 1 + m\angle 3$. Applying the addition property of equality and adding $-m\angle 1$ to both sides of the equation results in $m\angle 2 = m\angle 3$. By the definition of congruent angles, $\angle 2 \cong \angle 3$. Therefore the supplements of angles with equal measure are congruent.

5.9 Assume that the following statement is false: "If x and y are integers and $3x + 12y = 450$, then x is an even integer." Thus "x and y are integers and $3x + 12y = 450$" is true and "x is an even integer" is false. If x is not an even integer, then x must be an odd integer (since x is an integer). Because the product of two odd integers is an odd integer, it follows that $3x$ is an odd integer. No matter what y is, $12y$ is an even integer. Since the sum of an odd integer and an even integer is an odd integer, $3x + 12y$ is an odd integer. This contradicts the fact that the expression $3x + 12y$ must equal 450, which is an even integer. Therefore, the assumption is false and the original statement is true.

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Answers to Summary Assessment (page 240)

1. Sample response: There are no correct digits in the fourth guess. Thus, 3, 4, 5, and 6 are not used in the number. In the first two guesses, the correct digits are 1 and 2, since two digits are correct in both, with one in the correct position each time. For these guesses to result in one hot clue each, 1 must be the first digit and 2 the third digit. From the last guess, 9 must be either the second digit or the fourth digit. Based on the fifth guess, 9 must be the fourth digit. By comparing the sixth and seventh guesses, 7 must be the second digit. Therefore, the number is 1729.

2.
a. The statement is true. If $3 | m$, then there is an integer c so that $m = 3c$. Multiplying both sides of the equation by n results in $mn = (3c)n$. Rewriting the equation yields $nm = 3(nc)$. Since nc is an integer, $3 | nm$.

b. The statement is false. For example, $3 | (5 + 4)$ is true, but $3 | 5$ is false.

c. The statement is true for all integers m because $m = m \bullet 1$.

d. The statement is false. For example, $9 | 6^2$ is true, but $9 | 6$ is false.

3. Sample response: Assume ΔABC does have two angles each with measure 90˚. For example, let $m\angle A = 90^\circ$ and $m\angle B = 90^\circ$. All triangles have three angles whose sum is 180˚; therefore, $m\angle A + m\angle B + m\angle C = 180^\circ$. Substituting yields $90^\circ + 90^\circ + m\angle C = 180^\circ$. Solving this equation results in $m\angle C = 0^\circ$. If this is true, then ΔABC would no longer exist. This contradiction means the assumption is false. Therefore, the original statement is true.

4. Sample response: Assume the statement is false; hence, $2(3x - 5) = 6x + 3$ is true. Solving this equation results in $-10 = 3$. Because $-10 = 3$ is false, the assumption is false. Therefore, the original statement is true.

Module Assessment

1. Four athletes are seated around a square table. One is a swimmer, one a distance runner, one a gymnast, and one a tennis player. Two of the athletes are women, Bonnie and Kyara, and two are men, Luke and Qinfang. The swimmer is sitting on Bonnie's left. The gymnast is sitting across from Luke. Kyara and Qinfang are sitting next to each other. A woman is sitting on the distance runner's left.

Who is the tennis player? Describe the reasoning you used to determine your response.

2. Use the following statement to complete the questions below: "When the power goes out, my computer quits."

a. Write the statement in if-then form. Underline the hypothesis and place parentheses around the conclusion.

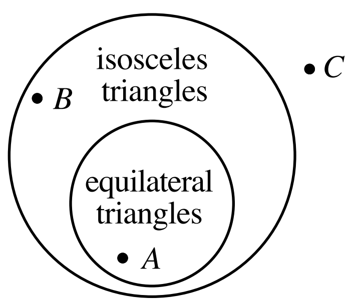
b. Write the converse of the statement in Part **a**.

c. Write the inverse of the statement in Part **a**.

d. Write the contrapositive of the statement in Part **a**.

e. Which of the statements in Parts **b**, **c**, and **d** are logically equivalent to the statement in Part **a**?

3. In the diagram below points A, B, and C represent triangles.



Use this diagram to determine which points satisfy each of the following conditions. Explain your reasoning for each response.

a. Two sides of the triangle are congruent.

b. All sides of the triangle are congruent.

c. None of the sides of the triangle are congruent.

d. The triangle does not have three congruent sides.

e. The triangle does not have two congruent sides.

4. Use a counterexample to prove that each of the following statements is false.

a. For all integers a, b, and c, $a^2 + b^2 = c^2$.

b. For any numbers a and b, $(a + b)^2 = a^2 + b^2$.

c. A statement and its converse are logically equivalent.

d. If Madeline is in the city of Troy, then she is in the state of New York.

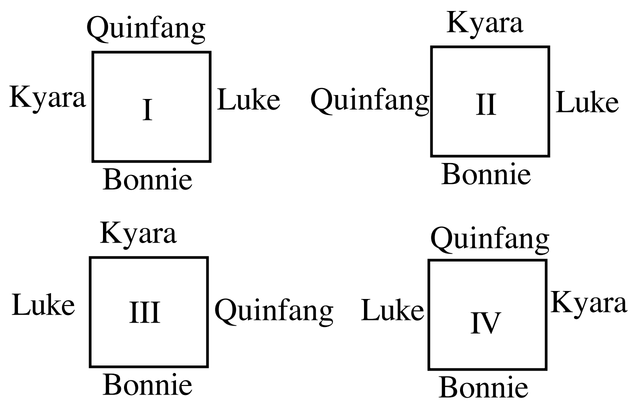
5. Prove the following statement indirectly: "If $n^2 + 6 = 32$, then $n \neq 5$."

6. Prove the following statement: "Complements of congruent angles are congruent."

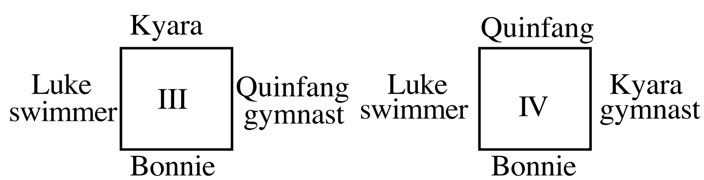
7. Prove that a collection of dimes and quarters worth 95 cents must contain an odd number of quarters.

Answers to Module Assessment

1. Bonnie is the tennis player. Sample response: Since Kyara and Qinfang are sitting next to each other, there are four possible seating arrangements:



Since the swimmer is sitting on Bonnie's left, and the gymnast is sitting across from Luke, arrangements I and II are eliminated.



Since a woman is sitting on the distance runner's left, arrangement III is eliminated, and Qinfang must be the distance runner. Therefore, Bonnie is the tennis player.

2.
- **a.** If <u>the power goes out</u>, then (my computer quits).
- **b.** If my computer quits, then the power goes out.
- **c.** If the power does not go out, then my computer does not quit.
- **d.** If my computer does not quit, then the power does not go out.
- **e.** The statement in Part **d**, the contrapositive, is logically equivalent to the conditional.

3.
- **a.** *A* and *B*
- **b.** *A*
- **c.** *C*
- **d.** *B* and *C*
- **e.** *C*

4.

a. Sample response: $1^2 + 2^2 \neq 3^2$.

b. Sample response: $(3+4)^2 \neq 3^2 + 4^2$.

c. Sample response: If it is raining, then it is cloudy. If it is cloudy, then it is raining.

d. Sample response: Madeline is in Troy, Michigan. (There are also towns named Troy in Ohio, Montana, and other states. The ancient city of Troy was located in northwest Asia Minor.)

5. Sample response: Assume that the original statement is false. In this case, $n^2 + 6 = 32$ is true and $n \neq 5$ is false. Thus, $n = 5$ is true. If $n = 5$, then $n^2 = 25$. If $n^2 = 25$, then $n^2 + 6 = 31$. This contradicts the hypothesis, $n^2 + 6 = 32$. Therefore, the assumption is false and the original statement is true.

6. Sample response: If $\angle 1$ and $\angle 2$ are complementary angles and $\angle 1$ and $\angle 3$ are complementary angles, then by using the definition of complementary angles, $m\angle 1 + m\angle 2 = 90°$ and $m\angle 1 + m\angle 3 = 90°$. By transitivity, $m\angle 1 + m\angle 2 = m\angle 1 + m\angle 3$. Applying the addition property of equality and adding $-m\angle 1$ to both sides of the equation results in $m\angle 2 = m\angle 3$. By the definition of congruent angles, $\angle 2 \cong \angle 3$. Therefore, complements of congruent angles are congruent.

7. Sample response: Assume that a collection of quarters and dimes worth 95 cents contains an even number of quarters. Then it must have 2 quarters, since that is the only even number of quarters less than 95 cents. If it contains 2 quarters, then it must have 45 cents in dimes, which is a contradiction of the value of a dime. Therefore, the assumption must be false. If the assumption is false, then the original statement must be true. Therefore, a collection of quarters and dimes worth 95 cents must have an odd number of quarters.

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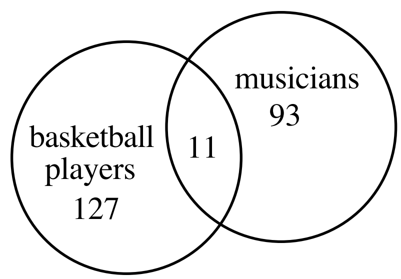
Flashbacks

Activity 1

1.1 The members of the senior class at Washington High School are taking their college entrance exams. Of the 417 students in the class, 217 are registered for the ACT and 182 are registered for the SAT. There are 93 students registered for both exams.

Draw a Venn diagram to illustrate this situation.

1.2 The following Venn diagram represents the number of Washington High students who participate in basketball and music. Describe the information shown in this diagram.



Activity 2

2.1 Determine whether each of the following statements is true or false. In each case, describe how you determined your response.

- **a.** Squares are polygons that have four sides.
- **b.** Through every two points in a plane, there is more than one line.
- **c.** All rectangles are squares.
- **d.** Saint Bernards are dogs.

2.2 Revise each false statement in Problem **2.1** to make it a true statement.

2.3 Write the following statement in if-then form: "All rainbow trout are fish."

Activity 3

3.1 Write each of the following statements in if-then form. Underline the hypothesis and place parentheses around the conclusion.

a. Every segment has two endpoints.

b. All my students love math.

3.2 Determine whether each of the following statements is true or false. In each case, describe how you determined your response.

a. If a person lives in New York City, then that person lives in the state of New York.

b. If a person lives in the state of New York, then that person lives in New York City.

c. If a person is in the state of Michigan, then that person is a resident of Michigan.

Activity 4

4.1 Predict the next two numbers in each of the following sequences.

a. 1, 2, 4, 8, 16, ...

b. 22, 33, 43, 52, 60, …

c. 1, 4, 9, 16, …

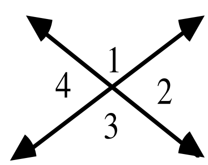
4.2 Sketch a diagram to represent each of the following:

a. a perpendicular bisector of a line segment

b. a set of parallel lines intersected by a transversal

c. a pair of vertical angles.

4.3. Describe all the pairs of angles in the diagram below.



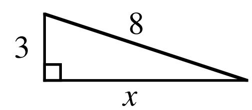
Activity 5

5.1 If $x = a + b$ and $y = a + b$, what can you conclude about x and y?

5.2 Write the negation of the statement, "Luella has solved the riddle."

5.3 Write the contrapositive of the statement, "If today is Thursday, then tomorrow is Friday."

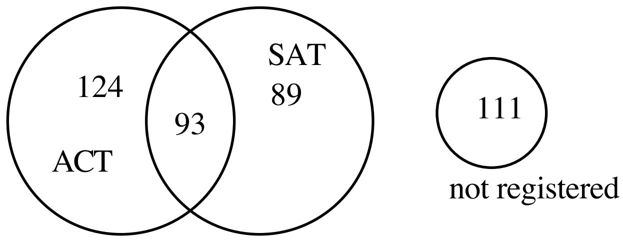
5.4 Solve for x in the following diagram:



Answers to Flashbacks

Activity 1

1.1 Sample Venn diagram:



1.2 Sample response: There are 138 students who participate in basketball and 104 who participate in music. A total of 11 students participate in both basketball and music. There are 127 students who participate in basketball but not music, and 93 students who participate in music but not basketball. There is a total of 231 students who participate in music or basketball.

Activity 2

2.1 **a.** This statement is true. A square is defined as a polygon with four congruent sides and four right angles.

b. This statement is false. Through every two points in a plane, there is exactly one line.

c. This statement is false. Only some rectangles are squares.

d. This statement is true. Saint Bernards are dogs.

2.2 **b.** Through every two points in a plane, there is exactly one line.

c. All squares are rectangles.

2.3 If an animal is a rainbow trout, then it is a fish.

Activity 3

3.1 **a.** If <u>a figure is a segment</u>, then (it has two endpoints).

b. If <u>a person is a student of mine</u>, then (that student loves math).

3.2 **a.** This statement is true.

b. This statement is false. The person may live in another city in the state of New York, such as Albany or Buffalo.

c. This statement is false. The person may be visiting Michigan.

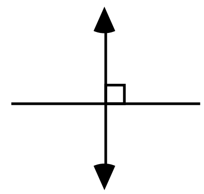
Activity 4

4.1 **a.** Sample response: 32, 64

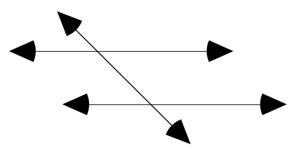
b. Sample response: 67, 73

c. Sample response: 25, 36

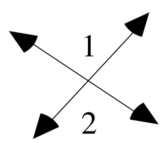
4.2 **a.** Sample diagram:



b. Sample diagram:



c. In the sample diagram below, angles 1 and 2 are a pair of vertical angles:



4.3 The pairs of vertical angles are 1 and 3, and 2 and 4. The pairs of adjacent and supplementary angles are 1 and 2; 2 and 3; 3 and 4; and 4 and 1.

Activity 5

5.1 By the transitive property, $x = y$.

5.2 Sample response: Luella has not solved the riddle.

5.3 Sample response: If tomorrow is not Friday, then today is not Thursday.

5.4 Using the Pythagorean theorem, $x = \sqrt{55} \approx 7.42$.

Template A

Hank is guilty.	**Hank is guilty.**	**Hank is guilty.**
Bill is guilty.	**Bill is guilty.**	**Bill is guilty.**
Paula is guilty.	**Paula is guilty.**	**Paula is guilty.**
Hank is innocent.	**Hank is innocent.**	**Hank is innocent.**
Bill is innocent.	**Bill is innocent.**	**Bill is innocent.**
Paula is innocent.	**Paula is innocent.**	**Paula is innocent.**
Hank is guilty.	**Hank is guilty.**	**Hank is guilty.**
Bill is guilty.	**Bill is guilty.**	**Bill is guilty.**
Paula is guilty.	**Paula is guilty.**	**Paula is guilty.**
Hank is innocent.	**Hank is innocent.**	**Hank is innocent.**
Bill is innocent.	**Bill is innocent.**	**Bill is innocent.**
Paula is innocent.	**Paula is innocent.**	**Paula is innocent.**

Template B

If Bill was snowbound, then he couldn’t leave Aspen until January 5.

If Bill didn’t buy the sword, then he didn’t own the murder weapon.

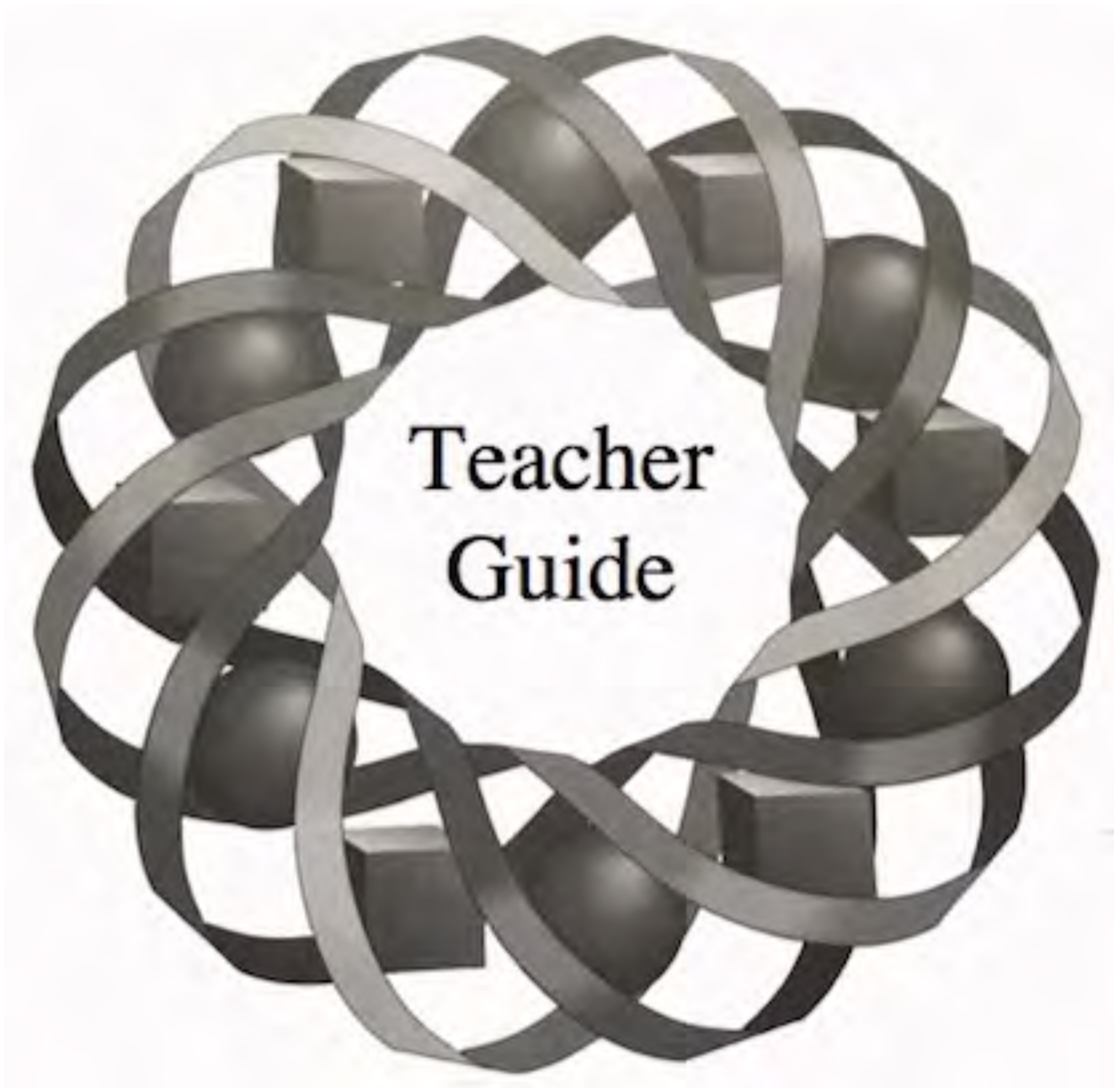
If Bill couldn’t leave Aspen until January 5, then he couldn’t be in Hillside on January 4.

If Bill didn’t own the murder weapon, then he didn’t commit the murder.

If Bill was in Aspen on January 2, 1997, then he was snowbound.

If Bill couldn’t be in Hillside on January 4, then he couldn’t buy the sword.

Fly the Big Sky with Vectors



What do swimming across a river, landing a helicopter, and refueling a plane in midair have in common? In this module, you explore how vectors can be used to model these situations and more.

Franklin Lund • Darlene Pugh • Michael Sinclair
Byron Anderson • Janet Kuchenbrod

Teacher Edition

Fly the Big Sky with Vectors

Overview

This module introduces vectors, the law of cosines, and the law of sines. Students use vectors to model displacement, velocity, and force. They add vectors graphically using the tip-to-tail method and investigate component vectors on a Cartesian coordinate system. **Note:** This module considers vectors as models in a physical setting. Mathematically, vectors are part of a larger structure called a *vector space*. While vector space is not addressed here, the properties of vectors examined in this module are the result of the properties within a vector space.

Objectives

In this module, students will:

- define vectors
- add vectors
- multiply vectors by scalars
- use the laws of cosines and sines
- resolve vectors into components
- add vectors using components.

Prerequisites

For this module, students should know:

- the Pythagorean theorem
- the sine, cosine, and tangent ratios
- the sum of the measures of the interior angles of a triangle
- the properties of parallel lines and associated angles
- how to use inverse trigonometric functions to determine angle measures.

Time Line

Activity	Intro.	1	2	3	4	Summary Assessment	Total
Days	1	2	2	2	3	1	11

Materials Required

Materials	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
fan	X					
meterstick	X	X		X	X	X
scissors	X					
paper clips	X					
helicopter template	X					
target template	X					
centimeter graph paper		X			X	
protractor		X		X	X	X
compass		X				X
map of United States		X				
map of Michigan				X		
map of Montana						X

Teacher Note

Blackline masters for the templates and maps appear at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
geometry utility		X	X	X	X	
symbolic manipulator				X		

Fly the Big Sky with Vectors

Introduction (page 247)

Beginning with a scenario in which two people attempt to swim across a river, students explore their intuitive understanding of vector quantities.

Materials List

- helicopter template (one per group; a blackline master appears at the end of the teacher edition for this module)
- target template (one per group; a blackline master appears at the end of the teacher edition for this module)
- scissors (one pair per group)
- paper clip (one per group)
- meterstick (one per group)
- small fan with adjustable settings (one per group)

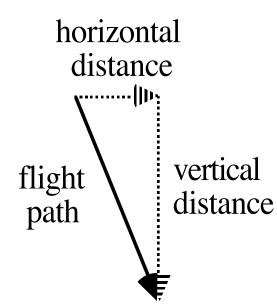
Exploration (page 247)

In this exploration, students examine the effect of wind on the flight path of a model helicopter.

a. The model helicopter will fly better if weight is added by attaching a paper clip to the base. Bending the paper "rotor" slightly may also help.

b. Instead of the target template provided at the end of the teacher edition for this module, you may wish to use hula hoops or circles of string or chalk as targets. Students should drop the helicopters from a consistent height while making their observations of the helicopter's path. Any air movement in the room, such as that created by a heating or cooling system, will affect the flight paths.

c. Student diagrams should resemble the one below.



d. Students repeat Parts **b** and **c** for flights in windy conditions.

Discussion (page 248)

a. The primary force acting on the helicopter is the force of gravity. Some students may also mention the force of friction created by air resistance.

b. Sample response: If the air in the room were perfectly still, the helicopter would drop straight down.

c. The flow of air from the fan moves the helicopter horizontally.

d. Sample response: Turning the fan on high should increase the horizontal distance traveled by the helicopter.

(page 248)

Activity 1

This activity introduces both vector addition and scalar multiplication. In the exploration, students describe displacement vectors on a map of the continental United States. They then investigate vector addition using the tip-to-tail method.

Materials List

- map of continental United States (one per student; a blackline master appears at the end of the teacher edition for this module)
- meterstick (one per student)
- protractor (one per student)
- compass (optional; one per student)
- graph paper (one sheet per student)

Technology

- geometry utility (optional)

Exploration (page 249)

Students measure and describe vectors for flights between several U.S. cities.

a–c. The map provided in the template has a scale of approximately 1 cm to 200 km. Sample responses:

1. 600 km south
2. 720 km northwest
3. 260 km north
4. 2240 km northeast
5. 1320 km east southeast

6. 360 km north
7. 340 km north
8. 600 km north
9. 960 km north

d. Sample response: The trip from Salt Lake to Seattle can be described by using the vectors from Salt Lake City to Butte, then from Butte to Seattle.

Discussion

(page 249)

a. Sample response: The instruction "fly 3 km" does not give a direction.

b. The magnitudes of corresponding vectors should be approximately the same. The descriptions of a vector's direction may take on a variety of forms.

c. The following sample responses are based on the template which appears at the end of the teacher edition for this module:

1. 600 km at a bearing of 188°
2. 720 km at a bearing of 298°
3. 260 km at a bearing of 2°
4. 2240 km at a bearing of 54°
5. 1320 km at a bearing of 260°
6. 360 km at a bearing of 354°
7. 340 km at a bearing of 348°
8. 600 km at a bearing of 8°
9. 960 km at a bearing of 354°

d. Sample response: No. There are no equivalent vectors since none of them have the same magnitude and same direction. **Note:** Some students may answer yes due to rounding of measurements. For example, the vector from Knoxville to Cincinnati is approximately equivalent to the vector from Tallahassee to Atlanta.

e. Sample response: Yes. The magnitude of the vectors from Butte to Salt Lake City and from Salt Lake City to Butte are the same. The bearings of the two vectors are opposite in direction. The bearing from Butte to Salt Lake City is 188°, while the bearing from Salt Lake City to Butte is 8°.

f. The trip from Salt Lake City to Seattle could be described by the sum of the vectors from Salt Lake City to Butte and Butte to Seattle. The resultant is the vector from Salt Lake City to Seattle.

g. The magnitude of the vector from Tallahassee to Cincinnati is roughly 3 times the magnitude of the vector from Tallahassee to Atlanta. The direction of the two vectors is the same. Therefore, if vector **c** represents the flight to Cincinnati and vector **a** the flight to Atlanta, $\mathbf{c} \approx 3\mathbf{a}$.

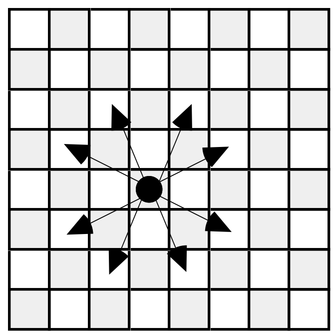
Assignment

(page 252)

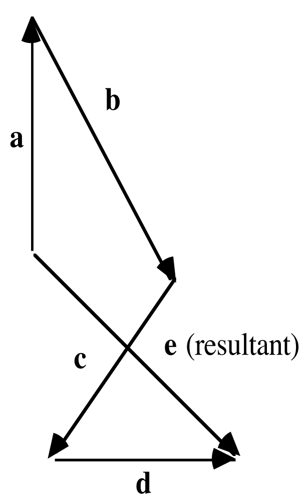
1.1 **a.** Sample response: Three examples of vector quantities are a displacement of 5 km northwest, a velocity of 16 km/hr at a bearing of 180°, and a force of 12 N at an angle of elevation of 45°. These quantities all have both magnitude and direction.

b. Sample response: Quantities which have only magnitude, such as 5 m, 6 hr, or 16 mL, cannot be modeled with vectors.

1.2 As shown in the diagram below, there are eight possible spaces to which the knight can move.



***1.3** **a.** Students should draw a scale diagram. Sample response:



b. The order in which vectors are added does not change the resultant. Vector addition is commutative and associative.

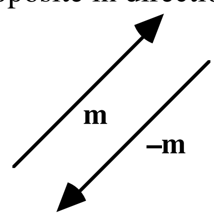
1.4 **a.** **1.** $\mathbf{m}+\mathbf{m}+\mathbf{m}=3\mathbf{m}$

2. $\mathbf{m}+0\mathbf{m}=\mathbf{m}$

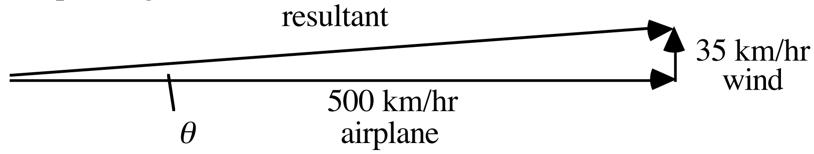
3. $\mathbf{m}+(-3\mathbf{m})=-2\mathbf{m}$

b. Sample response: In Part **a1**, the added vector has twice the magnitude as **m** and the same direction. In Part **a2**, the added vector has zero magnitude and the direction is irrelevant. In Part **a3**, the added vector has 3 times the magnitude and the opposite direction of the original vector.

c. To obtain a resultant with a magnitude of 0, add a vector equal in magnitude but opposite in direction.



1.5 **a.** Sample diagram:



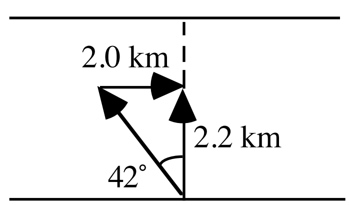
b. The velocity is increased slightly. The resultant velocity is approximately 501 km/hr.

c. The value of θ in the diagram can be found using the inverse tangent function, as indicated below:

$$\tan\theta = 35/500$$
$$\theta = \tan^{-1}(35/500)$$
$$\theta \approx 4^\circ$$

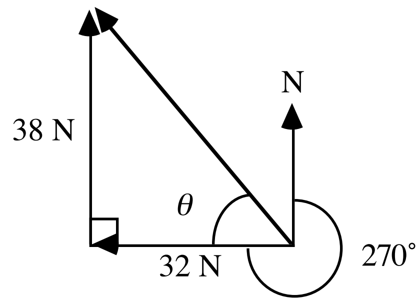
Therefore, the bearing of the resultant is $90^\circ - 4^\circ$ or 86°.

1.6 Christopher and Alee should swim upstream at an angle of about 42° from the path directly across the stream. This can be determined using the inverse tangent as follows:



$$\tan\theta = 2/2.2$$
$$\theta = \tan^{-1}(2/2.2)$$
$$\theta = 42.3^\circ$$

1.7 The resultant force is 49.7 N at a bearing of 320°. Sample diagram:



Some students may find the magnitude and direction of the resultant by measuring a scale diagram. Others may calculate the magnitude using the Pythagorean theorem as follows:

$$x^2 = 38^2 + 32^2$$
$$x = \sqrt{2468}$$
$$\approx 49.7 \text{ N}$$

The value of θ may be determined as shown below:

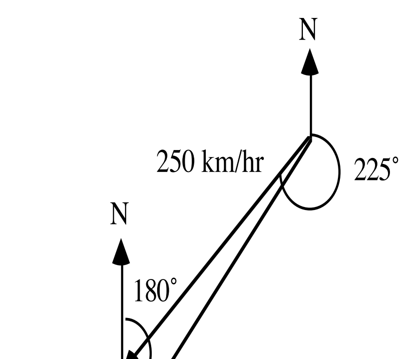
$$\tan\theta = 38/32$$
$$\theta = \tan^{-1}(38/32)$$
$$\theta \approx 50^\circ$$

The bearing is $270^\circ + 50^\circ = 320^\circ$.

1.8 **a.** The resultant velocity is about 30.4 km/hr.

b. The ship is forced about 9.5° off its original course.

***1.9** The plane's resultant velocity is 272 km/hr at a bearing of 221°. Sample diagram:



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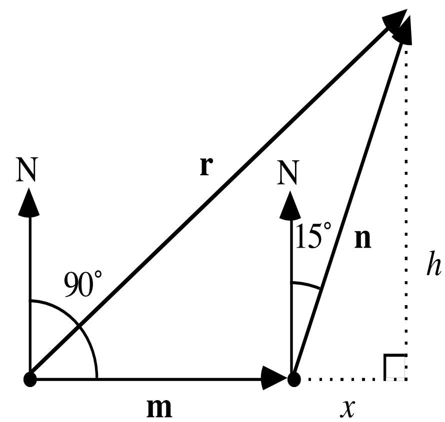
1.10 **a.** The eagle flies 75 km at a bearing of 55°.

b. The eagle flies 15 km at a bearing of 235°.

1.11 The resultant displacement is 22 km at a bearing of approximately 42°. Some students may draw a diagram like the one below and use the Pythagorean theorem or trigonometry to find h and x, then **r**. Other students may measure the resultant from a scale drawing.



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(page 254)

Activity 2

Students develop the law of cosines using the skills acquired in Activity **1**. They then apply this relationship to a variety of situations involving vector addition.

Materials List

- none

Technology

- geometry utility

Teacher Note

Students should use a geometry utility to complete the exploration as well as many of the assignment problems. (Some drawings will be much easier to develop if the software features a coordinate system.)

Exploration 1

(page 254)

a. Students create a triangle in which the angle measures can be changed, but two of the sides have fixed lengths.

b. Students create a square on each side of the triangle.

c–d. Students should observe that the sum of the areas of the squares on the two legs of the right triangle equals the area of the square on the hypotenuse. This provides a demonstration of the Pythagorean theorem.

e. Students vary the location of vertex A to create obtuse and acute triangles. In an obtuse triangle, the sum of the areas of the squares on the two shorter sides is less than the area of the square on the longest side.

In an acute triangle, the sum of the areas of the squares on any two sides of the triangle is greater than the area of the square on the third side.

Discussion 1 (page 255)

a.

1. When $m\angle ACB > 90^\circ$, the sum of the areas of the squares on the two smaller sides is less than the area of the square on the longest side.
2. When $m\angle ACB < 90^\circ$, the sum of the areas of the squares on any two sides of the triangle is greater than the area of the square on the third side.

b. In an obtuse triangle, the sum of the squares of the lengths of the shorter two sides is less than the square of the length of the longest side.

In an acute triangle, the sum of the squares of the lengths of any two sides is greater than the square of the length of the third side.

c. Sample response: If the sum of the squares of the lengths of the two shorter sides is equal to the square of the length of the longest side, then the triangle is a right triangle.

If the sum of the squares of the lengths of the two shorter sides is greater than the square of the length of the longest side, then the triangle is an acute triangle.

If the sum of the squares of the lengths of the two shorter sides is less than the square of the length of the longest side, then the triangle is an obtuse triangle.

Exploration 2 (page 256)

a–c. Students create an obtuse triangle, then construct an altitude from A to side $\overline{BC}$ to create two right triangles.

d.

1. $x = b\cos\angle BCA$
2. $h^2 = b^2 - (b\cos\angle BCA)^2$
3. $h^2 = c^2 - y^2$
4. $y = a - x$
5. $h^2 = c^2 - (a - b\cos\angle BCA)^2$

e. **1.** Students use the two expressions for h^2 to find the law of cosines.

$$c^2 - (a - b\cos\angle BCA)^2 = b^2 - (b\cos\angle BCA)^2$$
$$c^2 = b^2 - (b\cos\angle BCA)^2 + (a - b\cos\angle BCA)^2$$
$$c^2 = b^2 - (b\cos\angle BCA)^2 + a^2 - 2ab\cos\angle BCA + (b\cos\angle BCA)^2$$
$$c^2 = b^2 + a^2 - 2ab\cos\angle BCA$$

2. Sample response: The equation resembles the Pythagorean theorem adjusted by the quantity $-2ab\cos\angle BCA$.

Discussion 2

(page 257)

a. **1.** When $\angle BCA$ is acute, the sign of $\cos\angle BCA$ is positive.

2. When $\angle BCA$ is obtuse, the sign of $\cos\angle BCA$ is negative.

b. The equations should be equivalent.

c. In this case,

$$a^2 = 8.8^2 + 23.1^2 - 2(8.8)(23.1)\cos 44^\circ$$
$$\approx 77.44 + 533.61 - 292.45$$
$$\approx 318.6$$
$$a = \sqrt{318.6}$$
$$a \approx 17.85 \text{ cm}$$

d. Sample response: To find $m\angle C$, you can substitute the values for the lengths in the law of cosines as follows:

$$23.1^2 = 8.8^2 + 17.85^2 - 2(8.8)(17.85)\cos\angle C$$

Once this equation has been solved for $\cos\angle C$, you can use the inverse cosine function to determine $m\angle C$. Since the sum of the measures of the interior angles of a triangle is 180°, $m\angle B$ can be found as follows:

$$m\angle B = 180 - m\angle C - 44$$

e. Sample response: Yes. Since $\cos 90^\circ = 0$, then the law of cosines becomes $c^2 = a^2 + b^2 - 2ab(0) = a^2 + b^2$, which is the Pythagorean relationship.

f. Sample response: The measure of $\angle BAC$ is $105^\circ - 20^\circ = 85^\circ$. Using the north-south lines and transversal $\overline{AB}$, alternate interior angles are formed. Thus $m(\angle ABC) = 20^\circ + (180^\circ - 120^\circ) = 80^\circ$. The measure of $\angle ACB$ is $180^\circ - (85^\circ + 80^\circ) = 15^\circ$.

Assignment (page 258)

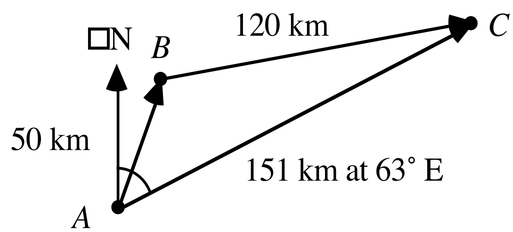
2.1 **a.** Using the law of cosines:

$$b^2 = 3^2 + 4^2 - 2 \bullet 3 \bullet 4 \bullet \cos 52^\circ$$
$$b^2 \approx 10.2$$
$$b \approx 3.2$$

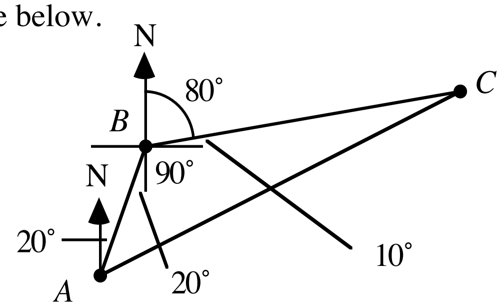
b. Sample response:

$$2.5^2 = 3.2^2 + 1^2 - 2 \bullet 3.2 \bullet 1 \bullet \cos X$$
$$X \approx 38.8^\circ$$
$$1^2 = 2.5^2 + 3.2^2 - 2 \bullet 2.5 \bullet 3.2 \bullet \cos Z$$
$$Z \approx 14.5^\circ$$
$$Y = 180^\circ - (38.8^\circ + 14.5^\circ) = 126.7^\circ$$

2.2 **a–b.** Sample drawing:



c. To find $m\angle ABC = 120^\circ$, students will need to use the bearings of the two given vectors, properties of parallel lines, and their knowledge of the angle measures in a compass rose, as shown in the figure below.



They must then apply the law of cosines twice. First, to find AC, as shown below:

$$AC^2 = 50^2 + 120^2 - 2(50)(120)\cos 120^\circ$$
$$AC \approx 151.3$$

Next, to find $m\angle BAC$:

$$120^2 = 50^2 + 151.3^2 - 2(50)(151.3)\cos\angle BAC$$
$$m\angle BAC \approx 43.4^\circ$$

Therefore, the magnitude is approximately 151 km and the bearing is approximately $43 + 20 = 63^\circ$.

***2.3** **a.** 0.2 hr or 12 min

b. **1.** The pilot should maintain an airspeed of approximately 181 km/hr. Using the law of cosines:

$$c^2 = 15^2 + 175^2 - 2 \bullet 15 \bullet 175 \bullet \cos 110^\circ$$
$$c \approx 180.7$$

2. The pilot should fly at a bearing of about 86˚. Using the law of cosines:

$$15^2 = 180.7^2 + 175^2 - 2 \bullet 180.7 \bullet 175 \bullet \cos \angle C$$
$$m\angle C \approx 4.5^\circ$$

The corresponding bearing is $90^\circ - 4.5^\circ = 85.5^\circ$.

3. At an average ground speed of 175 km/hr, the pilot should reach the clearing in about 13.7 min.

c. The wind's new magnitude is about 37.1 km/hr. Using the law of cosines:

$$c^2 = 200^2 + 210^2 - 2 \bullet 200 \bullet 210 \bullet \cos 10^\circ$$
$$c \approx 37.1$$

The wind's new bearing is about 201˚. Using the law of cosines:

$$210^2 = 200^2 + 37.1^2 - 2 \bullet 200 \bullet 37.1 \bullet \cos \angle C$$
$$m\angle C \approx 100.6^\circ$$

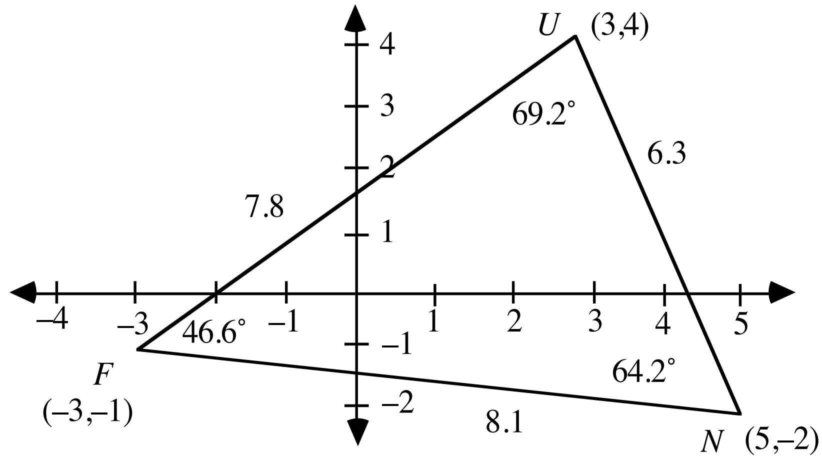
The corresponding bearing is $100.6^\circ + 90^\circ + 10^\circ = 200.6^\circ$

* * * * *

2.4 The bridge should be about 3.7 km long. Using the law of cosines:

$$c^2 = 8^2 + 5^2 - 2 \bullet 8 \bullet 5 \bullet \cos 20^\circ$$
$$c^2 \approx 13.8$$
$$c \approx 3.7$$

2.5 Sample sketch:



The lengths of the sides can be calculated as follows:

$$FU = \sqrt{(3-(-3))^2 + (4-(-1))^2} \approx 7.8$$

$$UN = \sqrt{(5-3)^2 + (-2-4)^2} \approx 6.3$$

$$NF = \sqrt{(-3-5)^2 + (-1-(-2))^2} \approx 8.1$$

The law of cosines can then be used to determine the measures of the angles:

$$6.3^2 = 7.8^2 + 8.1^2 - 2 \bullet 7.8 \bullet 8.1 \bullet \cos F$$

$$F \approx 46.6^\circ$$

$$8.1^2 = 7.8^2 + 6.3^2 - 2 \bullet 7.8 \bullet 6.3 \bullet \cos U$$

$$U \approx 69.2^\circ$$

$$N = 180^\circ - 46.6^\circ - 69.2^\circ = 64.2^\circ$$

* * * * * * * * * *

(page 260)

Activity 3

In this activity, students explore the law of sines.

Materials List

- protractor (one per student)
- meterstick (one per student)
- map of Michigan (one per student; a blackline master appears at the end of the teacher edition for this module)

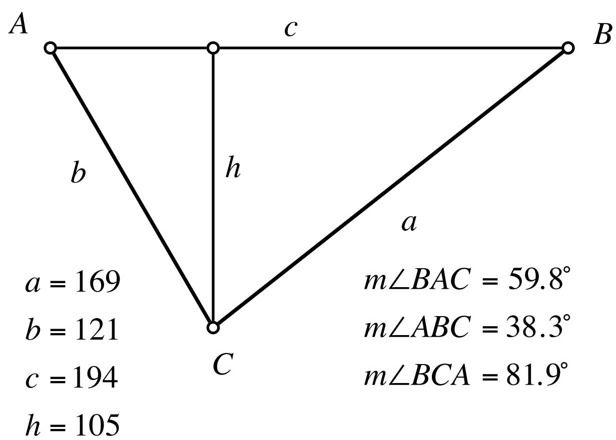
Technology

- geometry utility
- symbolic manipulator (optional; for research project)

Exploration

(page 261)

a–c. Sample sketch:



d. $\sin(\angle CAB) = h/b$; $\sin(\angle ABC) = h/a$

e. $h = b\sin(\angle CAB)$; $h = a\sin(\angle ABC)$

f. Using the sample sketch given in Parts **a–c**:

$$h = a\sin(\angle ABC) = 169 \bullet \sin(38.3) \approx 105$$

$$h = b\sin(\angle CAB) = 121 \bullet \sin(59.8) \approx 105$$

This is the same value as reported by the geometry utility.

g. The calculated values and the measured altitude should remain equal.

h. Students should obtain the following equation:

$$\frac{a}{\sin(\angle CAB)} = \frac{b}{\sin(\angle ABC)}$$

i. Students should observe similar results. The resulting equation is:

$$\frac{a}{\sin(\angle CAB)} = \frac{c}{\sin(\angle ACB)}$$

Discussion

(page 261)

a. See response to Part **h** above. All students should obtain the same equation.

b. Students should find that these ratios are equal.

c. Since $m\angle A = 38^\circ$ and $m\angle B = 85^\circ$, $m\angle C = 180 - 85 - 38 = 57^\circ$. Since a is known, b can be found using the equation below:

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B}$$

Similarly, c can be found using the following equation:

$$\frac{a}{\sin\angle A} = \frac{c}{\sin\angle C}$$

d. In order to solve a triangle, three parts, in which one part is a side, must be given. So any triangle with less information cannot be solved by either the law of sines or the law of cosines.

Since the law of sines is a set of three equations, then it theoretically can be used to solve for the three missing parts. However, it is only practical to use the law of sines for the situation in which two angles and a side are given.

The law of sines can also be used to solve for the missing parts when two sides and a non-included angle are given. Under certain conditions, however, this situation could lead to a pair of solutions (an ambiguous case).

e. Theoretically, both the law of sines and the law of cosines are sets of three equations and can be used to solve for three unknowns when a solution exists. Practically, the law of sines is used as described in Part **d** above, while the law of cosines is used to find a missing value when the givens are two sides and the included angle or three sides.

Teacher Note

Students will need a copy of the map of Michigan to complete Problem **3.5**. A blackline master appears at the end of the teacher edition for this module.

Assignment

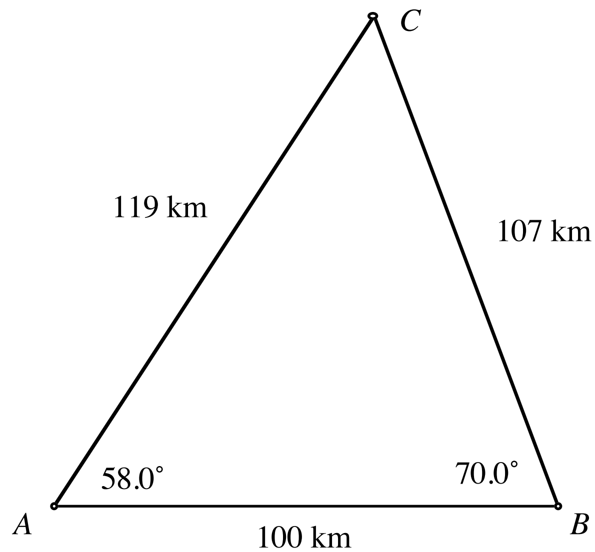
(page 263)

3.1 Using the law of sines:

$$\frac{\sin 36^\circ}{2} = \frac{\sin 75^\circ}{s} \qquad s \approx 3.28 \text{ cm}$$

$$\frac{\sin 36^\circ}{2} = \frac{\sin 69^\circ}{n} \qquad n \approx 3.18 \text{ cm}$$

3.2 **a.** In the following sample sketch, 1 cm represents 10 km:



Using the law of sines:

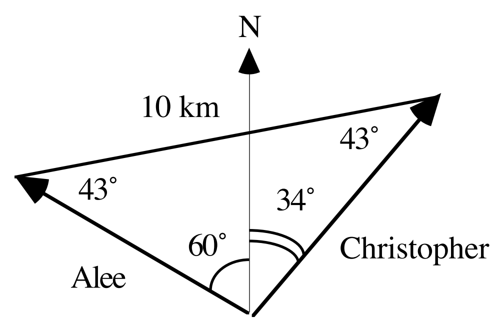
$$\frac{\sin 52^\circ}{100 \text{ km}} = \frac{\sin 58^\circ}{x} = \frac{\sin 70^\circ}{y}$$

Solving for these distances verifies the measurements in the sample sketch: $x \approx 107.6$ km; $y \approx 119.2$ km .

b. $119.2/500 = 0.2384$ hr or 14.3 min

c. $107.6/0.2384 \approx 451.3$ km/hr

***3.3** As shown in the diagram below, the base angles of the isosceles triangle formed by the two paths measure 43°.



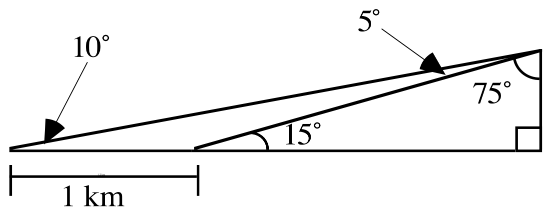
The lengths of the two congruent sides can be found as follows:

$$\frac{\sin 94^\circ}{10} = \frac{\sin 43^\circ}{x}$$

$$x \approx 6.84 \text{ km}$$

At a speed of 4 km/hr, Christopher and Alee can walk this distance in about 1 hr and 43 min.

3.4 Since the sum of the measures of the interior angles of a triangle is 180°, the measures of the other angles in the diagram can be determined as shown below.



The distance from the near side of the lake to the peak can be found as follows:

$$\frac{\sin 5^\circ}{1} = \frac{\sin 10^\circ}{x}$$
$$x \approx 2.0 \text{ km}$$

The height of the peak from the level of the lake is:

$$\frac{\sin 90^\circ}{2} = \frac{\sin 15^\circ}{x}$$
$$x \approx 0.5 \text{ km or } 500 \text{ m}$$

Since the lake's elevation is 3600 m, the peak's elevation is $3600 + 500 = 4100$ m.

* * * * *

3.5 **Note:** A blackline master of a map of Michigan appears at the end of the teacher edition for this module.

a. The vector should have its tail on Kalamazoo and its tip on Flint. It has a magnitude of 169 km at a bearing of 58°.

b. about 0.89 hr or 53.4 min

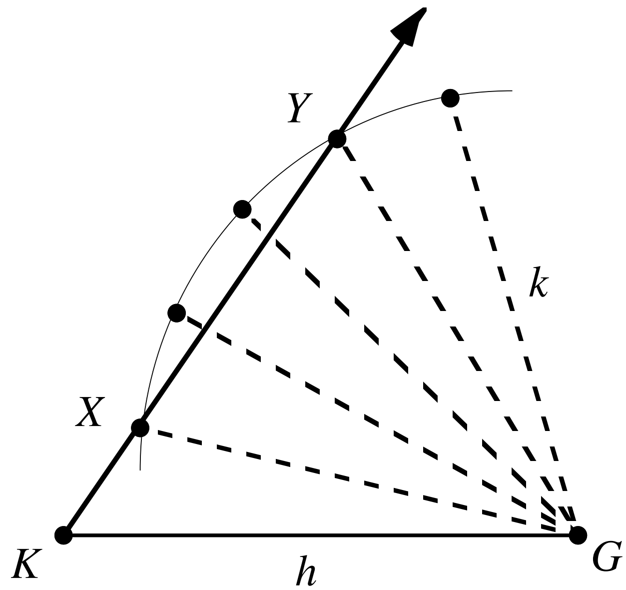
c. Using the law of cosines, the helicopter should fly about 175 km/hr at a bearing of 46°.

* * * * * * * * * *

Research Project (page 264)

The law of cosines is used to find parts of triangles given the lengths of three sides (SSS) or the lengths of two sides and the measure of the included angle (SAS). The law of sines is used to find parts of triangles given the length of one side and the measures of two angles (ASA or AAS).

In this research project, students explore the situation in which the lengths of two sides and the measure of a non-included angle (SSA or side-side-angle) are known. As shown in the diagram below, this can produce the ambiguous cases ΔKGX and ΔKGY, where $\overline{KX}$ and $\overline{KY}$ each represent the side opposite $\angle G$.



a. Using the law of cosines, $k^2 = g^2 + h^2 - 2gh\cos\angle K$. Students should substitute for k, h, and $m\angle K$, then solve for g using a symbolic manipulator.

1. Using the law of cosines:

$$4.3^2 = g^2 + 5^2 - 2 \bullet g \bullet 5\cos 55°$$
$$0 = g^2 - 2 \bullet 5 \bullet \cos 55° \bullet g + 6.51$$

There are two possible triangles in this situation, $g = 4.2$ cm, the length of $\overline{KY}$, or $g = 1.6$ cm, the length of $\overline{KX}$.

2. Using the law of cosines:

$$2^2 = g^2 + 5^2 - 2 \bullet g \bullet 5\cos 55°$$
$$0 = g^2 - 2 \bullet 5 \bullet \cos 55° \bullet g + 21$$

Since there are no real solutions, it is not possible to create a triangle from the given information.

3. Again using the law of cosines:

$$6^2 = g^2 + 5^2 - 2 \bullet g \bullet 5 \cos 55^\circ$$
$$0 = g^2 - 2 \bullet 5 \bullet \cos 55^\circ \bullet g - 11$$

There are two solutions: $g = 7.3$ and $g = -1.5$. However, since g represents the length of a side of a triangle, only the positive root makes sense.

4. The intent here is to show the case in which the side opposite *G* is tangent to a circle of radius *k* with center at point *G*. This forms a right triangle, where $g \approx 2.868$ cm. Using the law of cosines:

$$4.1^2 = g^2 + 5^2 - 2 \bullet g \bullet 5 \cos 55^\circ$$
$$0 = g^2 - 2 \bullet 5 \bullet \cos 55^\circ \bullet g + 8.22$$

b. Students should observe that using the law of sines to find an angle measure can be risky, since the sine of an acute angle and its supplement have the same value.

(page 265)

Activity 4

In this activity, students resolve vectors into horizontal and vertical components, then add components to determine the resultant.

Materials List

- centimeter graph paper (one sheet per student)
- protractor (one per student)
- ruler (one per student)

Technology

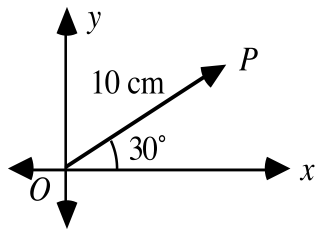
- geometry utility (optional)

Teacher Note

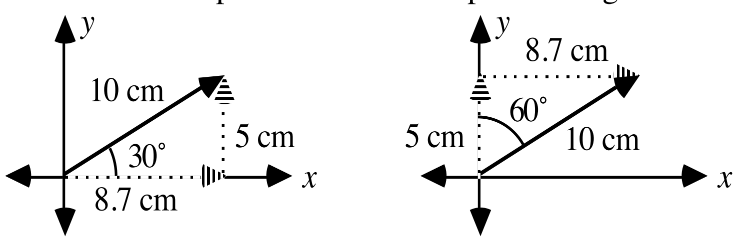
You may wish to remind students that magnitudes of vector quantities are always positive. Since a vector is a directed length, a negative value signifies direction.

Exploration 1 (page 265)

a–b. Students create a vector whose tail is located at the origin and whose tip is located in the first quadrant. Sample response:



c–e. Students draw the horizontal and vertical components of vector **r**. They then use right-triangle trigonometry to determine the magnitudes of the component vectors. The following diagram shows two possible sketches of the components for the sample vector given above.



f. Students repeat Parts **b–e** for vectors in the second, third, and fourth quadrants.

Discussion 1 (page 266)

a. Sample response: When the horizontal component coincides with the *x*-axis, the angle is the same as the direction of the original vector.

Note: When the vertical component coincides with the *y*-axis and the horizontal component is parallel to the *x*-axis, then the angle is the difference between 90° and the direction of the original vector.

b. Sample response: A vector and its two components form a right triangle. The magnitude of the horizontal component equals the magnitude of the original vector times the cosine of the angle representing the direction measured counterclockwise from the positive *x*-axis. The magnitude of the vertical component equals the magnitude of the original vector times the sine of the same angle.

c. Sample response: The horizontal component is equal to the *x*-coordinate. The vertical component is equal to the *y*-coordinate.

d. Sample response: A vector's horizontal and vertical components are not affected by its location. Placing the vector's tail at the origin, however, makes it easier to determine the components.

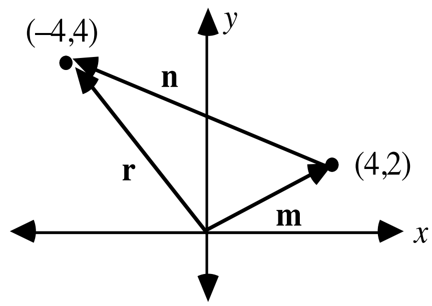
e. In this situation, the negative signs indicate direction.

f. Sample response: Draw the vector with its tail positioned at the origin. A bearing of 260° corresponds with an angle formed with the positive *x*-axis of 190°. Given this angle measure, the components can be determined using trigonometric ratios. The horizontal component is $355\cos 190°$. The vertical component is $355\sin 190°$.

Exploration 2

(page 267)

a. Sample graph:



b. The components in the following table correspond to the sample vectors given above.

Vector	Horizontal Component	Vertical Component
m	4	2
n	–8	2
r	–4	4

c.

1. Students should observe that the horizontal component of **r** is the sum of the horizontal components of **m** and **n**.
2. Students should observe that the vertical component of **r** is the sum of the vertical components of **m** and **n**.

d. Students repeat the exploration for a sum involving three vectors.

Discussion 2

(page 268)

a. Sample response: The sum of the horizontal components of the vectors in a sum is equal to the horizontal component of the resultant vector. The sum of the vertical components of the vectors in the sum is equal to the vertical component of the resultant vector.

b. Sample response: You can determine the vertical and horizontal components of the vectors involved in the sum from their directions and magnitudes. The components of the resultant can be found by adding the components of the vectors involved in the sum.

c. The magnitude of **r** can be found using the Pythagorean theorem as follows:

$$(|\mathbf{r}|)^2 = (\mathbf{r}_x)^2 + (\mathbf{r}_y)^2$$

For the sample vector in Exploration **2**, $\mathbf{r}_x = -4$ and $\mathbf{r}_y = 4$. Therefore,

$$\begin{aligned}(|\mathbf{r}|)^2 &= (-4)^2 + (4)^2 \\ &= 32 \\ |\mathbf{r}| &= \sqrt{32}\end{aligned}$$

The horizontal component equals the magnitude of the resultant vector times the cosine of the angle formed by the resultant and the positive *x*-axis. Therefore,

$$\begin{aligned}-4 &= \sqrt{32}\cos\theta \\ -4/\sqrt{32} &= \cos\theta \\ -4/\sqrt{32} &\approx \cos\theta \\ -0.707 &\approx \cos\theta \\ \cos^{-1}(-0.707) &\approx \cos^{-1}(\cos\theta) \\ 135^\circ &= \theta\end{aligned}$$

Note: There are two values for θ over the interval $[0^\circ, 360^\circ]$ whose cosine is approximately equal to –0.707. However, only 135° results in a vector in the second quadrant where the horizontal component is negative and the vertical component is positive.

Assignment

(page 268)

4.1 The vertical component can be calculated as the length of the side opposite the 40° angle of a right triangle in which the length of the hypotenuse is 75:

$$\begin{aligned}\sin 40^\circ &= \mathbf{u}_y/75 \\ \mathbf{u}_y &= 48.2 \text{ km/hr}\end{aligned}$$

Similarly, the horizontal component $\mathbf{u}_x = 57.5 \text{ km/hr}$.

4.2 **a.** $20\cos 45^\circ + 15\cos 105^\circ + 35\cos 200^\circ + 45\cos 300^\circ \approx -0.13 \text{ N}$

b. $20\sin 45^\circ + 15\sin 105^\circ + 35\sin 200^\circ + 45\sin 300^\circ \approx -22.31 \text{ N}$

c. (–0.2,–22.4)

4.3 Since each x-coordinate represents the horizontal component of the corresponding vector, the sum of the x-coordinates represents the horizontal component of the resultant: $4 + (-4) + 7 + (-6) = 1$. Similarly, the sum of the y-coordinates represents the vertical component of the resultant: $5 + 13 + (-4) + 8 = 22$. When the tail of the resultant is placed at the origin, the coordinates of the tip are (1,22).

4.4 The magnitude of **m** can be found using the Pythagorean theorem, as shown below:

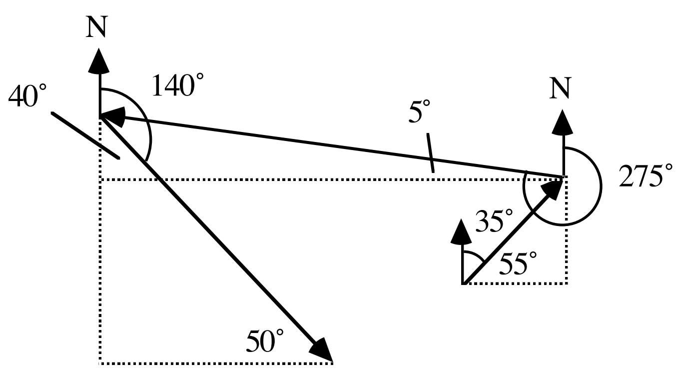
$$|\mathbf{m}|^2 = 15^2 + 20^2 = 625$$

$$|\mathbf{m}| = 25 \text{ m/sec}$$

The direction of **m**, measured counterclockwise from the positive x-axis, can be found using the inverse tangent:

$$\begin{aligned} \tan\theta &= 20/15 \\ \theta &= \tan^{-1}(20/15) \\ &\approx 53^\circ \end{aligned}$$

*__4.5__ To add these vectors using components, students must convert each bearing to the appropriate angle measure. Sample diagram (not to scale):



The horizontal component of the resultant can be found as follows:

$$350\cos 55^\circ + (-1250\cos 5^\circ) + 1000\cos 50^\circ \approx -402$$

Similarly, the vertical component of the resultant is:

$$350\sin 55^\circ + 1250\sin 5^\circ - 1000\sin 50^\circ \approx -370$$

Using the Pythagorean theorem, the magnitude of the resultant is:

$$\sqrt{(-402)^2 + (-370)^2} \approx 546 \text{ paces}$$

The bearing is:

$$270 - \tan^{-1}(-370/-402) \approx 227^\circ$$

* * * * *

4.6 The horizontal component of the resultant is:

$$25\cos 65^\circ + 30\cos 340^\circ + 10\cos 150^\circ \approx 30 \text{ N}$$

The vertical component of the resultant is:

$$25\sin 65^\circ + 30\sin 340^\circ + 10\sin 150^\circ \approx 17 \text{ N}$$

The magnitude of the resultant is $\sqrt{30^2 + 17^2} \approx 34.5$ N. Its direction with respect to the positive *x*-axis is $\tan^{-1}(17/30) \approx 30^\circ$.

4.7 **a.** $\mathbf{r}_x = 50\cos 60^\circ - 14\cos 10^\circ + 15\cos 85^\circ + 25\cos 15^\circ \approx 36.6 \text{ m}$

b. $\mathbf{r}_y = 50\sin 60^\circ - 14\sin 10^\circ - 15\sin 85^\circ - 25\sin 15^\circ \approx 19.5 \text{ m}$

c. The magnitude of the resultant is $\sqrt{36.6^2 + 19.5^2} \approx 41.5$ N. Its bearing is $90 - \tan^{-1}(36.6/19.5) \approx 28^\circ$.

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Teacher Note

Students will need a copy of the map of Montana to complete Problem **3**. A blackline master appears at the end of the teacher edition for this module.

Answers to Summary Assessment (page 270)

1. Sample response: Starting from camp, travel 0.58 km at a bearing of 239.0˚ to checkpoint 1. From checkpoint 1, travel 0.92 km at a bearing of 102.5˚ to checkpoint 2. From checkpoint 2, travel 1.22 km at a bearing of 305˚ to checkpoint 3. From checkpoint 3, travel 0.85 km at a bearing of 118.1˚ to checkpoint 4. From checkpoint 4, travel 0.52 km at a bearing of 16.7˚ to checkpoint 5. From checkpoint 5, travel 0.85 km at a bearing of 225˚ back to camp.

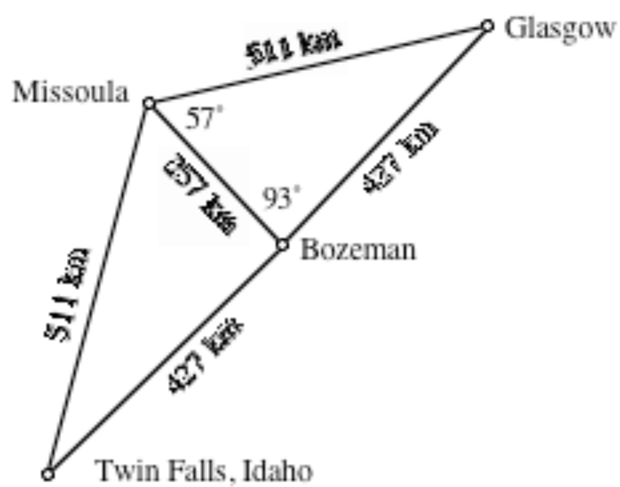
2. Students should draw a scale diagram of the vectors involved. Using the law of cosines, the wind is blowing 26.3 km/hr at a bearing of 37.7˚.

3. The distance between Missoula and Bozeman is about 257 km. Using the differences in time and speed, students can calculate the distances traveled by the waves from the epicenter to the two seismographs.

The distance from Missoula to the epicenter is $146 \text{ sec} \bullet 3.5 \text{ km/sec} = 511 \text{ km}$. The distance from Bozeman to the epicenter is $122 \text{ sec} \bullet 3.5 \text{ km/sec} = 427 \text{ km}$.

To determine the location of the epicenter, some students may use a compass to draw circles of radii 511 km and 427 km on the map. The points of intersection for these two circles indicate possible locations for the epicenter. In this case, one point of intersection is near Glasgow, Montana, while the other is outside the state.

Other students may use the law of cosines to find the measures of the angles in the two oblique triangles formed from the known distances, as shown in the diagram below. Once these angle measures are known, students could construct these triangles on the map.



Module Assessment

1. Determine three vectors whose resultant has a magnitude of 15 m at a bearing of 135°. Use components to verify your response.

2. Write the vector with a magnitude of 25 m/sec and a bearing of 340° in terms of its horizontal and vertical components.

3. The forces represented by the following vectors, where direction is measured counterclockwise from the positive x-axis, are acting simultaneously on an object: 55 N at 60°, 132 N at 90°, 25 N at 120°, and 45 N at 280°. What is the resultant force on the object?

4. To arrive at the desired destination, a pilot must maintain a ground speed of 450 km/hr at a bearing of 345°. If the wind is blowing at 30 km/hr with a bearing of 65°, at what airspeed and in what direction should the pilot fly?

Answers to Module Assessment

1. Answers will vary. Sample response: Three displacement vectors which produce the desired resultant are 3 m at 180°, 7.6 m at 180°, and 10.6 m at 90°. The sum of the *x*-components is –10.6 m; the sum of the *y*-components is 10.6 m. Using the Pythagorean theorem, the resultant has a magnitude of 15 m. Using the inverse tangent, its bearing is 135°.

2. $\mathbf{v}_x = 25 \text{ m/sec} \bullet \cos 110^\circ = -8.5 \text{ m/sec}$;
$\mathbf{v}_y = 25 \text{ m/sec} \bullet \sin 110^\circ = 23.5 \text{ m/sec}$

3. The components of each vector are as follows:

$\mathbf{a}_x = 55 \text{ N} \bullet \cos 60^\circ = 27.5 \text{ N}$ and $\mathbf{a}_y = 55 \text{ N} \bullet \sin 60^\circ \approx 47.6 \text{ N}$

$\mathbf{b}_x = 132 \text{ N} \bullet \cos 90^\circ = 0 \text{ N}$ and $\mathbf{b}_y = 132 \text{ N} \bullet \sin 90^\circ = 132 \text{ N}$

$\mathbf{c}_x = 25 \text{ N} \bullet \cos 120^\circ = -12.5 \text{ N}$ and $\mathbf{c}_y = 25 \text{ N} \bullet \sin 120^\circ \approx 21.7 \text{ N}$

$\mathbf{d}_x = 45 \text{ N} \bullet \cos 280^\circ \approx 7.8 \text{ N}$ and $\mathbf{d}_y = 45 \text{ N} \bullet \sin 280^\circ \approx -44.3 \text{ N}$

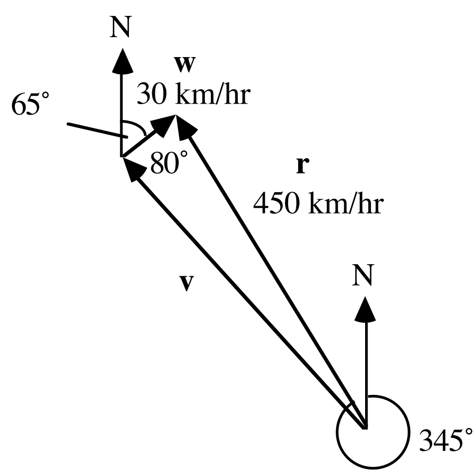
The sums of the components are:

$\mathbf{r}_x = 27.5 \text{ N} + 0 \text{ N} + (-12.5 \text{ N}) + 7.8 \text{ N} = 22.8 \text{ N}$

$\mathbf{r}_y = 47.6 \text{ N} + 132 \text{ N} + 21.7 \text{ N} + (-44.3 \text{ N}) = 157.0 \text{ N}$

Using the Pythagorean theorem, the magnitude of **r** is 158.6 N. Using the inverse tangent, its direction is about 81.7°.

4. Sample diagram:



Using the law of cosines to find the magnitude of **v**:

$$v^2 = 30^2 + 450^2 - 2 \bullet 30 \bullet 450 \bullet \cos 80^\circ$$
$$v^2 = 198711.5$$
$$v = 445.8 \text{ km/hr}$$

The measure of the angle between **v** and **r** can be found using the law of sines as follows:

$$\frac{30}{\sin x} = \frac{445.8}{\sin 80^\circ}$$
$$0.066 = \sin x$$
$$\sin^{-1} 0.066 = x$$
$$3.8^\circ \approx x$$

Since the bearing of **r** is 345°, the bearing of **v** is $345^\circ - 3.8^\circ = 341.2^\circ$.

Selected References

de Lange, J. *Flying through Math*. Scotts Valley, CA: WINGS for Learning, 1991.

Dolan, S., ed. *Newton's Laws of Motion*. New York: Cambridge University Press, 1989.

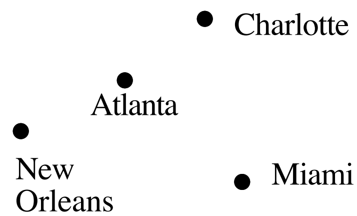
Jeppesen, S., ed. *Private Pilots Manual*. New York: Jeppesen Publishing, 1991.

1995 Road Atlas. Heathrow, FL: American Automobile Association, 1995.

Flashbacks

Activity 1

1.1 Use arrows to show a route an airplane might take for a flight that begins and ends in Miami and makes stops at Charlotte, Atlanta, and New Orleans.



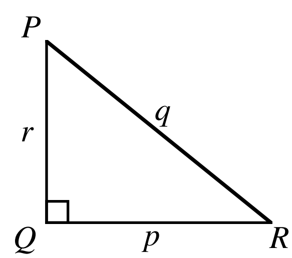
1.2 Sketch a map of the route you travel to and from school every day, including approximate distances and directions.

1.3 Describe the differences between the following two statements:

a. A car traveled to Washington, D.C., from Alexandria, Virginia, at 60 km/hr.

b. A car traveled 100 km to Washington, D.C., from Alexandria, Virginia.

1.4 The diagram below shows right triangle *PQR*.



Describe each of the following trigonometric ratios in terms of p, q, and r.

a. $\sin\angle RPQ$

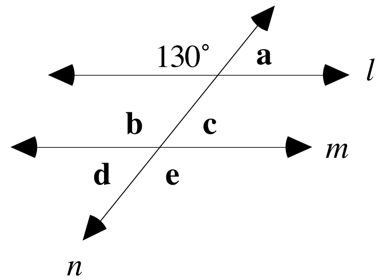
b $\cos\angle RPQ$

c. $\tan\angle QRP$

1.5 Using ΔPQR in Flashback **1.4**, find the $m\angle RPQ$ if q is 13 and r is 8.

Activity 2

2.1 Find the measure of each angle in the following diagram. Line l is parallel to line m.



2.2 Find the distance between the points with coordinates (–2,3) and (4,–6).

2.3 Solve for the variable in each of the following equations.

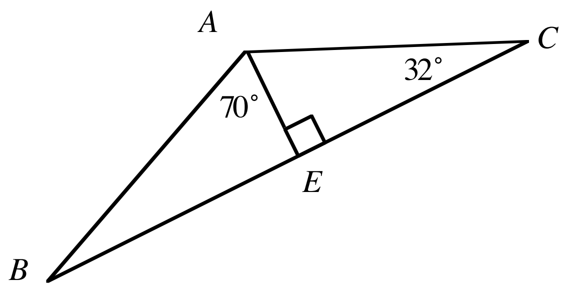
a. $6^2 + b^2 = 10^2$

b. $x^2 = 12^2 + 11^2 - 22(\cos 35)$

c. $5^2 = 3^2 + 6^2 - 36(\cos\theta)$

Activity 3

3.1 Find the unknown angle measures in the following triangles:



3.2 Solve for x in the equation below:

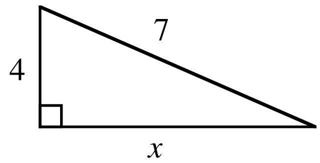
$$\frac{15}{\sin 32^\circ} = \frac{x}{\sin 64^\circ}$$

3.3 How long would it take a car traveling 80 km/hr to cover 200 km?

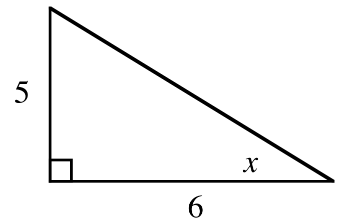
Activity 4

4.1 Determine the value of x in each of the following triangles.

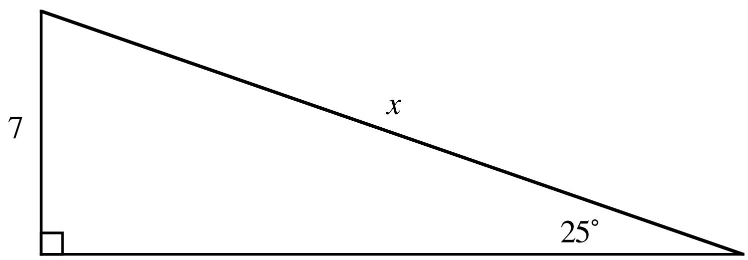
a.



b.



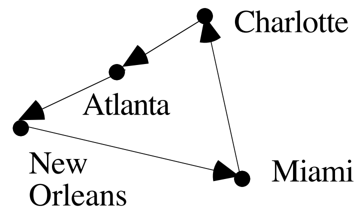
c.



Answers to Flashbacks

Activity 1

1.1 Sample response:



1.2 Answers will vary.

1.3 Sample response: The first statement describes the rate at which the car traveled between the cities, while the second statement describes the distance between the cities.

1.4 **a.** p/q

b. r/q

c. r/p

1.5 $m\angle RPQ = \cos^{-1}(8/13) \approx 52^\circ$

Activity 2

2.1 **a.** 50°

b. 130°

c. 50°

d. 50°

e. 130°

2.2 Using the Pythagorean theorem, $d = \sqrt{(4-(-2))^2 + (-6-3)^2} = \sqrt{117}$.

2.3 **a.** $b = 8, -8$

b. $x = 15.7$

c. $\theta \approx 56.3^\circ$

Activity 3

3.1 $m\angle ABE = 20^\circ$; $m\angle EAC = 58^\circ$; $m\angle BEA = 90^\circ$

3.2 $x \approx 25.4$

3.3 2.5 hr

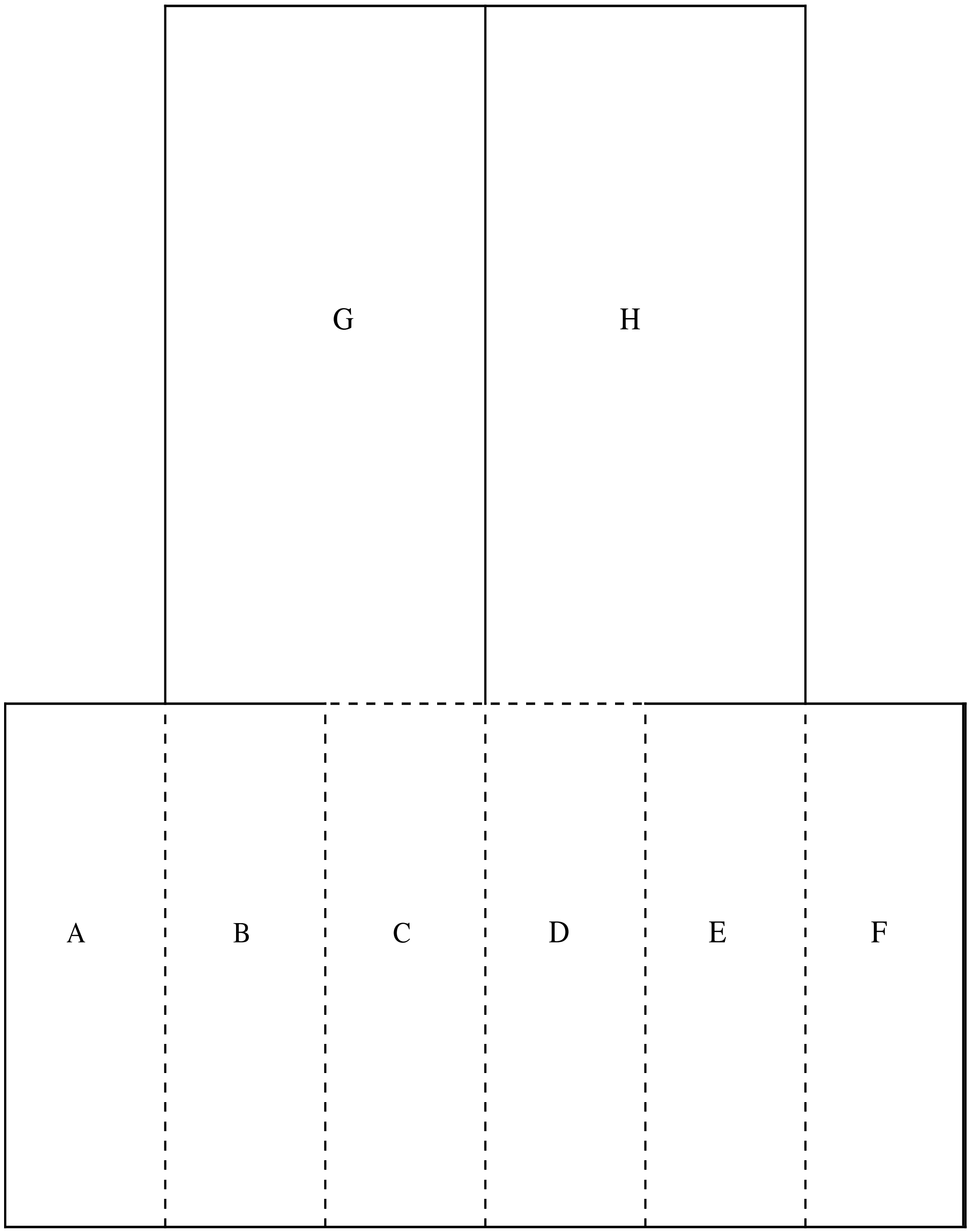
Activity 4

4.1 **a.** $x = \sqrt{7^2 - 4^2} \approx 5.7$

b. $x = \tan^{-1}(5/6) \approx 39.8^\circ$

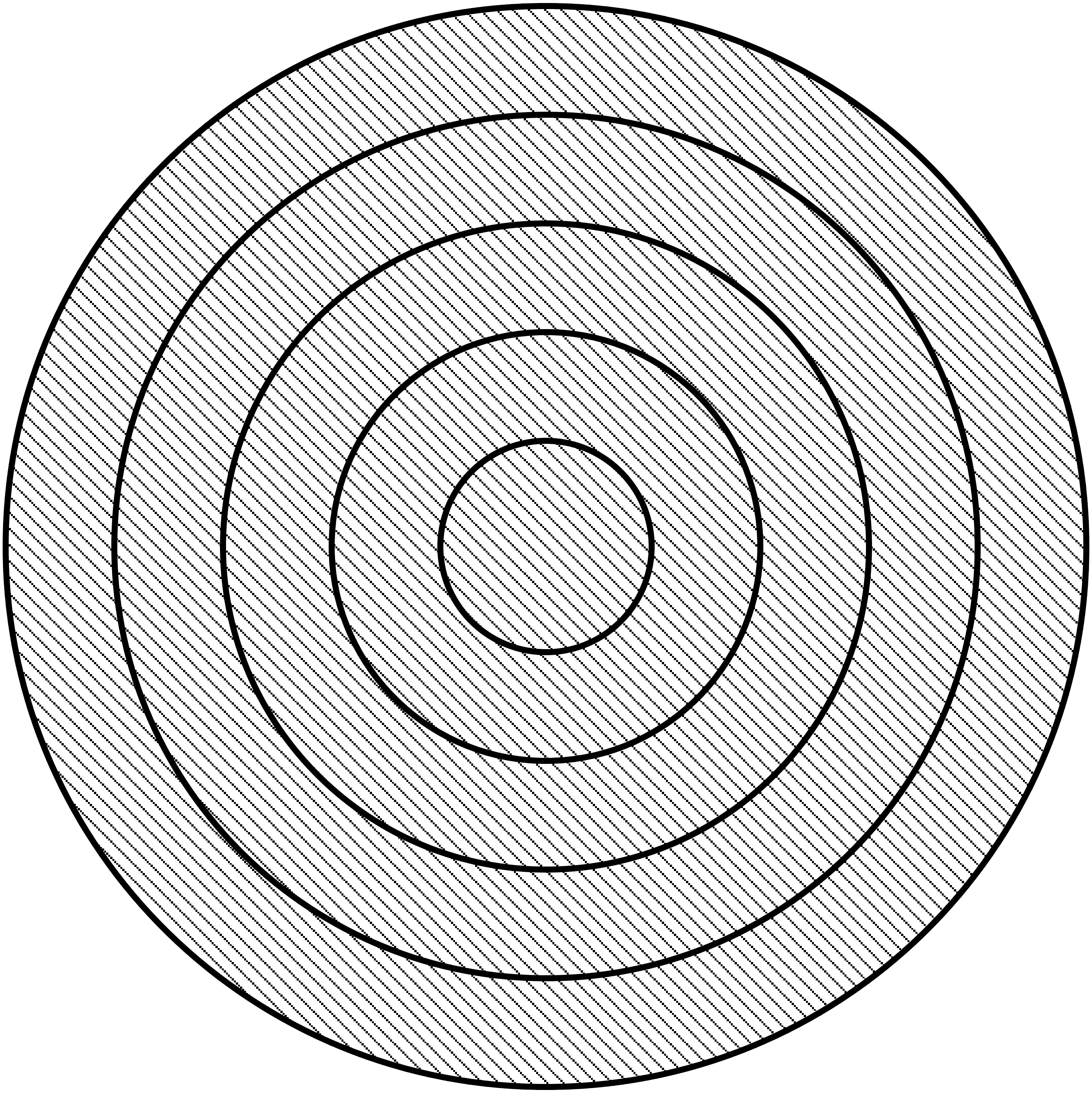
c. $x = 7/\sin 25^\circ \approx 16.6$

Template for Helicopter

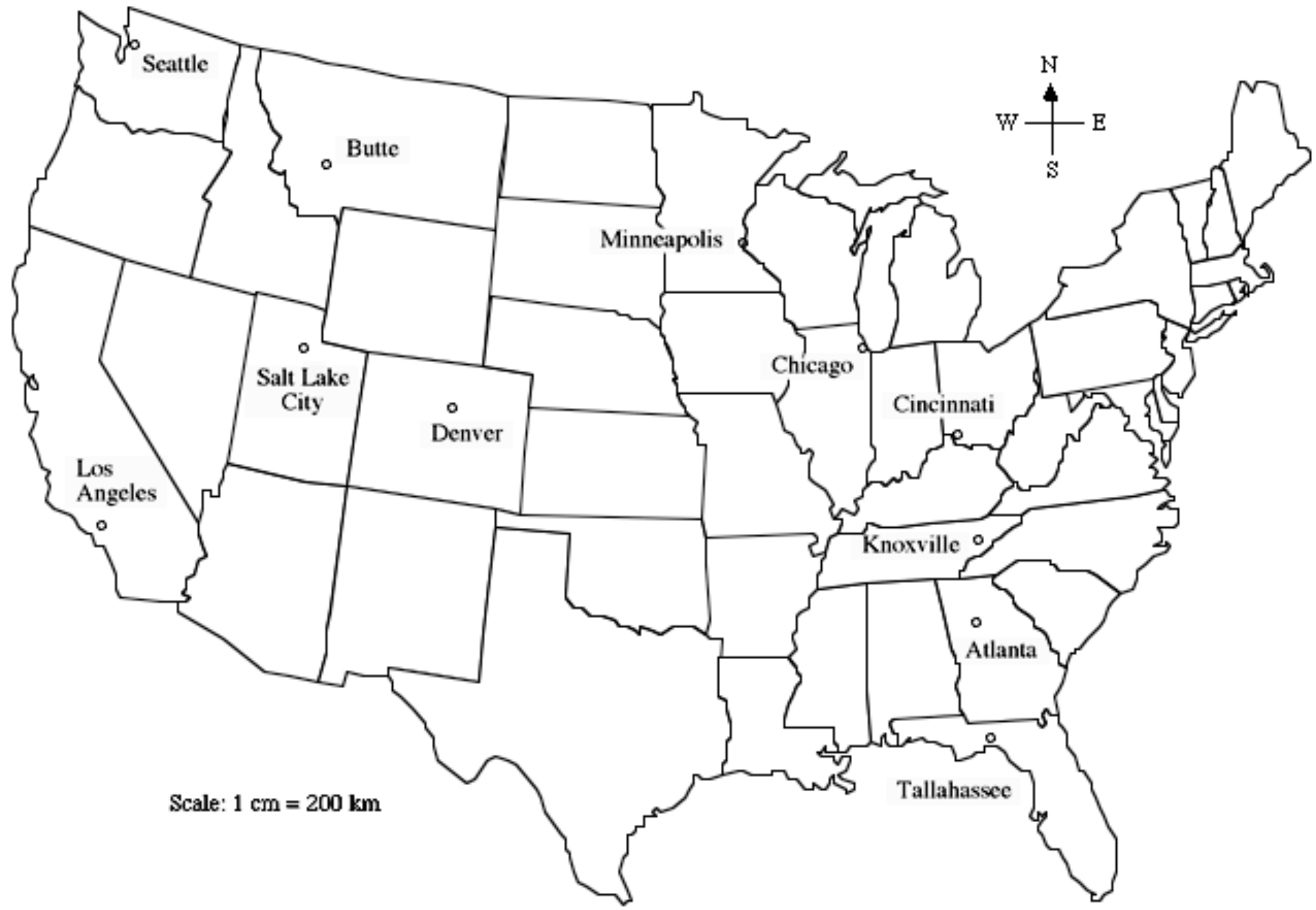


1. Cut along solid lines.
2. Fold along dashed lines.
3. Fold A onto B, then onto C.
4. Fold F onto E, then onto D.
5. Fold G and H in opposite directions.
6. Staple at the top and bottom of the seam where A and F meet.

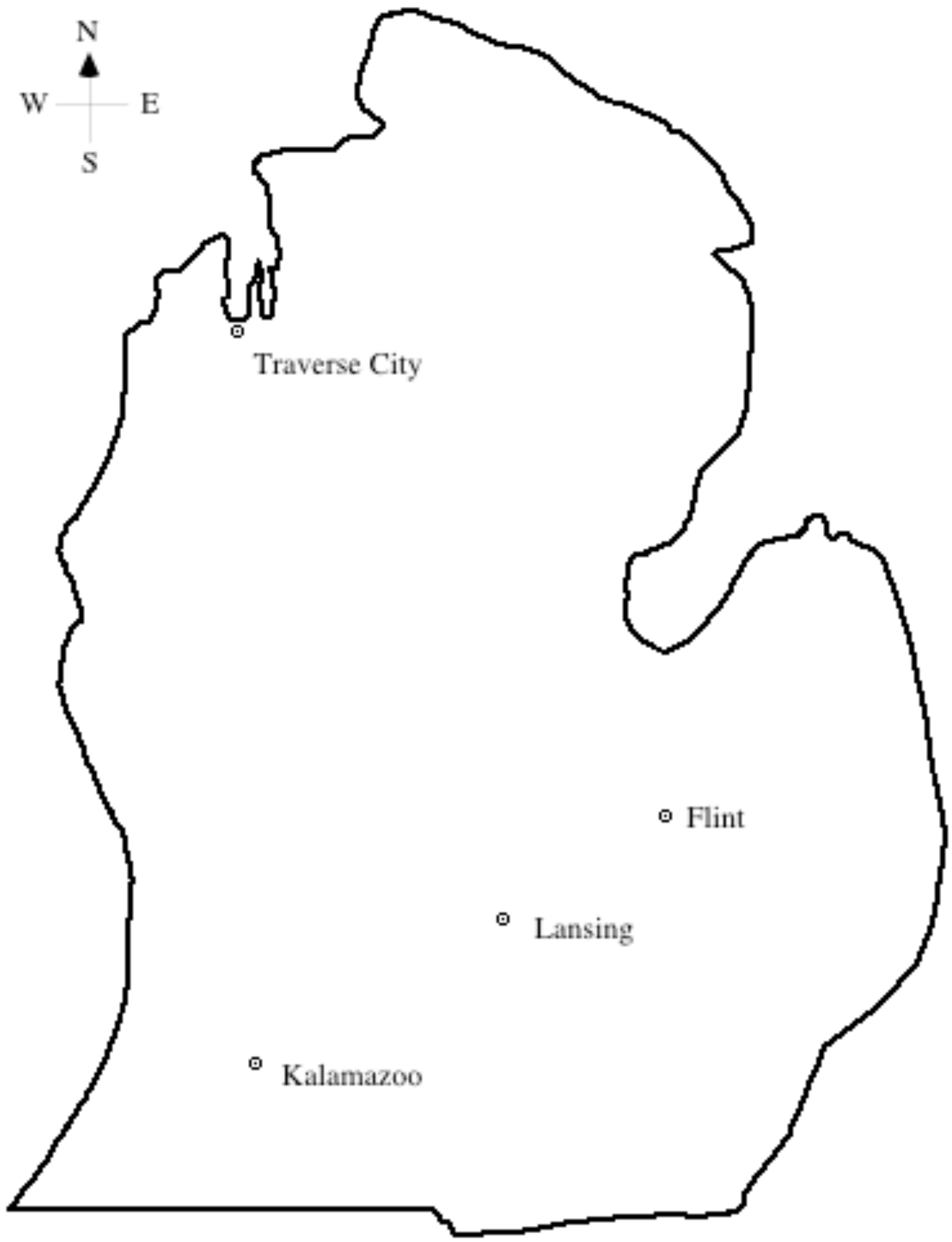
Template for Target



Map of United States

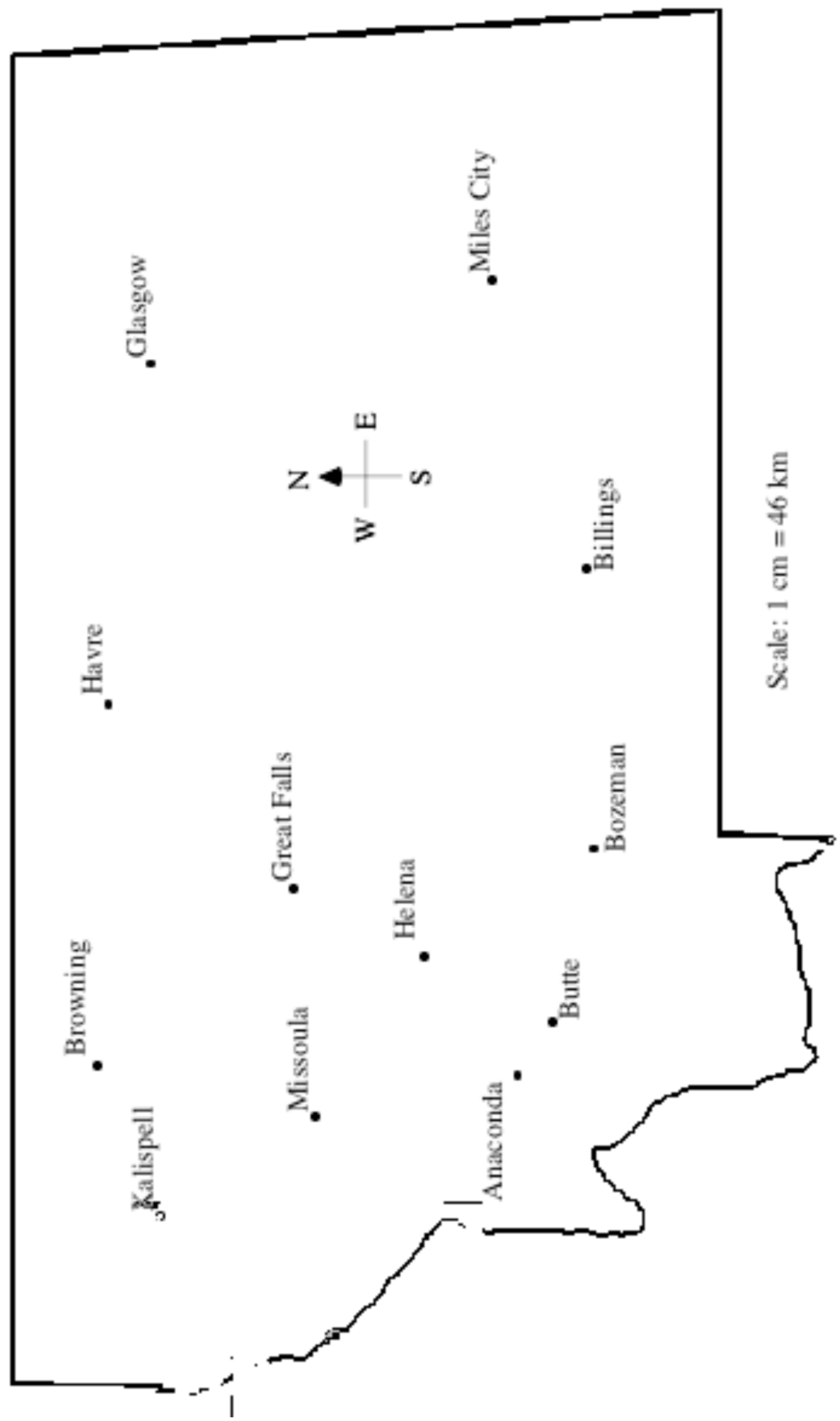


Map of Southern Michigan



Scale: 1 cm = 26 km

Map of Montana



Everyone Counts

The grand opening of a shopping mall promises lots of new choices for customers. How many choices? In this module, you investigate some different ways to count your options.

Art Perleberg • Deanna Turley • Dan West

Teacher Edition

Everyone Counts

Overview

The usefulness of combinatorics has expanded rapidly since the introduction of computers. Massive problems that once were unthinkable can now be solved quickly and efficiently with technology. For computers to accomplish these feats, however, programmers must have knowledge of counting algorithms.

In this module, students investigate the fundamental counting principle, permutations, and combinations. The setting for these investigations is a new shopping mall.

Objectives

In this module, students will:

- review factorial notation and the fundamental counting principle
- use tree diagrams, lists, and charts to organize information and solve problems
- develop and use a formula for permutations
- develop and use a formula for combinations.

Prerequisites

For this module, students should know:

- the fundamental counting principle
- how to use tree diagrams
- the meaning and notation of factorials
- a technique for generating Pascal's triangle
- how to determine simple probabilities.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	2	1	9

Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
combination lock	X			
keys	X			
template A			X	

Teacher Note

A blackline master for template A appears at the end of the teacher edition FOR THIS MODULE.

Everyone Counts

Introduction
(page 275)

In this module, students encounter counting problems that could arise in the development of a mall. **Note:** The businesses named in the memo appear in assignment problems throughout the module.

(page 275)

Activity 1

In this activity, students review basic theoretical probability, factorial notation, and the fundamental counting principle. Tree diagrams and lists are used to organize information.

Materials List

- rotary combination lock (optional; one per group)
- pin-lock key (optional; one per student or group)

Teacher Note

You may wish to allow students to try to "crack" a rotary combination lock. This should illustrate the large number of possible combinations and encourage students to consider ways to systematically count them.

Exploration
(page 276)

Students investigate rotary combination and keyed pin locks.

a.

1. Students should recognize that many combinations are possible. The actual number is $40 \bullet 40 \bullet 40 = 64{,}000$.

2. Students should observe that the probability that a randomly selected combination will open the lock is:

$$\frac{1}{\text{total number of combinations}}$$

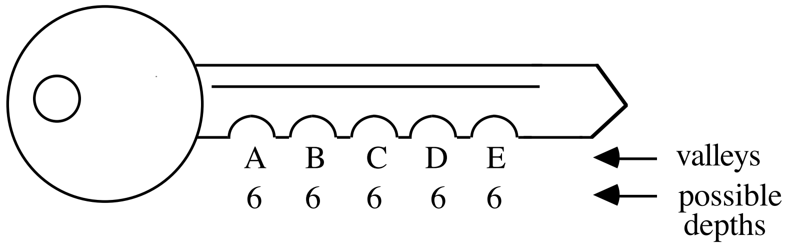
The actual probability is $1/64{,}000$.

3. Answers will vary. Sample response: List the combinations in a sequenced pattern: 0–0–0, 0–0–1, 0–0–2, 0–0–3, …, 39–39–37, 39–39–38, 39–39–39. **Note:** Some students may recall the fundamental counting principle from the Level 1 modules "AIDS: The Preventable Epidemic" and "Going in Circuits."

b. **1.** There are $40 \bullet 40 \bullet 40 = 64,000$ possible combinations.

2. The probability that a randomly selected combination will open the lock is $1/64,000$.

c. **1.** Sample drawing:



If 6 different depths are available for each of the 5 valleys, there are $6 \bullet 6 \bullet 6 \bullet 6 \bullet 6 = 6^5 = 7776$ possible keys.

2. The maximum number of keys is $10^6 = 1,000,000$.

3. The probability that a randomly selected key will open the lock is $1/1,000,000$.

Discussion

(page 278)

a. **1.** Answers will vary. Listing each combination in a sequenced arrangement of number patterns works well. Using this form, students may recognize a method of counting the total number of combinations.

2. Sample response: Yes, numbers in a lock combination can be repeated as one turns to the right or left.

3. Sample response: Because there are so many possibilities, it would be an enormous task to make a complete list of all of them.

b. Sample response: To increase the probability that the promotion will last a long time, use a pin key with 6 different valleys and 10 different depths. The probability of opening the lock on any one attempt is smaller than for the combination lock. This might be very expensive, however, since the mall would have to make a million keys or devise a computer program for a simulation.

c. Sample response: For a given lock, each person who tries has the same probability of opening the safe. Although it's possible that the first person will choose the right key or combination, it is highly unlikely because of the small probability for any combination.

Assignment (page 278)

1.1 **a.** There are $10 \bullet 10 \bullet 10 \bullet 10 = 10^4 = 10,000$ possible combinations.

b. The probability that a combination chosen at random will open the lock is $1/10,000$.

1.2 Answers will vary. Sample response: I would use the digital combination lock because the probability that a single combination will unlock it is $1/10,000$. If 500 people enter the contest each day, the probability that the safe will be cracked on any given day is about $1/20$. The probabilities of opening the other locks are too small and not practical if the contest is to have a winner in a reasonable period of time. **Note:** Since there is a small chance that more than one contestant in a day may select a particular combination, the actual probability in the sample response above is:

$$1 - \left(\frac{9,999}{10,000}\right)^{500} \approx 0.0488$$

***1.3** **a.** Sample response: A Title Identifier with 2 digits will allow the manager to publish 10^2 or 100 books.

b. Sample response: Since there are 8 digits shared between the Publisher and Title Identifiers, if 2 are designated for the Title Identifier, there are 6 available for the Publisher Identifier. Applying the fundamental counting principle, there are $10^6 = 1,000,000$ possibilities.

c. Sample response: A large publishing company that keeps many different books in print will want fewer digits for its Publisher Identifier so that it can have more digits for the Title Identifier. For example, with a two-digit Publisher Identifier, the company will have a million different possibilities for Title Identifier.

d. Sample response: Since there are 8 digits shared between the Publisher and Title Identifiers, if 3 are designated for the Publisher Identifier, there are 5 available for the Title Identifier. Applying the fundamental counting principle, there are $10^5 = 100,000$ possibilities.

***1.4** **a.** Sample response: The letters Q and Z are missing.

b. **1.** There are $24^3 = 13,824$ possible 3-letter prefixes.

2. There are $8^3 = 512$ possible 3-digit prefixes.

3. Sample response: There are fewer 3-digit prefixes because only 1 digit is assigned to each key. For example, on a telephone keypad, the prefixes ADG and ADH can be dialed using the same 3 digits.

c. **1.** If no letter may repeat, there are $24 \cdot 23 \cdot 22 = 12{,}144$ possible 3-letter prefixes.

2. $24 \cdot 23 \cdot 22 = \dfrac{24!}{21!}$

* * * * *

1.5 **a.** There are $26^3 = 17{,}576$ possible 3-letter prefixes.

b. If no letter may repeat, there are $26 \cdot 25 \cdot 24 = 15{,}600$ possible 3-letter prefixes.

1.6 **a.** There are $6^3 = 216$ possible arrangements.

b. The probability is $6/216 \approx 0.28$.

1.7 **a.** The number of possible license plates is $26^3 \cdot 10^3 = 17{,}576{,}000$.

b. The number of possible license plates is $10^4 \cdot 26^2 = 6{,}760{,}000$.

c. Sample response: Since the option described in Part **a** allows for more growth, it would be the better choice. **Note:** In 1994, the population of Alaska was about 606,000.

d. Sample response: The population of California in 1994 was about 31.4 million. Even if only half of these people owned cars, neither of the options would be acceptable.

* * * * * * * * * *

(page 281)

Activity 2

Students explore counting situations in which repetition is not allowed. The formula for permutations is developed using the fundamental counting principle.

Materials List

- none

Teacher Note

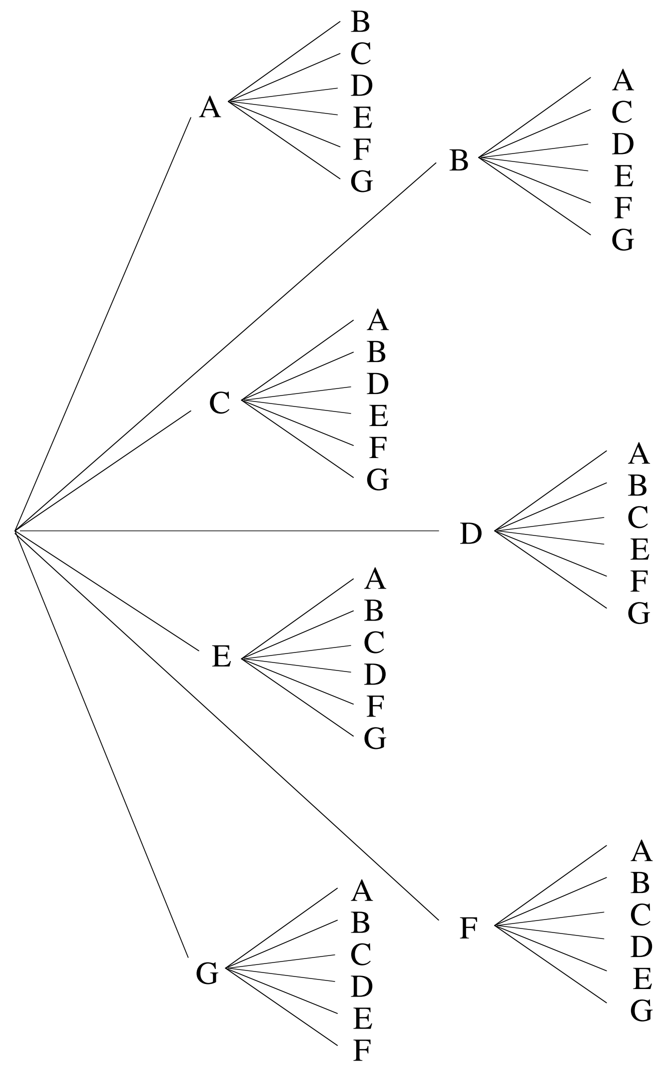
You may wish to wait until the assignment to introduce students to the ${}_nP_r$ button on their calculators. Students may better appreciate the capabilities of technology (and the corresponding formulas) after they have worked with large numbers.

Exploration

(page 281)

In this exploration, students use the fundamental counting principle to determine the number of ways in which distinct events can occur.

a-b. There are $7 \bullet 6 = 42$ arrangements, as shown in the tree diagram below.



c. There are $5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 5! = 120$ arrangements.

d. There are $(7 \bullet 6) \bullet (5 \bullet 4 \bullet 3 \bullet 2 \bullet 1) = 42 \bullet 120 = 5040$ arrangements.

e. **1.** There are $7! = 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 5040$ arrangements.

2. The number of arrangements is the same.

f. Sample response:

$$7 \bullet 6 = \frac{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}$$

g. Sample response:

$$\frac{n(n-1)(n-2)\bullet\cdots\bullet 3\bullet 2\bullet 1}{(n-r)(n-r-1)(n-r-2)\bullet\cdots\bullet 3\bullet 2\bullet 1}$$

Teacher Note

You may wish to point out that the process described in Parts **a–d** of the exploration leads to the general equation $P(n,r)\bullet(n-r)! = n!$, which implies that

$$P(n,r) = \frac{n!}{(n-r)!}$$

the formula for the number of permutations of *n* things taken *r* at a time.

Discussion

(page 282)

a. $n\bullet(n-1)\bullet(n-2)\bullet\cdots\bullet 3\bullet 2\bullet 1 = n!$

b. $(n-r)(n-r-1)(n-r-2)\bullet\cdots\bullet 3\bullet 2\bullet 1 = (n-r)!$

c. $n!/(n-r)!$

d. Sample response: The ratio $n!/(n-r)!$ can be written as follows:

$$\frac{n(n-1)(n-2)\bullet\cdots\bullet 3\bullet 2\bullet 1}{(n-r)(n-r-1)(n-r-2)\bullet\cdots\bullet 3\bullet 2\bullet 1}$$

Since $n > r$, there is some point where $(n-r)$ is one of the factors in $n!$ Therefore, $n!$ can be expressed as shown below:

$$n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\bullet 2\bullet 1$$

Canceling the common factors leads to the following equation:

$$\frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

e. Sample response:

$$P(n,r) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Therefore, 0! must be defined as 1 if the formula is to hold true when $r = n$.

Assignment

(page 283)

2.1 **a.** Sample response: Yes, because a batting order is an ordered arrangement of players' names.

b. 10! or 3,628,800

c. $P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720$

2.2. **a.** Sample response: The arrangement of songs on a tape is a permutation because the order is important.

b. $P(11,6) = \frac{11!}{(11-6)!} = \frac{11!}{5!} = 332,640$

c. $P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$

d. $P(11,11) = \frac{11!}{(11-11)!} = \frac{11!}{0!} = \frac{11!}{1} = 11! = 39,916,800$

e. They are the same: $332,640 \bullet 120 = 39,916,800$.

f. $P(11,6) = \frac{P(11,11)}{P(5,5)}$

2.3 **a.** Because the order in which a ticket is drawn determines the prize, this is a permutation.

$$P(400,3) = \frac{400!}{(400-3)!} = \frac{400!}{397!} = 63,520,800$$

b. Since two prizes have been given, only 398 tickets remain. The probability is $1/398$ or about 0.0025.

***2.4** **a.** $3! = 6$ ways

b. $3! = 6$ ways

c. $6 \bullet 6 = 36$ ways

d. $6! = 720$ ways

e. Sample response: Because there are only 36 ways out of 720 that the incumbents' names can appear in the first three positions on the ballot, the challenger's complaint is probably valid. The probability that such an arrangement will occur by chance is $36/720 = 0.05$.

***2.5** **a.** $40 \bullet 40 \bullet 40 = 64{,}000$

b. $P(40,3) = \frac{40!}{(40-3)!} = \frac{40!}{37!} = 59{,}280$

c. $\frac{64{,}000 - 59{,}280}{64{,}000} \approx 7.4\%$

d. Sample response: This is a change of $4720/64{,}000 \approx 0.074$, or about 7.4%. The probability of guessing the correct combination increases from $1/64{,}000$ to $1/59{,}280$.

* * * * *

2.6 $P(12,8) = \frac{12!}{(12-8)!} = \frac{12!}{4!} = 19{,}958{,}400$

2.7 **a.** $P(14,10) = \frac{14!}{(14-10)!} = \frac{14!}{4!} = 3{,}632{,}428{,}800$

b. If Jean must play catcher, there are 13 people left to fill 9 positions:

$$P(13,9) = \frac{13!}{(13-9)!} = \frac{13!}{4!} = 259{,}459{,}200$$

c. $1/14$ or about 0.071

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Research Project

(page 285)

Students may prepare their reports in written form or as class presentations. Your local Social Security office should be a helpful source of information (ask for "Your Social Security Number," SSA Publication 05-10002). **Note:** Social security numbers are not assigned entirely at random. For example, the first three digits generally indicate the state of residence at the time the person applies.

(page 285)

Activity 3

Students explore counting situations in which repetition is not allowed and order does not matter. The exploration leads to the development of the formula for combinations without repetition.

Materials List

- template A (optional; a blackline master appears at the end of the teacher edition for this module)

Exploration

(page 286)

a. The permutation of 5 distinct things taken 3 at a time is $P(5,3) = 60$.

b. **1.** The 6 that include L, O, and P are: LOP, LPO, OLP, OPL, PLO, and POL.

2. In permutation notation, $6 = P(3,3)$.

c. You may wish to distribute copies of template A to help students count the 10 different committees. They are: LMN, LMO, LMP, NLO, OMN, LNP, MPN, OMP, LOP, OPN. **Note:** Each column in the table below represents a different committee of three.

LMN	LMO	LMP	NLO	OMN	LNP	MPN	OMP	LOP	OPN
MLN	MLO	MLP	OLN	MON	PLN	PMN	PMO	OPL	NOP
NLM	OLM	PML	LNO	NOM	LPN	NPM	OPM	POL	PON
LNM	LOM	PLM	LON	ONM	NPL	PNM	POM	PLO	NPO
MNL	MOL	LPM	NOL	MNO	PNL	MNP	MOP	LPO	ONP
NML	OML	MPL	ONL	NMO	NLP	NMP	MPO	OLP	PNO

d. Sample response: The number of permutations of 5 letters chosen 3 at a time is the product of the number of committees (10) and the number of permutations of 3 letters chosen 3 at a time:

$$P(5,3) = 10 \bullet P(3,3)$$

e. Sample response:

$$P(5,3) = C(5,3) \bullet P(3,3)$$

f. $C(5,3) = \dfrac{P(5,3)}{P(3,3)}$

Discussion

(page 287)

a. Sample response: When spaces were assigned to the stores, the order of the assignments made a difference in the arrangement of stores. In a committee, the order doesn't matter because all members are considered equal.

b. Sample response: Using a permutation would allow LOP and PLO to be different committees when, in fact, they are the same. In a combination, the duplicates are removed.

c. Sample response: The permutation of 3 things taken 3 at a time represents the different ways you can choose the same 3 people for a 3-member committee. The letters of each committee could be arranged in a total of 6 ways.

d. Sample response: In a combination for a lock, the order of the numbers is important and repetition is allowed. Neither of these facts fits the definition of a mathematical combination.

e. Sample response: By dividing both sides of the equation $P(n,r) = C(n,r) \bullet P(r,r)$ by $P(r,r)$, it follows that

$$C(n,r) = \frac{P(n,r)}{P(r,r)}$$

Assignment

(page 288)

3.1 The number of committees of 2 possible from a group of 4 is $C(4,2) = 6$. Using the letters given, these are HI, HJ, HK, IJ, IK, and JK.

3.2 In the formula for a combination, $r!$ represents the number of ways r items can be ordered in groups of r.

3.3
- **a.** Answers will vary. Students may argue for groups of any size.
- **b.** There are 66 combinations of 2 employees chosen from a group of 12. James and Alice can expect to work together about every 2 months.

3.4 Sample response: Order is not important for the 4 crayons and there is no repetition, so this is a combination. There are $C(12,4)$ or 495 different assortments possible. It would not be reasonable to produce that many different kits. Toddler's Haven should create the kits as needed.

***3.5**
- **a.** Sample response: The volume of the mall is

  $$5000 \text{ cm} \bullet 12{,}000 \text{ cm} \bullet 500 \text{ cm} = 3 \bullet 10^{10} \text{ cm}^3$$

  This means that it contains approximately

  $$3 \bullet 10^{10} \text{ cm}^3 \left(\frac{3 \bullet 10^{19} \text{ molecules}}{1 \text{ cm}^3} \right) = 9 \bullet 10^{29} \text{ molecules of air}$$

- **b.** This problem asks "What is n such that $C(n,30) \geq 9 \bullet 10^{29}$?" Through trial and error, students should find $n \geq 135$. The minimum number of cards in the deck is 135.

***3.6** Responses will vary.
- **a.** There are $3 \bullet 3 = 9$ choices considering crust and size only.
- **b.** The number of choices of topping combinations can be found as follows: $C(10,0) + C(10,1) + C(10,2) + \cdots + C(10,10) = 1024$.
- **c.** There are $9 \bullet 1024 = 9216$ different choices.

***3.7** **a.** The 10th row of Pascal's triangle is:

1 10 45 120 210 252 210 120 45 10 1

b. The sum of the terms in the 10th row of Pascal's triangle is:

$1+10+45+120+210+252+210+120+45+10+1=1024$

c. This is the same result as obtained in Problem **3.6b**.

d. The numbers of combinations possible for the numbers of toppings are equivalent to the individual terms in row 10 of Pascal's triangle. In other words, $C(10,0)=1$, $C(10,1)=10$, $C(10,2)=45$, and so on.

e. Using ${}_nC_r$ notation, the 10th row of Pascal's triangle is:

${}_{10}C_0 \quad {}_{10}C_1 \quad {}_{10}C_2 \quad {}_{10}C_3 \quad {}_{10}C_4 \quad {}_{10}C_5 \quad {}_{10}C_6 \quad {}_{10}C_7 \quad {}_{10}C_8 \quad {}_{10}C_9 \quad {}_{10}C_{10}$

f. Using ${}_nC_r$ notation, the first five rows are:

$$
\begin{array}{c}
{}_0C_0 \\
{}_1C_0 \quad {}_1C_1 \\
{}_2C_0 \quad {}_2C_1 \quad {}_2C_2 \\
{}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 \\
{}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4
\end{array}
$$

3.8 **a.** The sums of the terms in the first five rows of Pascal's triangle are:

Row 0: $1=2^0$

Row 1: $2=2^1$

Row 2: $4=2^2$

Row 3: $8=2^3$

Row 4: $16=2^4$

One possible formula is $2^{(\text{row number})}=\text{sum of elements in row}$.
Note: Although it is possible to prove that

$$\sum_{i=0}^{n} C(n,i)=2^n$$

students are not expected to provide such a proof.

b. Using the sample formula given above $2^{10}=1024$.

c. Using the sample formula above, 1024 represents the total number of pizzas that can be made using 10 toppings, 10 represents the number of available toppings, and 2 represents the choices for each topping (either put it on the pizza, or do not put it on the pizza).

d. Sample response: When buying a pizza with a choice of 10 toppings, each topping can either be chosen or not chosen. Thus, the problem can be modeled with the fundamental counting principle using 10 boxes to represent the 10 toppings. Each box contains a 2, representing the choices of whether or not the pizza has that topping. In other words, there are 10 factors of 2, or $2^{10} = 1024$ different pizzas.

* * * * *

3.9 **a.** The number of distinct subsets of a set with *n* elements is 2^n.

b. The number of distinct subsets of a 4-element set is 16.

c. The subsets of R are: { }, {1}, {3}, {8}, {11}, {1, 3}, {1, 8}, {1, 11}, {3, 8}, {3, 11}, {8, 11}, {1, 3, 8}, {1, 3, 11}, {1, 8, 11}, {3, 8, 11}, and {1, 3, 8, 11}.

d. $C(4,0) + C(4,1) + C(4,2) + C(4,3) + C(4,4) = 1 + 4 + 6 + 4 + 1 = 16$

3.10 The number of triangles is $C(8,3) = 56$.

3.11 A committee of 5 can be selected from a group of 20 in $C(20,5) = 15{,}504$ ways.

3.13 The number of 5-card hands in a 52-card deck is $C(52,5) = 2{,}598{,}960$

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Answers to Summary Assessment (page 291)

1. **a.** The number of different arrangements is 15! (over 1 trillion).

   b. This number is larger than many calculators will display or compute: $(13 \bullet 2 + 16 \bullet 3)! = 74! \approx 3.3 \bullet 10^{107}$.

2. **a.** Since there are 26 letters in the alphabet, it would require $26^2 = 676$ different mugs.

   b. Sample response: No, because using middle initials increases the number of possibilities to $26^3 = 17,576$. That is too many mugs for most stores to stock.

3. **a.** The number of ways is $C(800, 20) \approx 3.8 \bullet 10^{39}$.

   b. Since there are 780 entries remaining, this second group can be selected in $C(780, 30) \approx 1.2 \bullet 10^{54}$ ways.

   c. Since there are 750 entries remaining, the grand prize winner can be selected in $C(750, 1) \approx 750$ ways.

   d. The probability that any single remaining entry will win is $1/750$.

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Module Assessment

1. **a.** In the Apple Lottery game, players must choose two different integers from 1 to 6, inclusive. To select the winning numbers, Apple Lottery officials draw two different integers from 1 to 6 at random. In this game, an ordered pair can be used to represent each way that a player can select two different integers. For example, three possible ordered pairs are (1,2), (2,1) and (5,6). How many ordered pairs would it take to represent all the possibilities?

b. In another version of the Apple Lottery, players can select any pair of integers from 1 to 6. In other words, the two integers need not be different. How many ordered pairs would it take to represent all the possibilities in this version of the game?

c. When determining a winning ticket, the order in which integers are picked in the Apple Lottery is not important. How many different ways are there to choose 2 different integers from 6?

2. **a.** In the Apple Lottery described in Problem **1c**, a player who matches neither integer wins a red apple. How many ways are there to win a red apple?

b. A player who matches both integers wins a yellow apple. How many ways are there to win a yellow apple?

c. A player who matches one of the two integers wins a green apple. How many ways are there to win a green apple?

3. In one state lottery, players select 4 of 24 available integers. For example, two possible selections of 4 integers could be 9, 19, 6, 11 and 6, 11, 9, 19. Depending on the rules of the game, these two arrangements could be counted as different choices or as the same choice.

a. How many different arrangements of four integers would include the numbers 6, 9, 11, and 19?

b. To win the jackpot, a player must match all 4 integers drawn by the lottery. Show why counting all the arrangements in Part **a** as the same selection would be advantageous to players.

Answers to Module Assessment

1. Students may recall the Apple Lottery from the Level 1 module "I'm Not So Sure Anymore."

- **a.** Because order is important and there is no repetition, the possibilities can be counted using the formula for permutations without repetition. There are $P(6,2) = 30$ ordered pairs.
- **b.** Because order is important and repetition is allowed, the number of possibilities is $6 \bullet 6 = 36$.
- **c.** Because order is not important and repetition is not allowed, the number of possibilities can be counted as follows: $C(6,2) = 15$.

2.

- **a.** In order for a player to win a red apple, the Lottery's numbers must be drawn from the four non-matching numbers. There are $C(4,2) = 6$ ways to win a red apple.
- **b.** In order for a player to win a yellow apple, the Lottery's numbers must be drawn from the two matching numbers. There is $C(2,2) = 1$ way to win a yellow apple.
- **c.** In order for a player to win a green apple, the Lottery must draw a number that matches one of the player's two numbers and a second number that does not match either of the player's two numbers. There are $C(2,1) \bullet C(4,1) = 2 \bullet 4 = 8$ ways to win a green apple.

3.

- **a.** If order is important, there are $P(4,4) = 24$ possible arrangements of four numbers.
- **b.** If order is important, there are $P(24, 4) = 255,024$ different choices in the lottery.

  If order is not important, there are $C(24,4) = 10,626$ different choices. A player's chances of winning with any one ticket improve by a factor of $P(4,4) = 24$ if order is not important.

Flashbacks

Activity 1

1.1 Consider an experiment that consists of flipping a fair coin and tossing an ordinary die. Construct a tree diagram that shows all the possible outcomes.

1.2 The menu at a local restaurant features 5 beverages, 12 main courses, and 7 desserts. In how many ways can you select a meal with one beverage, one main course, and one dessert?

1.3 An urn contains 15 marbles. Of these marbles, 2 are yellow, 5 are green, and 8 are purple.

a. What is the probability of randomly selecting a yellow marble?

b. What is the probability of randomly selecting a purple marble?

Activity 2

2.1 Evaluate 5!

2.2 If 9! is 362,880, find 10!

2.3 Evaluate $82!/81!$

2.4 Is the following equation true?

$$(3!)(6!) = 9!$$

Justify your response.

Activity 3

3.1. The first three rows of Pascal's triangle are shown below:

1

1 2 1

1 3 3 1

a. List the elements of the next row.

b. Describe how the elements in the triangle are determined.

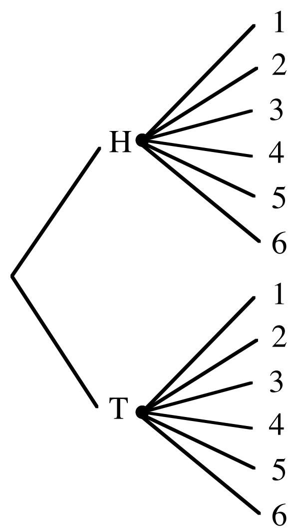
3.2 Evaluate $P(9, 7)$.

3.3 If each person can serve in only one position, in how many ways can a president, vice president, and secretary be selected from a group of 8 candidates?

Answers to Flashbacks

Activity 1

1.1 Sample tree diagram:



1.2 The meal can be selected in $5 \bullet 12 \bullet 7 = 420$ ways.

1.3 **a.** The probability of selecting a yellow marble is $2/15$.

b. The probability of selecting a purple marble is $8/15$.

Activity 2

2.1 $5! = 120$

2.2 $10! = 10 \bullet 9! = 10 \bullet 362,880 = 3,628,800$

2.3 $82!/81! = 82$

2.4 The expressions are not equal. If $3! \bullet 6! = 9!$, then $3! = 9!/6! = 9 \bullet 8 \bullet 7$. Thus $3 \bullet 2 \bullet 1 = 9 \bullet 8 \bullet 7$. Since this is not true, the assumption is false.

Activity 3

3.1 **a.** The elements of the next row are 1, 4, 6, 4, and 1.

b. Sample response: For each successive row, you begin with 1. Each of the next elements is the sum of the two diagonally above. The last element in each row also is 1.

3.2 The permutation of 9 distinct items taken 7 at a time is 181,440.

3.3 The officers can be selected in $P(8,3) = 336$ ways.

Template A

LMN	LMO	LMP	LNM	LNO	LNP
MLN	MLO	MLP	MNL	MNO	MNP
NLM	NLO	NLP	NML	NMO	NMP
OLM	OLN	OLP	OML	OMN	OMP
PLM	PLN	PLO	PML	PMN	PMO
LOM	LON	LOP	LPM	LPN	LPO
MOL	MON	MOP	MPL	MPN	MPO
NOL	NOM	NOP	NPL	NPM	NPO
ONL	ONM	ONP	OPL	OPM	OPN
PNL	PNM	PNO	POL	POM	PON

It's All in the Family

When trying to fit a model to a set of data, it can help you to know the characteristics of different types of mathematical functions. In this module, you examine and compare the graphs of several families of functions.

Masha Albrecht • Gary Bauer • Art Perleberg

Teacher Edition

It's All in the Family

Overview

In this module, students examine families from the following types of functions: exponential, logarithmic, rational, and periodic. Students discover how each family of functions is related to a parent function and analyze these relationships as transformations. They categorize functions, develop a graphing tool kit for some general functions, and use this knowledge to interpret data sets.

Objectives

In this module, students will:

- recognize parent functions for selected exponential, logarithmic, rational, and periodic functions
- develop skills for transforming functions
- categorize functions into families
- find equations to model data sets.

Prerequisites

For this module, students should know:

- how to graph and recognize the following types of functions: exponential, logarithmic, rational, and periodic
- how to transform simple functions
- how to model data using curve-fitting techniques.

Time Line

Activity	Intro.	1	2	3	Summary Assessment	Total
Days	1	2	3	2	1	9

Materials Required

Materials	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
graph paper		X			
cardboard or plastic disks	X				
chalk	X				
meterstick				X	
40-watt lamp				X	

Technology

Software	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
graphing utility		X	X	X	
science interface device				X	
light sensor				X	
symbolic manipulator		X	X	X	

It's All in the Family

Introduction
(page 297)

Students construct graphs of periodic functions by simulating the path of a pebble on a bicycle tire.

Materials List

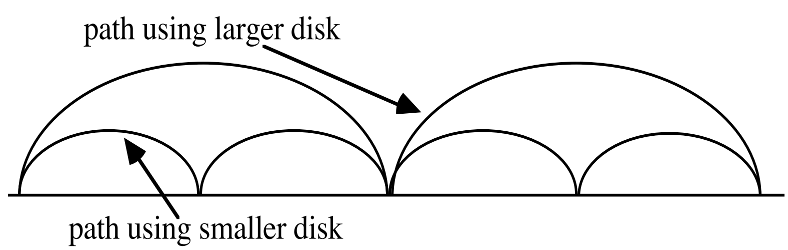
- chalk (one stick per group)
- disks cut from stiff cardboard or plastic lids (two per group)

Teacher Note

The function examined in the exploration is a cycloid. Students are not expected to name this function—only to make observations and answer questions about it. The parametric equations for the graphs in Parts **c** and **d** are: $x = a(t - \sin t)$ and $y = a(1 - \cos t)$, where a is the radius of the disk.

Exploration
(page 297)

a–d. Students simulate the path of a pebble stuck in the tread of a tire using two different disks. The paths should resemble cycloids. Sample sketches:



Discussion
(page 298)

a. Each of the sketched graphs should represent a function, whose domain is the set of distances the tires have rolled and whose range is the set of locations of the pebble.

b. The period of each graph is the circumference of the disk. The amplitude of each graph equals the radius of the disk.

c. Sample response: The graphs are periodic, appear to involve arcs, and have finite height or amplitude. The domain is the set of real numbers; the range is the set of non-negative real numbers.

d. Sample response: It depends on the characteristics that define the class. Since the sine and cosine functions are also periodic, they might belong in the same grouping.

(page 299)

Activity 1

Students determine how members of a family of functions are related to a parent function through transformations. **Note:** Any member of a family can be treated as a parent function. In this module, students use the simplest form of a set of related functions as the parent.

Materials List

- graph paper

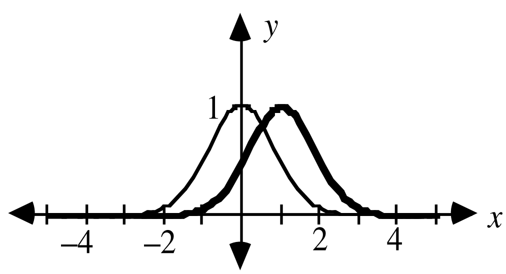
Technology List

- graphing utility
- symbolic manipulator (optional)

Exploration

(page 299)

a–c. Sample graph:



d. The translated graph is represented by equation **3**: $y = 2^{-(x-1)^2}$.

e. $y = 2^{-(x+2)^2}$

f.
1. Sample response: The graph of the second function is a horizontal shrink of the parent function.
2. Sample response: The graph of the second function is a vertical stretch of the parent function.

g.
1. Sample response: If *d* is positive, the graph is translated up *d* units. If *d* is negative, the graph is translated down *d* units.
2. Sample response: If *c* is positive, the graph is translated to the right *c* units. If *c* is negative, the graph is translated to the left *c* units.

3. Sample response: If $|b| < 1$, the graph is a horizontal stretch of the parent function. If $|b| > 1$, the graph is a horizontal shrink of the parent. The sign of *b* does not influence the graph because of squaring.

4. Sample response: If $|a| > 1$, the graph is a vertical stretch of the parent function. If $|a| < 1$, the graph is a vertical shrink of the parent function. If *a* is negative, the graph is reflected in the *x*-axis.

Discussion

(page 300)

a.

1. The domain is the set of real numbers, $(-\infty,\infty)$; the range is $[0,\infty)$.
2. Sample response: Add 2 to the right-hand side of the equation.
3. Sample response: The domain stays the same, while the range changes to $[2,\infty)$.

b.

1. Sample response: Subtract 2 from *x*.
2. Sample response: Neither the domain nor the range are affected.

c.

1. The graph is a reflection in the *x*-axis.
2. The domain does not change; the range is $(-\infty,0]$.

d.

1. Sample response: Multiply the right-hand side of the equation by a number with an absolute value greater than 1.
2. Sample response: If the right-hand side of the equation is multiplied by 3, for example, the domain and range do not change.

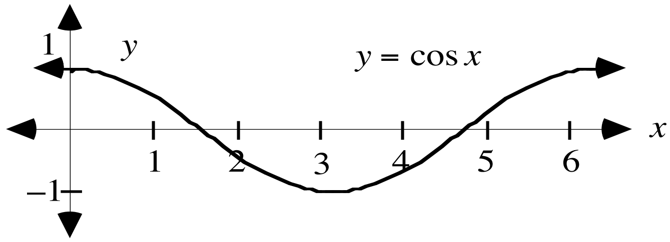
e.

1. Sample response: Multiply the *x*-value of the equation by a number with an absolute value less than 1 but greater than 0.
2. Sample response: If the *x*-value is multiplied by 0.5, for example, then neither the domain nor the range are affected.

f. Sample response: Using the function $f(x) = x^2$ where $b = 2$, the ordered pair (4,16) corresponds to the ordered pair (8,16) on the new function. Because $4 = \frac{1}{2} \bullet 8$, the new function is $g(x) = f(0.5x) = (0.5x)^2$.

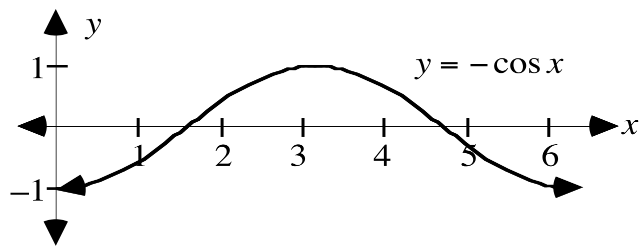
Assignment (page 302)

1.1 **a.** **1.** Sample graph:



2. The domain is the set of real numbers; the range is [–1, 1].

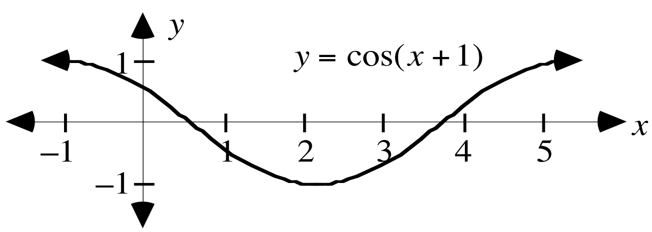
b. **1.** Sample graph:



2. The domain is the set of real numbers; the range is [–1, 1].

3. $y = -\cos x$

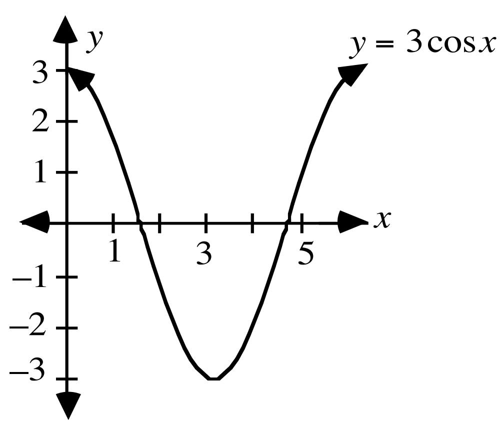
c. **1.** Sample graph:



2. The domain is the set of real numbers; the range is [–1, 1].

3. $y = \cos(x+1)$

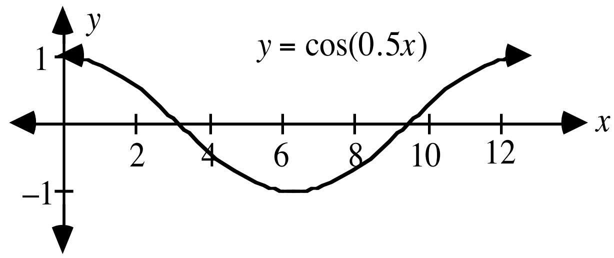
d. **1.** Sample graph:



2. The domain is the set of real numbers; the range is [–3, 3].

3. $y = 3\cos x$

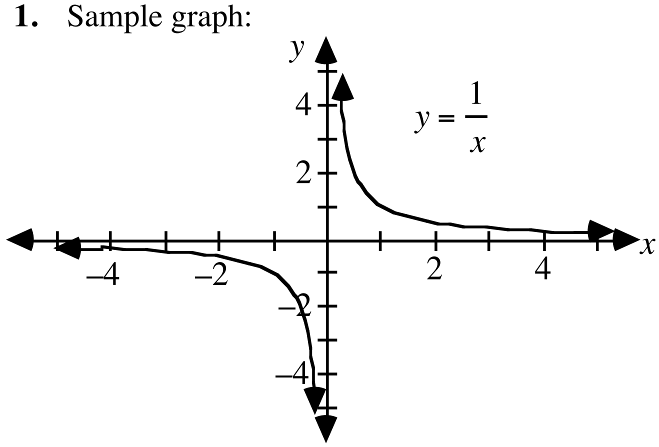
e. 1. Sample graph:



2. The domain is the set of real numbers; the range is [–1, 1].

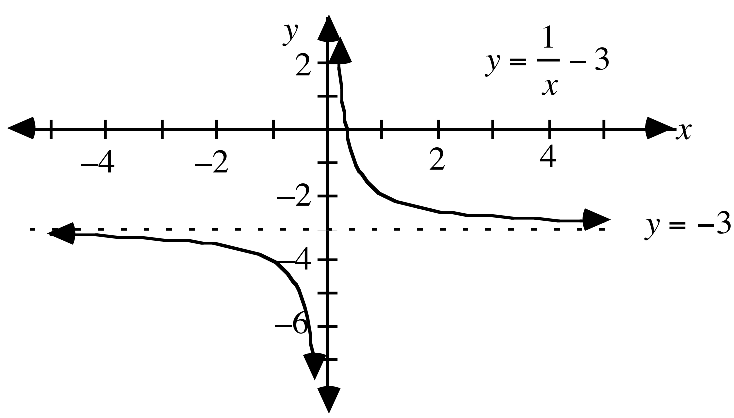
3. $y = \cos(0.5x)$

1.2 a. 1. Sample graph:



2. The range is the set of all nonzero real numbers, $(-\infty, 0) \cup (0, \infty)$.

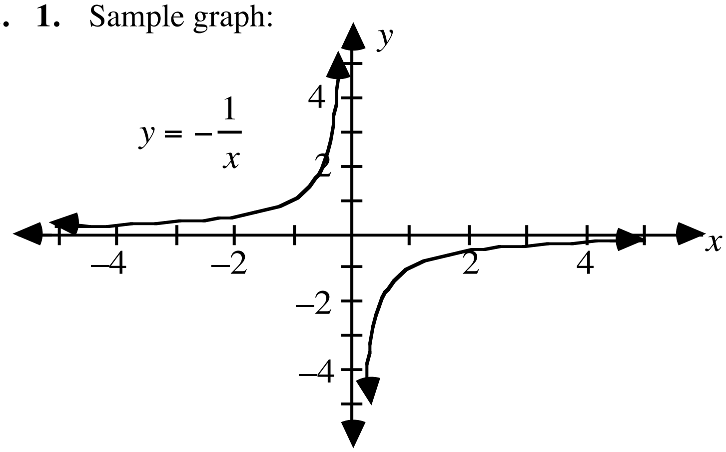
b. 1. Sample graph:



2. The domain is the set of all nonzero real numbers; the range is the set of all real numbers not equal to –3, $(-\infty, -3) \cup (-3, \infty)$.

3. $y = \frac{1}{x} - 3$

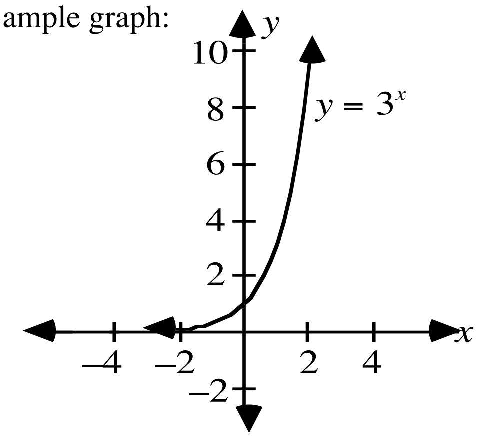
c. **1.** Sample graph:



2. The domain is the set of all nonzero real numbers; the range is the set of all nonzero real numbers.

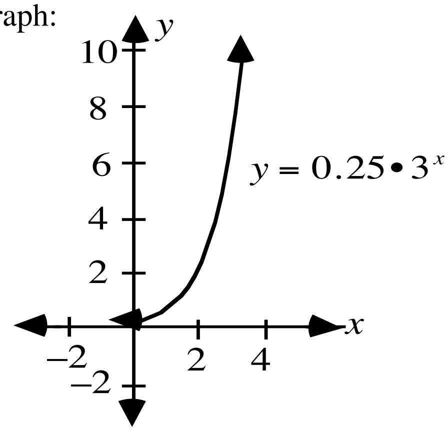
3. $y = -1/x$

1.3 **a.** **1.** Sample graph:



2. The range is the set of positive real numbers, $(0, \infty)$.

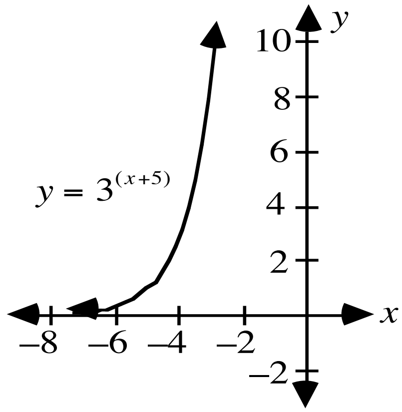
b. **1.** Sample graph:



2. The domain is the set of real numbers; the range is the set of positive real numbers.

3. $y = 0.25 \bullet 3^x$

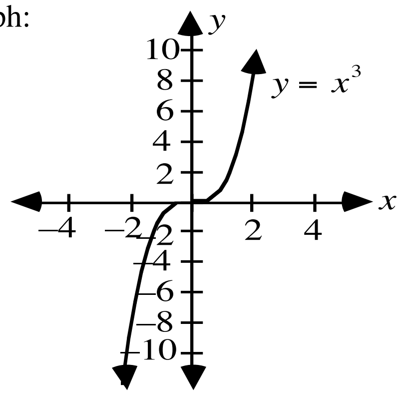
c. **1.** Sample graph:



2. The domain is the set of real numbers; the range is the set of positive real numbers.

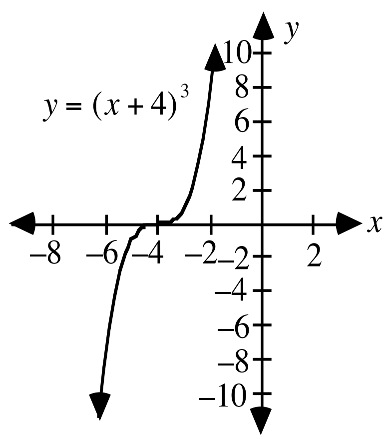
3. $y = 3^{(x+5)}$

***1.4** **a.** **1.** Sample graph:



2. The range is the set of real numbers.

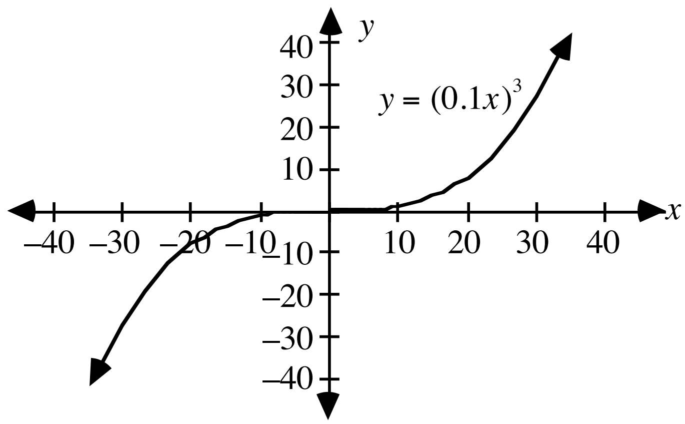
b. **1.** Sample graph:



2. The domain and the range are the set of real numbers.

3. $y = (x + 4)^3$

c. 1. Sample graph:



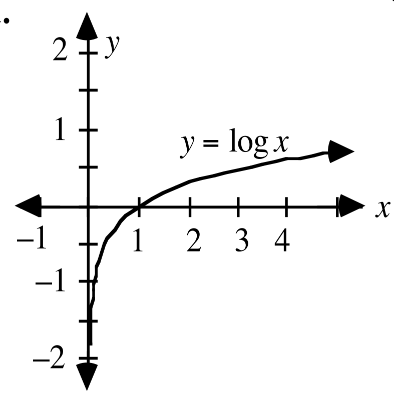
2. The domain and the range are the set of real numbers.

3. $y = (0.1x)^3$

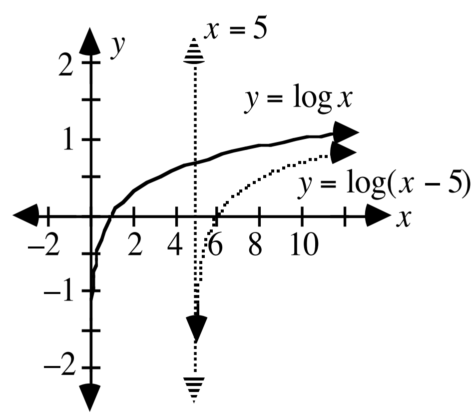
* * * * *

1.5 Sample graphs:

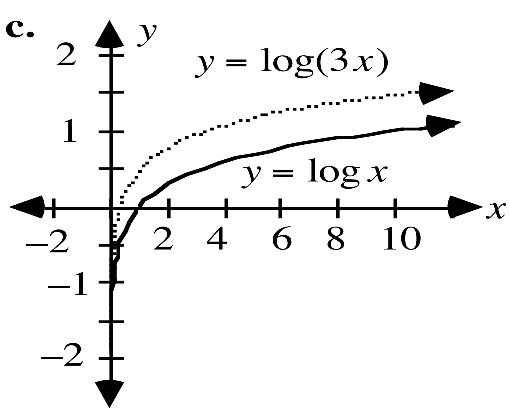
a.



b.

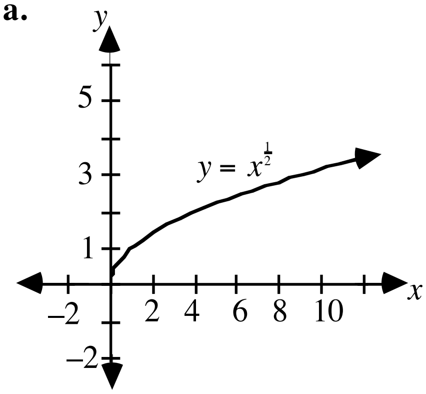


c.

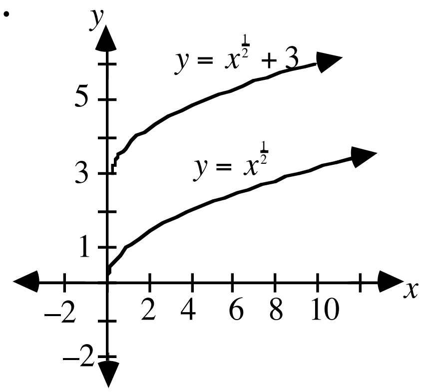


1.6 Sample graphs:

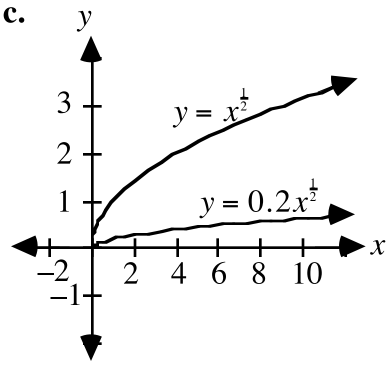
a.



b.

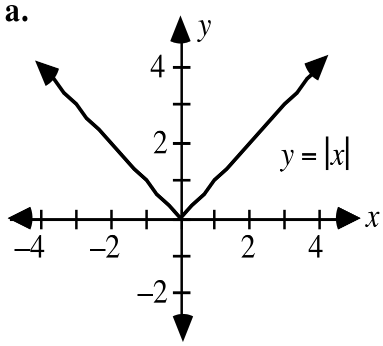


c.

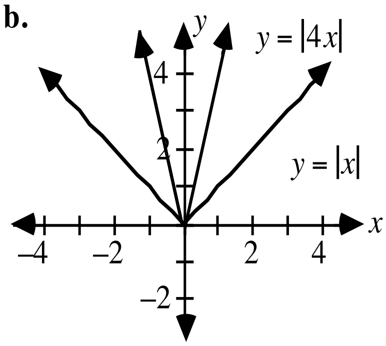


1.7 Sample graphs:

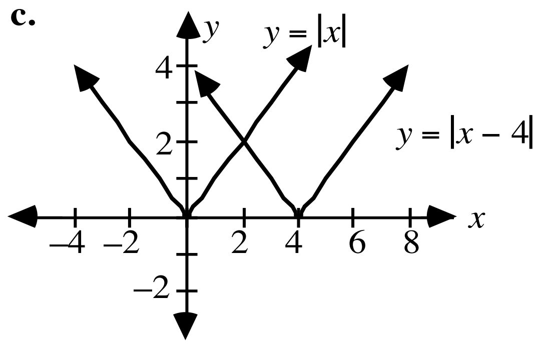
a.



b.



c.



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Research Project (page 304)

Circular functions such as $y = \sin x$ can be used to model sound waves. In fact, such functions are involved in the electronic synthesis of musical notes. Raising the pitch of a note produces a shorter period (lower frequency). A louder note corresponds to a higher amplitude. **Note:** Some students may wish to experiment with sound waves using a tuning fork and science interface device.

(page 304)

Activity 2

This activity extends the algebra of transformations to multiple transformations performed on a parent function to obtain a new family member.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator (optional)

Teacher Note

You may wish to have a group of students work with only one parent function, then compare results with a group that worked with the other function.

Depending on the software used, students may have to solve for *y* first in functions **2**, **4**, and **7**.

Exploration (page 304)

Students examine the transformations between a function and its parent with the aid of a graphing utility. They may record their observations in a table like the one below.

	Parent $y = 1/x$	Parent $y = x^2$	Transformations
1.	$y = \dfrac{1}{x-3}$	$y = (x-3)^2$	Translated 3 units horizontally.
2.	$y - 2 = 1/x$	$y - 2 = x^2$	Translated 2 units vertically.
3.	$y = -(1/x)$	$y = -(x^2)$	Reflected in the x-axis
4.	$y - 4 = -\left(\dfrac{1}{x+1}\right)$	$y - 4 = -(x+1)^2$	Reflected in the x-axis; translated –1 unit horizontally; translated 4 units vertically.
5.	$y = \dfrac{1}{2x}$	$y = (2x)^2$	Shrunk horizontally by 0.5.
6.	$y = \dfrac{1}{(0.5x)}$	$y = (0.5x)^2$	Stretched horizontally by 2.
7.	$y + 1 = -\left(\dfrac{1}{3(x-2)}\right)$	$y + 1 = -\left(3(x-2)^2\right)$	Reflected in the x-axis; translated 2 units horizontally; translated –1 unit vertically; shrunk horizontally by $1/3$.

Discussion (page 305)

a. **1.** $y = f(x-3) - 5$

2. $y = -4f(0.5x)$

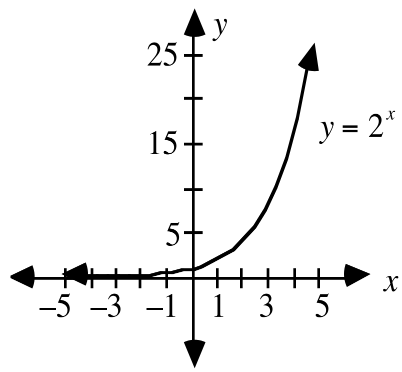
b. Sample response: Given $(y - d) = f(b(x - c))$, constant b is the reciprocal of the horizontal stretching or shrinking factor, constant c indicates a horizontal translation of c units, and constant d indicates a vertical translation of d units.

Assignment (page 305)

2.1 The graph is translated 2 units horizontally, 5 units vertically, and is stretched vertically by 3.

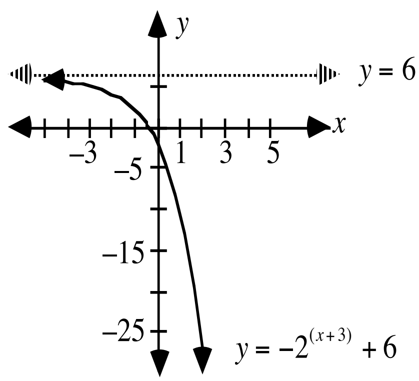
2.2 Students' responses will vary. The following sample responses use $y = 2^x$ as the parent function.

a. Sample graph:



b. Sample response: $y = -2^{(x+3)} + 6$

c. Sample graph:



***2.3** **a.** The parent is $y = \sin x$. The function represents a horizontal translation of $\pi/2$ units and a horizontal shrink by $1/3$.

b. The parent is $y = x^3$. The function represents a vertical translation of –90 units and a vertical stretch by 2.

c. The parent is $y = 10^x$. The function represents a horizontal translation of 8 units.

d. The parent is $y = 1/x^2$. The function represents a horizontal translation of –100 units, a reflection in the *x*-axis, and a vertical stretch by 100.

e. The parent is $y = \log x$. The function represents a reflection in the *y*-axis, a horizontal translation of 2.3 units, and a vertical translation of –4.7 units.

2.4 **a**. $y = \sin\left(x - \frac{\pi}{2}\right)$

b. $y = 2\sin\left(x - \frac{\pi}{2}\right)$

c. $y = \sin x + 1$

d. $y = \sin(2x) + 2$

2.5 **a**. The graphs in Problems **2.4a** and **b** show phase shifts of $\pi/2$ units to the right.

b. $y = 3\cos\left(x + \frac{\pi}{6}\right)$

2.6 **a.** This is an exponential function. One possible equation is $y = 2^x + 1$. Any base greater than 1 and less than 3 is acceptable.

b. This is a rational function. One possible equation is:

$$y = \frac{1}{x - 3} + 3$$

c. This is a polynomial function. One possible equation is $y = x(x + 2)(x - 3)$. **Note:** The sample response is based on the zeros of the function. Students may not recognize how the parent $y = x^3$ could be transformed to obtain this graph. You may wish to discuss why equations of this type do not easily lend themselves to the transformation model.

d. This is a circular function. Some possible equations are: $y = 3\cos x + 1$, $y = 3\sin\left(\frac{\pi}{2} - x\right) + 1$, and $y = 3\sin\left(x + \frac{\pi}{2}\right) + 1$.

* * * * *

2.7 Sample response: The function $y = x^2 - 3$ is a vertical translation of 3 units of $y = x^2$. The function $y = (x - 3)^2$ is a horizontal translation of 3 units of $y = x^2$. The function $y = (x - 2)^2 + 3$ is a horizontal translation of 2 units and a vertical translation of 3 units of $y = x^2$. The function $y = -x^2$ is a reflection of $y = x^2$ in the *x*-axis. The function $y = 3x^2$ is a vertical stretch of $y = x^2$.

2.8 **a.** $y = \log(x + 2)$

b. $y = \log(x) - 3$

c. $y = -\log x$

d. $y = \log(10x)$

2.9 **a–c.** The constant a stretches or shrinks the function vertically. If a is negative, it reflects the function in the x-axis. The constant b stretches or shrinks the function horizontally. The constant c translates the graph horizontally; the constant d translates the function vertically.

***2.10** Sample responses:

a. $y = 1/x^2$ or $y = \tan x$

b. $y = 10^x$ or $y = 0.5^x$

c. $y = x^2$ or $y = \sin x$

d. $y = \cos x$ or $y = \tan x$

e. $y = \tan x$ or $y = 1/x$

f. $y = x^2$ or $y = \sin x$

* * * * * * * * * *

(page 308)

Activity 3

Students use functions to model sets of real-world data. **Note:** Students are not expected to find regression equations.

Materials List

- meterstick
- 40-watt lamp

Technology

- light sensor
- science interface device
- graphing utility
- symbolic manipulator

Exploration

(page 309)

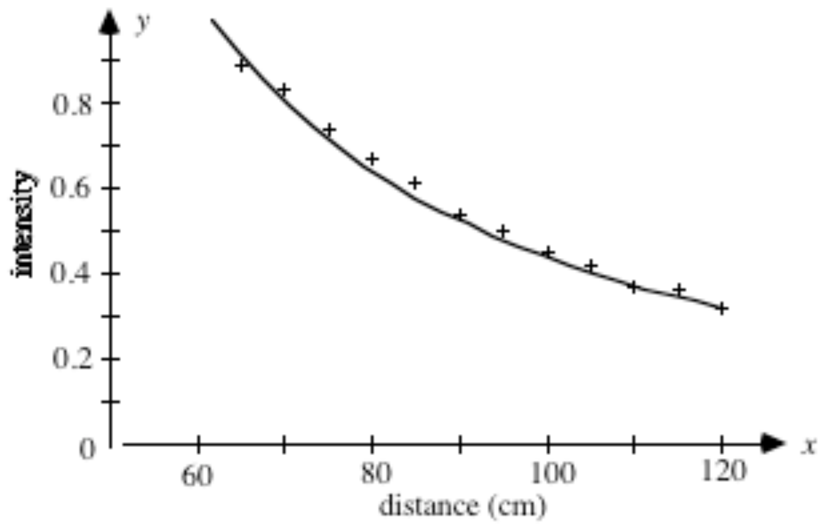
Students investigate the relationship between light intensity and distance. Since light intensity varies inversely as the square of the distance, a graph of the data should show roughly an inverse square relationship.

a–b. If students use more powerful bulbs, they should start collecting data farther from the light. The sample data below, for example, was collected using a 60-watt bulb beginning at a distance of 65 cm.

c. Sample data:

Distance from Light (cm)	Intensity ($watts/cm^2$)	Distance from Light (cm)	Intensity ($watts/cm^2$)
65	0.89	95	0.50
70	0.83	100	0.45
75	0.74	105	0.42
80	0.67	110	0.37
85	0.61	115	0.36
90	0.54	120	0.32

d. One model that fits the sample data well is the function $y = 1100x^{-1.7}$.
Sample graph:



Discussion

(page 310)

a. Sample response: The shape of the graph shows that light intensity and distance appear to be inversely proportional.

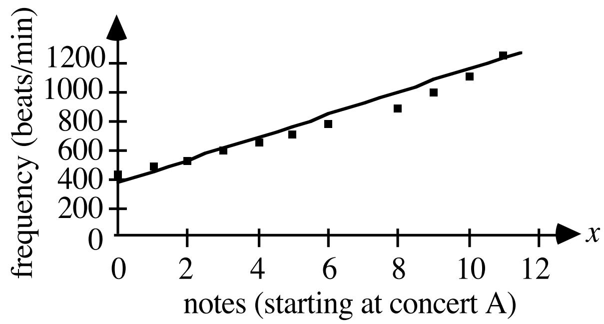
b. Students may suggest polynomial (quadratic), exponential, or logarithmic functions.

c. Sample response: The graph looks like a rational function, possibly $y = k/x$ or $y = k/x^2$. If you substitute data into $y = k/x^2$, it seems to be the better of the two models. As the distance increases, the light intensity will continue to decrease. The graph should have a horizontal asymptote. It also appears to have a vertical asymptote.

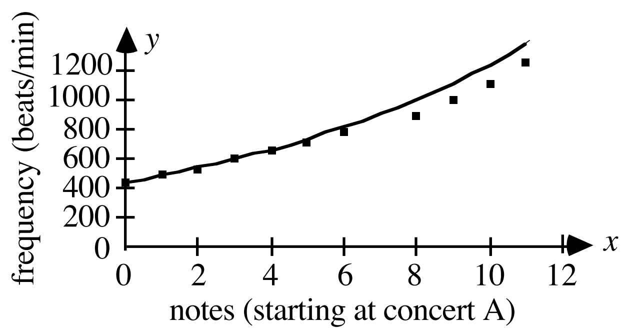
Assignment

(page 310)

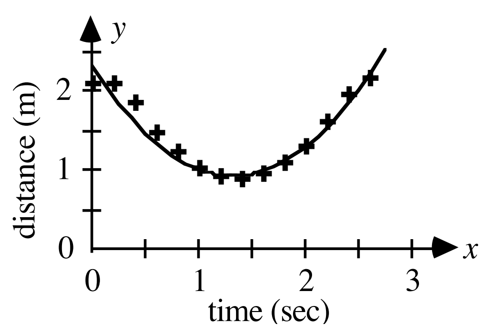
3.1. a–c. In the following sample graph, note 0 corresponds to concert A. The data is modeled with the linear function $y = 78.07x + 373.8$.



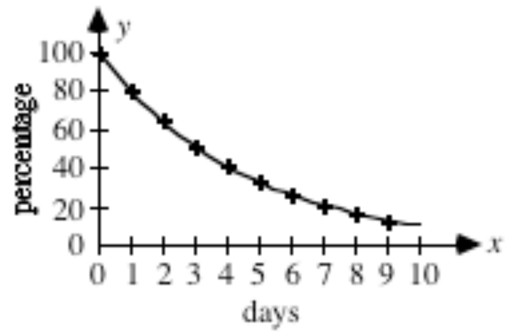
The data also can be modeled with an exponential function, such as $y = 432.99(1.11)^x$.



3.2 a–c. A quadratic equation, such as $y = 0.79(x - 1.33)^2 + 0.93$, is an appropriate model in this situation.

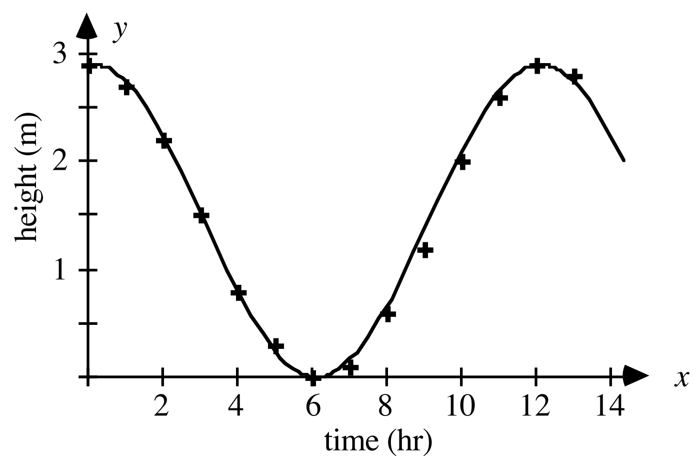


3.3 **a–c.** This data can be modeled by an exponential equation, such as $y = 100 \bullet 0.8^x$.



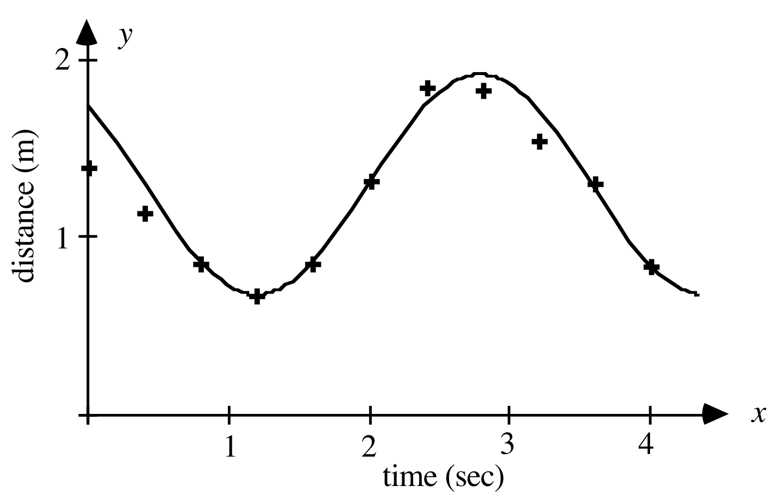
***3.4** **a–c.** The data can be modeled by a periodic function, such as

$$y = 1.45\left(\cos\left(\frac{\pi}{6}(x - 0.12)\right)\right) + 1.45$$

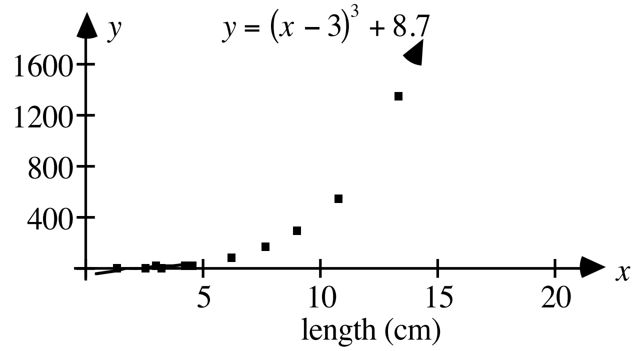


* * * * *

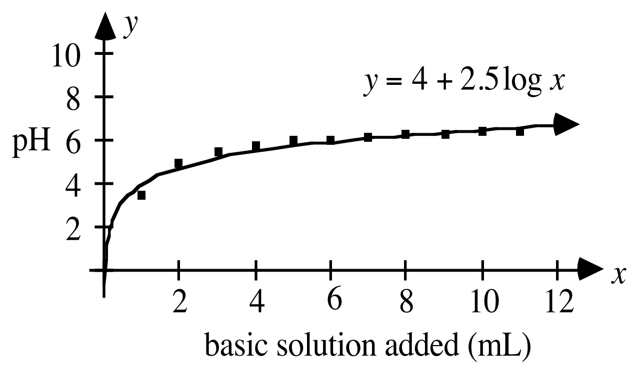
3.5 **a–c.** This data can be modeled using a periodic function, such as $y = -0.62\cos(1.98(x - 1.2)) + 1.3$.



3.6 **a–c.** Students may recall this context from the Level 1 module "Are You Just a Small Giant?" Since volume (and hence mass) should be approximately proportional to the cube of the length, a cubic function may provide an appropriate One possible cubic model is $y = (x-3)^3 + 8.7$



3.7 **a–c.** This data can be modeled using a logarithmic function, such as $y = 4 + 2.5 \log x$.



Note: Students should recall the logarithmic nature of the pH scale from the Level 4 module "Log Jam."

* * * * * * * * * *

Answers to Summary Assessment (page 314)

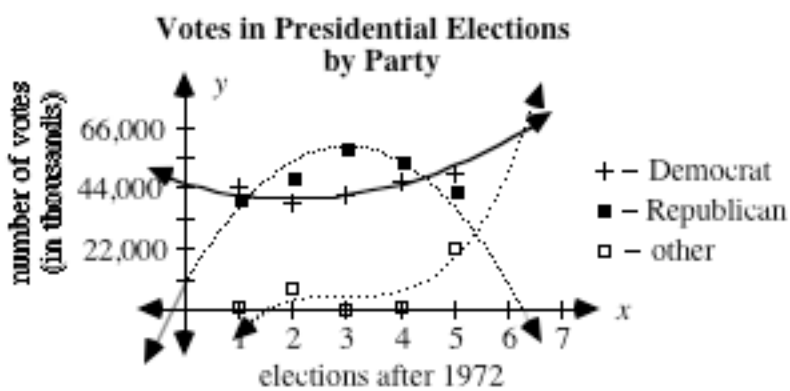
1. The cosine function can be obtained from the sine function by translating it $-\pi/2$ units horizontally. Therefore, the function

$$y = \sin\left(x + \frac{\pi}{2}\right)$$

is the same as $y = \cos x$.

2. **a–b.** **Note:** The equations given here are sample responses only. Many others may be used to approximate the data. Students should use their knowledge of transformations and the general shape of functions to create their models, not the regression capabilities of technology.

Sample response: The scatterplot for the Democrats resembles a parabola. One equation that fits the data is $y = 1200(x - 2)^2 + 36500$. The scatterplot for the Republicans resembles an inverted parabola. One equation that fits the data is $y = -500(x - 3)^2 + 54000$. The scatterplot for the other parties resembles the graph of a cubic equation. One equation that fits the data is $y = 1500(x - 3)^3 + 4500$. A graph of the scatterplots and models is shown below:



c. Sample response: Based on the models in Part **b**, the Democrats should win the next election by a narrow margin over the other party's candidate. The Republicans will finish a distant third.

According to the same models, the third party will easily win every election after the year 2000. The Democrats will continue to lose more and more ground and the Republican party will not receive any votes.

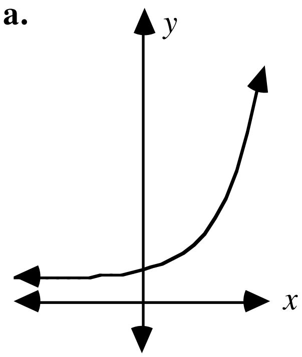
d. Sample response: As with any model, there are dangers in projecting beyond the collected data. The 1992 election was an extremely successful one for a third-party candidate, but this was an exception historically. This one exception had a great effect on the choice of the model. Political trends, however, tend to be somewhat cyclic and are based on many outside influences.

3. Answers will vary. The curves are described by the exponential functions $y = 0.1^x$ and $y = -0.1^x$. Students may use any similar pair of functions with a base between 0 and 1. Others may suggest the rational functions $y = 1/x$ and $y = -1/x$.

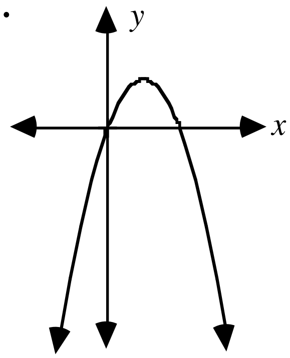
Module Assessment

1. Identify a possible parent function for each of the following graphs:

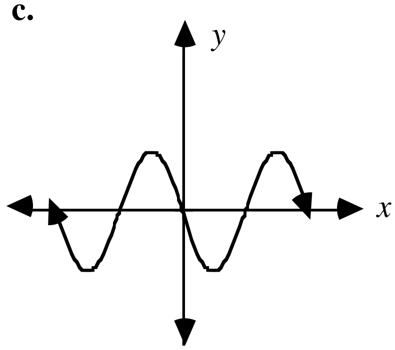
a.



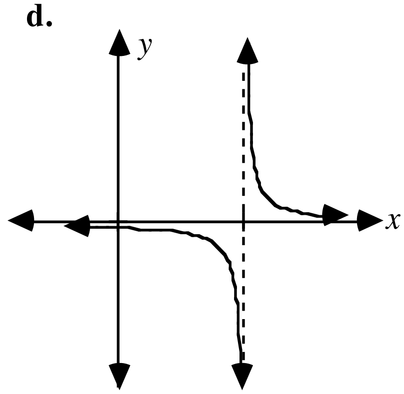
b.



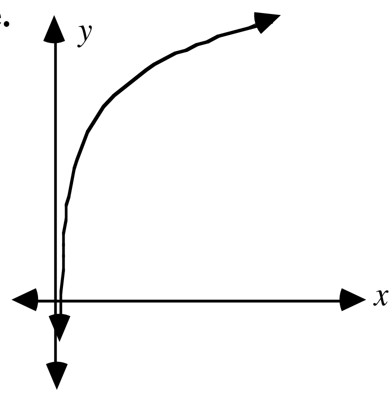
c.



d.



e.



2. Write the equation of the function that results from performing the following transformations on the parent $f(x) = x^2$: a translation –3 units vertically, a translation –5 units horizontally, a reflection in the x-axis, and a vertical stretch by 8.

3. Write an equation that fits the following data set reasonably well. Describe the additional information you would need to determine whether or not the equation represents a good model.

x	$f(x)$	x	$f(x)$
–4.9	–7.4	–2.0	–2.1
–4.5	–4.7	–1.6	–4.0
–4.1	–2.3	–1.4	–5.2
–3.5	–0.4	–0.9	–8.4
–2.6	–0.3	–0.7	–10.1

Answers to Module Assessment

1. Sample responses:

- **a.** $f(x) = 2^x$
- **b.** $f(x) = x^2$
- **c.** $f(x) = \cos x$
- **d.** $f(x) = 1/x$
- **e.** $f(x) = \log x$

2. The resulting function would have the equation $g(x) = -8(x+5)^2 - 3$.

3. Sample response: A scatterplot of the data resembles a parabola. One possible model is $y = -2(x+3)^2$. Whether it is a good model or not depends on the situation from which the data was collected and on how the model is going to be used.

Selected References

Mays, R. "Computer Use in Algebra: And Now, the Rest of the Story." *Mathematics Teacher* 86 (October 1993): 538–41.

Owens, J. E. "Families of Parabolas." *Mathematics Teacher* 85 (September 1992): 477–79.

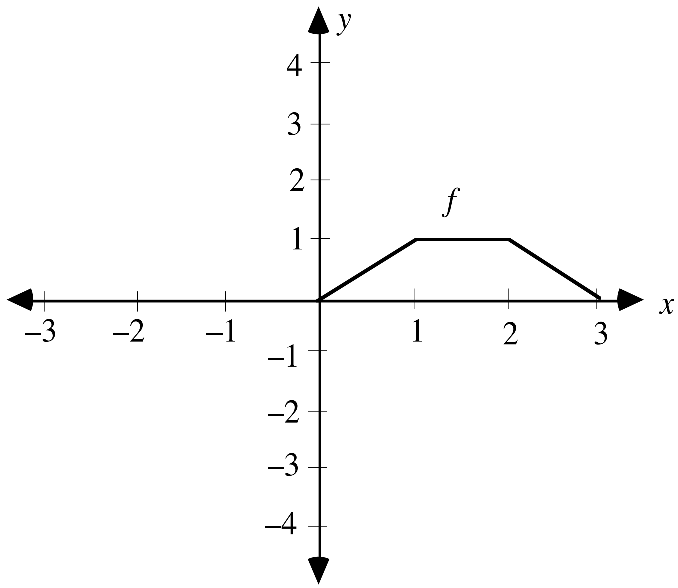
Wells, T. *The Technique of Electronic Music*. New York: Schirmer Books, 1981.

Flashbacks

Activity 1

1.1 Graph $f(x) = \cos(x)$ over the domain $-2\pi \le x \le 2\pi$.

1.2 The diagram below shows a graph of the function *f*.



Perform the transformations described below on *f*:

a. a reflection in the *y*-axis

b. a translation 2 units horizontally, followed by a translation –4 units vertically, followed by a reflection in the *x*-axis.

1.3 Consider the function $f(x) = -2x^2 - 3x$. Determine each of the following values:

a. $f(-4)$

b. $f(-2a)$

c. $f(x + a)$

1.4 Describe the domain and range of the following function:

$$f(x) = \frac{1}{(x-3)}$$

1.5 Sketch a graph of the function $g(x) = 2^x$.

Activity 2

2.1 **a.** Sketch a graph of $y = \cos x$ over the domain $[0, 2\pi]$.

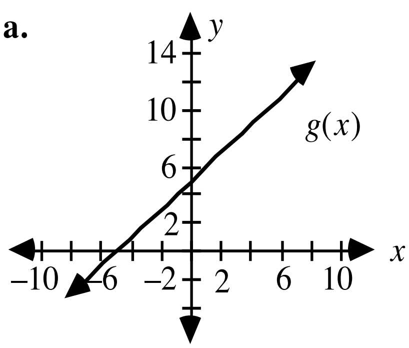
b. How can the graph of $y = \cos x + 4$ over the domain $[0, 2\pi]$ be obtained from the graph in Part **a**?

c. How can the graph of $y = 2\cos x$ over the domain $[0, 2\pi]$ be obtained from the graph in Part **a**?

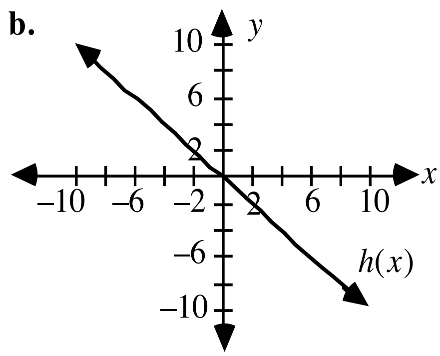
d. How can the graph of $y = \cos(x - \pi/2)$ over the domain $[\pi/2, 5\pi/2]$ be obtained from the graph in Part **a**?

2.2 Each of the following graphs was obtained by performing a transformation on the graph of $f(x) = x$. In each case, determine the equation of the resulting graph.

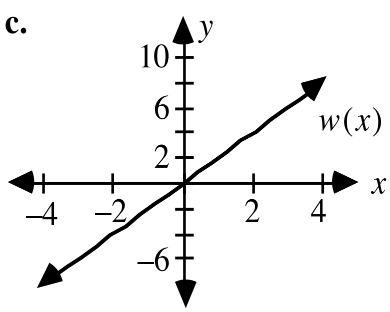
a.



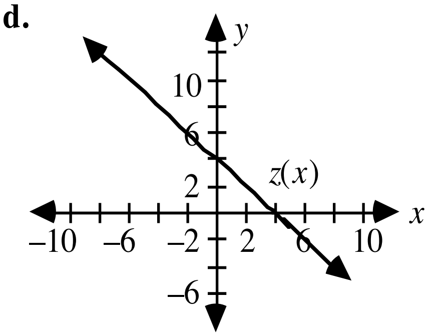
b.



c.



d.



Activity 3

3.1 **a.** Create a scatterplot for the following set of data.

x	0.7	2.5	3.4	5.4	–0.7	–1.6	–3.7	–4.4
y	7.9	2.84	–2.2	6.05	2.2	–4.4	–4.6	9.89

b. Draw a smooth curve through the points on the scatterplot.

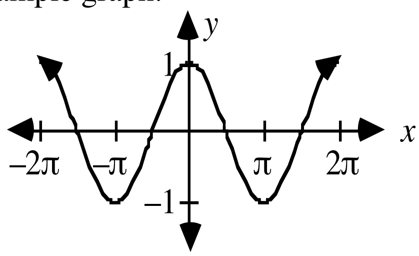
c. Estimate the zeros of the function whose graph is the curve drawn in Part **b**.

d. Use your response to Part **c** to determine a possible model for the data.

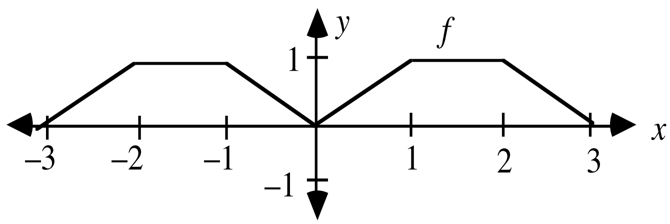
Answers to Flashbacks

Activity 1

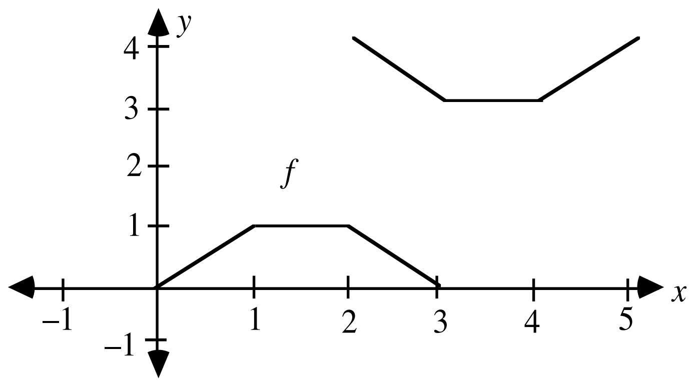
1.1 Sample graph:



1.2 **a.** Sample graph:



b. Sample graph:



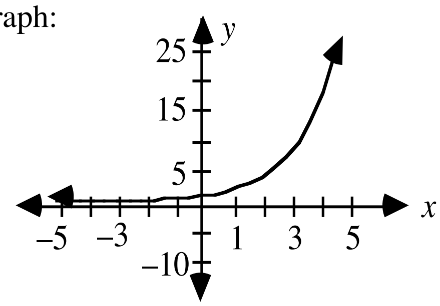
1.3 **a.** –20

b. $-8a^2 + 6a$

c. $-2(x + a)^2 - 3(x + a) = -2x^2 - 4ax - 2a^2 - 3x - 3a$

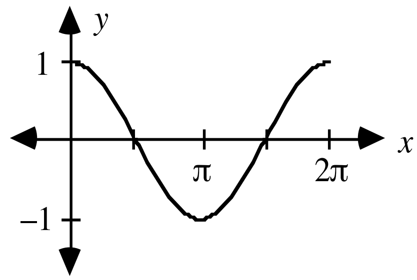
1.4 The domain is the set of real numbers with the exception of 3. The range is the set of real numbers with the exception of 0.

1.5 Sample graph:



Activity 2

2.1 **a.** Sample graph:



b. Sample response: A vertical translation of 4 units in the positive direction will result in this graph.

c. Sample response: The graph is stretched vertically by 2.

d. Sample response: The graph is translated $\pi/2$ units to the right.

2.2 **a.** $g(x) = x + 5$

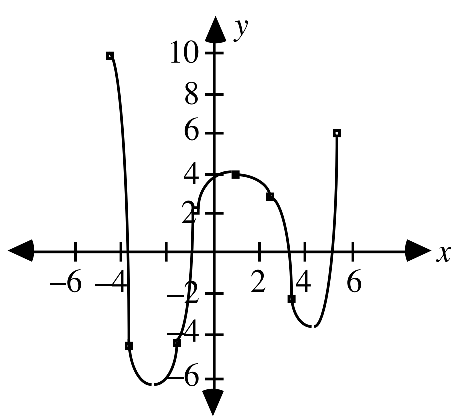
b. $h(x) = -x$

c. $w(x) = 2x$

d. $z(x) = -x + 4$

Activity 3

3.1 **a–b.** Sample response:



c. Sample response: The zeros are approximately –1, –3.8, 3, and 5.

d. Sample response: $f(x) = (x + 1)(x + 3.8)(x - 3)(x - 5)$.

Confidence Builder

When a light-bulb company reports a mean life expectancy for its products, how confident can you be that their bulbs won't leave you in the dark?

Danny Jones • Kate Riley • John Knudson-Martin

Teacher Edition

Confidence Builder

Overview

In this module, students review normal distributions and sampling. Using statistics generated by simulation, they examine the law of large numbers and the central limit theorem. They also investigate some characteristics of a normal curve, use normal distributions to model the distribution of sample means, and determine confidence intervals for sample means.

Objectives

In this module, students will:

- use the law of large numbers
- use the central limit theorem in connection with sample means
- apply the 68–95–99.7 rule of normal distributions
- use sample statistics to create confidence intervals.

Prerequisites

For this module, students should know:

- how to determine mean, median and standard deviation
- the definitions of the terms *sample*, *population*, *statistic*, *parameter*, and *simple random sample*
- some basic sampling techniques
- the basic properties of normal distributions.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	2	1	9

Materials Required

Materials	1	2	3	4	Summary Assessment
measuring tape	X				
template of fish lengths		X			

Teacher Note

A blackline master of the template of fish lengths appears at the end of the teacher edition FOR THIS MODULE.

Technology

Software	1	2	3	4	Summary Assessment
statistics package	X	X	X	X	X
spreadsheet	X	X	X	X	X
graphing utility	X	X			

Confidence Builder

Introduction

(page 319)

Students review the terms and concepts introduced in the Level 2 module "And the Survey Says . . . ." These include *population*, *parameter*, *sample*, *statistic*, and *simple random sample*.

Discussion

(page 319)

a. A statistic is used to describe a sample and a parameter is used to describe a population.

b. Sample response: One example of a population in a school is the entire student body. Some examples of populations in the community are the names in the phone book and the residents of a retirement home.

c. Sample response: Theater owners may be interested in the number of movies attended by high school students monthly. A senior citizens' organization may be interested in the monthly income or cost of living for retirement home residents. A telemarketing firm may be interested in the shopping habits of people listed in the phone book.

d.

1. Sample response: The junior class or a group of students randomly selected by their identification numbers could be samples of the student population. Residents of the retirement home that have lived there more than five years could be a sample of the retirement home and every hundredth person in the phone book could be a sample from the phone book.
2. Sample response: The group of students randomly selected by identification numbers is a simple random sample. None of the other sampling methods mentioned produce simple random samples.

(page 319)

Activity 1

Students collect data on height, then calculate sample mean and standard deviation and population mean and standard deviation. They also draw relative frequency histograms, review normal curves, and examine the law of large numbers.

Materials List

- measuring tape (one per group)

Technology

- statistics package
- graphing utility
- spreadsheet (optional)

Teacher Note

Classes with few students may not provide sample sizes large enough to give good estimates of the population mean. In such cases, you may wish to sample from a larger population (for example, an elementary school).

If listing heights is impractical or problematic, the exploration may be modified to examine, for example, arm length, foot length, or the length from the knee to the sole of the foot.

Exploration

(page 320)

a–b. One sample of 5 heights from the data given in Part **c** below is 150, 165, 171, 173, and 183. Using this sample, $\bar{x} = 168.4$ cm and $s \approx 12.16$ cm.

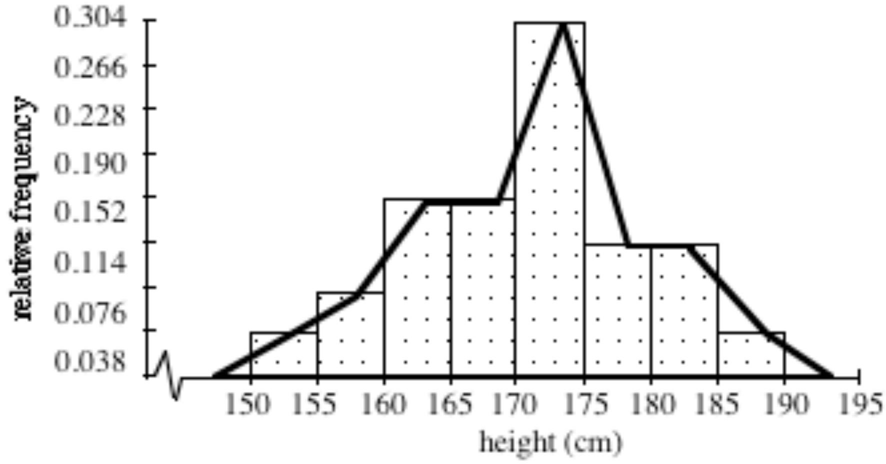
c. Sample data:

Student Heights (in cm)			
150	165	172	178
155	165	172	180
156	169	173	180
162	169	174	183
163	170	174	186
163	171	175	
164	172	175	

d. Using the data given in Part **c** above, $\mu \approx 169.85$ cm and $\sigma \approx 8.47$ cm.

e. **Note:** Students should recall how to create a relative frequency polygon from the Level 4 module "Nearly Normal." First, the midpoints of the tops of consecutive bars of the histogram are connected with segments. Then the point on the horizontal axis one-half the width of the interval to the right of the last bar is connected to the midpoint of that bar. Similarly, the point on the horizontal axis one-half the width of the interval to the left of the first bar is connected to the midpoint of that bar. The base of the polygon is formed by the segment connecting the extreme points on the horizontal axis.

Sample relative frequency histogram and polygon:



Discussion

(page 321)

a.

1. Using the data given in Parts **b** and **d** of the exploration, the sample mean of 168.4 cm is relatively close to the population mean of 169.85 cm.
2. Answers will vary. Students may observe that a disproportionate number of taller or shorter students were randomly included in the sample. If the sample size was small, it may not have been representative of the population.
3. Sample response: No. Since the students in this classroom do not represent all grade levels, ages, or levels of physical maturity, the class probably is not representative of the entire school.

b. Students may observe that the graph looks like a mountain or mountains. It may look roughly like a normal curve.

c. Sample response: The relative frequency polygon may look about the same but it would move to the right on the *x*-axis.

d.

1. Sample response: In Figure **2**, curve 1 has a smaller standard deviation than curve 2. The curve is therefore narrower and taller.
2. Sample response: In Figure **3**, curve 1 is farther to the right. This indicates that its mean is greater than the mean in curve 2.

e. Sample response: The value of σ is a measure of the variance or spread of the data. The larger the standard deviation, the wider the curve. The smaller the standard deviation, the narrower the curve.

f. Sample response: The value of μ indicates the middle of the normal curve. As μ increases, the normal curve will move farther to the right on the *x*-axis.

Assignment

(page 323)

1.1 **a.** Sample response:

Family	**No. in Household**
Aarstad	1
Trujillo	2
Dobrowski	5
Zanani	3
Estrada	4

b. Using the data given in Part **a**, $\overline{x} = 3$ people and $s \approx 1.58$.

c. **1.** Sample response: Everyone in my hometown didn't have an equal chance of being picked since I only chose families I knew.

2. Sample response: The families selected all came from one region. They are probably not representative of the whole nation.

3. Sample response: The families selected were only from the United States. They are probably not representative of the world population.

1.2 **a.** $\mu = 15.5$

b. Sample response: The mean of 30 and 13 is 21.5.

c. Sample response: The mean of 12, 26, 5, 29, and 3 is 15. The mean of 16, 28, 12, 9, 18, 10, 18, 26, 22, 21, 16, 9, 13, 29, 8, 7, 15, 13, 2, and 8 is 15.25.

d. Answers will vary. Some students may find that the sample mean approaches the population mean for larger sample sizes. Other students may obtain conflicting results.

***1.3** **a.** Sample response: The sample of 10 numbers below has a mean of 55.8.

71	81	62	35	34
15	54	83	51	72

The sample of 20 numbers below has a mean of approximately 56.6.

2	90	95	86	91
23	93	30	84	28
76	47	49	42	7
95	49	18	84	44

The sample of 80 numbers below has a mean of approximately 54.1.

41	90	9	74	35	81	51	72
21	67	22	36	35	76	31	98
11	6	25	12	67	92	67	92
26	92	31	68	66	92	97	15
60	73	18	23	14	88	29	43
56	50	7	65	62	15	97	80
97	67	79	51	18	4	79	77
48	52	18	76	17	97	70	76
60	23	64	65	45	95	40	81
99	8	10	74	29	76	53	30

b. Students repeat Part **a** at least three more times.

c. In general, students should find that as sample size increases, the sample mean is likely to be closer to the population mean.

* * * * *

1.4 **a.** $\mu \approx \$1032.11$; $\sigma \approx \$716.12$

b. **1.** Sample response: \$1228, \$1408, \$187, \$451, \$314.

2. Using the sample given in Part **a**, $\bar{x} \approx \$717.60$ and $s \approx \$559.61$.

c. **1.** Answers will vary, depending on the five values selected. Some sample means will provide good estimates; others will not.

2. Sample response: Select a larger sample size. The larger the sample size, the closer the sample mean will probably be to the population mean.

* * * * * * * * * *

(page 324)

Activity 2

This activity introduces the central limit theorem. Students compare estimates obtained from sample statistics to actual population parameters. They simulate sampling from a given fish population and explore the distribution of sample means.

Materials List

- template of fish lengths (one per group; a blackline master appears at the end of the teacher edition for this module)

Teacher Note

The distribution of fish lengths on the template can be approximated by a normal curve.

Technology

- statistics package
- graphing utility
- spreadsheet (optional)

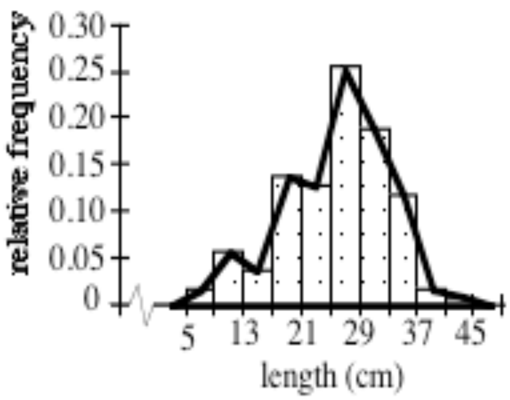
Exploration

(page 325)

Students perform a sampling experiment on a known population.

a. **1.** $\mu \approx 24.72$ cm; $\sigma \approx 7.68$ cm

2. Sample relative frequency polygon:



b. As an alternative to sampling by hand, students may enter the 100 fish lengths in a statistics package, then randomly select a sample of 30 values from the data set. (Some spreadsheets also feature a sampling tool.)

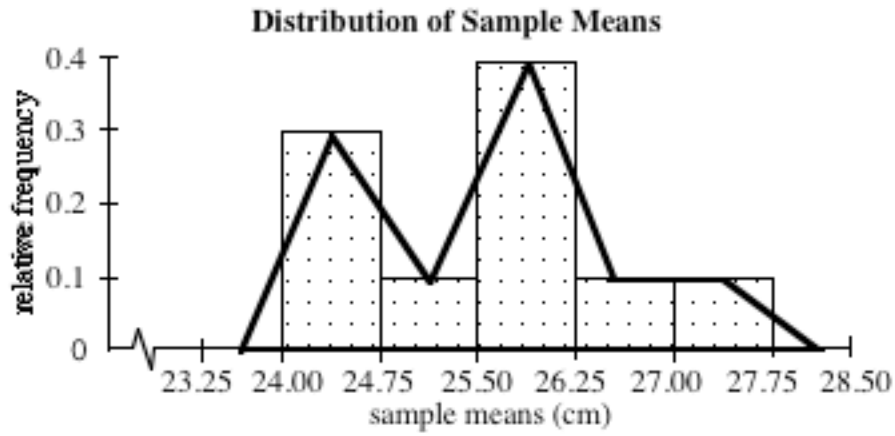
Note: This exploration uses sampling with replacement. Some technology may offer a choice of sampling with or without replacement.

c–e. Sample data:

Sample No.	Sample Mean	Sample Standard Deviation
1	26.17	4.92
2	24.93	6.44
3	25.58	8.41
4	24.00	8.85
5	27.36	6.71
6	26.45	7.42
7	25.69	7.43
8	26.14	7.85
9	24.72	9.37
10	24.72	6.86

f. Using the data given above, the mean of the sample means is approximately 25.58 cm. The standard deviation of the sample means is approximately 1.0 cm. **Note:** In this case, the 10 sample means are considered a sample rather than a population as the class will compile their results in Part **h**.

g. Sample relative frequency polygon:



h. Students compile the class data and make a relative frequency polygon representing all the sample means. This polygon should more closely approximate a normal curve than the one in Part **g**.

Discussion

(page 325)

a. Students compare their results to those of their classmates. **Note:** You also may wish to ask these related questions: Are there any outliers in the data? What are some possible reasons for the similarities and differences you observe?

b. Using the sample data given in Part **e** of the exploration, the mean of the 10 sample means, 25.58 cm, is slightly higher than the mean of the population, 25.07 cm. The standard deviation of the 10 sample means, 1.0 cm, indicates there was less spread in the sample means than in the population, where $\sigma \approx 7.68$ cm.

c. Sample response: Yes, because the mean of the sample means was relatively close to 25 cm.

d. Most of the relative frequency polygons should be roughly bell-shaped.

e. The graph containing the class statistics will generally approximate a normal curve. The mean of the class data should be close to 25.

f. Sample response: The larger the sample size, the narrower the normal curve. As the sample size gets larger, $\sigma_{\bar{x}}$ get smaller. Since the value of $\sigma_{\bar{x}}$ measures the spread about the mean μ, a smaller $\sigma_{\bar{x}}$ indicates a narrower normal curve.

g. Sample response: The researcher would want to have an unbiased sample so a simple random sample would probably be used. In order to get a more accurate reading, the researcher would probably pick a large sample size. The law of large numbers states that as the sample size increases, the sample mean tends to be a better estimator of the population mean.

Assignment

(page 327)

***2.1** **a.** Sample response: Curve 1 represents the distribution of sample means. It is taller and narrower than curve 2 because the standard deviation of the sample means, $\sigma_{\bar{x}}$, is less than σ, the population standard deviation, since $\sigma_x = \sigma/\sqrt{n}$. This means the sample means are grouped more closely around the population mean than the population as a whole. As a result, the frequencies of values close to the mean are higher.

b. Using curve 2, the mean of the population is approximately 20. The standard deviation is approximately $(40 - 20)/3 \approx 6.67$ since 99.7% of the data is within 3 standard deviations.

c. The mean of the sample means is also approximately 20. Based on the estimate made in Part **b**, the standard deviation is:

$$\frac{6.67}{\sqrt{16}} \approx 1.67$$

2.2 **a.** $\mu = 166$ cm

b. ${}_6C_2 = 15$

c–d. The 15 possible samples of two heights and their respective means are shown in the table below.

Sample	Sample Mean	Sample	Sample Mean
145, 156	150.5	156, 188	172
145, 163	154	163, 170	166.5
145, 170	157.5	163, 174	168.5
145, 174	159.5	163, 188	175.5
145, 188	166.5	170, 174	172
156, 163	159.5	170, 188	179
156, 170	163	174, 188	181
156, 174	165		

e. Answers will vary. The mean of the sample means of the five samples below is 162.7 cm.

145, 156	145, 163	156, 174	156, 188	170, 174

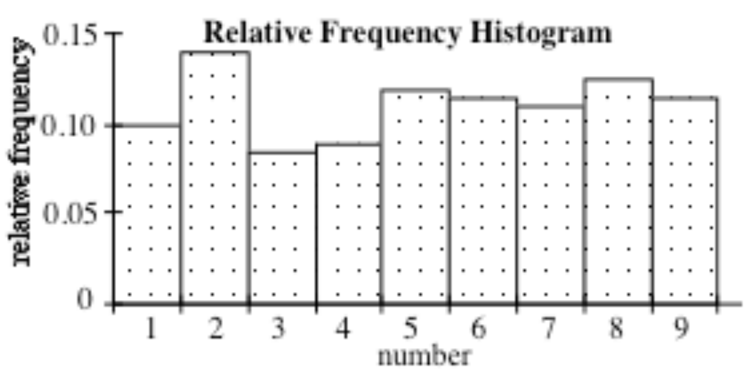
f. The mean of all the sample means is 166 cm.

g. The mean of 5 randomly chosen sample means should be close to the population mean (although it could be as much as 16 cm above or below the population mean). The mean of all the sample means should approach the population mean.

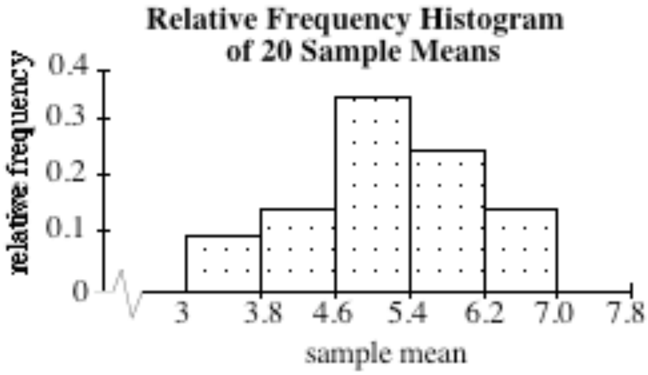
2.3 a–b. The table below shows 20 samples of 10 randomly generated numbers from 1 to 9 and their respective means.

Sample No.	10 Random Numbers										Sample Means
1	5	2	2	1	2	5	3	7	7	2	3.6
2	5	4	9	5	2	7	5	6	9	7	5.9
3	9	2	9	9	7	2	5	5	6	6	6
4	7	1	2	3	5	7	7	1	9	1	4.3
5	1	8	2	7	6	6	2	3	9	2	4.6
6	6	6	2	1	5	1	5	7	4	5	4.2
7	8	4	1	9	3	6	1	8	3	7	5
8	3	9	7	6	9	2	4	8	8	8	6.4
9	8	8	5	7	9	4	2	9	2	9	6.3
10	8	9	5	3	6	5	9	1	9	8	6.3
11	9	4	5	9	7	6	5	5	8	3	6.1
12	4	3	1	1	8	3	9	5	9	9	5.2
13	4	8	8	5	8	1	2	3	8	7	5.4
14	4	2	5	3	4	2	5	4	5	7	4.1
15	2	8	2	2	6	6	4	4	6	8	4.8
16	6	4	1	6	4	9	5	1	4	8	4.8
17	4	6	2	3	2	2	5	1	3	9	3.7
18	4	8	3	8	2	6	6	7	6	8	5.8
19	1	6	6	8	2	7	2	8	1	6	4.7
20	7	3	3	1	7	8	2	1	7	7	4.6

c. Using the data given in Part **b**, the heights of the bars on the histogram are nearly the same. Some students may recall from "Nearly Normal" that this approximates a uniform distribution.



d. Using the data given in Part **b**, the histogram of the sample means looks more bell-shaped than the histogram of the population.



e. The histogram of the sample means approximates a normal distribution, while the histogram of the population approximates a uniform distribution. This is an example of the central limit theorem. The sample means of a population will be normally distributed regardless of the distribution of the population.

***2.4** The mean of the sample means should equal the population mean, 355 mL; $\sigma_{\bar{x}} = 2/\sqrt{4} = 1$ mL.

* * * * *

2.5 **a.** $\sigma \approx 29$

b. $\sigma_{\bar{x}} \approx 29/\sqrt{30} \approx 5.3$

c. Sample response: The standard deviation of the numbers in the population is approximately 29 while the standard deviation of the sample means is approximately 5.3. The standard deviation of the population is much larger than the standard deviation of the sample means since $\sigma_{x} = \sigma/\sqrt{n}$. This indicates that the sample means are grouped much closer to the population mean than are the actual numbers in the population.

2.6 **a.** $\sigma_{\bar{x}} = 3750/\sqrt{100} = 375$ km

b. Sample response: The company could increase the sample size.

* * * * * * * * * *

(page 329)

Activity 3

In this activity, students examine the distribution of the area under a normal curve.

Materials List

- none

Technology

- statistics package
- spreadsheet (optional)

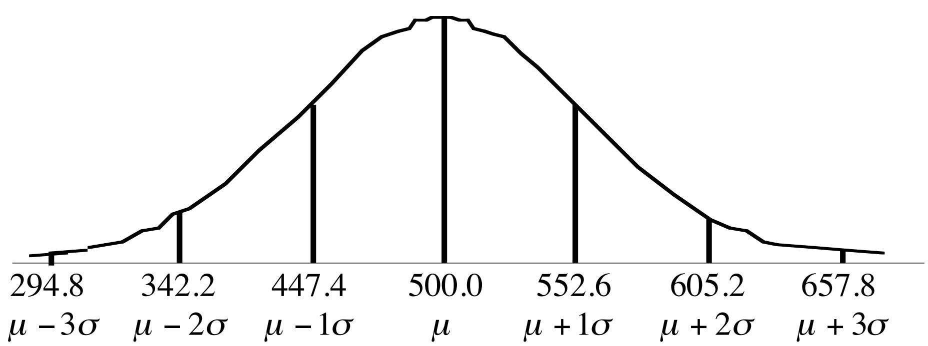
Exploration

(page 329)

In this exploration, students sample from the population of whole numbers from 1 through 999. They then examine the distribution of sample means and determine the percentages that fall within 1, 2, and 3 standard deviations of the population mean.

a. $\sigma_{\bar{x}} \approx 288/\sqrt{30} \approx 52.6$

b–c. The normal curve should have a mean of 500 and a standard deviation of 52.6.



d. A statistics package can be used to generate the numbers from 1 to 999 as patterned data. Students may then take random samples of 30 numbers from this population.

Alternately, students may use the random number generator in a spreadsheet to generate a random number in a cell of a column. They can then fill down 30 cells to obtain a random sample of 30 numbers. The mean of the sample can be found using spreadsheet formulas.

Some calculators also can be used to complete this part of the exploration. For example, the following program was written for the TI-92 calculator. It takes random samples from the whole numbers from 1 to 999.

```
PROGRAM:SAMPMEAN

:ClrHome
:ClrList L1
:Input "Random Number Generator Seed", R
:RandSeed R
:Input "SAMPLE SIZE?",S
:Input "NO. OF TRIALS?", T
:1→B
:Lbl N
:0→A
:1→C
:Lbl M
:int (999*Rand()+1)+A→A
:C+1→C
:If C≤S
:Goto M
:A/S→L1[B]
:B+1→B
:If B≤T
:Goto N
:ClrHome
:Disp "YOU CAN VIEW SAMPLE MEANS IN L1 AND"
:Disp "SET UP A HISTOGRAM OF THE DATA."
:Stop
```

e–f. Students should determine the number of sample means that fall within each interval.

1. Although the percentage should be close to 68%, individual results may vary considerably.
2. Although the percentage should be close to 95%, individual results may vary considerably.
3. Although the percentage should be close to 99.7%, individual results may vary considerably.

g. The class results should provide a good approximation of the 68–95–99.7 rule.

Discussion (page 330)

a. The results obtained by most students should come reasonably close to the 68–95–99.7 rule.

b. The class percentages should be closer to the rule than the individual percentages, due to the larger number of samples.

c. The percentages obtained in the exploration are based on a small number of samples. The normal curve in the mathematics note is based on all possible values of $\bar{x}$.

d. Sample response: Increase the number of samples taken, resulting in a larger number of sample means.

Assignment (page 331)

3.1 **a.** Sample response: Since the interval is 1 standard deviation on either side of the mean, about 68% of the tires should have life spans between 58,800 km and 61,200 km.

b. Sample response: The bounds of the interval are 57,600 and 62,400. This is found by adding $2 \bullet 1200$ to 60,000 to find the upper bound and by subtracting $2 \bullet 1200$ from 60,000 to find the lower bound.

***3.2** **a**. Sample response: The interval represents the area to the right of 1 standard deviation above the mean, or one-half the area outside of 1 standard deviation. Since the area outside of 1 standard deviation is $100\% - 68\% = 32\%$ and $0.5 \bullet 32\% = 16\%$, about 16% should last longer than 1090 hr.

b. Sample response: Since the interval is the area to the left of 2 standard deviations below the mean, only about 2.5% or 25 light bulbs should burn out before 865 hr. (The area outside of 2 standard deviations is $100\% - 95\% = 5\%$ and $0.5 \bullet 5\% = 2.5\%$.)

3.3. **a.** Sample response: The engineer should expect the masses of 68% of the population to lie in the interval [60.1, 89.9].

b. Approximately 64% of the data lies within the given interval.

c. Sample response: Since the masses given represent only one sample from the population, some difference in percentages is likely.

3.4 **a.** The percentage of sample means that lie within the interval calculated in Problem **3.3a** is 95%.

b. Sample response: The percentage of sample means that fall in the interval is significantly larger than the percentage of masses from a single sample. Since the distribution of sample means has a smaller standard deviation than the population, you would expect more of them to fall within a given interval about the mean.

* * * * *

3.5 **a.** [264, 1536]

b. 97.5%

c. 16%

3.6 **a.** About 95% of the reported tips should be between $7.01 and $47.69.

b. Sample response: Technically, the interval for 99.7% should be [–$3.16, $57.86]. Since a waiter or waitress cannot make a negative amount of tips, the interval should be adjusted to be [$0, $57.86].

c. About 84% of the staff should make more than $17.18 in tips per shift.

d. Sample response: No. Typically, some shifts are busier than others and staff who work the busy shifts earn more tips. Also, the more courteous and helpful servers typically receive bigger tips.

* * * * * * * * * *

(page 333)

Activity 4

In this activity, students investigate confidence intervals.

Materials Required

- none

Technology

- statistics package
- spreadsheet (optional)

Exploration

(page 333)

Students use the numbers from 1 through 999 as a model population. They take samples of size 30 from this population and generate 95% confidence intervals.

a. Sample response: $\bar{x} \approx 547.1$.

b. Sample response: $s \approx 293.2$.

c. **1.** Sample response:

$$\sigma_{\bar{x}} \approx \frac{293.2}{\sqrt{30}} \approx 53.5$$

2. Sample response:

$$\bar{x} - 2\left(\frac{s}{\sqrt{n}}\right) \le \mu \le \bar{x} + 2\left(\frac{s}{\sqrt{n}}\right)$$

$$547.1 - 2\left(\frac{293.2}{\sqrt{30}}\right) \le \mu \le 547.1 + 2\left(\frac{293.2}{\sqrt{30}}\right)$$

$$440.3 \le \mu \le 654.4$$

d. Most students should find that μ is contained in their intervals.

e. Since these are 95% confidence intervals, most students should report that from 1 to 4 intervals did not contain the population mean.

Discussion

(page 333)

a. Since these are 95% confidence intervals, one would expect the population mean to fall outside the interval about 5% of the time.

b. Sample response: I would expect that 95% of the intervals contain the population mean and 5% do not contain it.

c. Sample response: Yes, our results were close to 95%.

d. Sample response: You would add and subtract 1 standard deviation of the sample means instead of 2 from the sample mean.

e. Sample response: You could increase the size of the sample. Since

$$\sigma_{\bar{x}} \approx \frac{\sigma}{\sqrt{n}}$$

the larger the value of n, the smaller $\sigma_{\bar{x}}$ becomes. The width of a confidence interval is determined by the size of the standard deviation of the sample means.

Assignment

(page 335)

4.1 **a.** Sample response: No, unless there is very little spread in the population, it would be unlikely for two different samples to have the same mean and standard deviation because the samples are randomly selected.

b. Sample response: Since $0.95 \bullet 20 = 19$, I would expect about 19 of the 20 intervals to contain the population mean.

4.2 **a.** Sample response: A 95% confidence interval encompasses 2 standard deviations of the sample means on either side of μ for a total of 4 standard deviations. A 99.7% confidence interval encompasses 3 standard deviations on either side of μ for a total of 6 standard deviations. Therefore, the 99.7% confidence interval is wider.

b. $\bar{x} - 3\left(\frac{s}{\sqrt{n}}\right) \le \mu \le \bar{x} + 3\left(\frac{s}{\sqrt{n}}\right)$

4.3 **a.** $14/\sqrt{100} = 1.4 \text{ g}$

b. Sample response:

$$84 - 2\left(\frac{14}{\sqrt{100}}\right) \le \mu \le 84 + 2\left(\frac{14}{\sqrt{100}}\right)$$
$$81.2 \le \mu \le 86.8$$

c. Sample response: This means that about 95% of the intervals constructed as in Part **b** will contain the actual population mean.

d. Sample response: It should be about half as large, since $\sqrt{400} = 2\sqrt{100} = 20$.

e. Sample response: The sample size determines the size of the standard deviation of all possible sample means. The value of $\sigma_{\bar{x}}$ determines the width of the confidence interval. Smaller deviations result in narrower widths. If the company could advertise a high gain in mass with a narrow 95% confidence interval, this would make their claim more persuasive.

***4.4** **a.** A 68% confidence interval does not contain the figure claimed by the car maker:

$$13.5 - \frac{1.6}{\sqrt{16}} \le \mu \le 13.5 + \frac{1.6}{\sqrt{16}}$$
$$13.1 \le \mu \le 13.9$$

b. A 95% confidence interval contains the car maker's claim:

$$13.5 - 2\left(\frac{1.6}{\sqrt{16}}\right) \le \mu \le 13.5 + 2\left(\frac{1.6}{\sqrt{16}}\right)$$
$$12.7 \le \mu \le 14.3$$

c. Sample response: The manufacturer's claim is just outside the 68% confidence interval, but within the 95% confidence interval. Because the sample size is small, it's hard to draw a strong conclusion. It is possible that the true mean consumption rate may be slightly different than the claim.

4.5 **Note:** You may wish to remind students that, when entering this data in some graphing calculators or spreadsheets, 1 must be entered four times, 2 must be entered five times, and so on.

a. Sample response: $\bar{x} \approx 3.48$ hr and $s \approx 1.72$ hr. Using the sample standard deviation as an estimate of σ,

$$\sigma_{\bar{x}} \approx \frac{s}{\sqrt{n}} \approx \frac{1.72}{\sqrt{33}} \approx 0.30 \text{ hr}$$

The 95% confidence interval is:

$$3.48 - 2(0.3) \leq \mu \leq 3.48 + 2(0.3)$$
$$2.88 \leq \mu \leq 4.08$$

b. Sample response: It indicates that about 95% of the intervals constructed as in Part **a** will contain the actual population mean.

***4.6** **a.** Since a 95% confidence interval has a width of 4 standard deviations, and $450 - 410 = 40$, then 1 standard deviation is $40/4 = 10$ hr. Since $\bar{x}$ is the center of the interval, $\bar{x} = 430$. The 99.7% confidence interval is:

$$430 - 3(10) \leq \mu \leq 430 + 3(10)$$
$$400 \leq \mu \leq 460$$

b. Sample response: It indicates that about 99.7% of the intervals constructed as in Part **a** will contain the actual population mean.

4.7 Sample response: They should consider how confident they need to be that the interval does in fact contain the population mean. Since higher confidence levels result in larger intervals, they should also consider the size of samples they are willing to take in order to reduce the width of the confidence interval.

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4.8 **a.** Sample response:

$$340.8 - 2\left(\frac{2.6}{\sqrt{100}}\right) \leq \mu \leq 340.8 + 2\left(\frac{2.6}{\sqrt{100}}\right)$$
$$340.28 \leq \mu \leq 341.32$$

b. Sample response:

$$340.2 - 2\left(\frac{2.1}{\sqrt{100}}\right) \leq \mu \leq 340.2 + 2\left(\frac{2.1}{\sqrt{100}}\right)$$
$$339.78 \leq \mu \leq 340.62$$

c. **1.** The two intervals overlap between 340.28 and 340.62. The first has a width of approximately 1.0 while the second has a width of 0.84.

2. Sample response: If you calculated an interval for each day of data, you would get five different confidence intervals.

d. Sample response: No. The process creates an interval that only contains the population mean 95% of the time. However, while the process can create intervals that do not contain the population mean, it would be highly unlikely for all five to not contain it.

e. Sample response: While it appears highly likely that the mean is at least 340 mL, it is also possible that many individual bottles contain less than 340 mL.

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Research Project (page 337)

Students should select a well-defined population, such as the members of the junior class.

Sample sizes will vary but should be at least 30 (in order to use the sample standard deviation as an estimate of the population standard deviation).

Reports should include the sample size, the data collected, the sample mean, the sample standard deviation, and at least one confidence interval.

Answers to Summary Assessment (page 338)

Check student work for the reasonableness of their simulations and their understanding of the mathematics involved. They should record all data generated and describe how this information was used to obtain their statistics.

Module Assessment

1. Describe the implications that the law of large numbers has for researchers and scientists.

2. Describe a step-by-step process that researchers could use to determine a 95% confidence interval for the mean annual income of graduates of a local college. Include a description of a 95% confidence interval that explains its significance in this setting.

3. The numbers in the table below were selected randomly from a population consisting of the whole numbers from 1 through 999.

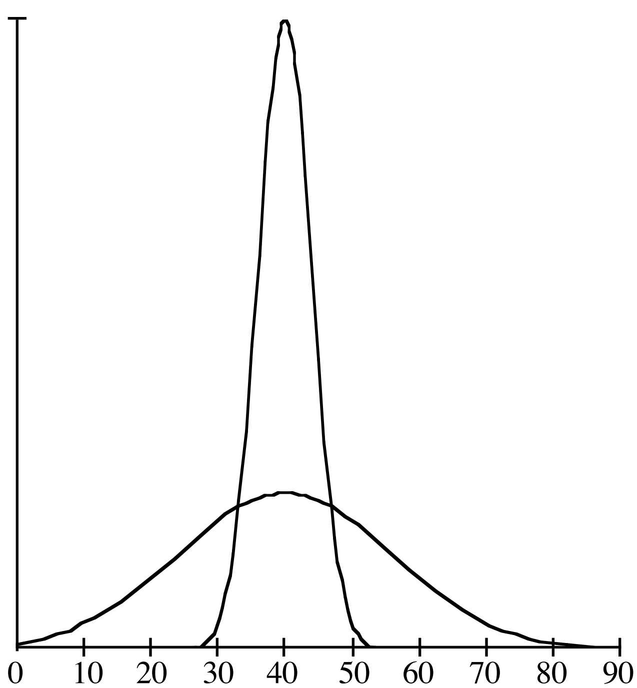
410	154	622	831	339	809	543	354	506	716
2	225	762	946	910	895	926	472	493	277
45	302	492	179	68	256	842	419	840	443

a. Determine the mean of this sample.

b. Determine the standard deviation of this sample.

c. Determine a 95% confidence interval for the mean of the population.

4. The normal curves shown in the following diagram model the distribution of a population and the distribution of sample means for samples of size n taken from that population.



a. Use the appropriate curve to estimate σ.

b. Use the appropriate curve to estimate $\sigma_{\bar{x}}$.

c. Use your responses to Parts **a** and **b** to approximate the size of the samples taken from the population.

d. Describe how the normal curve representing the distribution of the sample means would change if the sample size were $1/4$ as large as the original sample size.

Answers to Module Assessment

1. Sample response: The law of large numbers states that sample means tend to be better estimators of the population mean as the size of the samples increase. In order to get the best possible approximation of the population mean, researchers should make their sample sizes as large as possible.

2. Sample response: The researcher should follow the steps described below.

- Decide on a reasonable sample size that would give reliable statistics with which to approximate the population mean.
- Determine a way of selecting a random sample of the selected size.
- Take the sample.
- Determine the sample mean and the sample standard deviation.
- Estimate the standard deviation of the sample means.
- Determine a 95% confidence interval by adding and subtracting 2 standard deviations of the sample means from the sample mean.

About 95% of the intervals produced by this procedure will contain the population mean.

3. **a.** $\overline{x} = 502.6$

b. $s \approx 289.4$

c. Using the sample standard deviation as an estimate of the population standard deviation:

$$\sigma_{\overline{x}} = \frac{289.4}{\sqrt{30}} = 52.8$$

The 95% confidence interval is therefore:

$$502.6 - 2(52.8) \leq \mu \leq 502.6 + 2(52.8)$$
$$397 \leq \mu \leq 608.2$$

4. **a.** Sample response: The normal curve representing the population has a width of approximately 84. Since 99.7% of the population falls within 3 standard deviations of μ, or a width of 6 standard deviations, the population standard deviation, σ, is about 14.

b. Sample response: The normal curve representing the sample means has a width of about 26, so the standard deviation of the sample means, $\sigma_{\overline{x}}$, is approximately 4.3.

c. Sample response:

$$\sigma_{\bar{x}} = \sigma/\sqrt{n}$$
$$4.3 = 14/\sqrt{n}$$
$$n \approx 11$$

d. If the sample size were $1/4$ as large as the original, then the standard deviation of the sample means would be

$$\frac{\sigma}{\sqrt{n/4}} = \frac{\sigma}{\sqrt{n}/2} = \frac{2\sigma}{\sqrt{n}}$$

Therefore the standard deviation of the sample means would be twice as large. The normal curve would be wider and not as tall.

Selected References

Brase, C. P., and C. H. Brase. *Understandable Statistics: Concepts and Methods*. Lexington, MA: D. C. Heath, 1991.

Miller, I., J. E. Freund, and R. A. Johnson. *Probability and Statistics for Engineers*. Princeton, NJ: Prentice-Hall, 1990.

Moore, D. S. *Statistics: Concepts and Controversies*. San Francisco, CA: W. H. Freeman and Co., 1979.

Neter, J., W. Wasserman, and G. A. Whitmore. *Applied Statistics*. Boston, MA: Allyn and Bacon, 1993.

Rowntree, D. *Statistics Without Tears: A Primer for Non-Mathematicians*. New York: Scribner's, 1981.

Flashbacks

Activity 1

1.1 **a.** Find the relative frequency for each of the scores in the table below.

Score	Frequency
29	1
35	2
43	2
45	1
55	3
58	3
63	2
67	1
76	2
80	1

b. Find the mean and standard deviation of the scores.

c. Use appropriate intervals to create a relative frequency histogram of the data.

d. Sketch the relative frequency polygon for the data on the histogram from Part **c.**

1.2 Describe how you could take a simple random sample of size 5 from the scores in Problem **1.1**.

Activity 2

2.1 Describe the differences between μ and $\bar{x}$.

2.2 How would the graph of a normal curve with a small standard deviation compare to the graph of a normal curve with a large standard deviation?

Activity 3

3.1 A certain population is normally distributed with $\mu = 16$ and $\sigma = 0.5$. Make a sketch of a normal curve that models this distribution. Label the scale on the *x*-axis.

3.2 A population has $\mu = 26.8$ and $\sigma = 3.6$. Calculate the standard deviation of the sample means $(\sigma_{\bar{x}})$ for samples of size 40 taken from this population.

3.3 When plotted on the same axis, why is the graph of a normal curve for a population wider than the graph of a normal curve for a group of sample means from the same population?

Activity 4

4.1 A population is normally distributed with a mean of 365 and a standard deviation of 16. Use the 68–95–99.7 rule to find intervals centered at the mean that will include 68%, 95%, and 99.7% of the population.

4.2 A population is normally distributed with a mean of 98 and a standard deviation of 3.5. Find the approximate percentage of the population that lies above 105.0.

Answers to Flashbacks

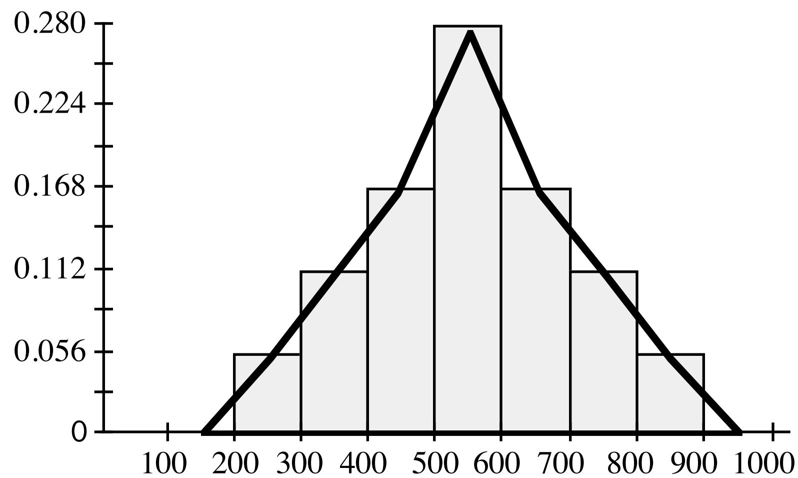
Activity 1

1.1 **a.** The relative frequencies are shown in the following table.

Score	Frequency	Relative Frequency
29	1	0.06
35	2	0.11
43	2	0.11
45	1	0.06
55	3	0.17
58	3	0.17
63	2	0.11
67	1	0.06
76	2	0.11
80	1	0.06

b. $\mu \approx 55.22$; $\sigma \approx 14.28$

c–d. Sample graph:



1.2 A simple random sample could be taken by placing 18 pieces of paper, each with 1 of 18 scores written on it, into a container. The sampling can be done by drawing out 1 piece of paper, recording the score written on the paper, replacing the paper in the container, then stirring the contents of the container. This is repeated 4 more times.

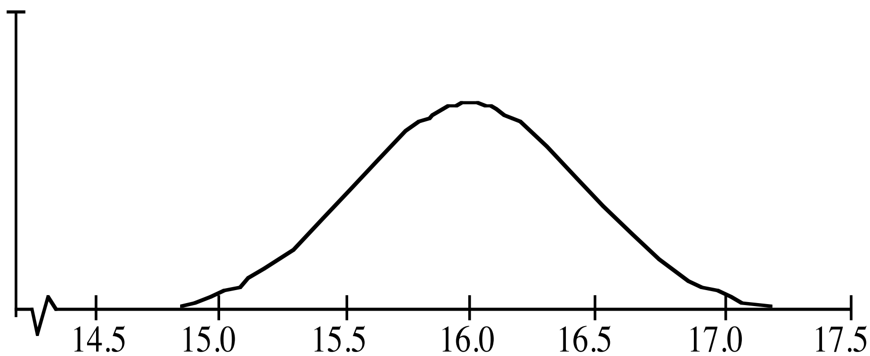
Activity 2

2.1 Sample response: μ is the population mean and $\bar{x}$ is the sample mean. The sample mean is an estimator of the population mean.

2.2 Sample response: The standard deviation of a normal curve describes its spread. A normal curve with a small standard deviation is less wide than a normal curve with a large standard deviation. A normal curve with a small standard deviation has more of the values clustered around the mean, while a normal curve with a large standard deviation has more values farther from the mean.

Activity 3

3.1 Sample graph:



3.2 $\sigma_{\bar{x}} = 3.6/\sqrt{40} \approx 0.57$

3.3 Sample response: The normal curve of a population encompasses all of the population including its outliers. The normal curve of the sample means includes only the means. When calculating the means, the influence of any outliers is balanced by the rest of the sample. Therefore, the sample means tend to lie closer to the population mean.

Activity 4

4.1 The 68% confidence interval is [349, 381]. The 95% confidence interval is [333, 397]. The 99.7% confidence interval is [317, 413].

4.2 approximately 2.5%

Template for Fish Lengths
(in centimeters)

26	24	30	26	35	30	28	24	18	26
18	28	23	30	30	20	21	38	18	27
25	27	26	16	17	25	19	29	30	39
21	26	25	17	33	10	24	9	29	33
20	11	28	26	11	25	33	25	42	31
32	19	22	33	26	25	31	26	11	29
23	23	26	27	17	31	19	35	36	8
24	31	31	20	22	13	32	35	27	18
32	35	33	25	16	36	21	8	16	30
23	33	28	23	29	26	30	20	34	25

Transmitting Through Conics

What does watching television have to do with conic sections? In this module, you'll see how conics influence your favorite telecasts—from signal to satellite dish.

Anne Merrifield • Pete Stabio

Teacher Edition

Transmitting Through Conics

Overview

Students study conic sections from a geometric point of view and develop the standard form of the algebraic equations for them.

Objectives

In this module, students will:

- investigate conic sections as loci of points
- develop algebraic equations for conic sections
- use a symbolic manipulator to show that different forms of a given equation are equivalent.

Prerequisites

For this module, students should know:

- the Pythagorean theorem
- the distance formula
- the definition of an asymptote
- how to transform the graphs of functions.

Time Line

Activity	Intro.	1	2	3	4	Summary Assessment	Total
Days	0.5	1.5	3	3	3	1	12

Materials Required

Materials	Activity					
	Intro.	1	2	3	4	Summary Assessment
pencil	X		X			
graph paper	X	X	X		X	
compass		X				
tacks			X			
cardboard			X			
string			X			

Technology

Software	Activity					
	Intro.	1	2	3	4	Summary Assessment
geometry utility		X	X	X	X	X
graphing utility		X	X	X	X	X
symbolic manipulator		X	X	X	X	X

Transmitting Through Conics

Introduction
(page 345)

Students examine the shapes of four conic sections and discuss some familiar objects that contain those shapes.

Materials List

- paper (at least one sheet per student)
- pencil (one per student)

Exploration
(page 345)

a. While holding the midpoint in one hand, students should use the other hand to rotate the end of the pencil.

b. Students should describe the locus of points (or lines) that form a cone.

Note: Many students use the term *cone* to describe only one nappe of a conical surface. The cone described in the exploration involves both nappes.

Discussion
(page 346)

a. The locus of points traced by the pencil's tip is an approximation of a circle. Students should recognize that while the tip is tracing one figure, the other end of the pencil is tracing a congruent figure. **Note:** If the pencil is held at a point other than its midpoint, the figures would be similar.

b. The locus is a single point.

c. Sample response: The locus of points forms a cone that extends indefinitely in both directions.

d. Sample response: The tip-off area on a basketball court is a circle. A cross section of an aluminum can could be circular or elliptical. The opening of an ice cream cone is circular. A cross section of an eyeglass lens can be parabolic or elliptical. The orbits of planets about the sun are elliptical. A cross section of a riding saddle could be parabolic or hyperbolic. Arches for concrete bridges may be parabolic.

e. Sample response: A circle is shaped like a ring. An ellipse is shaped like an oval or an elongated circle. A parabola is shaped like a rainbow. A hyperbola is shaped like the middle section of an hourglass.

f. When the plane intersects the cone only at the apex, the intersection is a point. When the plane intersects the cone in a line generating the cone, the intersection is a line. When the plane contains the vertical axis of symmetry, the intersection forms two intersecting lines.

(page 347)

Activity 1

Students examine a circle both geometrically and algebraically.

Materials List

- graph paper (optional; one sheet per student)
- compass (optional)

Technology

- geometry utility
- graphing utility
- symbolic manipulator

Teacher Note

If your geometry utility does not feature a coordinate grid, you may wish to ask students to complete this exploration on graph paper.

Exploration

(page 347)

Students draw circles on a graphing utility, find their equations, then relate these equations to the radii and centers of the circles.

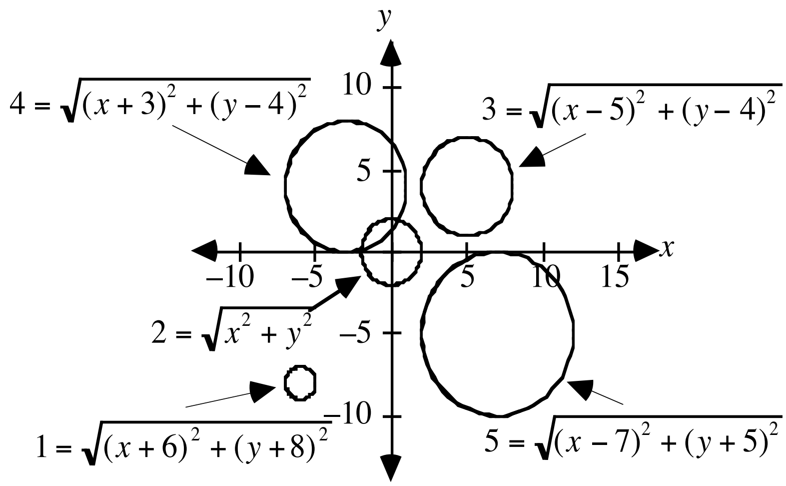
a. By locating the center of the circle at an intersection of grid lines, students can use integers to describe its coordinates. This simplifies the exploration.

b. Students may select any distance. See sample graph in Part **e**.

c. Students determine the equation of the circle using the distance formula. If the selected center is the origin, for example, the equation is of the form below, where r is the radius:

$$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

d. Sample graphs:



Discussion

(page 347)

a. Sample response: Substituting r for d and (h,k) for (x_1, y_1) and allowing (x_2, y_2) to be the general point with coordinates (x,y) yields the equation below:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

b. Squaring both sides of the equation in Part **a** above yields the following: $r^2 = (x-h)^2 + (y-k)^2$. **Note:** You may wish to point out that the process outlined in the student edition is not reversible. For example, $x^2 + y^2 = 16$ does not guarantee that $\sqrt{x^2 + y^2} = 4$. The square root also could be equal to –4. This fact is addressed in Part **e** of this discussion.

c. Sample response: The standard form is the square of the equation found using the distance formula. **Note:** Some geometry utilities will report the equation of a circle given its center and radius. You may wish to encourage students to experiment with this feature.

d. Sample response: To determine if a point lies on a circle, substitute the point into the equation and see if it satisfies the equation.

e. Sample response: Yes, since the symbol $\sqrt{\ }$ indicates the positive square root. However, it is not always true that $x = 4$ since $\sqrt{(-4)^2} = 4$.

f. **1.** $y = \sqrt{16 - x^2}$ or $y = -\sqrt{16 - x^2}$

2. The domain for both functions is the real-number interval [–4, 4]. The range for $y = \sqrt{16 - x^2}$ is the interval [0, 4]; while the range for $y = -\sqrt{16 - x^2}$ is the interval [–4, 0].

3. Yes. All of the points on the circle are included in the graph of the two functions.

Assignment

(page 348)

1.1 **a.** $x^2 + y^2 = 9$

b. $(x + 1.3)^2 + (y - 8.9)^2 = 21$

c. $(x - a)^2 + (y - b)^2 = c$

1.2 Sample response: The following equations describe three circles with the same center and different radii: $(x - 5)^2 + (y + 2)^2 = 13$, $(x - 5)^2 + (y + 2)^2 = 17$, and $(x - 5)^2 + (y + 2)^2 = 21$.

1.3 **a.** center (30,120); radius 17

b. center (–21,73); radius = $\sqrt{141} \approx 11.9$

c. center (0,0); radius 11

***1.4** **a.** **1.** Students may list any point (x, y) such that $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius.

2. Students may list any point (x, y) such that $(x - h)^2 + (y - k)^2 < r^2$.

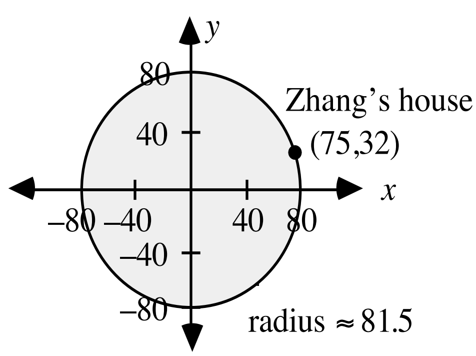
3. Students may list any point (x, y) such that $(x - h)^2 + (y - k)^2 > r^2$.

b. Sample response: When the coordinates are substituted in the equation, the value of the expression $(x - h)^2 + (y - k)^2$ must be greater than r^2.

* * * * *

1.5 $(x + 2)^2 + (y + 8)^2 = 212$

1.6 **a.** Sample response: Since $75^2 + 32^2 \approx 81.5^2$, the radius is approximately 81.5 km.



b. $x^2 + y^2 = 6649$

c. Answers will vary. Sample response: (0 km east, 81.5 km north), (60 km east, 55.2 km north), (75 km east, 32 km south), and (75 km west, 32 km north).

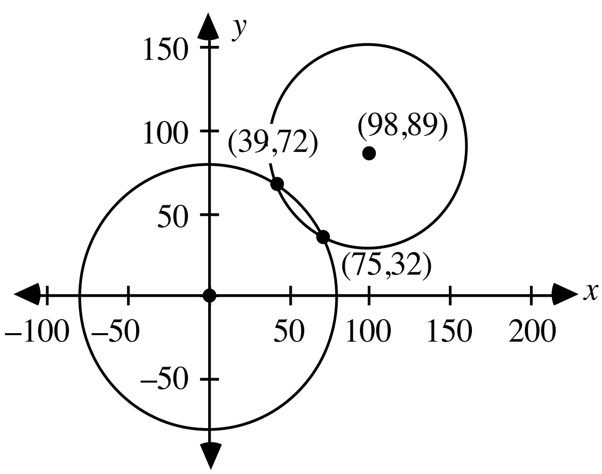
1.7 **a.** Sample response: If KIZY's transmitter is at (0,0), then KZME is at (98,89).

b. Sample response: Since $(98-75)^2 + (89-32)^2 \approx 61.5^2$, the radius is approximately 61.5 km.

c. $(x-98)^2 + (y-89)^2 = 3778$

d. Sample response: Using KIZY as the origin, the following points all lie on the locus of KZME's maximum broadcast range: (75 km east, 146 km north), (121 km east, 32 km north), (121 km east, 146 km north), and (98 km east, 150.5 km north).

e. Sample sketch.



f. (75,32) and (39,72)

g. Sample response: Zhang's home is one of the intersection points.

h. Sample response: The area that is covered by both circles represents the region where homes receive the signals of both radio stations.

* * * * * * * * * *

(page 350)

Activity 2

Students investigate the ellipse from a geometric point of view. They then develop the standard form of the equation of an ellipse with center at the origin and foci on the x-axis. **Note:** It is possible to develop the conic sections by beginning with a cone, slicing it with a plane, then applying geometric theorems. This approach is not taken in this module.

Materials List

- tacks (two per student)
- string (at least 15 cm per student)
- graph paper (at least three sheets per student)
- cardboard (one sheet per student)

Technology

- geometry utility
- graphing utility
- symbolic manipulator (optional)

Exploration 1

(page 350)

a. Students use a piece of string to create a circle whose radius is half the length of the string.

b. Students create an ellipse using a procedure similar to that described in Part **a**, in which the two tacks represent the foci of the ellipse.

c. Students move the pencil to one of the endpoints of the minor axis. They should note that the string, along with the segment joining the foci, forms an isosceles triangle. In the following discussion, students will use this triangle to identify the relationships among the foci, the major axis, and the minor axis.

d. Students label the foci and locate and label the center of the ellipse. They compare the distance between the two knots in the string to the length of the segment that contains the foci and whose endpoints are on the ellipse (the major axis). They should find that these lengths are the same.

e. Students relocate the tacks and repeat Parts **b–d**, observing how changing the distance between the foci affects the resulting ellipse.

Discussion 1

(page 352)

a.

1. The locus of points is a circle whose radius is half the distance between the knots in the string.
2. Any line passing through the circle's center is an axis of symmetry.

b. The radius of the circle is half the distance between the knots in the string used to create the locus of points.

c. Sample response: As the pencil is moved about the locus of points, the sum of the distance from the two tacks (the foci) is always equal to the distance between the knots in the string. This length remains constant throughout the construction. So any point on the locus of points satisfies the definition of an ellipse.

d. An ellipse is symmetrical about the lines containing the major and minor axes.

e. **1.** The distance *d* between the knots in the string equals $2a$, the length of the major axis.

2. Sample response: As the tacks are moved farther apart, the ellipse becomes more elongated. As the tacks are moved closer together, the ellipse becomes more circular.

f. From the Pythagorean theorem:

$$OP = \sqrt{PF_1^2 - OF_1^2}$$

g. The distance between the knots in the string is $PF_1 + PF_2$.

h. Students should find some form of the following equation:

$$\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

Exploration 2

(page 353)

a. Equations will vary. The sample responses given in Parts **b–f** are based on the circle defined by $x^2 + y^2 = 16$.

b. Sample response:

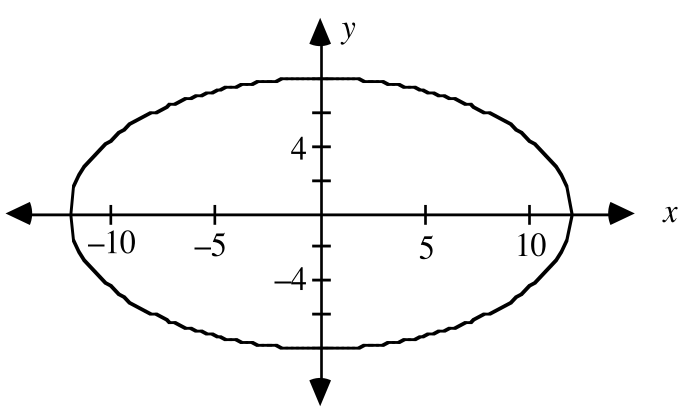
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 16$$

c. **1–2.** Sample response:

$$\frac{x^2}{12^2} + \frac{y^2}{8^2} = 1$$

Note: You may wish to allow students to use a symbolic manipulator to simplify these expressions.

d. **1.** If their graphing utility plots only functions, students must graph the ellipse in parts. Sample graph:



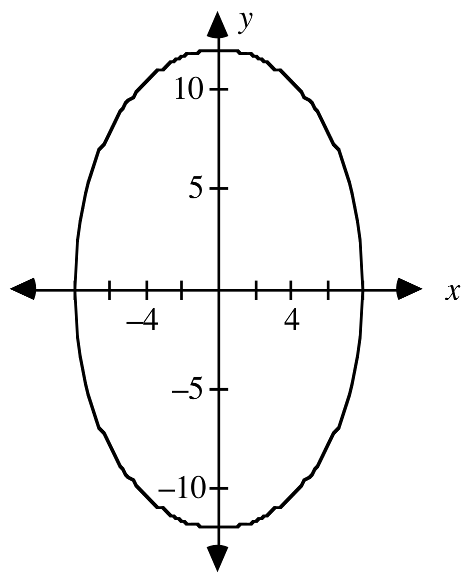
2. Sample response: (12,0), (–12,0), (0,8), and (0,–8).

e. Students repeat the exploration, stretching the original circle horizontally by 2 and vertically by 3. Sample equation:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 16$$

$$\frac{x^2}{8^2} + \frac{y^2}{12^2} = 1$$

The graph of this ellipse intersects the x- and y-axes at (–8,0), (8,0), (0,–12), and (0,12). Sample graph:



Discussion 2

(page 354)

a. The graphs are ellipses. The major axis of the graph in Part **d** is along the x-axis. The major axis of the graph in Part **e** is along the y-axis.

b. The coordinates of the intersections of the ellipse and the x-axis are $(a,0)$ and $(-a,0)$. The coordinates of the intersections of the ellipse and the y-axis are $(b,0)$ and $(-b,0)$.

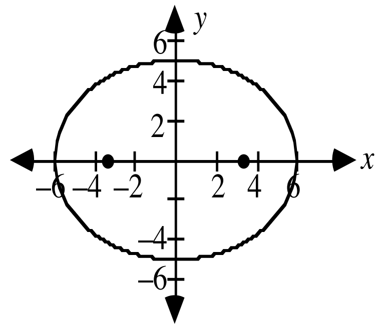
c. If $a > b$, then the x-axis contains the major axis and the y-axis contains the minor axis. The length of the major axis is $2a$ and the length of the minor axis is $2b$.

If $b > a$, then the y-axis contains the major axis and the x-axis contains the minor axis. The length of the major axis is $2b$ and the length of the minor axis is $2a$.

d. The coordinates of the foci are $\left(\sqrt{a^2 - b^2},0\right)$ and $\left(-\sqrt{a^2 - b^2},0\right)$.

Assignment (page 355)

2.1 **a.** Sample graph



b. Using the Pythagorean theorem, the foci are located at $(\sqrt{11},0)$ and $(-\sqrt{11},0)$.

c. $\sqrt{\left(x-\sqrt{11}\right)^2+\left(y-0\right)^2}+\sqrt{\left(x+\sqrt{11}\right)^2+\left(y-0\right)^2}=12$

2.2 **a**. Sample response: The equation does not represent a function since some x-coordinates are paired with two different y-coordinates.

b. Sample response: If the graphing utility only plots functions, it is necessary to graph the ellipse in two parts on the same coordinate system. The functions that describe these parts can be found by solving the equation for y and graphing the solutions. Using a TI-92 calculator, for example, the symbolic manipulator gives the following solutions for y:

$$y=\frac{3\sqrt{-(x^2-16)}}{4};\ x^2-16\le 0 \text{ and } y=\frac{-3\sqrt{-(x^2-16)}}{4};\ x^2-16\le 0$$

Graphing both parts of the solution results in a graph of the ellipse.

c. The coordinates of the vertices are (0,3), (0,–3), (4,0), and (–4,0).

d. For any point on the ellipse, $PF_1+PF_2=2a=8$.

2.3 **a**. $PF_1=\sqrt{(x+3)^2+(y-0)^2}$ and $PF_2=\sqrt{(x-3)^2+(y-0)^2}$

b. $\sqrt{(x+3)^2+(y-0)^2}+\sqrt{(x-3)^2+(y-0)^2}=16$

c. The vertices are $(0, \sqrt{55})$ and $(0, -\sqrt{55})$. The *y*-coordinates can be found as follows:

$$\sqrt{(x+3)^2 + (y-0)^2} + \sqrt{(x-3)^2 + (y-0)^2} = 16$$

$$\sqrt{(0+3)^2 + (y-0)^2} + \sqrt{(0-3)^2 + (y-0)^2} = 16$$

$$\sqrt{9+y^2} + \sqrt{9+y^2} = 16$$

$$2\sqrt{9+y^2} = 16$$

$$9 + y^2 = 64$$

$$y^2 = 55$$

$$y = \pm\sqrt{55}$$

d. Sample response: The major axis has a length of 16, which implies that $2a = 16$ and $a = 8$. The *x*–intercepts are (8,0) and (–8,0).

e. $\frac{x^2}{64} + \frac{y^2}{55} = 1$

2.4 **a.** Using a symbolic manipulator:

$$\sqrt{(x+2)^2 + (y-0)^2} + \sqrt{(x-2)^2 + (y-0)^2} = 10$$

$$\sqrt{(x+2)^2 + (y-0)^2} = 10 - \sqrt{(x-2)^2 + (y-0)^2}$$

$$x^2 + 4x + y^2 + 4 = -20\sqrt{x^2 - 4x + y^2 + 4} + x^2 - 4x + y^2 + 104$$

$$8x - 100 = -20\sqrt{x^2 - 4x + y^2 + 4}$$

$$\left(\frac{8x-100}{-20}\right)^2 = x^2 - 4x + y^2 + 4$$

$$64x^2 + 1600x + 10{,}000 = 400x^2 - 1600x + 400y^2 + 1600$$

$$8400 = 336x^2 + 400y^2$$

$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

b. Both forms give the same solutions in terms of *y*:

$$y = \sqrt{\frac{-21(x^2 - 25)}{5}};\ x^2 \le 25 \text{ and } y = -\sqrt{\frac{-21(x^2 - 25)}{5}};\ x^2 \le 25$$

***2.5**. **a.** The center is (8,6) and the equation is:

$$\frac{(x-8)^2}{25} + \frac{(y-6)^2}{9} = 1$$

b. **1.** Since $a = 5$, the coordinates are (3,6) and (13,6).

2. Since $b = 3$, the coordinates are (8,3) and (8,9).

c. Sample response:

$$\begin{aligned} b^2 &= a^2 - c^2 \\ 3^2 &= 5^2 - c^2 \\ c^2 &= 25 - 9 \\ c &= 4 \end{aligned}$$

The coordinates of the foci are (4,6) and (12,6).

d. The equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

2.6. **a.** Sample response:

$$\frac{2.99 \bullet 10^8}{2} - 2.99 \bullet 10^6 \approx 1.47 \bullet 10^8 \text{ km}$$

b. Sample response:

$$\frac{2.99 \bullet 10^8}{2} + 2.99 \bullet 10^6 \approx 1.52 \bullet 10^8 \text{ km}$$

* * * * *

2.7 Using a symbolic manipulator to solve the first equation for *y* yields the following:

$$\sqrt{(x+4)^2 + (y-0)^2} + \sqrt{(x-4)^2 + (y-0)^2} = 34$$

$$\sqrt{(x+4)^2 + (y-0)^2} = 34 - \sqrt{(x-4)^2 + (y-0)^2}$$

$$x^2 + 8x + y^2 + 16 = -68\sqrt{x^2 - 8x + y^2 + 16} + x^2 - 8x + y^2 + 1172$$

$$16x - 1156 = -68\sqrt{x^2 - 8x + y^2 + 16}$$

$$y = \frac{\pm\sqrt{-273(x^2 - 289)}}{17}$$

Solving the equation in standard form for *y* produces the same results:

$$\frac{x^2}{289} + \frac{y^2}{273} = 1$$

$$y = \frac{\pm\sqrt{-273(x^2 - 289)}}{17}.$$

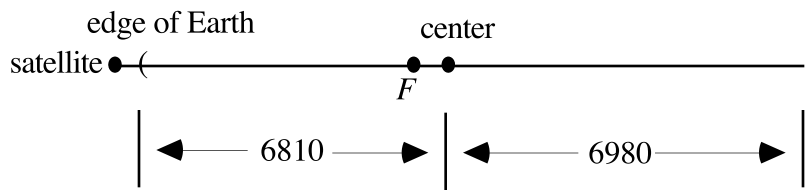
2.8 **a.** Sample response:

$$\frac{x^2}{6980^2}+\frac{y^2}{6980^2-410^2}=1$$

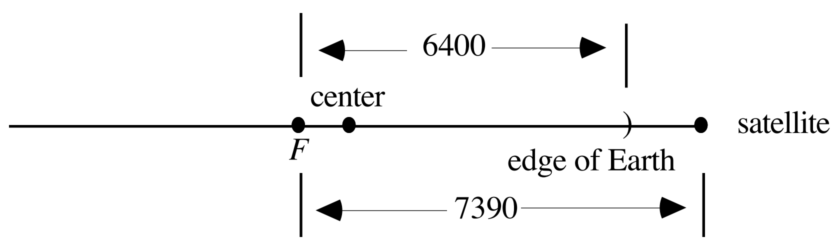
$$\frac{x^2}{48,720,400}+\frac{y^2}{48,552,300}=1$$

b. Sample response: Since the denominators are very close in size, the ellipse is almost circular in shape.

c. Sample response: Half of the major axis is 6980 km. It is 410 km from the center of the ellipse to the focus where Earth is located. The distance from the center of the ellipse to the surface of Earth nearer to the satellite is $6400 + 410 = 6810$ km. Therefore, the satellite is $6980 - 6810 = 170$ km from Earth.



d. Sample response: When the satellite is at its farthest point, it is $6980 + 410 = 7390$ km from Earth's center to the satellite. Therefore, the satellite is 990 km from the surface of the Earth.



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(page 358)

Activity 3

Students describe a hyperbola both geometrically and algebraically. They develop the equation of a hyperbola using the distance formula.

Materials List

- none

Technology

- geometry utility
- graphing utility
- symbolic manipulator

Exploration 1 (page 358)

Students use a geometry utility to generate circles and ellipses. They will use the same construction to generate a hyperbola in the next exploration.

a–g. As P_1 is moved around the circle, the path traced by X produces an ellipse.

h. When P is located at the center O, a circle is formed.

Discussion 1 (page 359)

a. Sample response: Any time P is in the circle, but not at the center, the figure generated is an ellipse. As P is moved farther from the center of the circle, the ellipse becomes more elongated. When P is located at the center, a circle is generated.

b.

1. Points O and P are the foci of the ellipse.
2. The sum $OX + XP$ must be a constant in order to generate an ellipse.
3. They are equal.
4. Sample response: It is a radius of the circle. No matter where the radius is located, it is the same length.
5. Since $OX + XP_1 = OP_1$ and $XP_1 = XP$, $OX + XP = OP_1$.
6. The point X is a point on the generated figure. The sum of the distances from X to point O and from X to point P is a constant. Therefore, X is a point on an ellipse by the definition of an ellipse.

Exploration 2 (page 360)

a. Students generate hyperbolas using the construction created in Exploration **1**. A hyperbola results when P is located outside the circle.

b. Students repeat Part **a** for other locations of P. When P is on the circle, the result is a line, one of the degenerate conic sections.

Discussion 2

(page 361)

a. Sample response: When P is located on the circle, a line is formed (one of the degenerate conic sections). As P is moved outside the circle, a hyperbola is formed. The farther from the circle P is located, the "flatter" the hyperbola appears.

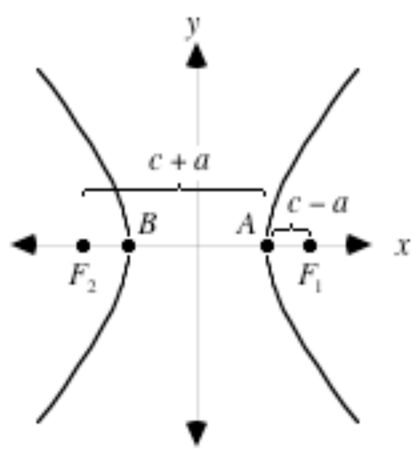
b.

1. Points O and P appear to be the foci.
2. The length of $\overline{OP_1}$ remains the same because $\overline{OP_1}$ represents a radius of a circle.
3. They are equal.
4. Since $OX + OP_1 = P_1X$ and $P_1X = PX$, $PX = OX + OP_1$.
5. Subtracting OX from both sides of the equation $PX = OX + OP_1$ results in $PX - OX = OP_1$.
6. The point X is a point on the generated figure. The difference between the distances from X to point O and from X to point P is a constant, equal to the radius of the circle. Therefore, X is a point on a hyperbola by the definition of a hyperbola.

c. Sample response: A hyperbola is symmetrical with respect to two axes—the line through the foci and the perpendicular bisector of the segment connecting the foci. It is also symmetric about the center.

d.

1. $(-c, 0)$
2. As shown in the diagram below, $c + a - (c - a) = 2a$.



3. Sample response: Yes, because from the definition of a hyperbola, the difference of the distances from H to F_1 and H to F_2 is constant for any point H on the locus.
4. Sample response: The constant difference for the hyperbola is $2a$, where a is the distance from each vertex to the origin.

e. **1.** Sample response: The equation in standard form for a hyperbola is not a function because, for many of the values in the domain, there is more than one corresponding value in the range.

2. Sample response: The equation for the hyperbola could be solved for y. This results in two solutions, each one a function. By plotting the two solutions on the same coordinate system, a graph of the hyperbola is obtained.

Assignment

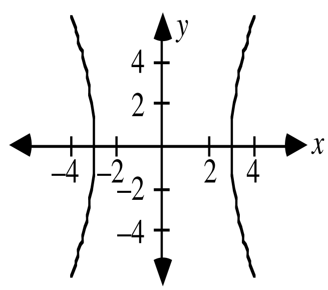
(page 362)

3.1 **a.** Sample response:

$$\sqrt{(x-(-4))^2+(y-0)^2}-\sqrt{(x-4)^2+(y-0)^2}=6$$
$$\sqrt{(x+4)^2+y^2}-\sqrt{(x-4)^2+y^2}=6$$

b. Sample response: You must solve for y and graph both solutions.

$$y=\frac{\sqrt{7(x^2-9)}}{3};\ 4x-9\geq 0 \text{ and } x^2\geq 9$$
$$y=-\frac{\sqrt{7(x^2-9)}}{3};\ 4x-9\geq 0 \text{ and } x^2\geq 9$$



3.2 **a.** Sample response: A constant sum of distances between a point on the locus and the foci defines an ellipse, while the constant difference of these distances defines a hyperbola.

b. Using a symbolic manipulator to solve the first equation for y yields the following:

$$\sqrt{(x-4)^2+(y-0)^2}-\sqrt{(x+4)^2+(y-0)^2}=6$$

$$\sqrt{(x-4)^2+(y-0)^2}=6+\sqrt{(x+4)^2+(y-0)^2}$$

$$x^2-8x+y^2+16=12\sqrt{x^2+8x+y^2+16}+x^2+8x+y^2+52$$

$$-16x-36=12\sqrt{x^2+8x+y^2+16}$$

$$y=\frac{\pm\sqrt{-7(x^2-9)}}{3}$$

Solving the standard form of the equation for y produces the same results:

$$\frac{x^2}{9}-\frac{y^2}{7}=1$$

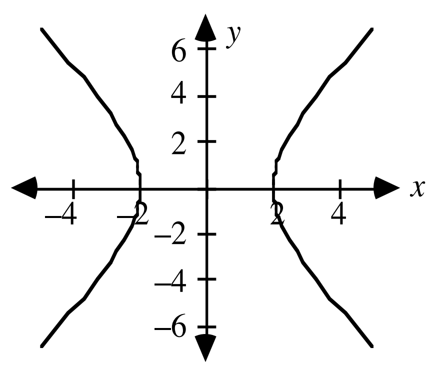
$$y=\frac{\pm\sqrt{-7(x^2-9)}}{3}$$

3.3 **a.** The vertices of the hyperbola are (–2,0) and (2,0).

b. **1.** Sample response:

$$y=\frac{3}{2}\sqrt{x^2-4} \text{ and } y=-\frac{3}{2}\sqrt{x^2-4}$$

2. Sample graph:



3. As $|x|$ increases, the value of $\sqrt{(x^2-4)}$ approaches x.

4. The functions are equations of lines:

$$y=\frac{3}{2}x \text{ and } y=-\frac{3}{2}x$$

5. Sample response: No, but as $|x|$ increases, the closer the y-values of the hyperbola are to the y-values of the lines.

c. Sample response: Yes, the two lines are asymptotes. As $|x|$ increases, the graph of the hyperbola gets closer and closer to the lines defined by the asymptotes.

d. Sample response: The denominators in the standard form of the equation for the hyperbola can be used to determine the equations of the asymptotes as follows:

$$y \approx \pm x\sqrt{\frac{9}{4}} = \pm\frac{3}{2}x$$

e. Sample response:

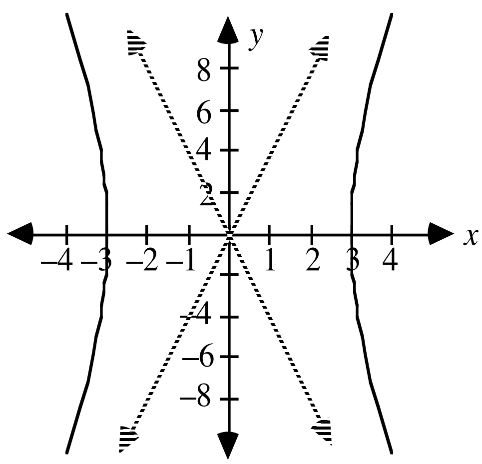
$$y = \frac{b}{a}x \text{ and } y = -\frac{b}{a}x$$

***3.4** For this hyperbola, $a^2 = 9$, $b^2 = 144$, and $c^2 = 153$.

a. The vertices of the hyperbola are (–3,0) and (3,0). The coordinates of the foci are $(-\sqrt{153},0)$ and $(\sqrt{153},0)$, or approximately (–12.4,0) and (12.4,0).

b. The equations of the asymptotes are $y = 4x$ and $y = -4x$.

c. Sample graph:



d. Sample response: Substituting 6 for x in the standard equation yields $y = \sqrt{432}$ or $y = -\sqrt{432}$. Thus, the coordinates of the two points on the hyperbola are $(6,-\sqrt{432})$ and $(6,\sqrt{432})$ or approximately (6,–20.8) and (6,20.8).

3.5 **a.** The equation of the translated hyperbola is:

$$\frac{(x+4)^2}{25}-\frac{(y-3)^2}{4}=1$$

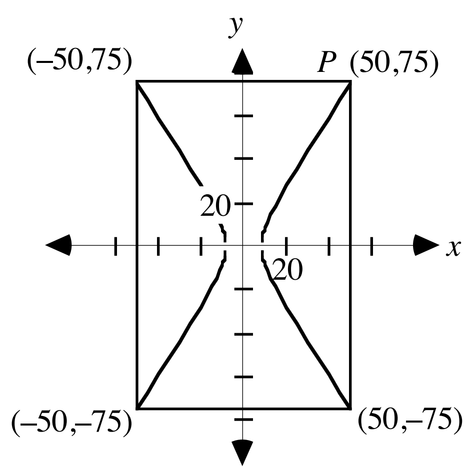
b. **1.** The coordinates of the center are (–4,3).

2. Sample response: The vertices for the original hyperbola are (5,0) and (–5,0). They are translated to the left 4 units and up 3 units, resulting in the points (–9,3) and (1,3). Since $c^2=a^2+b^2=25+4=29$, the foci are $\sqrt{29}$ units to the left and right of them. Thus, the coordinates of the foci are $\left(-9-\sqrt{29},3\right)$ and $\left(1+\sqrt{29},3\right)$.

3. The equations of the new asymptotes can be found by translating the equations of the original asymptotes 3 units vertically and –4 units horizontally, as follows:

$$\begin{aligned} y &= \frac{2}{5}(x+4)+3 & y &= -\frac{2}{5}(x+4)+3 \\ &= \frac{2}{5}x+\frac{8}{5}+\frac{15}{5} & &= -\frac{2}{5}x-\frac{8}{5}+\frac{15}{5} \\ &= \frac{2}{5}x+\frac{23}{5} & &= -\frac{2}{5}x+\frac{7}{5} \end{aligned}$$

***3.6** **a.** Answers will vary. In the following sample design, the foci are 30 cm apart. Therefore, $c=15$ and the coordinates of the foci are (15,0) and (–15,0).



b. For the sample design given in Part **a**:

$$\sqrt{(50-(-15))^2+75^2}-\sqrt{(50-15)^2+75^2}=2a$$
$$8.24 \approx a$$

Therefore, $a^2 \approx 67.9$ and $c^2 = 275$, resulting in $b^2 = c^2 - a^2 \approx 157.1$. The equation for the sample design is:

$$\frac{x^2}{67.9}-\frac{y^2}{157.1}=1$$

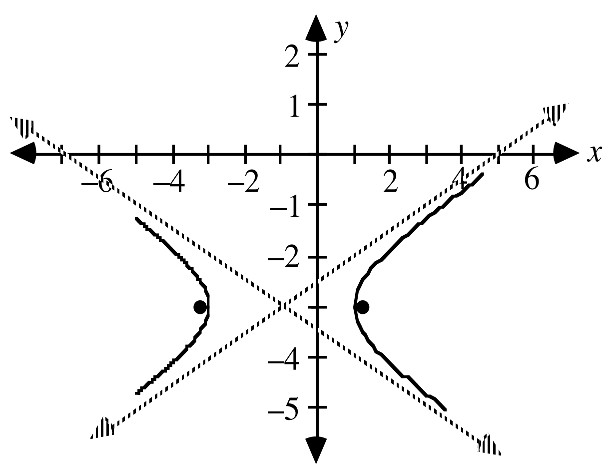
c. The sample response in Part **a** uses a hyperbolic curve with foci at (–15,0) and (15,0), vertices at (–8.24,0) and (8.24,0), and a constant difference of 16.5.

* * * * *

3.7 The center of the hyperbola has been translated from the origin to the point (–1,–3). Since $a = 2$ and $b = 1$, $c^2 = 5$ and $c = \sqrt{5}$. The vertices are at the points (1,–3) and (–3,–3). The foci are located at the points $\left(-1+\sqrt{5},-3\right)$ and $\left(-1-\sqrt{5},-3\right)$. By translation, the equations of the asymptotes are:

$$y=\frac{1}{2}(x+1)-3 \qquad\qquad y=-\frac{1}{2}(x+1)-3$$
$$=\frac{1}{2}x+\frac{1}{2}-\frac{6}{2} \qquad\qquad =-\frac{1}{2}x-\frac{1}{2}-\frac{6}{2}$$
$$=\frac{1}{2}x-\frac{5}{2} \qquad\qquad =-\frac{1}{2}x-\frac{7}{2}$$

Sample graph:



3.8 Using a symbolic manipulator to solve the first equation for *y* yields the following:

$$\sqrt{(x-5)^2+(y-0)^2}-\sqrt{(x+5)^2+(y-0)^2}=8$$
$$\sqrt{(x-5)^2+(y-0)^2}=8+\sqrt{(x+5)^2+(y-0)^2}$$
$$x^2-10x+y^2+25=16\sqrt{x^2+10x+y^2+25}+x^2+10x+y^2+89$$
$$-20x-64=16\sqrt{x^2+10x+y^2+25}$$
$$y=\frac{\pm3\sqrt{x^2-16}}{4}$$

Solving the equation in standard form for *y* produces the same results.

* * * * * * * * * *

Research Project (page 365)

a. Sample proof:

$$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$
$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$
$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$
$$x^2+2cx+c^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2$$
$$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$$
$$cx-a^2=-a\sqrt{(x-c)^2+y^2}$$
$$c^2x^2-2a^2cx+a^4=a^2(x^2-2cx+c^2+y^2)$$
$$c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2$$
$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$
$$x^2(c^2-a^2)-a^2y^2=a^2(c^2-a^2)$$
$$\frac{x^2(c^2-a^2)-a^2y^2}{a^2(c^2-a^2)}=1$$
$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$
$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

b. Sample proof:

$$\sqrt{(x+c)^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2=4a^2+4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2$$

$$4cx-4a^2=4a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=a\sqrt{(x-c)^2+y^2}$$

$$c^2x^2-2a^2cx+a^4=a^2(x^2-2cx+c^2+y^2)$$

$$c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2$$

$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$x^2(c^2-a^2)-a^2y^2=a^2(c^2-a^2)$$

$$\frac{x^2(c^2-a^2)-a^2y^2}{a^2(c^2-a^2)}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

(page 365)

Activity 4

Students investigate the parabola from a geometric point of view. They then develop the standard equation for a parabola with a vertical line of symmetry.

Materials List

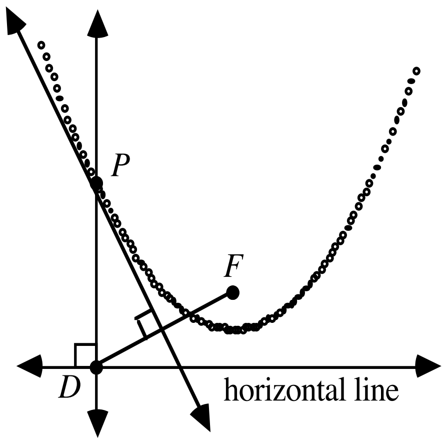
- graph paper (one sheet per student)

Technology

- geometry utility
- symbolic manipulator
- graphing utility

Exploration (page 366)

a–g. Student constructions should resemble the one shown in the diagram below. **Note:** A horizontal line is used so that the parabola generated represents a function.



h. Students create a scatterplot of the coordinates of P and graph a quadratic equation that approximates the points on the same coordinate system. **Note:** Some graphing utilities can determine polynomial regression equations for data sets. You may wish to allow students to use this feature to find an equation that fits the scatterplot.

i. Students repeat Parts **g** and **h** for at least two other locations of point F. When F is located below the directrix, the parabola opens downward.

Discussion (page 367)

a. Sample response: The definition of a parabola requires that any point on the parabola be the same distance from a line and a point not on the line. Since P is a point on the perpendicular bisector of $\overline{DF}$, $\overline{DP} \cong \overline{FP}$. Therefore, the figure is a parabola because the distance from the point on the parabola to the line (DP) is equal to the distance from the point on the parabola to the focus (FP).

b. Sample response: A parabola is symmetric to the line that passes through the focus and is perpendicular to the directrix.

c. Sample response: Both the focus and the vertex are located on the axis of symmetry. Therefore, when $\overline{DP}$ and $\overline{FP}$ lie on the same line, P is the vertex.

d. Sample response: If the distance is increased, the parabola gets wider. If the distance is decreased, the parabola gets narrower.

e.
1. Sample response: The parabola opened down instead of up.
2. The lead coefficient is negative when the focus is below the directrix.

f. Sample response: For a parabola that opens to the left, the directrix would be a vertical line and the focus would be to the left of the directrix. For a parabola that opens to the right, the directrix would be a vertical line and the focus would be to the right of the directrix.

g.
1. Sample response: The y-axis passes through the focus and is perpendicular to the directrix. The x-axis passes through the vertex, the point where the parabola and the y-axis intersect.
2. Sample response: The graph has the same shape as the parent but opens downward. It is symmetrical about the y-axis and has a maximum y-value at the origin.
3. Sample response: The graph has the same shape as $y = x^2$ but is translated vertically 3 units. It has a minimum y-value at (0,3), its vertex. It is symmetrical about the y-axis.
4. Sample response: The graph has the same shape as $y = x^2$ but is translated to the right 2 units. It has a minimum y-value of 0 at its vertex. It is symmetrical about the line $x = 2$.

h. Sample response: Change the coefficient of x^2 to a number not equal to 1.

i. Sample response: The graph is narrower, symmetrical about the line $x = 4$ instead of the y-axis, and has its vertex at (4,5) instead of (0,0).

Assignment

(page 368)

4.1

a. $y_2 = -4$

b. (0,0)

c. $x_1 = 2$

d. $\sqrt{(0-2)^2 + (4-y_1)^2} = \sqrt{(2-2)^2 + (-4-y_1)^2}$

e. Sample response:

$$\sqrt{(0-2)^2 + (4-y_1)^2} = \sqrt{(2-2)^2 + (-4-y_1)^2}$$
$$4 + y_1^2 - 8y_1 + 16 = y_1^2 + 8y_1 + 16$$
$$4 = 16y_1$$
$$y_1 = 1/4$$

4.2

a. Sample response: The x-coordinates do not change on a vertical line.

b. $\sqrt{(0-x)^2 + (4-y)^2} = \sqrt{(x-x)^2 + (-4-y)^2}$

c. Sample response:

$$\sqrt{(0-x)^2+(4-y)^2}=\sqrt{(x-x)^2+(-4-y)^2}$$
$$x^2+y^2-8y+16=y^2+8y+16$$
$$x^2-8y=8y$$
$$x^2=16y$$
$$y=\frac{1}{16}x^2$$

d. $(1/16)\bullet(2)^2=1/4$

4.3 **a.** Sample response: The graph has a vertex at (h,k) instead of (0,0). In other words, the parent function was translated h units horizontally and k units vertically.

b. The parent function undergoes a horizontal shrink or stretch by a.

c. Sample response: Both R_1R_2 and a determine the width of the parabola.

d. The two quantities are inversely related and can be described by the following equation, where c is a constant:

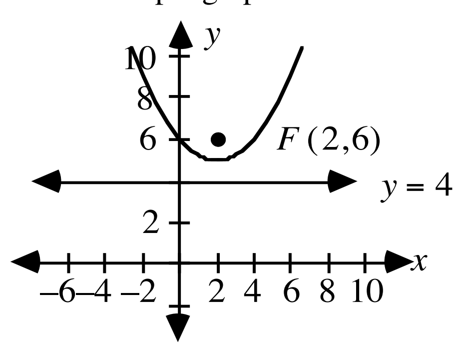
$$a=\frac{c}{R_1R_2}$$

e. The quantities $4p$ and a can be described by the equation below, where c is a constant:

$$a=\frac{c}{4p}$$

Note: You may wish to point out that, to simplify problems, the constant c is often given the value of 1 while a is adjusted accordingly.

***4.4** **a.** The vertex is located at (2,5) and the distance from the vertex to the focus is 1. Sample graph:



b. Sample response:

$$y=\frac{1}{4\bullet 1}(x-2)^2+5$$
$$=\frac{1}{4}(x-2)^2+5$$

c. By inspection, students should observe that both (4,6) and (0,6) are equidistant from the focus and the directrix.

d. These coordinates satisfy the equation from Part **b**.

$$y = \frac{1}{4}(4-2)^2 + 5 \qquad y = \frac{1}{4}(0-2)^2 + 5$$
$$= \frac{1}{4}(2)^2 + 5 \qquad = \frac{1}{4}(-2)^2 + 5$$
$$= 6 \qquad = 6$$

* * * * *

4.5 The parabola is concave down because the directrix is above the focus. The distance from the focus to the vertex is 1.5; the coordinates of the vertex are (–1,3.5). The equation is:

$$y = -\frac{1}{6}(x+1)^2 + 3.5$$

4.6 **a.** Sample response: Since $a = 1/20$, then

$$\frac{1}{20} = \frac{1}{4p}$$

Solving for *p* results in $p = 5$. Since *p* is the distance from the focus to the vertex, the bulb should be 5 units above the vertex at (0,5).

b. Sample response: The maximum values of *y* occur when $x = 20$ and $x = -20$. Substituting these values into the equation of the parabola, $y = 20$ cm.

4.7 **a.** If students place the vertex at (0,0) with the *y*-axis as the line of symmetry, then the equation of the parabola will have the form $y = ax^2$. The maximum values of *y* occur when $x = 90$ and $x = -90$. Students can then experiment with values of *a* and analyze the resulting graphs to determine an appropriate equation for the satellite dish.

Students also may choose a reasonable depth for the satellite dish and solve for the corresponding value of *a*. For example, based on a depth of 60 cm, an equation can be determined in the following manner:

$$y = ax^2$$
$$60 = a(90)^2$$
$$60 = 8100a$$
$$a \approx 0.0074$$

Therefore, one possible equation is $y = 0.0074x^2$.

b. For the sample response given in Part **a**:

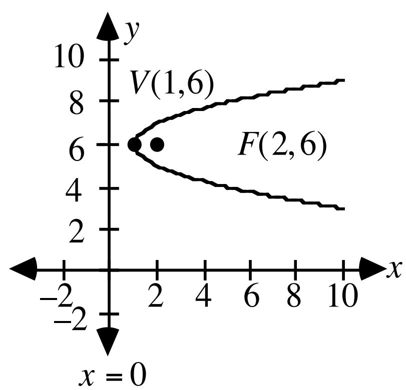
$$0.0074 \approx \frac{1}{4p}$$
$$4p \approx 135$$
$$p \approx 33.8$$

Since p is the distance from the focus to the vertex, the receiver should be placed about 33.8 cm above the vertex.

4.8 **a.** Sample response: The vertex is the point (1,6). Using the standard form of the equation:

$$x = a(y-6)^2 + 1\frac{1}{4p}$$
$$= \frac{1}{4p}(y-6)^2 + 1$$
$$= \frac{1}{4 \bullet 1}(y-6)^2 + 1$$
$$= \frac{1}{4}(y-6)^2 + 1$$

b. Sample graph:



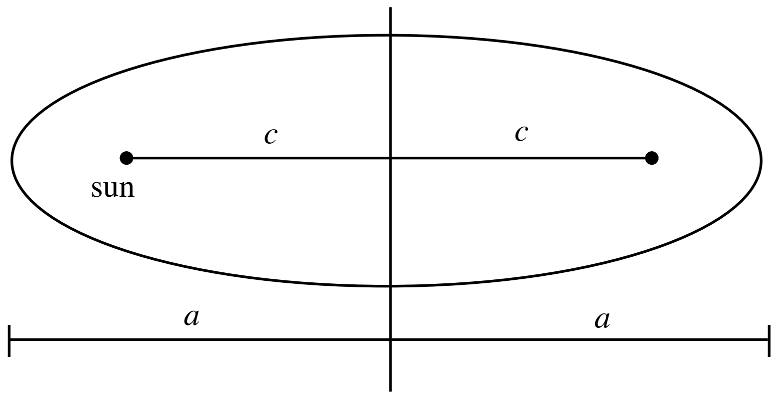
c–d. Both the points (2,4) and (2,8) are equidistant from the focus and the directrix. These coordinates satisfy the equation from Part **a**.

$x = \frac{1}{4}(y-6)^2 + 1$	$x = \frac{1}{4}(y-6)^2 + 1$
$= \frac{1}{4}(4-6)^2 + 1$	$= \frac{1}{4}(8-6)^2 + 1$
$= \frac{1}{4}(-2)^2 + 1$	$= \frac{1}{4}(2)^2 + 1$
$= 2$	$= 2$

* * * * * * * * * *

Answers to Summary Assessment (page 373)

1. Sample response: The closest approach of the comet is $9.00 \bullet 10^6$ km, while its farthest distance is $1.79 \bullet 10^9$ km. As shown in the diagram below, this means that $a - c = 9.00 \bullet 10^6$ and $a + c = 1.79 \bullet 10^9$.



This means that $a = 9.00 \bullet 10^8$ and $c = 8.91 \bullet 10^8$. Since $b = \sqrt{a^2 - c^2}$, $b = 1.27 \bullet 10^8$. The equation of the ellipse in standard form is therefore:

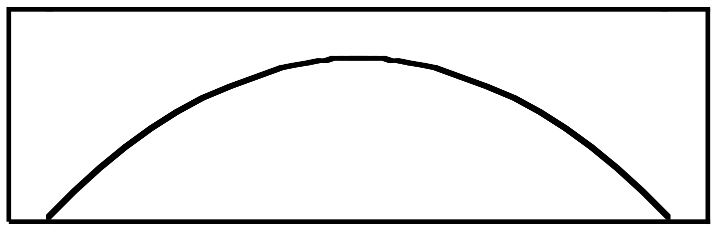
$$\frac{x^2}{8.1 \bullet 10^{17}} + \frac{y^2}{1.6 \bullet 10^{16}} = 1$$

2. **a.** Answers will vary. Students may use various conic sections to model the curves found in each bridge design. Arches may be represented as semicircles, semi-ellipses, or parabolas. The cables on a suspension bridge may be represented by parabolas or hyperbolas. **Note:** Many of the curves found in bridge designs are actually catenaries. However, they can be closely approximated by conic sections.

b. Answers will vary. Students should describe how they determined each equation. One method might involve tracing the curve found in the bridge design onto a coordinate plane. Students may then identify the approximate coordinates of points in the graph and use them to determine an equation for the conic.

Module Assessment

1. Imagine that you are a construction engineer. As shown in the diagram below, you have been asked to design a highway tunnel in the shape of a parabolic arch.



A two-lane road will pass through the tunnel. Each lane should be at least 9.0 m wide. The tunnel entrance should have a maximum height of 4.2 m.

a. Draw a sketch of your tunnel on a two-dimensional coordinate system.

b. Determine an equation for a parabola that will satisfy the given conditions.

c. In your design, how wide is each lane?

d. Consider a truck that is 3.0 m wide. Assuming that the truck remains on its own side of the center line, how tall can it be and still pass through the tunnel?

2. Determine an equation for each figure described in Parts **a–e**.

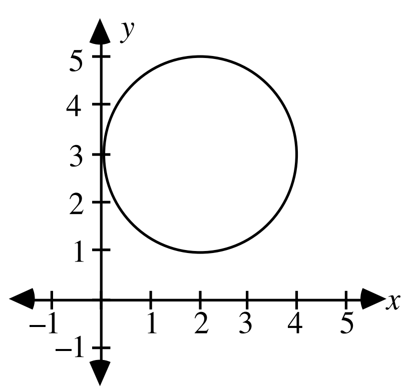
a. an ellipse with vertices at (–5,0), (–4,0), (5,0), and (4,0)

b. a parabola passing through (2,5), (1,6), and (3,6)

c. a circle with a radius of 7 passing through (–3,2) and (4,9)

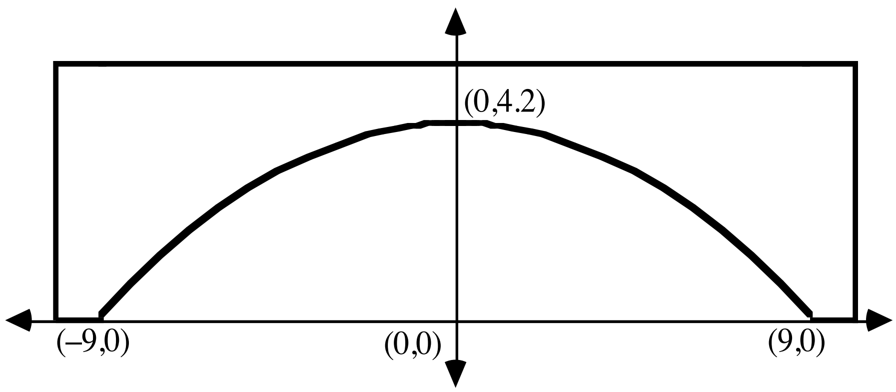
d. a hyperbola with x-intercepts of ± 4 and a focus at (5,0)

e. the figure shown in the graph below:



Answers to Module Assessment

1. **a.** Sample sketch:



b–c. Sample response: From the sketch, $y = ax^2 + 4.2$. By substituting the point (9,0) into the equation, you can determine the value of *a* needed to assure a lane width of 9 m.

$$0 = a(9)^2 + 4.2$$
$$-4.2 = 81a$$
$$a = \frac{-4.2}{81} \approx -0.05185$$

One possible value for *a* could be –0.05. By substituting 0 for *y*, you can check to see if this assures a width of at least 9 m:

$$0 = -0.05x^2 + 420$$
$$-4.2 = -0.05x^2$$
$$x = \sqrt{84}$$
$$x \approx 9.2$$

The resulting width is approximately 9.2 m. Therefore, one equation for the tunnel could be $y = -0.05x^2 + 4.2$.

d. Using the sample equation given above,

$$y = -0.05(3 + 0.45)^2 + 4.2 \approx 3.6$$

The truck can be no more than 3.6 m high.

2. **a**. $\frac{x^2}{25}+\frac{y^2}{16}=1$

b. $y=(x-2)^2+5$

c. $(x-4)^2+(y-2)^2=49$

d. $\frac{x^2}{16}-\frac{y^2}{9}=1$

e. $(x-2)^2+(y-3)^2=4$

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Broughton, P. "Halley's Comet in the Classroom." *Mathematics Teacher* 79 (February 1986): 85–89.

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Flashbacks

Activity 1

1.1 Determine the distance between each of the following pairs of points:

a. (1,5) and (–6,5)

b. (3,–2) and (3,–9)

c. (4,3) and (–6,–4)

1.2 Rewrite the following equation so that its left side is $x^2 + y^2$.

$$20x^2 + 4 = -20y^2 - 6$$

1.3 Rewrite the following equation in the form $Ax + By = C$ where A, B, and C are not fractions.

$$y = \frac{5}{3}x + \frac{3}{2}$$

Activity 2

2.1 Solve the following equation for x:

$$\sqrt{x+2} = 5$$

2.2 Find the distance between $(a,-2)$ and $(-3,b)$.

2.3 Solve for the radical in the following expression:

$$x^2 - 4x + 4 + y^2 = -12\sqrt{x^2 + 4x + 4 + y^2} + x^2 + 4x + y^2 + 40$$

Activity 3

3.1 Consider the equation below:

$$y = \frac{x+2}{x-2}$$

a. Identify the domain and range of the function.

b. Create a graph of the function.

c. Write equations for the asymptotes.

Activity 4

4.1 Identify each of the following transformations of the parent function $y = |x|$.

a. $y = |x + 3|$

b. $y = |x| + 3$

c. $y = -|x|$

d. $y = 3|x|$

Answers to Flashbacks

Activity 1

1.1 **a.** $d = 7$

b. $d = 7$

c.
$$\begin{aligned} d &= \sqrt{(-6-4)^2 + (-4-4)^2} \\ &= \sqrt{100+64} \\ &= \sqrt{164} \\ &\approx 12.8 \end{aligned}$$

1.2 Sample response:

$$\begin{aligned} 20x^2 + 4 &= -20y^2 - 6 \\ 20x^2 + 20y^2 &= -10 \\ \frac{20x^2 + 20y^2}{20} &= -\frac{10}{20} \\ x^2 + y^2 &= -\frac{1}{2} \end{aligned}$$

1.3 Sample response:

$$\begin{aligned} y &= \frac{5}{3}x + \frac{3}{2} \\ 6y &= 6\left(\frac{5}{3}x + \frac{3}{2}\right) \\ 6y &= 10x + 9 \\ -10x + 6y &= 9 \end{aligned}$$

Activity 2

2.1 Sample response:

$$\begin{aligned} \sqrt{x+2} &= 5 \\ \left(\sqrt{x+2}\right)^2 &= 5^2 \\ x + 2 &= 25 \\ x &= 23 \end{aligned}$$

2.2 Using the distance formula:

$$\sqrt{(a-(-3))^2+(-2-b)^2}=d$$

$$\sqrt{a^2+6a+b^2+4b+13}=d$$

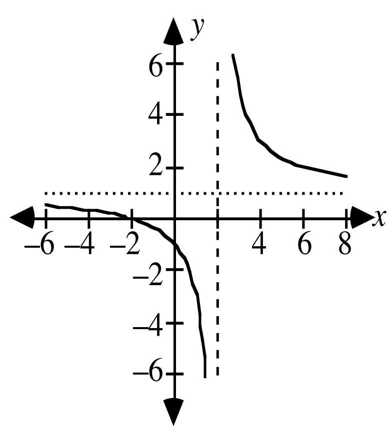
2.3 The simplified equation is:

$$\left(\frac{-8x-36}{-12}\right)=\sqrt{x^2+4x+4+y^2}$$

Activity 3

3.1 **a.** The domain is the set of real numbers except 2. The range is the set of real numbers except 1.

b. Sample graph:



c. The equation of the horizontal asymptote is $y=1$; the equation of the vertical asymptote is $x=2$.

Activity 4

4.1 **a.** a horizontal translation of –3 units

b. a vertical translation of 3 units

c. a reflection in the *x*-axis

d. a vertical stretch by 1/3

Controlling the Sky with Parametrics

Air traffic controllers must know the precise locations of all nearby planes at any given time. In this module, you discover how mathematics can help model these situations.

Sherry Horyna • Ed Sisolak • Dan West

Teacher Edition

Controlling the Sky with Parametrics

Overview

In this module, students model linear and circular motion using parametric equations.

Objectives

In this module, students will:

- use parametric equations and their graphs to model linear and circular paths
- convert parametric equations into nonparametric form $y = f(x)$
- establish appropriate intervals for the domains and ranges of parametric graphs.

Prerequisites

For this module, students should know:

- the difference between functions and relations
- the distance formula
- how to solve systems of linear equations
- how to represent coordinates on a unit circle using trigonometric functions
- the standard form of the equation for a circle.

Time Line

Activity	**1**	**2**	**3**	**Summary Assessment**	**Total**
Days	2	2	2	2	8

Materials Required

Materials	1	2	3	Summary Assessment
graph paper (optional)	X	X	X	X

Technology

Software	1	2	3	Summary Assessment
graphing utility	X	X	X	X
spreadsheet (optional)			X	

Controlling the Sky with Parametrics

Introduction

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This module uses the context of airplanes and airports to explore parametric equations.

(page 379)

Activity 1

In this activity, students use parametric equations to model linear motion.

Materials Required

- graph paper (optional)

Technology

- graphing utility

Teacher Note

When graphing equations that represent the motions of two or more objects, students should set their graphing utilities to plot all graphs simultaneously, if possible. (Some graphing utilities may not have this feature. In that case, equations typically are graphed one at a time, in the same sequential order they are entered into the utility.)

Exploration 1

(page 379)

a. As they enter the screen, the small plane is 1000 m from the tower, while the jet is 1500 m from the tower.

b. Student methods for determining these equations may vary. For example, they may use ordered pairs to determine that the slope of each line is 0, then substitute this value and the coordinates of a point into the point-slope form of a line. The linear equation representing the jet's path is $y = 0x + 1500$. The linear equation representing the small airplane's path is $y = 0x + 1000$.

c. **1.** Sample response: After 1 sec, the coordinates that represent the small plane's location are (55,1000). The coordinates that represent the jet's location are (90,1500).

2. Sample response: After 2 sec, the coordinates that represent the small plane's location are (110,1000). The coordinates that represent the jet's location are (180,1500).

3. The time that the small plane remains on the screen is:

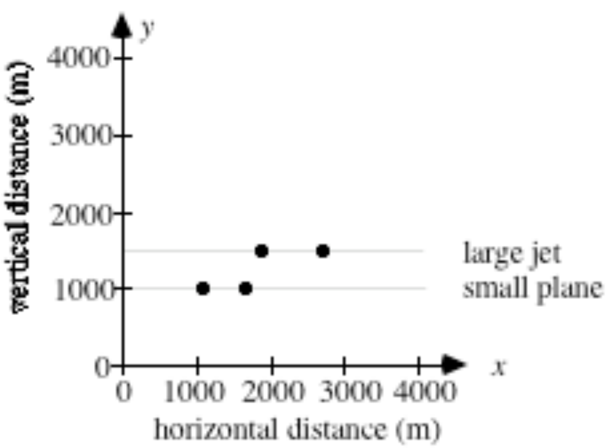
$$\frac{4000 \text{ m}}{55 \text{ m/sec}} \approx 72.7 \text{ sec}$$

The time that the jet remains on the screen is:

$$\frac{4000 \text{ m}}{90 \text{ m/sec}} \approx 44.4 \text{ sec}$$

d. **1.** Using the boundaries of radar screen, the domain for both functions is [0, 4000]. The range for the function modeling the small plane is 1000. The range for the function modeling the jet is 1500.

2. The locations after 20 sec are (1100,1000) and (1800,1500) for the small plane and jet, respectively. The locations after 30 sec are (1650,1000) and (2700,1500) for the small plane and jet, respectively. Sample graph:



Discussion 1

(page 380)

a. Sample response: The ordered pairs represent a plane's distances east and north of the tower.

b. Sample response: The controller would know neither the speeds nor the altitudes of the planes.

c. Sample response: Just by looking at the equations and the graphs from Exploration **1**, you would not be able to determine a plane's location at a specific time. Since distance = rate • time, the x-coordinate of each plane n sec after it passes the tower can be found by multiplying the speed of each plane by the desired time.

Teacher Note

In this activity, it may be helpful for students to express parametric equations with 0s in the appropriate positions. For example, the equations

$$\begin{cases} x = 2t \\ y = 3 \end{cases}$$

can also be written as

$$\begin{cases} x = 0 + 2t \\ y = 3 + 0t \end{cases}$$

This indicates that the position at $t = 0$ is (0,3), a rate of change in the *x*-component of 2, and a rate of change in the *y*-component of 0.

Exploration 2

(page 380)

a. Sample table:

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0	0	1000
1	$55 \bullet 1 = 55$	1000
2	$55 \bullet 2 = 110$	1000
3	$55 \bullet 3 = 165$	1000
⋮	⋮	⋮
10	$55 \bullet 10 = 550$	1000
⋮	⋮	⋮
t	$55 \bullet t$	1000

b. Sample table:

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0	0	1500
1	$90 \bullet 1 = 90$	1500
2	$90 \bullet 2 = 180$	1500
3	$90 \bullet 3 = 270$	1500
⋮	⋮	⋮
10	$90 \bullet 10 = 900$	1500
⋮	⋮	⋮
t	$90 \bullet t$	1500

c. Sample parametric equations:

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases}$$

d. Sample parametric equations:

$$\begin{cases} x = 0 + 90t \\ y = 1500 + 0t \end{cases}$$

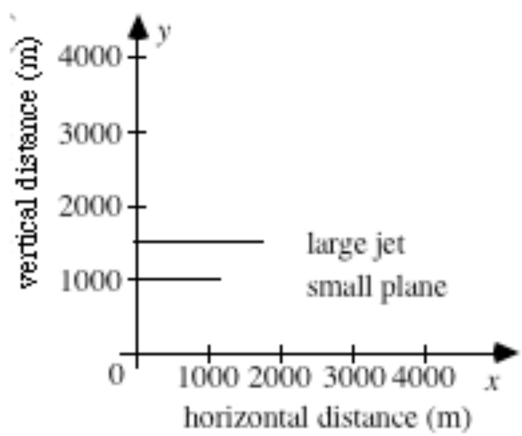
e. **1.** Using the TI-82, TI-85, or TI-92 calculators, for example, students should set the mode to "parametric" before entering the equations.

Students should understand that there are no "correct" intervals or increments for the parameter. Satisfactory intervals and increments allow the user of a graphing utility to interpret information from the graph. An interval that is too small will not graph enough pertinent information; an interval that is too large will produce a graph outside the appropriate intervals for *x* and *y*.

An increment that is too small may cause the utility to plot the graph very slowly (since the utility plots more points). On the other hand, a relatively large increment may result in a graph that is plotted very quickly, but without enough detail.

Using a TI-92 calculator, for example, an appropriate interval and increment for *t* are $t_{min} = 0$, $t_{max} = 73$, and $t_{step} = 0.1$, since at least one of the planes remains on the radar screens for approximately 73 sec.

Sample graph:



Discussion 2

(page 382)

a. Sample response: When both pairs of parametric equations in Exploration **2** are graphed simultaneously, the rates that the lines are drawn are different. The line representing the motion of the large jet is drawn more quickly than the line representing the motion of the small airplane. The graphs in Exploration **2** better model the scenario.

b. **1.** Sample response: Changing the increment for t affects the speed at which the graph is drawn. It also changes the increments of the trace feature. **Note:** Students should recognize that the utility plots the graph more slowly as the increment decreases because the number of ordered pairs increases. The actual speed of the object being modeled remains the same.

2. Sample response: If t really represented time on the graphing utility, each point should be plotted more quickly as the increment decreases. The total time to plot the graph should remain the same.

c. Sample response: Yes. Using the settings of $t_{\min} = 0$, $t_{\max} = 73$, and $t_{\text{step}} = 0.1$ along with the trace feature on a TI-92 calculator, we were able to determine each airplane's position every 0.1 sec for the first 73 sec after they flew past the tower.

d. Sample response: The interval for t would have to include negative numbers.

e. Sample response: The parameter t is the independent part of the parametric equations and therefore defines the domain. The ordered pair (x,y) is the dependent part and therefore defines the range. **Note:** This means that the parametric function $f(t) = (x,y)$. This is defined as the mapping of the real numbers onto the set of real numbers crossed with the set of real numbers, denoted $\Re \to \Re \times \Re$. In other words, the function f pairs each real number with an ordered pair of real numbers.

f. **1.** Sample response: The equations representing the small plane remain the same. The equation for x in the parametric equations representing the jet would be changed to $x = 4000 - 90t$.

2. Sample response: Setting the equations representing x in each set of parametric equations equal to each other and solving for t results in the time at which the two planes would pass each other.

Teacher Note

For the remainder of this teacher edition, the sample responses will list appropriate intervals for the domain and range on a TI-92 as follows:

$$t_{\min},\ t_{\max},\ t_{\text{step}}$$
$$x_{\min},\ x_{\max}$$
$$y_{\min},\ y_{\max}$$

The interval $[t_{\min},\ t_{\max}]$ represents the domain. The range is constructed from the ordered pairs (x,y) where x is an element of $[x_{\min}, x_{\max}]$ and y is an element of $[y_{\min}, y_{\max}]$.

Assignment

(page 383)

1.1 **a.** Using the parametric equations below to model the motion of the small airplane puts the small plane at the coordinates $(440, 1000)$ after 8 seconds.

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases}$$

Using the parametric equations below to model the motion of the large jet puts the jet at the coordinates $(720, 1500)$ after 8 seconds.

$$\begin{cases} x = 0 + 90t \\ y = 1500 + 0t \end{cases}$$

b. **1.** By tracing the graph, students should find that the small airplane travels the first 100 m in 1.8 sec. Sample intervals for domain and range for a TI-92:

$$t_{\text{min}} = 0,\ t_{\text{max}} = 8,\ t_{\text{step}} = 0.1$$
$$x_{\text{min}} = 0,\ x_{\text{max}} = 500$$
$$y_{\text{min}} = 0,\ y_{\text{max}} = 1200$$

2. The time required can be found algebraically using the equation for x for the small airplane. Since the small airplane's speed is 55 m/sec, the time required to travel 100 m can be found as follows:

$$100 = 0 + 55t$$
$$t \approx 1.8 \text{ sec}$$

c. **1.** By tracing the graph, students should find that the large jet travels the first 100 m in 1.1 sec. Sample intervals for domain and range for a TI-92:

$$t_{\text{min}} = 0,\ t_{\text{max}} = 8,\ t_{\text{step}} = 0.1$$
$$x_{\text{min}} = 0,\ x_{\text{max}} = 800$$
$$y_{\text{min}} = 0,\ y_{\text{max}} = 1600$$

2. The time required can be found algebraically using the equation for x for the jet. Since the jet's speed is 90 m/sec, the time required to travel 100 m can be found as follows:

$$100 = 0 + 90t$$
$$t \approx 1.1 \text{ sec}$$

d. After 8 sec, the small plane's position is (440,1000). The large jet's position is (720,1500). The distance between the planes can be found using the Pythagorean theorem:

$$\sqrt{(280)^2 + (500)^2} \approx 573 \text{ m}$$

1.2 **a.** Students may graph the given parametric equations using the following settings:

$$t_{min} = 0 \qquad t_{max} = 73 \qquad t_{step} = 1$$
$$x_{min} = 0 \qquad x_{max} = 4000$$
$$y_{min} = 0 \qquad y_{max} = 4000$$

b. By tracing, the planes pass each other at approximately 27 sec.

c. The planes pass approximately 27 sec after they enter the radar screen. This can be found by solving the following system of equations for t:

$$\begin{cases} x = 0 + 55t \\ x = 4000 - 90t \end{cases}$$

1.3 **a.** Sample parametric equations:

$$\begin{cases} x = 1000 + 0t \\ y = 0 + 55t \end{cases}$$

b. Sample parametric equations:

$$\begin{cases} x = 1500 + 0t \\ y = 0 - 90t \end{cases}$$

***1.4** **a.** To produce a collision, the parametric equations for the small airplane and the large jet, respectively, should be:

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases} \text{ and } \begin{cases} x = a + 0t \\ y = 0 + 90t \end{cases}$$

where a represents the unknown horizontal distance from the tower to the jet. Students may estimate the value of a by experimenting with a graphing utility. By tracing the graphs, a collision occurs when a is approximately 610 m.

Solving algebraically, a collision occurs at the point where the small plane travels a meters and the large jet travels 1000 m. These distances can be represented by the equations $55t = a$ and $1000 = 90t$. Solving for t yields the following equations:

$$t = \frac{a}{55} \text{ and } t = \frac{1000}{90}$$

For a collision to occur, the time for each plane to reach that location must be the same:

$$\frac{a}{55} = \frac{1000}{90}$$
$$a \approx 611$$

b. Students may change the speed or the initial position of either plane. Sample response: Since the large jet should be 25 m away when the small plane reaches the collision point, the large jet should pass the control tower 25 m to the west of its initial position in Part **a**, or at about 586 m. The resulting parametric equations are:

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases} \text{ and } \begin{cases} x = 586 + 0t \\ y = 0 + 90t \end{cases}$$

* * * * *

1.5 Students may select any reasonable value for the distance between the two sets of tracks. **Note:** In the following sample response, distances are expressed in kilometers, while time is expressed in hours. Students may choose to express these quantities in other units.

a. Sample parametric equations for the blue train and the red train, respectively:

$$\begin{cases} x = 5 - 55t \\ y = 0.03 + 0t \end{cases} \text{ and } \begin{cases} x = 0 + 60t \\ y = 0.01 + 0t \end{cases}$$

To help visualize the situation, students also may wish to draw a vertical line to represent the location of the switch. Sample parametric equations for this line are given below:

$$\begin{cases} x = 3 - 0t \\ y = 0 + t \end{cases}$$

b. Sample response: No. The blue train will reach the switch before the red train. If it is switched, the blue train will be on the same track as the red train and heading toward it.

c. Answers will vary. Sample response: Slowing the blue train to 35 km/hr allows the red train to pass the switch before the blue train reaches it. The new equations for the blue train would be:

$$\begin{cases} x = 5 - 35t \\ y = 0.03 + 0t \end{cases}$$

In this case, the blue train will reach the switch at $t \approx 0.06 \text{ hr} \approx 3.6 \text{ min}$. This can be found by solving the equation $3 = 5 - 35t$ for t.

When $t = 0.06$, the red train will be 3.6 km from its starting position or 0.6 km from the switch. Therefore, the switch can be made safely.

1.6 **a.** Sample parametric equations for the red train and the blue train, respectively:

$$\begin{cases} x = 0 + 60t \\ y = 2.8 + 0t \end{cases} \text{ and } \begin{cases} x = 4 + 0t \\ y = 0 + 50t \end{cases}$$

b. Sample response: No. The blue train will reach the intersection when $y = 2.8$. The red train will reach the intersection when $x = 4$. The value of t when the blue train reaches the intersection is found by solving the equation

$$2.8 = 0 + 50t$$
$$t \approx 0.06 \text{ hr}$$

When $t = 0.06$, the red train is at $x = 3.6$. This means that the red train is still 0.4 km from the intersection.

The value of t when the red train reaches the intersection can be found as follows:

$$4 = 0 + 60t$$
$$t \approx 0.07 \text{ hr}$$

When $t = 0.07$, the blue train is at $y = 3.5$. This means that the blue train is 0.7 km past the intersection.

* * * * * * * * * *

(page 385)

Activity 2

In this activity, students use parametric equations to model linear motion along oblique paths.

Materials List

- graph paper (optional)

Technology

- graphing utility

Exploration

(page 385)

Students model the motion of two airplanes traveling on nonperpendicular and nonparallel paths.

a. **1.** The initial position of the Boeing 747 is (2000,0); the initial position of the Piper Cub is (0,500).

2. After 1 sec, the position of the Boeing 747 is (1850,100); the Piper Cub is at (60,540).

3. Since students know two points on the path of each plane, they should determine the following linear equations for the 747 and the Piper Cub, respectively:

$$y = -\frac{2}{3}x + \frac{4000}{3} \text{ and } y = \frac{2}{3}x + 500$$

b. **1.** In 1 sec, the Boeing 747 travels 150 m west. Therefore, the horizontal component of its velocity is –150 m/sec. This value is negative, because west on this coordinate system is in the negative direction.

2. In 1 sec, the Boeing 747 travels 100 m north. Therefore, the vertical component of its velocity is 100 m/sec.

3. The ratio is $100/(-150) = -2/3$.

4. The slope of the path of the 747 can be found using the points (2000,0) and (1850,100):

$$\frac{100-0}{1850-2000} = -\frac{2}{3}$$

Hence, the slope of the path is the same as the ratio in Step **3**.

c. Sample tables:

Table 3: The Boeing 747's position

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0	2000	0
1	$2000 - 150 = 1850$	$0 + 100 = 100$
2	$2000 - 150 \bullet 2 = 1700$	$0 + 100 \bullet 2 = 200$
3	$2000 - 150 \bullet 3 = 1550$	$0 + 100 \bullet 3 = 300$
⋮	⋮	⋮
7	$2000 - 150 \bullet 7 = 950$	$0 + 100 \bullet 7 = 700$
⋮	⋮	⋮
t	$2000 - 150t$	$0 + 100t$

Table 4: The Piper Cub's position

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0	0	500
1	$0 + 60 = 60$	$500 + 40 = 540$
2	$0 + 60 \bullet 2 = 120$	$500 + 40 \bullet 2 = 580$
3	$0 + 60 \bullet 3 = 180$	$500 + 40 \bullet 3 = 620$
⋮	⋮	⋮
7	$0 + 60 \bullet 7 = 420$	$500 + 40 \bullet 7 = 780$
⋮	⋮	⋮
t	$0 + 60t$	$500 + 40t$

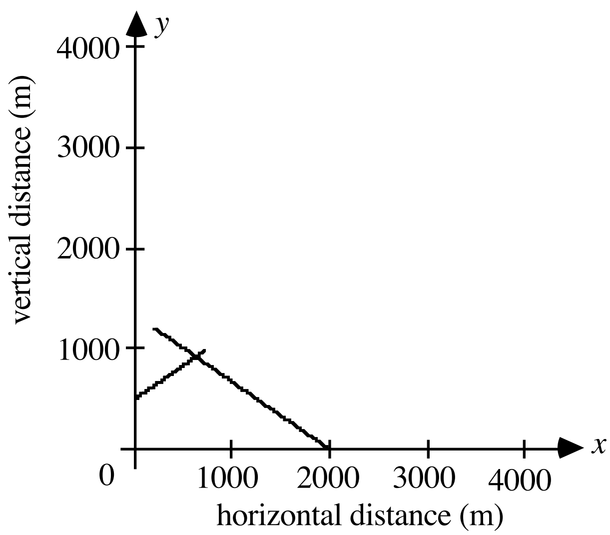
d. Sample parametric equations for the Boeing 747 and Piper Cub, respectively:

$$\begin{cases} x = 2000 - 150t \\ y = 0 + 100t \end{cases} \text{ and } \begin{cases} x = 0 + 60t \\ y = 500 + 40t \end{cases}$$

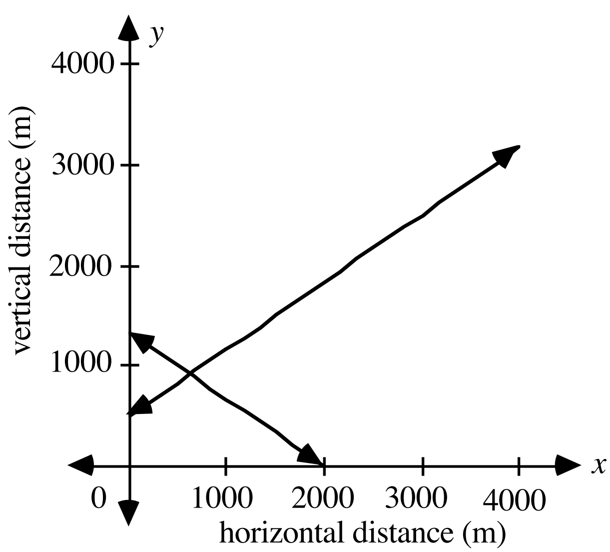
e. Sample intervals for domain and range for a TI-92:

$$t_{\min} = 0,\ t_{\max} = 12,\ t_{step} = 0.1$$
$$x_{\min} = 0,\ x_{\max} = 4000$$
$$y_{\min} = 0,\ y_{\max} = 4000$$

Sample graph:



f. **1.** Sample graph:



2. Sample response: The parametric graphs are segments whose lengths are determined by the parameter interval. The equations in slope-intercept form are lines that contain those segments.

g. Students may trace to find the intersection or solve the system of equations. The coordinates of the point of intersection are approximately (625,917).

h. Sample parametric equations for the Boeing 747 and Piper Cub, respectively:

$$\begin{cases} x = 1700 - 150t \\ y = 200 + 100t \end{cases} \text{ and } \begin{cases} x = 120 + 60t \\ y = 580 + 40t \end{cases}$$

The graphs are segments whose lengths are determined by the parameter interval. Like the graphs from Part **e**, these segments are contained in the lines from Part **a**:

$$y = -\frac{2}{3}x + \frac{4000}{3} \text{ and } y = \frac{2}{3}x + 500$$

Discussion

(page 387)

a. Sample response: The variable *t* represents time and the coordinates (x,y) represent the location of the airplane at a given time. The constant terms in the parametric equations represent the coordinates of the plane's initial position. For the Piper Cub, this location is (0,500) or 500 m north of the tower. The coefficients for *t* represent the horizontal and vertical components of the velocity. The Piper Cub is moving at a rate of 60 m/sec east and at a rate of 40 m/sec north.

b.

1. Sample response: Changing the initial positions of the planes changed the constant terms in the parametric equations.
2. Sample response: Using the same interval for *t*, the graphs in Part **h** appear shorter than the graphs from Part **e**. However, the graphs are contained in the same lines.

   Note: You may wish to point out that, using the domain of all real numbers, the graphs of both sets of parametric equations would be identical to the graphs of the linear equations in Part **a**.

c. Any line with parametric equations

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

has slope d/b. This can be found by solving for *t* in each equation, setting the first expression equal to the second expression, then solving for *y*.

Solving each equation for *t* yields

$$\frac{x-a}{b} = t \text{ and } \frac{y-c}{d} = t$$

Solving for y gives the equation of the line in slope-intercept form:

$$\frac{x-a}{b} = \frac{y-c}{d}$$

$$\frac{d}{b}x + \left(\frac{cb-ad}{b}\right) = y$$

d. Sample response: Since there is an infinite number of points on the line, there is an infinite number of possible x- and y-coordinates which can be represented as the x- and y-values when $t = 0$. Likewise, there is an infinite number of pairs of numbers whose ratio is equal to the slope.

e. **1.** Sample response: No. The 747 reaches the point where the paths cross at $t \approx 9.2$ sec. The Piper Cub doesn't reach this point until $t \approx 10.4$ sec.

2. Sample response: The graph of the parametric equations shows where the planes are located at each time increment. This makes it possible to see that the planes are not at the point of intersection at the same time.

f. Sample response: Find the coordinates of the second plane at the time that the first plane reaches the point of intersection. You can then find how far apart the two planes are using the distance formula.

Assignment

(page 387)

2.1 **a.** Sample parametric equations, where the interval for t is [0, 4]:

$$\begin{cases} x = -7 + 2t \\ y = -9 + 3t \end{cases}$$

b. Sample response, where the interval for t is [0, 4]:

$$\begin{cases} x = 0 + 2t \\ y = 1.5 + 3t \end{cases}$$

c. Sample response, where the interval for t is [0, 2]:

$$\begin{cases} x = -7 + 4t \\ y = -9 + 6t \end{cases}$$

d. Sample response: Since there are any number of starting points and velocities, there are an infinite number of possibilities for the parametric equations.

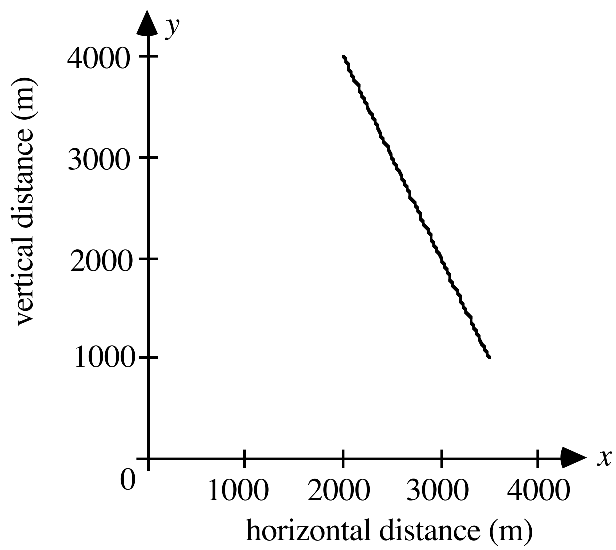
2.2 **a.** The Boeing 747 passes the point of intersection, (625,917), approximately 9.2 sec after it enters the radar screen. At $t = 9.2$, the coordinates of the Piper Cub are approximately (552,868). Using the distance formula, the distance between the two planes is approximately 85.6 m.

b. The Piper Cub passes the point of intersection, (625,917), approximately 10.4 sec after it enters the radar screen. At $t = 10.4$, the coordinates of the Boeing 747 are approximately (440,1040). Using the distance formula, the distance between the two planes is approximately 222.2 m.

2.3 **a.** Sample intervals for domain and range for a TI-92:

$$t_{\min} = 0,\ t_{\max} = 50,\ t_{step} = 0.5$$
$$x_{\min} = 0,\ x_{\max} = 4000$$
$$y_{\min} = 0,\ y_{\max} = 4000$$

Sample graph:



b. The airplane was first detected at the coordinates (3500,1000).

c. The slope of the line segment is $60/(-30) = -2$.

d. The equation of the line is $y = -2x + 8000$.

***2.4** **a.** **1.** If x and y represent distances in meters, the coordinates of the lower end of the ramp are (0,1).

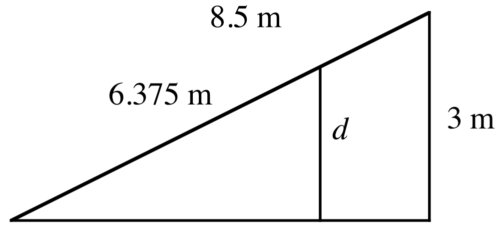
2. The coordinates of the higher end of the ramp are approximately (8,4). (The x-coordinate of the higher end of the ramp can be found using the Pythagorean theorem.)

b. The slope of the ramp is approximately $3/8$.

c. Sample response, where the interval for t is [0, 2]:

$$\begin{cases} x = 0 + 4t \\ y = 1 + 1.5t \end{cases}$$

d. Sample response: Since the ramp is 8.5 m long, the suitcase must be $0.75 \bullet 8.5 = 6.375$ up the ramp.



Using similar triangles, the proportion

$$\frac{d}{6.375} = \frac{3}{8.5}$$

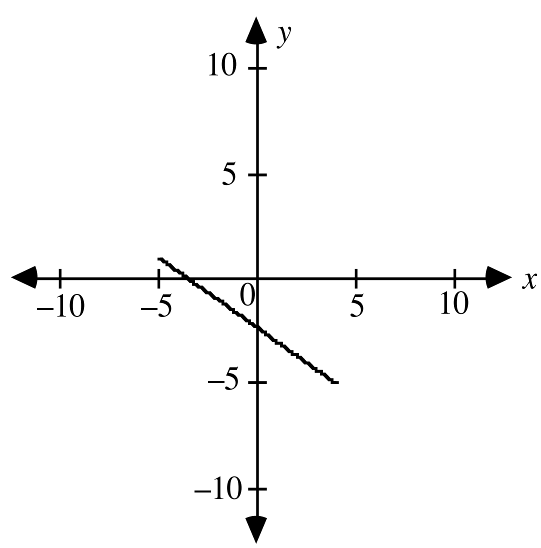
can be solved to find that the suitcase's height from the bottom of the ramp is 2.25 m. Adding 1 m to this distance accounts for the height of the lower end of the ramp from the ground. The total distance is 3.25 m.

* * * * *

2.5 **a.** Sample intervals for the domain and range for a TI-92:

$$t_{\min} = 0,\ t_{\max} = 3,\ t_{step} = 0.1$$
$$x_{\min} = -10,\ x_{\max} = 10$$
$$y_{\min} = -10,\ y_{\max} = 10$$

Sample graph:



b. The object's initial position is (4,–5).

c. The coefficients of the variable t in the parametric equations yields a slope of $-2/3$.

d. Sample response:

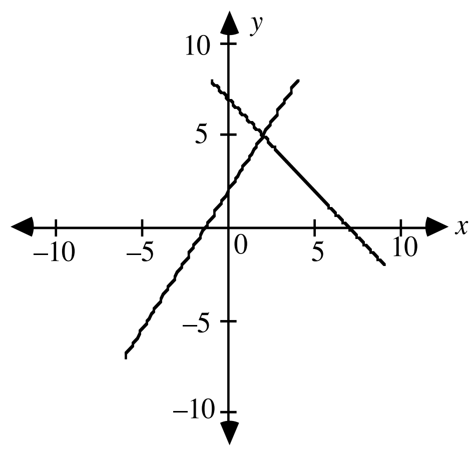
$$y - y_1 = m(x - x_1)$$
$$y + 5 = \frac{2}{3}(x - 4)$$
$$y = \frac{2}{3}x - \frac{23}{3}$$

e. After 2.5 sec, the object's position is (–3.5,0). Using this point and the initial point in Part **b**, the Pythagorean theorem yields a distance of approximately 9 m.

2.6 **a.** Sample intervals for the domain and range for a TI-92:

$$t_{\min} = 0,\ t_{\max} = 5,\ t_{step} = 0.1$$
$$x_{\min} = -10,\ x_{\max} = 10$$
$$y_{\min} = -10,\ y_{\max} = 10$$

Sample graph:



b. Given the point $(-6,-7)$ and the slope $3/2$, the parametric equations that model the motion of object A can be expressed as a linear equation as follows:

$$y - (-7) = \frac{3}{2}[x - (-6)]$$
$$y = \frac{3}{2}x + 2$$

Similarly, the parametric equations that model the motion of object B can be expressed as a linear equation as shown below:

$$y - (-2) = -1(x - 9)$$
$$y = -1x + 7$$

Solving these two linear equations simultaneously results in a point of intersection of (2,5).

c. Sample response: By substituting one of the coordinates of the point of intersection from Part **b** into the parametric equations for object A and object B, the time t required to reach the point of intersection can be found. It takes object A 4 sec to reach this point, while it takes object B only 3.5 sec. This means object B reaches the point of intersection first.

d. Object A reaches the intersection point $4.0 - 3.5 = 0.5$ sec after object B.

e. Sample response: Object A takes 4 sec to reach the point of intersection. Substituting this value for t, the initial position for object B, and the coordinates of the point of intersection (2,5) into the parametric equations below will yield the necessary values for b and d.

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases} \Rightarrow \begin{cases} 2 = 9 + b(4) \\ 5 = -2 + d(4) \end{cases}$$

Solving these equations:

$$b = -1.75$$

$$d = 1.75$$

The new parametric equations for object B should be:

$$\begin{cases} x = 9 - 1.75t \\ y = -2 + 1.75t \end{cases}$$

[Note that these values for b and d change the speed of object B without affecting the slope of the line.]

* * * * * * * * * *

(page 389)

Activity 3

In this activity, students use parametric equations to model the circular path of a suitcase on a luggage carousel.

Materials List

- graph paper (optional)

Technology

- graphing utility
- spreadsheet (optional)

Teacher Note

On some graphing utilities, the graph of a circle may appear to be shaped like an ellipse. Students should find appropriate intervals for the domain and range to adjust for this discrepancy. Some graphing calculators include a feature that can "square" the viewing window.

Exploration

(page 390)

Note: Students should recall how to represent the coordinates of any ordered pair on the unit circle from the Level 4 module "Can It!"

a. Any ordered pair (x,y) on a unit circle can be represented by $(\cos\theta, \sin\theta)$, where θ is the measure of a central angle in radians.

b. Some students may wish to complete the following table on a spreadsheet:

Angle Measure (radians)	***x*-coordinate**	***y*-coordinate**
0	1	0
π/6	0.866	0.5
π/3	0.5	0.866
π/2	0	1
2π/3	–0.5	0.866
5π/6	–0.866	0.5
π	–1	0
7π/6	–0.866	–0.5
4π/3	–0.5	–0.866
3π/2	0	–1
5π/3	0.5	–0.866
11π/6	0.866	–0.5
2π	1	0

c. The parametric equations for the unit circle are shown below, where *t* represents an angle measure in radians:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

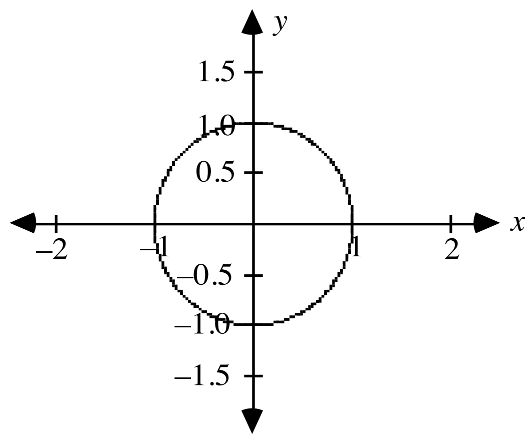
d. Sample intervals for domain and range for a TI-92:

$$t_{min} = 0,\ t_{max} = 6.28,\ t_{step} = 0.1$$

$$x_{min} = -2,\ x_{max} = 2$$

$$y_{min} = -2,\ y_{max} = 2$$

Sample graph:



Students should trace the circle on the graphing utility and verify each pair of coordinates in Table **5**.

e. The equations for a circle with a radius of 2 and center at the origin are:

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

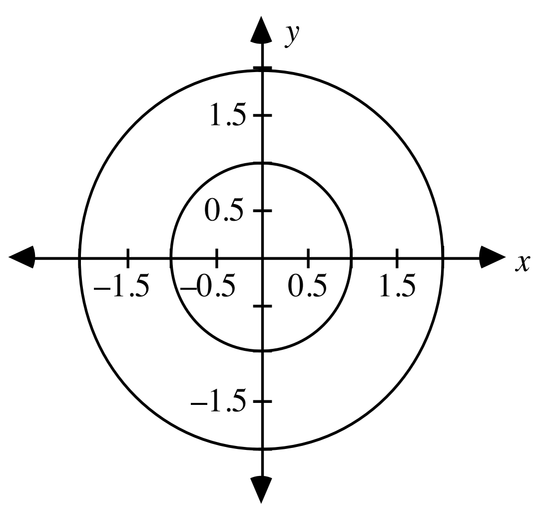
Sample intervals for domain and range for a TI-92:

$$t_{\text{min}} = 0,\ t_{\text{max}} = 6.28,\ t_{step} = 0.1$$

$$x_{\text{min}} = -2,\ x_{\text{max}} = 2$$

$$y_{\text{min}} = -2,\ y_{\text{max}} = 2$$

Sample graph:



f. The parametric equations for a circle with a radius of 1 and center at (2,3) are:

$$\begin{cases} x = 2 + \cos t \\ y = 3 + \sin t \end{cases}$$

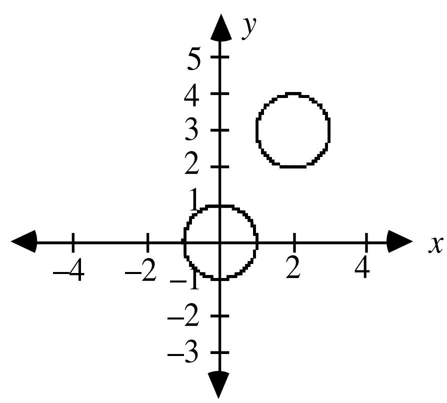
Sample intervals for domain and range for a TI-92:

$$t_{\min} = 0,\ t_{\max} = 6.28,\ t_{step} = 0.1$$

$$x_{\min} = -4,\ x_{\max} = 4$$

$$y_{\min} = -3,\ y_{\max} = 5$$

Sample graph:



Discussion

(page 391)

a. In the exploration in Activity **3**, the parameter *t* represents an angle measure.

b. Sample response: Varying the coefficients of $\cos t$ and $\sin t$ in the parametric equations for a circle changes the radius of the circle. The graph remains a circle as long as those coefficients are equal.

c. Sample response: The constant terms in the parametric equations for a circle represent the center of the circle. They also can indicate the translation of a circle centered at the origin to a new location.

d. Sample response: Parametric equations allow you to graph relations as well as functions. On a TI-92 graphing calculator, for example, a circle graphed in parametric form is complete and continuous with no breaks. Graphing a circle in the function mode requires separate equations representing the top half and bottom half of the circle.

e. Students developed the standard form of the equation for a circle in the previous module, "Transmitting Through Conics." Sample response:

$$\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases} \Rightarrow \begin{cases} x - h = r\cos t \\ y - k = r\sin t \end{cases} \Rightarrow \begin{cases} (x-h)^2 = r^2(\cos t)^2 \\ (y-k)^2 = r^2(\sin t)^2 \end{cases}$$

Therefore,

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2(\cos t)^2 + r^2(\sin t)^2 \\ &= r^2\left((\cos t)^2 + (\sin t)^2\right) \\ &= r^2(1) \\ (x-h)^2 + (y-k)^2 &= r^2 \end{aligned}$$

Assignment

(page 392)

3.1 **a.** Sample parametric equations:

$$\begin{cases} x = 100\cos\theta \\ y = 100\sin\theta \end{cases}$$

b. The parameter *t* represents an angle measure in radians.

c. Sample response: The dependent variables are the ordered pairs (x, y). They represent locations on a circle with the radar transmitter at its center. The independent variable is *t,* which represents an angle measured in radians.

***3.2** **a.** Sample parametric equations:

$$\begin{cases} x = 2500 + 500\cos t \\ y = 3500 + 500\sin t \end{cases}$$

Sample intervals for the domain and range for a TI-92:

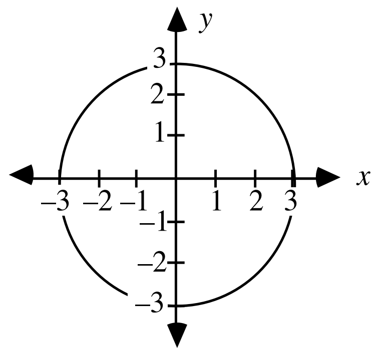
$$t_{min} = 0,\ t_{max} = 6.28,\ t_{max} = 0.1$$
$$x_{min} = 0,\ x_{max} = 4000$$
$$y_{min} = 0,\ y_{max} = 4000$$

b. One-fifth of a revolution is $2\pi/5 \approx 1.3$ radians. Graphically, the ordered pair may be found by tracing to the point where $t = 1.3$. Algebraically, the ordered pair may be found by substituting $t = 2\pi/5$ into the parametric equations from Part **a**. The resulting location is (2654,3975).

c. The airplane completes 1 revolution every 5 min. After 9 min, the airplane will have completed 1.8 revolutions. Since it began circling counterclockwise, it would have traveled $1.8 \bullet 2\pi = 3.6\pi$ radians. By tracing on the graph or substituting $t = 3.6\pi$ in the parametric equations, the coordinates of the plane's location are (2655,3024).

d. Since the distance traveled in one revolution is the circumference of the circle or $2\pi(500)$, the distance traveled in 1.8 revolutions is $(1.8)2\pi(500) \approx 5655\text{ m}$.

3.3 **a.** Sample graph:



b. Sample response: The first point plotted was (3,0). The circle was plotted in the counterclockwise direction.

c. Sample response: The parametric equations $x = 3\sin t$ and $y = 3\cos t$ plot the circle in the clockwise direction, but change the starting point to the positive *y*-axis.

d. Sample response: The parametric equations $x = 3\cos t$ and $y = -3\sin t$ plot the circle in the clockwise direction from the positive *x*-axis.

e. Sample response: The parametric equations $x = -3\sin t$ and $y = 3\cos t$ plot the circle in the counterclockwise direction from the positive *y*-axis.

* * * * *

3.4 **a.** Using the Pythagorean theorem, the distance from the tower to the radar transmitter is 2.5 km.

b. Sample parametric equations:

$$\begin{cases} x = 1.5 + 100\cos t \\ y = 2.0 + 100\sin t \end{cases}$$

c. **1.** The location of the plane is (–75,–10).

2. Using the Pythagorean theorem, the distance from the plane to the radar transmitter is 77.4 km.

3. Sample parametric equations:

$$\begin{cases} x = 1.5 + 77.4\cos t \\ y = 2.0 + 77.4\sin t \end{cases}$$

* * * * * * * * * *

Answers to Summary Assessment (page 395)

1\.

a. The origin of a rectangular coordinate system is placed at the lodge. Jolene is at the top of the left peak, and Michael is at the top of the right peak. The differences in elevation from the lodge are 301 m and 411 m, respectively. Given the lengths of the ski runs, the Pythagorean theorem can be used to find the horizontal distances from the lodge. Jolene's location is (–3344,301); Michael's location is (4110,411).

b. **1.** The ratio of the vertical change in distance to the horizontal change in distance on the west slope is $301/3344$. Jolene drops 0.8 m vertically every second. To find the horizontal distance covered in 1 sec, students should solve the following proportion:

$$\frac{301}{3344} = \frac{0.8}{x}$$
$$x = 8.9 \text{ m}$$

2. The ratio of the vertical change in distance to the horizontal change in distance on the east slope is $411/4110$. Michael drops 0.8 m vertically every second. To find the horizontal distance covered in 1 sec, students should solve the proportion below:

$$\frac{411}{4110} = \frac{0.8}{x}$$
$$x = 8.0 \text{ m}$$

c. Sample parametric equations for Jolene's motion:

$$\begin{cases} x = -3344 + 8.9t \\ y = 301 - 0.8t \end{cases}$$

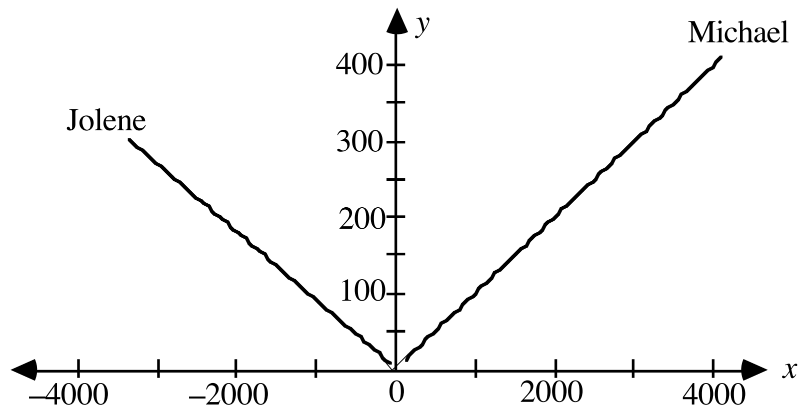
Sample parametric equations for Michael's motion:

$$\begin{cases} x = 4110 - 8t \\ y = 411 - 0.8t \end{cases}$$

d. Sample intervals for domain and range for a TI-92:

$$t_{\text{min}} = 0,\ t_{\text{max}} = 520,\ t_{\text{step}} = 1$$
$$x_{\text{min}} = -4000,\ x_{\text{max}} = 4200$$
$$y_{\text{min}} = 0,\ y_{\text{max}} = 420$$

Sample graph:



e. Jolene arrives first, after a time of approximately 375 sec. Michael arrives at the lodge in approximately 514 sec.

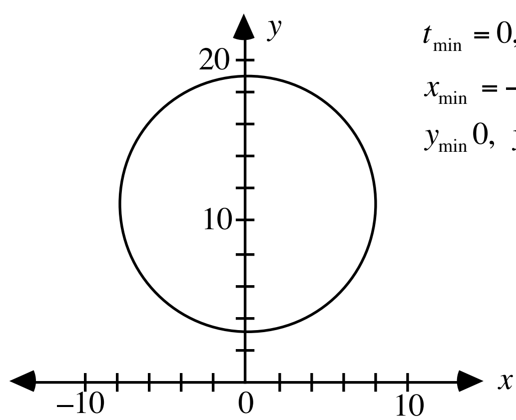
f. The difference in the arrival times is 139 sec.

g. When Jolene arrives at the lodge, Michael's horizontal and vertical distances from the lodge are 1110 m and 111 m, respectively. Using the Pythagorean theorem, his distance from the lodge is 1115 m.

2. **a.** Answers may vary. Sample response: The radius of the Ferris wheel is 8 m. The wheel turns in a counterclockwise direction with center at (0,11). Anton and Julia are at the top of the Ferris wheel when the ride starts. The following equations can be used to model the path of their chair:

$$\begin{cases} x = 0 - 8\sin t \\ y = 11 + 8\cos t \end{cases}$$

Using the following intervals for domain and range produces the graph below:



$t_{\text{min}} = 0,\ t_{\text{max}} = 6.28,\ t_{\text{step}} = 0.1$

$x_{\text{min}} = -10,\ x_{\text{max}} = 10$

$y_{\text{min}} 0,\ y_{\text{max}} = 20$

b. Since the wheel completes 4 revolutions in 1 min, 1 revolution takes 15 sec. After 30 sec, Anton and Julia will have completed 2 revolutions. The distance traveled is $2(2\pi r) = 32\pi \approx 101$ m.

c. After 4.75 min, the Ferris wheel makes 19 complete revolutions. The distance traveled is $19(2\pi r) = 304\pi \approx 955$ m.

d. If students solve this problem by tracing on the graph, they should adjust the interval for t to $[0, 4\pi]$ in order to see 2 full revolutions.

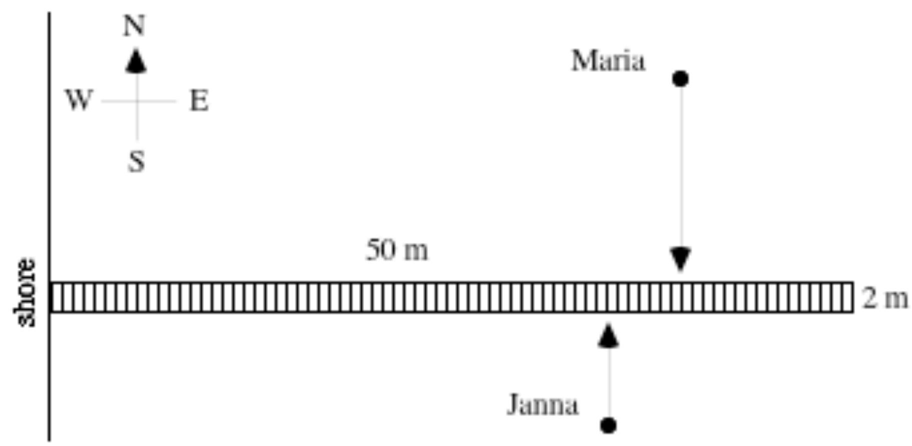
To solve the problem algebraically, the times must be converted to the angle of rotation described by the chair. At a rate of 4 revolutions per minute, any one point on the wheel travels at rate of 64π radians/min or 3.35 radians/sec. After 10 sec, the height of the seat can be found by substituting 33.5 for t in the equation $y = 11 + 8\cos t$.

After 10 sec, the seat is approximately 7 m off the ground. After 20 sec, the seat is approximately 7 m off the ground. After 30 sec, the seat is approximately 19 m off the ground.

Module Assessment

1. Maria and Janna are swimming in a lake. As shown in the diagram below, they are approaching a dock from opposite directions. The dock is 2 m wide and extends 50 m into the lake from the west shoreline.

Maria swims at a constant speed of 0.8 m/sec while Janna swims at a constant speed of 0.6 m/sec . At the same moment that Maria is 20 m north of the dock and 40 m east of the shoreline, Janna is 12 m south of the dock and 35 m east of the shore.



a. Use parametric equations and technology to model this situation.

b. List the intervals for the domain and range that you used to model the scenario. Include the increment for the parameter.

c. Determine who reaches the dock first, and at what time (to the nearest 0.1 sec).

d. At that time, how far is the other swimmer from the dock?

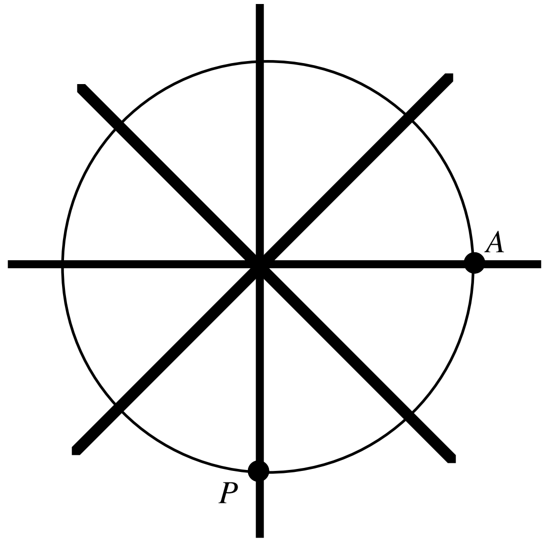
2. Two soccer players are running at the same speed toward a ball. The positions of the two players can be modeled on a rectangular coordinate system. At the same moment that Player A has coordinates (0,15), Player B has coordinates (4,3). The ball is not moving. Its location can be modeled with coordinates (8,11).

a. Write parametric equations to model the two soccer players running toward the ball.

b. Use a graphing utility to graph the parametric equations from Part **a**. List the intervals used for domain and range.

c. Determine which player wins the race to the soccer ball and find the winning time (to the nearest 0.1 unit).

3. A riverboat is powered by a paddle wheel with a diameter of 10 m. At full speed, the wheel makes 1 revolution counterclockwise every 8 sec.



a. Assume that point P (which is at water level) lies at the origin of a rectangular coordinate system and point A is its image under a counterclockwise rotation of 90°, as shown in the diagram above.

Write parametric equations to describe the path of point A as the wheel turns.

b. Through how many radians has point A rotated after the paddle wheel has been turning for 41 sec?

c. How far is point A from the x-axis (water level) after the paddle wheel has rotated for 41 sec?

Answers to Module Assessment

1. Students should model the motions of the swimmers with vertical lines. Some students may also model the position of the dock with horizontal lines. The following sample responses place the origin of a rectangular coordinate system in the lower left-hand corner of the diagram. Using this system, Janna's initial position is (0,35). Since Maria is 20 m north of the dock, the dock is 2 m wide, and Janna is 12 m south of the dock, her initial position is (34,40).

a. Sample parametric equations for Maria's motion:

$$\begin{cases} x = 40 + 0t \\ y = 34 - 0.8t \end{cases}$$

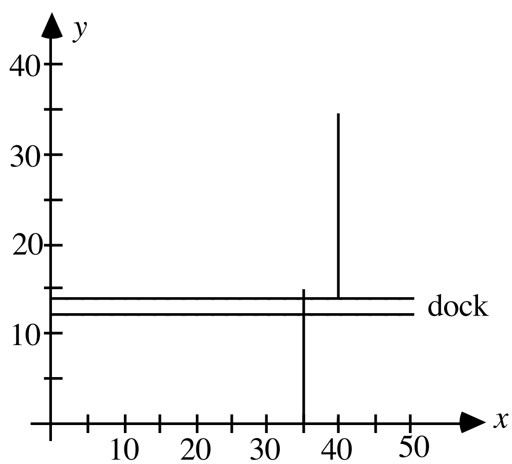
Sample parametric equations for Janna's motion:

$$\begin{cases} x = 35 + 0t \\ y = 0 + 0.6t \end{cases}$$

b. Sample intervals for the domain and range for a TI-92:

$$t_{\text{min}} = 0,\ t_{\text{max}} = 25,\ t_{\text{step}} = 0.1$$
$$x_{\text{min}} = 0,\ x_{\text{max}} = 50$$
$$y_{\text{min}} = 0,\ y_{\text{max}} = 40$$

Sample graph:



c. By tracing, students can determine that Janna reaches the dock first, with a time of 20.0 sec.

d. By tracing, students can determine that at $t = 20$ sec, Maria is still 4 m from the dock.

2. **a.** The line that models the path of Player A to the ball is $y = -0.5x + 15$. The slope of this line is –0.5. The line that models the path of Player B to the ball is $y = 2x - 5$. The slope of this line is 2.

Sample parametric equations for Player A's motion:

$$\begin{cases} x = 0 + 2t \\ y = 15 - t \end{cases}$$

Sample parametric equations for Player B's motion:

$$\begin{cases} x = 4 + t \\ y = 3 + 2t \end{cases}$$

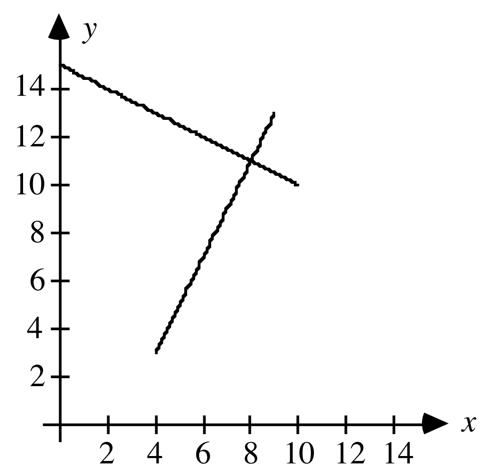
b. Sample intervals for the domain and range for a TI-92:

$t_{min} = 0,\ t_{max} = 5,\ t_{step} = 0.1$

$x_{min} = 0,\ x_{max} = 15$

$y_{min} = 0,\ y_{max} = 15$

Sample graph:



c. Sample response: There is no clear winner of the race to the soccer ball. Both players reach the ball at precisely 4.0 units of time. **Note:** No specific units of either time or length are given in this problem. Students may argue that the only unit of time that makes sense in the context of the problem is seconds.

3. **a.** Sample parametric equations:

$$\begin{cases} x = 0 + 5\cos t \\ y = 5 + 5\sin t \end{cases}$$

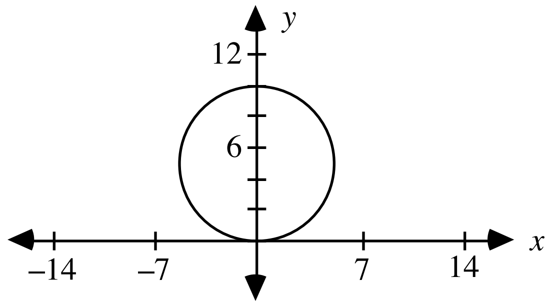
Sample intervals for the domain and range for a TI-92:

$t_{min} = 0,\ t_{max} = 6.28,\ t_{step} = 0.01$

$x_{min} = -14,\ x_{max} = 14$

$y_{min} = 0,\ y_{max} = 12$

Sample graph:



b. Sample response: After 41 sec, the wheel has made $5\frac{1}{8}$ revolutions. Point *A* will travel through an angle of $5\frac{1}{8}(2\pi) \approx 32.2$ radians.

c. Since the *x*-axis represents the water level, the *y*-coordinate represents distance above the water level. Substituting 32.2 radians for *t* in the equation $y = 5 + 5\sin t$ yields $y \approx 8.53$ m. Students can also find this distance by tracing the circle on the graphing utility.

Selected References

Cieply, J. F. "Parametric Equations: Push 'Em Back, Push 'Em Back, Way Back!" *The Mathematics Teacher* 86 (September 1993): 470–74.

Mathematical Association of America (MAA). *Teaching Mathematics with Calculators: A National Workshop*. Providence, RI: MAA, 1994.

Vonder Embse, C. "Visualization in Precalculus: Making Connections Using Parametric Graphs." Lecture presented at the Third International Conference on Technology in Collegiate Mathematics at The Ohio State University in Columbus, OH. November 10, 1990.

Flashbacks

Activity 1

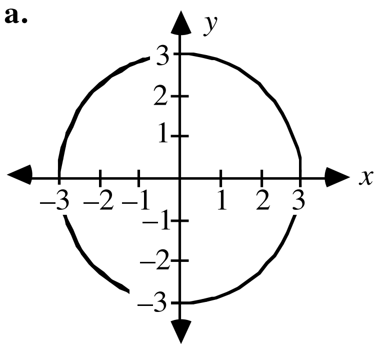
1.1 Write the equation of the line through each of the following pairs of points:

a. (2,3) and (4,3)

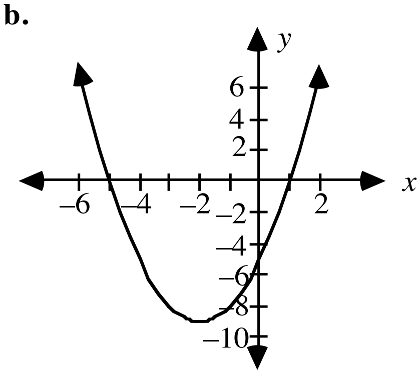
b. (–4,1) and (–4,–2)

1.2 Describe the domain and range for each graph below and determine whether or not the graph represents a function.

a.



b.



Activity 2

2.1 Graph the linear function below:

$$y = \frac{3}{4}x - 2$$

2.2 Solve the following system of equations:

$$\begin{cases} y = 0.5x + 2 \\ y = -3x + 9 \end{cases}$$

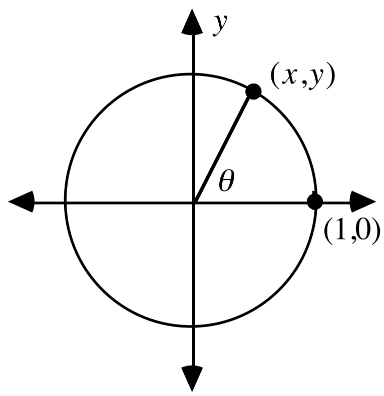
2.3 **a.** Determine the distance between the points (–3,1), and (5,–2).

b. Find the equation of the line that contains the points in Part **a**.

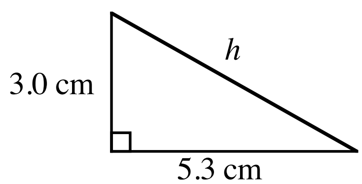
2.4 Determine the average speed of an object that moves 20 m in 6 sec.

Activity 3

3.1 Find the ordered pair (x,y) on the unit circle below, where $\theta = 1.25$ radians.



3.2 Determine the length of the hypotenuse in the right triangle below:



Answers to Flashbacks

Activity 1

1.1 **a.** $y = 3$

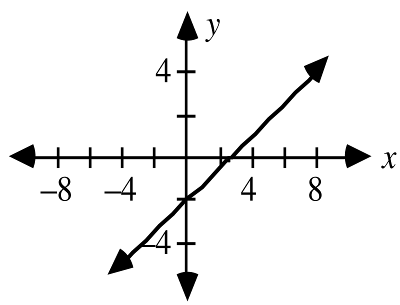
b. $x = -4$

1.2 **a.** The domain is [–3, 3] and the range is [–3, 3]. The graph of a circle is not a function because it does not pass the vertical line test.

b. The domain is $(-\infty, \infty)$ and the range is $[-9, \infty)$. The graph of the parabola is a function since it passes the vertical line test.

Activity 2

2.1 Sample graph:



2.2 The solution to the system of equations is (2,3).

2.3 **a.** The distance between the points is $\sqrt{(-3-5)^2 + (1-(-2))^2} \approx 8.54$.

b. The equation of the line in slope-intercept form is:

$$y = -\frac{3}{8}x - \frac{1}{8}$$

2.4 The average speed of the object is:

$$\frac{20 \text{ m}}{6 \text{ sec}} \approx 3.3 \text{ m/sec}$$

Activity 3

3.1 Any ordered pair (x,y) on the unit circle can be represented by the ordered pair $(\cos\theta, \sin\theta)$. Given a radian measure of 1.25, the ordered pair is (0.315,0.949).

3.2 Using the Pythagorean theorem, the length of the hypotenuse is approximately 6.1 cm.

Having a Ball

News flash! The sum of the measures of a triangle's interior angles is not always 180˚. In this module, you discover how—and why—this can occur.

Monty Brekke • Janet Kuchenbrod • Tim Skinner

Teacher Edition

Having a Ball

Overview

In this module, students investigate the basic properties of spherical geometry through hands-on exploration and comparison with Euclidean geometry.

Objectives

In this module, students will:

- examine properties of lines on a sphere
- investigate the sum of the measures of the interior angles of a triangle on a sphere
- examine properties of quadrilaterals on a sphere
- consider why similar polygons on spheres are congruent
- compare properties of Euclidean geometry with properties of spherical geometry.

Prerequisites

For this module, students should know:

- that two points determine a line in a plane
- the meaning of betweenness for points in a plane
- the notions of parallel and perpendicular lines in a plane
- properties of quadrilaterals in a plane
- how to find the sum of the measures of the interior angles of a triangle
- how to determine the measure and length of an arc
- the definition of similarity.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	1	1	8

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
ruler	X	X	X	
straightedge	X	X		
protractor	X	X	X	
string	X	X		
tape	X	X		
spheres	X	X		
large rubber bands	X	X		
scissors	X	X		

Teacher Note

You may wish to collect an assortment of model spheres, from as small as a softball to as large as a basketball. The physical education department at your school can be a valuable resource. (It is difficult to make the appropriate measurements on spheres smaller than a softball).

Having a Ball

Introduction

(page 401)

Students are introduced to the historical background of Euclidean and non-Euclidean geometries. **Note:** Euclid's five postulates are listed in the research project that appears at the end of Activity **1**.

(page 402)

Activity 1

In this activity, students investigate some familiar properties of plane geometry on a sphere.

Materials List

- rulers (one per group)
- scissors (one pair per group)
- tape (one roll per group)
- straightedge (one per group)
- protractors (one per group)
- spheres of various sizes (one per group)
- rubber bands large enough to fit around the spheres (three per group)
- string

Exploration 1

(page 402)

In this exploration, students investigate great circles on a sphere.

a. Students should approximate the circumference by finding the greatest distance around the sphere.

b. Students should place one end of the string on one of the points, then wrap the string around the sphere, forming a circle that uses the whole length of the string and passes through the second point. They should find that only one great circle contains each pair of points.

c. As long as the two points are not on opposite ends of a diameter, students should conclude that any two points determine a unique circle with a circumference equal to the circumference of the sphere.

d. When the two points are endpoints of a diameter, then an infinite number of great circles contain them.

Discussion 1

(page 403)

a. Sample response: The cross section of the sphere made by cutting along the path of the circle contains the center of the sphere.

b. Sample response: Yes, this is possible if the two points coincide or if they are the endpoints of a diameter of the sphere. In either case, there is an infinite number of great circles containing the two points.

c.
1. In spherical geometry, a minimum of two distinct points are required to determine a unique line.
2. No. If the two points happen to be located at the ends of a diameter of the sphere, an infinite number of lines are possible.

d. Sample response: All lines of longitude are great circles. Only one line of latitude is a great circle: the equator. The remaining lines of latitude are circles whose centers are not the center of the sphere.

e.
1. The fractional part of the great circle represented by arc *AB* is:
$$\frac{m\angle ACB}{360^\circ}$$
2. Sample response: Using the response to Step **1**, you can write the following proportion,
$$\frac{\text{length of arc } AB}{\text{circumference of circle}} = \frac{m\angle ACB}{360^\circ}$$
Therefore, $m\angle ACB$ equals the length of arc *AB* multiplied by 360° and divided by the circumference of the great circle.

f. Sample response: Since both planes are perpendicular to the same line, they must be parallel.

g.
1. The two angles are congruent.
2. Since $m\angle AOB = m\angle A'PB'$ and $m\angle APB = m\angle A'PB'$, $m\angle AOB = m\angle APB$.

h. Sample response: The length of arc *AB* is equal to $1/4$ of the circumference of the sphere. This is true because when $m\angle AOB = m\angle APB = 90^\circ$, the measure of arc *AB* is $90/360$ or $1/4$ of the circumference of the sphere.

i. Using the sphere shown in Figure **4** of the student edition, since $\overline{PQ}$ is perpendicular to the great circle containing point *A*, $\angle AOP \cong \angle AOQ$. Since equal angles intercept equal arcs, arc *AP* is congruent to arc *AQ*. Therefore, points *P* and *Q* are equidistant from point *A*.

j. Sample response: Measure along a great circle containing each side of the angle a distance equal to $1/4$ the circumference of the sphere. Mark each point and name them *D* and *E*. This locates the great circle that is equidistant between the endpoints of the diameter $\overline{BF}$. This great circle is parallel to the plane which defines the measure of the angle on the sphere. Since $m\angle ABC = m\angle DBE = m\angle DOE$, the measure of the angle can be found by measuring the length of arc *DE*, then solving the following proportion for *x*.

$$\frac{\text{length of arc } DE}{\text{circumference of circle}} = \frac{x}{360^\circ}$$

Exploration 2

(page 405)

a.
1. In the Euclidean plane, two distinct lines may intersect in zero points or exactly one point.
2. In spherical geometry, two distinct lines intersect in two points.

b.
1. In the Euclidean plane, there is exactly one perpendicular line from a point not on a line to the line.
2. In spherical geometry, there may not be a unique perpendicular from a point not on a line to the line. If the point in question is one of the endpoints of a diameter of the sphere and the diameter is perpendicular to the plane containing the line, then there are infinitely many perpendicular lines through the point to the line. If the point does not satisfy these requirements, there is a unique perpendicular.

c. In the Euclidean plane, two lines perpendicular to the same line do not intersect and, therefore, are parallel. In spherical geometry, two lines perpendicular to the same line intersect in two points.

d. In the Euclidean plane, there are no points equidistant from every point on a line. In spherical geometry, there are two points equidistant from every point on a line. The two points are the endpoints of a diameter that is perpendicular to the plane containing the great circle representing the line.

Discussion 2

(page 406)

a. Sample response: Yes. Since Earth is shaped roughly like a sphere, the equator can be thought of as a great circle of the sphere. The earth's north and south poles are the two poles defined by the equator.

b. If polar points are not considered, there is only one line perpendicular to a line from a point not on the line.

c. Sample response: The lengths of the arcs are equal. Since point *A* lies on the great circle that is perpendicular to the diameter $\overline{JK}$, it is equidistant from the poles.

d. Sample response: The intersection of two lines perpendicular to a third line on a sphere are the poles of the third line.

e. Sample response: Yes, every point is at one end of a diameter of the sphere. Endpoints of diameters are poles for a unique line on the sphere.

f. The measure of an angle on a sphere equals the measure of the minor arc of the great circle for which the vertex of the angle is a polar point.

g. Sample response: No, two distinct great circles always intersect in two points.

Assignment

(page 407)

1.1 **a.** Sample response: No, lines of latitude are not great circles, with the exception of the equator. The intersection of a Euclidean plane with a line of latitude other than the equator will not pass through the center of the sphere.

b. A circle or a single point are the only figures that can be formed by the intersection of a Euclidean plane and a sphere.

1.2 In spherical geometry, lines are great circles. The circumference of a great circle is equal to the circumference of the sphere. Therefore, in spherical geometry, lines have a finite length.

1.3 **a.** The angle measure between the two cities is $78.5^\circ + 9.5^\circ = 88^\circ$.

b. Sample response: The circumference of the earth is about $\pi \bullet 12,756 \approx 40,074$ km. Therefore, the distance between the two cities is:

$$\frac{x}{40,074 \text{ km}} = \frac{88^\circ}{360^\circ}$$
$$x \approx 9796 \text{ km}$$

1.4 **a.**
1. There is one line perpendicular to line *AM* through point *A*.
2. There is one line perpendicular to line *AM* through point *B*.
3. There are an infinite number of perpendicular lines to line *AM* through point *P*.

b. Sample response: If a point is not a polar point, there is exactly one line through it perpendicular to the given line. If a point is a polar point, there are infinitely many lines through it perpendicular to the given line.

1.5 **a.** Sample response: The circumference of the earth is about $\pi \bullet 12,756 \approx 40,074$ km. Therefore, the greatest possible distance between two cities on earth is about half the circumference or 20,037 km.

b. Two locations are possible: one is the midpoint of the major arc connecting *B* and *C*; the other is the midpoint of the minor arc.

c. Sample response: An infinite number of locations are possible. These locations are points forming a great circle perpendicular to the great circle containing cities *B* and *C* at the midpoint of the minor arc joining *B* and *C*.

d. In both cases, the number of points is infinite. In a plane, however, the set of points form a straight line instead of a circle.

***1.6** Sample response: Using the Euclidean definition, *B* is between *A* and *C* only when arc AC is a minor arc. If the sum of *AB* and *BC* is less than or equal to half the circumference of the great circle, the equation $AB + BC = AC$ is true. If the sum of *AB* and *BC* is greater than half the circumference of the great circle, the equation is false because the length *AC* describes the length of a minor arc.

* * * * *

1.7 **a.** The equator is perpendicular to every line of longitude.

b. The distance from each pole to any point on the equator is $1/4$ the circumference of Earth.

1.8 Students responses should include, but not be limited to, the following concepts:

- In both spherical and Euclidean geometries, any two points determine a unique line. In spherical geometry, a line is a great circle.
- In both spherical and Euclidean geometries, the distance between any two points is the shortest distance between them. In spherical geometry, that distance is the length of the minor arc of the great circle determined by the two points.
- In Euclidean geometry, lines cannot be measured because they have infinite length. In spherical geometry, a line is a great circle and has length equal to the circumference of the sphere.
- In both spherical and Euclidean geometries, any three or more points that lie on the same line are considered collinear points.
- In Euclidean geometry, two distinct lines may intersect in exactly one point or be parallel. In spherical geometry, there are no parallel lines. Two distinct lines always intersect in two points.

- In Euclidean geometry, there is exactly one perpendicular line from a point not on a line to the line. In spherical geometry, if the point in question is one of the endpoints of a diameter of the sphere and the diameter is perpendicular to the plane containing the line, then there are infinitely many perpendicular lines through the point to the line. If the point does not satisfy these requirements, there is a unique perpendicular.
- In Euclidean geometry, two lines perpendicular to the same line do not intersect and, therefore, are parallel. In spherical geometry, two lines perpendicular to the same line intersect in two points.

* * * * * * * * * *

Research Project (page 409)

Euclid's most famous work is the *Elements*. The five postulates, as translated by Thomas L. Heath (New York: Dover Publications, 1956), are listed below:

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles.

In more familiar terms, these five postulates also can be interpreted as follows:

1. For any two distinct points *P* and *Q* there exists a unique line *l* that passes through (or contains) *P* and *Q*.
2. For every segment *AB* and for every segment *CD* there exists a unique point *E* such that *B* is between *A* and *E* and segment *CD* is congruent to segment *BE*.
3. For every point *O* and every point *A* not equal to *O* there exists a circle in the plane with center *O* and radius *OA*.
4. All right angles are congruent to each other.
5. For every line *l* and for every point *P* that does not lie on *l* there exists a unique line *m* through *P* that is parallel to *l*.

(page 409)

Activity 2

In this activity, students explore properties of triangles and quadrilaterals on the surface of a sphere.

Materials list

- spheres of different sizes (one per group)
- large rubber bands (three per group)
- straightedge (one per group)
- ruler (one per group)
- protractor (one per group)
- scissors (one pair per group)
- tape
- string

Teacher Note

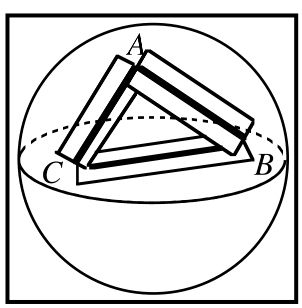
To allow students to observe how a sphere's circumference affects the measures of a triangle's interior angles, you may wish to provide spheres of several different sizes.

You also may wish to use radians instead of degrees to measure angles. If so, then students should consider that, in a plane, the sum of the measures of the interior angles of a triangle is π.

Exploration 1

(page 409)

a–c. Students use paper strips to form a triangle on the surface of a sphere, as shown below:



Note: To verify that each side of a triangle lies on a great circle, students may wish to measure the circumference of the circle with string. The circumference of the circle should equal the circumference of the sphere.

d–f. The sample data in the table below were collected using a volleyball as a sphere.

Perimeter of Triangle (cm)	$m\angle ACB$	$m\angle ABC$	$m\angle BAC$	Sum of Angle Measures
20	49.3°	78.8°	60.2°	188.3°
24	46.2°	79.9°	63.8°	189.9°
27	50.3°	78.8°	62.2°	191.3°

The sample data in the following table were collected using a softball as a sphere.

Perimeter of Triangle (cm)	$m\angle ACB$	$m\angle ABC$	$m\angle BAC$	Sum of Angle Measures
20	126.0	47.2	40.5	213.7
24	84.4	100.1	96.8	281.3
27	92.3	115.9	104.6	312.8

g. On a given sphere, as the perimeter of a triangle increases, the sum of the measures of its interior angles increases. As the circumference of the sphere increases, the sum of the measures of the interior angles for a triangle of given perimeter decreases.

Discussion 1

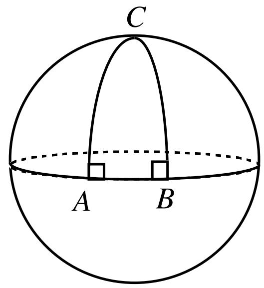
(page 411)

a. Sample response: On a sphere, the sum of the measures of a triangle's interior angles is greater than 180°. In a plane, the sum of the measures of these angles is always 180°. In a plane, the sides of a triangle are straight line segments, while on a sphere they are arcs.

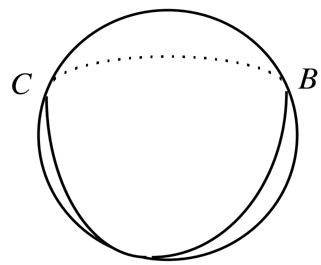
b. Sample response: Yes. As the perimeter increases, the sum of the angles also appears to increase.

c. Sample response: As the circumference of a sphere increases, the surface of a given portion of the sphere becomes flatter. As a result, the sum of the angle measures for a triangle with a given perimeter approaches 180°.

d. Sample response: Yes, it is possible to construct a triangle on a sphere that has two right angles. One example is a triangle on the globe formed by two lines of longitude and the equator, as shown below.



e. Sample response: Yes, it is possible to construct a triangle with three obtuse angles on a spherical surface. The measure of each angle can approach 180°. The appearance of the triangle will approach that of a great circle as the angle measures approach 180°.



f. If two poles are considered as vertices, it appears that a two-sided polygon can be formed. However, because sides of polygons are minor arcs of great circles and the semicircles between the poles are not minor arcs, the figure is not a polygon. **Note:** Some texts refer to a figure formed by two semicircles intersecting at poles of a sphere as *biangles*.

Exploration 2 (page 412)

The sample responses given below were found using a basketball as a sphere.

a–b. Sample response for a quadrilateral *EFGH*, where base *EF* has a length of 6 cm and sides *EG* and *FH* have lengths of 3 cm: $m\angle E = 90^\circ$, $m\angle F = 90^\circ$, $m\angle G \approx 95.5^\circ$, $m\angle H \approx 95.5^\circ$; the length of side *GH* is 5.7 cm.

c. Sample response for a quadrilateral *EFGH*, where base *EF* has a length of 12 cm and sides *EG* and *FH* have lengths of 6 cm: $m\angle E = 90^\circ$, $m\angle F = 90^\circ$, $m\angle G \approx 110.9^\circ$, $m\angle H \approx 110.9^\circ$; the length of side *GH* is 10.5 cm.

d. Sample response: A quadrilateral with sides of lengths 12.1 cm, 11.2 cm, 5.3 cm, and 3.1 cm had angle measures of 117.4°, 95°, 93.6°, and 84.8°.

e. For the sample quadrilaterals given above, these sums are 371° in Part **b**, 401.8° in Part **c**, and 390.8° in Part **d**, respectively.

Discussion 2 (page 412)

a.
1. Sample response: Nothing is known about the angle measures except that their sum is greater than 180°.

2. The other two angles are congruent and obtuse.

b. Students should realize that the construction of this quadrilateral in a Euclidean plane yields a rectangle.

1. The opposite side is congruent to the base. Its length would be 6 cm.

2. The angles opposite the base would be right angles.

c. Sample response: The side opposite the base is shorter than the base. This will always hold true when the quadrilateral is placed on the sphere as described in Exploration **2**.

d. Sample response: No, because in Euclidean geometry, rectangles have parallel sides and there are no parallel lines on a sphere.

e. On a sphere, the sum of the measures of the interior angles of a quadrilateral is greater than 360°.

Assignment (page 413)

***2.1** **a.** The upper bound for the measure of an individual angle is 180°.

b. The upper bound for the sum of the measures of a triangle's interior angles is $3(180°) = 540°$.

2.2 According to the triangle inequality theorem in Euclidean geometry, the sum of the lengths of any two sides of a triangle is greater than the length of the third side. This also holds true in spherical geometry. Students may demonstrate this property by modeling the problem on a ball.

2.3 On a sphere, the sides of a triangle increase in length as the measures of its interior angles increase. When the sum of the measures of its interior angles approaches 540° (the maximum), the triangle approaches the shape of a great circle. Thus, the sum of the lengths of the sides approaches the circumference of the sphere.

2.4 **a.** Sample response: A quadrilateral can be divided into two triangles. The sum of the measures of the interior angles in the quadrilateral is equal to the sum of the measures of the interior angles in the two triangles. On a sphere, the sum of the measures of the interior angles of a triangle is greater than 180°. Therefore, the sum of the measures of the interior angles in a quadrilateral is greater than 360°.

b. Sample response: On a sphere, the upper bound for the sum of the measures of a quadrilateral's interior angles is 720°, since the measure of each angle in the quadrilateral can approach 180°.

2.5 The lower bound for the sum of the measures of a hexagon's interior angles is 720°, because a hexagon can be divided into four triangles and the sum of the measures of the interior angles of each triangle has a lower bound of 180°. The upper bound for the sum is 1080°, since each of the six interior angles in the hexagon can approach 180°.

***2.6** **a.** Sample response: The sum of the measures of the interior angles of an n-sided polygon has a lower bound of $(n-2) \bullet 180°$. This is because $n-2$ triangles can be drawn from one vertex of the polygon and the sum of the measures of the interior angles of each triangle has a lower bound of 180°.

b. Sample response: The upper bound for the measures of the interior angles of an n-sided polygon is $n \bullet 180°$ because each of the n angles of the polygon has an upper bound of 180°.

2.7 Sample response: Since there are no parallel lines on a sphere, there can be no trapezoids, parallelograms, rectangles, squares, or rhombi.

* * * * *

2.8 **a.** The length of each side of the triangle must be one-fourth the circumference of Earth, or $(1/4)\pi(12,756) \approx 10,019$ km.

b. Each hemisphere is tessellated by four of these triangles. The entire sphere is tessellated by eight of these triangles. Thus, each triangle's area is $1/8$ of Earth's surface area.

2.9 Sample response: In Euclidean geometry, the sum of the measures of the interior angles of a triangle is always 180°. In spherical geometry, this sum can vary within the interval (180°, 540°). In Euclidean geometry, the measure of only one angle in a triangle can be greater than or equal to 90°. In spherical geometry, the measures of one or more angles can be as great as 180°.

In Euclidean geometry, the sum of the measures of the interior angles of a quadrilateral is always 360°. In spherical geometry, this sum can vary within the interval (360°, 720°). Since there are no parallel lines on a sphere, trapezoids and parallelograms do not exist.

* * * * * * * * * *

(page 414)

Activity 3

In this activity, students discover that, on a sphere, the only similar polygons are congruent polygons.

Materials List

- ruler (optional; one per group)
- protractor (optional; one per group)

Discussion

(page 414)

a. Sample response: Yes. As long as their corresponding angles are congruent, and the ratio of corresponding sides is a constant, the triangles are similar.

b. As the perimeter of the triangle on a sphere increases, so does the sum of the measures of the interior angles.

c. Sample response: No. If the lengths of the sides of two triangles are different, they are either not proportional or the perimeter of one triangle is greater than that of the other. If the latter is true, then the sums of the measures of the interior angles are different. Therefore, the measures of the individual angles in each triangle cannot be equal to the corresponding angles in the other triangle. In either case, the triangles cannot be similar.

d. Sample response: The only time two triangles can be similar on a sphere is when the triangles are congruent. This is because a change in the lengths of the sides changes the measures of the interior angles.

Assignment

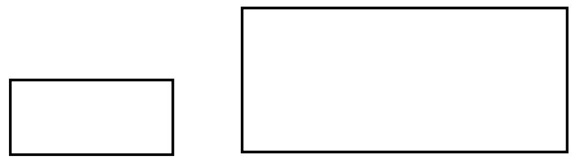
(page 414)

3.1 **a.** Sample response: No, because the corresponding acute angles may have different measures.

b. Sample response: Yes, because all angles are right angles and the ratio of corresponding sides is equal.

c. Sample response: No. Although all angles are right angles, the ratio of corresponding sides can vary.

3.2 **a.** Sample response:



b. Sample response: No. The corresponding angle measures of two quadrilaterals on a sphere would not be equal unless the quadrilaterals were congruent.

3.3 The surface of a soccer ball has 10 congruent pentagons and 20 congruent hexagons. Since they are congruent, they are also similar.

***3.4** **a.** The remaining sides have lengths of 7.2 cm and 9.6 cm. The three angles have measures of approximately 90˚, 37˚, and 53˚.

b. Sample response: It depends on the size of the sphere. The sphere could be so small that the triangles would not fit. If they do fit, the sum of the measures of the interior angles would be greater for the larger triangle.

* * * * *

3.5 **a.** Sample response: Since all squares are similar, the artist can use any scale that results in a square that will fit on the card. One possibility is to allow 1 cm to represent 20 feet. The resulting scale drawing would be a square 4.5 cm on a side. The pitcher's mound would be 3.025 cm from home plate.

b. Sample response: A square cannot be drawn on a sphere since there are no parallel lines on a sphere.

3.6 **a.** There are an infinite number of triangles that can be drawn with identical angle measures but different perimeters. The triangles would be similar.

b. Sample response: None. If two triangles on a sphere have the same angle measures, they are congruent. Therefore, their perimeters are equal.

* * * * * * * * * *

Answers to Summary Assessment (page 416)

1. **a.** The measure of an exterior angle is equal to the sum of the measures of the nonadjacent interior angles. Using the diagram given in the student edition:

$$m\angle CAB + (m\angle ABC + m\angle BCA) = 180^\circ$$
$$m\angle DAC + m\angle CAB = 180^\circ$$
$$\therefore m\angle DAC = m\angle ABC + m\angle BCA \text{ by substitution}$$

b. The exterior angle theorem does not hold true in spherical geometry. Any example will show that the measure of the exterior angle is less than the sum of the measures of the two non-adjacent interior angles.

For example, given a triangle *ABC* on the surface of a sphere: $m\angle DAC + m\angle CAB = 180$ and $m\angle CAB + m\angle CBA + m\angle ACB > 180$. Therefore, by substitution, $m\angle CAB + m\angle CBA + m\angle ACB > m\angle DAC + m\angle CAB$, which leads to the conclusion that $m\angle CBA + m\angle ACB > m\angle DAC$.

2. **a.** Sample response: Degrees of longitude are measured with the equator acting as a scaled ruler. Lines of longitude (also called meridians) are great circles perpendicular to the equator. The longitude of a particular point is the number of degrees in the arc of the equator cut off by the line of longitude through the point under consideration and the prime meridian (0° longitude).

Degrees of latitude are measured on the line of longitude through the point under consideration. The latitude of the point is the number of degrees in the arc of the line of longitude from the equator to the point.

b. Sample response: One such triangle is formed by arcs from two great circles that form an angle measuring 90° at the north pole and the intercepted part of the equator. Since all three interior angles measure 90°, their sum is 270°.

c. Sample response: A line of latitude is not a great circle unless it is the equator. As a result, they generally do not satisfy the definition of *line* in spherical geometry.

3. **a.** Sample response: A circle on a sphere is the set of all points at a given distance from a fixed point, the center. This is any circle formed by the intersection of a plane and the sphere. The plane must be the perpendicular bisector of the segment joining the center of the circle and the center of the sphere. Using Earth as an example, the Tropic of Cancer could be considered a circle with its center at the south pole.

b. Sample response: A circle on a sphere can be line if and only if it is a great circle.

Module Assessment

1. **a.** On the surface of a sphere, what is the upper bound for the sum of the measures of the interior angles of a pentagon? Explain your response.

b. On the surface of a sphere, what is the lower bound for the sum of the measures of the interior angles of a pentagon? Explain your response.

2. If two pentagons on the same sphere are similar, what can you conclude about them?

3. Consider a set of polygons constructed on spheres of different sizes. The polygons all have the same number of sides and the lengths of corresponding sides are equal. As the spheres increase in size, the sum of the measures of the interior angles of the polygons has a lower bound of 1080˚. How many sides do the polygons have? What is the upper bound for the sum of the measures of their interior angles?

4. Three cities—Alpha, Beta, and Costa—all are located along the same great circle of a spherical planet with a radius of approximately 6667 km. In terms of distance, what conditions must exist if regularly scheduled flights between Alpha and Costa pass over Beta?

5. In a paragraph, describe at least eight differences between Euclidean geometry and spherical geometry.

Answers to Module Assessment

1. **a.** On a sphere, the sum of the measures of the interior angles in a pentagon has an upper bound of 900°, because each of the five angles has an upper bound of 180°.

b. The lower bound for the sum of the measures of the interior angles in a pentagon is 540°, because a pentagon can be divided into three triangles by drawing diagonals from one vertex and each triangle has a lower bound of 180° for the sum of the measures of its angles.

2. If two pentagons on the same sphere are similar, the pentagons are congruent.

3. From Problem **2.6**, the sum of the measures of the interior angles of an n-sided polygon has a lower bound of $(n-2) \bullet 180^\circ$. This is because $n-2$ triangles can be drawn from one vertex of the polygon and the sum of the measures of the interior angles of each triangle has a lower bound of 180°. Since the lower bound is 1080°, $n = 8$.

The upper bound for the sum of the measures of the angles is $180 \bullet 8 = 1440^\circ$.

4. The distance between two locations on a great circle is the length of the minor arc between the two points. Half the circumference of the planet is approximately 20,945 km. Assuming that flights between Alpha and Costa take the shortest route, they would pass over Beta if and only if the following relationship is true: $AB + BC \leq 20,945$ km.

5. Answers will vary. The table below lists nine differences between the two geometries.

Euclidean Geometry	**Spherical Geometry**
Two distinct lines intersect in one point or are parallel.	Two distinct lines intersect in two points.
The sum of the measures of the interior angles of a triangle is 180°.	The sum of the measures of the interior angles of a triangle is greater than 180° and less than 540°.
The sum of the measures of the interior angles of a quadrilateral is 360°.	The sum of the measures of the interior angles of a quadrilateral is greater than 360° and less than 720°.
Parallel lines exist.	Parallel lines do not exist.
There is exactly one perpendicular from a point not on the line to the line.	There are an infinite number of perpendiculars from a polar point to a line; otherwise only one perpendicular exists from a point not on the line to the line.
Lines are infinite and have no length,	Lines have a finite length.
Rectangles and parallelograms exist.	Rectangles and parallelograms do not exist.
Similarity does not necessarily imply congruence.	Similarity implies congruence.
The sum $AB + BC = AC$ when B is between A and C.	The sum $AB + BC = AC$ when B is between A and C as long as AC is a semicircle or a minor arc.

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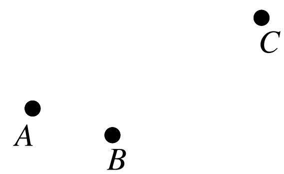
Flashbacks

Activity 1

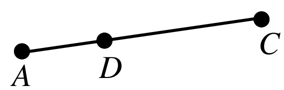
1.1 **a.** How many lines may be created in a plane using the points *A*, *B*, and *C* in the diagram below?



b. How many rays may be created in a plane using the points *A*, *B*, and *C*?

c. Describe the differences between a line and a ray.

1.2 **a.** How could you describe the length of $\overline{AC}$ using point *D*?

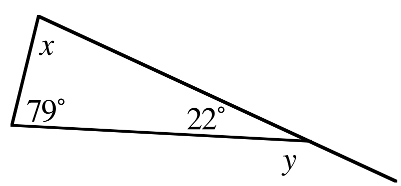


b. What is the relationship of *D* to *A* and *C*?

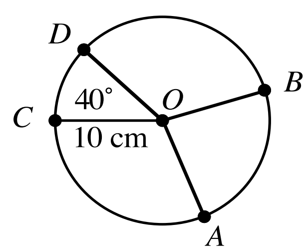
1.3 **a.** What must be true for two lines to be parallel?

b. What must be true for two lines to be perpendicular?

1.4 Determine the values of *x* and *y* in the diagram below.



1.5 **a.** Determine the length of arcs *CD* in the circle below.



b. If arc *AB* were 14 cm long, what would be the approximate measure of $\angle AOB$?

Activity 2

2.1 Draw all diagonals from one vertex of a polygon. What is the relationship between the number of diagonals and the number of triangles formed?

2.2 Describe the characteristics of each of the following:

a. a quadrilateral

b. a parallelogram

c. a rectangle

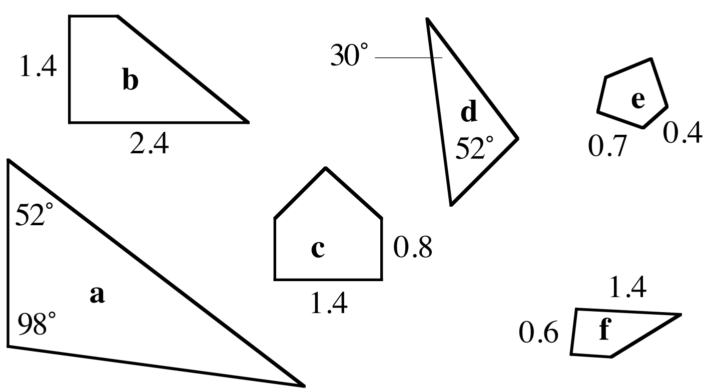
d. a trapezoid

e. a rhombus

f. a square

Activity 3

3.1 Determine which of the following polygons are similar. Justify your responses.



Answers to Flashbacks

Activity 1

1.1 **a.** Three lines may be created using A, B, and C: lines AB, BC, and AC.

b. Six rays may be created: $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, and $\overrightarrow{CA}$

c. Sample response: Through any two points A and B in a plane, one distinct line containing an infinite number of points may be created. A ray AB contains an endpoint, A, and extends without end through B. It may be said that ray AB is a part of line AB.

1.2 **a.** The length of $\overline{AC}$ may be described as follows: $AD + DC = AC$.

b. Point D is between A and C.

1.3 **a.** The two lines must not intersect.

b. The two intersecting lines must form right angles.

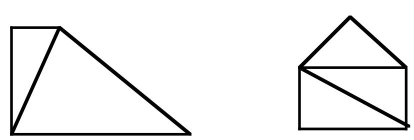
1.4 $x = 79^\circ$; $y = 158^\circ$

1.5 **a.** approximately 7.0 cm

b. approximately 80°

Activity 2

2.1 For n diagonals from one vertex in a polygon, there are $n + 1$ triangles. This is true for any convex polygon. Sample sketches:



2.2 Answers may vary. Sample responses are given below.

a. A quadrilateral is a simple closed four-sided figure.

b. A parallelogram is a quadrilateral in which opposite sides are parallel.

c. A rectangle is a parallelogram with one right angle.

d. A trapezoid is a quadrilateral with one pair of parallel sides.

e. A rhombus is a parallelogram with equal adjacent sides.

f. A square is a rectangle with equal adjacent sides.

Activity 3

3.1 Figure **a** is similar to Figure **d** since all three angles are congruent.