SIMMS Integrated Mathematics:

A Modeling Approach Using Technology

Level 5 Volumes 1-3

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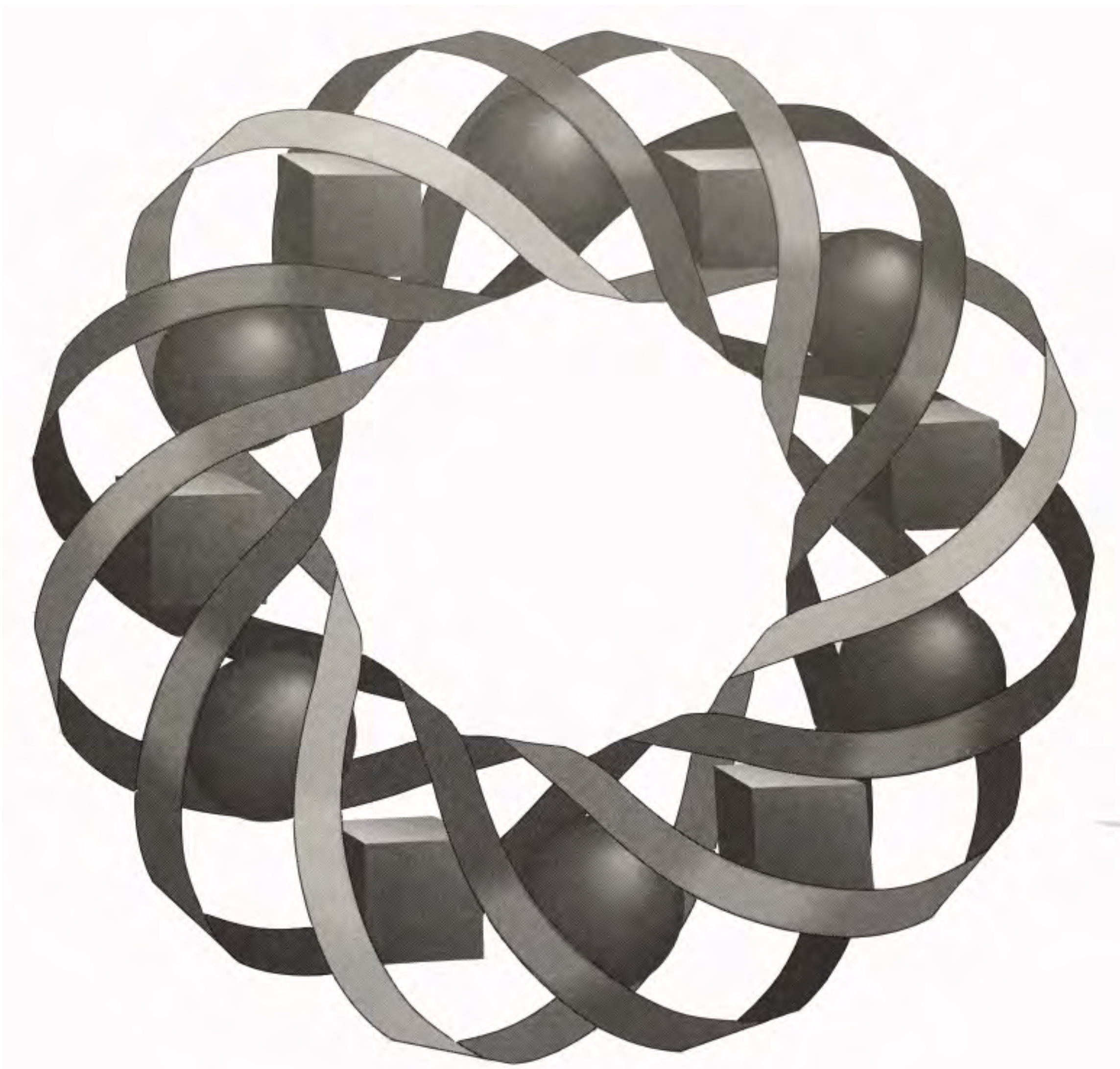
Reinvent the Wheel



If wheels weren't round, would they still roll smoothly? In this module, you investigate some other types of curves that might supply a smooth ride.

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Reinvent the Wheel

Introduction

The invention of the wheel took place several thousand years ago, sometime during the Bronze Age (about 3500–1000 B.C.). Since then, circular shapes have been incorporated in an almost endless list of useful inventions, from drill bits to medicine bottles. But have you ever stopped to wonder if a circle provides the best design for these products?

The circle's popularity may have discouraged inventors from examining the usefulness of other shapes. Are there other curved shapes that might work as well—or better—in some applications?

Activity 1

When building the pyramids, the ancient Egyptians may have transported the immense blocks of stone from quarry to construction site using a system of rollers. Today, cylindrical rollers help workers move heavy loads in a wide variety of situations—everywhere from shipyards to supermarkets. In this activity, you model the movement of an object using rollers with different shapes.

Exploration 1

a. Place a flat object on two cylinders with the same radius, as shown in Figure **1**. Roll the object and cylinders along a level surface.

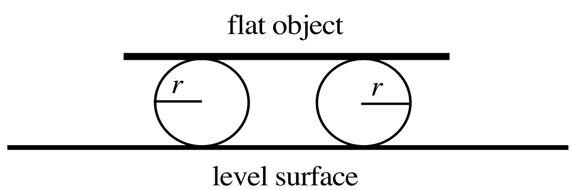


Figure 1: Two cylindrical rollers between object and surface

b. Observe the ride of the object on the rollers. Identify this motion as either "level" or "uneven."

c. Cut out each of the shapes in template A (seven in all) as accurately as possible. Predict which of the shapes, when used as a roller, might provide a level ride.

Mathematics Note

A **line of support** is a line that intersects a curve such that all points of the curve not on the line lie on one side of the line. The perpendicular distance between a pair of parallel lines of support is a **width** (w) of the curve.

A **curve of constant width** has the same width between every pair of parallel lines of support.

Figure **2** shows one example of a curve of constant width w with two parallel lines of support, line n and line m.

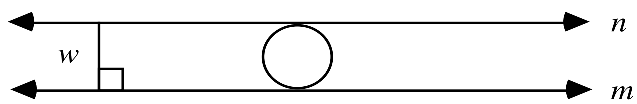


Figure 2: Curve of constant width with lines of support

d. Imagine that each of the seven shapes from template A represents a cross section of a roller. To simulate the motion of an object using rollers of each type, complete the following steps.

1. On a sheet of paper, draw a line m to represent a level surface (see Figure **1**).
2. Position shape A from the template so that line m is a line of support for it.
3. Draw a line parallel to line m to represent a flat object to be moved by rolling shape A. Label this line n; it should also be a line of support for shape A.
4. Simulate the motion of the flat object by “rolling” shape A along line m. If both m and n remain lines of support for shape A at all times (in other words, if shape A is always between, and touching, both lines), identify the motion as “level.” If not, identify the motion as “uneven.”
5. Repeat Steps **1–4** using shapes B through G.

e. Determine which of the seven shapes have constant widths.

Discussion 1

a. Is a line of support always a tangent line? Explain your response.

b. In Part **d** of Exploration **1**, why are lines m and n drawn parallel to each other?

c. Which shapes are curves of constant width? Justify your response.

d. **1.** What characteristics are common to the shapes that provided "level" rides?

2. What characteristics can cause shapes to provide "uneven" rides?

e. Is every circle a curve of constant width? Explain your response.

Exploration 2

In Exploration **1**, one of the figures that has a constant width is shape E. This shape is a **Reuleaux** (pronounced "rue-low") **triangle**, named after the German mathematician and high school teacher Franz Reuleaux (1829–1905). Figure **3** shows another example of a Reuleaux triangle.

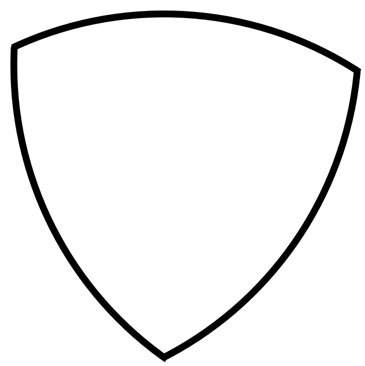


Figure 3: A Reuleaux triangle

In this exploration, you construct several **Reuleaux polygons**, then determine whether or not each one is a curve of constant width.

a. Construct a Reuleaux triangle by completing the following steps.

1. Construct an equilateral triangle with sides measuring 5 cm.

2. Construct an arc that has its center at one vertex and its endpoints at the two other vertices, as shown in Figure **4**.

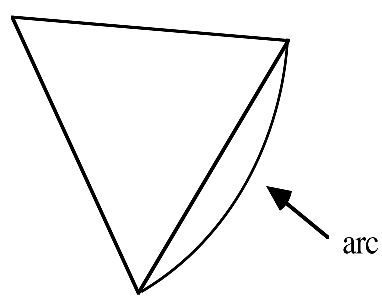


Figure 4: Arc connecting two vertices with center at third vertex

3. Construct similar arcs with centers at the other two vertices. The figure formed by the three arcs is a Reuleaux triangle.

4. Carefully cut out the shape of the Reuleaux triangle.

b. The constructions described in Part **a** are not possible for all regular polygons.

1. Attempt to repeat Part **a** using a square, a regular pentagon, a regular hexagon, and a regular heptagon.
2. Record which regular polygons did not allow the constructions described in Part **a**.

c. Using the method described in Part **d** of Exploration **1**, determine which of the Reuleaux polygons you created are curves of constant width.

d. Record the width of each Reuleaux polygon that is a curve of constant width.

Discussion 2

a.
1. Which regular polygons do not allow the constructions described in Part **a** of Exploration **2**? Explain your response.
2. In general, what types of polygons can be used to construct Reuleaux polygons?

b. Do all Reuleaux polygons appear to be curves of constant width? Explain your response.

c. How many pairs of parallel lines of support are there for a Reuleaux polygon? Explain your response.

Mathematics Note

A **Reuleaux polygon** is a curve of constant width constructed from a regular polygon with an odd number of sides. For example, Figure **5** shows a Reuleaux triangle with a constant width w.

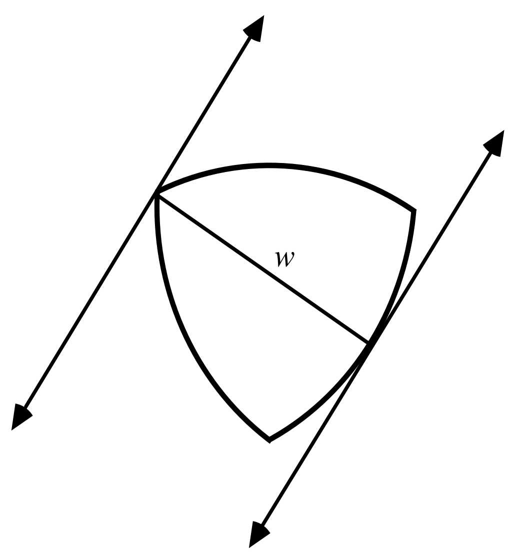


Figure 5: A Reuleaux triangle and two parallel lines of support

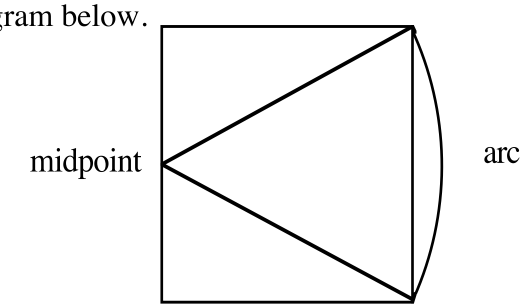
Assignment

1.1 Does a Reuleaux triangle fit the definition of a triangle in Euclidean geometry? Explain your response.

1.2 Use the following algorithm to construct a Reuleaux-like polygon from a regular polygon with an even number of sides.

a. Construct a regular polygon with an even number of sides, each measuring 5 cm.

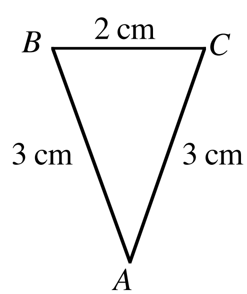
b. Construct an arc that has its center at the midpoint of one side and its endpoints at the vertices on the opposite side, as shown in the sample diagram below.



c. Construct similar arcs with centers at the midpoints of the remaining sides. Erase the original regular polygon.

d. Is the resulting figure a curve of constant width? Justify your response.

1.3 Two students, Anton and Jennifer, think that they can make a curve of constant width from isosceles triangle *ABC*, shown below.



a. If Anton and Jennifer are right, predict the measure of the constant width for the curve generated by ΔABC.

b. To test Anton and Jennifer's prediction, complete Steps **1–8**.

1. Create a triangle with the same dimensions as ΔABC above.

2. Draw $\overrightarrow{BC}$ about 2 cm past *C*.

3. Draw an arc with its center at *B* and one endpoint at *A* so that it intersects $\overrightarrow{BC}$. Label the point where the arc intersects $\overrightarrow{BC}$ as point *F*.

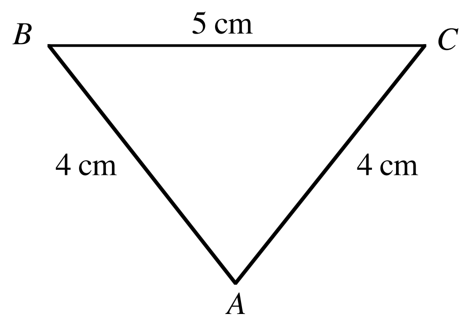
4. Draw $\overrightarrow{CB}$ about 2 cm past *B*.

5. Draw an arc with its center at C and one endpoint at A so that it intersects $\overrightarrow{CB}$. Label the point where the arc intersects $\overrightarrow{CB}$ as point E.
6. Construct a circle with its center at C that contains F. Construct another circle with its center at B that contains E.
7. Construct $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Label a point on $\overrightarrow{AB}$ that is not between A and B as point K. Label a point on $\overrightarrow{AC}$ that is not between A and C as point L.

   Label the point of intersection of circle B and $\overrightarrow{BK}$ as point G. Label the point of intersection of circle C and $\overrightarrow{CL}$ as point H.
8. Construct an arc with its center at A and endpoints at G and H.

c. Does figure *AFHGE* represent a curve of constant width? Were Anton and Jennifer correct? Explain your responses.

1.4 After attempting to create a curve of constant width from an isosceles triangle whose congruent sides are longer than the base side, Anton and Jennifer decided to try using an isosceles triangle whose base side is longer than the congruent sides. They started with ΔABC shown below.



a. First, the two friends constructed a line perpendicular to $\overline{BC}$ through point A. Then, they labeled a point A' on the perpendicular line so that $\Delta A'BC$ was equilateral.

Beginning with these same two steps, recreate Jennifer and Anton's construction.

b. Is the result in Part **a** a curve of constant width? If so, describe its width. If not, explain why not.

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1.5 Many compact disc (CD) players feature a disc changer that holds more than one disc. Disc changers typically come in one of two formats: a stack or a single drawer.

a. Why might the single drawer of a three-disc changer be shaped like a Reuleaux triangle, rather than an equilateral triangle or a circle?

b. Why would the single drawer of a four-disc changer not be shaped like a square?

1.6 Describe a curve of constant width that might provide the best shape for a single-drawer CD changer that holds five discs.

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Activity 2

In addition to their applications in compact disc players, Reuleaux triangles also provide the basic design for drill bits used to create "square" holes. In this activity, you examine how and why these drill bits work.

Exploration

In this exploration, you model the process of drilling a square hole.

a. Create a square cardboard frame with outside edges that measure 12 cm each, as shown in Figure **6**. The inside of the frame should measure 8 cm on each side.

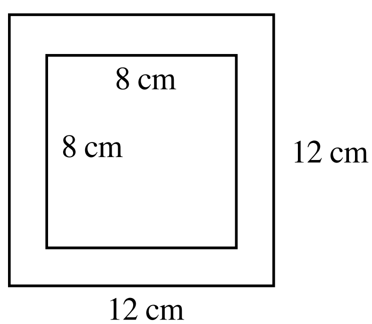


Figure 6: Square cardboard frame

b. Tape the frame to a sheet of white paper, then tape the white paper to a hard surface.

c. On another sheet of cardboard, construct a Reuleaux triangle using an equilateral triangle with sides measuring 8 cm.

Mathematics Note

A **median** is a segment that connects the vertex of a triangle with the midpoint of the opposite side. The intersection of the medians of a triangle is the **centroid**.

Consider an equilateral triangle inscribed in a Reuleaux triangle, as shown in Figure **7**. If the medians of the inscribed triangle are extended to intersect the Reuleaux triangle, they form the medians of the Reuleaux triangle. Since the intersection of the two sets of medians is the same, the centroid of the inscribed triangle is also the centroid of the Reuleaux triangle.

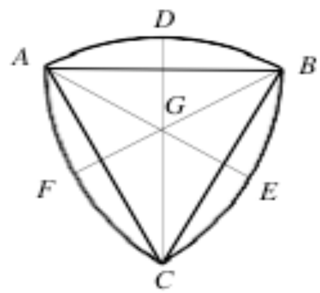


Figure 7: Reuleaux triangle with medians and centroid

In Figure **7**, the three medians of Reuleaux triangle ABC are $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. The centroid is located at point G:

d. Construct the three medians and centroid of your Reuleaux triangle from Part **c**. Punch a small hole through the centroid.

e. Cut out the Reuleaux triangle and place it in the square frame, as shown in Figure **8**.

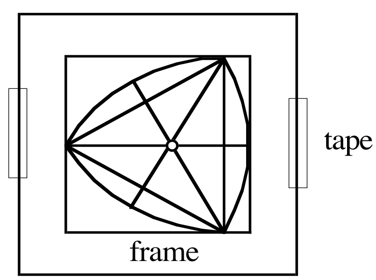


Figure 8: Reuleaux triangle in square frame

f. Place the point of a pencil in the hole at the centroid.

g. With the tip of the pencil, trace the path of the centroid as you rotate the Reuleaux triangle inside the square frame. (This may require two people: one to rotate the triangle and another to hold the pencil.)

Discussion

a. How do the results of the exploration confirm that a drill bit in the shape of a Reuleaux triangle can create a “square” hole?

b. Describe the path of the centroid in Part **g** of the exploration.

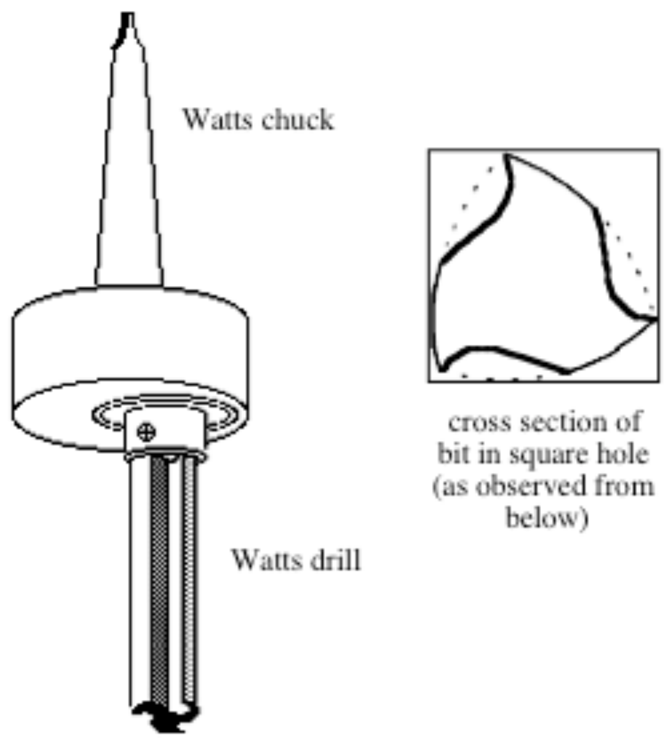
c. What does the path of the centroid represent for a drill bit?

d. Suppose that the path of the centroid were a single point. What is the shape of the hole produced by this drill?

e. Why do you think that Reuleaux polygons are not used as wheels on vehicles?

Assignment

2.1 The actual bit that drills square holes was invented by Harry Watts in 1914. The figure below shows a Watts drill, the Watts chuck that allows the unusual motion of the centroid, and a cross section of the bit. (A metal guide is necessary for accurate drilling.)



a. In which direction should this bit rotate to drill a hole?

b. Explain why the cross section of this bit does not show the shape of an entire Reuleaux triangle.

2.2 For a woodworking project, Ebdul wants to drill a square hole 4 cm on each side. Describe the size of the drill bit required to drill this hole.

2.3 **a.** Consider a drill bit based on the shape of a Reuleaux pentagon. If the centroid follows a path similar to the one in Part **g** of the exploration, what will be the shape of the hole? Hint: You may want to repeat the exploration using a Reuleaux pentagon.

b. As the number of sides increases in the Reuleaux polygon that determines the shape of the bit, what shape will the hole approach?

2.4 **a.** Draw all the lines of symmetry for an equilateral triangle.

b. Is every median of an equilateral triangle contained in a line of symmetry? Justify your response.

c. Is every median of a Reuleaux triangle contained in a line of symmetry? Justify your response.

* * * * *

2.5 **a.** Draw several regular polygons, other than equilateral triangles, each with a different number of sides. For each regular polygon, draw all the lines of symmetry.

b. Do you think the lines of symmetry for every regular polygon always intersect at one point? Justify your response.

c. Do you think that the lines of symmetry for every Reuleaux polygon always intersect at one point? Justify your response.

2.6 **a.** Draw several triangles, including at least one scalene, one isosceles, and one equilateral. For each triangle, draw the perpendicular bisector of each side. Extend the perpendicular bisectors until they intersect. This point of intersection is the **circumcenter**.

b. For each triangle, draw a circle with its center at the circumcenter and radius equal to the distance from the circumcenter to one of the triangle's vertices.

c. Based on your observations in Parts **a** and **b**, suggest a definition for a circumcenter.

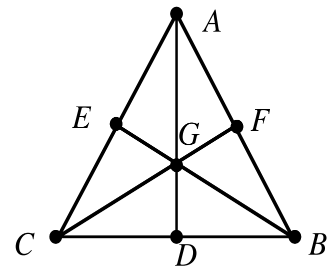
d. Use your responses to Parts **a–c** to explain why the following statement is true: "For any three noncollinear points in a plane, there is a circle that contains all three of them."

2.7 **a.** Draw several triangles, including at least one scalene, one isosceles, and one equilateral. For each triangle, draw the angle bisector of each angle. Extend the angle bisectors until they intersect. This point of intersection is the **incenter**.

b. For each triangle, draw a circle with its center at the incenter and radius equal to the perpendicular distance from the incenter to one of the triangle's sides.

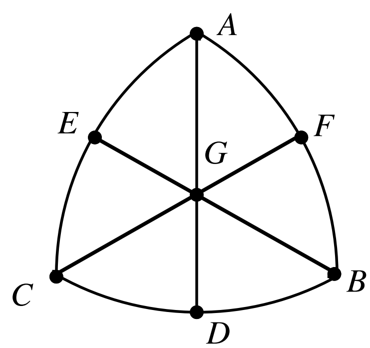
c. Based on your observations in Parts **a** and **b**, suggest a definition for an incenter.

2.8 **a.** The diagram below shows an equilateral triangle *ABC*, its medians, and its centroid. What do you think is the ratio of *AG* to *AD*? Use a geometry utility to test your prediction.



b. Prove that the centroid of an equilateral triangle is $2/3$ the distance from any vertex to the opposite side. Hint: Show that $AG/AD = 2/3$, $BG/BE = 2/3$, and $CG/CF = 2/3$.

2.9 The diagram below shows a Reuleaux triangle *ABC*, its medians, and its centroid. Determine the ratio AG/AD for a Reuleaux triangle.



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Research Project

Reuleaux triangles have been incorporated in a number of inventions, including the rotary engine. Developed by Felix Wankel in the 1950s, the rotary engine weighs about one-third as much as a piston engine of comparable power. Figure **9** shows a diagram of the rotor in this engine.

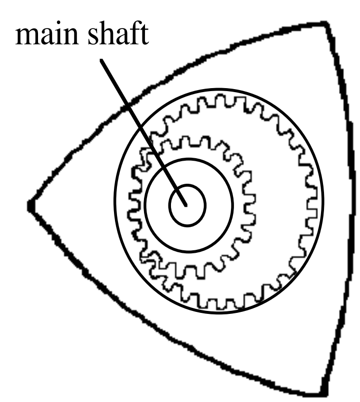


Figure 9: Rotor for a rotary engine

In the 1970s, the Mazda Company introduced two automobiles powered by rotary engines: the RX-2 and the RX-3. The popularity of these cars was limited, due in part to their poor fuel economy. By 1995, Mazda's RX-7 was one of the few automobiles with a rotary engine remaining on the U.S. market.

Investigate the development of the rotary engine. Your report should include:

- a description of the engine's origins
- an explanation of how it works
- a comparison with piston-powered engines
- a summary of its applications.

Activity 3

Since the sides of a Reuleaux polygon are curved, they cannot be measured in the same way as the sides of an ordinary polygon. In this activity, you derive a formula for the perimeter of a Reuleaux triangle, then generalize this formula for all Reuleaux polygons. You also determine how to calculate the area of Reuleaux polygons.

Exploration 1

Each side of a Reuleaux polygon is an arc of a circle. In this exploration, you determine a method for finding the length of an arc given the measure of its central angle and the radius of the circle.

Figure **10** below shows a regular polygon, along with one side ($\overset{\frown}{AB}$) of the corresponding Reuleaux polygon. This construction involves two different central angles: the central angle of the regular polygon ($\angle ADB$) and the central angle that determines the side of the Reuleaux polygon ($\angle ACB$). The measures of these central angles depend on the number of sides in the Reuleaux polygon.

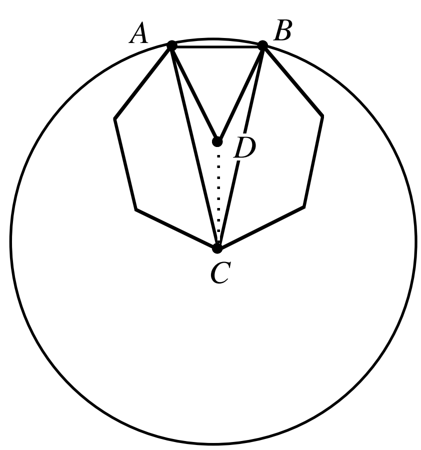


Figure 10: Construction of a Reuleaux polygon

a. Create a table with headings like those in Table **1** below.

Table 1: Angles used to create a Reuleaux polygon

Reuleaux Polygon	$m\angle ADB$	$m\angle ADC$	$m\angle ACB$	$\frac{m\angle ACB}{360^\circ}$
triangle	120°	120°	60°	1/6
pentagon		144°		
heptagon			$25\frac{5}{7}^\circ$	
nonagon				
n-gon				

b.
1. Create a regular pentagon and one arc of the corresponding Reuleaux polygon.
2. Construct $\angle ADB$, $\angle ADC$, and $\angle ACB$, as shown in Figure **10**.
3. Use your construction to complete the appropriate row in Table **1**.

c. Repeat Part **b** for a regular heptagon and a regular nonagon.

d. Use any patterns you observe to complete Table **1** for a regular polygon with n sides.

Mathematics Note

The **measure of an arc** is the measure of its central angle.

The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circumference of the circle by the fractional part of the circle that the arc represents.

For example, Figure **11** shows a circle with center at O and a radius of 5 cm.

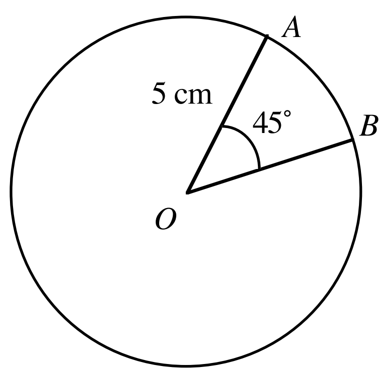


Figure 11: An arc of a circle

The measure of $\overset{\frown}{AB}$ is 45°. Since the central angle of the entire circle is 360°, the length of $\overset{\frown}{AB}$ can be found as follows:

$$\frac{45}{360} \bullet 2\pi(5) \approx 4 \text{ cm}$$

e. Given a constant width of 5 cm for each Reuleaux polygon, complete Table **2** below.

Table 2: Perimeter of Reuleaux polygons with $w = 5$ cm

Reuleaux Polygon	Side Length	Perimeter
triangle		
pentagon		
heptagon		
nonagon		
n-gon		

Discussion 1

a. When considering the arcs that form the sides of Reuleaux polygons, how is the radius of the circle that contains the arc related to the width of the Reuleaux polygon?

b. What is the relationship between the number of arcs used to construct a Reuleaux polygon and its perimeter?

c. Suggest a formula for the perimeter of a Reuleaux polygon with n sides and a constant width w.

Mathematics Note

The perimeter (P) of any curve of constant width w can be calculated by the formula $P = \pi w$.

For example, a Reuleaux triangle with a constant width of 10 cm has a perimeter of 10π cm.

d. Why doesn't the formula for the perimeter of a Reuleaux polygon include the number of sides?

e. A circle is a curve of constant width. Does the formula for the perimeter of a curve of constant width apply to a circle? Explain your response.

f. Recall that a central angle is an angle whose vertex is at a circle's center. An **inscribed angle** is an angle whose vertex is on the circle. For example, Figure **12** shows three different inscribed angles.

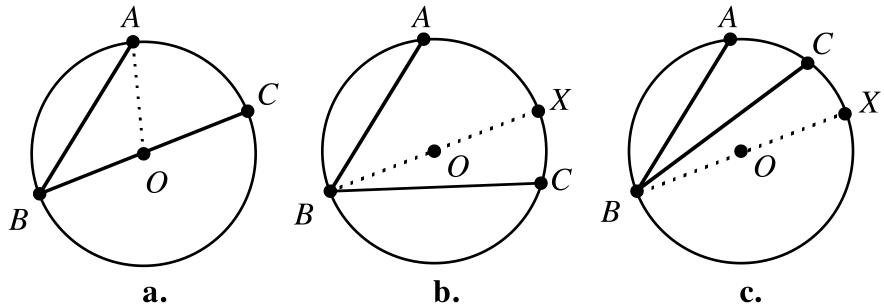


Figure 12: Three inscribed angles

1. In Figure **12a**, one of the sides of inscribed angle *ABC* passes through the circle's center *O*. What is the relationship between an inscribed angle of this type and the measure of its intercepted arc? Explain your response.
2. Is the relationship you described in Part **f1** also true when the center *O* lies in the interior of inscribed angle *ABC*, as shown in Figure **12b**? Justify your response.
3. Does this relationship also hold true when *O* lies outside inscribed angle *ABC*, as shown in Figure **12c**?

g. Figure **13** below shows part of the construction of a Reuleaux polygon.

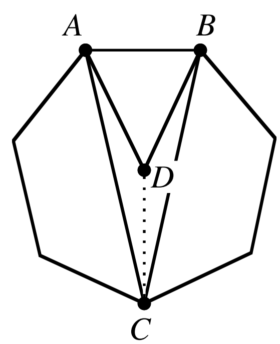


Figure 13: Construction of a Reuleaux polygon

In Exploration **1**, you discovered that

$$m\angle ACB = \frac{1}{2} m\angle ADB$$

How are these two angles related in terms of the circle with center at *D* and radius *DA*?

Exploration 2

In this exploration, you use your results from Exploration **1** to develop a method for calculating the areas of Reuleaux polygons.

Mathematics Note

A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. In Figure **12**, for example, the shaded region represents the sector defined by $\angle AOB$.

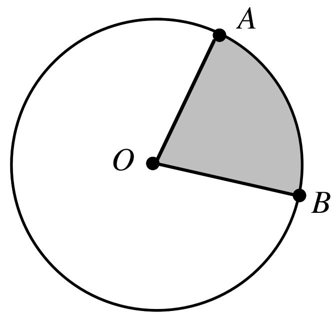


Figure 12: A sector of circle *O*

A **segment of a circle** is a region bounded by a chord and the arc that shares the same endpoints as the chord. For example, Figure **13** shows the segment defined by chord AB and $\overset{\frown}{AB}$.

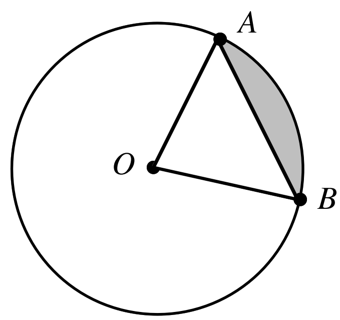


Figure 13: A segment of circle *O*

a. Create a table with headings like those in Table **3**.

Table 3: Area of segments for Reuleaux polygons with w = 5 cm

Reuleaux Polygon	Area of Sector	Area of Segment
triangle		
pentagon		
heptagon		
nonagon		
n-gon		

b. The area of a sector can be found by multiplying the area of a circle by the fractional part of the circle that the sector represents. Use the values you recorded in the appropriate column of Table **1** to find the area of the sector that corresponds with each Reuleaux polygon in Table **3**.

c. Use the area of each sector in Part **b** to determine the area of the corresponding segment of a circle for each Reuleaux polygon. Record these values in Table **3**.

Hint: Use the area of the triangle formed by the chord and the central angle to help determine the area of the segment. You may need to use trigonometry to find the lengths of sides.

d. Given a set of Reuleaux polygons with the same constant width, which one do you think will have the least area?

Discussion 2

a. Describe how you found the area of the sector that corresponds to a Reuleaux triangle for which $w = 5$ cm .

b. Describe how you found the area of the segment that corresponds to a Reuleaux triangle for which $w = 5$ cm .

c. In general, how does the area of a Reuleaux polygon compare with the area of the inscribed regular polygon?

Assignment

3.1 Calculate the perimeter of each of the following Reuleaux polygons:

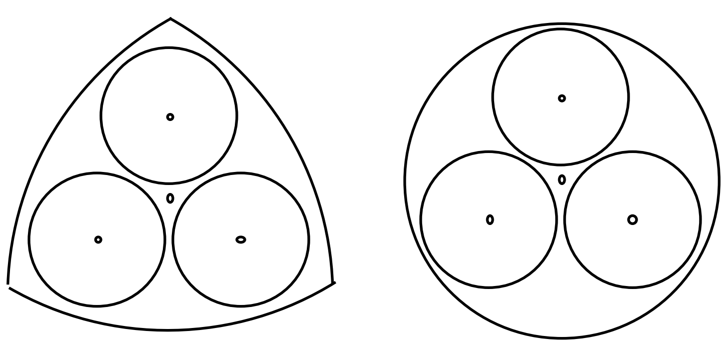
a. a Reuleaux triangle with a constant width of 11 cm

b. a Reuleaux 17-gon with a constant width of 15 cm

c. a Reuleaux pentagon with a constant width of 10 cm.

3.2 Calculate the area of a Reuleaux triangle with constant width of 7 cm.

3.3 The diagram below shows the changer trays for two different CD players. The width of the tray shaped like a Reuleaux triangle is 26.1 cm. The radius of the tray shaped like a circle is 13.8 cm. Compare the areas of the two trays.



3.4 Determine the area of each of the following Reuleaux polygons:

a. a Reuleaux pentagon with a constant width of 10 cm

b. a Reuleaux heptagon with a constant width of 10 cm.

* * * * *

3.5 Describe a method for determining the area of any Reuleaux polygon.

3.6 For its new conveyor system, a manufacturing company is considering rollers shaped like Reuleaux heptagons. An engineer says this design will reduce costs because these rollers require less material than circular rollers. What percentage of material will be saved by using rollers shaped like Reuleaux heptagons instead of circular rollers?

3.7 Describe the three-dimensional surface that results when a Reuleaux triangle is rotated around an axis of symmetry. Include a sketch in your description, and suggest some possible applications for the surface.

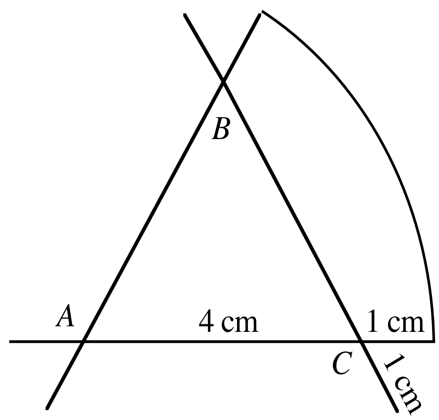
* * * * * * * * * *

Summary Assessment

1. In some applications, it may be desirable to modify the shape of a Reuleaux polygon to obtain more "rounded" corners.

a. To construct a Reuleaux triangle with rounded corners, complete the following steps.

- Construct an equilateral triangle *ABC* with sides measuring 4 cm each.
- Extend the sides of the triangle 1 cm past each vertex.
- Construct an arc with its center at one vertex of the triangle and a radius of 5 cm. Use the endpoints of two extended sides as the endpoints of the arc, as shown below.



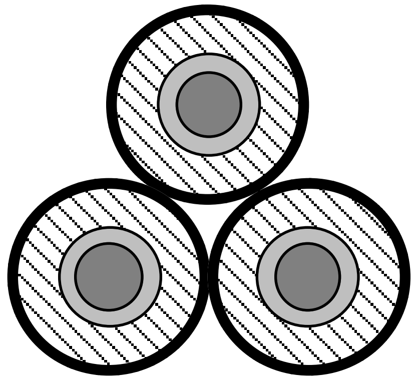
- Repeat the previous step for the two other vertices.
- To connect the endpoints of each pair of adjacent arcs, construct a smaller arc with its center at a vertex of the original triangle and a radius of 1 cm.

b. Is the shape you drew in Part **a** a curve of constant width? Justify your response.

c. Determine the perimeter of the shape.

d. Determine the area of the shape.

2. The circular cutting blades on a Norelco™ electric razor are arranged as shown in the diagram below.



What is the best shape for the head of this razor? Explain your response.

Module Summary

- A **line of support** is a line that intersects a curve such that all points of the curve not on the line lie on one side of the line.
- The perpendicular distance between a pair of parallel lines of support is a **width** (w) of the curve.
- A **curve of constant width** has the same width between every pair of parallel lines of support.
- A **Reuleaux polygon** is a curve of constant width constructed from a regular polygon with an odd number of sides.
- A **median** is a segment that connects the vertex of a triangle with the midpoint of the opposite side. The intersection of the medians of a triangle is the **centroid**.
- An **arc** of a circle is a part of the circle whose endpoints are the intersections, with the circle, of the sides of a central angle. The **measure of an arc** is the measure of its central angle.
- The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circle's circumference by the fractional part of the circle that the arc represents.
- The perimeter (P) of any curve of constant width w can be calculated by the formula $P = \pi w$.
- A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. The **area of a sector** can be found by multiplying the circle's area by the fractional part of the circle that the sector represents.
- A **segment of a circle** is a region bounded by a chord and the arc that shares the same endpoints as the chord.

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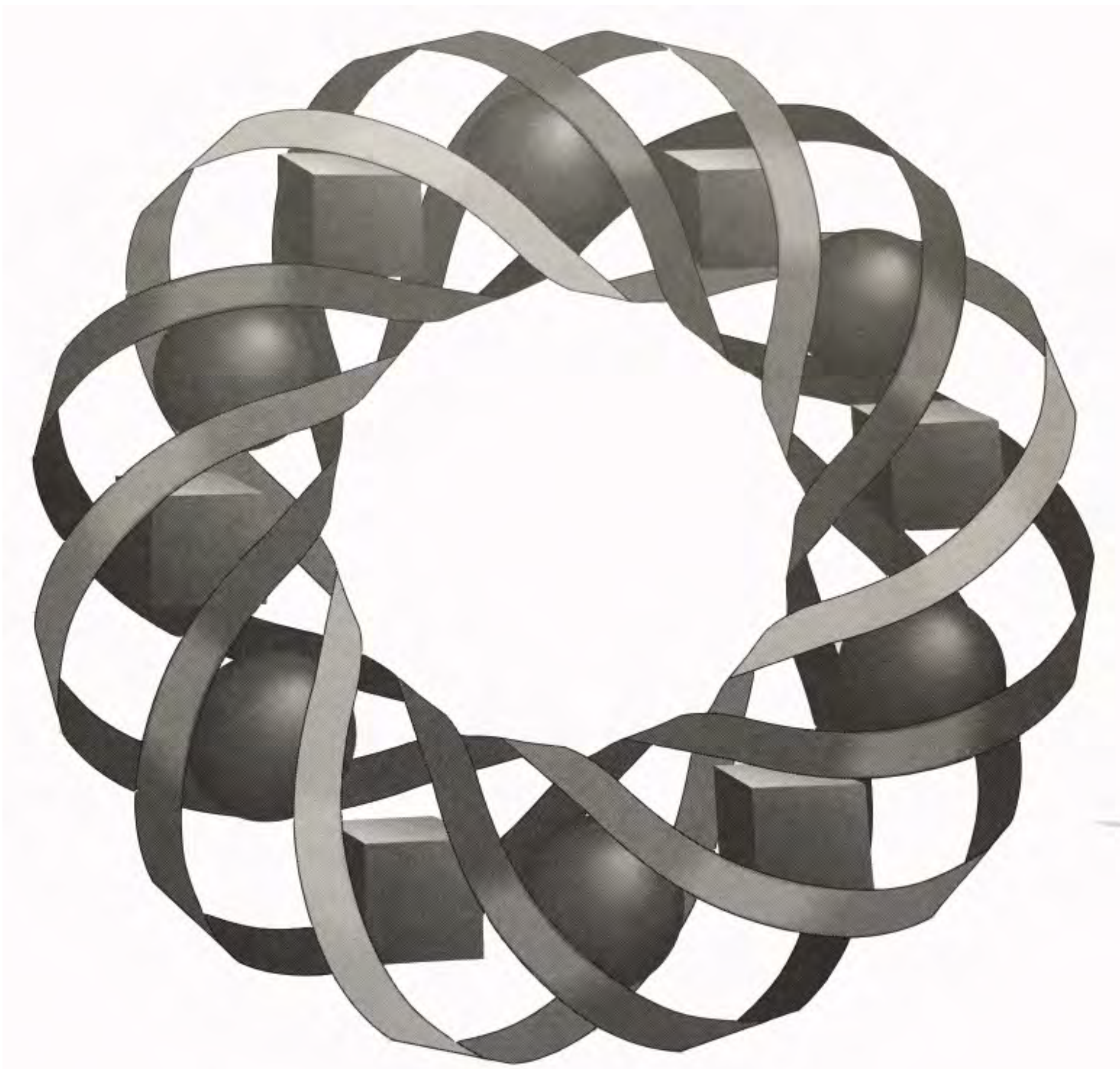
Sunken Sub



"This is the USS *Livingston*. We've lost ballast and are going down! Our coordinates are . . . " A submarine is sinking and you're in charge of the rescue. Can you save the crew?

John Knudson-Martin • Laurie Paladichuk

Sunken Sub

Introduction

Whenever a child is reported missing in the woods, a skier is buried in an avalanche, or a ship is lost at sea, rescue teams scramble to find them. But locating someone wandering in the wilderness or adrift on the ocean can be a daunting task. To increase the odds of a successful rescue, searchers use probability to devise search patterns.

In this module, you follow the search for the USS *Livingston*, a fictional submarine lost during a training cruise. First, you use probability to determine where the submarine might have gone down; then, you design a search strategy. Finally, you determine the chances of finding the *Livingston* in time to save the crew.

History Note

The U.S. Navy prides itself on the safety of its ships. But going below the waves in a submarine is an inherently risky business. As the depth of the water increases, the pressure on the ship's hull also increases. Despite the Navy's stringent safety precautions, two nuclear submarines have been lost at sea.

On April 10, 1963, the USS *Thresher* was on a test cruise off Cape Cod after a long period in dry dock for repairs. At shallow depths, the *Thresher* performed perfectly. The captain then took the ship and its 129-member crew deeper.

Later, a surface vessel monitoring the *Thresher*'s dive received the message "attempting to blow." This meant that the *Thresher* was blowing out its ballast tanks and coming to the surface. But why did the radio message say "attempting"? As no further contact was ever made with the *Thresher*, the answer to this question remains unknown. The wreckage of the submarine was found on the ocean floor, 2500 m below the surface.

Five years after the *Thresher* disaster, the USS *Scorpion* with a crew of 99 was returning home to Norfolk Naval Base in Virginia. On May 21, the *Scorpion* made routine radio contact with Norfolk, reporting its position as 75 km south of the Azores Islands in the mid-Atlantic. Two days later, however, the Scorpion had neither arrived in port nor reported another position. The wreckage was eventually pinpointed by an anti-submarine listening post located off the Azores.

Submarines have improved since the losses of the *Thresher* and the *Scorpion*. They now have stronger hulls and can withstand greater underwater pressures. To help retrieve the crew of a disabled submarine, the Navy has developed a Deep Submergence Rescue Vehicle.

Activity 1

The USS *Livingston* left its base on a training mission at 0800 hours. At 1100 hours, the submarine sent the following message: "This is the USS *Livingston*. We're going down! The crew is secure in the forward compartments. Our coordinates are . . .". Here the message was interrupted by the crackle of static. The radio operator immediately called the base commander.

The base commander ordered all officers to the control center. They began reviewing the facts and working on a rescue plan.

The base is located on the northern shore of an island in the Pacific Ocean. The island is roughly circular, with a radius of about 10 km. Within 50 km of the shoreline, the water is relatively shallow. Beyond 50 km, the ocean bottom drops off to a depth of 1000 m. During training missions, submarines can go no farther than 75 km from the base.

To attempt to locate the *Livingston*, the rescue team decided to send out submarines equipped with special sonar devices. A sonar technician informed the officers that if the submarine went down in shallow water and the entire region is searched, they have about a 70% chance of detecting it. If the *Livingston* went down in deep water, the rescuers have only about a 10% chance of detecting it.

Exploration

a.

1. Draw a sketch of the island, the submarine base, and the training area.
2. Determine the surface area of the region in which the submarine may have gone down, and record this value on your sketch.

b.

1. Draw the region of relatively shallow water around the island. Determine the surface area of this region and record it on your sketch.
2. Determine the surface area of the region of deep water that lies within the training area and record it on your sketch.

Mathematics Note

When the outcomes in an event can be modeled geometrically, the theoretical probability of the event may be found using the ratio below:

$$P(\mathrm{E}) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

Some examples of geometric measures are lengths, angle measures, areas, and volumes.

For example, suppose that you have lost a gold earring on a lawn measuring 10 m by 15 m. You plan to search the lawn with an instrument that can detect metal within a radius of 0.5 m. As shown in Figure **1**, one pass with the metal detector along the width of the lawn will cover an area of approximately $10\ \text{m}^2$.

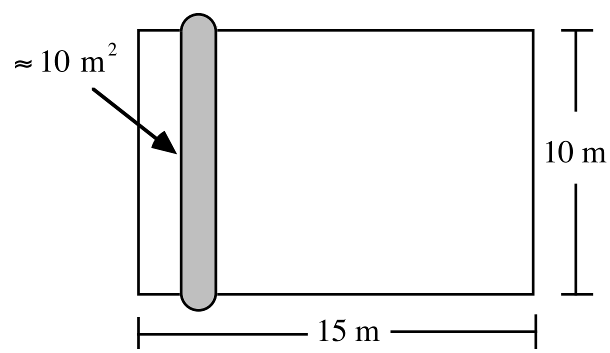


Figure 1: Area covered by metal detector on lawn

The entire lawn has an area of $10 \bullet 15 = 150\ \text{m}^2$. Assuming that the earring will be found if the metal detector passes over it, the probability that you will find it on one pass is approximately equal to the ratio of areas:

$$\frac{10}{150} \approx 0.067 = 6.7\%$$

c. Assume that the chances of the *Livingston* sinking in any one point in the training area are uniform.

1. Determine the probability that it went down in relatively shallow water.
2. Determine the probability that it went down in deep water.

d. Recall that **conditional probability** is the probability of an event occurring given that an initial event, or **condition** occurs. The probability that event B occurs given that event A occurs is denoted by the expression $P(\text{B}|\text{A})$.

According to the sonar technician, the conditional probability of finding the *Livingston*, given that it went down in deep water and the entire region is searched, is 10%. What is the probability that the *Livingston* will not be found after one search of the area, given that it went down in deep water?

Discussion

a. Describe how you used a geometric model to determine the probability that the *Livingston* went down in shallow water.

b. Describe two ways to determine the probability that the *Livingston* went down in deep water.

c. In a **multistage experiment**, one event is followed by one or more other events. In a multistage experiment involving conditional probabilities, the probability of event A followed by event B is found by multiplying the probability of A by the conditional probability of B, given A has already occurred. This can be denoted mathematically as shown below:

$$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$$

Describe the differences that exist between the following two probabilities:

1. the probability of finding the *Livingston*, given that it went down in deep water
2. the probability that the *Livingston* went down in deep water and will be found.

d. Without knowing where the sub went down, should the rescue team begin their search in shallow water or in deep water? Explain your response.

Assignment

1.1 If the *Livingston* went down in shallow water and the entire region is searched one time, the rescuers have about a 70% chance of detecting it. If the *Livingston* went down in deep water, the rescuers have about a 10% chance of detecting it with one complete search.

a. Create a tree diagram to show the possible outcomes of the search for the *Livingston*. Label each branch with the appropriate probability.

b. Describe how to use the tree diagram to determine the probability that the *Livingston* went down in deep water and will be found.

1.2 Use your tree diagram from Problem **1.1** to determine each of the following:

a. the probability that the *Livingston* went down in shallow water and will be found

b. the probability that the *Livingston* went down in deep water and will not be found

c. the total probability that the *Livingston* will be found.

1.3 Jamal and Laurie are enjoying a cross-country ski trip in Yellowstone National Park. As they pass under a cornice of snow, a small avalanche occurs. Laurie is swept down the hill and buried.

Jamal rushes to search for his friend. The debris from the avalanche is spread over a rectangular region 12 m by 10 m. Laurie could be buried anywhere in this area. Jamal estimates that the snow is no more than 2 m deep. Using the cord from his hood, he lashes two ski poles together to form a pole about 2.5 m long. This enables him to probe the full depth of the snow.

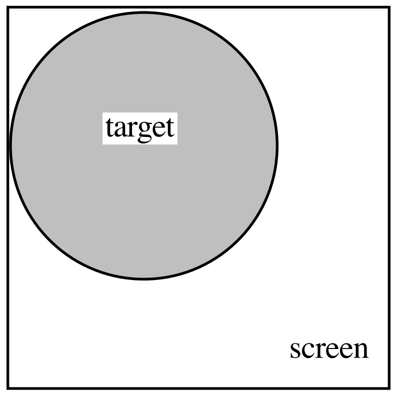
a. Assume that Laurie is lying face down under the snow. Using your own estimate of Laurie's body size, determine the probability that Jamal will find Laurie on his first probe with the pole.

b. If Jamal can search an area of 6 m^2 every minute, how long will it take him to search the entire region of the avalanche?

c. Assume that if Jamal probes the location in which Laurie is buried, he will find her. If Jamal finds Laurie within the first 10 min of the search, there is an 80% chance that she will survive. If he finds her from 10 to 20 min after the avalanche, her chances of survival fall to about 55%. What is the probability that Laurie will survive her ordeal?

1.4 In one game at the local arcade, a player selects a circular target area on a square video screen. Once the target zone has been chosen, the player presses a button that randomly illuminates one pixel on the screen. If the illuminated pixel is in the target zone, the player wins a prize.

The video screen measures 800 pixels by 800 pixels, while the target has a radius of 280 pixels. The diagram below shows one possible position for the target on the screen.

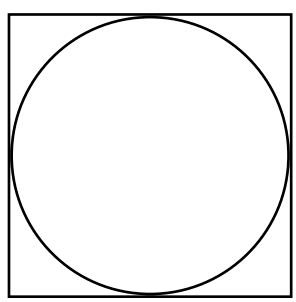


a. What is the probability that the lighted pixel is inside the target area?

b. Are the chances of winning improved by moving the target to another location? Explain your response.

* * * * *

1.5 Laurie challenged Jamal to a game similar to the one described in Problem **1.4**. As shown below, she drew a circle inscribed in a square on a sheet of paper. She then placed the paper on the floor, blindfolded Jamal, and asked him to drop a dart onto this target. If the dart falls inside the square, but outside the circle, Jamal wins.

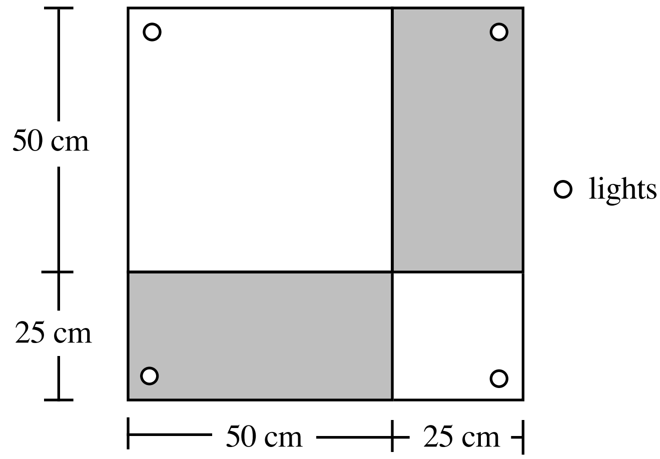


a. Given that the dart lands inside the square, what is the probability that it lands inside the circle? Explain your response.

b. Given that the dart lands inside the square, what is the probability that it lands exactly on the circle? Explain your response.

c. Given that the dart lands inside the square, what is the probability that Jamal wins the game?

1.6 In one game at the school carnival, a player tosses a spinning top onto a board. As shown below, the board is divided into four regions, two of which are shaded. Each region has a light beneath it. These lights are turned on and off at random so that only one light is lit at any one time. For a player to win the game, the top must stop in a shaded region and a light must be on in a shaded region.



a. What is the probability that the top lands in a shaded region?

b. What is the probability that a light is on in a shaded region??

c. What is the probability of winning the game?

d. In a situation involving conditional probabilities, $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$. If A and B are independent events, however, then $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B})$. (In other words, when A and B are independent, $P(\text{B}|\text{A}) = P(\text{B})$.)

   1. Consider the event that the top lands in a shaded region and the event that a light is on in a shaded region. Are these independent events? Explain your response.

   2. Consider the event that the top lands in a shaded region and the event that a player wins the game. Are these independent events? Explain your response.

* * * * * * * * * *

Activity 2

Unfortunately for the crew of the *Livingston*, the search is proceeding at a painfully slow pace. Two other submarines from the base are combing the training region using sonar. As shown in Figure **2**, each sub is able to scan a circular region on the ocean floor 2 km in diameter. In order to search the bottom effectively, however, the two subs are forced to limit their cruising speed to 10 km/hr.

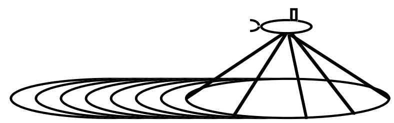


Figure 2: Rescue submarine searching for the USS *Livingston*

The crew of the *Livingston* was equipped with enough oxygen and supplies to last five days. Can the rescuers search the entire area before time runs out?

Exploration

The region of shallow water around the submarine base has a surface area of about 10,996 km^2. Within a 75-km radius of the base, the region of deep water has a surface area of 6361 km^2. If the rescue team passes over the lost submarine in shallow water, there is a 70% chance they will detect it. If they pass over the lost submarine in deep water, there is only a 10% chance they will detect it.

a. Find the surface area of the region that the two rescue submarines can cover if they search without a break for five days. (Recall that their cruising speed is limited to 10 km/hr.)

b. **1.** Assume that the USS *Livingston* went down in shallow water. If the subs confine their search to this region, determine the probability that the rescue submarines pass over it in five days.

2. Assume that the USS *Livingston* went down in deep water. If the subs confine their search to this region, determine the probability that the rescue submarines pass over it in five days.

c. **1.** Create a tree diagram to show the possible outcomes of the search for the *Livingston* if both subs search the shallow water. Label each branch with the appropriate probability.

Your diagram should show the probabilities that the sub went down in deep water or shallow water, the probabilities that the rescue team will pass over the lost submarine in the five-day period, and the probabilities that the lost submarine will be detected.

2. Create a tree diagram to show the possible outcomes of the search for the *Livingston* if both subs search the deep water. Label each branch with the appropriate probability. **Note:** Save both tree diagrams for use in the assignment.

Discussion

a. One officer predicted that the two rescue submarines could make one complete search of the region of shallow water in five days. Do you agree with this estimate?

b. How many days would it take the two submarines to search the entire training region?

c. **1.** How many submarines would be needed to search the entire training region in five days?

2. Would involving this many subs in the search guarantee that the *Livingston* will be found?

d. Suppose that the base commander is able to obtain enough submarines to search the entire training region two times in five days. Compare the probability of finding the *Livingston* after one complete search with the probability of finding it after two complete searches.

Assignment

2.1 In order to find the *Livingston*, the team must pass over the lost submarine and detect it. Use your tree diagrams from Part **c** of the exploration to complete Parts **a–c** below.

a. If both submarines search the shallow water, determine the probability that the *Livingston* will be found in time to save the crew.

b. If both submarines search the shallow water, determine the probability that the *Livingston* will not be found in time to save the crew.

c. If both submarines search the deep water, determine the probability that the *Livingston* will not be found in time to save the crew.

2.2 At the beginning of the search operation's third day, three more submarines arrive. The commander assigns them to search the deep water around the island. (Recall that if a rescue team passes over the *Livingston* in deep water, there is only a 10% chance they will detect it.)

a. What percentage of the region of deep water can the three submarines cover in the remaining three days?

b. Create a new tree diagram to show the possible outcomes of the search for the *Livingston*, given that two submarines search in shallow water for five days and three submarines search in deep water for three days.

c. What is the probability that the three new submarines will find the *Livingston*?

d. What is the total probability that *Livingston* will be found within five days?

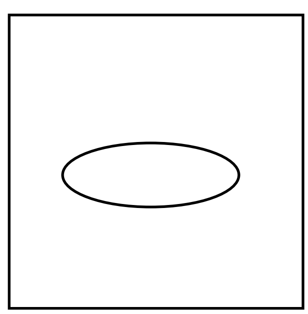
2.3 Given that two submarines search in shallow water for five days and three submarines search in deep water for three days, what is the probability that the rescue team will not scan the area where the *Livingston* went down? Justify your response.

2.4 What is the probability that the five submarines will pass over the *Livingston* but not detect it? Justify your response.

2.5 Imagine that the first two submarines were assigned to search the region of deep water, while the three submarines that arrived on the third day were assigned to search the shallow water. What is the probability that the *Livingston* will be found using this strategy? Justify your response.

* * * * *

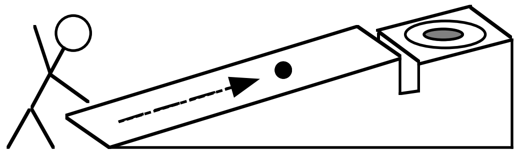
2.6 Laurie and Jamal are playing another game like the one described in Problem **1.6**. In this game, the target is an ellipse on a square sheet of paper 30 cm on each side. While blindfolded, Laurie drops a dart onto the paper. If the dart lands inside the ellipse, she wins.



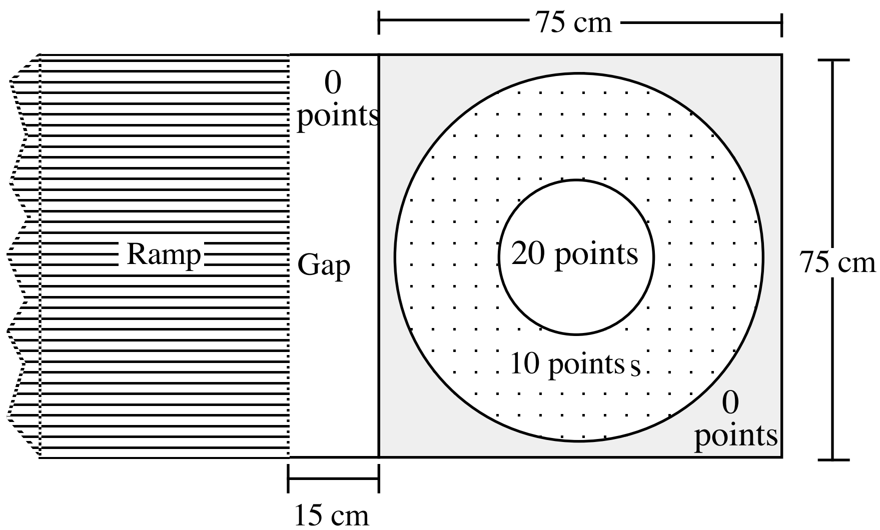
30 cm

After dropping a dart onto the paper 100 times, Laurie has a total of 12 wins. Use this information to estimate the area of the ellipse.

2.7 At the school carnival, one of the most popular games is called "Roller Ball." As shown in the diagram below, players try to roll a small ball up a ramp, over a gap, and into a circular hole. Points are awarded depending on the region in which the ball lands.



The following diagram shows a top view of the game and the number of points awarded for each region. The radius of the larger circle is 30 cm. The radius of the inner circle is 10 cm.



a. Each player receives two rolls. After two rolls, what total scores are possible?

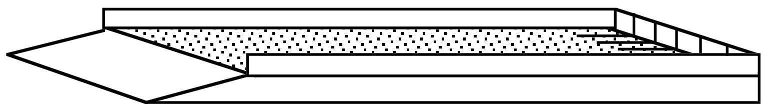
b. In how many different ways can each of the scores in Part **a** be earned?

c. Assuming that the ball is equally like to land at any given point, determine the probability that a ball rolled up the ramp will land in each region shown above.

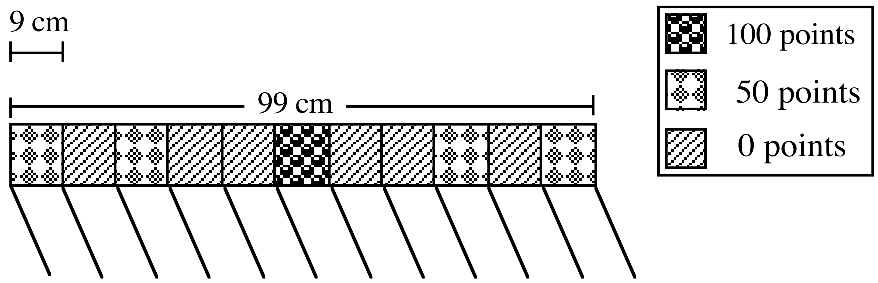
d. Using your responses to Parts **a–c**, determine the probability that a player will win each prize in the table below.

Total Points	**Prize**
40	large stuffed animal
20–30	poster
10 or less	piece of candy

2.8 Another game at the school carnival is called "Golf-O-Rama." In this game, players hit a golf ball up a ramp and into 1 of 11 slots at the end of a runway. Each slot is worth a different number of points. A side view of the game is shown below.



The following figure shows the dimensions of the end of the runway, along with the number of points awarded for each slot.



Each player receives two balls. Assuming that a ball is equally likely to land in any slot, determine the probability that a player's total score is:

a. exactly 100

b. 150 or more

c. 50 or less.

2.9 Rafael has misplaced his car keys. He estimates that the probability that they are somewhere on his cluttered desk is 0.7. However, he might also have left them in a more obvious place. The probability that they are in his coat pocket is 0.2, while the probability that they are in the glove compartment is 0.1.

If the keys are in his coat pocket or the glove compartment, he will find them on the first search of either location. If they are on the desk, the probability of finding them after one complete search is 0.7.

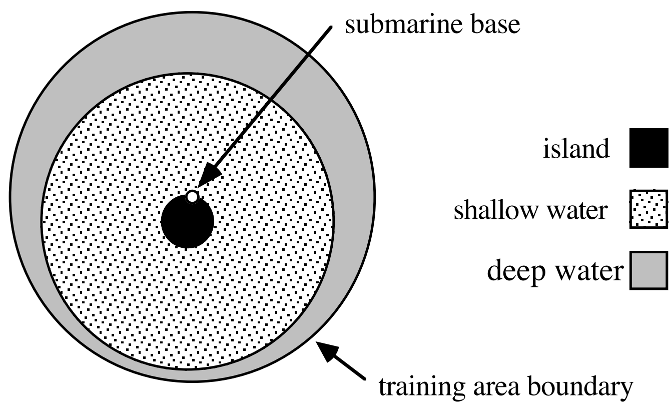
Design a search strategy for Rafael that minimizes the expected number of searches.

* * * * * * * * * *

Summary Assessment

A few months after the successful rescue of the *Livingston*, another training mission loses contact with the submarine base.

As shown in the diagram below, the base is located on an island surrounded by a roughly circular region of shallow water. The total area of the training region is $17,357 \text{ km}^2$. The area of the region of shallow water is $10,996 \text{ km}^2$, while the area of the region of deep water is 6361 km^2.



The base commander orders eight submarines to participate in the search. Each is equipped with a sonar device capable of scanning a circular area 2 km in diameter. In order to search the bottom effectively, the subs must limit their cruising speed to 10 km/hr. If the searchers pass over the lost submarine in shallow water, there is a 70% chance it will be detected. If they pass over the lost submarine in deep water, there is only a 10% chance it will be detected.

Devise an effective search strategy for the rescue of the missing submarine. Your response should include the calculations of all probabilities used to determine the strategy.

Module Summary

- When the outcomes in an event can be modeled geometrically, the theoretical probability of the event may be found using the ratio below:

$$P(\text{E}) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

- **Conditional probability** is the probability of an event occurring given that an initial event, or **condition**, has already occurred. The probability that event B occurs given that event A has occurred is denoted by the expression $P(\text{B}|\text{A})$.

- In a **multistage experiment**, one event is followed by one or more other events. In a multistage experiment involving conditional probabilities, the probability of event A followed by event B is found by multiplying the probability of A by the conditional probability of B, given A has already occurred. This can be denoted mathematically as shown below:

$$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$$

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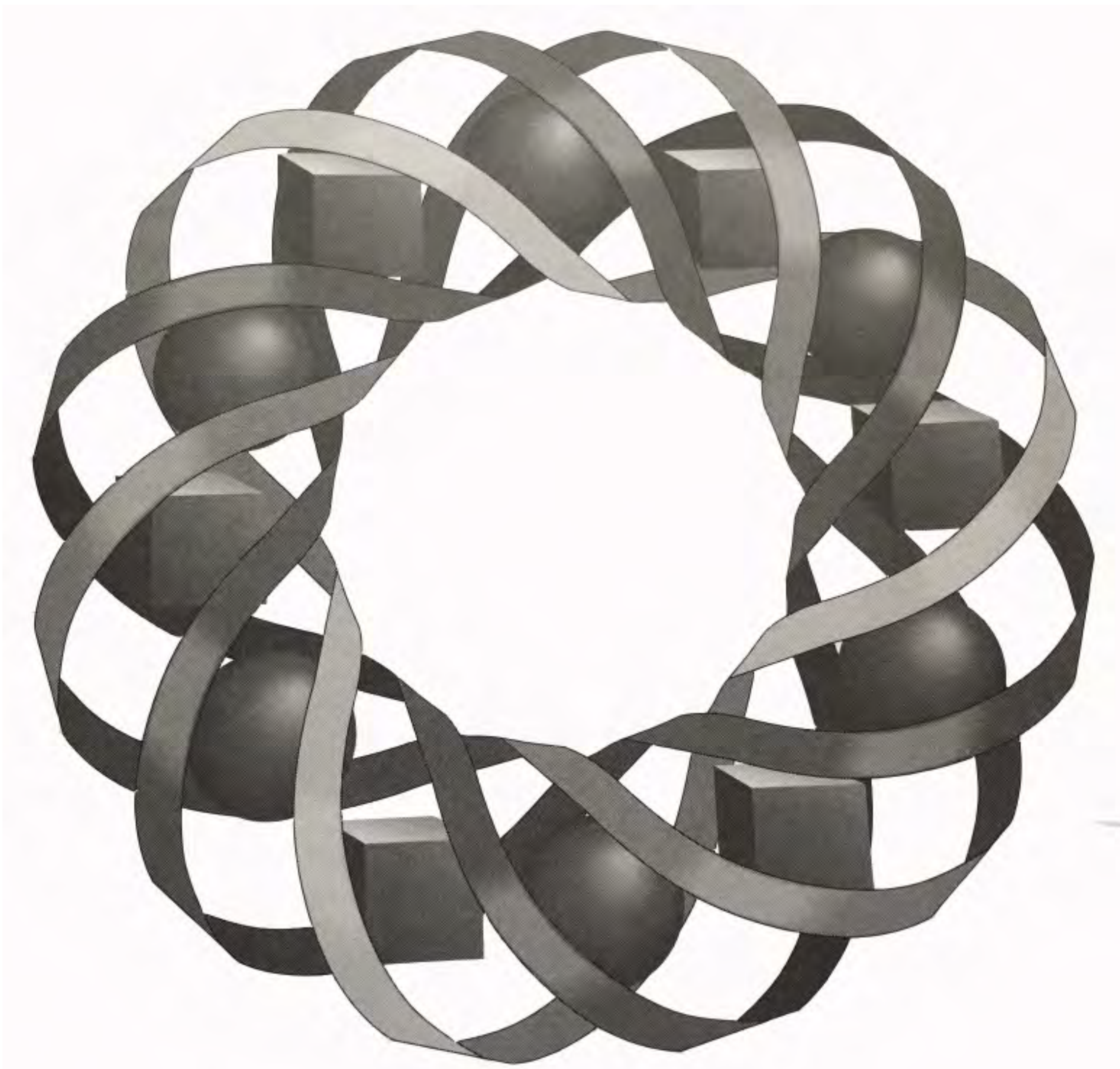
How Long Is This Going to Take?



When building a house, how does a contractor estimate the time required to complete the job? In this module, you explore how to design a schedule for a complicated project.

Russ Killingsworth • Verne Schlepp • Dan West

How Long Is This Going to Take?

Introduction

Imagine that your school is participating in a community service program that builds affordable housing. Your mathematics class has been asked to help plan a major construction project. There are many phases to this project, including obtaining materials and scheduling contractors, as well as the actual construction itself. In this module, you examine some organizational strategies to help minimize the time and money required to complete the project.

Activity 1

Three local firms—United Drywall, Empire Plumbing, and J&C Hardware—have donated materials for the project. To haul these materials to the building site, a freight company has offered the use of a flatbed truck for one day. Your only expense is fuel for the truck, at a cost of $0.18 per kilometer.

Exploration 1

In this exploration, your job is to find the shortest possible route to the three firms. You must start at the building site, visit each firm, then return to the site. A **weighted graph** of the four locations is shown in Figure **1**. (Recall that in a weighted graph, each edge is assigned a numerical value). In this graph, *B* represents the building site, *U* represents United Drywall, *E* represents Empire Plumbing, and *J* represents J&C Hardware. The distances given are in kilometers.

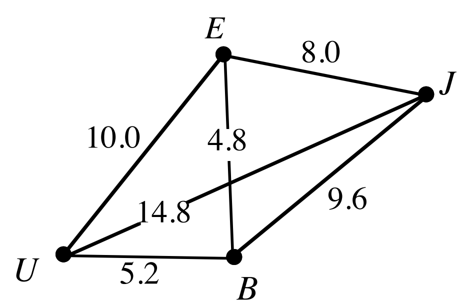


Figure 1: Graph of four locations

a. Using the graph in Figure **1**, determine one route that could be used to pick up all the materials.

b. Recall that a **Hamiltonian circuit** is a closed path that starts at one vertex, visits every other vertex in a graph exactly once, and returns to the starting vertex. Determine the total number of Hamiltonian circuits possible in Figure **1**.

c. List all the possible Hamiltonian circuits that start at the building site.

d. Considering all the circuits you listed in Part **c**, find the shortest route that could be used to pick up all the materials. Compare this route to the one you identified in Part **a**.

e. Determine the minimum cost of picking up the donated materials from the three firms.

Discussion 1

a. A **complete graph** is a graph in which each pair of vertices is connected by exactly one edge. Is the graph in Figure **1** a complete graph? Explain your response

b. When selecting a route for picking up materials, you may or may not have considered the direction of travel. In what types of situations might direction be important?

Exploration 2

Another local firm, AAA Lumber, also has agreed to donate materials to the building project. Now you must add another destination to your route. Figure **2** shows a complete graph of the four locations from Exploration **1**, along with AAA Lumber, represented by vertex *A*. As in Figure **1**, distances are given in kilometers.

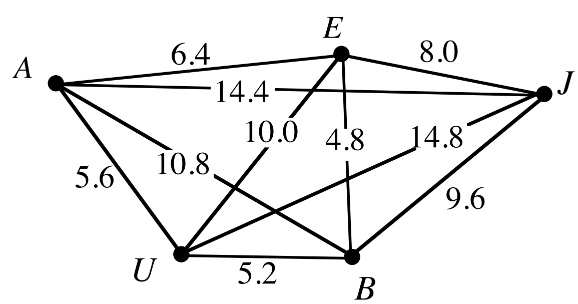


Figure 2: Graph of five locations

To spend as little as possible on fuel, you would like to minimize the total distance traveled. However, identifying the total distance for each possible route from the building site to the four companies could be very time consuming. To save time when designing routes to multiple locations, planners may use an algorithm, such as the **nearest neighbor algorithm**.

Using such an algorithm may not identify the shortest route possible. Nevertheless, it does provide an efficient method of determining a reasonably short route.

Mathematics Note

Figure **3** below shows a weighted graph with four vertices. The following steps describe the use of the **nearest neighbor algorithm** to identify a reasonably efficient Hamiltonian circuit for this graph.

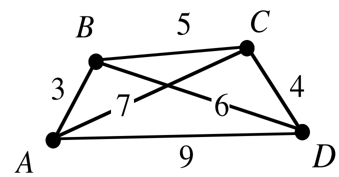


Figure 3: A weighted graph

- Starting with any vertex, select the edge to its "nearest" vertex. In other words, select the edge that has the least weight. In Figure **4**, for example, the starting vertex is *A*. Since the edge connecting *A* and *B* has a weight less than those connecting *A* to *C* or to *D*, it is selected.

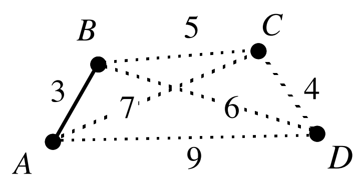


Figure 4: Selecting an edge between the starting vertex and its "nearest" vertex

- Continue this process from the second vertex, selecting the edge to the next nearest vertex not yet visited, and so on, until all vertices have been visited. For example, Figure **5a** shows the selection of the edge from *B* to *C*, while Figure **5b** shows the selection of the edge from *C* to *D*.

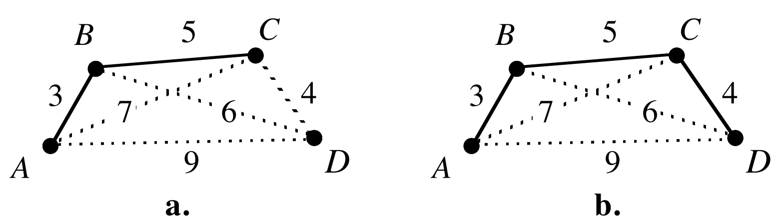


Figure 5: Continuing the nearest neighbor algorithm

- To complete a Hamiltonian circuit, return to the original vertex. For example, Figure **6** shows the edge marked from *D* to *A*.

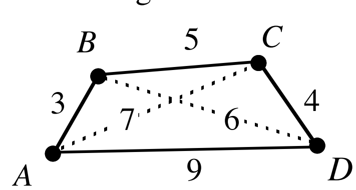


Figure 6: Completing a Hamiltonian circuit

a. Determine the number of Hamiltonian circuits possible for the graph in Figure **2.**

b. Starting at the building site (vertex *B*), use the nearest neighbor algorithm to find a Hamiltonian circuit for the graph in Figure **2**.

c. If the cost of fuel is $0.18 per kilometer, calculate your expenses for the route identified in Part **b**.

d. Repeat Parts **b** and **c** using each vertex in the graph as the starting vertex.

Discussion 2

a. On a given graph, will the nearest neighbor algorithm always produce a circuit that uses the same edges, regardless of the starting point? Explain your response.

b. The circuit that starts at AAA Lumber, then visits United Drywall, the building site, Empire Plumbing, and J&C Hardware, before returning to AAA Lumber, is 38 km long. Is it possible to use this route when starting at the building site? Explain your response.

c. Describe some advantages and disadvantages of using the nearest neighbor algorithm to determine an efficient route.

d.
1. Compare the numbers of possible Hamiltonian circuits for a complete graph with four vertices and a complete graph with five vertices.
2. Suppose that 20 businesses wished to donate building materials to the school project. Would it be feasible to determine the shortest possible route connecting all these businesses? Explain your response.

Mathematics Note

The nearest neighbor algorithm is just one of many that can be used to select a reasonably efficient Hamiltonian circuit. In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each step, regardless of starting and stopping points. Individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that does not visit all the vertices in the graph, then the next best action is taken. When a Hamiltonian circuit is found, the algorithm is complete.

For example, consider the complete weighted graph in Figure **7** below.

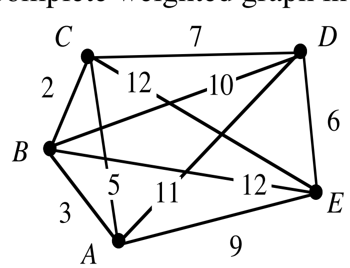


Figure 7: A weighted graph with five vertices

Figure **8** shows the first two steps in drawing a Hamiltonian circuit for this graph using the cheapest link algorithm. Since the edge between *B* and *C* has the least weight of any edge in the graph, it is selected first.

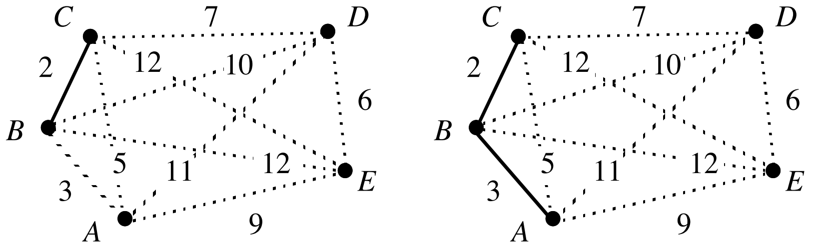


Figure 8: Using the cheapest link algorithm

Figure **9** shows the next step in drawing a Hamiltonian circuit for the complete graph in Figure **7**. Notice that the edge between *D* and *E* is selected even though the edge between *C* and *A* has a lesser weight. This occurs because selecting the edge between *A* and *C* would complete a circuit that does not visit all the vertices.

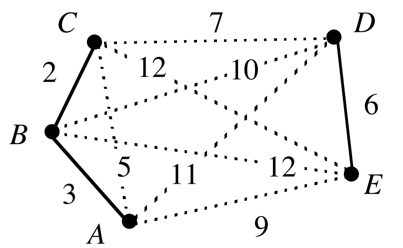


Figure 8: Continuing the cheapest link algorithm

Figure **9** shows the final two steps necessary to complete a Hamiltonian circuit for the entire graph.

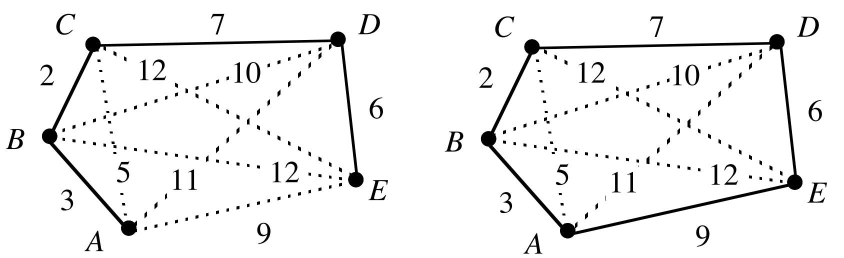


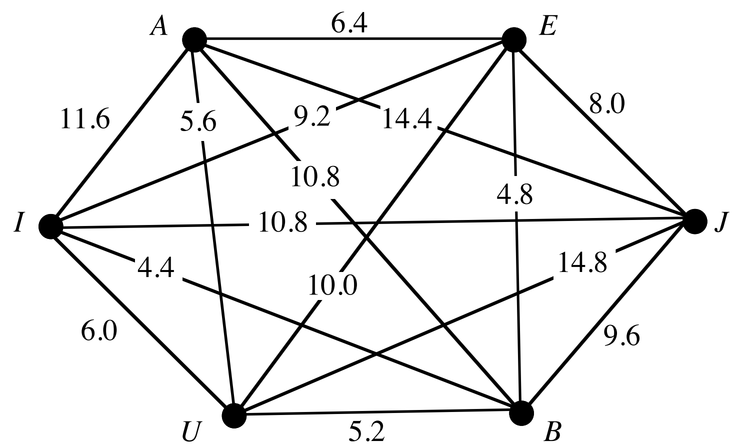
Figure 9: Completing a Hamiltonian circuit

e. **1.** Describe how to use the cheapest link algorithm to determine a route among the five locations described in Figure **2**.

2. How does the resulting route compare to the ones you found using the nearest neighbor algorithm?

Assignment

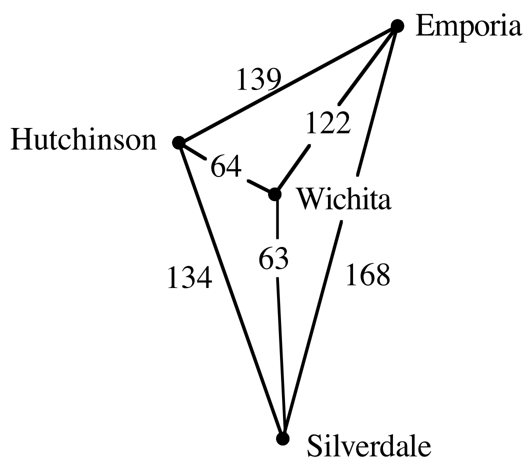
1.1 The Independent Roofing Company also offers to donate materials to the building project. You must now pick up materials from five locations before returning to the building site. The figure below shows a complete weighted graph of all these locations, with distances given in kilometers.



a. Find an efficient route for picking up materials from the five businesses. Describe the method you used to select the route.

b. What is the cost of picking up the materials using your route?

1.2 A small airline has just purchased a new plane. The company plans to use this plane to offer daily flights connecting four cities in Kansas: Emporia, Hutchinson, Silverdale, and Wichita. Since the company's headquarters are in Wichita, they would like each day's flight schedule to begin and end there. The following weighted graph shows the air distance in kilometers between the four cities.

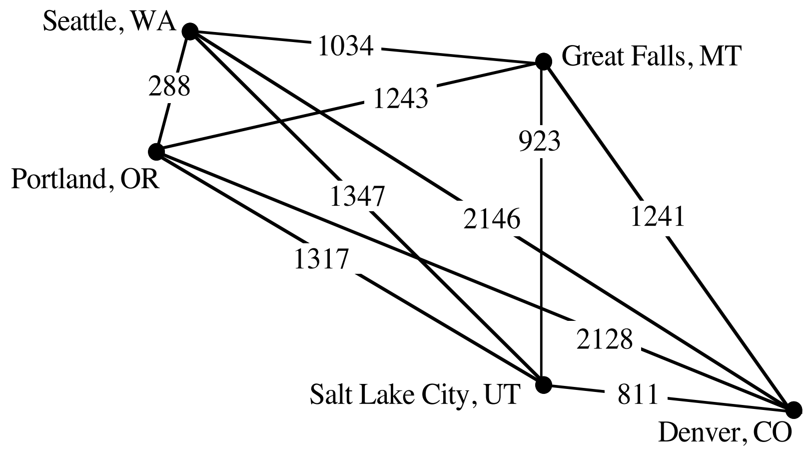


Find a Hamiltonian circuit that could help the airline design a schedule for the plane using each of the following algorithms.

a. the nearest neighbor algorithm

b. the cheapest link algorithm

1.3 Imagine that you own a trucking company based in Seattle, Washington. Your company delivers goods to four other cities in the western United States. The map below shows the distances in kilometers between each city.

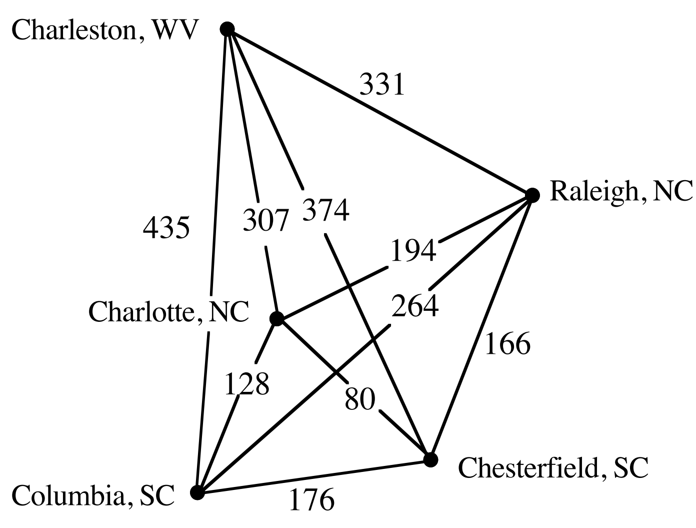


a. Use the cheapest link algorithm to find a Hamiltonian circuit that connects the five cities.

b. Use the nearest neighbor algorithm to find a Hamiltonian circuit that connects the five cities.

c. Can you be sure that either of the circuits you found in Parts **a** and **b** represents the shortest possible circuit connecting the five cities? Explain your response.

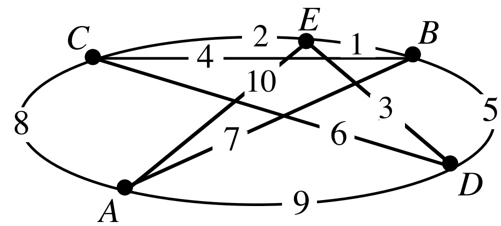
1.4 **a.** The following weighted graph shows the air distances in kilometers between five U.S. cities. Use any method you wish to find an efficient Hamiltonian circuit for the graph.



b. Calculate the total air distance for the circuit found in Part **a**.

* * * * *

1.5 Consider the following weighted graph.



a. Use the nearest neighbor algorithm to identify a Hamiltonian circuit for this graph.

b. Use the cheapest link algorithm to identify a Hamiltonian circuit for this graph.

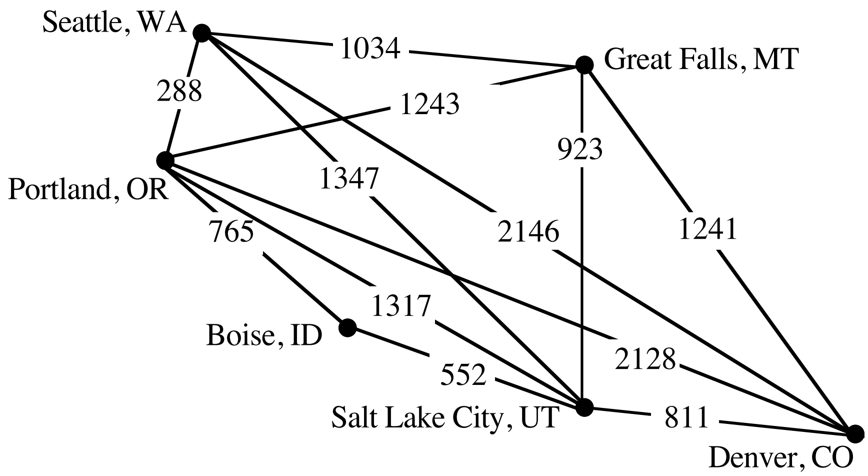
1.6 Imagine that you are creating a computer network with four workstations—A, B, C, and D—and one network server. The network's cable must connect all the computers in a closed circuit. The following table shows the distances among the workstations and the server. An X in the table means that no cable can be connected between these two stations.

	server	**D**	**C**	**B**
A	5 m	10 m	X	3 m
B	4 m	X	4 m	
C	2 m	9 m		
D	8 m			

a. Draw a graph to represent the computer network.

b. Since the connecting cable is expensive, you would like to keep the total length of cable used in the network reasonably low. How much cable should you purchase? Explain your response.

1.7 The trucking company in Problem **1.3** decides to add Boise, Idaho, to its delivery route. The map below shows the distances in kilometers between the original cities, as well as the distances between Boise and Portland and Boise and Salt Lake City.



a. Does the map above represent a complete weighted graph? Explain your response.

b. Use the cheapest link algorithm to find a Hamiltonian circuit that connects the six cities.

c. Use the nearest neighbor algorithm to find a Hamiltonian circuit that connects the six cities.

* * * * * * * * * *

Activity 2

Now that the materials are at the building site, the actual construction can begin. A local contractor has given your class a list of the jobs necessary to proceed with the project. Table **1** shows each task, its approximate completion time in hours, and any prerequisite tasks.

Table 1: Tasks in a construction project

	Task	**Hours**	**Prerequisite Tasks**
A	construct floor joists and subfloor	20	none
B	frame and raise walls	16	A
C	install trusses and roof	27	B
D	install siding, windows, and doors	26	B
E	partition walls	6	C
F	install wiring and plumbing	16	E
G	finish floor	12	E
H	install insulation	6	D
I	install drywall	19	F

Exploration

In this exploration, you investigate how to organize the construction schedule in the most efficient manner.

a.
1. Cut nine 1-cm wide strips from a piece of stiff paper.
2. Using an appropriate scale, cut and label a strip to represent the time required for each task in Table **1**.

b. Use your paper strips to create a model that represents the maximum possible completion time for the project. Record the maximum completion time.

c. To complete the project more quickly, some tasks can be done at the same time. Use your strips of paper to create a model that represents a more efficient schedule.

Before starting any particular task, make certain that all prerequisite tasks are finished. For example, job B must follow job A; while jobs C and D must both follow B. This portion of the schedule can be modeled by placing the strips as shown in Figure **10** (not drawn to scale).

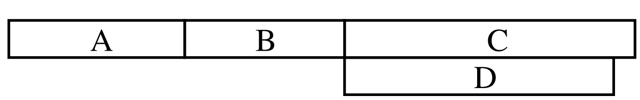
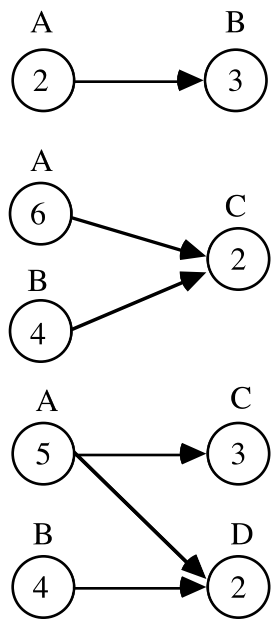


Figure 10: A paper-strip model

d. Determine the minimum completion time for the project. Draw a sketch of your model.

Mathematics Note

A **network diagram** (or order requirement digraph) can be used to represent a scheduling problem. In a network diagram, prerequisite tasks are connected with arrows. For example, Figure **11** shows three different network diagrams and their interpretations. The direction of the arrows shows the order in which the tasks must be completed. The number in each circle represents the time required to complete the task.



Task A, requiring 2 hrs, must be completed before Task B, requiring 3 hrs, can begin.

Tasks A (6 hr) and B (4 hr) can be done concurrently, but both must be completed before Task C (2 hr) can begin.

Tasks A (5 hr) and B (4 hr) can be done concurrently, but both must be completed before Task D (2 hr) can begin. Tasks C (3 hr) and D can be done concurrently, but Task A must be completed before Task C can begin.

Figure 11: Three network diagrams

The **critical path** is the longest set of prerequisite-linked tasks in a project. The length of the critical path is the minimum time required for the project. When a project is represented in a network diagram, the critical path is the longest path in the diagram. In the network diagram shown in Figure **12**, for example, the critical path P–Q–S is 21 units long.

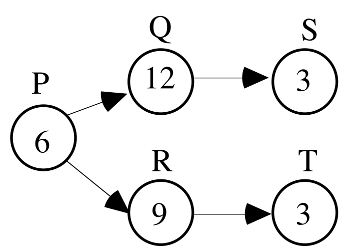


Figure 12: A network diagram

e. **1.** Draw a network diagram based on the tasks listed in Table **1**.

2. Determine the critical path for your network diagram.

Discussion

a. Describe how you used your model to determine the minimum completion time for the tasks in Table **1**.

b. Compare your results in Part **d** of the exploration with those of others in the class. Are all the schedules with the same completion time exactly the same?

c. How does the sketch of your paper-strip model compare to the network diagram?

d. How does the minimum time you found in Part **d** of the exploration compare to the critical path of your network diagram? Explain why this relationship occurs.

e. In Figure **12**, what is indicated by the lack of arrows between Q and T ?

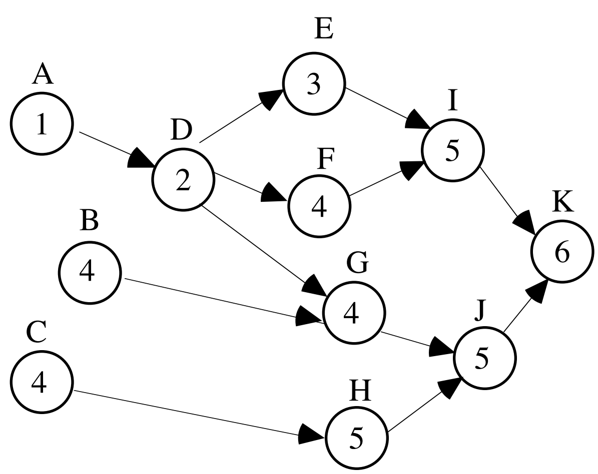
Assignment

2.1 After considering the tasks listed in Table **1**, the contractor realizes that one important job was forgotten: the installation of the heating and cooling system. This task will take about 11 hr. It cannot be started until the floor joists and subfloor are done and must be completed before the floor can be finished and the drywall installed.

a. Draw a diagram of a new work schedule for the project.

b. Determine the minimum time to complete the entire project, including this new task.

2.2 Find the length of the critical path in the network diagram below.



2.3 Imagine that you are the chief of ground operations for Fly-by-Night Airlines. During the time the company's planes are at the gate, 10 major tasks must be completed. These tasks are listed in the table below.

	Task	Time (min)	Prerequisite Tasks
A	unloading passengers	17	none
B	unloading cargo	25	none
C	loading passengers	23	A, J, F
D	loading cargo	25	B
E	refuel plane	15	none
F	stock food supplies	12	A
G	flush holding tank	6	none
H	wash cockpit windows	4	none
I	pre-flight check	12	E
J	clean plane interior	19	A

One of your responsibilities is to minimize the time each plane is on the ground. Create a schedule that completes all 10 tasks in the shortest amount of time.

2.4 After the building's shell is finished, the students working on the building will be divided into two teams for the remaining tasks. The contractor has set the following rules for the two work crews.

- The two crews cannot share a task.
- Once a crew begins a task, it must finish that task before starting another one.
- No crew can be idle while a task is available.

The table below lists each task in the final stage of the construction project, its prerequisites, and its completion time. Determine a schedule for the two crews that minimizes the time needed to complete all the tasks.

	Task	Time (hr)	Prerequisite Tasks
A	interior painting	20	none
B	exterior painting	15	none
C	pour sidewalks	21	none
D	finish carpentry	11	A
E	finish kitchen	6	A
F	wallpaper	5	D, E
G	landscaping	25	B, C, J
H	install carpeting	12	F
I	install linoleum	7	F
J	build deck	12	none

2.5 **a.** By adding one more student to each crew, the contractor believes that the time required for each task in Problem **2.4** can be reduced by 1 hr. Create a new work schedule for these larger crews. Record the hours required to complete the work.

b. As an alternative to the larger crews, one student suggests working on the deck and the landscaping at the same time. Which option would finish the project more quickly: adding one more student to each crew or building the deck during landscaping? Explain your response.

2.6 **a.** In another attempt to save time, one of the volunteers suggests dividing the students into three work crews. Assuming that the time required to complete each task remains the same as described in Problem **2.4**, write a work schedule that minimizes the time required for three crews to complete all the tasks.

b. Suppose that all the tasks must be completed in 45 hr, no matter what the cost. How many crews would it take to meet this deadline? Explain your response.

* * * * *

2.7 Zach, Vin, and Signe work at a company that makes winter coats. Zach takes 20 min to sew the inside lining of each coat. Vin requires 30 min to sew the outside of each coat. After the inside and outside are completed, Signe requires 40 min to sew the two parts together and inspect the finished product.

a. Assuming that each employee works only at a single task, create a network diagram of this situation. Identify the critical path and determine the time required to complete 5 coats.

b. Determine the time to complete 5 coats if Signe reduces the time required to finish her task by 50%.

2.8 Stacie, Chris, and Kodjo work for a company that prepares bulk mailings for businesses. At 1:00 P.M., they receive orders from 13 different customers. The time required to process each customer's mailing is listed in the following table. In order for all the mailings to be delivered on time, the processing must be finished by 3:00 P.M. that same day.

Business	Time (min)	Business	Time (min)
A	12	H	18
B	26	I	25
C	45	J	35
D	24	K	40
E	15	L	30
F	25	M	20
G	10		

The manager assigns Stacie to businesses A, D, G, J, and M; Chris to businesses B, E, H, and K; and Kodjo to businesses C, F, I, and L.

a. Can all the mail be processed in time using the manager's work schedule? Explain your response

b. Stacie, Chris, and Kodjo would like a schedule that allows them to take a 10-min break and still finish the work on time. Write a schedule that will satisfy the employees.

2.9 The copy service for the Plainville School District promises to deliver all orders received before 8:00 A.M. by the following day.

The table below shows the number of copies requested by each school in the district.

School	Copies Requested
A	1000
B	2200
C	430
D	5300
E	6500
F	200
G	600
H	700
I	4500
J	5000
K	800

a. The copy service's lone photocopier can produce 1 copy per second. What is the minimum time required to complete these orders?

b. Suppose that the copy service purchases a second photocopier that operates at the same speed. Write a schedule that will allow the two copiers to finish the orders in the least amount of time.

c. After finishing a copying job, workers need about 10 min to package and label the copies, then prepare the photocopier for the next job. Repeat Part **b** given these new conditions.

* * * * * * * * * *

Activity 3

Because the schedule that your class developed for the construction project worked so well, the landscaping crew also has asked for your advice. To create borders for trees, bushes, and flower beds, they will need landscaping timbers of many different lengths, as shown in Table **2**. However, the lumberyard is only willing to donate timbers in 10-ft lengths.

Table 2: Quantities and measurements for landscaping timbers

Quantity	Length of Each (ft)	Quantity	Length of Each (ft)
3	7	2	6
6	4	1	8
2	9	3	2
1	5	3	10
2	3		

This situation can be analyzed in terms of fitting items of different lengths into a bin of a particular size. Such situations are often called **bin-packing** problems.

Exploration

In this exploration, you use a bin-packing approach to determine how many 10-ft timbers are required for the landscaping project.

a. **1.** What is the total length of all the landscaping timbers in Table **2**?

2. Is it possible to cut all the timbers from 10-ft lengths without wasting any wood?

b. Use a diagram or a paper-strip model to determine the minimum number of 10-ft timbers needed to complete the landscaping.

c. Draw a diagram that illustrates how each timber should be cut.

Discussion

a. How does the number of 10-ft timbers you determined in Part **b** of the exploration compare with those of the rest of the class?

b. Compare your diagram from Part **c** of the exploration with a classmate who obtained the same minimum number of timbers. Describe any differences you observe.

c. Describe the strategies you used to determine the minimum number of timbers required. Which strategy appears to be most efficient?

d. How does the total length of the timbers in Table **2** compare with the total length of 10-ft timbers needed to complete the landscaping?

Assignment

3.1 Suppose that the lumberyard offered to donate landscaping timbers only in 16-ft lengths.

a. How many 16-ft timbers would you need to satisfy the requirements listed in Table **2**? Use a diagram to justify your response.

b. How many feet of timber would remain unused?

3.2 Suppose that the school decided to purchase a combination of 10-ft and 12-ft timbers.

a. How many of each length would you need to satisfy the requirements listed in Table **2**? Use a diagram to justify your response.

b. How many feet of waste would there be in your order?

c. If landscaping timbers cost $0.60 per foot, what would be the total bill for your order?

3.3 The last task in finishing the construction project is building a deck. The students assigned to this job have created a design which requires boards of several different lengths. The quantities and measurements needed are shown in the table below.

Quantity	Length of Each (ft)
6	7
12	6
30	3

The lumber can be obtained in lengths of either 8 ft, 12 ft, 16 ft, or 20 ft. However, the lumberyard that is donating the wood has requested that the boards all be the same length. Determine which size will result in the least amount of waste.

* * * * *

3.4 In addition to his full-time job, Jake owns a small T-shirt printing shop. Five days before the opening of the county fair, he receives an emergency order. Because their regular shirt printer has unexpectedly gone out of business, the fair needs the shirts listed in the table below.

Print color	Quantity
black	300
gray	300
yellow	150
red	200
purple	100
navy	250
orange	150
green	100

Jake can print 60 shirts per hour. Because of his other job, however, he can only work 6 hr per day at the shirt shop. He prints one ink color at a time and does not start printing a color unless he can finish it that day.

a. Design a schedule that allows Jake to finish the shirts in 5 days.

b. The fair's purchasing agent calls Jake back to tell him that they also will need 50 shirts printed in pink and 75 shirts in teal. Jake promises that he can finish the entire order before the fair begins. Can Jake print the additional shirts in time? Justify your response.

3.5 Imagine that you are an advertising executive for a television station. Your responsibilities include scheduling the advertising for an upcoming sports event. Your clients have purchased two 60-sec ads, four 45-sec ads, four 30-sec ads, and seven 15-sec ads. All the commercial breaks must be the same length, and no longer than 90 sec. What is the minimum number of commercial breaks needed to air these advertisements?

3.6 There are several well-known algorithms used to pack items into bins efficiently. Two examples are the next-fit and first-fit algorithms. Use each method, as described below, to determine the number of 10-ft timbers needed to provide the materials listed in the following table.

Quantity	Length of Each (ft)	Quantity	Length of Each (ft)
2	10	2	5
3	9	4	4
2	8	2	3
4	7	7	2
4	6		

a. In the next-fit algorithm, each item is placed in the first bin that has room for it. Once a bin has been passed over, it is not used again. In the case of timbers, once a cut piece is too short for the next piece in the list, the cut piece is discarded.

b. In the first-fit algorithm, each item is placed in the first bin that has room for it. If no bin has room for an item, a new bin is opened. Unlike the next-fit algorithm, no bin is closed until all items are packed. In the case of timbers, no cut piece is discarded until all pieces are used.

c. Compare your results in Parts **a** and **b**.

* * * * * * * * * *

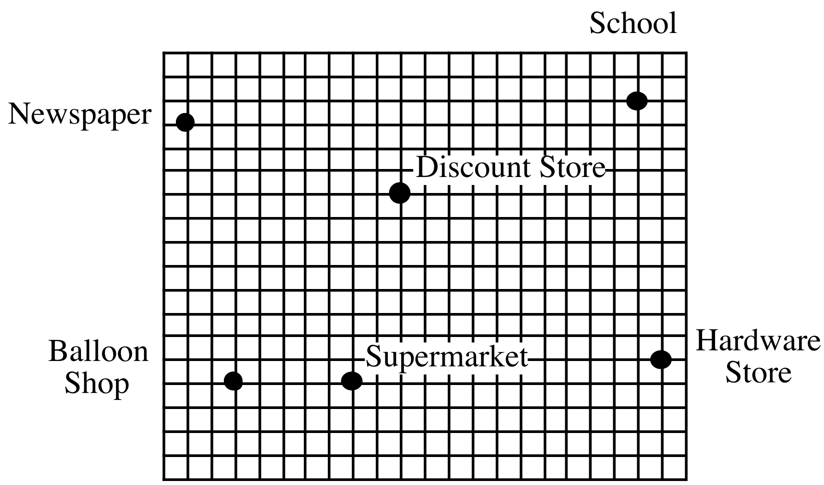
Research Project

The nearest neighbor and cheapest link algorithms are just two of many so-called **greedy algorithms**. Graphs and greedy algorithms are often used by communications and transportation companies to determine how to minimize costs or distances. One such algorithm, first suggested by Joseph Kruskal of AT&T Bell Laboratories, results in a subset of a complete graph called a **spanning tree**. Find out more about the algorithm that produces a spanning tree. Your report should include a description of the algorithm as well as some examples of how spanning trees are used by businesses.

Summary Assessment

1. The theme of this year's spring dance is "Egyptian Oasis." As chair of the student dance committee, your job is to make sure that the gym is decorated before the morning of the dance.

To pick up the decorating materials, you must stop at each location on the following map. The scale of the map is = 1 city block.



a. Draw a weighted graph to represent these locations. **Note:** All paths must follow the grid lines.

b. Find an efficient route that begins at school, visits each business, and returns to school. Describe the method you used to determine the route.

2. You have three evenings to decorate the gym. The school administration has given the dance committee access to the gym from 4:00 P.M. to 11:00 P.M. on Wednesday and Thursday and from 4:00 P.M. to midnight on Friday.

The following table shows the list of required tasks, along with the average time to complete each one using a crew of five students.

	Task	Hours
A	inflate 300 balloons	3
B	build frames for pyramids	3
C	decorate pyramids with crepe paper	2
D	build large cardboard sphinx	2
E	apply gold foil to sphinx	2
F	decorate ceiling with balloons and streamers	6
G	drape plastic sheeting at end of gym	3
H	cover bleachers with plastic sheeting	3
I	decorate stage with balloons and install sphinx	2
J	draw and cut out designs for walls	6
K	build palm trees	4
L	set up platform for disc jockey	2
M	decorate walls and plastic sheeting	2
N	set up palm trees on floor	3
O	set up snack bar	1
P	set up tables and chairs	3

a. Create a table that lists each task along with its prerequisites.

b. If no tasks are begun before Wednesday at 4:00 P.M., is it possible for two crews of five students each to complete the decorations by midnight on Friday? Justify your response.

c. If necessary, tasks J and K can be completed outside the gym before Wednesday evening. Design a schedule that allows two crews to complete the decorations by midnight on Friday.

3. During one part of the dance, the disc jockey plans to play music in 5-min portions. The songs she has picked have the following lengths (in sec): 60, 60, 60, 90, 90, 90, 120, 120, 120, 150, 150, 180, 180, 180, 180, 180, 210, 210, 210, 240, 240, 240, and 270.

What is the minimum number of 5-min time slots needed? Justify your response.

Module Summary

- A **complete graph** is a graph in which each pair of vertices is connected by exactly one edge.
- In a **weighted graph**, each edge is assigned a numerical value.
- A **Hamiltonian circuit** is a closed path that starts at one vertex, visits every other vertex in a graph exactly once, and returns to the starting vertex.
- To use the **nearest neighbor algorithm** to draw a Hamiltonian circuit on a weighted graph, start with any vertex and select an edge to its nearest vertex. In other words, select the edge that has the least weight. Continue this process from the second vertex to the next nearest vertex not yet visited, and so on, until all vertices have been visited. To complete a Hamiltonian circuit, return to the original vertex.
- In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each stage, regardless of starting and stopping points. When using this algorithm to draw a Hamiltonian circuit on a weighted graph, individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that is not Hamiltonian, then the next best action is taken. When a Hamiltonian circuit is formed, the algorithm is complete.
- A **network diagram** (or order requirement digraph) can be used to represent a scheduling problem. In a network diagram, prerequisite tasks are connected with arrows. The direction of the arrows shows the order in which the tasks must be completed. The number in each circle represents the time required to complete the task.
- The **critical path** is the longest set of prerequisite-linked tasks in a project. The length of the critical path is the minimum time required for the project. When a project is represented in a network diagram, the critical path is the longest path in the diagram.
- A **bin-packing** approach to a problem analyzes the situation in terms of fitting items of different lengths into a bin of a particular size.

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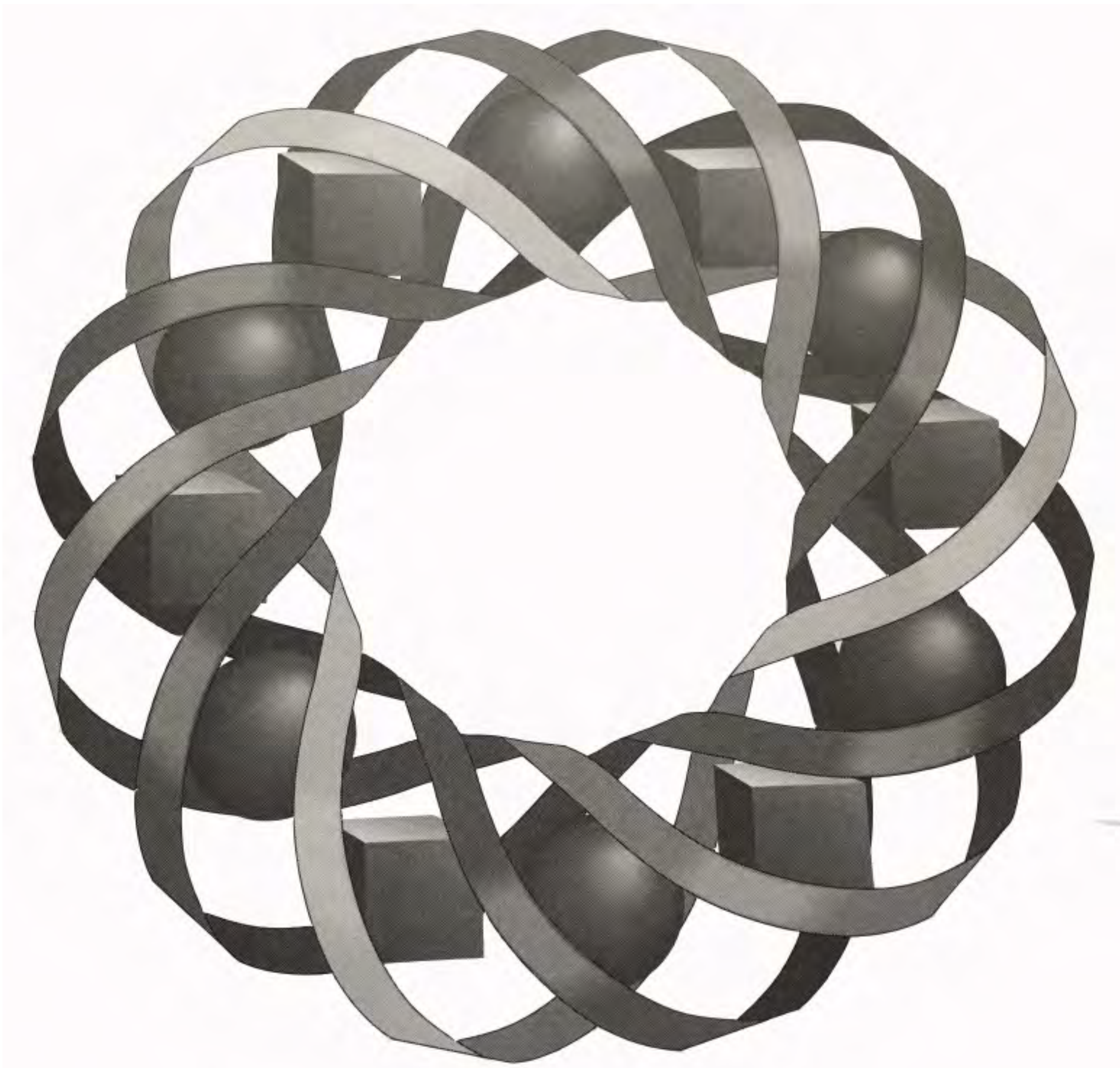
A Ride with Markov



At Boards Incorporated, the first priority is giving customers a quality ride—on their skateboards, of course. In this module, you examine how a process developed by Russian mathematician Andrei Markov can help the company control quality.

Douglas H. Mack • Laurie Paladichuk • Arthur R. Perleberg

A Ride with Markov

Introduction

Welcome to Boards Incorporated, skateboard manufacturer. The company makes about 1000 skateboards every week. As with any production process, skateboard manufacturing has its ups and downs. Not every board that rolls off the assembly line is good enough to carry the company name. In the past, the defective rate has been approximately 20%.

Currently, the quality-control department is trying to develop a sampling strategy to monitor the number of defective products. The sampling process must be cost effective. It must be repeated enough times to ensure quality, yet not so many times that the cost of sampling becomes too great.

What might an effective sampling strategy look like? How can the cost of the strategy be predicted? What factors will serve to indicate that the number of products sampled needs to be changed? In this module, you use the Markov process—named after Andrei Andreyevich Markov (1856–1922)—and binomial experiments to answer these questions.

Activity 1

You may recall from previous modules that a **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions (trials) of the same action.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure.
- The probability of a success remains the same from trial to trial.
- The total number of successes is observed.

The sampling done for quality control typically does not satisfy all the conditions for a binomial experiment, since a defective product is not returned to the population before sampling the next item. However, a binomial experiment approximates the process fairly well if the sample size is small compared to the total population.

Exploration

The quality-control process at Boards Incorporated consists of three states: "ordinary," "relaxed," and "heightened." In the ordinary state, a sample of 30 boards is examined once each week. If the number of defectives in the sample is too large, then the process moves into the heightened state. If the number of defectives is small enough, then the process moves into a relaxed state. One hundred boards per week are sampled in the heightened state, while only 5 boards per week are sampled in the relaxed state.

How should the company decide when to change the sample size? In this exploration, you use a probability distribution to examine this question.

a. At Boards Incorporated, a 20% defective rate is considered acceptable. To model this situation, create a population of 150 beans in which 20% have a distinguishing mark on them.

b. In the ordinary state, quality-control specialists sample 30 skateboards from the population. Given a 20% defective rate, how many defectives would you expect to find in each sample?

c.
1. Take a random sample of 30 beans from the population. Record the number of defectives (or marked beans) in the sample, then return the sample to the container and mix thoroughly.
2. Repeat the sampling process 49 more times, for a total of 50 samples.

d. Recall that the **frequency** of an item in a data set is the number of times that item is observed. A **frequency histogram** consists of bars of equal width whose heights indicate the frequencies of items or intervals.

Create a frequency histogram of the numbers of defectives found in your samples. Represent frequency on the y-axis and numbers of marked beans on the x-axis.

e. Calculate the mean number of defectives for your 50 samples.

f.
1. Combine the results of your 50 samples with those of the rest of the class.
2. Determine the mean number of defectives for the class data.
3. Display the class data in a frequency histogram.
4. Connect the midpoints of the tops of the bars to create a **frequency polygon**, as shown in Figure **1**.

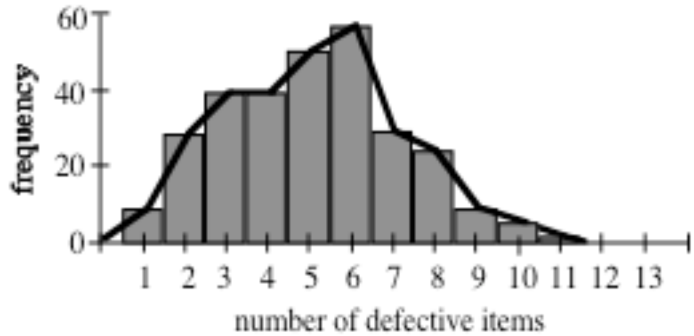


Figure 1: A frequency polygon

Mathematics Note

The probability distribution for a binomial experiment is a **binomial distribution**. The mean of a binomial distribution is the product of the number of trials and the probability of a success. In other words, the mean μ can be found as follows, where n is the number of trials and p is the probability of a success:

$$\mu = np$$

The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

For example, consider an experiment that consists of tossing a six-sided die 10 times and observing the number of times that a 6 appears. In this case, $n = 10$ and the probability of a success is $1/6$. Therefore, the mean of the corresponding binomial distribution is $10(1/6) = 10/6 \approx 1.67$. The standard deviation is $\sqrt{10(1/6)(5/6)} = \sqrt{50/36} \approx 1.18$.

g. Determine the mean μ and standard deviation σ for the binomial distribution that models the sampling of 30 skateboards from a population with a 20% defective rate.

h. On the histogram from Part **f**, draw and label vertical lines that represent each of the following:

1. the mean μ

2. values 1 standard deviation σ on either side of the mean

3. values 2 standard deviations on either side of the mean

4. values 3 standard deviations on either side of the mean.

i. Estimate the percentage of samples that lie within each of the following intervals on the histogram:

1. $[\mu + \sigma,\ \mu - \sigma]$

2. $[\mu + 2\sigma,\ \mu - 2\sigma]$

3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

Discussion

a. As noted previously, a binomial experiment provides a reasonable model for the sampling in the exploration if the number of items removed is small compared to the total population. Why is this requirement necessary?

b.
1. How does the mean of your 50 samples compare with the mean μ of the corresponding binomial distribution?

2. How does the mean of the class data compare with μ ?

3. In general, how would you expect the mean of a very large set of samples to compare with μ ?

c. What patterns did you observe in your histograms from the exploration?

d.
1. What would the histograms have looked like if the percentage of defectives had been 40% instead of 20%?

2. What would the histograms have looked like if twice as many samples were taken?

Mathematics Note

A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.

In this situation, the probabilities of these outcomes can be represented graphically by the area enclosed by the x-axis, a specific real-number interval, and a distribution curve. The sum of the non-overlapping areas that cover the entire interval is 1.

One continuous probability distribution is the **normal distribution**. As shown in Figure **2**, the graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.

As in all continuous probability distributions, the total area between the horizontal axis and a normal curve is 1. In a normal distribution, approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

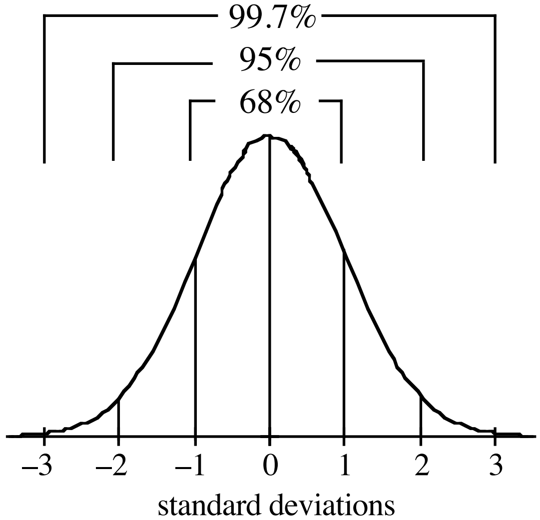


Figure 2: A normal curve and the 68–95–99.7 rule

Normal distributions can be used to model a wide variety of data sets. When this distribution provides a reasonable model, the 68–95–99.7 rule can help you characterize a population. For example, if a population appears to be normally distributed with a mean of 100 and a standard deviation of 10, then you would expect about 68% of the observations to fall between 90 and 110, 95% of the observations to fall between 80 and 120, and 99.7% of the observations to fall between 70 and 130.

e. How does the shape of the frequency histogram for the class data compare with a normal curve?

f. How does the percentage of samples that fell in each of the following intervals compare with the 68–95–99.7 rule?

1. $[\mu + \sigma,\ \mu - \sigma]$

2. $[\mu + 2\sigma,\ \mu - 2\sigma]$

3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

g. What percentage of all samples of a given size would you expect to find in each of the following regions under a normal curve:

1. to the right of μ ?

2. to the right of $\mu + 1\sigma$?

3. to the left of $\mu - 1\sigma$?

4. to the right of $\mu - 3\sigma$?

5. to the left of $\mu + 2\sigma$?

Assignment

1.1 Determine whether or not each of the following procedures is a binomial experiment. If the procedure is not a binomial experiment, can it be reasonably modeled by one? Explain your responses.

- **a.** You select a random sample of 6 computer chips from a batch of 20, replacing each chip before selecting the next one. The total number of defectives is recorded.
- **b.** You select a random sample of 6 computer chips from a batch of 20, without replacement, and record the total number of defectives.
- **c.** You select a random sample of 500 American teenagers and record their favorite brand of tennis shoes from a list of 10 brands.
- **d.** You select a random sample of 5 electric motor shafts from an assembly line that produces 2000 shafts a day and determine if the shaft diameter is between 1.71 cm and 1.73 cm.
- **e.** You roll a pair of dice 5 times and record the number of times their sum is greater than 8.

1.2 At Boards Incorporated, the ordinary quality-control process involves a sample of 30 boards per week. As long as the number of defectives is within 1 standard deviation of the mean, the quality-control process remains in the ordinary state.

- **a.** Given a 20% defective rate, what interval for the number of defectives in a sample of 30 would keep the quality-control process in the ordinary state?
- **b.** If the number of defectives is less than $\mu - 1\sigma$, the quality-control process moves to the relaxed state. What is the greatest number of defectives that would cause this move?
- **c.** If the number of defectives is more than $\mu + 1\sigma$, the quality-control process moves to the heightened state. What is the least number of defectives that would cause this move?
- **d.** Describe the numbers of defectives for which the quality-control process would stay in the ordinary state, move from ordinary to heightened, and move to relaxed from ordinary.

1.3 When the quality-control process moves into the heightened state at Boards Incorporated, the sample size increases to 100. If the number of defectives found is no more than 1 standard deviation above the mean, the quality-control process moves back to the ordinary state.

- **a.** Assuming a defective rate of 20%, determine the mean number of defectives for samples of size 100.
- **b.** What is the greatest number of defectives that would cause the quality-control process to move back into the ordinary state?

c. Given a population in which the actual defective rate is 20%, what percentage of samples of size 100 would keep the quality-control process in the heightened state?

d. Describe the numbers of defectives for which the quality-control process would stay in the heightened state or move from the heightened state to the ordinary state.

1.4 When the quality-control process moves into the relaxed state, the sample size decreases to 5. If the number of defectives found is greater than 1 standard deviation above the mean, the quality-control process moves back to the ordinary state.

a. Assuming a defective rate of 20%, determine the mean number of defectives for samples of size 5.

b. What is the least number of defectives that would cause the quality-control process to move back into the ordinary state?

c. Given a population in which the actual defective rate is 20%, what percentage of samples of size 5 would keep the quality-control process in the relaxed state?

d. Describe the numbers of defectives for which the quality-control process would stay in the relaxed state or move from the relaxed state to the ordinary state.

* * * * *

1.5 Imagine that you are conducting an experiment using an unfair coin. The probability of obtaining a head on any one toss is 0.4.

a. Are the formulas for the mean and standard deviation of a binomial distribution appropriate in this situation? Explain your response.

b. Determine the mean and standard deviation for the number of heads given each of the following numbers of trials:

1. 20
2. 40
3. 60.

c. Repeat Part **b** when the probability of obtaining a head on any toss is 0.2.

1.6 The mean score on an examination was 72 with a standard deviation of 9. Students whose scores fall in the top 2.5% will receive a scholarship to the college of their choice. If the scores are normally distributed, what is the minimum score required to receive a scholarship?

* * * * * * * * * *

Activity 2

The potential cost of their quality-control strategy is important to Boards Incorporated. The company's managers realize that more sampling means higher costs. They also know that if the quality-control process remains in a heightened state for a long time, the assembly line must be shut down and the source of the increased defects identified and repaired.

To estimate costs, the quality-control specialist has been asked to predict what portion of time will be spent in each state. In this activity, you examine one method for making such predictions.

Exploration

In Activity **1**, you identified the numbers of defectives that cause the quality-control process to move from one state to another. For example, if the number of defectives found in the ordinary state is less than 4, the process moves to the relaxed state. If the number of defectives is greater than 8, the process moves to the heightened state.

In the relaxed state, if the number of defectives is greater than 1, the process returns to the ordinary state. The process returns to the ordinary state from the heightened state when 24 or fewer defectives are found.

In this exploration, you determine the probabilities of moving from one state to another, assuming a defective rate of 20%. Since the quality-control process can be modeled by a binomial experiment, these probabilities can be determined using the **binomial probability formula**.

Mathematics Note

The **binomial probability formula** can be used to determine the probability of obtaining *r* successes in *n* trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where *p* is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

For example, if 25% of a population of computer disks are defective, then $(1 - 25\%) = 75\%$ are not. The theoretical probability that exactly 4 defective disks will occur in a sample of 10 is:

$$\begin{aligned} P(4 \text{ successes in } 10 \text{ trials}) &= C(10,4) \bullet (0.25)^4 \bullet (0.75)^{10-4} \\ &= 210 \bullet (0.25)^4 \bullet (0.75)^6 \\ &\approx 0.15 \end{aligned}$$

a. Assuming that the defective rate is 20%, complete Table **1**. This is the probability distribution for the possible numbers of defective boards in a sample of size 30.

Table 1: Probability distribution

No. of Defectives (n)	$P(n)$
0	
1	
2	
$\vdots$	
29	
30	

b. Recall that given two events A and B, the theoretical probability of either A or B occurring can be found as follows:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

1. Use your probability distribution to determine the probability that the quality-control process will move from the ordinary state to the heightened state.

2. Determine the probability that the quality-control process will move from the ordinary state to the relaxed state.

3. Determine the probability that the quality-control process will remain in the ordinary state.

c. **1.** Repeat Part **a** for the possible numbers of defective boards in a sample of size 5.

2. Determine the probability that the quality-control process will move from the relaxed state to the ordinary state.

3. Determine the probability that the quality-control process will move from the relaxed state to the heightened state.

4. Determine the probability that the quality-control process will remain in the relaxed state.

d. Table **2** below shows the probability distribution for the possible numbers of defective boards in a sample of size 100.

n	***P*(*n*)**	***n***	***P*(*n*)**
0	0.00000	22	0.08490
1	0.00000	23	0.07198
2	0.00000	24	0.05773
3	0.00000	25	0.04388
4	0.00000	26	0.03164
5	0.00001	27	0.02168
6	0.00006	28	0.01413
7	0.00020	29	0.00877
8	0.00058	30	0.00519
9	0.00148	31	0.00293
10	0.00336	32	0.00158
11	0.00688	33	0.00081
12	0.01275	34	0.00040
13	0.02158	35	0.00019
14	0.03353	36	0.00009
15	0.04806	37	0.00004
16	0.06383	38	0.00002
17	0.07885	39	0.00001
18	0.09090	40	0.00000
19	0.09807	41	0.00000
20	0.09930	$\vdots$	$\vdots$
21	0.09457	100	0.00000

1. Using Table **2**, determine the probability that the quality-control process will move from the heightened state to the ordinary state.
2. Determine the probability that the quality-control process will move from the heightened state to the relaxed state.
3. Determine the probability that the quality-control process will remain in the heightened state.

e. Complete Table **3** using the probabilities found in Parts **b–d**. Each entry represents the probability of moving from the present state to another state. These probabilities can be denoted as follows:

$$P(\text{new state}\,|\,\text{present state})$$

For example, the probability that the quality-control process will move to the relaxed state, given that it is presently in the relaxed state, is:

$$\begin{aligned} P(\text{R}|\text{R}) &= P(0) + P(1) \\ &= C(5,0) \bullet 0.2^0 \bullet 0.8^5 + C(5,1) \bullet 0.2^1 \bullet 0.8^4 \\ &\approx 0.7373 \end{aligned}$$

Round each probability to the nearest thousandth.

Table 3: Probabilities of quality-control states

	Next State			
Present State	relaxed	ordinary	heightened	**Total**
relaxed	0.737			
ordinary				
heightened				

Mathematics Note

The process of moving from one **state**, or outcome, to another is a **transition**.

For example, the weather may be described by one of three states: cloudy, rainy, or sunny. The process of moving from one state to another—from sunny to cloudy, for example—is a transition.

A **transition diagram** is a convenient way to display the possible changes among states. In such diagrams, each state is represented by a vertex, while each transition is represented by a directed edge labeled with a probability.

For example, Figure **3** shows a transition diagram for three states of weather: cloudy (C), sunny (S), and rainy (R).

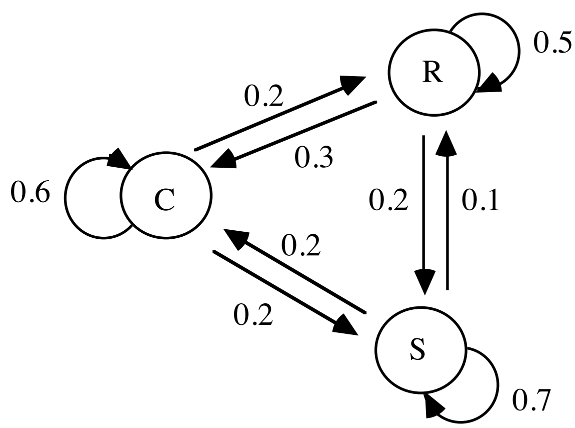


Figure 3: A transition diagram for the weather

The directed edge from R to C in Figure **3**, labeled 0.3, represents the probability of a transition from a rainy day to a cloudy day, or $P(\mathrm{C}|\mathrm{R}) = 0.3$. The loop about C, labeled 0.6, represents the probability of transition from a cloudy day to another cloudy day, or $P(\mathrm{C}|\mathrm{C}) = 0.6$.

f. Draw a transition diagram for the three states of the quality-control process, using the information in Table **3**. **Note:** Save a copy of Table **3** and the corresponding transition diagram for use later in the module.

Discussion

a. **1.** What does the entry in each cell in Table **3** represent?

2. How is the entry in each cell illustrated in the transition diagram you created in Part **f** of the exploration?

b. **1.** What does an entry of 0 in any cell of Table **3** represent?

2. How is an entry of 0 in a cell represented in a transition diagram?

c. **1.** What does the total for each row in Table **3** represent?

2. How is each row total represented in a transition diagram?

Assignment

2.1 According to the U.S. Bureau of Labor Statistics, 65% of the graduating class of 1996 attended college in the fall of that year. Consider a random sample of 25 students from this population.

a. Create a probability distribution for the number of students who attended college in a sample of 25 from this population.

b. What is the probability that more than half the students in a sample of 25 attended college?

2.2 **a.** For each row in Table **3** from the exploration, draw a tree diagram. Label each branch with the corresponding probability.

b. Do your tree diagrams from Part **a** represent transition diagrams? Explain your response.

2.3 The managers of Boards Incorporated suggest that the quality-control process use only two states: ordinary and heightened. If 0 to 8 defective boards are found in a sample of 30, the ordinary state would continue. If 9 or more are found, the quality-control process would move into the heightened state. The process would return to the ordinary state if 24 or fewer defective boards are found in a sample of 100.

a. Create a transition diagram to represent this situation.

b. Does it appear that this change will result in more frequent occurrences of the heightened state? Explain your response.

c. Because of the smaller sample size, the relaxed state costs less to administer than the ordinary state. Similarly, the ordinary state costs less than the heightened state. Given this fact, would you recommend that the company use a quality-control process with three states or with two states? Justify your response.

* * * * *

2.4 The table below shows the probabilities of movement in the U.S. population from one region to another in 1991.

	To			
From	**Northeast**	**Midwest**	**South**	**West**
Northeast	0.9818	0.0034	0.0111	0.0037
Midwest	0.0013	0.9866	0.0075	0.0046
South	0.0022	0.0049	0.9886	0.0043
West	0.0014	0.0035	0.0077	0.9874

Source: U.S. Bureau of the Census, 1993.

a. Explain what is represented by the entry in the second row, third column of the table.

b. Describe the meaning of the entries in the principal diagonal (the diagonal from the upper left-hand corner to the lower right-hand corner).

c. Create a transition diagram for the information in this table.

2.5 Imagine that you own 50 shares of stock in a company that manufactures electric cars. The table below shows the movement in the stock's price for the past 30 business days. The entry for day 1, for example, indicates that the stock price went up during that day.

Day	**Change**	**Day**	**Change**	**Day**	**Change**
1	up	**11**	down	**21**	up
2	up	**12**	up	**22**	up
3	no change	**13**	up	**23**	down
4	down	**14**	down	**24**	down
5	no change	**15**	no change	**25**	no change
6	no change	**16**	no change	**26**	no change
7	up	**17**	up	**27**	up
8	up	**18**	up	**28**	up
9	down	**19**	down	**29**	no change
10	down	**20**	down	**30**	down

a. Using the information given, create a list of ordered pairs in which the first element indicates the movement in stock price for a given day and the second element represents its movement on the following day. For example, the first ordered pair in the list should be (U,U), where U represents up.

b. Use the ordered pairs in Part **a** to complete the following table of probabilities. For example, there are 12 days on which the stock price was up. Of the 12 following days, the stock price went up again on 6. Thus, $P(\text{U,U}) = 6/12 = 0.5$.

	Following Day		
Given Day	up	no change	down
up	0.500		
no change			
down			

c. Assuming that the data is representative of the stock's long-term behavior, create a transition diagram.

d. Use the information from Parts **a–c** to predict what will happen on day 32, given that the stock price is down on day 31.

* * * * * * * * * *

Activity 3

Overall, the managers of Boards Incorporated are satisfied with their quality-control process. However, they are concerned about what to expect in the future. Is there some way to predict how often the state will change in the long run, given that the process begins in the ordinary state?

Exploration

As you discovered in the previous activity, the probabilities for the states in the company's quality-control process can be represented in a transition diagram. This transition diagram is shown in Figure **4**.

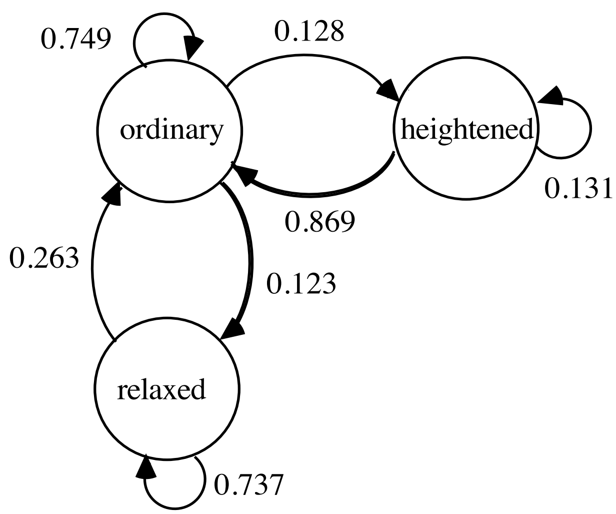


Figure 4: Transition diagram for quality control

In this exploration, you examine how transition diagrams can be used to predict future trends.

a. Assume that the quality-control process at Boards Incorporated is presently in the ordinary state. Draw a tree diagram that shows all the possible changes that can occur after one transition. Label each branch with the corresponding probability.

b. Extend this tree diagram, complete with probabilities, for a second transition. (Assume that the probabilities of moving from one state to another remain the same as in Figure **4**.)

c. Given that the quality-control process was originally in the ordinary state, calculate the probability that the process will be in each of the following states after two transitions:

1. the ordinary state

2. the relaxed state

3. the heightened state

Mathematics Note

A **Markov chain** is a model for predicting the probability of moving from one state (or outcome) to other states, given that there are a finite number of states and the probability of being in one state depends only on the state before the move.

The technique of predicting the probabilities of these transitions is the **Markov process**.

The probabilities of moving among states can be displayed in a **transition matrix**. A transition matrix **T** for a Markov chain has the following characteristics.

- The matrix is square with dimensions $m \times m$, where m represents the number of states.
- All the elements in the matrix are between 0 and 1, inclusive.
- The sum of the elements in any row is 1.

The element in row i, column j in the matrix $\mathbf{T}^n$ represents the probability of moving from state i to state j after n transitions.

For example, consider the transition diagram for the weather shown in Figure **5**.

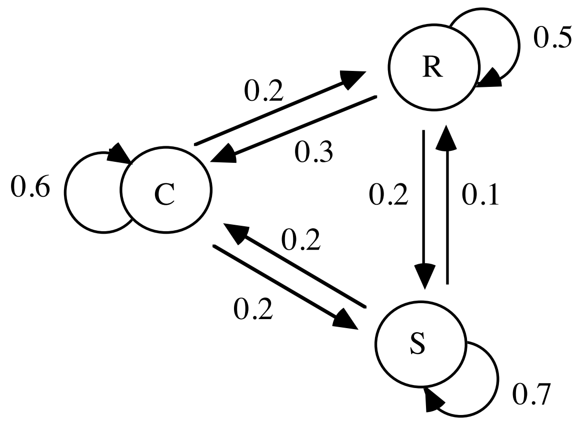


Figure 5: A transition diagram for the weather

The corresponding transition matrix, where C represents cloudy days, R represents rainy days, and S represents sunny days, is shown below.

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

This matrix has all the characteristics of a transition matrix for a Markov chain—it is square, has dimensions 3×3, and the sum of the elements in any row is 1. The element in row S, column R represents the probability of the transition from a day of sunny weather to a day of rainy weather (or the proportion of rainy days that follow sunny days).

d. Create the corresponding transition matrix **T** for the information in Figure **4**. **Note:** Save this matrix for use in the assignment.

e. Calculate the matrix $\mathbf{T}^2$.

f. Calculate each matrix $\mathbf{T}^n$, for values of n from 1 to 50. Describe any patterns you observe in these matrices.

Discussion

a. According to the previous mathematics note, a transition matrix must be square. Explain why this is true.

b. Why are the elements of a transition matrix always between 0 and 1, inclusive?

c. Why is the sum of the elements in a row of a transition matrix always 1?

d. Explain why each of the following matrices does not represent a Markov chain.

1. $\begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ **3.** $\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.2 & 0.9 \\ 0.3 & 0.7 & 0 \end{bmatrix}$

e. **1.** How do the probabilities you determined in Part **c** of the exploration compare with the elements in the second row of $\mathbf{T}^2$? Justify your response by comparing multistage probability to matrix multiplication.

2. What do the elements in the remaining rows of $\mathbf{T}^2$ represent?

f. What do the elements in each of the following matrices represent?

1. $\mathbf{T}^3$

2. $\mathbf{T}^n$

Mathematics Note

A transition matrix **T** is **regular** if for some n, all of the elements in the matrix $\mathbf{T}^n$ are positive. A Markov chain is **regular** if its transition matrix is regular.

For example, the transition matrix for the weather given in the previous mathematics note is regular because all of its elements are positive. Consequently, the Markov chain for that situation also is regular.

g. Does the information in matrix **T** in the exploration describe a regular Markov chain?

h. Describe what you observed about $\mathbf{T}^n$ in the exploration as n increased. Such matrices are referred to as **stable** (or **steady**) **state matrices**.

Mathematics Note

A **stable** (or **steady**) **state matrix** is formed by raising a transition matrix to some power such that the difference between any two elements in the same column is very small.

For example, consider the following transition matrix **T** for the weather:

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

When **T** is raised to a large power, such as 100, the matrix stabilizes to the steady state matrix **S**:

$$\mathbf{T}^{100} = \mathbf{S} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

i. What information is provided by a stable state matrix?

Assignment

3.1 In Problem **2.3**, the managers of Boards Incorporated suggested a quality-control process using only two states: ordinary and heightened. The probability of moving from the ordinary state to the heightened state was 0.128. The probability of moving from the heightened state back to the ordinary state was 0.869.

a. Create the transition matrix for this situation.

b. Determine the probability that the quality-control process is in the heightened state after three transitions.

c. **1.** Does the matrix from Part **a** represent a Markov chain? Explain your response.

2. Does it represent a regular Markov chain? Explain your response.

3.2 The table below shows the movement in the U.S. population, by region, in 1991.

	To			
From	Northeast	Midwest	South	West
Northeast	0.9818	0.0034	0.0111	0.0037
Midwest	0.0013	0.9866	0.0075	0.0046
South	0.0022	0.0049	0.9886	0.0043
West	0.0014	0.0035	0.0077	0.9874

Source: U.S. Bureau of the Census, 1993.

a. Create a transition matrix for this information.

b. Does this transition matrix represent a regular Markov chain? Explain your response.

c. Assuming that changes in residence depend only on the present residence, determine the probability of each of the following:

1. living in the West after two moves, given that the person originally lived in the Northeast
2. living in the Northeast after five moves, given that the person originally lived in the Midwest
3. living in the South after three moves, given that the person originally lived in the West.

d. Do you think that the assumption made in Part **c** is a reasonable one? In other words, do an individual's future moves depend only on the present? Explain your response.

3.3 Assuming that the acceptable defective rate remains at 20%, the quality control specialist at Boards Incorporated suggests another sampling strategy consisting of three states. In the ordinary state, a sample of 20 skateboards is taken each week. The heightened state requires a sample of 30 boards per week, while the relaxed state requires a sample of 10 boards.

If 0 to 2 defectives are found when sampling in the ordinary state, the process is moved into the relaxed state. If 6 or more defectives are found, the process is moved into the heightened state. Otherwise the quality-control process continues as usual.

Four or more defectives found when sampling in the relaxed state moves the process back to the ordinary state. When sampling in the heightened state, 9 or fewer defectives moves the process back to the ordinary state. A transition between the heightened and relaxed states is not permitted.

a. Create a transition matrix for this information.

b. Can this matrix form a stable state matrix? Explain your response.

3.4 A polling organization compiles ratings for three television networks: CBA, CBN, and SBC. In its most recent survey, the organization found that after 1 hr of television, CBA viewers continued watching CBA 50% of the time, switched to CBN 20% of the time, and changed to SBC 30% of the time.

After the same period, CBN viewers continued watching CBN 30% of the time, switched to SBC 10% of the time, and turned to CBA 60% of the time. Viewers of SBC, on the other hand, continued watching SBC 10% of the time, changed to CBA 60% of the time, and switched to CBN 30% of the time.

Use this information to predict the probability of each of the following:

a. watching CBA after 4 hr, given that a viewer was originally tuned to SBC

b. watching SBC after 5 hr, given that a viewer was originally tuned to CBN

c. watching CBA after 10 hr, given that a viewer was originally tuned to CBA.

* * * * *

3.5 At the end of basketball practice, the coach asks a pair of players to stand at the free-throw line. One player shoots free throws, while the other rebounds. The first player gets to continue shooting until she misses. The players then change positions. The coach wants the pair to shoot a total of 150 free throws.

a. Suppose that the first player's free-throw percentage is 80%, while the second player's free-throw percentage is 60%. Create a transition matrix for this situation.

b. How many shots would you predict each player to take? Explain your reasoning.

c. How many free throws would you expect each player to make?

3.6 Consider the transition matrix **A** shown below, where a, b, and c are not equal to 0.

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Determine the values of a, b, and c after many transitions.

* * * * * * * * * *

Activity 4

Besides planning an overall quality-control process, the managers of Boards Incorporated also must consider day-to-day quality issues. Does knowing the probability of moving from today's state to another help predict what to expect tomorrow or next week? In this activity, you examine how to use the Markov process to predict the probability of future events.

Mathematics Note

The **initial state vector** $\mathbf{X}_0$ of a population is represented by a matrix with a single row. Each element of the matrix represents the probability of a state before any transitions.

The **state vector** after one transition, $\mathbf{X}_1$, is determined by multiplying the initial state vector $\mathbf{X}_0$ by the transition matrix **T**. In other words, $\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1$. The order of the states in $\mathbf{X}_0$ must match the order of the states in the corresponding transition matrix. Each element of the state vector $\mathbf{X}_1$ represents the probability of a state after one transition.

Similarly, when $\mathbf{X}_1$ is multiplied by **T**, the result is the state vector $\mathbf{X}_2$. Each of its elements represents the probability of a state after two transitions. In general, $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T}$ and each element in $\mathbf{X}_n$ represents the probability of a state after *n* transitions.

For example, consider the following initial state vector for the weather:

$$\mathbf{X}_0 = \begin{array}{c} \begin{array}{ccc} \mathrm{C} & \mathrm{R} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

The elements in this matrix indicate a 20% chance of clouds, a 20% chance of rain, and a 60% chance of sunny weather, respectively, on a given day. To predict the weather for the next day, you could multiply $\mathbf{X}_0$ by the corresponding transition matrix **T** to obtain $\mathbf{X}_1$, as shown below.

$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1$$

$$\begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \bullet \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \end{bmatrix}$$

The state vector $\mathbf{X}_1$ indicates that there is a 30% chance of clouds, a 20% chance of rain, and a 50% chance of sunshine for the next day. To predict the weather in two days, the state vector $\mathbf{X}_1$ can be multiplied by **T** to obtain $\mathbf{X}_2$, and so on.

Exploration

How does the current state affect the quality-control process in upcoming weeks? In this exploration, you examine how different initial state vectors affect predictions for future events.

a. As with any row of a transition matrix, the sum of the probabilities within an initial state vector must total 1. Suppose Boards Incorporated has decided to begin the sampling process in the ordinary state. Write an initial state vector that reflects this decision in the form below.

$$\begin{array}{cc} & \begin{array}{ccc} R & O & H \end{array} \\ \mathbf{X}_0 = & [\qquad\qquad] \end{array}$$

b. The quality-control process under consideration allows transitions among three states. It is possible to represent this process using the following transition matrix:

$$\mathbf{T} = \begin{array}{c} \\ R \\ O \\ H \end{array} \begin{array}{c} \begin{array}{ccc} R & O & H \end{array} \\ \begin{bmatrix} 0.737 & 0.263 & 0.000 \\ 0.123 & 0.749 & 0.128 \\ 0.000 & 0.869 & 0.131 \end{bmatrix} \end{array}$$

Use this matrix to predict the probability of each state after the following numbers of transitions:

1. 1
2. 2
3. 3–30.

c.
1. Repeat Parts **a** and **b** assuming that the quality-control process began in the relaxed state.
2. Repeat Parts **a** and **b** assuming that the quality-control process began in the heightened state.

d. Create an initial state vector of your choice and repeat Part **b**.

e.
1. Multiply your initial state vector from Part **d** by $\mathbf{T}^{30}$.
2. Compare the resulting product matrix with the matrix for $\mathbf{X}_{30}$ obtained in Part **d**.

f. Using the symbols $\mathbf{X}_0$ and $\mathbf{T}^n$, write a general formula for finding $\mathbf{X}_n$.

Discussion

a. The sum of the elements in an initial state vector must be 1. Explain why this true.

b. Compare the distributions in the state vectors for 3–30 transitions. Describe any patterns you observe.

c. In general, $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T}$. How does this formula compare with the one you wrote in Part **f** of the exploration?

d.
1. What impact does $\mathbf{X}_0$ have on the resulting state vector $\mathbf{X}_n$ after a large number of transitions?
2. In Part **e** of the exploration, how did the elements of $\mathbf{T}^{30}$ compare with the elements of $\mathbf{X}_{30}$?

Mathematics Note

Multiplying any state vector by the stable state matrix results in a **stable** (or **steady**) **state vector**. This vector consists of one row of the stable state matrix.

For example, consider the following state vector $\mathbf{X}_2$ for the weather:

$$\begin{array}{cc} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{X}_2 = & \begin{bmatrix} 0.3 & 0.1 & 0.6 \end{bmatrix} \end{array}$$

Multiplying $\mathbf{X}_2$ by the stable state matrix **S** results in the stable state vector **P**:

$$\begin{array}{cc} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{P} = & \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

The entries in this stable state vector indicate that, over the long term, there is a 40% chance of clouds, a 20% chance of rain, and a 40% chance of sunshine on any given day.

All regular Markov chains have a stable state matrix and vector. Markov chains that are not regular may or may not have a stable state matrix and vector.

Assignment

4.1 Determine the stable state matrix, if it exists, for each of the following transition matrices.

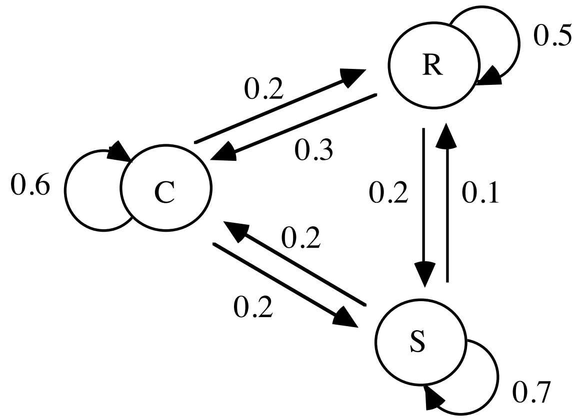
a. $\begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.9 & 0.0 & 0.1 \end{bmatrix}$

b. $\begin{bmatrix} 0.0 & 0.2 & 0.8 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.6 & 0.4 \end{bmatrix}$

c. $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.7 & 0.1 & 0.2 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$

d. $\begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.4 & 0.0 & 0.6 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}$

4.2 In the transition diagram below, C represents cloudy weather, R represents rainy weather, and S represents sunny weather. Use this diagram to predict the numbers of rainy, sunny, and cloudy days in any one year.



4.3 In Problem **2.3**, the managers of Boards Incorporated suggested a quality-control using only two states. The probability of moving from the ordinary state to the heightened state was 0.128. The probability of moving from the heightened state back to the ordinary state was 0.869.

a. Find the stable state matrix for this information, if one exists.

b. Find the stable state vector for this information, if it exists.

c. After 25 transitions, what proportion of the sampling would you predict will be done in each state?

4.4 In Problem **3.3**, a quality control specialist suggested another sampling strategy involving three states. In the ordinary state, a sample of 20 skateboards is taken each week. The heightened state requires a sample of 30 boards per week, while the relaxed state requires a sample of 10 boards. The transition matrix for this strategy is given below:

$$\mathbf{T} = \begin{array}{c} \\ R \\ O \\ H \end{array} \begin{array}{c} \begin{array}{ccc} R & O & H \end{array} \\ \begin{bmatrix} 0.879 & 0.121 & 0.000 \\ 0.206 & 0.598 & 0.196 \\ 0.000 & 0.129 & 0.871 \end{bmatrix} \end{array}$$

a. Determine the probability of each of the following:

1. being in the ordinary state after two transitions, given that the initial state was ordinary.
2. being in the ordinary state after five transitions, given that the initial state was relaxed.
3. being in a relaxed state after three transitions, given that the initial state was ordinary.

b. Determine the probability of each state after 52 weeks of sampling.

c. Describe the effects of the initial state vector on the resulting state vector as the number of transitions increases.

4.5 Consider a carnival game in which players toss a ball through a hole in a board. Players win prizes based on the number of successful tosses in a row: the more successful tosses in a row, the bigger the prize.

After any successful toss, a player can decide that the game is over and collect a prize. The player also may continue the game, as long as the previous toss was successful. After any unsuccessful toss, the game is over and no prize is awarded.

a. After making a successful toss, players make another successful toss 25% of the time, are unsuccessful 65% of the time, and quit 10% of the time. Create a transition matrix for this situation.

b. Assuming that the initial toss is successful, predict the probability that a player will make a successful sixth toss.

c. Predict the probability of each state over the long run.

d. This Markov chain has a state called the **absorbing state**. What do you think this term means?

* * * * *

4.6 A large ski resort has decided to upgrade the mass transit system it uses to bring skiers from a nearby city to the mountain. Currently, the resort uses school buses. They would like to change to motor coaches. At present, 28% of skiers take the mass transit system, while 72% drive their own vehicles.

To help predict the percentage of skiers who will use the upgraded transit system, the resort owners conducted a survey. The results of the survey are shown in the following transition matrix.

$$\begin{array}{cc} & \text{Next Year} \\ & \begin{array}{cc} \text{Mass Transit} & \text{Own Vehicle} \end{array} \\ \text{This Year} \begin{array}{r} \text{Mass Transit} \\ \text{Own Vehicle} \end{array} & \begin{bmatrix} 0.65 & 0.35 \\ 0.25 & 0.75 \end{bmatrix} \end{array}$$

a. Assuming that the total number of skiers remains constant, what percentage do you predict will use the new transit system after each of the following numbers of years?

1. 1
2. 2
3. 5

b. What percentage of skiers do you predict will use the mass transit system in the long run?

* * * * * * * * * *

Research Project

The game Tug O'Spot can be played by one or two people. The game is played by placing a marker on a board, then moving it according to the outcome of a roll of dice. As shown in Figure **6** below, there are five available starting positions. Depending on where you start, the probabilities of winning change.

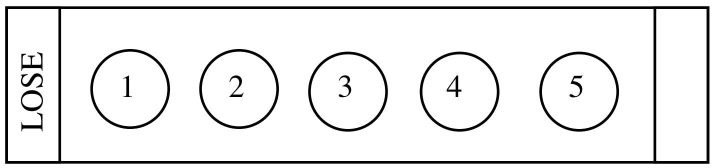


Figure 6: Tug O'Spot game board

To determine the marker's starting position in the two-person version of the game, one player rolls a die. If a 6 shows, the player rolls again. If any other number shows, the player places his marker on the starting position that corresponds to the number on the die.

Once the marker has been placed, each player tosses a die. If the two rolls are the same, the marker is moved two positions toward WIN. If the two rolls differ by 1, then the marker is moved one position toward WIN. In all other cases, the marker is moved one position toward LOSE. When the marker reaches either end, the game is over. The player who placed the marker is credited with either a win or a loss.

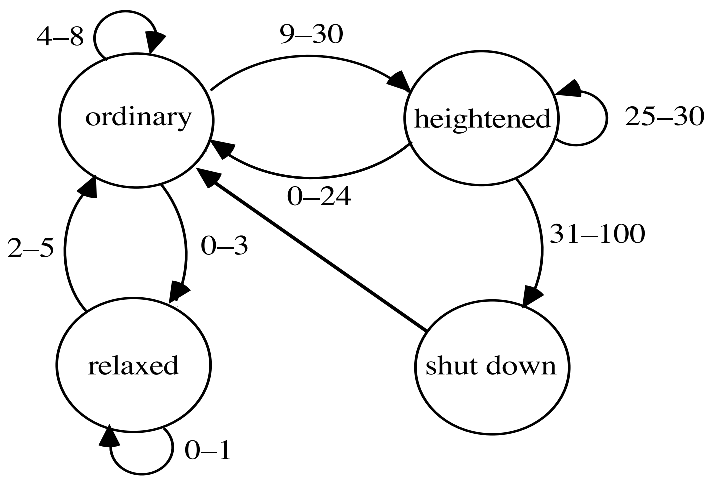
The one-person version is played in a similar manner, with the player tossing two dice each time.

Use your knowledge of Markov processes to determine the probabilities of winning from each of the five possible starting positions.

Summary Assessment

Boards Incorporated has decided to make one more adjustment to the company's quality-control process. If 31 or more defective skateboards are found while sampling in the heightened state, the plant will be shut down for one week while the source of the defects is identified. In the following week, sampling will continue in the ordinary state.

The quality-control process now contains four states. The diagram below shows how many defective items result in the various transitions in the process.



It costs the company about $15 to test each skateboard sampled. The estimated cost of a shut down is $2000 per week.

1. Predict how much it will cost to follow the approved quality-control process for one year. Your response should contain a detailed explanation of how you arrived at your estimate, including any assumptions you made.

2. The quality-control process described above is based on an assumed defective rate of 20%. Explain what you would expect to occur if the actual rate of defective items was greater than 20%.

Module Summary

- A **binomial experiment** has the following characteristics:
  1. It consists of a fixed number of repetitions (trials) of the same action.
  2. The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
  3. Each trial has only two possible outcomes: a success or a failure.
  4. The probability of a success remains the same from trial to trial.
  5. The total number of successes is observed.
- The probability distribution for a binomial experiment is a **binomial distribution**.
- The mean μ of a binomial distribution is the product of the number of trials and the probability of a success. In other words, $\mu = np$, where n is the number of trials and p is the probability of a success.
- The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

- The **binomial probability formula** can be used to determine the probability of obtaining r successes in n trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

- A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.
- A **normal distribution** is a continuous probability distribution. The graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.

  As in all continuous probability distributions, the total area between the horizontal axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

- The process of moving from one **state**, or outcome, to another is a **transition**.
- A **transition diagram** is a convenient way to display the possible changes among states. In such diagrams, each state is represented by a vertex, while each transition is represented by a directed edge labeled with a corresponding probability.
- A **Markov chain** is a model for predicting the probability of moving from one state (or outcome) to other states, given that there are a finite number of states and the probability of being in one state depends only on the state before the move.
- The technique of predicting the probabilities of these transitions is the **Markov process**.
- The probabilities of moving among states can be displayed in a **transition matrix**. A transition matrix **T** for a Markov chain has the following characteristics.

  1. The matrix is square with dimensions $m \times m$, where m represents the number of states.

  2. All the elements in the matrix are between 0 and 1, inclusive.

  3. The sum of the elements in any row is 1.

- The element in row i, column j in the matrix $\mathbf{T}^n$ represents the probability of moving from state i to state j after n transitions.
- A transition matrix **T** is **regular** if for some n, all of the elements in the matrix $\mathbf{T}^n$ are positive. A Markov chain is **regular** if its transition matrix is regular.
- The **initial state vector** $\mathbf{X}_0$ of a population is represented by a matrix with a single row. Each element of the matrix represents the probability of a state before any transitions.
- In general, the **state vector** $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^n$. Each element of $\mathbf{X}_n$ represents the probability of a state after n transitions.

  Each element of $\mathbf{T}^n$ represents the probability of moving from one state to another state after n transitions.

- A **stable** (or **steady**) **state matrix** is formed by raising a transition matrix to some power such that the difference between any two elements in the same column is very small.
- Multiplying any state vector by the stable state matrix results in a **stable** (or **steady**) **state vector**. This vector consists of one row of the stable state matrix.
- All regular Markov chains have a stable state matrix and vector. Markov chains that are not regular may or may not have a stable state matrix and vector.

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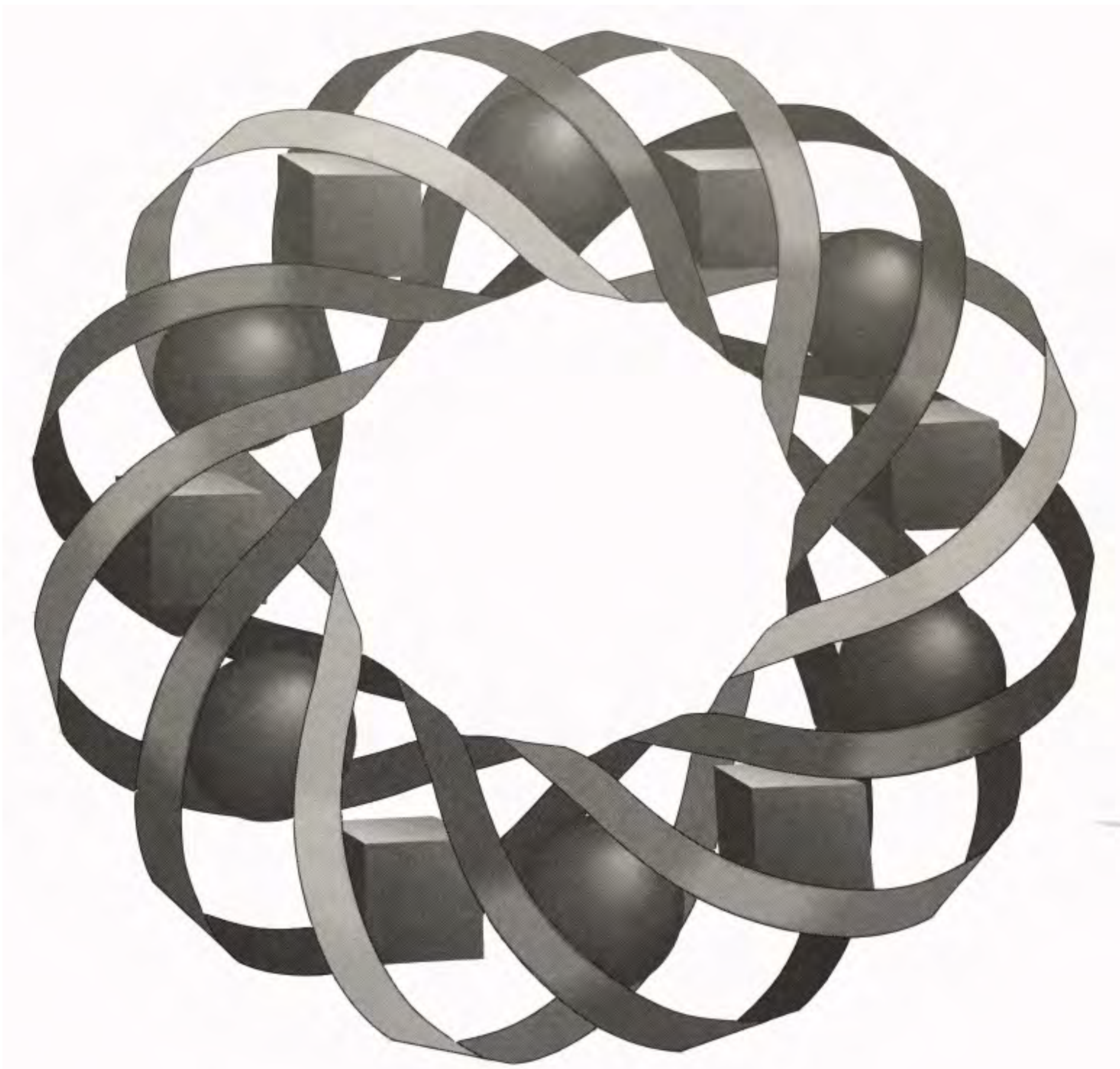
Let There Be Light



Why are circles, ellipses, parabolas, and hyperbolas so important in our technological world? In this module, you answer this question by studying a group of shapes called conics.

Gary Bauer • Sherry Horyna • John Knudson-Martin

Let There Be Light

Introduction

At a stadium in Pasadena, California, two soccer teams stand ready. In Rome, Italy, fans anxiously await the beginning of the match. It's the World Cup final—the world's most watched sporting event! Millions of soccer fans expect to view every second of the game, live on television. But how can a television signal be received at countless locations, all over the globe, all at the same time?

The answer involves parabolic reflectors. First, a reflector in the shape of a parabola focuses a signal from a broadcasting station near the stadium, transmitting it to a satellite 36,000 km above the earth's surface. After receiving this signal, the satellite broadcasts it back towards earth. Parabolic dishes throughout the world receive and concentrate the signal. Local networks then transmit it to the television screens of eager fans.

In this module, you investigate the reflective qualities of parabolas and three other unique shapes, which together are known as the **conic sections**.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a double-napped cone. Depending on the slope of the plane, the intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.

Figure **1** shows the four conic sections and the plane-cone intersections that produce them. Note that circles and ellipses are closed figures, while parabolas and hyperbolas are not closed. Notice also that a hyperbola has two parts or **branches**.

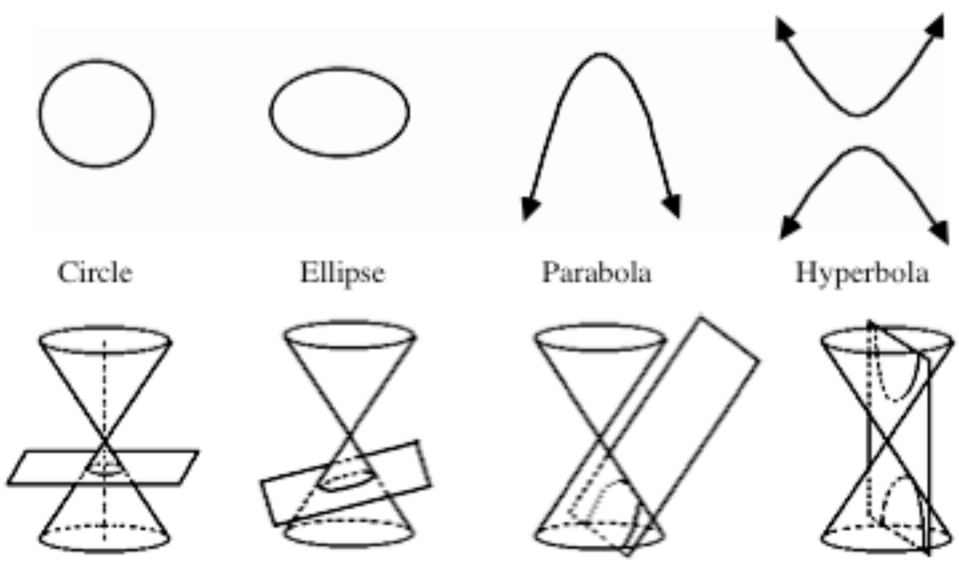


Figure 1: The conic sections

Activity 1

A satellite dish reflects television signals, a concert hall or band shell reflects sound, and a space heater reflects light and warmth. Each of these objects, shaped like a different conic section, reflects in a different way. The reflective properties of each conic are closely linked to a point (or points) known as the **focus** (or **foci**).

Mathematics Note

Figure **2** shows a diagram of the four conics. Each dot in the diagram indicates the location of a **focus** (plural **foci**) for a particular conic. Hyperbolas and ellipses each have two foci. Circles and parabolas each have one focus.

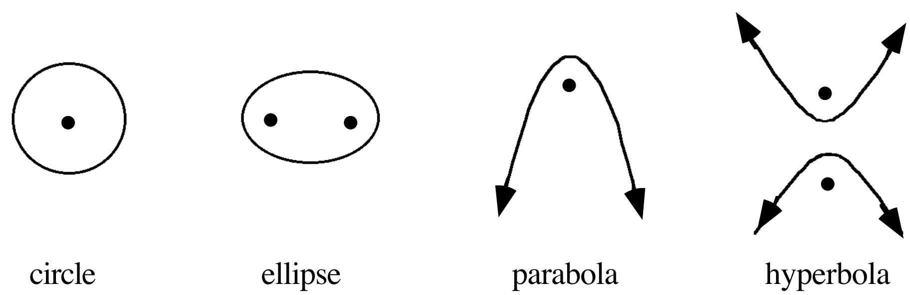


Figure 2: The foci of the conic sections

Each conic divides the plane that contains it into regions. The **interior** of a conic is the region (or regions) that contains at least one focus. The **exterior** of a conic is the region that does not contain a focus.

Exploration

In this exploration, you experiment with the reflective properties of conics by passing light rays through their foci.

a. Partially cover a flashlight lens with black electrical tape, leaving a slit 1–2 mm wide, as shown in Figure **3**.

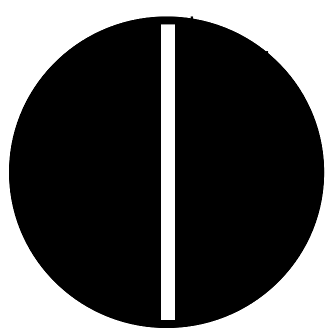


Figure 3: Flashlight lens partially covered with tape

b. Obtain an ellipse template, a sheet of cardboard, and some reflective material from your teacher.

1. Tape or glue the template to the sheet of cardboard.
2. To create a slot for the reflective material, cut through the cardboard along approximately 2/3 of the outline of the ellipse.
3. As shown in Figure **4** below, insert a strip of reflective material in the slot from Step **2**.

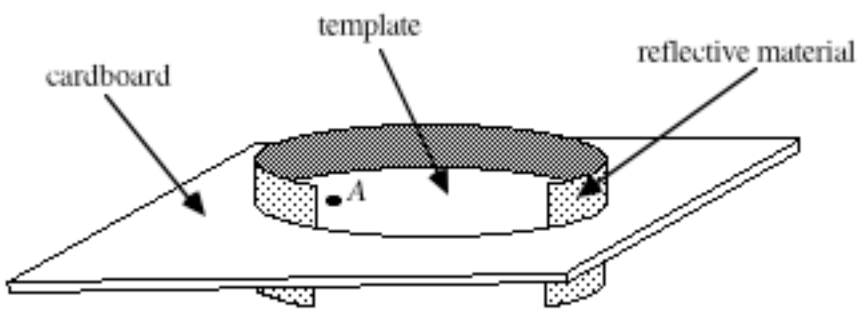


Figure 4: Model of ellipse with reflective material

c.
1. Hold the flashlight so that the long axis of the slit you created in Part **a** is vertical. As shown in Figure **5**, shine the beam of light through a focus and onto the reflective material.

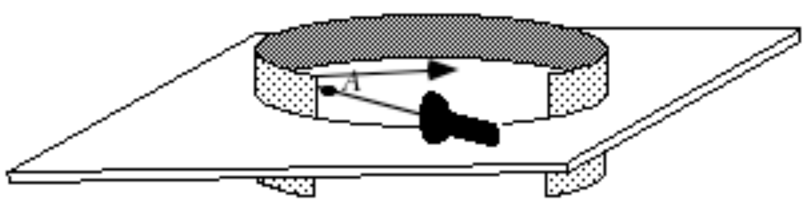


Figure 5: Flashlight beam passing through a focus

2. Use a pencil to trace the path of the reflected light.
3. Move the flashlight so that the beam of light, after passing through the focus, strikes the reflective material at several different locations. Trace the path of the reflected light in each case.
4. Move the flashlight so that the beam of light does not pass through the focus. Trace the path of reflected light. **Note:** Save the template for use in the assignment.

d. Repeat Parts **b** and **c** using the template of a circle.

e. Repeat Parts **b** and **c** using the template of a parabola.

Discussion

a. Each conic section has at least one axis of symmetry. Describe the locations of the axes for a circle, an ellipse, a parabola and a hyperbola.

b. Describe the reflective properties of each of the following conic sections:

1. an ellipse

2. a circle

3. a parabola.

Mathematics Note

The reflective properties of conics are used in many real-world applications. In most cases, the actual reflectors are the three-dimensional counterparts of the conic sections. Figure **6** shows one example of each of these shapes: a sphere, a paraboloid, a hyperboloid, and an ellipsoid.

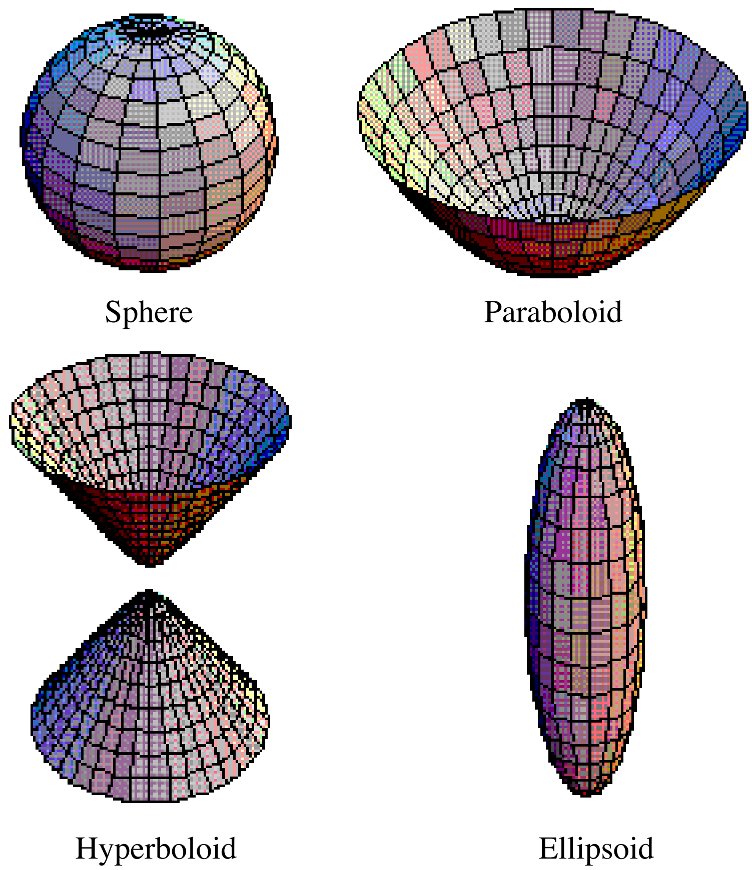


Figure 6: Three-dimensional conic reflectors

A **sphere** is generated by rotating a circle around any axis of symmetry.

A **paraboloid** is generated by rotating a parabola around its axis of symmetry.

A **hyperboloid** is generated by rotating a hyperbola around the axis of symmetry that contains the foci.

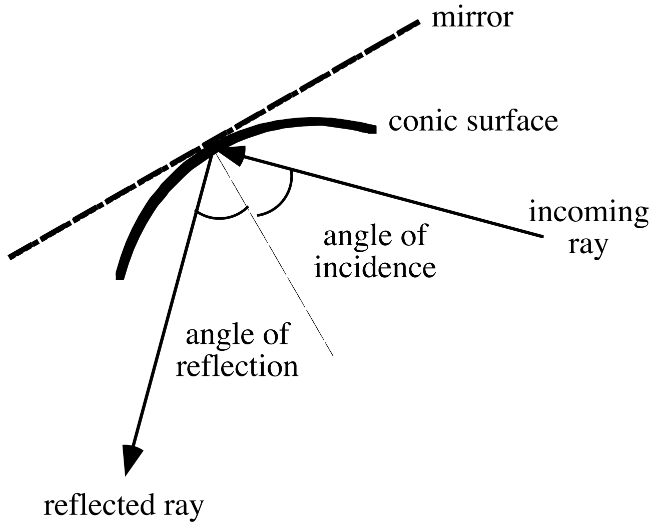
An **ellipsoid** is generated by rotating an ellipse around the axis of symmetry that contains the foci.

Light passing through a focus of one of these three-dimensional shapes is reflected in the same manner as light reflected from the corresponding two-dimensional conic. For example, light rays passing through one focus of an ellipsoid are reflected through the other focus.

c. Consider a mirror shaped like a paraboloid. If a ray of light enters the mirror parallel to the paraboloid's axis of symmetry, in what direction would you expect it to be reflected?

Assignment

1.1 Light rays reflecting off a flat mirror form an angle of incidence that is congruent to the angle of reflection. As shown in the diagram below, when light is reflected off a point on a conic (or any other curve), it behaves as if it were reflecting off a flat mirror tangent to the curve at the point of reflection.



a. 1. On the template of the ellipse from the exploration, locate an angle formed by an incoming light ray and its reflected ray.

2. Bisect the angle to find the angle of incidence and the angle of reflection.

3. To draw the line represented by the flat mirror in the diagram above, construct a line perpendicular to the angle bisector at the point of reflection. This line is tangent to the curve at the point of reflection.

b. Consider a light ray traveling toward a focus of an ellipse from its exterior.

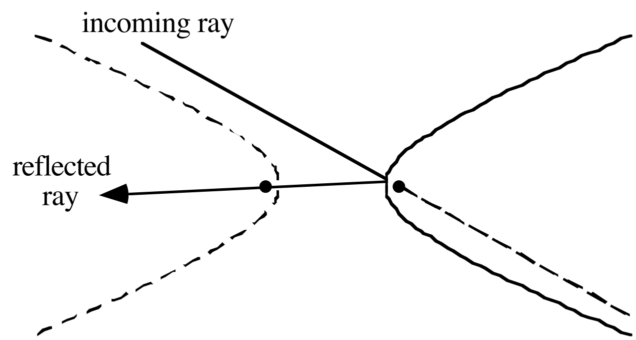
1. Sketch one such ray that intersects the ellipse at the point of reflection from Part **a**.

2. Draw the reflected ray for the incoming ray in Step **1**. (Recall that the angle of incidence and the angle of reflection are congruent.)

c. Describe the path that a ray of light traveling toward the focus of a circle would take if it were reflected off the exterior of the circle. Draw a diagram to illustrate your conclusions.

d. Repeat Part **c** for a parabola.

1.2 When light rays traveling toward a focus from the exterior of a hyperbolic mirror reflect off the surface, they are directed towards the other focus, as shown in the diagram below.



a. Obtain a copy of this diagram from your teacher. At the point of reflection, construct a line tangent to the hyperbola. (Recall that the angle of incidence is congruent to the angle of reflection.)

b. Consider a light ray traveling through the focus from the interior of the hyperbolic reflector towards the same point of reflection as Part **a**.

1. Sketch the paths of the incoming and reflected rays.

2. Draw the lines containing these rays.

c. Repeat Parts **a** and **b** for several other light rays.

d. In a brief paragraph, describe the reflective properties of a hyperbola.

1.3 **a.** Consider a paraboloid mirror with a light bulb placed at its focus. Make a sketch of a cross section of this paraboloid. On your sketch, show how a ray of light from the bulb would be reflected off the mirror.

b. A car headlight consists of a parabolic reflector with a bulb at the focus. Why is this an appropriate design?

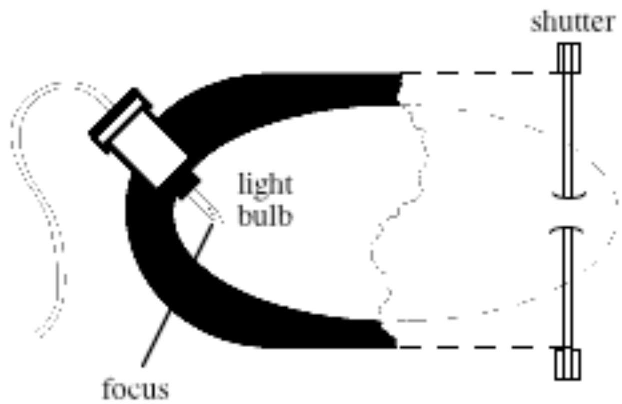
c. The dish for a satellite television antenna is a paraboloid. The receiver for the television signal is located at the focus of the paraboloid. Why is this an appropriate design?

* * * * *

1.4 Why does cupping your hand around your ear make it easier to hear faint sounds?

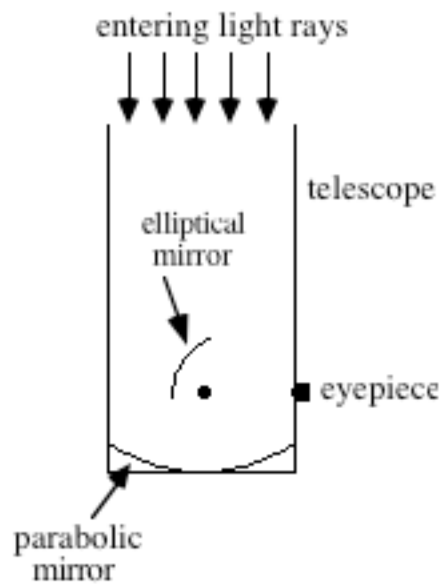
1.5 The Statuary Hall in the U.S. Capitol was built in the shape of an ellipse. By standing at a particular location, you can overhear the whispers of another person across the room. Explain why this is possible.

1.6 The following diagram shows a cross section of a spotlight with an ellipsoidal reflector. The light bulb is located at one focus of the ellipse.



On this type of spotlight, the shutter creates a circular beam of light. Through what point do all the light rays pass before continuing on through the hole in the shutter?

1.7 Some telescopes are constructed so that the focus of a parabolic mirror and the focus of an elliptical mirror occur at the same point. The eyepiece is located at the other focus of the ellipse. Describe what happens to a ray of light entering the telescope.



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Activity 2

When designing reflective surfaces, engineers often use the **standard form** of the equations for a conic sections. In this activity, you examine the standard form of the equation for an ellipse.

Exploration

a. Construct a rectangular coordinate system with its origin near the center of a sheet of graph paper.

1. Label the *x*- and *y*-axes and graph two points F_1 (–4,0) and F_2 (4,0). These points represent the foci of an ellipse.
2. On a string with a length of about 25 units (in your coordinate system), mark the endpoints of a segment 10 units long.

   When held straight, without stretching, your segment should extend from (–5,0) to (5,0) along the *x*-axis.

b. Place the two points marked on the string so that they coincide with the foci, F_1 and F_2. As shown in Figure **7**, use the edge of a ruler to hold these points firmly in position.

Trace the portion of an ellipse above the ruler with a pencil, making sure to keep the string taut. Reposition the string and ruler to complete the lower portion of the ellipse.

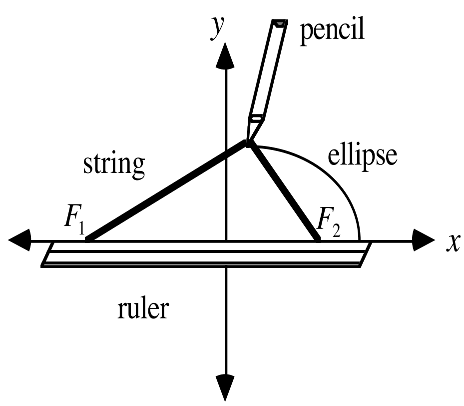


Figure 7: Drawing an ellipse

c. Use your graph from Part **b** to complete Table **1** for points on the ellipse. Estimate each coordinate as accurately as possible.

Table 1: Coordinates of points on an ellipse

	Foci: $F_1(-4,0)$ and $F_2(4,0)$											
x	0	0			1	1	2	2	–1	–1	–2	–2
y			0	0								

d. For each point P in Table **1**, find PF_1, the distance from point P to the focus F_1, and PF_2, the distance from P to F_2.

e. Calculate $PF_1 + PF_2$ for each point and describe any patterns you observe.

f. Determine the length of the segment that contains the foci and whose endpoints are on the ellipse.

g. Using the same sheet of graph paper, repeat Parts **b–f** for each of the following pairs of foci:

1. (–1,0) and (1,0)

2. (0,–4) and (0,4)

Discussion

a. Describe the symmetry of the three ellipses you drew in the exploration.

b. **1.** What happens to the shape of an ellipse as the distance between the foci decreases?

2. What shape would result if the distance between the foci was 0?

c. What happens to the shape of an ellipse as the distance between the foci increases?

d. How does an ellipse with its foci located on the y-axis compare with an ellipse with its foci located on the x-axis?

e. Suppose that the length of the segment marked on the string had been 12 units instead of 10 units. In this case, what would have been your response to Part **e** of the exploration?

f. In the exploration, you examined the sum of the distances from a point on the ellipse to the foci. You also found the length of the segment that contains the foci and whose endpoints are on the ellipse. What is the relationship between these two lengths?

Mathematics Note

An **ellipse** is a set of points in the plane such that the sum of the distances from each point to two foci is a constant. For example, Figure **8** shows an ellipse with foci at F_1 and F_2. For any point *P* on the ellipse, $PF_1 + PF_2$ is a constant.

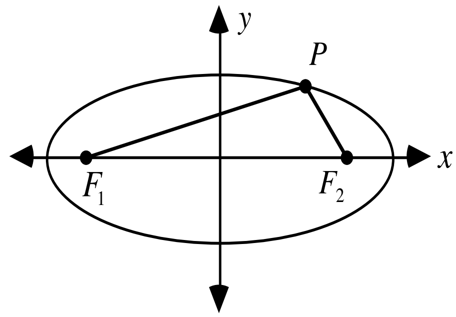


Figure 8: An ellipse with its foci on the *x*-axis

The **major axis** of an ellipse is the segment, with endpoints on the ellipse, that contains the foci. The **minor axis** is the segment, with endpoints on the ellipse, contained in the perpendicular bisector of the major axis. The major and minor axes intersect at the **center** of the ellipse. The endpoints of the major and minor axes are the **vertices** of the ellipse.

For example, Figure **9** shows an ellipse with foci at F_1 and F_2. In this case, $\overline{MN}$ is the major axis and $\overline{TW}$ is the minor axis. The origin *O* is the center of the ellipse. Points *M*, *N*, *W*, and *T* are the vertices of the ellipse.

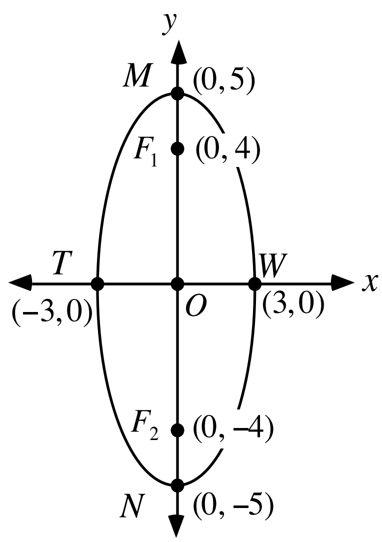


Figure 9: An ellipse with its foci on the *y*-axis

g. What are the vertices of the three ellipses you created in the exploration?

h. Figure **10** shows the graph of an ellipse with foci at $(-c,0)$ and $(c,0)$.

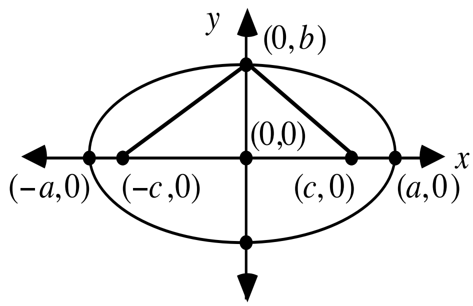


Figure 10: An ellipse with foci at $(-c,0)$ and $(c,0)$

1. Using this graph, describe how the length *s* of the segment marked on the string and the locations of the foci determine the locations of the vertices contained in the major axis. Hint: Examine the distances from the vertices to the center and from the vertices to the foci.
2. Describe how the length *s* and the locations of the foci determine the locations of the vertices contained in the minor axis.

Mathematics Note

The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *x*-axis is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where *a* is half the length of the major axis and *b* is half the length of the minor axis.

The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *y*-axis is:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

For example, the ellipse in Figure **9** has a major axis 10 units long and a minor axis 6 units long. The equation of this ellipse is shown below:

$$\frac{x^2}{9}+\frac{y^2}{25}=1$$

i. Describe the equation in standard form of each ellipse constructed in the exploration.

j. **1.** Figure **10** shows an ellipse with foci at $(-c,0)$ and $(c,0)$. The length of its major axis is $2a$. Use this figure to write an equation that describes the relationship among a, b, and c.

2. Given the equation of an ellipse with center at the origin, describe how you could use the relationship among a, b, and c to determine the coordinates of the foci.

k. Describe the shape of an ellipse in which $a = b$.

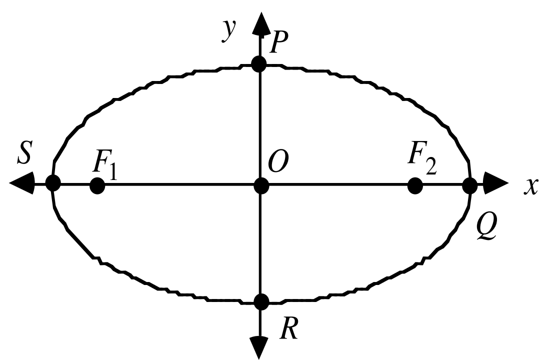
Assignment

2.1 Describe how the definition of an ellipse given in the mathematics note is illustrated by the string constructions you made in the exploration.

2.2 Describe a method for determining the intercepts of an ellipse with center at the origin, given its equation in the form below:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2.3 The figure below shows an ellipse with foci at F_1 and F_2 and center at the origin.



The equation of this ellipse is:

$$\frac{x^2}{10^2} + \frac{y^2}{8^2} = 1$$

a. Find the coordinates of the vertices P, Q, R, and S and the foci F_1 and F_2.

b. Determine each of the following distances: PR, SQ, PO, PF_2, and OQ.

c. Write an equation that describes the relationship among the length of the major axis, the length of the minor axis, and the distance between the foci.

2.4 In Part **c** of the exploration, you constructed an ellipse with foci at (–4,0) and (4,0) in which the length of the major axis was 10.

a. **1.** Write the equation of the ellipse in standard form.

2. Test your equation using the coordinates of points in Table **1**.

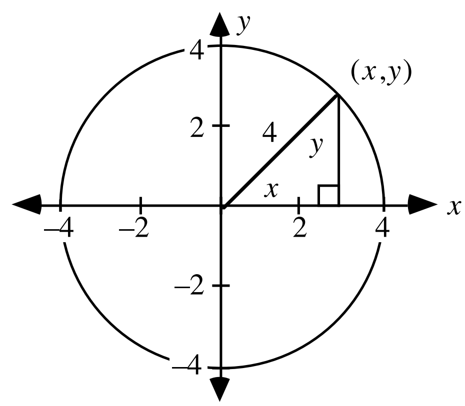
b. Consider a point P with coordinates (x,y) on an ellipse with center at the origin and foci at $(-c,0)$ and $(c,0)$. Using the definition of an ellipse and the distance formula, the length $(2a)$ of the major axis can be described as follows:

$$2a = \sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2}$$

1. Substitute the values for a and c from the ellipse in Part **a** into this equation.

2. Test the resulting equation using the coordinates of points in Table **1**.

2.5 The circle shown in the figure below is the set of all points whose distance from the origin is 4.



a. Given that (x,y) is any point on the circle, use the Pythagorean theorem to write an equation that shows the relationship among x, y, and 4. This is the standard form of the equation of a circle with center at the origin and radius 4.

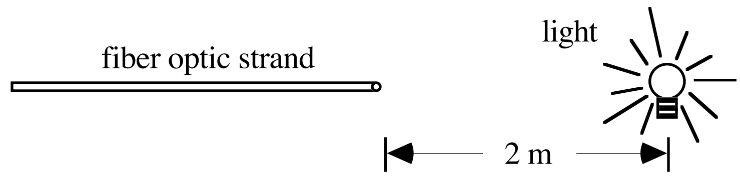
b. Use a symbolic manipulator to solve the following equation for y:

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} = 1$$

c. Use a graphing utility to graph the result in Part **b**.

d. Show algebraically that the equation you wrote in Part **a** is equivalent to the equation given in Part **b**.

2.6 Imagine that you are an engineer at a fiber optics company. As part of an ongoing project, you must design a reflector to focus the light from a bulb onto the end of a fiber optic strand 2 m away.



a. Design an appropriate reflector. Describe the locations of the bulb and the end of the fiber optic strand in relation to the reflector and explain why you chose these locations.

b. Make a scale drawing of the reflector and label its dimensions.

c. Write an equation that would allow a computer to draw a cross section of your reflector.

* * * * *

2.7 The equation $9x^2 + 16y^2 = 144$ describes an ellipse.

a. Graph this equation.

b. Determine the x- and y-intercepts of the ellipse.

c. Find the coordinates of the foci.

2.8 Consider a satellite traveling in an elliptical orbit directly above the earth's equator. One focus of the ellipse is at the earth's center. The distance from each focus to the center of the ellipse is 500 km. The length of the major axis is 16,000 km.

a. Write an equation that describes the satellite's orbit.

b. How does the shape of the satellite's orbit compare to a circle?

c. The radius of the earth at the equator is about 6400 km.

1. How far is the satellite from the earth's surface when the orbit is nearest the earth ?

2. How far is the satellite from the earth's surface when the orbit is farthest from the earth?

2.9 **a.** Graph the following two ellipses on the same coordinate system.

$$16x^2 + 64y^2 = 1024$$

$$49x^2 + 25y^2 = 4900$$

b. The formula for the area of an ellipse is $A = \pi ab$, where a and b are half the lengths of the major and minor axes, respectively. Determine the areas of the two ellipses in Part **a**.

* * * * * * * * * *

Activity 3

Some applications of reflectors require the scattering of heat, light, or sound. For example, an effective wall heater disperses heat uniformly in a room, while a well-designed band shell reflects sound so that the entire audience can hear.

The conic whose reflective properties are well-suited for these applications is the hyperbola. Like an ellipse, a hyperbola can be defined as a set of points that satisfy a common rule.

Mathematics Note

A **hyperbola** is a set of points in the plane such that the difference of the distances from each point to two foci is a constant.

For example, Figure **11** shows a hyperbola with foci F_1 and F_2. For any point P on the hyperbola, the difference between PF_1 and PF_2 is a constant.

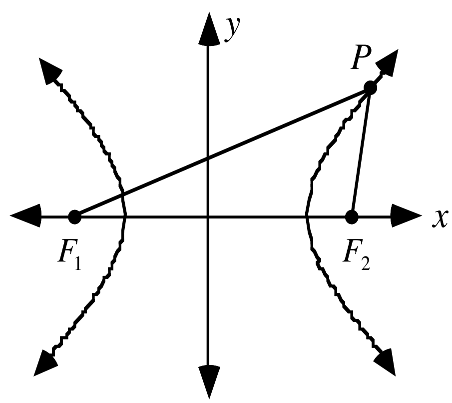


Figure 11: A hyperbola with center at the origin

The **vertices** of a hyperbola are the intersections of the hyperbola and the segment that joins the foci. The **transverse axis** is the line that passes through the vertices. The **conjugate axis** is perpendicular to the transverse axis and bisects the segment joining the vertices at the **center** of the hyperbola.

In Figure **12**, for example, the vertices of the hyperbola are $(-a,0)$ and $(a,0)$. The transverse axis is the *x*-axis and the conjugate axis is the *y*-axis. The center is at the origin, the midpoint of $(-a,0)$ and $(a,0)$.

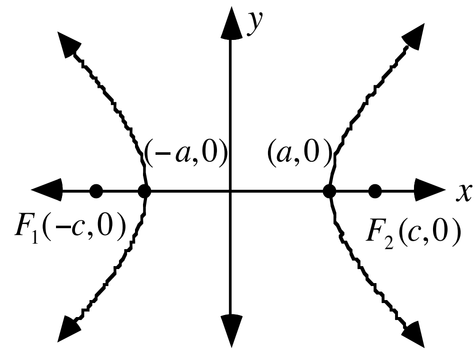


Figure 12: A hyperbola with foci on the *x*-axis

The **standard form** of the equation of a hyperbola with center at the origin and foci at $(-c,0)$ and $(c,0)$ is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a is the distance from the center to a vertex and $b^2 = c^2 - a^2$.

For example, consider a hyperbola with foci at (–5,0) and (5,0) and vertices at (–3,0) and (3,0). In this case, $c = 5$, $a = 3$, and $b = \sqrt{25-9} = 4$. The equation of this hyperbola, in standard form, is

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$$

Exploration

a. Select values for a and b in the following equation of a hyperbola with center at the origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

b.
1. Determine the coordinates of the vertices.
2. Determine the coordinates of the foci.
3. Determine the distance between each focus and the center.

c. Solve the equation in Part **a** for y and graph both parts of your solution.

d.
1. Select a point P on the hyperbola in the first quadrant. Find $|PF_1 - PF_2|$, where F_1 and F_2 are the foci.
2. Select a point Q on the hyperbola in the second quadrant. Verify that $|QF_1 - QF_2|$ equals the constant difference found in Step **1**.
3. Determine the distance between the vertices of the hyperbola.
4. Describe the relationship between the constant difference and the distance between the vertices.

e. The hyperbola in Part **a** is not a function because each x-value in the domain, other than the vertices, has two corresponding y-values.

Complete Table **2** for points on the hyperbola, using the given x-values. Describe any trends or patterns you observe.

Table 2: Coordinates of points on a hyperbola

x	10	10	100	100	1000	1000	10,000	10,000
y								
y/x								

f. Determine the values of b/a and $-b/a$ for the hyperbola. Compare these values with the ratios for y/x in Table **2**.

g. In Part **f**, you found that as the value of x increases, the ratio y/x appears to approach b/a or $-b/a$. If $b/a = y/x$, then $y = (b/a)x$.

1. Graph the lines $y = (b/a)x$ and $y = -(b/a)x$ on the same coordinate system as the hyperbola in Part **c**.
2. Compare the y-values for points on the hyperbola with those of the corresponding points on the lines.

h. On your graph from Part **g**, draw a rectangle with center at the origin, a horizontal length of $2a$, and a vertical length of $2b$. Draw the diagonals of the rectangle and note any relationships you observe between the rectangle and the hyperbola.

i.

1. Replot the graphs of $y = (b/a)x$, $y = -(b/a)x$, and the hyperbola using intervals for the domain and range that are 10 times those in Part **g**. Compare the y-values for points on the hyperbola with those of the corresponding points on the lines.
2. Repeat Step **1** using intervals for the domain and range that are 100 times those in Part **g**.

j. Using the values you chose for a and b in Part **a**, repeat Parts **b** and **c** for the following equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line approaches zero as the distance from P to the origin increases without bound, where P is on a suitable part of the curve.

The asymptotes for a hyperbola with an equation of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

are the lines $y=(b/a)x$ and $y=-(b/a)x$.

For example, Figure **13** shows a graph of the hyperbola

$$\frac{x^2}{3^2}-\frac{y^2}{4^2}=1$$

The equations of the asymptotes are $y=(4/3)x$ and $y=-(4/3)x$.

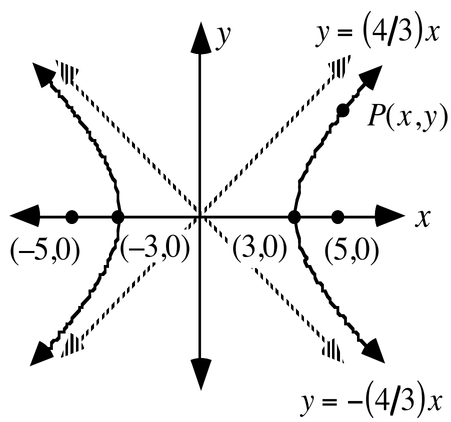


Figure 13: A hyperbola and its asymptotes

Discussion

a. Compare the graph of your hyperbola in Part **c** with those of others in the class. What role do the values of a and b in the equation below have in determining the shape of the hyperbola?

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

b. How are the values of a and b in the standard form of the equation of a hyperbola related to the distance between a focus and the center?

c. Given an equation of a hyperbola in standard form, how could you find the coordinates of the foci?

d. Consider a rectangle with center at the origin and side lengths of $2a$ and $2b$. Describe how this rectangle can be used to quickly sketch a graph of the hyperbola with the equation below:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

e. Compare the graphs of the hyperbolas in Parts **c** and **j** of the exploration.

f. Describe the equation, in standard form, of a hyperbola with vertices located on the y-axis.

g. How could you determine the equations of the asymptotes for a hyperbola with its foci on the y-axis?

Assignment

3.1 **a.** **1.** Write an equation in standard form for a hyperbola with center at the origin and foci on the x-axis where $a = 10$ and $b = 5$.

2. Determine the coordinates of the foci.

3. Create a graph of the hyperbola.

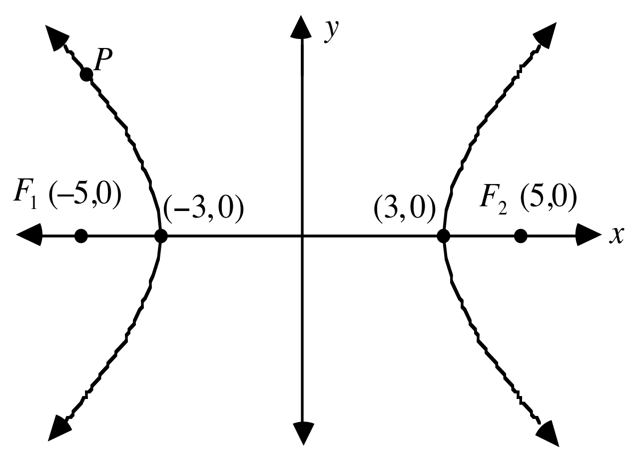
b. Repeat Part **a** for $a = 4$ and $b = 5$.

3.2 **a.** Graph the hyperbola with the equation below.

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

b. Write the equations of its asymptotes and determine the coordinates of its foci and vertices.

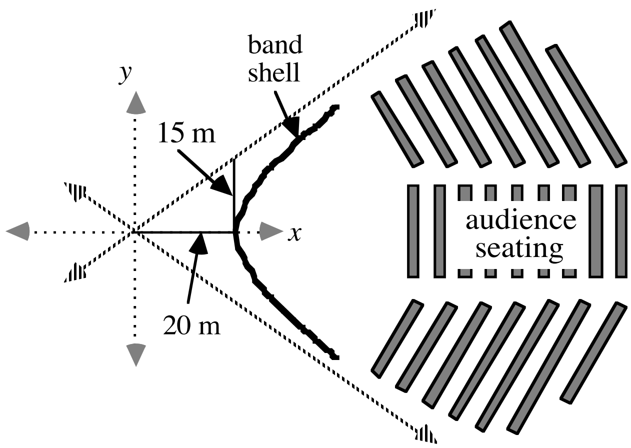
3.3 **a.** Determine the equation, in standard form, of the hyperbola shown in the following graph.



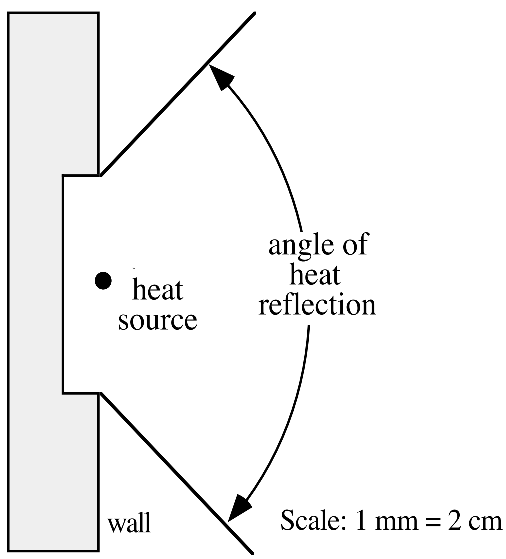
b. Consider a light ray that passes through F_1 and reflects off the hyperbola at P. Describe the direction in which the ray is reflected.

3.4 The conductor of a community band is designing a band shell shaped like one branch of a hyperbola. In this application, the asymptotes of the hyperbola describe the area within which sound waves will scatter when reflected off the band shell.

Write an equation, in standard form, for the hyperbolic band shell shown in the diagram below.



3.5 An engineer is designing a wall-mounted heater. To help disperse heat around the room, she wants to place a hyperbolic reflector behind the heat source. The diagram below shows a preliminary sketch of her design.



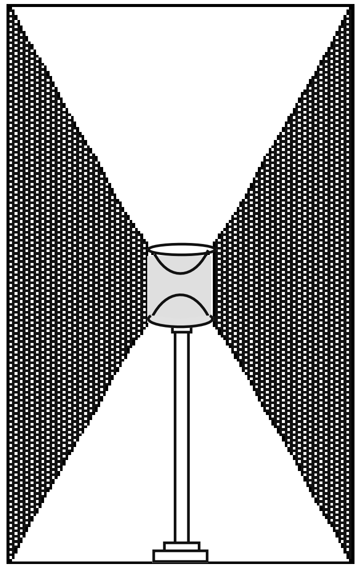
a. On a copy of this diagram, sketch the shape of a hyperbolic reflector that would help disperse heat.

b. Use your sketch to determine the lengths of a and b in centimeters.

c. Write an equation, in standard form, for your hyperbolic reflector.

* * * * *

3.6 The light cast on the wall by the lamp in the illustration below forms a hyperbola.



Trace a copy of this hyperbola onto a sheet of graph paper. Assuming that the center is located at the origin, write an equation that approximately describes the hyperbola.

3.7 Two hyperbolas are **conjugates** if they have the same asymptotes. The following hyperbolas are conjugates of each other.

$$\frac{x^2}{25} - \frac{y^2}{144} = -1 \text{ and } \frac{x^2}{25} - \frac{y^2}{144} = 1$$

a. Graph these hyperbolas on the same coordinate system.

b. Compare the coordinates of the foci of the conjugate hyperbolas.

3.8 The equation $9x^2 - 16y^2 = 144$ describes a hyperbola.

a. Graph this equation.

b. Determine the *x*- and *y*-intercepts of the hyperbola.

c. Find the coordinates of the foci.

d. Rewrite the equation of this hyperbola in standard form.

* * * * * * * * * *

Activity 4

In Activity **1**, you examined the reflective properties of the parabola. In this activity, you investigate how these properties can be used to design a radio telescope.

Mathematics Note

A **parabola** is the set of all points in a plane that are the same distance from a fixed line and a fixed point not on the line. The line is the parabola's **directrix** and the point is its **focus**.

A parabola is symmetric about the line perpendicular to the directrix and passing through the focus. The point where the axis of symmetry intersects the parabola is the **vertex**.

For example, Figure **14** shows a parabola with its focus at point F and vertex at the origin. Its axis of symmetry is the y-axis. For any point P on the parabola, $PQ = PF$.

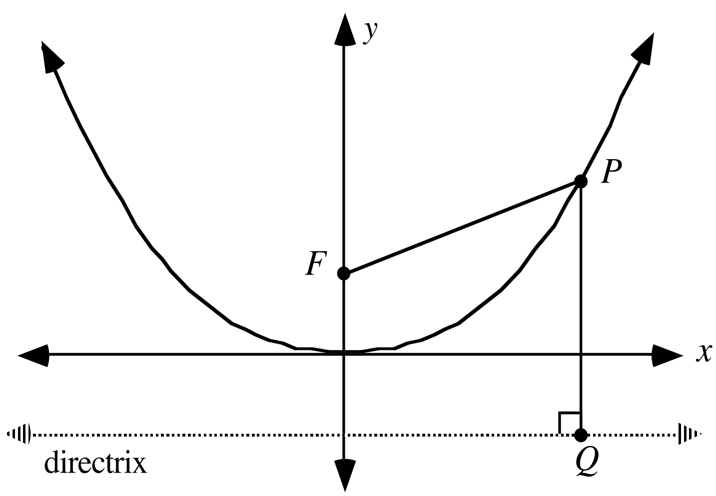


Figure 14: Parabola with vertex at origin

The general equation for a parabola with its vertex at the origin and the y-axis as its axis of symmetry is $y = ax^2$.

Exploration

Imagine that you are an astronomer. Your observatory has just received a grant to build a new radio telescope. A radio telescope consists of a collecting antenna and a radio receiver. Because of the reflective properties of a parabola, you have decided to use a huge collecting dish, 30 m across, in the shape of a paraboloid.

To determine where to place the radio receiver, you must locate the focus of the paraboloid. In this exploration, you investigate a method for relating the general equation of a parabola to the location of its focus.

a. Figure **15** shows a parabola with focus at (0,2) and vertex at the origin.

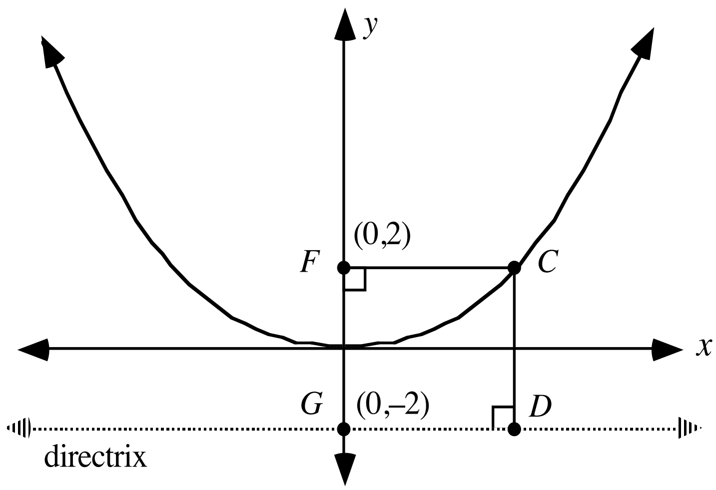


Figure 15: Parabola with focus at (0,2)

1. Given that the coordinates of F are (0,2) and the coordinates of G are (0,–2), determine the coordinates of point C.
2. As mentioned in the previous mathematics note, the general equation for a parabola with its vertex at the origin and the y-axis as its axis of symmetry is $y = ax^2$.

   Substitute the coordinates of point C into the equation $y = ax^2$ and solve for a.
3. Write an equation for the parabola in Figure **15**.

b. Repeat Part **a** given that the coordinates of F are $(0, p)$ and the coordinates of G are $(0, -p)$.

Discussion

a. Given a parabola described by an equation of the form $y = ax^2$, describe how to find the coordinates of the focus.

b. If the coordinates of the focus are $(0, p)$, what happens to the value of p as the value of a changes?

c. Describe what happens to the value of p as the distance between the focus and the vertex increases.

d. Describe what happens to the shape of the parabola as the distance between the focus and the vertex increases.

e. The equation of a parabola is often a function. What type of function defines a parabola?

Assignment

4.1. **a.** Create graphs of the equation $y = ax^2$ for four different values of a. Include two negative and two positive values for a.

b. How does the value of a affect the shape of a parabola?

4.2 Determine the coordinates of the focus for each of the following parabolas.

a. $y = 2x^2$

b. $y = 0.5x^2$

c. $y = -10x^2$

4.3 The equation $x = 3y^2$ describes a parabola that opens to the right.

a. Sketch a graph of this parabola.

b. Determine the coordinates of the focus.

c. Find the equation of the directrix.

d. Is this equation a function? Explain your response.

4.4 Imagine that you are building a radio telescope with a parabolic dish 10 m deep and 30 m in diameter. The radio receiver will be located at the focus of the paraboloid.

a. To identify the receiver's position, complete Steps **1–5** below.

1. Make a scale drawing of a cross section of the dish on a two-dimensional coordinate system. Place the vertex at the origin.
2. Label a point B on the edge of the dish and identify its coordinates.
3. Substitute the coordinates of B into the following equation and solve for p.

$$y = \frac{1}{4p}x^2$$

4. Write an equation, in standard form, for the parabola that describes the cross section of the dish.
5. Determine the coordinates of the focus.

b. During construction of the telescope, cost overruns require you to change the size of the collecting dish. To save money, you must reduce the depth of the dish to 5 m. The diameter of 30 m remains unchanged. Repeat Part **a** for this new design.

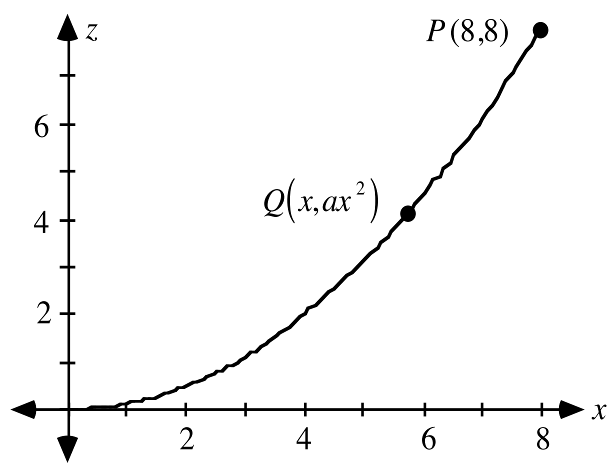
4.5 When soccer fans watch a game on television, they hear the referee's whistles on the field—but not the cries of popcorn vendors in the stands. This is because the camera crew uses directional microphones to pick up the sounds of the game.

a. A directional microphone uses a paraboloid dish to collect sounds at a focus. Sketch a cross section of a paraboloid dish. Identify the location of the microphone. Use your diagram to show how the sound waves are collected.

b. The directional microphones used at sporting events are often small enough to be carried by hand. On your sketch from Part **a**, suggest an appropriate width and depth for the paraboloid dish.

c. Using the width and depth suggested in Part **b**, determine the distance from the vertex to the focus.

4.6 The following graph shows a portion of the parabola $z = ax^2$, where $0 \le x \le 8$.



a. **1.** Determine the value of a.

2. Identify the coordinates of the focus.

b. When this curve is rotated about the z-axis, its path describes a paraboloid. Determine the coordinates, in the form (x,y,z), of the vertex and focus of this paraboloid.

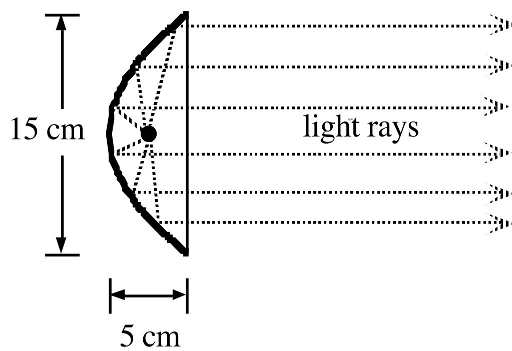
c. As the curve is rotated about the z-axis, the rim of the paraboloid is described by the path of point P. Given that the coordinates of P are (8,8), what is the radius of the rim?

d. The coordinates of point Q are $\left(x, ax^2\right)$. Describe the set of points defined by the path of Q as the curve is rotated about the z-axis.

e. Use a three-dimensional graphing utility to graph the equation $z = a\left(x^2 + y^2\right)$, where a is the value determined in Part **a**. How does this graph appear to be related to the paraboloid in Part **b**?

* * * * *

4.7 You have been asked to design a headlight for a new car. As shown in the diagram below, the headlight consists of a halogen bulb mounted in a parabolic reflector. The reflector should be 5 cm deep, and must produce a beam of parallel light rays 15 cm wide.



a. When the base of the bulb is mounted on the surface of the reflector, the light-producing filament is located 2 cm from the reflective surface. Explain why this location will not satisfy the manufacturer's constraints.

b. How would you redesign the headlight to satisfy the manufacturer's requirements?

4.8 Imagine that you are an engineer working on a new research project. Your job involves gathering radio waves from deep space in an attempt to find signs of other intelligent life. To collect radio waves, you have built a paraboloid dish 80 m in diameter and 25 m deep.

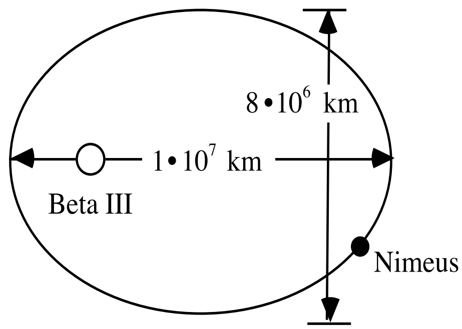
a. The radio receiver must be located at the focus of the dish. How far from the vertex of the paraboloid should you place the receiver?

b. Write an equation, in standard form, for the parabola that describes a cross section of the dish.

* * * * * * * * * *

Summary Assessment

1. The distant star Beta III is orbited by a single planet, Nimeus. Like most planets in the universe, Nimeus follows an elliptical orbit, with its star located at one focus of the ellipse. The Beta III system is shown in the diagram below.



a. Describe the location at which Nimeus is farthest from Beta III.

b. What is the maximum distance between Nimeus and Beta III? What is the minimum distance?

2. In this age of worldwide telecommunications, orbiting satellites reflect television signals back to large sections of the earth. Many of these satellites are located about 36,000 km above the equator, in what is known as the Clarke Belt. This region is named for Arthur C. Clarke, author of *2001: A Space Odyssey*. Written in 1945, this popular novel described television signals bouncing off satellites and back to earth more than 10 years before any country launched an object into space.

a. The antennas that broadcast signals from earth to a satellite are shaped like paraboloids. This allows the signal to be transmitted in focused parallel waves. Explain why this is true.

b. What disadvantages would there be to using a paraboloid reflector to direct television signals from the satellite back to earth?

c. What conic section would you choose to reflect the television signal back to earth? Defend your response.

d. Assume that the focus of the satellite's reflector is 3 m from its vertex. If television signals can be received at any point on the earth with an unobstructed line of sight to the satellite, determine an equation that describes the curve of the reflector. (The earth's diameter at the equator is about 13,000 km.)

Module Summary

- A **conic section** can be formed by the intersection of a plane with a double-napped cone. Depending on the slope of the plane, the intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.
- Each conic divides the plane into regions. The **interior** of a conic is the region (or regions) that contains at least one focus. The **exterior** of a conic is the region that does not contain a focus.
- **Spheres**, **paraboloids**, **hyperboloids**, and **ellipsoids** are the three-dimensional counterparts of the conic sections.
- An **ellipse** is a set of points in the plane such that the sum of the distances from each point to two foci is a constant.
- The **major axis** of an ellipse is the segment, with endpoints on the ellipse, that contains the foci. The **minor axis** is the segment, with endpoints on the ellipse, contained in the perpendicular bisector of the major axis. The major and minor axes intersect at the **center** of the ellipse. The endpoints of the major and minor axes are the **vertices** of the ellipse.
- The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *x*-axis is:

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

  where a is half the length of the major axis and b is half the length of the minor axis.
- The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *y*-axis is:

  $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

  where a is half the length of the major axis and b is half the length of the minor axis.
- The **standard form** of the equation of a circle with center at the origin and radius r is $x^2 + y^2 = r^2$.
- A **hyperbola** is a set of points in the plane such that the difference of the distances from each point to two foci is a constant.

- The **vertices** of a hyperbola are the intersections of the hyperbola and the segment that joins the foci. The **transverse axis** is the line that passes through the vertices. The **conjugate axis** is perpendicular to the transverse axis and bisects the segment joining the vertices at the **center** of the hyperbola.
- The **standard form** of the equation of a hyperbola with center at the origin and foci at $(-c, 0)$ and $(c, 0)$ is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

  where a is the distance from the center to a vertex and $b^2 = c^2 - a^2$.
- An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line decreases to zero as the distance from P to the origin increases without bound, where P is on a suitable part of the curve.
- The asymptotes for a hyperbola with an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

  are the lines $y = (b/a)x$ and $y = -(b/a)x$.
- A **parabola** is the set of points in a plane that are the same distance from a fixed line and a fixed point not on the line. The line is the parabola's **directrix** and the point is its **focus**.
- A parabola is symmetric about the line perpendicular to the directrix and passing through the focus. The point where the axis of symmetry intersects the parabola is the **vertex**.
- The **standard form** of the equation of a parabola with vertex at the origin and focus on the y-axis is:

$$y = \frac{1}{4p}x^2$$

  where p is the directed distance from the vertex to the focus.

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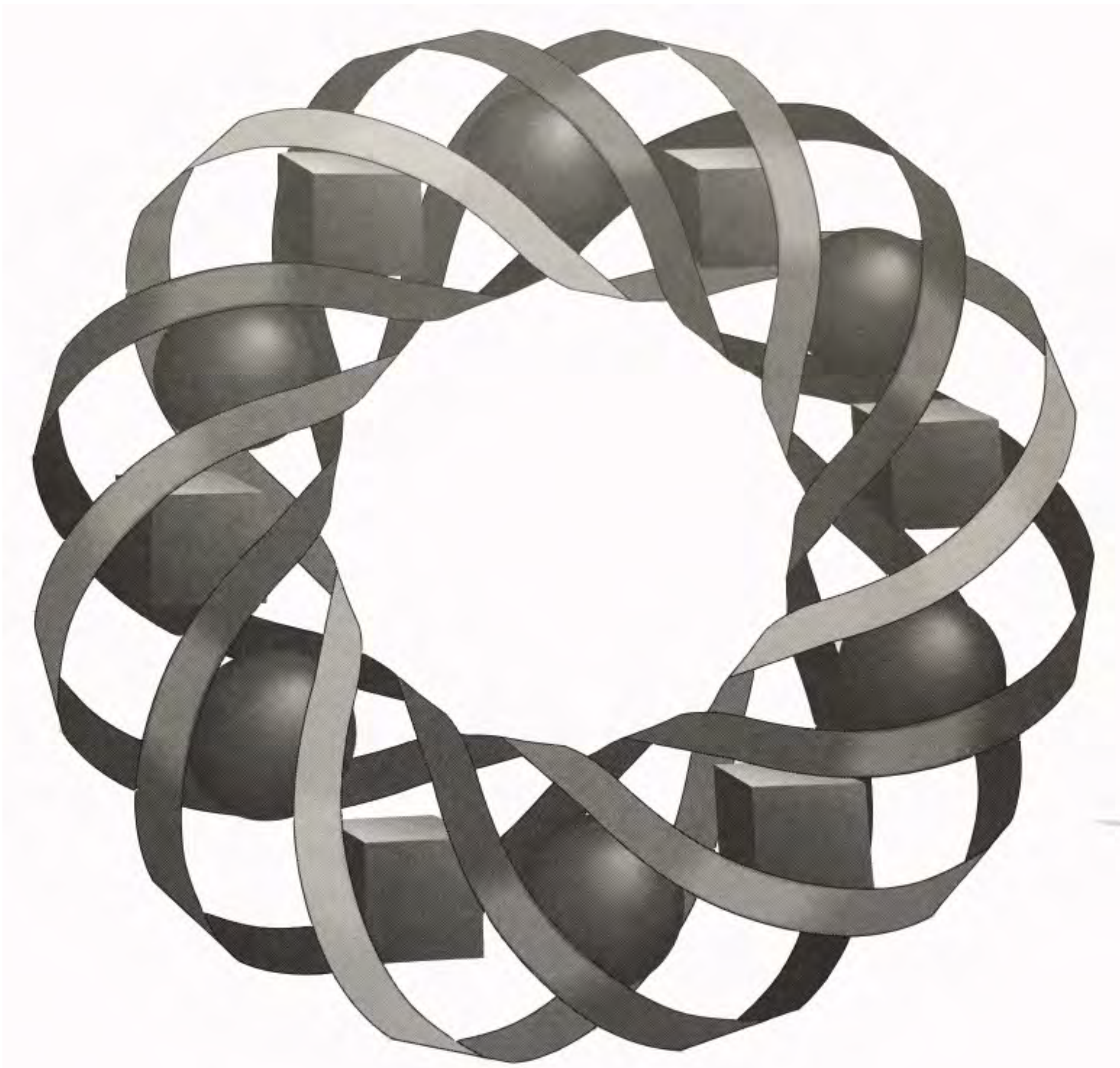
Risky Business



Why buy automobile insurance? The rising costs of accidents—and an examination of their probabilities—can help explain why people choose to share the economic risks of driving their cars.

Randy Carspecken • Todd Fife

Risky Business

Introduction

Since cars and driving play prominent roles in our society, automobile accidents are everyone's concern. The annual cost of auto accidents amounts to billions of dollars. Often, the cost of even a single accident can be too high for the average family budget. Insurance companies can be used to help with these costs.

> **Business Note**
>
> Insurance companies provide protection against the costs of accidents in exchange for a fee. The fee for this service is an insurance **premium**, and the contract with the insurance company is an **insurance policy**. The people who pay insurance premiums are **policyholders**.

Figure **1** shows an example of a premium renewal notice for an automobile driven by an individual in the 16–20 age category.

SIMMS Insurance Company

Car Policy Number: 542–45983325

Continuous Renewal Policy				
Effective		Expires	Classification	1121
05-08	to	11-08	Term (Month)	06
12:01 A.M. Standard Time			Acc/Viol points	0/0
			Territory	14

Coverage is subject to the terms of the policy*

Coverage	**Limit of Liability**		**Premium**
Bodily Injury Liability	Each Person	$100,000	$95.30
	Each Occurrence	$300,000	
Property Damage Liability	Each Occurrence	$ 50,000	$46.20
Medical Payments	Each Person	$ 5,000	$23.60
Comprehensive	Actual cash value / $100 deductible		$40.50
Collision	Actual cash value / $250 deductible		$95.20
Uninsured Motor Vehicle	Each Person	$ 25,000	$15.70
Bodily Injury	Each Accident	$ 50,000	
		Total Premium	$316.50
		Previous Balance	$ 0.00
		Premium Charge	$316.50
		Payment Credit	$ 0.00
		Current Balance	$316.50
			*************

*Your premium is based on the following: Principal driver under 21.
If not correct, please contact your agent.

Figure 1: Premium renewal notice

Discussion

a. How would you calculate the annual premium of the policy in Figure **1**?

b. Does this premium seem reasonable?

c. How do you think this premium would compare with the premium for a driver in the 35–44 age category?

d. If the policyholder who received this renewal notice has an accident, do you think the premium will change on the next renewal notice?

e. What factors do you think are considered by an insurance company when determining an individual's premium?

Business Note

Bodily injury liability insurance pays for any individual who is injured as a result of the negligent operation of the insured vehicle.

Collision insurance covers the cost of repairing the insured vehicle when the damage is due to its negligent operation.

Comprehensive insurance covers natural damage (by floods, hail, or storms), theft, or vandalism to the insured car.

Property damage liability insurance covers the cost of damages to another person's property caused by the negligent operation of the insured vehicle.

f. Give an example of an accident that would be covered by each type of insurance described in the business note above.

g. What additional information appears on the renewal notice? What do you think this information means?

Activity 1

Driving an automobile involves risk. Billboards, news stories, and magazine articles provide daily reminders of the hazards of the road. In this activity, you use statistics and probability to simulate the risks of driving.

Exploration

To determine the probability of a policyholder having an accident, insurance companies use accident statistics like those in Table **1**. In this exploration, you investigate the importance of considering the number of policyholders when making predictions based on such probabilities.

Table **1** summarizes U.S. automobile accident statistics for 1992. The entries in the table are given per 100,000 licensed drivers. For example, the entry in the right-hand column of the third row indicates that there were 6951 accidents for every 100,000 licensed drivers in the 25–34 age category. Of this total, 2531 were the bodily-injury category, and 4420 were the property-damage-only category.

Table 1: Accidents per 100,000 licensed drivers in 1992

Age	Bodily Injury (fatal and non-fatal)	Property Damage Only	Total
16–20	5753	10,022	15,775
21–24	3688	6687	10,375
25–34	2531	4420	6951
35–44	1945	3580	5525
45–54	1757	3432	5189
55–64	1322	2587	3909
65–69	1235	2119	3354
older than 69	1408	2353	3761

Source: National Highway Traffic Safety Administration, 1992.

a. Use the data in Table **1** to estimate the probability that a randomly selected driver in the 16–20 age category was involved in an automobile accident in 1992. Round the probability to the nearest hundredth.

b. Use a random number generator to design a simulation that predicts the driving record of a person in the 16–20 age category for one year.

c. Suppose an insurance company has *n* policyholders in the 16–20 age category. While it is not possible to predict exactly how many of these drivers will be involved in an accident during the year, insurance companies need to know the approximate percentage of their policyholders who will have accidents.

1. Use the simulation developed in Part **b** to model the driving records of different numbers of policyholders for 1 year. Record the numbers of accidents in a table with headings like in Table **2**.

Table 2: Simulation results

Number of Policyholders	Number of Accidents	Accident Rate
5		
25		
50		
100		
250		
500		
1000		

2. For each row in Table **2**, determine the ratio of the number of accidents to the number of policyholders. This is the accident rate. Record these ratios, in decimal form, in Table **2**.

d. The percentage of drivers in a specific age group who have accidents will vary from year to year. Assume that each class member's simulation represents a different year for the same age group.

To see how the accident rate can vary, collect the class data and determine the difference between the smallest and largest accident rates obtained for each value of n.

Discussion

a. Why is it reasonable to use the data in Table **1** to determine the probability that a randomly selected driver in the 16–20 age category has an accident?

b. Based on the data in Table **1**, how do the driving records of groups in the various age categories compare?

c. What are some limitations of the simulation you developed in the exploration for modeling actual driving records?

d. Compare the accident rates for different numbers of policyholders obtained in Part **c** of the exploration.

e. Describe the trend in the differences between the smallest and largest accident rates obtained in Part **d** of the exploration as the number of policyholders increases.

f. Compare the accident rates for each number of policyholders with the probability calculated in Part **a** of the exploration.

g. If you continued this simulation using larger numbers of policyholders, what number do you think the accident rate would approach?

Mathematics Note

The **law of large numbers** indicates that, if a large random sample is taken from a population, the sample proportion has a high probability of being very close to the population proportion.

For example, it would not be uncommon for a random sample of 10 children from a population with 50% females to contain 8 females and 2 males (a sample proportion of 0.8). On the other hand, it is highly unlikely that a random sample of 10,000 children from this population would contain 8000 females and 2000 males. For large sample sizes, it is much more likely that the proportion of females in the sample would be close to 0.5, the population proportion.

h. Would it be practical for an insurance company to use statistics based on a sample of 100 drivers to determine the probability of 1 driver being involved in an accident?

i. Based on the data in Table **1**, the probability that a randomly selected driver in the 16–20 age category had an accident during 1992 is approximately 0.16. Do you think this is a reliable statistic? Explain your response.

Assignment

1.1 Assume that the probability that a randomly selected newborn baby is a girl is 0.5.

a. Design a simulation that predicts the gender of a randomly selected newborn baby.

b. Use your simulation to complete the following table. Express the proportion of girls to total births to the nearest hundredth.

Number of Births	Proportion of Girls
10	
20	
40	
80	
100	
200	
300	
400	
500	

c. **1.** Create a connected scatterplot of the proportion of girls versus the number of births.

2. Graph the line $y = 0.5$ on the same coordinate system. This line represents the probability that a randomly selected newborn is a girl.

d. Does your graph in Part **c** illustrate the law of large numbers? Explain your response.

1.2 An actuary is a mathematician employed by an insurance company to predict events on which policy premiums are based.

a. Why is the accident rate important when determining premiums?

b. Why would an actuary consider a larger group a better basis for making predictions than a smaller group?

1.3 Insurance companies compile accident data from previous years. This historical data is used to determine the probabilities of drivers being involved in accidents in the future. To make predictions, insurance companies treat these probabilities as theoretical probabilities for the current year.

What are some limitations of using the previous year's accident statistics to make predictions about the current year?

1.4 **a.** The following table shows the probabilities that a driver in the 16–20 age category has a bodily-injury accident, a property-damage-only accident, or no accident. All the values in the first row are based on information contained in Table **1**. Explain how these probabilities were calculated.

Accident Probabilities				
Age	**Bodily Injury**	**Property Damage Only**	**No Accident**	**Total**
16–20	0.06	0.10	0.84	1
21–24				
25–34				
35–44				
45–54				
55–64				
65-69				
older than 69				

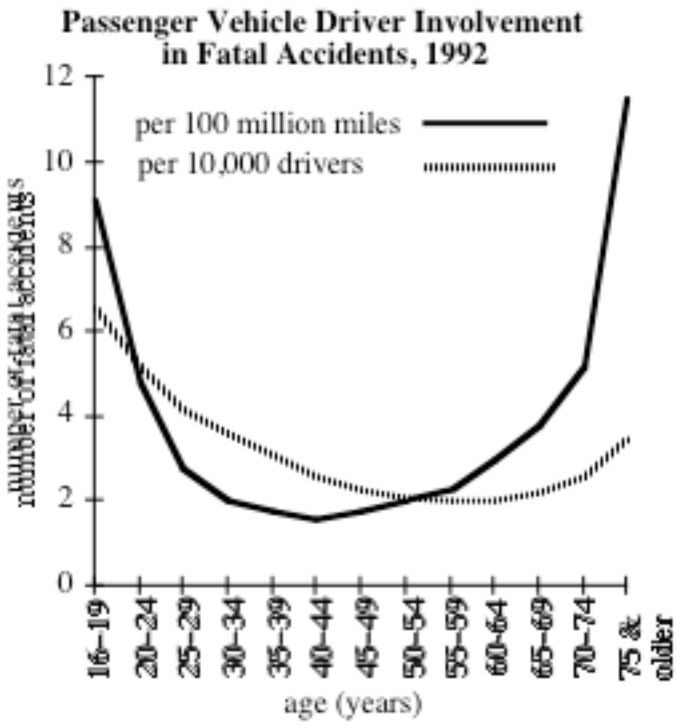
b. What is represented by the 1 in the column with the heading "Total"?

c. Copy and complete the table using the data from Table **1.**

d. Write a paragraph comparing the probabilities of having a bodily-injury or property-damage-only accident as the age of the driver increases. Discuss some possible reasons for the differences in these probabilities.

1.5 As shown in the following graph, the number of fatal automobile accidents in 1992 varied among age categories. One curve displays the number of fatal accidents for every 100 million miles driven. The other illustrates the number of fatal accidents for every 10,000 licensed drivers.

For example, drivers in the 16–19 age category had about 9 fatal accidents for every 100 million miles driven, and about 6 fatal accidents for every 10,000 licensed drivers. In the 65–69 age category, there were about 4 fatal accidents for every 100 million miles driven, and about 2 fatal accidents per 10,000 drivers.



Source: Insurance Institute for Highway Safety, 1992.

a. What statistic in the graph seems to indicate that drivers in the "75 and older" category are better drivers than teenage drivers?

b. What statistic in the graph seems to indicate that teenage drivers are better drivers than those in the "75 and older" category?

c. **1.** Depending on the statistics you select, there are two different age categories with the best driving records. Identify those categories and explain why this is possible.

2. Based on the information in the graph, which age category do you believe has the best driving skills? Explain your response.

d. Using the statistics in the graph, describe the risks encountered over a lifetime of driving. Your answer should include some possible explanations for the changes in risk over time.

* * * * *

1.6 **a.** Use the following statistics to estimate the probability that a randomly selected member of the population from which the samples were taken will have the given characteristic.

1. In a random sample of 1084 adults, 813 indicated that they believed there is too much violence on television.

2. In a nationwide survey of vehicle accidents, a random sample of 64,000 accidents included 41,600 involving property damage of $800 or less.

3. Among a sample of 90 randomly selected hospital patients, 41 had type O blood.

4. In a random sample of 750 taxpayers with incomes under $100,000, the Internal Revenue Service had audited 25.

b. **1.** Which estimate in Part **a** do you believe is the most reliable? Explain your response.

2. Which estimate do you believe is the least reliable? Explain your response.

1.7 **a.** If a coin was flipped 5 times and came up heads 4 times, would you suspect that the coin was unfair? Explain your response.

b. Create a simulation that models 1000 flips of a coin. Run this simulation 10 times and record the number of times 800 or more heads are counted.

c. If the coin from Part **a** was flipped 10,000 times and 8000 of the tosses were heads, would you suspect the coin was unfair? Explain your response.

* * * * * * * * * *

Activity 2

The cost of an automobile accident can range from a few dollars to well over a million. Typically, insurance companies pay part of these costs for insured drivers. To obtain benefits, policyholders must file an insurance **claim**. A claim reports the details and costs of an accident and requests payment. The claim value is the amount paid by the company.

If you were involved in an accident, how much might you expect your claim value to be? To answer this question, you will first examine some historical data.

Exploration 1

Insurance companies consider all claims made in a year and use the information from these claims to set premium costs for future years. In this exploration, you examine a method used by insurance companies to compile this information.

a. Table **3** below shows 40 bodily injury claims filed with an insurance company in 1990.

Table 3: Insurance claims in 1990, in dollars

3,300	15,236	15,461	570
562	175	4,589	1,131
7,200	1,802	4,200	250
888	52	3,563	111
3,900	3,250	2,252	23,898
1,402	290	700	8,550
89	4,300	4,602	388
12,889	5,300	1,244	20,500
20,336	995	13,202	94
410	23,555	4,001	1,452

1. Sort these claim values from least to greatest.
2. Divide the sorted claim values into 10 sets of 4 claim values each. These 10 sets are **deciles**.

b. Calculate the mean of each decile. Record the means in a table with headings like those in Table **4** below.

Table 4: Mean claim amounts for 1990, by decile

Decile	Claim Value ($)
1	
2	
3	
⋮	
10	

c. From 1985 to 1990, the consumer price index increased at an average annual rate of 4.1%. Assuming that this trend continues, estimate the mean claim amounts, by decile, for 1992.

Discussion 1

a. Suppose that an insurance company compiles data for 25 million claims in a given year. Describe the process of finding the mean decile values for that year.

b. If each decile represented 1 million claims, how could you determine the total number of claims in the data set?

c. Describe the model you used to estimate the mean claim amounts for 1992.

Exploration 2

After collecting and analyzing historical data, insurance companies must determine the amount in claims they can expect per policyholder.

a. The information in Table **5** summarizes data from approximately 25 million bodily-injury claims and 120 million property-damage claims.

Table 5: Mean claim amounts in 1992, by decile

Decile	Bodily Injury ($)	Property Damage ($)
1	125	47
2	378	150
3	753	266
4	1400	389
5	2300	527
6	3550	699
7	5400	936
8	8000	1313
9	13,250	2061
10	24,380	3462

1. Determine the number of property-damage claims represented in each decile.
2. Determine the probability that a randomly selected bodily-injury claim falls in the third decile.

Mathematics Note

A **random variable** *X* is a variable that takes on each of its possible values with a specific probability. Given possible values for *X* of $x_1, x_2, \ldots, x_k$, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.

For example, consider an insurance company that pays $6000 for a bodily-injury accident, $1000 for a property-damage-only accident, and $0 for no accident. A random variable *X* could be used to represent claim values as follows: $x_1 = \$6000$, $x_2 = \$1000$, and $x_3 = \$0$.

A **probability distribution** for a random variable *X* assigns probabilities $p_1, p_2, \ldots, p_k$ to the values $x_1, x_2, \ldots, x_k$ for *X*.

For example, if the probabilities for a bodily-injury accident, property-damage-only accident, and no accident in a given year are 0.08, 0.02, and 0.90, respectively, then the probability distribution for *X* is shown in Table **6**.

Table 6: Probability distribution for *X*

Outcome	Bodily Injury	Property Damage	No Accident
Value of X	\$6000	\$1000	\$0
Probability	0.08	0.02	0.90

The **expected value** or **mean** of a random variable *X*, denoted $E(X)$, is the sum of the products of each possible value of *X* and its respective probability:

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_kp_k$$

Using the probability distribution in Table **6**, for example, the expected value of the claims per insured driver during one year can be found as follows:

$$E(X) = \$6000(0.08) + \$1000(0.02) + \$0(0.90) = \$500$$

b. Table **7** shows the probabilities of a bodily-injury accident, a property-damage-only accident, or no accident for a driver in the 16–20 age category. (These are the values you found in Problem **1.4** using the information from Table **1**.)

Table 7: Accident probabilities for the 16-20 age category

Bodily Injury	Property Damage Only	No Accident
0.06	0.10	0.84

A claim value selected at random is equally likely to fall in any given decile. Therefore, the probability of a driver having a claim value for a bodily-injury accident in any given decile is $0.06(1/10) = 0.006$. The probability of a driver having a claim value for a property-damage-only accident in any given decile is $0.10(1/10) = 0.01$.

Using the 1992 claim values from Table **5**, create a probability distribution for the random variable *C*, where *C* is assigned to the set of claim values for a driver in the 16–20 age category. As noted in Table **8** below, the claim value is \$0 if the driver does not have an accident.

Table 8: Probability distribution for *C*

	Bodily Injury		Property Damage	
Decile	**Claim (\$)**	**Probability**	**Claim (\$)**	**Probability**
1		0.006		0.01
2		0.006		0.01
⋮		⋮		⋮
10		0.006		0.01
	No Accident			
	\$0	0.84		

c. Determine the expected value of *C*.

Discussion 2

a. Describe the significance to an insurance company of the value found in Part **c** of the exploration.

b. In Part **b** of Exploration **2**, how many possible values are there for the random variable C?

c. Explain how an insurance company might use the expected value of C to determine an annual premium for drivers in a given age category.

d. A weighted mean is a representative value for a set of numbers in which each number may be assigned a different relative importance.

For example, suppose that a teacher has decided that each of three research projects should count for $1/5$ of a student's final grade, while each of four test scores determine $1/10$ of the grade. If a student receives scores of 76, 87, and 92 on the research projects, and 88,75, 82, and 96 on the tests, the final grade could be calculated as follows:

$$0.2(76)+0.2(87)+0.2(92)+0.1(88)+0.1(75)+0.1(82)+0.1(96)\approx 85$$

Compare the expected value of a random variable to a weighted mean.

Assignment

2.1 **a.** The data in Table **5** summarizes approximately 25 million bodily-injury claims and 120 million property-damage claims. Estimate the total value of these claims.

b. In 1992, the U.S. population was approximately 256 million. Use your estimate from Part **a** to determine the cost per person represented by the claims in Table **5**.

2.2 **a.** From 1992 to 1997, the consumer price index increased at an average annual rate of 2.6%. Given this fact, determine a model you could use to predict claim values in future years.

b. Use your model and the data in Table **5** to predict mean decile values for the current year. **Note:** Save your work for use in Problems **2.3** and **2.4**.

c. The table below shows the probabilities of a bodily-injury accident, a property-damage-only accident, or no accident for a driver in the 21–24 age category.

Bodily Injury	Property Damage Only	No Accident
0.04	0.07	0.89

Use this table and the predicted values from Part **b** to create a probability distribution, then find the expected claim value for a driver in the 21–24 age category.

2.3 **a.** Use the predicted decile values from Problem **2.2b** to find a mean bodily-injury claim value and a mean property-damage-only claim value for the current year.

b. Let the random variable C represent the claim values for a driver in the 21–24 age category during a given year. Use the mean claim values from Part **a** to complete the probability distribution for C.

Probability Distribution for *C*			
Outcome	**Bodily Injury**	**Property Damage**	**No Accident**
Value of *C*			\$0
Probability	0.04	0.07	0.89

c. Determine the expected value of C and describe what it represents to an insurance company.

2.4 In Problem **2.3c**, expected value was found by multiplying each of the decile values by its corresponding probability. Each of these probabilities was determined by dividing a given probability by 10, the number of equally likely outcomes in that category. This method can be represented using the following equation:

$$E(C) = x_1 \bullet \left(\frac{p_1}{10}\right) + x_2 \bullet \left(\frac{p_1}{10}\right) + \cdots + x_{10} \bullet \left(\frac{p_1}{10}\right)$$
$$+ x_{11} \bullet \left(\frac{p_2}{10}\right) + x_{12} \bullet \left(\frac{p_2}{10}\right) + \cdots + x_{20} \bullet \left(\frac{p_2}{10}\right)$$
$$+ 0 \bullet p_3$$

In Problem **2.3c**, expected value was found by averaging the claim values for each category, then multiplying each mean by the given probability. This method can be represented using the following equation:

$$E(C) = \left(\frac{x_1 + x_2 + \cdots + x_{10}}{10} \bullet p_1\right) + \left(\frac{x_{11} + x_{12} + \cdots + x_{20}}{10} \bullet p_2\right) + 0 \bullet p_3$$

Explain why these two methods yield the same value.

2.5 In a given year, the probability that a driver in the 65–69 age category is involved in a bodily-injury accident is about 0.01. The probability that a driver in the same age category has a property-damage-only accident is about 0.02. The probability that a driver is not involved in an accident is 0.97.

a. Use the predicted mean values from Problem **2.3a** to create a probability distribution table for drivers in the 65–69 age category.

b. Let the random variable C represent the claim values for a driver in the 65–69 age category during a given year. Determine the expected value of C and describe what it represents to an insurance company.

c. Use your results in Problems **2.3c** and **2.5b** to explain why drivers in the 21–24 age category typically pay higher insurance premiums than drivers in the 65–69 age category.

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2.6 Consider a baseball player with a lifetime batting average of 0.315.

a. Assume that the probability this player gets a hit on the next at-bat is $P(\mathrm{H}) = 0.315$. Determine the probability that the player does *not* get a hit, or $P(\mathrm{N})$.

b. In an upcoming game, the player will have four turns at bat. One possible outcome in this situation is HNHH. List all the possible outcomes for the four at-bats and, assuming that they are independent events, determine the corresponding probabilities.

c. Complete the following probability distribution table for the player's next four at-bats.

No. of Hits	Probability
0	
1	
2	
3	
4	

d. How many hits do you predict that this player will get in the next four at-bats? Explain your response.

2.7 According to one math teacher's grading system, group work counts for 20% of the final grade, projects 30%, and tests 50%.

a. A student's scores for the class are shown in the table below. Use this information to determine the final grade.

Group Work	Projects	Tests
78	85	92
95	75	88
82	99	95
75	89	79
98	82	
83	65	
85		
90		
74		

b. Compare this teacher's method of determining final grades to the calculation of expected value.

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Activity 3

Whenever you choose to drive, there is a risk of having an accident. The potential costs of an accident—as well as the law, in many states—motivates most drivers to insure themselves. By insuring a large number of drivers, insurance companies provide a service that allows policyholders to share these financial risks with others.

Exploration

By simulating the operation of an insurance company, you may gain insight into some issues concerning insurance premiums.

Suppose that your class decides to create its own insurance company in which every member of the class is insured for both bodily injury and property damage.

Using statistics from previous years, the class determines that the probability of having a bodily-injury claim for a policyholder in the 16–20 age category is 0.06. The probability of having a claim for property damage only is 0.1. Since the total probability of filing a claim is $0.06 + 0.1 = 0.16$, the probability of not filing a claim is $1 - 0.16 = 0.84$.

To simplify the operation of the company, the class has decided to limit claim payments to the 20 values listed in Table **9**. If no claim is filed, the claim value is $0.

Table 9: Probability distribution table for *C*

	Bodily Injury		**Property Damage Only**	
Decile	**Claim Value ($)**	**Probability**	**Claim Value ($)**	**Probability**
1	208	0.006	22	0.01
2	572	0.006	88	0.01
3	1105	0.006	196	0.01
4	2050	0.006	319	0.01
5	3079	0.006	453	0.01
6	4501	0.006	615	0.01
7	6861	0.006	835	0.01
8	9771	0.006	1189	0.01
9	15,266	0.006	1909	0.01
10	26,642	0.006	3567	0.01
	No Accident			
	0	0.84		

a. The class wants to keep insurance premiums as low as possible. To analyze this situation, they use the random variable C to represent the 21 possible claim values.

Since the expected value of C represents the mean annual claim per policyholder, the class decides to use this value as the annual premium. Determine $E(C)$.

b. Design a simulation of the class insurance company for a one-year period. For each policyholder, your simulation should complete the following sequence of steps:

1. Determine if the policyholder files a claim during the year.

2. If a claim is filed, determine if it is for bodily injury or for property damage only. Hint: Consider the ratios $0.06/0.16 = 0.375$ and $0.1/0.16 = 0.625$.

3. If the claim is for bodily injury, determine which of the 10 claim amounts the company will pay. Similarly, if the claim is for property damage only, determine which of the 10 claim amounts the company will pay.

4. Record the cost of the driver's claim, if any, for the year.

c. Run the simulation once to simulate your driving record for the year.

d. **1.** Collect the class data and determine the sum of the claim values for the entire class.

2. Calculate the claims cost per policyholder for the year.

3. Using the annual premium from Part **a**, determine the profit or loss per policyholder for the year.

4. Determine the total profit or loss for the year for the class insurance company.

e. Repeat Part **c** nine more times, to obtain data for a total of 10 years. Compile the class data for each year. Determine the mean value of each of the following for the 10 simulations:

1. the claims cost per policyholder

2. the profit or loss per policyholder

3. the total profit or loss for the year

Note: Save your work for use in the assignment.

f. Suppose the number of policyholders increases to 100. To examine how this might affect premiums, use your simulation to model the claims of 100 policyholders over 10 years. For each year, determine the mean values described in Part **e** above.

Note: Save the results for use in the assignment.

Discussion

a. What are some of the limitations of your simulation in terms of modeling the operation of an actual insurance company?

b.
1. Why is it not reasonable for an insurance company to set its premiums equal to the expected value of claims per policyholder?
2. Would it be reasonable for the company to set premiums much higher than the expected value of claims?

c. How did your company's ability to pay claims for the class compare with its ability to pay claims for 100 policyholders?

d. A typical insurance company insures a relatively large group of people. What are the advantages and disadvantages of insuring small groups like those in the exploration?

e. Compare the means calculated for the two different groups in Parts **e** and **f** of the exploration. How would you expect this statistic to change for 10,000 policyholders?

f. How does the law of large numbers relate to the data you collected in the exploration?

g. Based on your results in the exploration, what annual premium would you recommend to insure your class only? to insure a group of 100 policyholders?

Assignment

3.1 Imagine that you are employed by an insurance company. The chief executive officer (CEO) requests a report on the predictability of insurance claims. Write a report for the CEO, using the results of the exploration to support your explanation.

3.2 Insurance companies typically use 65% of the premiums they collect to cover claim costs. This part of the premium is the *pure premium*. Another 30% covers operating costs, or *overhead*. The remaining 5% of the premiums collected represents profit.

- **a.** Explain why the annual premium used in your simulation is lower than what an actual insurance company would charge.
- **b.** What premium would a typical insurance company charge in a situation in which the claim cost per policyholder was $512?
- **c.** Estimate the annual profit for a typical insurance company providing coverage for your class.

3.3 An insurance company's *loss ratio* is defined as follows:

$$\frac{\text{total claims paid}}{\text{total premiums collected}}$$

a. Would an insurance company want its loss ratio to be large or small? Explain your response.

b. Use your responses to Problem **3.2** to explain why an insurance company's loss ratio is typically 65%.

c. Use the results of the simulations in the exploration and an annual premium of $788 to find the largest and smallest loss ratio for each of the following:

1. your class

2. 100 people

3. 1 million people (estimated).

d. Use the law of large numbers and your responses from Part **c** to describe some considerations insurance companies may have regarding numbers of policyholders.

3.4 Consider an insurance company with 3500 policyholders. The average annual claim is $707. The probability that a policyholder will file a claim is 9%. Considering the percentages described in Problem **3.2**, suggest an appropriate premium for this insurance.

3.5 The following table shows the annual number of accidents per 100,000 licensed drivers of eight different age groups.

Number of Accidents per 100,000 Drivers			
Age	**Bodily Injury**	**Property Damage Only**	**Total**
16–20	5753	10,022	15,775
21–24	3688	6687	10,375
25–34	2531	4420	6951
35–44	1945	3580	5525
45–54	1757	3432	5189
55–64	1322	2587	3909
65–69	1235	2119	3354
older than 69	1408	2353	3761

The table below shows predicted decile values for insurance claims in the same population.

Decile	Bodily Injury Claim ($)	Property Damage Claim ($)
1	150	100
2	442	255
3	873	361
4	1622	474
5	2591	613
6	3925	795
7	5975	1049
8	8725	1450
9	14,125	2225
10	25,415	3513

a. An insurance company plans to offer coverage for both bodily injury and property damage to this population. Use the data given, along with the percentages in Problem **3.2**, to suggest an annual insurance premium for a 17-year-old driver. Describe how you determined your response.

b. Repeat Part **a** for a 67-year-old driver.

c. Compare the premium of the 67-year-old with the premium of the 17-year-old.

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3.6 A life insurance company sells term insurance policies to 20-year-old males that pay $40,000 if the policyholder dies within the next 5 years. The company collects an annual premium of $200 from each policyholder.

The following table shows the probability distribution for X, the random variable that represents the company's income (or loss) per policyholder. In this case, x_1 represents the income if the policyholder dies during the first year (at age 20), while x_6 represents the income if the policyholder dies after the policy expires (at age 26 or older).

Income (or Loss)	Probability
$x_1 = -\$39,800$	0.00175
$x_2 = -\$39,600$	0.00181
$x_3 = -\$39,400$	0.00184
$x_4 = -\$39,200$	0.00189
$x_5 = -\$39,000$	0.00191
$x_6 = \$1000$	p_6

a. Determine the probability that the policyholder will die after the policy expires.

b. Calculate $E(X)$, the company's expected income per policyholder.

c. **1.** Explain why it might be risky for this company to insure only one policyholder.

2. Explain why it is not as risky for the company to insure 10,000 policyholders.

3. How much income would the company expect from insuring 10,000 policyholders?

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Research Project

Many insurance companies give discounts for drivers or cars that fit certain criteria. For example, since a car with airbags is considered safer than a comparable car without airbags, a company might offer lower premiums for this feature.

a. Compile a list of the discounts that might apply to you as a driver.

b. Determine the amount that premiums are typically reduced for each type of discount in Part **a**.

c. Comment on the relationship between each discount and the probability of having a claim or the potential value of the claim.

d. Determine the annual premium you would pay—including any applicable discounts—for comprehensive coverage on each of the following types of vehicles:

1. a sports car

2. a luxury car

3. a family car.

Summary Assessment

Cars are not the only possessions that can be expensive to repair or replace. As a result, many people buy insurance for their homes, jewelry, and appliances. Because of the potential for water damage, some even buy insurance for their waterbeds. In fact, it's possible to insure almost anything—from a champion race horse to a concert pianist's hands.

For example, contact lenses are especially easy to lose or damage. Imagine that you own an insurance business that provides coverage for contact lens. Before setting your premiums, you do some research on contact lens claims. Here are the statistics:

- An average of 10 out of every 100 people with contact lens insurance files a claim for damage or loss each year.
- The first decile of claims averages $75.00, deciles 2–7 average $125.00, and deciles 8–10 average $300.

Continue your analysis by completing the following steps.

1. Develop a simulation for annual contact lens claims. Your simulation should utilize a random variable and a probability distribution table. It also should identify mean claim values and be able to model different numbers of policyholders.

2. Use your simulation to compare the mean claim cost per policyholder, as well as the range in claim costs per policyholder, for different numbers of policyholders.

3. Suggest a pure premium and a full premium for this insurance coverage and describe how you determined these values.

4. Use the results of your simulation and the law of large numbers to describe how the number of policyholders affects your ability to earn a predictable profit.

Module Summary

- The **law of large numbers** indicates that, if a large random sample is taken from a population, the sample proportion has a high probability of being very close to the population proportion.
- A **random variable** X is a variable that takes on each of its possible values with a specific probability. Given possible values for X of $x_1, x_2, \ldots, x_k$, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.
- A **probability distribution** for a random variable X assigns probabilities $p_1, p_2, \ldots, p_k$ to the values $x_1, x_2, \ldots, x_k$ for X.
- The **expected value** or **mean** of a random variable X, denoted as $E(X)$, is the sum of the products of each value of X and its respective probability:

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_kp_k$$

- Insurance companies provide protection against part of the cost of an accident in exchange for a fee. The fee for this service is an insurance **premium**, and the contract with the company is called an **insurance policy**. The people who pay insurance premiums are **policyholders**.
- **Bodily injury liability insurance** pays for any individual who is injured as a result of the negligent operation of the insured vehicle.
- **Collision insurance** covers the cost of repairing the insured vehicle when the damage is due to its negligent operation.
- **Comprehensive insurance** covers natural damage (by floods, hail, or storms), theft, or vandalism to the insured car.
- **Property damage liability insurance** covers the cost of damages to another person's property caused by the negligent operation of the insured vehicle.
- To obtain benefits, policyholders must file an insurance **claim**. The claim reports the details and costs of an accident and requests payment.
- The percentage of insurance premiums used to cover claim costs is the **pure premium**. The operating costs of a business are its **overhead**.
- An insurance company's **loss ratio** is defined as

$$\frac{\text{total claims paid}}{\text{total premiums collected}}$$

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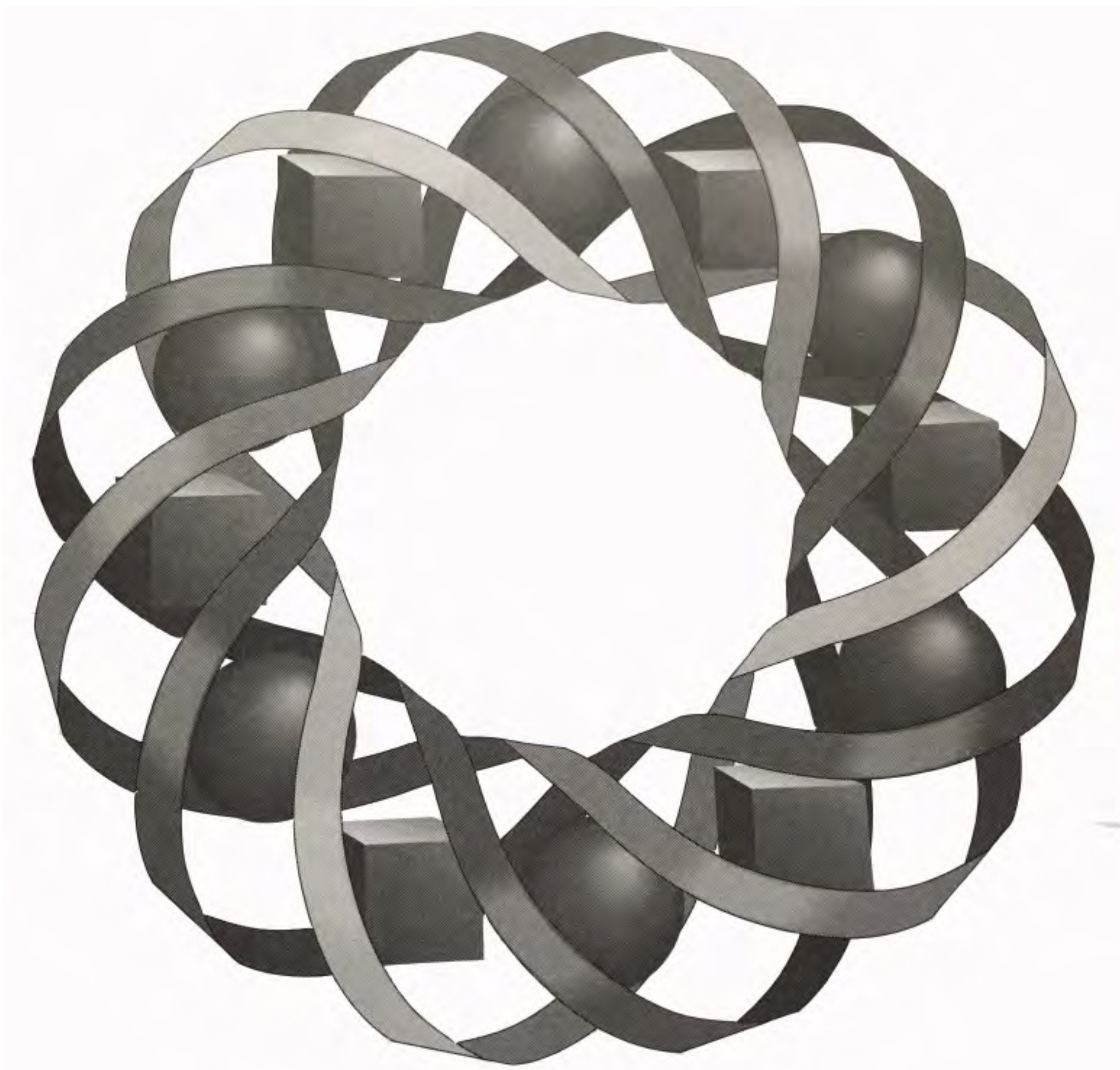
Wilderness Wanderings



What do a hike through the forest, a ride down a country road, and the thrust of a jet engine have in common? They all can be modeled by vectors. In this module, you learn how to use vectors to represent displacement, velocity, and force.

Wendy Driscoll • Darlene Pugh • Peter Rasmussen

Wilderness Wanderings

Introduction

Imagine that you are a member of the Wilderness Club. The club is planning a backpacking trip in the Northwest Territories, a remote region of Canada between the Yukon and Hudson Bay.

Since the forest in the Northwest Territories is dense, with few distinguishable landmarks, you must follow a planned route. The leaders of a previous trip have recorded this route using **displacement vectors**.

Mathematics Note

A **vector** is a quantity with both magnitude and direction. A vector can be represented by an arrow as shown in Figure **1**. The vector's length from tip to tail indicates its magnitude. The orientation of the arrowhead indicates its direction.

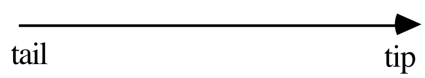


Figure 1: A vector

A vector is typically symbolized using a bold lowercase letter (**r**), or in handwritten work, by an arrow over a letter ($\vec{r}$). The magnitude of a vector **r** is represented by $|\mathbf{r}|$.

A **displacement vector** indicates both the distance moved by an object and the direction of its change in position. For example, the vectors in Figure **2** represent changes in position of 20 cm and 38 cm at azimuths of 250˚ and 75˚, respectively, where the **azimuth** is an angle measured clockwise from north.

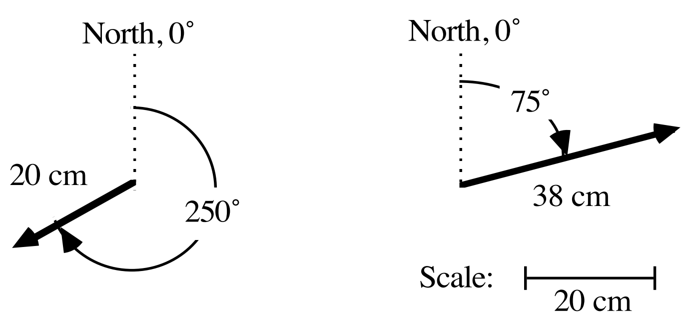


Figure 2: Displacement vectors and azimuths

Vectors can be represented using ordered pairs of numbers in many different ways. For example, the first number could represent the vector's magnitude, while the second represents its azimuth. Using this method, the vectors in Figure **2** can be represented by the ordered pairs (20,250) and (38,75), respectively.

Discussion

a. Describe how a displacement vector could be used to identify the straight-line path from your desk to your teacher's desk.

b. What other measurements, besides the azimuth, could be used to represent a vector's direction?

Activity 1

Your planned route to the final campsite has five sections. Table **1** describes each section as a displacement vector written as an ordered pair. The magnitudes are given in kilometers and the azimuths in degrees. In the following exploration, you use these vectors to model your route.

Table 1: Five sections of the route written as ordered pairs

Section of Hike	Vector (km,degrees)
1	(4,90)
2	(2,215)
3	(3,150)
4	(3,200)
5	(5,80)

Exploration

a. The first section of your route begins at the base camp. To represent this camp's location, draw a point near the upper left-hand corner of a sheet of centimeter grid paper.

b. Using an appropriate scale (and with true north located at the top of the page), construct a vector to represent the first section of the route.

c. Starting at the tip of the first vector, construct a vector to represent the second section of the route. **Note:** Make sure to use the same scale.

d. Continue this process until you have drawn a vector to represent each section of the route.

Mathematics Note

Equal vectors have the same magnitude and direction. **Opposite vectors** have the same magnitude but opposite directions. In Figure **3**, for example, vectors **a** and **c** are equal vectors, while vector **b** is the opposite of both **a** and **c**.

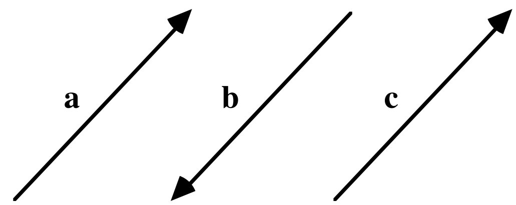


Figure 3: Three vectors

A **zero vector** has a magnitude of 0.

Vector addition can be performed geometrically by the **tip-to-tail** method. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.

The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum. Figure **4** shows the addition of vectors **a** and **b**, along with their resultant vector **r**.

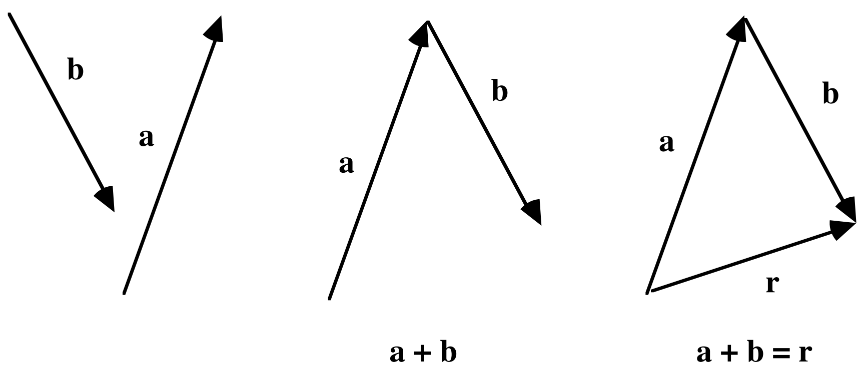


Figure 4: The tip-to-tail method of vector addition

e. Use the tip-to-tail method to add the five vectors you drew in Parts **b–d**. Write the resultant vector as an ordered pair. This vector models the straight-line path from the base camp to the final campsite.

f.
1. Construct the five vectors in Table **1** in an order other than the one given in Parts **b–d**. For example, you could begin at the base camp with section 5 and end with section 1.
2. Use the tip-to-tail method to find the resultant vector.

g. After reaching your final campsite, you decide to fill your canteens at a nearby spring. Select a point somewhere on your diagram to represent the spring's location.

1. Draw one vector to represent the route from the campsite to the spring, and a second vector to represent the route from the spring to the campsite.
2. Write an ordered pair to describe each vector in Step **1**.
3. Add the two vectors you drew in Step **1**.

Discussion

a.
1. Compare the resultant vectors you identified in Parts **e** and **f**.
2. Do you think that you would have found the same final campsite regardless of the order in which you hiked the five sections of the route? Explain your response.

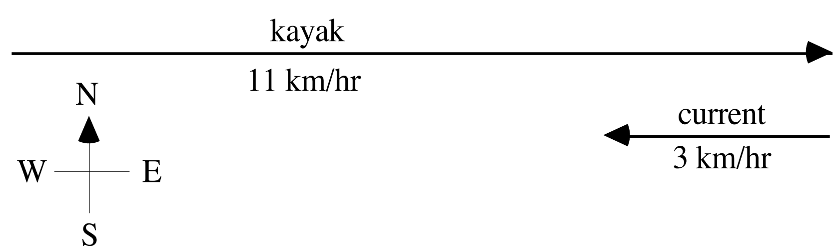
b.
1. Do you think that vector addition is commutative? In other words, does $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$?
2. Do you think that vector addition is associative? In other words, does $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$?

c.
1. Compare the ordered pairs you wrote for the two opposite vectors in Part **g** of the exploration.
2. Describe the resultant vector.

Assignment

1.1 Consider a kayaker who can paddle at a speed of 11km/hr on still water. How fast can she move on a river flowing to the west with a current of 3 km/hr? That depends on the direction in which she paddles.

Since velocity has both magnitude and direction, it can be represented by vectors. For example, the vectors in the following diagram represent the velocities of the kayaker and the current described above when the kayaker paddles directly upstream.



a. Describe the directions of the kayaker and the current in terms of azimuths measured clockwise from north.

b. The sum of the two vectors in the diagram describes the kayaker's velocity with respect to the river bank. Determine this sum.

c. What would the kayaker's resultant velocity be if she turned around and paddled downstream? Explain your response.

d. Describe the kayaker's resultant velocity if she paddles in the current at a speed of 11 km/hr at an azimuth of 235°.

e. Describe the kayaker's resultant velocity if she paddles in the current at a speed of 1 km/hr at an azimuth of 235°.

1.2 As part of his daily workout, a jogger runs the following routes, in any order: 400 m due east, 100 m due north, and 400 m due west.

a. List all the different orders possible for running these routes.

b. Sketch a vector diagram for each possible order.

c. For each vector diagram in Part **b**, write a displacement vector that describes the jogger's change in position from the beginning of the run to the end of the run.

d. What do your results in Part **c** demonstrate about vector addition?

1.3 Janine loves to sail on Crystal Lake. One afternoon, as she finished her lunch on Pelican Island, a strong wind began to blow from the north. To sail back to her parents' cabin, she had to follow a zigzag course. This procedure is called tacking. On her trip home, Janine had to tack four times. Her first tack covered 3.0 km at an azimuth of 45°, her second covered 5.5 km at 300°, and her third 2.5 km at 55°.

a. Obtain a map of Crystal Lake from your teacher. Use vectors to make a scale drawing of the displacements for Janine's first three tacks.

b. Draw the resultant vector that represents her change in position after the first three tacks. Write this vector as an ordered pair (distance,azimuth).

c. Draw the vector that represents the final tack Janine must sail to return to the dock. Write this vector as an ordered pair.

1.4 Using a map of the United States, determine the displacement vectors that describe a direct flight from your home to Seattle, Washington, followed by a direct flight to San Francisco, California, and a final direct flight back to your home.

* * * * *

1.5 During a hike, Daniel used displacement vectors to keep track of his route. He wrote the following vectors, recording the magnitudes in kilometers and the degrees azimuth: (2,65), (1.5,0), (2.4,300), and (4,205).

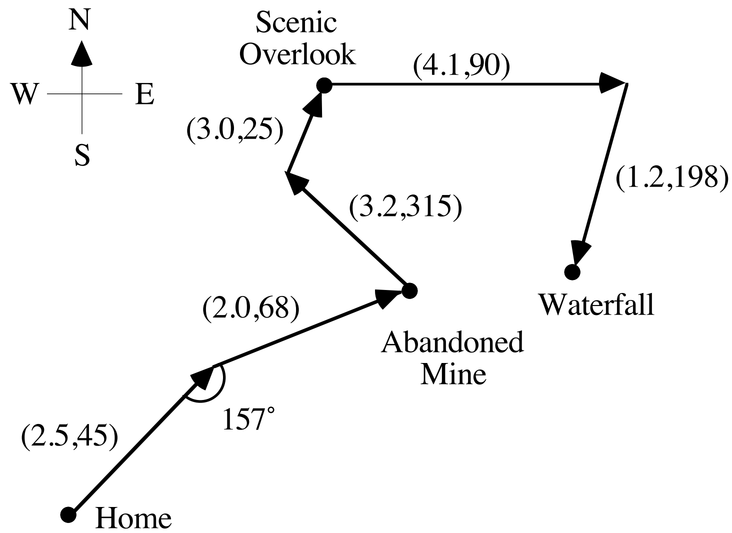
a. Make a scale drawing of Daniel's hike.

b. Write a resultant vector that describes his change in position.

c. Write a displacement vector that describes the straight-line route needed to return Daniel to his starting point.

d. What is the relationship between the vectors you described in Parts **b** and **c**? Explain your response.

1.6 After spending a day exploring in a park near his home, Manuel returned with the following map.



Manuel plans to take some friends to visit the mine, the overlook, and the waterfall. To determine shorter routes to each site, he wants to find a direct path from home to the mine, from the mine to the overlook, and from the overlook to the waterfall.

Since he has not drawn his map to scale, Manuel cannot draw and measure the appropriate resultant vectors. Instead, he decides to use his knowledge of the relationships among the lengths of the sides and the measures of the angles in a triangle.

Recall that in a triangle ABC, the law of sines states that

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}$$

while the law of cosines states that

$$c^2 = a^2 + b^2 - 2ab\cos \angle C$$

a. Manuel begins his calculations by identifying the angle between the first two sections of his old route: 157˚. Describe how he could find this angle measure.

b. Determine a displacement vector that represents each of the following:

1. a direct route from Manuel's house to the mine
2. a direct route from the mine to the scenic overlook
3. a direct route from the scenic overlook to the waterfall.

* * * * * * * * * *

Activity 2

In Activity **1**, you used the tip-to-tail method to add vectors geometrically. In this activity, you investigate a method for adding vectors algebraically.

Exploration 1

a. Draw a two-dimensional coordinate system on a sheet of centimeter grid paper, with the origin located near the center of the sheet. Tape the paper to a table.

b. **Force** is a physical quantity that can affect the motion of an object. Like displacement and velocity, force has both magnitude and direction. The metric unit of force is the newton (N). The magnitude of a force can be measured using spring scales.

Hook two spring scales to a ring and pull them in opposite directions until each scale reads 12 N. Position the ring over the origin of your coordinate system, with the scales extending into the first and third quadrants, as shown in Figure **5**.

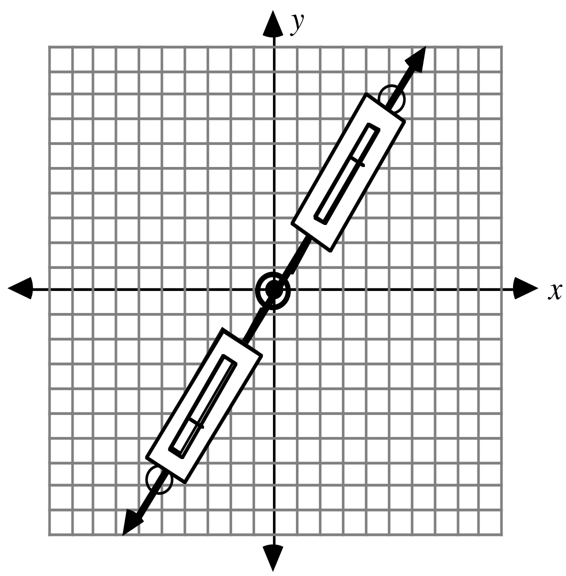


Figure 5: Scales and ring over graph paper

c. Mark a line on the grid paper that shows the angle at which the scales cross the x-axis. Measure and record this angle.

d. Draw a vector on the grid paper to represent the force on each scale, letting 1 cm represent 1 N. Locate the tail of each vector at the origin and label the magnitudes. Your grid paper should now resemble Figure **6**. Note that vectors **a** and **b** are opposite vectors.

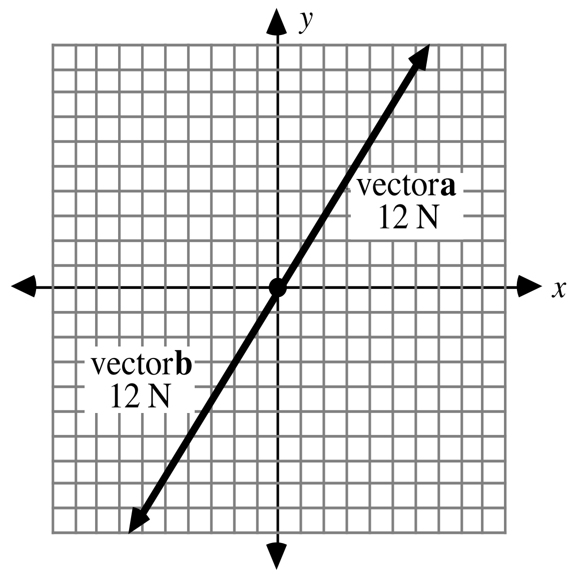


Figure 6: Graph paper with vectors

e. Hook a third scale to the ring. Position one scale directly over vector **b**. Position the other two scales so that they are directly over the axes, as shown in Figure **7**. Keeping the ring centered over the origin, pull on the three scales until the scale over vector **b** reads 12 N.

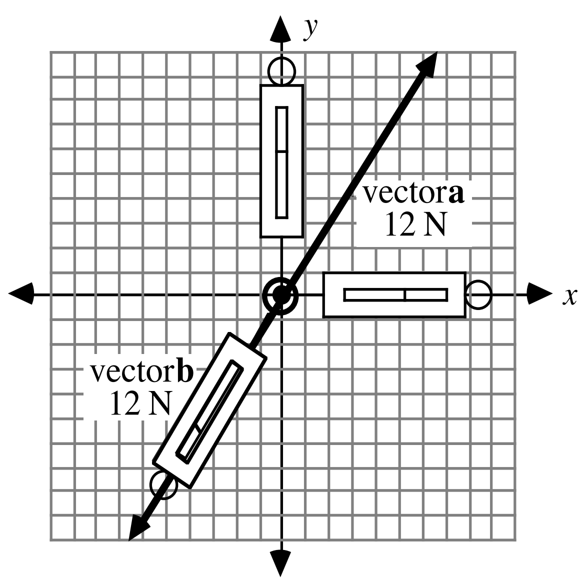


Figure 7: Positioning three scales

f. Record the readings of the scales positioned over the *x*- and *y*-axes. These values are the magnitudes, in newtons, of the corresponding force vectors.

g. To represent the forces on these scales, draw vectors **c** and **d** on the grid paper. The grid paper should now resemble Figure **8**.

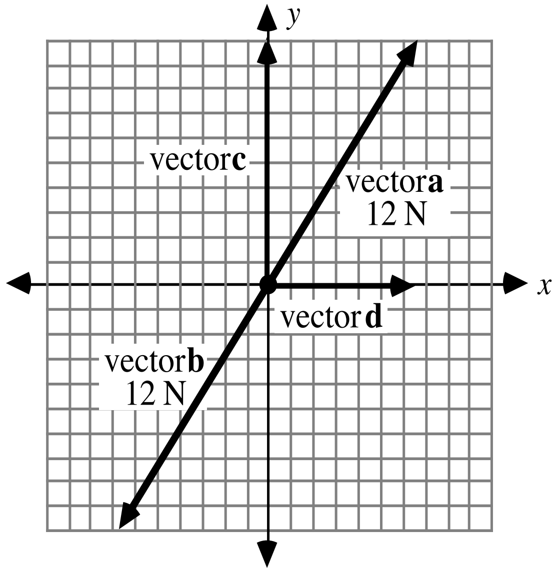


Figure 8: Graph paper with force vectors

h. Sketch a vector diagram representing the sum of vectors **c** and **d**. Compare the resultant vector with vector **a**.

i. Use right-triangle trigonometry, the magnitude of vector **a**, and the angle that vector **a** forms with the horizontal vector **d** to determine the magnitudes of vectors **c** and **d**.

j. Repeat Parts **a–i** one more time, varying the angle formed by the two scales in Part **b** and the *x*-axis.

Mathematics Note

The **components** of a vector are the pair of horizontal and vertical vectors that, when added, result in the given vector. The horizontal component of vector **r** is denoted by $\mathbf{r}_x$ (read "**r** sub *x*"), while its vertical component is $\mathbf{r}_y$.

The angle measure between a vector **r** and its horizontal component $\mathbf{r}_x$ is the **reference angle**.

For example, Figure **9** shows a graph of vector **r** with its tail located at the origin, along with its components. In this case, the reference angle is θ.

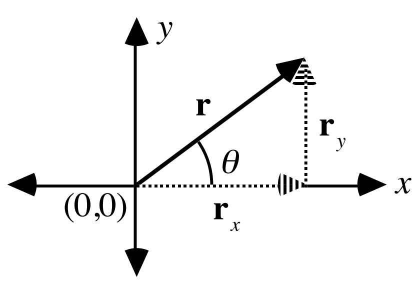


Figure 9: A vector and its components

Since a vector and its components form a right triangle, the magnitudes of the components can be found using right-triangle trigonometry. By convention, the direction of a component vector is indicated by its sign. A positive $\mathbf{r}_x$ is in the direction of the positive x-axis, while a negative $\mathbf{r}_x$ is in the direction of the negative x-axis. A similar rule applies to the vertical component $\mathbf{r}_y$.

For example, consider vector **s** shown in Figure **10** below.

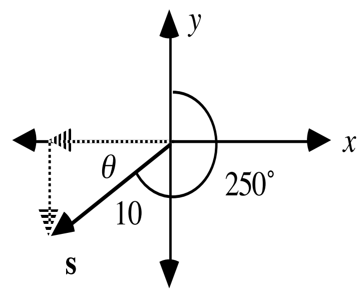


Figure 10: A vector and its components

Vector **s** has a magnitude of 10 and a direction of 250°, measured clockwise from the positive y-axis. To express **s** in terms of its components, you might complete the following steps.

- First, determine the measure of the reference angle. In this case, $\theta = 20^\circ$.
- Next, find the magnitude of the horizontal component $\mathbf{s}_x$:

$$\begin{aligned}|\mathbf{s}_x| &= |\mathbf{s}| \bullet \cos\theta \\ &= 10 \bullet \cos 20^\circ \\ &\approx 9.4\end{aligned}$$

  Since $\mathbf{s}_x$ is in the direction of the negative x-axis, its sign is negative. As a result, $\mathbf{s}_x \approx -9.4$.

- Next, find the magnitude of the vertical component $\mathbf{s}_y$:

$$\begin{aligned}|\mathbf{s}_y| &= |\mathbf{s}| \bullet \sin\theta \\ &= 10 \bullet \sin 20^\circ \\ &\approx 3.4\end{aligned}$$

  Since $\mathbf{s}_y$ is in the direction of the negative y-axis, its sign also is negative. As a result, $\mathbf{s}_y \approx -3.4$.

- Vector **s** now can be expressed as the sum of the following components: $\mathbf{s}_x \approx -9.4$ and $\mathbf{s}_y \approx -3.4$.

Discussion 1

a. In Part **e** of Exploration **1**, how does the total force exerted on the two scales positioned along the axes compare with the force exerted on the third scale? Explain your response.

b. How can you find the magnitude of a resultant given its horizontal and vertical components?

c. How can you find the measure of the reference angle given the horizontal and vertical components?

Exploration 2

In this exploration, you express the displacement vector for each section of the hike in Activity **1** in terms of its components. Assume that north is in direction of the positive *y*-axis and that east is in the direction of the positive *x*-axis.

a. The displacement vectors for the route from the base to the campsite, in order, were (4,90), (2,215), (3,150), (3,200), and (5,80), where the magnitudes are given in kilometers and the azimuths in degrees. Draw a vector diagram of this route.

b. Draw the horizontal and vertical components of each vector on the diagram, with the tail of each horizontal component at the tail of the corresponding vector.

1. Use trigonometry to determine the magnitude of each component.
2. Determine the appropriate sign for each component.

c. The displacement vector that describes a direct route from the base to the camp is (10.3,127). Determine the horizontal and vertical components for this resultant vector.

d.
1. Compare the horizontal component of the resultant vector to the sum of the horizontal components of the five individual vectors.
2. Compare the vertical component of the resultant vector to the sum of the vertical components of the five individual vectors.

Discussion 2

a. In Exploration **2**, you assumed that north is in the direction of the positive *y*-axis and that east is in the direction of the positive *x*-axis.

1. For which interval of azimuths is a vector's vertical component negative and its horizontal component positive?
2. For which interval of azimuths is a vector's vertical component positive and its horizontal component negative?

b. Describe how you determined the magnitude of the components of each vector in Exploration **2**.

c. Could you have determined the components of these vectors without drawing them? Explain your response.

d. Describe how you could determine the horizontal and vertical components of a resultant vector using the horizontal and vertical components of each vector in the sum.

e. Given the reference angle for a vector, how can you determine its azimuth?

f. What are the advantages of expressing a vector in terms of its components?

Assignment

2.1 To help locate some interesting sights near your camp in the Northwest Territories, a helicopter flies over the area. The pilot locates a waterfall 10 km from camp at an azimuth of 320°.

- **a.** Make a diagram of the area around your camp, using a displacement vector to show the direct route to the waterfall.
- **b.** Determine the reference angle of the displacement vector.
- **c.** Determine the vector's horizontal and vertical components.

2.2 On your second day at the campsite, you and some friends go on a hike. At first, you walk 2.5 km at an azimuth of 60°. Next, you walk 6.3 km at an azimuth of 85°. Because dusk is only two hours away, you would like to hike straight back to camp.

- **a.** Using the tip-to-tail method, determine the displacement vector that represents a direct route back to camp.
- **b.** Express each vector in component form, then use these components to justify your response to Part **a** algebraically.
- **c.** If you walk at approximately 4 km/hr, will you make it back to camp before it gets dark?

2.3 The speed of a plane or helicopter with respect to the air is its airspeed. But pilots seldom fly in still air. When determining the proper airspeed and azimuth for their destination, they must consider the effect that wind will have on their movement with respect to the ground.

In terms of vectors, $\mathbf{a} + \mathbf{w} = \mathbf{v}$, where **a** is the velocity with respect to the air, **w** is the wind velocity, and **v** is the velocity with respect to the ground.

Suppose that a helicopter leaves the base camp to pick up supplies. To reach the nearest town, the pilot would like to fly with a groundspeed of 200 km/hr at an azimuth of 336°. If the wind is blowing to the north at a speed of 12 km/hr, at what airspeed and azimuth should the pilot fly? Justify your response algebraically.

2.4 While gathering berries near the campsite, you leave camp and walk 400 m at an azimuth of 315°. You then turn due east and walk 320 m.

a. How far are you from camp?

b. What azimuth should you take to return directly to camp?

2.5 The velocity of a plane can be represented by the vector (800,203), where the magnitude is given in kilometers per hour and the azimuth in degrees.

a. Draw a diagram of this vector, including its horizontal and vertical components.

b. Determine the magnitudes and signs of the horizontal and vertical components. Justify your answers algebraically.

2.6 A plane leaves the airport with an airspeed of 480 km/hr at an azimuth of 210°. The wind is blowing to the east, resulting in an actual azimuth of 205°.

a. What is the speed of the plane with respect to the ground?

b. What is the speed of the wind?

* * * * *

2.7 A cruise ship travels at 30 km/hr and an azimuth of 303° for 10 hr. It then turns and travels at 35 km/hr and an azimuth of 229° for another 10 hr.

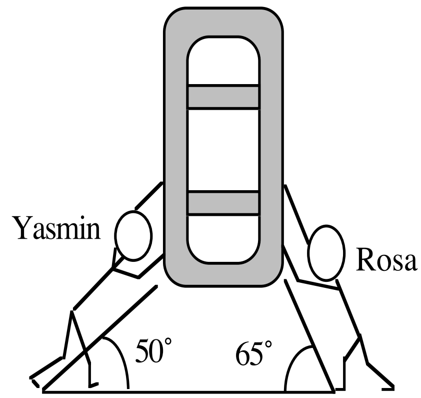
a. Sketch the displacement vectors that correspond to this situation.

b. Determine the horizontal and vertical components of the resultant displacement vector.

c. Determine the magnitude and azimuth of the resultant displacement vector.

2.8 A ferry leaves the dock at a speed of 28 km/hr and an azimuth of 20°. Just off shore, it encounters a current flowing to the east, which results in an actual azimuth of 40°. Determine the speed of the current, as well as the actual speed of the ferry with respect to the shore.

2.9 Yasmin and Rosa are pushing their inflatable raft off a sandbar. Yasmin pushes with a force of 90 N, while Rosa pushes with a force of 80 N. These forces are directed as shown in the diagram below.

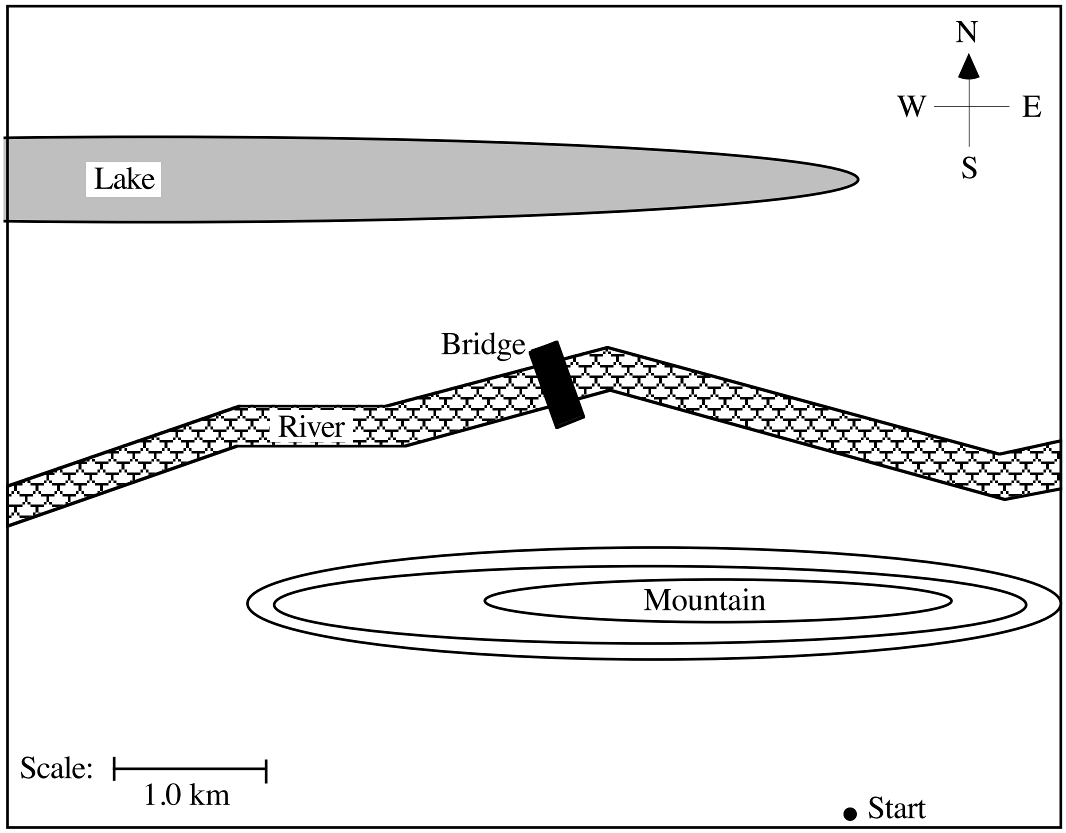


Determine the magnitude and direction of the resultant force.

* * * * * * * * * *

Summary Assessment

As a participant in a route-finding contest, you are given the following map, which identifies the starting position and several natural obstacles.



1. Your destination for the contest can be described by the displacement vector (5.9,332), where the magnitude is in kilometers and the azimuth in degrees. The tail of this vector is on the starting position and its tip is on the destination point. Find the destination point on the map.
2. Your route must travel west of the mountain, then across the bridge, and then east of the lake. Draw a series of displacement vectors to model your route from the start to the destination. Write an ordered pair to represent each of these vectors.
3. Use trigonometry to find the horizontal and vertical components of each vector you drew in Problem **2**. Verify algebraically that the sum of these vectors equals the displacement vector (5.9,332).
4. How long will it take you to walk the route at a rate of 3 km/hr?

Module Summary

- A **vector** is a quantity with both magnitude and direction. A vector can be represented graphically by an arrow. The vector's length from tip to tail indicates its magnitude. The orientation of the arrowhead indicates its direction.
- A vector is typically symbolized using a bold lowercase letter (**r**), or in handwritten work, by an arrow over a letter ($\overrightarrow{\text{r}}$). The magnitude of a vector **r** is represented by $|\mathbf{r}|$.
- An **azimuth** is an angle measured clockwise from north.
- **Displacement** is a change in position in a particular direction. A displacement vector indicates both the distance moved and the direction.
- **Equal vectors** have the same magnitude and direction.
- **Opposite vectors** have the same magnitude but opposite directions.
- A **zero vector** has a magnitude of 0.
- Vector addition can be performed geometrically by the **tip-to-tail** method. Using this method, each vector is drawn so that its tail coincides with the tip of the previous vector.
- The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.
- The **velocity** of an object is its speed in a specific direction. Like displacement, velocity has both magnitude and direction.
- **Force** is a physical quantity that can affect the motion of an object. Like displacement and velocity, force has both magnitude and direction. The metric unit of force is the newton (N).
- The **components** of a vector are the pair of horizontal and vertical vectors that, when added, result in the given vector. The horizontal component of vector **r** is denoted by $\mathbf{r}_x$ (read "**r** sub *x*"), while its vertical component is $\mathbf{r}_y$.
- The angle measure between a vector **r** and its horizontal component $\mathbf{r}_x$ is the **reference angle**.

- Since a vector and its components form a right triangle, the magnitudes of the components can be found using right-triangle trigonometry. If θ is the reference angle, then $\left|\mathbf{r}_x\right| = |\mathbf{r}| \bullet \cos\theta$ and $\left|\mathbf{r}_y\right| = |\mathbf{r}| \bullet \sin\theta$.

  By convention, the direction of a component vector is indicated by its sign. A positive $\mathbf{r}_x$ is in the direction of the positive *x*-axis, while a negative $\mathbf{r}_x$ is in the direction of the negative *x*-axis. A similar rule applies to the vertical component $\mathbf{r}_y$.

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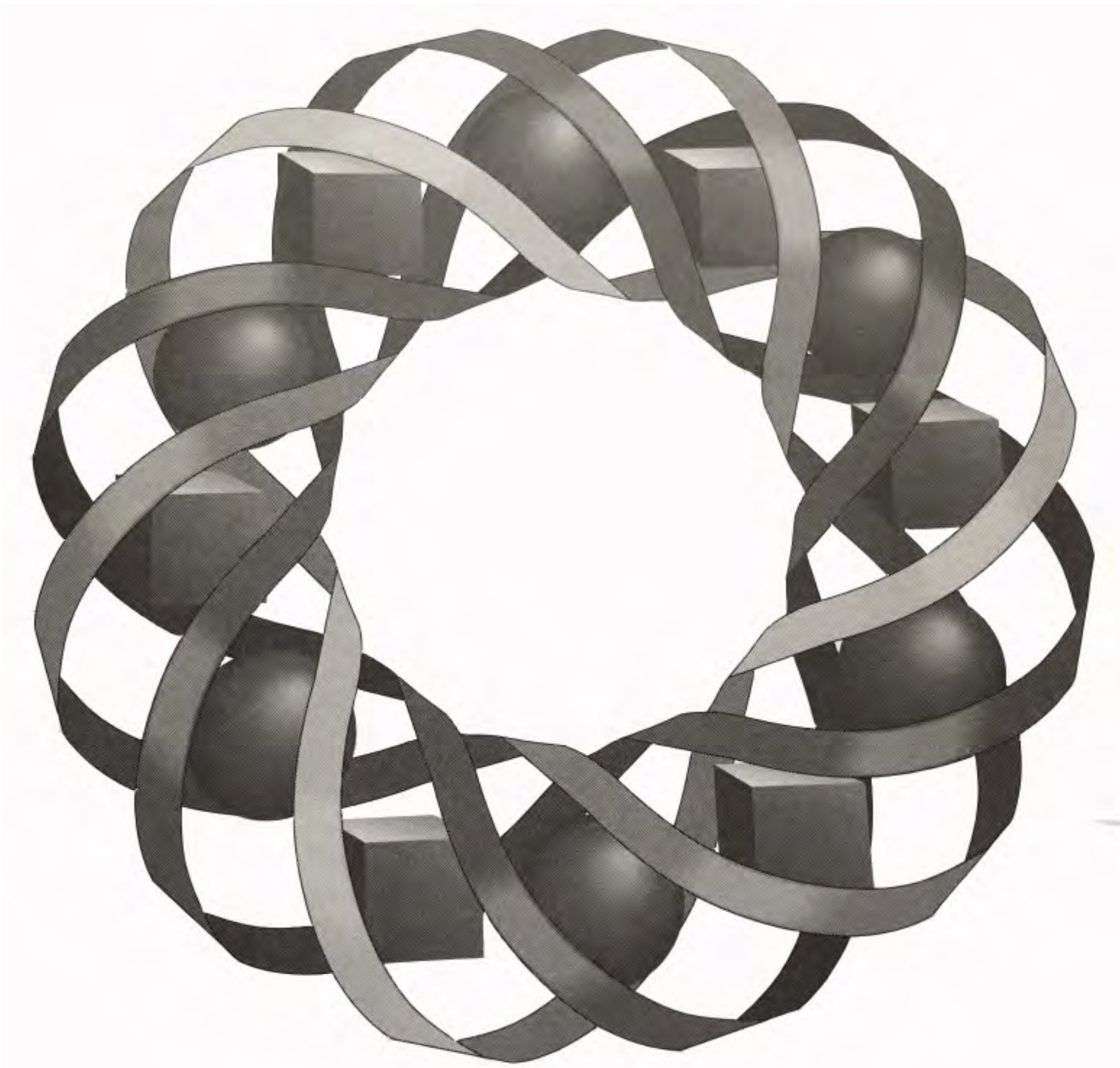
Making Cents of Your Income



What's the average income in your state? If a researcher claimed that the mean is $30,000 after surveying 100 residents, would you find this claim believable? In this module, you investigate how to use sampling to test claims about populations.

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Making Cents of Your Income

Introduction

How could you determine the average income of a high school graduate in your state? Since it is not practical to contact every high school graduate, you might survey a sample of this population. In this case, you would want to contact enough graduates to obtain an accurate estimate, but not more than is necessary. How many are enough?

Using a sample to obtain accurate information about a population can be a complicated process. In this module, you investigate how sample size affects the reliability of an estimate, determine the confidence you should have in an estimate, and use samples to test claims about a population.

Activity 1

In order to decide how large a sample you need to make reasonable predictions about a population, you must first understand how information gained from samples of different sizes can vary.

Exploration 1

In this exploration, you draw samples of different sizes from a population, then compare the means and standard deviations of each sample.

Mathematics Note

Standard deviation is a measure of the spread in a data set. The standard deviation of a population, often denoted by σ, can be calculated using the formula below:

$$\sigma = \sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_N-\mu)^2}{N}}$$

where $x_1, x_2, \cdots, x_N$ represent all the individual values in the population, μ represents their mean, and N represents the population size.

The standard deviation of a sample is the **sample standard deviation**, denoted by s. Statisticians often use s to approximate σ. The formula for calculating s is:

$$s = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$$

where $x_1, x_2, \cdots, x_n$ represent the individual values in the sample, $\bar{x}$ represents the sample mean, and n represents the sample size.

For example, suppose that you selected a sample of five cars during a study of the ages of motor vehicles. The ages of the cars, in years, were 1, 4, 6, 8, and 11. In this case, the sample mean $\bar{x}$ can be found as follows:

$$\bar{x} = \frac{1+4+6+8+11}{5} = 6 \text{ years.}$$

The sample standard deviation s can be found as shown below:

$$s = \sqrt{\frac{(1-6)^2+(4-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5-1}} \approx 3.8 \text{ years}$$

When n is large, $\bar{x}$ approximates μ and the effect caused by subtracting 1 from n in the formula for s is very small. In such cases, the value of s approximates σ.

a. Obtain a population of pennies from your teacher and place them in a container. To obtain a sample of size 5 from this population, complete the following steps:

1. Draw one member of the population at random from the container.
2. Record its age.
3. Return it to the container and mix the population thoroughly.
4. Repeat Steps **1–3** until the desired sample size has been reached.

b. Determine the mean age $\bar{x}$ and standard deviation s of this sample. Record these values in a table with headings like those in Table **1**.

Table 1: Statistics for samples of five pennies

Sample Number	$\bar{x}$	s
1		
2		
3		
⋮		
10		

c. Repeat Parts **a** and **b** nine more times. **Note:** Save a copy of Table **1** for use in Activity **2**.

d. Create a frequency histogram of the means of your 10 samples.

e. Collect all the sample means from each group in your class. Create a frequency histogram of these sample means. **Note:** Save the class data for use in Exploration **2**.

Discussion 1

a. In Part **b** of Exploration **1**, why were you instructed to use s for the standard deviation rather than σ?

b. Compare the frequency histogram of your 10 sample means with the one for the class data.

c. Describe how you could use a frequency histogram to estimate the mean age of the population in Exploration **1**.

d. Which histogram should provide a better estimate of the population mean: your histogram for 10 sample means or the one for the class data?

Exploration 2

In this exploration, you use a simulation to investigate how sample size can affect the results of a sampling.

a. With the help of technology, you can examine many samples from the population in Exploration **1** in a relatively short time.

1. Use the simulation provided by your teacher to obtain 10 samples of size 20 from the population.
2. Determine the mean age and standard deviation of each sample and record these values in a table with headings like those in Table **1**. **Note:** Save this data for use in Activity **2**.
3. Create a frequency histogram of the means of your 10 samples.
4. Collect all the sample means from each group in your class. Create a frequency histogram of these sample means.

b. Repeat Part **a** for samples of size 40.

c. Using the class data for samples of size 5 from Exploration **1**, determine the mean and standard deviation for the sample means. Record these values in a table with headings like those in Table **2**.

Table 2: Sample sizes, mean, and standard deviation

Sample Size (n)	Mean of Sample Means	Standard Deviation of Sample Means
5		
20		
40		

d. Repeat Part **c** for sample sizes of 20 and 40.

e. Estimate the mean age of the population of pennies.

Discussion 2

a. Describe how the frequency histograms in Exploration **2** change as the sample size increases.

b. Describe how the means and standard deviations recorded in Table **2** change as the sample size increases.

c. Why would you expect an increase in sample size *n* to produce a decrease in the standard deviation of the sample means?

d.
1. How did you estimate the mean age μ for the population of pennies?
2. How accurate do you think your estimate is? Explain your response.

e.
1. Why should the mean of one large sample from a population be approximately the same as the mean of a large number of sample means from the same population?
2. Why should the standard deviation *s* of one large sample from a population be approximately the same as the standard deviation μ of the entire population?

Mathematics Note

The **sampling distribution of sample means** contains the means ($\bar{x}$) of *all* possible samples of size *n* from a population.

The mean of the sampling distribution of sample means, denoted by $\mu_{\bar{x}}$, equals the population mean μ.

The standard deviation of the sampling distribution, denoted by $\sigma_{\bar{x}}$, equals $\sigma/\sqrt{n}$, where σ is the population standard deviation and *n* is the sample size. When σ is unknown, the standard deviation of the sample (*s*) may be used as an estimate of σ.

For example, consider a jar containing three pennies, A, B, and C with ages 2 yr, 6 yr, and 6 yr, respectively. Table **3** shows the mean ages, in years, of all possible samples of size 2 that can be taken from this population, drawn one at a time with replacement.

Table 3: A sampling distribution of sample means

Sample	AA	AB	AC	BA	BB	BC	CA	CB	CC
$\bar{x}$	2	4	4	4	6	6	4	6	6

In this case,

$$\mu_{\bar{x}} = \frac{2+4+4+4+6+6+4+6+6}{9} \approx 4.66 \text{ years.}$$

Since $\mu_{\bar{x}} = \mu$, the mean also can calculated as follows: $(2 + 6 + 6)/3 \approx 4.66$ yr.

Using the formula for the standard deviation of a population, the standard deviation of all possible sample means is approximately 1.33 yr. This also can be calculated as shown below:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \approx \frac{1.89}{\sqrt{2}} \approx 1.33 \text{ yr.}$$

f. Obtain the actual mean and standard deviation of the population of pennies used in Explorations **1** and **2**. Use the formulas $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n}$ to determine $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$ for sample sizes of 5, 20, and 40. Compare your results with the values you recorded in Table **2**.

g. How does sample size affect the spread of all the possible sample means about the population mean?

h. Taking large samples from a population can be expensive. In what situations might it be worth the cost of collecting a very large sample?

Assignment

1.1 The following table shows the ages, in years, of a sample of pennies found in a piggy bank:

28	8	6	5	1	1	3	4	2	2
14	1	2	3	6	6	12	7	5	1
12	7	1	6	1	2	1	8	1	8
16	10	5	2	3	10	16	3	4	7
3	5	2	4	8	3	4	1	4	5

a. Determine the mean age $\bar{x}$ of these pennies.

b. Determine the sample standard deviation s.

c. Estimate the mean age μ for the population of pennies from which this sample came.

1.2 To determine the mean annual income in a large city, a research group analyzed 1000 random samples of 40 adults from this population. The results of the study are shown in the following table.

Sample Mean (to nearest $1000)	Frequency
18,000	1
19,000	11
20,000	45
21,000	111
22,000	184
23,000	220
24,000	197
25,000	131
26,000	66
27,000	24
28,000	6
29,000	2
30,000	1
31,000	1

a. Use the mean of these sample means to estimate the mean $\mu_{\bar{x}}$ of all possible sample means (to the nearest thousand dollars) using samples of size 40.

b. Use the standard deviation of these sample means to estimate the standard deviation $\sigma_{\bar{x}}$ of all possible sample means using samples of size 40.

c. Estimate the mean μ and standard deviation σ for the population from which these samples came.

d. How accurate do you think your estimate is for μ? Explain your response.

e. Using the given data, estimate the probability that a random sample of 40 adults from this city has an average income that is:

1. more than $29,000

2. less than $21,000

3. between $21,000 and $29,000, inclusive.

1.3 When trying to estimate the average income of high school graduates in your state, Andreas claims that using a sample size of 240 instead of 120 would reduce the standard deviation of all possible sample means by half. Likewise, using a sample size of 480 instead of 240 would reduce $\sigma_{\bar{x}}$ again by half. Defend or refute Andreas' claim.

* * * * *

1.4 At Lincoln High School, many students take the Scholastic Aptitude Test (SAT). The following table shows the scores on the mathematics portion of the SAT for the students in one classroom.

410	770	430	420	400	780	440	420
610	630	400	440	430	500	430	450
680	500	720	450	520	470	440	740
440	580	550	590	770	400	500	610

a. Determine the mean score $\bar{x}$ (to the nearest whole number).

b. Determine the sample standard deviation s.

c. Estimate the mean score μ for the entire population of Lincoln High School students who took the exam.

d. Why might you hesitate to use the results of this sample to estimate the mean for the entire population?

1.5 The table below shows the ages of a sample of high school students in a community.

Age	**Frequency**
13	100
14	150
15	120
16	93
17	157
18	82
19	51

a. Determine the mean age, to the nearest year, of students in this sample.

b. Determine the standard deviation of this sample.

c. Estimate the mean μ for the population from which this sample came.

d. Estimate the proportion of students in the community that are:

1. older than 16

2. younger than 17

3. between 15 and 17, inclusive.

e. How confident are you that the estimates made from this sample are accurate? Explain your response.

1.6 The following table shows the numbers of persons per household for a sample of households taken in a large city:

Number in Household	Frequency
1	15
2	20
3	37
4	23
5	14
6	4
7	2

a. Determine the mean number of persons per household for this sample.

b. Determine the standard deviation of this sample.

c. Estimate the mean number of persons per household for the city from which this sample came.

d. Estimate the proportion of households in this city that have:

1. more than 4 persons

2. less than 4 persons.

e. How confident are you that the estimates made from this sample are accurate? Explain your response.

* * * * * * * * * *

Activity 2

In Activity **1**, you examined how sample means can vary by taking many samples of different sizes. In the real world, however, researchers typically do not collect a large number of samples. In fact, they often take only one.

In this activity, you investigate the probability that the mean from a single sample accurately estimates the population mean.

Exploration

a. Use the data you compiled in Activity **1** for samples of size 5 to complete a table with headings like those in Table **4**. To estimate $\sigma_{\bar{x}}$, use the value of s for each sample to approximate σ, then use the formula $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

Table 4: Statistics for samples of five pennies

Sample Number	$\bar{x}$	s	Estimate of $\sigma_{\bar{x}}$
1			
2			
3			
⋮			
10			

b.

1. For each sample in Table **4**, find the interval $[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$.
2. Graph each interval above a number line as a line segment with the midpoint $\bar{x}$ indicated as shown in Figure **1** below.

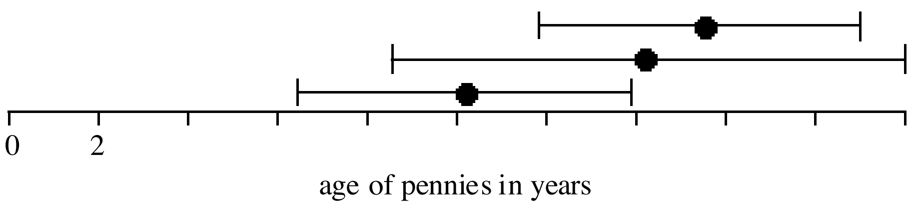

Figure 1: Intervals for three samples of pennies

c.

1. Obtain the mean age μ of all the pennies in the population used in Activity **1**.
2. Draw a line on your graph to represent the population mean, as shown in Figure **2**:

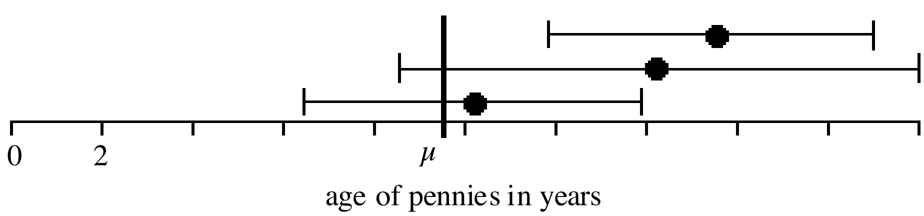

Figure 2: Graph of intervals with line at μ

d. Determine the percentage of intervals that contains the population mean. In Figure **2**, for example, approximately 67% of the intervals contain the population mean. Record the result in a table with headings like those in Table **5**.

Table 5: Percentage of intervals containing μ 7

Samples sizes *n*	**Interval**	**Percentage that Contained μ**
5	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	
20	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	
40	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	
5	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	
20	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	
40	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	
5	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	
20	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	
40	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	

e. Repeat Parts **a–d** for sample sizes of 20 and 40 pennies.

f. For each sample size, determine the percentage of intervals of $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ that contain the population mean. Record your results in your copy of Table **5**.

g. For each sample size, determine the percentage of intervals of $[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$ that contain the population mean. Record your results in your copy of Table **5**.

Discussion

a. What appears to be the relationship between the number of standard deviations used to create the interval and the percentage of intervals that contain the population mean? Explain your response.

Mathematics Note

The **central limit theorem** states that, regardless of the population, as the sample size increases, the sampling distribution of sample means approaches a **normal distribution**.

Figure **3** below shows a graph of a normal distribution. The curve that describes the shape of the graph is the **normal curve**. As in all continuous probability distributions, the total area between the *x*-axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

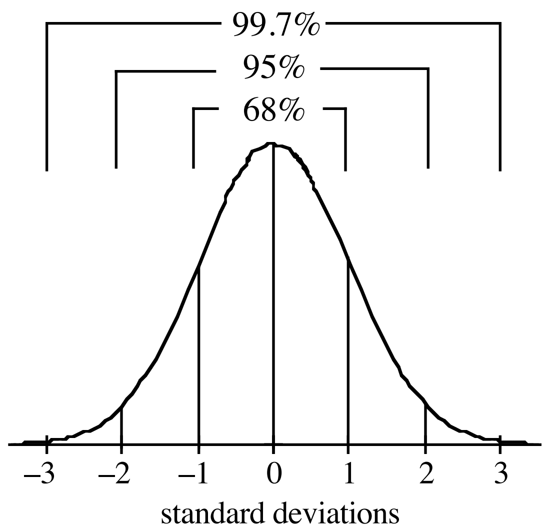

Figure 3: A normal curve and the 68–95–99.7 rule

As a rule of thumb, samples of size $n \geq 30$ are large enough to assume that the sampling distribution of sample means approaches a normal distribution.

b. How do your results in the exploration compare with the percentages predicted by the 68–95–99.7 rule?

c. As the sample size changes, what happens to the sizes of the corresponding intervals? Explain your response.

Mathematics Note

A **confidence interval** for a population mean μ is an interval of numbers in which you would expect to find the value of μ. The 68–95–99.7 rule implies the following:

- For approximately 68% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$ contains the population mean μ.
- For approximately 95% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ contains the population mean μ.
- For approximately 99.7% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$ contains the population mean μ.

For example, consider a sample of 40 pennies with a mean age $\bar{x}$ of 9 yr and a standard deviation s of 3 yr. In this case, $\sigma_{\bar{x}}$ can be estimated by $3/\sqrt{40}$=0.47, Using the 68–95–99.7 rule, you can be 68% confident that the population mean μ is in the interval $[9 - 0.47, 9 + 0.47]$, 95% confident μ is in the interval $[9 - 2(0.47), 9 + 2(0.47)]$, and 99.7% confident μ is in the interval $[9 - 3(0.47), 9 + 3(0.47)]$.

d. What is meant by a "95% confidence interval?"

e. How often would you expect a 95% confidence interval to not contain the population mean? Explain your response.

f. Suppose that a researcher surveys a random sample of 100 high school graduates and calculates their mean annual income. What other information is necessary to give a reasonable estimate of how close the sample mean is to the population mean?

Assignment

2.1 Imagine that you have taken 20 samples of size 30 from a population of pennies. For each sample, you determine the mean age and estimate the corresponding 95% confidence interval. Approximately how many of these intervals do you think will include the population mean?

2.2 For a random sample of 100 eggs, the mean mass was 67 g, with a standard deviation of 45 g.

a. Estimate the standard deviation $\sigma_{\bar{x}}$ of all sample means for $n = 100$.

b. Construct the interval $[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$.

c. How confident should you be that the population mean falls in the interval in Part **b**? Justify your response.

2.3 The Shiny Bright Company manufactures light bulbs. To determine the mean life expectancy of their 60-watt bulbs, the company sampled 1000 bulbs. They found that the mean life expectancy of the sample was 827 hr, with a sample standard deviation of 424 hr.

a. Estimate $\sigma_{\bar{x}}$ for these 60-watt bulbs.

b. Write a conclusion about the mean life expectancy at each of the following:

1. a 68% confidence level

2. a 95% confidence level

3. a 99.7% confidence level.

2.4 Ken sampled 20 bags of a certain brand of candy. He discovered that the mean mass was 52 g, with a standard deviation of 4 g.

a. Write a conclusion about the mean mass of all bags of this candy.

b. Considering the sample size, how sure should you be of your conclusion in Part **a**?

c. How could you make the results of this experiment more conclusive?

2.5 As a quality control specialist, you have been asked to determine the mean volume of soft drinks packaged in 2-L bottles. The table below shows the data (in liters) collected from a sample of 30 bottles. Use this information to write a conclusion about the mean volume of soft drinks in 2-L bottles.

1.90	1.95	1.97	1.98	2.00	1.94
1.91	2.04	2.02	1.99	1.97	1.96
1.92	2.00	1.97	1.96	1.94	1.94
1.93	2.05	2.04	2.00	1.98	1.97
2.00	1.98	1.97	1.94	2.01	1.99

2.6 A random sample of the savings account balances of 500 people banking at Third National Bank resulted in $\bar{x} = \$720$ and $s = \$217$.

a. Estimate the standard deviation $\sigma_{\bar{x}}$ of all sample means for this sample size.

b. Construct the interval $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$.

c. How confident are you that the population mean falls in the interval from Part **b**?

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2.7 The Rolling-On Company manufactures automobile tires. To determine the mean life expectancy of their four-ply tires, the company sampled 800 tires. The mean life expectancy of this sample was 197,124 km, with a standard deviation of 79,678 km. Assuming that the lives of these tires are normally distributed, write a conclusion about the mean life expectancy at:

a. a 68% confidence level

b. a 95% confidence level

c. a 99.7% confidence level.

2.8 To determine the mean life expectancy of their tennis shoes, the Brand 10 company surveyed a random sample of 32 people. The results of the survey are shown in the table below.

Life of Tennis Shoes (in months)							
35	39	6	41	23	61	45	18
16	3	13	8	21	13	15	27
27	40	26	18	22	23	6	40
25	42	17	41	38	29	13	12

Assuming that the lives of these tires are normally distributed, write a conclusion about the mean life of Brand 10 shoes, using:

a. a 68% confidence level

b. a 95% confidence level

c. a 99.7% confidence level.

2.9 After obtaining a sample of size 1000, a researcher constructed the following 95% confidence interval: [1250, 1420]. Use this information to identify $\bar{x}$ and $\sigma_{\bar{x}}$.

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Activity 3

Researchers, pollsters, and statisticians sample populations in order to gain information about the population. Since statistics gathered by sampling only provide estimates of the population parameters, errors are possible. In this activity, you examine how to quantify and minimize the risk of making mistakes.

Exploration 1

Since finding the actual mean of a large population can be difficult, it is often necessary to use statistics to make predictions about a population. In this exploration, you investigate the use of confidence intervals in estimating a population mean.

Imagine that your school business club is completing a survey of the average incomes of high school graduates in your state. The club has budgeted enough money to conduct a telephone survey of 30 former students. The information gathered in the survey is shown in Table **6**:

Table 6: A sample of 30 incomes (in dollars)

4000	17,400	10,900	15,600	8000
9800	10,000	19,400	14,400	42,000
32,000	4500	46,000	90,400	24,400
112,000	19,600	25,400	7000	16,000
16,800	23,600	26,200	19,600	28,400
14,800	19,400	13,500	18,200	6600

a. Construct 68%, 95%, and 99.7% confidence intervals of the sample mean of the data. (Estimate $\sigma_{\bar{x}}$ using the sample standard deviation s.)

b. Graph the confidence intervals above a number line as in Activity **2**. **Note**: Save this graph for use in Exploration **2**.

c. Make a statement about where you would expect to find the actual mean income of high school graduates in your state at a:

1. 68% confidence level
2. 95% confidence level
3. 99% confidence level.

Discussion 1

a. Which confidence interval is most likely to include the actual average income of high school graduates in your state? Explain your response.

b. Which confidence level has the least range of estimates of the actual average income of the graduates? Explain your response.

c. What is the relationship between interval size and the probability of making an error?

d. How could the confidence intervals be narrowed without increasing the chances of making an error? Explain your response.

e. Describe some of the advantages and disadvantages of using:

1. a 99.7% confidence level
2. a 95% confidence level
3. a 68% confidence level.

f. Explain why it is just as important to use a random sample to estimate a confidence interval as it is to use a random sample to estimate the population mean.

Exploration 2

Roberto is writing an article for the school newspaper. Based on the study by the school business club, he wants to claim that the average income for high school graduates in your state is $36,000. However, the editor of the newspaper disagrees. She believes that a better value for the average income is $16,000. Who is right? In this activity, you use confidence intervals to test the claims of Roberto and his editor.

Mathematics Note

Statisticians often make hypotheses or claims about the parameters of a population, then use sampling techniques to test their claims. If a researcher assumes that a population parameter has a specific value, then a hypothesis can be formed about the consequences of that assumption. In statistical analysis, there are two types of hypotheses.

A **null hypothesis (H_0)** is a statement about one or more parameters. The **alternative hypothesis (H_a)** is the statement that must be true if the null hypothesis is false. The null hypothesis usually involves a claim of no difference or no relationship. In many situations, the null hypothesis and alternative hypothesis are negations of each other, but this is not necessarily the case.

For example, if a researcher wants to test the claim that the mean income of the population is $25,000, then the null hypothesis is that the mean income of the population equals $25,000. The alternative hypothesis is that the mean income of the population does not equal $25,000. Symbolically, this can be represented as shown below:

$$H_0: \mu = \$25{,}000$$
$$H_a: \mu \neq \$25{,}000$$

a. State the null and alternative hypotheses for Roberto's claims about the average income for high school graduates.

b. On your graph of the 68%, 95%, and 99.7% confidence intervals of the income data from Table **6** (from Part **b** of Exploration **1**), draw a vertical line to represent Roberto's claim about the average income.

c. Record the confidence intervals that include Roberto's predicted mean.

Mathematics Note

A **hypothesis test** may consist of the following steps.

- State null and alternative hypotheses about a parameter of a population.
- If the null hypothesis is true, predict what this implies about a sample of the population.
- Take a sample of the population and compare the results with your prediction.
- If the results are inconsistent with the prediction, then you can conclude, with some level of certainty, that the null hypothesis is false and, therefore, reject it.
- If the results are consistent with the prediction, you fail to reject the null hypothesis. The failure to reject the null hypothesis does not guarantee that the null hypothesis is true, only suggests that it might be true.

For example, to test the claim that the mean income of high school graduates is \$25,000 at the 95% confidence level, you would:

- State the null hypothesis: $H_0: \mu = \$25{,}000$.
- Sample the population, calculate the mean H_0 of the sample, and construct the 95% confidence interval around $\bar{x}$.
- If \$25,000 is not in the 95% confidence interval, then you would reject the null hypothesis and accept the alternative hypothesis $H_a: \mu \neq \$25{,}000$

. If \$25,000 is in the 95% confidence interval, then you would fail to reject the null hypothesis. You can only conclude that H_0 may or may not be true.

d. Determine whether you would reject or fail to reject the hypotheses from Part **a** at the 68%, 95%, and 99.7% confidence levels.

e. Repeat Parts **a–d** for the editor's claim about the average income for high school graduates.

Discussion 2

a. Given your results in Exploration **2**, what can you conclude about the claims of Roberto and his editor? Explain your response.

b. What is the difference between "failing to reject" a null hypothesis and "accepting" a null hypothesis?

c. Why does the rejection of the null hypothesis result in the acceptance of the alternative hypothesis?

d. Does failing to reject a null hypothesis prove that it is true? Explain your response.

e. Does rejecting a null hypothesis prove that it is false?

f. Scientists usually require at least a 95% confidence level to reject a hypothesis. Using this standard, state conclusions about the editor's and Roberto's hypotheses.

g. After Roberto discovers that his hypothesis cannot be rejected at a 99.7% confidence level, he exclaims, "This shows that there is a 99% chance that my hypothesis is right." Explain what is wrong with Roberto's reasoning.

Assignment

3.1 To test her null hypothesis, Roberto's editor conducted a survey of 30 acquaintances who recently graduated from high school. Their incomes (in dollars) are shown in the table below.

6200	9400	22,300	6200	18,000	38,000
17,100	21,500	17,300	27,500	11,900	23,200
19,000	11,000	13,200	16,500	13,800	13,400
15,500	8700	33,000	34,000	16,200	9500
14,000	11,000	32,000	14,700	6400	3400

a. The editor claimed that the mean income is $16,000. Use the data in the table above to test her null hypothesis at a confidence level of your choice.

b. Based on your test in Part **a**, state your conclusions.

c. Does this test prove that the editor's claim is correct? Explain your response.

d. Roberto complains that the editor's sample is not representative of the high school graduates in their state. Describe some possible sources of bias in the sample.

3.2 Roberto decides to survey his own sample of 30 graduates. He telephones the class presidents from each of the past 30 yr and records their incomes in the table below.

74,200	29,500	17,000	74,000	44,500	31,000
26,000	19,000	36,000	12,500	32,000	21,000
6100	27,700	34,000	29,000	72,100	30,500
93,000	12,000	19,000	31,000	26,200	33,200
34,000	31,000	23,400	46,300	35,400	6200

a. Recall that Roberto claimed that the mean income is $36,000. Use the data in the table above to test his claim at a 95% confidence level.

b. Based on your test in Part **a**, state your conclusions.

c. What possible sources of bias are there in Roberto's sample?

3.3 **a.** Make a claim about the mean income of graduates in your state.

b. Write null and alternative hypotheses for your claim.

c. Describe what group you would sample to test your hypothesis and how you would collect your information.

d. What level of confidence would you choose to test your hypothesis? Defend your choice.

3.4 In its advertisements, the Shiny Bright Company claims that its 60-watt bulbs have an average life expectancy of 1250 hr. They based this conclusion on a sample of 1000 light bulbs in which the mean life span was 827 hr, with a standard deviation of 424 hr.

According to the company, their advertised value is within 1 standard deviation of the sample mean. Therefore, their claim cannot be rejected. What is wrong with their logic?

3.5 In tests conducted by outside experts, Shiny Bright's "Best Bulb" had an average life expectancy of 2000 hr, with a standard deviation of 300 hr. The Hi-Glow Company claims their "Long-Life" bulbs are better because they last even longer. Both companies decide to test the hypothesis that the mean life expectancy of Long-Life bulbs is 2000 hr.

Using a random sample of 100 Long-Life bulbs, Shiny Bright found a mean life span of 2040 hr, with a standard deviation of 470 hr. The firm concluded that the mean life of Long-Life bulbs is the same as that of their Best Bulbs.

Hi-Glow tested 10,000 Long-Life bulbs and found a mean life span of 2010 hr, with a standard deviation of 400 hr. They concluded that the mean life span of their bulbs is not the same as that of Shiny Bright's bulbs. Perform a hypothesis test at the 95% confidence level to determine which company you think is right.

* * * * *

3.6 Cereal boxes usually display the disclaimer that the boxes are filled by weight, not by volume, and that some settling may occur during shipping. A high school statistics class wants to see if this is true or if the actual mass of cereal is significantly less than is claimed. To test a manufacturer's claims, students randomly sample forty 397-g boxes of the same brand of cereal. The table below shows the mass of each box of cereal, rounded to the nearest gram:

402	397	404	384	390	395	397	385	392	399
380	390	408	403	389	389	393	381	402	401
383	403	383	392	400	392	395	395	406	396
408	383	381	390	401	385	382	404	409	387

a. State the null and alternative hypotheses for this experiment.

b. Test the hypothesis at the 95% confidence level.

c. Decide whether to reject or fail to reject the null hypothesis. Justify your reasoning.

d. Explain what conclusion the class should reach about the net mass of the boxes of cereal.

3.7 Loretta has bowled in a league for years. Last season, her average score per game was 152. This season, she bought a new ball. Her weekly average scores for this season with the new ball are shown below.

194	153	171	199	141	146
151	190	171	168	150	128
166	166	160	183	141	210
169	172	132	126	195	191
127	155	204	191	129	170
180	150	155	167	192	168

Has Loretta's average score changed significantly?

a. Use the information in the table to construct a 68% confidence interval.

b. Test the claim that Loretta's average has changed significantly from last year's average of 152 at the 68% confidence level.

c. Based on your hypothesis test, determine whether or not Loretta's bowling average has changed significantly since she started using the new ball.

* * * * * * * * * *

Research Project

Design and implement a plan to determine the average age of a home in your community. Your report should include at least the following information:

a. A claim about the mean age of homes in your community.

b. The null and alternative hypotheses for your claim.

c. A description of your sample and how you collected the data.

d. The level of confidence you chose to test your hypothesis.

Summary Assessment

Whether they recognize this fact or not, anglers often use sampling to judge the waters in which they fish. For example, after catching several big fish, an angler may conclude that a lake contains a healthy population of large fish. On the other hand, anglers who have little success may grumble that, "There aren't many big fish in this lake."

In such informal evaluations, anglers rarely use confidence intervals or take the variability of their samples into account. Wildlife managers, however, need a more reliable method to describe a fish population.

Suppose you are a wildlife manager responsible for estimating the mean size of the fish in a "lake" provided by your teacher. You must evaluate your estimate by taking a sample of the fish and performing a hypothesis test.

1. a. After examining the fish in the lake, make null and alternative hypotheses about their average length.

   b. Select a confidence level with which to test your hypothesis.

   c. Sample the population using an appropriate sample size.

2. **a.** Construct the appropriate confidence interval and use it to test your hypothesis.

   b. Write a conclusion based on your test.

3. If you had the opportunity to sample the fish population again, what would you change about your technique?

Module Summary

- The **standard deviation** of a population, denoted by σ, is calculated using the formula below, where Type equation here. represent all the individual values in the population, μ represents their mean, and *N* represents the population size:

$$\sigma = \sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_N-\mu)^2}{N}}$$

- The standard deviation of a sample is the **sample standard deviation**, denoted by s. The formula for calculating s is:

$$s = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$$

  where $x_1, x_2, \cdots, x_n$ represent the individual values in the sample, $\bar{x}$ represents the sample mean, and *n* represents the sample size.

- The **sampling distribution of sample means** contains the means ($\bar{x}$) of *all* possible samples of size *n* from a population.

- The mean of the sampling distribution of sample means, denoted by $\mu_{\bar{x}}$, equals the population mean μ.

- The standard deviation of the sampling distribution, denoted by $\sigma_{\bar{x}}$, equals $\sigma/\sqrt{n}$, where σ is the population standard deviation and *n* is the sample size. When σ is unknown, the standard deviation of the sample (s) may be used as an estimate of σ.

- The **central limit theorem** states that, regardless of the population, as the sample size increases, the sampling distribution of sample means approaches a normal distribution. As a rule of thumb, samples of size $n \geq 30$ are large enough to assume that the sampling distribution of sample means approaches a normal distribution.

- The curve that describes the shape of a normal distribution is the **normal curve**. As in all continuous probability distributions, the total area between the *x*-axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

- A **confidence interval** for a sample mean is an interval of numbers in which you would expect to find the population mean. The 68–95–99.7 rule for normal distributions implies the following:
- For approximately 68% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$ contains the population mean μ.
- For approximately 95% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ contains the population mean μ.
- For approximately 99.7% of all sample means $\bar{x}$, the confidence interval $[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$ contains the population mean μ.
- In statistical analysis, there are two types of hypotheses. A **null hypothesis ($\boldsymbol{H_0}$)** is a statement about one or more parameters. The **alternative hypothesis ($\boldsymbol{H_a}$)** is the statement that must be true if the null hypothesis is false. The null hypothesis usually involves a claim of no difference or no relationship. In many situations, the null hypothesis and alternative hypothesis are negations of each other, but this is not necessarily the case.
- A **hypothesis test** may consist of the following steps.
- State null and alternative hypotheses about a parameter of a population.
- If the null hypothesis is true, predict what this implies about a sample of the population.
- Take a sample of the population and compare the results with your prediction.
- If the results are inconsistent with the prediction, then you can conclude, with some level of certainty, that the null hypothesis is false and, therefore, reject it.
- If the results are consistent with the prediction, you fail to reject the null hypothesis. The failure to reject the null hypothesis does not guarantee that the null hypothesis is true, only suggests that it might be true.

Selected References

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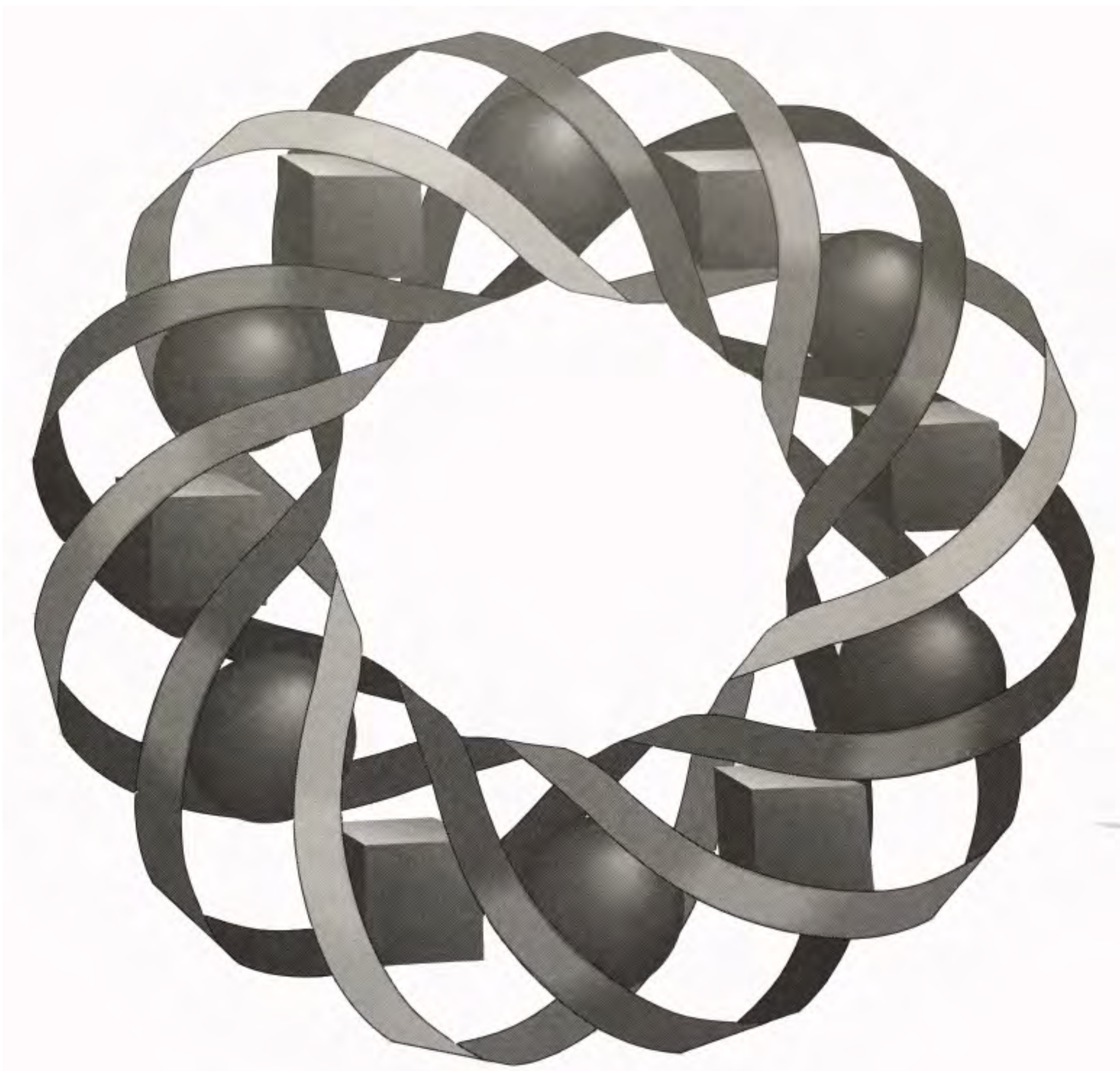
Is It Really True?



In this module, you continue your investigations of logical reasoning.

Terri Dahl • Laurie Paladichuk • Peter Rasmussen

Is It Really True?

Introduction

The August sun beats down on Colter's aching back. He has been working for Mykelti Construction on this job for the past 2 weeks, 7 days a week, 12 hours a day. When Colter asked his supervisor for a day off, she replied, "If you finish the foundation, then you're off for the weekend."

It is now Friday afternoon. Colter is sure that he won't finish the foundation by the end of the day. He considers the responsibilities of his job and of keeping the entire construction project on schedule. A dull pain throbs inside his head. Then he imagines two days of rest, an afternoon by the pond, and dinner with his family.

Discussion

a. How should Colter expect his supervisor to react if he takes the weekend off without finishing the foundation? Justify your response.

b. How should Colter expect his supervisor to react if he takes the weekend off after finishing the foundation? Justify your response.

Activity 1

Colter's company won a bid for a construction project. The project involves remodeling a dentist's office in a large office complex. The following guidelines are included in the bid.

- All materials used by Mykelti Construction will be purchased locally.
- Some of the wood used by Mykelti Construction is oak.
- No employees will smoke on the job site.

Exploration

To determine how closely the company follows these guidelines, Mykelti Construction decides to conduct a study. Model this study by completing the following exploration using two sets of chips, each a different color.

a. Recall that a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.

Consider the statement "All materials used by Mykelti Construction will be purchased locally." This statement implies that two subsets of materials could exist: one consists of materials that are purchased locally, and the other consists of materials that are not purchased locally.

1. The company plans to examine four recent purchases. Let one set of colored chips represent materials purchased locally, and the other set of chips represent materials not purchased locally. Use the different colored chips to represent all possible combinations of purchases.
2. Identify every combination of four chips that correctly represents the true statement "All materials used by Mykelti Construction will be purchased locally."

b. In previous modules, you have seen that a statement can be shown to be false by finding at least one counterexample. Identify every combination of four chips that represents a counterexample to the statement "All materials used by Mykelti Construction will be purchased locally."

c. A **quantifier** is a word or phrase that indicates quantity in a statement. Some commonly used quantifiers are *some*, *at least one*, *all*, *every*, and *none*. Use a quantifier to write a single statement that correctly describes all the counterexamples identified in Part **b**.

Mathematics Note

Two general statements are **logically equivalent** if one statement is true (or false) exactly when the other is true (or false). In other words, the truth of one statement implies the truth of the other in all cases, and vice versa. Logical equivalence is denoted by the symbol $\equiv$.

For example, consider the general statements "all p are q" and "no p are not q." These are logically equivalent because they are either both true or both false for all possible selections for p and q. Symbolically, this equivalence can be denoted as follows:

$$\text{all } p \text{ are } q \equiv \text{no } p \text{ are not } q$$

For instance, suppose that p represents squares and q represents rectangles. In this case, "all p are q" indicates "All squares are rectangles." The statement "no p are not q" represents "No squares are not rectangles." Both of these statements are true.

It also is possible to consider logical equivalence using sets. For example, Figure **1** shows a Venn diagram for two sets A and B. In this case, the statement "no A are not B" is logically equivalent to "all A are B," because they are either both true or both false, no matter what sets A and B represent.

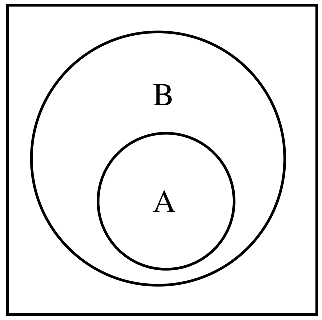


Figure 1: Venn diagram for sets A and B

d. Use a different quantifier to rewrite your statement in Part **c** without changing its truth value.

e. The company plans to use two different woods, oak and pine, in the remodeling project. Let one set of chips represent oak and the other represent pine. Identify every combination of four chips that correctly represents the true statement "Some of the wood used by Mykelti Construction is oak." Then repeat Parts **b–d** for this statement.

f. Four different employees will work on the remodeling project. Let one set of chips represent smokers and the other represent nonsmokers. Identify every combination of four chips that correctly represents the true statement "No employees will smoke on the job site." Repeat Parts **b–d** for this statement.

Mathematics Note

The **negation** of a statement *p* has the opposite truth value of *p*. In general, the negation of *p* is written as "not *p*," symbolized as "$\sim p$."

For example, consider the statement *p*, "All lawyers are accountants." In this case, $\sim p$ is "Not all lawyers are accountants."

In order to write clear and grammatically correct negations, it is often helpful to use logically equivalent statements. Table **1** shows three general statements, their negations, and some statements that are logically equivalent to each negation.

Table 1: Three general statements and their negations

Statement	Negation (and some Logical Equivalents)
All A are B.	Not all A are B. It is not the case that all A are B. Some A are not B. At least one A is not B.
Some A are B.	It is not the case that some A are B. No A are B. All A are not B.
No A are B.	It is not the case that no A are B. Some A are B. At least one A is B.

For example, suppose that A is the set of squares and B is the set of rectangles. In this case, the statement "all A are B" would represent "all squares are rectangles." This statement is true. Its negation is "not all squares are rectangles." This statement is false. Using the logically equivalent statements listed in Table **1**, the negation has the same truth value (false) as each of the following:

- It is not the case that all squares are rectangles. (false)
- Some squares are not rectangles. (false)
- At least one square is not a rectangle. (false)

g. Write the negation of each of the following statements.

1. All materials used by Mykelti Construction will be purchased locally.
2. Some of the wood used by Mykelti Construction is oak.
3. No employees will smoke on the job site.

Discussion

a. Is the statement you wrote in Part **c** of the exploration the negation of the original statement? Explain your response.

b. Consider the false statement "All houses are painted white."

1. Describe the negation of this statement.
2. Explain why the following true statement is not the negation of the original statement: "Some houses are painted white."

c. Use logically equivalent statements to express the negation of each of the following in two different ways.

1. All Saint Bernards are dogs.

2. Some presidents of the United States have been women.

3. None of my friends have been to Los Angeles.

4. Some puppies are canines.

d. Give an example of a sentence that doesn't have a negation and explain why this is the case.

Assignment

1.1 **a.** Use logically equivalent statements to rewrite each of the following with a different quantifier.

1. Some people are old.

2. No one can dance.

3. All buildings have elevators.

b. Write the negation of each statement in Part **a**.

1.2 **a.** Write the negation to the statement "All horses are not black."

b. Use a logical equivalent to rewrite the statement "All horses are not black."

1.3 Consider the statement "All police officers are single." Do any of the statements in Parts **a–f** represent the negation of this statement? Justify your response.

a. No police officers are single.

b. Some police officers are not single.

c. All police officers are not single.

d. Some police officers are divorced.

e. Some police officers are not married.

f. Not all police officers are single.

1.4 Use logically equivalent statements to rewrite each of the following.

a. All Mykelti construction workers are men.

b. Some college students major in engineering.

c. No professional football team includes female players.

1.5 The diagram below shows three houses, all of which are painted white. Use similar diagrams to demonstrate the difference between the statements "some houses are painted white" and "some houses are not painted white."

* * * * *

1.6 Describe how you could determine the truth value for each of the following statements.

a. Some people in your math class are on the football team.

b. All people in your math class are taking a science class this year.

c. No students in your math class ride the bus.

1.7 An advertisement for a department store states, "All sale items are 40% off." One of the sale items in the store, normally $25.00, is priced at $20.00. Is the advertisement true? Explain your response.

1.8 Write the negation of each of the following statements and identify its truth value.

a. All integers are real numbers.

b. Some integers are whole numbers.

c. No irrational numbers are rational numbers.

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Activity 2

Mykelti Construction is currently accepting applications for several new supervisory positions. In this activity, you investigate how different criteria for the jobs affect potential candidates.

Exploration

Mykelti Construction plans to promote one of its current employees to supervising electrician. The personnel director recommends that candidates be previously trained for a supervisory position and be nonsmokers. Her assistant, however, worries that these restrictions may result in too few potential candidates. He recommends that the person be previously trained for a supervisory position or be a nonsmoker. In this exploration, you examine the differences between these two recommendations.

a. Draw a Venn diagram to represent all the employees of the company. This diagram should show employees who have received supervisory training, those who are nonsmokers, those that are both trained and are nonsmokers, and those who are neither trained nor nonsmokers.

b. **1.** Shade the region that represents those employees who satisfy the personnel director's recommendation, in other words, those who have received supervisory training and are nonsmokers.

2. Identify the regions in the Venn diagram that represent employees who do not satisfy the director's recommendation.

3. Describe the employees who correspond with the regions identified in Step **2**.

4. Given the two requirements in Step **1**, each employee fits into one of four types. The suitability of each of these types for the supervisor's job can be analyzed using a table.

Create a table with headings like those in Table **2** below, showing each of the four types of employees. Use T to indicate that the column heading is true for a candidate and F to indicate that it is false. This type of table is referred to as a **truth table**.

Table 2: Employees to hire for supervising electrician

Nonsmoker	Trained	Possible Candidate

5. Using the personnel director's recommendation, the employees who are not qualified for the supervisor's job may be described by the following sentence: "These employees are not both trained and nonsmokers."

Recall that two statements can be joined into a **compound statement** using the connectives *and* and *or*. Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, a compound statement like the one above can be represented in general as follows: $\sim(p \text{ and } q)$.

Complete a truth table with headings like those in Table **3** for $\sim(p \text{ and } q)$.

Table 3: Truth table for $\sim(p \text{ and } q)$

p	q	p and q	$\sim(p$ and $q)$

6. Another way to describe the employees who are not qualified for the supervisor's job is the following sentence: "These employees-are not trained or are smokers."

Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows: $\sim p$ or $\sim q$.

Add the necessary columns to Table **3** for this statement, then complete the truth table. **Note:** In mathematics, the connective *or* is **inclusive**. The inclusive *or* is interpreted as meaning one statement or the other statement, or both statements.

c. 1. Make another copy of your Venn diagram from Part **a**. Shade the region that represents those employees who satisfy the assistant's recommendation, in other words, those who have received supervisory training or are nonsmokers.

2. Identify the regions in the Venn diagram that represent employees who do not satisfy the assistant's recommendation.

3. Describe the employees who correspond with the regions in Step **2**.

4. As in Part **b**, each employee fits into one of four types and the suitability of each type for the supervisor's job can be analyzed using a table. Complete a table with headings like those in Table **4** for each of the four types of employees.

Table 4: Employees to hire for supervising electrician

Nonsmoker	Trained	Possible Candidate

5. Using the assistant's recommendation, the employees who are not qualified for the supervisor's job may be described by the following sentence: "These employees are not both trained or nonsmokers."

Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows: $\sim(p$ or $q)$. Complete Table **5** for this statement.

Table 5: Truth table for $\sim(p$ or $q)$

p	q	p or q	$\sim(p$ or $q)$

6. Another way to describe the employees who are not qualified for the supervisor's job is the following sentence: "These employees are not trained and are smokers."

   Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows:
   $\sim p$ and $\sim q$.

   Add the necessary columns to Table **5** for this statement, then complete the truth table.

Discussion

a. Given your results in the exploration, explain how you could use truth tables to identify when two compound statements are logically equivalent.

b. Describe two different ways of expressing the negation of "*p* and *q*," where *p* and *q* represent statements.

c. Describe two different ways of expressing the negation of "*p* or *q*," where *p* and *q* represent statements.

Mathematics Note

De Morgan's laws, named in honor of British mathematician Augustus De Morgan (1806–1871), apply to negating statements that contain the connectives *and* and *or*.

First, the negation of the compound statement "*p* and *q*" is the compound statement "not (*p* and *q*)." This is logically equivalent to the compound statement "not *p* or not *q*."

For example, consider the compound statement "Joan is tall and she is a basketball player." One form of the negation of this statement is "It is not true that Joan is both tall and a basketball player." This has the same truth value as the statement "Joan is not tall or she is not a basketball player."

Second, the negation of the compound statement "*p* or *q*" is the compound statement "not (*p* or *q*)." This is logically equivalent to the compound statement "not *p* and not *q*."

For example, consider the statement "They want to go to the baseball game on Wednesday or Thursday night." One form of the negation of this statement is "It is not true that they want to go to the baseball game on Wednesday or Thursday night." This has the same truth value as the statement "They do not want to go to the baseball game on Wednesday and they do not want go to the baseball game on Thursday."

d. Describe how to use De Morgan's laws to rewrite the following compound statement in an equivalent form: "It is not the case that they are overworked and underpaid."

Assignment

2.1 When *or* is used as a connector in casual English, it is sometimes interpreted as one or the other, but not both. This is an **exclusive** *or*. For example, consider an intersection where the driver must turn left or right. In this case, turning left or turning right means you choose one or the other, but not both. Computer scientists use the letters *xor* to indicate that the exclusive *or* is desired.

a. Construct a truth table for the exclusive *or*.

b. How does this truth table differ from the one for the inclusive *or*?

2.2 **a.** Suppose you are in a restaurant. A waiter asks "Would you like coffee or tea?" Which interpretation of the connective *or* does this situation imply? Explain your response.

b. When you answer, "Coffee," the waiter then asks, "Cream or sugar?" Which interpretation of the connective *or* does this situation imply? Explain your response.

c. List two examples of questions, statements, or phrases that use the exclusive *or*.

d. Give two examples of questions, statements, or phrases that use the inclusive *or*.

2.3 In most two-door cars, the interior light comes on when at least one of the doors is open.

a. Create a truth table that describes when the interior light is on using the following two statements: "The driver's door is open" and "The passenger's door is open."

b. Which interpretation of the connective *or* does this situation imply? Explain your response.

2.4 Consider the statement "It is not the case that Julia's age is less than or equal to 5 yr."

a. Rewrite this statement using mathematical notation.

b. Use De Morgan's laws to rewrite the statement with the connective *and*.

2.5 Eileen is of average height. A newspaper writes the following sentence describing Eileen: "It is not true that she is too tall or too short."

a. Rewrite this statement using mathematical notation.

b. Use De Morgan's laws to rewrite the statement in an equivalent form.

c. Which sentence do you think is easier to interpret, the original or the one written in Part **b**?

* * * * *

2.6 Determine the truth value of each of the following statements. Then use De Morgan's laws to write the negation of the statement.

a. The number 3 is an even integer and the number 6 is an odd integer.

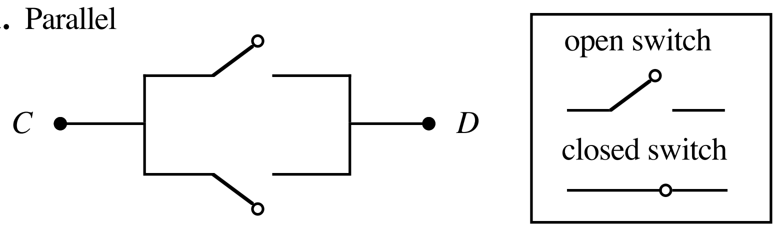
b. The Pacific is a river or the Pacific is an ocean.

c. $3 = 5$ or $4 \neq 5$

2.7 The diagrams in Parts **a** and **b** below show portions of two different electrical circuits. In both circuits, electricity flows from point C to point D. Both have switches that allow electricity to pass when they are closed, but do not allow electricity to flow when they are open. The switches in Part **a** are parallel, while the switches in Part **b** are in series.

Which circuit illustrates the connective *or*? Which illustrates the connective *and*? Explain your responses.

a. Parallel



b. Series

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Research Project

George Boole was instrumental in the development of symbolic logic. Find out more about Boole's life and describe the connection between Boolean algebra, symbolic logic, and the fields of electronics and engineering.

Activity 3

It's lunch time at the Mykelti construction site. Two workers are arguing about the relative merits of their dogs. One is the proud owner of a Chesapeake Bay retriever. The other has a Labrador retriever.

The Chesapeake's owner states, "No Labrador can retrieve better than my dog."

The Lab's owner replies, "If the dog is smart, then it's a Labrador."

In this activity, you explore the meaning of conditional statements and practice rewriting them in logically equivalent forms.

Exploration

In this exploration, you use Venn diagrams to illustrate the statements made by the two workers, then use logically equivalent statements to rewrite them.

a. Recall that a **conditional statement** may be written in the form "$p \rightarrow q$," where p represents the hypothesis and q represents the conclusion. Identify the hypothesis and the conclusion in the statement, " If the dog is smart, then it is a Labrador."

b. Figure **2** shows a Venn diagram that depicts the statement, "If the dog is smart, then it is a Labrador." Using the quantifiers *all, some*, and *no*, write a statement describing the dogs represented by each of the three regions of the Venn diagram. Do not use the words *if* or *then* in your three statements.

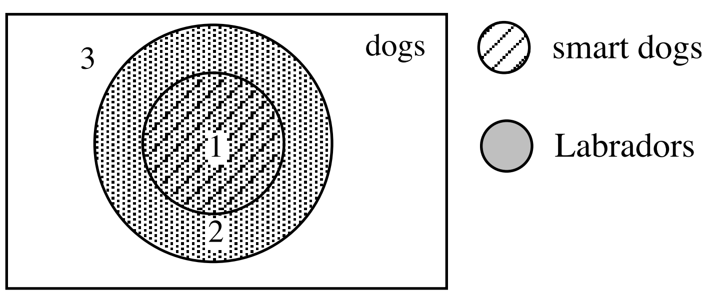


Figure 2: Venn diagram for a conditional statement

c. To disprove the Labrador owner's statement "If the dog is smart, then it is a Labrador," the Chesapeake owner considers some different types of dogs. If the worker finds a smart Labrador, then—for that dog—the hypothesis is true and the conclusion is true. In this case, there is no conflict with the Venn diagram in Figure **2**, since such a dog could be found in region 1. Therefore, the original statement is true.

Consider the other three types of dogs that the Chesapeake owner could find:

- a smart dog that is not a Labrador—(true hypothesis, but false conclusion)
- a dog that is not smart and is a Labrador—(false hypothesis, but true conclusion)
- a dog that is not smart and is not a Labrador—(false hypothesis and false conclusion)

To see if there are any types of dogs that the Chesapeake owner could use to dispute the original statement, copy and complete a table with headings like those in Table **6**.

Table 6: Possible cases of the Labrador owner's statement

Hypothesis	Conclusion	Conflict?	Statement
T	T	no	T
T	F		
F	T		
F	F		

d. Consider the truth table for the compound statement "p or q" shown in Table **7** below. Although this truth table and the one in Part **c** are not identical, they both have three cases where the statement is true and one case where the statement is false. This may suggest a relationship between a conditional statement "$p \rightarrow q$" and a compound statement with a connective *or*.

Table 7: Truth table for *p* or *q*

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

Consider three other cases of a statement with a connective *or*: "$\sim p$ or q," "p or $\sim q$," and "$\sim p$ or $\sim q$." Are any of these statements logically equivalent to the conditional "$p \rightarrow q$"? Use truth tables to verify your response.

Discussion

a. **1.** Consider the conditional statement "If the dog is smart, then it is a Labrador." If you find a Labrador that is not smart, is this statement necessarily false? Explain your response.

2. Is this conditional statement necessarily false if you find a smart dog that is not a Labrador? Explain your response.

b. Describe all circumstances that would make the conditional statement "if *p*, then *q*" false.

c. Explain how to rewrite the following statement using the connective *or*: "If a dog is smart, then it is a Labrador."

d. After the Labrador's owner finished speaking, the Chesapeake's owner retorted, "If the moon is made of green cheese, then all smart dogs are Labs." What can you conclude about this statement?

Assignment

3.1 Consider the conditional statement, "If the tree is an oak, then it is tall."

a. Construct a Venn diagram that models this statement.

b. Using the quantifiers *all*, *some*, and *no*, write three statements that describe the sets of trees in the Venn diagram. Do not use the words *if* or *then* in your three statements.

3.2 **a.** Identify the hypothesis and conclusion in the conditional statement "If the house was built by Mykelti Construction, then its materials were purchased locally."

b. When is this conditional statement false?

c. Use a quantifier such as *all*, *some*, or *no* to rewrite the statement in Part **a** in a logically equivalent form. Do not use the words *if* or *then* in your statement.

d. Use the connective *or* to rewrite the conditional statement in Part **a** in a logically equivalent form.

3.3 In the introduction, Colter's supervisor told him "If you finish the foundation, then you're off for the weekend."

a. Identify the hypothesis and conclusion of this conditional statement.

b. Describe the four possibilities Colter should consider when interpreting this statement.

c. If Colter doesn't finish the foundation, can he justify taking the weekend off? Explain your response.

3.4 Use the connective *or* to rewrite each of the following conditional statements in a logically equivalent form.

a. If it is fall, then the leaves are falling.

b. If the worker is not a carpenter, then she is an electrician.

c. If the person is happy, then he is not young.

d. If it is not snowing, then it is not winter.

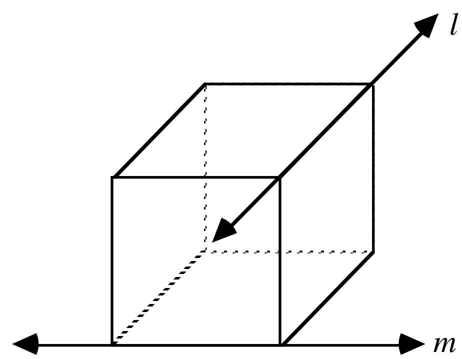
3.5 **a.** Write the following statement as a conditional: "All prime numbers greater than 2 are odd."

b. Is this statement true? Explain your response.

3.6 Consider the statement "If three angles of one triangle are congruent to three angles of another triangle, then the triangles are congruent." Is this statement true? Explain your response.

* * * * *

3.7 **Skew lines** are lines in space that lie in different planes. For example lines l and m contain the edges of a cube that lie in different planes. Although skew lines are not parallel, they do not intersect.



a. Construct a Venn diagram that models the conditional statement, "If two lines are parallel, then they never intersect."

b. Using the quantifiers *all*, *some*, and *no*, write three statements that describe the sets of lines in the Venn diagram. Do not use the words *if* or *then* in your three statements.

c. **1.** Where do skew lines fit in your Venn diagram from Part **a**?

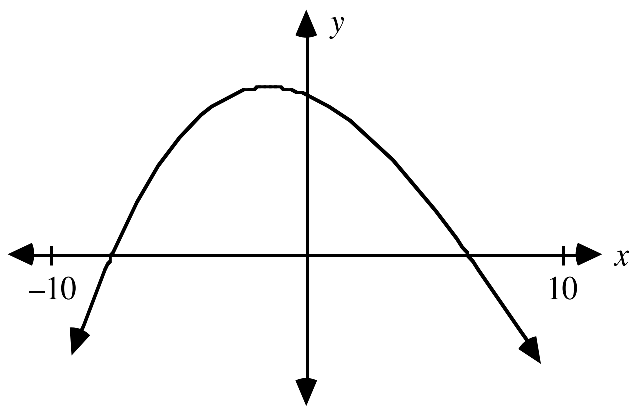
2. Do skew lines serve as a counterexample to the statement in Part **a**? Explain your response.

3.8 Consider the conditional statement "If the polygon is a triangle, then $a^2 + b^2 = c^2$, where a, b, and c are the lengths of the sides of the triangle and c is the length of the longest side."

a. Identify the hypothesis and the conclusion of this conditional.

b. Is this statement true? Explain your response.

3.9 The following graph shows a portion of a polynomial function.



a. One of your classmates makes the claim: "All the roots are between –10 and 10." Write this statement as a conditional.

b. Another classmate makes the claim "Some root is not between –10 and 10." Restate this claim using the connective *or*.

* * * * * * * * * *

Activity 4

Back at the Mykelti Construction site, the two dog owners are still arguing. While the rest of the crew listens, the Labrador owner repeats his claim: "If a dog is smart, then it's a Labrador."

One of his coworkers then asks, "Do you mean that if a dog is a Labrador, then it's smart?"

"No," a second worker chimes in. "He means that if a dog isn't smart, then it's not a Labrador."

"That's not it," cries a third. "What he really wants to say is if a dog's not a Labrador, then it's not smart."

In this activity, you determine which of these three statements has the same truth value as the owner's original claim.

Mathematics Note

The **converse** of a conditional statement is formed by interchanging the hypothesis and the conclusion. The converse of $p \to q$ can be represented by $q \to p$ or "if q, then p."

The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. The inverse of $p \to q$ can be represented by $\sim p \to \sim q$ or "if $\sim p$, then $\sim q$."

The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and the conclusion and negating both of them. The converse of $p \to q$ can be represented by $\sim q \to \sim p$ or "if $\sim q$, then $\sim p$."

For example, consider the conditional statement "If today is July 4, then it is summer in the northern hemisphere." Its converse is the conditional "If it is summer in the Northern hemisphere, then today is July 4."

Its inverse is the conditional "If today is not July 4, then it is not summer in the Northern hemisphere."

Its contrapositive is the conditional "If it is not summer, then today is not July 4 in the Northern hemisphere."

Exploration

In this exploration, you look for patterns in the truth tables for the converse, inverse, and contrapositive of the conditional $p \to q$.

a. Create a truth table for $p \to q$ and its converse, inverse, and contrapositive.

b. Describe any patterns or relationships you observe in the table.

Discussion

a. When a conditional statement is true, is its converse also true? Explain your response.

b. When a conditional statement is true, is its inverse also true? Explain your response.

c. When a conditional statement is true, is its contrapositive also true? Explain your response.

d. Which worker's statement best describes the Labrador owner's original claim? Justify your response.

Assignment

4.1 Consider the statement, "If a polygon is a square, then it is a quadrilateral."

a. Write the converse of this statement and explain whether or not the converse is true.

b. Write the inverse of the statement and explain whether or not the inverse is true.

c. Write the contrapositive of the statement and explain whether or not the contrapositive is true.

4.2 **a.** Write an example of a conditional statement in mathematics whose converse is true.

b. Write an example of a conditional statement in mathematics whose converse is false.

4.3 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw? Explain your responses.

a. If a house was built by Mykelti Construction, then its materials were purchased locally. The materials used in the house were not purchased locally.

b. If a house was built by Mykelti Construction, then its materials were purchased locally. The house was not built by Mykelti Construction.

4.4 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If it's raining, then the sidewalk is wet. It's raining.

b. If it's raining, then the sidewalk is wet. The sidewalk is wet.

c. If it's raining, then the sidewalk is wet. The sidewalk isn't wet.

d. If it's raining, then the sidewalk is wet. It's not raining.

4.5 Consider the statement, "If the diagonals of a quadrilateral are congruent, then the quadrilateral is a square."

a. Explain whether or not the converse of this statement is true.

b. Explain whether or not the inverse of this statement is true.

c. Explain whether or not the contrapositive of this statement is true.

* * * * *

4.6 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The polygon is a triangle.

b. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The sum of the measures of its exterior angles is 360˚.

c. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The polygon is not a triangle.

d. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The sum of the measures of its exterior angles is not 360˚.

4.7 A **tautology** is a statement that is always true. Use truth tables to determine which of the following statements are tautologies.

a. $\left[(p \to q) \text{ and } p\right] \to q$

b. p or $\sim p$

c. $(p \text{ and } \sim q) \text{ and } (\sim p \text{ or } q)$

d. $(p \text{ and } q) \text{ or } (\sim p)$

4.8 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If it is sunny, then I'll go for a bike ride. I'll go for a bike ride.

b. If it is sunny, then I'll go for a bike ride. I will not go for a bike ride.

c. If it is sunny, then I'll go for a bike ride. It is sunny.

d. If it is sunny, then I'll go for a bike ride. It is not sunny.

* * * * * * * * * *

Summary Assessment

1. The home of a wealthy businesswoman has been burglarized. Because of your expert knowledge of the rules governing logic, you have been called in to assist local detectives. So far, they have collected the following statements from various individuals.

a. The niece and not the butler committed the crime.

b. No silver was taken.

c. If the crime happened on Monday, then the woman's banker or lawyer committed the crime.

d. Some of the fingerprints at the crime are those of the woman's sister.

e. If no jewelry was taken, then the woman's husband committed the crime.

f. The thief took silver and no jewelry.

g. If the butler committed the crime, then no silver was taken.

h. If the crime happened on Monday, then the woman's sister committed the crime.

i. If the woman's husband did not commit the crime, then it was committed by her lawyer.

As a result of their investigations, the detectives believe that statements **a**, **b**, **d**, **f**, and **h** are false. Assuming they are correct, identify both the thief and the time of the theft. Justify your conclusions.

2. Hood Wink the magician has a deck of eight cards. Each card has a whole number printed on one side and a letter of the alphabet on the other side. Hood Wink claims, "In my deck of cards, if a card has an even number on one side, then it has a vowel on the other side." Hood Wink then deals the cards as shown in the diagram below.

8	A	D	17	P	3	10	E

Which card(s) would you need to turn over to make sure that Hood Wink's statement is true? Explain your response.

Module Summary

- A **statement** is a sentence that can be determined to be either true or false, but not both. The truth or falseness of a statement is its **truth value**.
- A **quantifier** is a word or phrase that indicates quantity in a statement. Some commonly used quantifiers are *some*, *at least one*, *all*, *every*, and *none*.
- Two general statements are **logically equivalent** if one statement is true (or false) exactly when the other is true (or false). In other words, the truth of one statement implies the truth of the other in all cases, and vice versa. Logical equivalence is denoted by the symbol $\equiv$.
- The **negation** of a statement p has the opposite truth value of p. In general, the negation of p is written as "not p," symbolized as "$\sim p$."
- **De Morgan's laws** apply to negating statements that contain the connectives *and* and *or*. First, the negation of the compound statement "p and q" is the compound statement "not (p and q)." This is logically equivalent to the compound statement "not p or not q."

  Second, the negation of the compound statement "p or q" is the compound statement "not (p or q)." This is logically equivalent to the compound statement "not p and not q."
- In mathematics, the connective *or* is **inclusive**. The inclusive *or* is interpreted as meaning one or the other or both. Sometimes when *or* is used as a connector in casual English, it is interpreted as one or the other, but not both. This is an **exclusive** *or*.
- A **conditional statement** consists of two parts: a **hypothesis** and a **conclusion**. It is usually written in the form "if p, then q," where p is the hypothesis and q is the conclusion. This can be represented as "$p \to q$."
- A conditional statement "if p, then q" is false only when the hypothesis (p) is true and the conclusion (q) is false.
- The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. The inverse of $p \to q$ can be represented by $\sim p \to \sim q$ or "if $\sim p$, then $\sim q$."
- The **converse** of a conditional statement is formed by interchanging the hypothesis and the conclusion. The converse of $p \to q$ can be represented by $q \to p$ or "if q, then p."
- The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and the conclusion and negating both of them. The converse of $p \to q$ can be represented by $\sim q \to \sim p$ or "if $\sim q$, then $\sim p$."
- A conditional statement and its contrapositive are logically equivalent.

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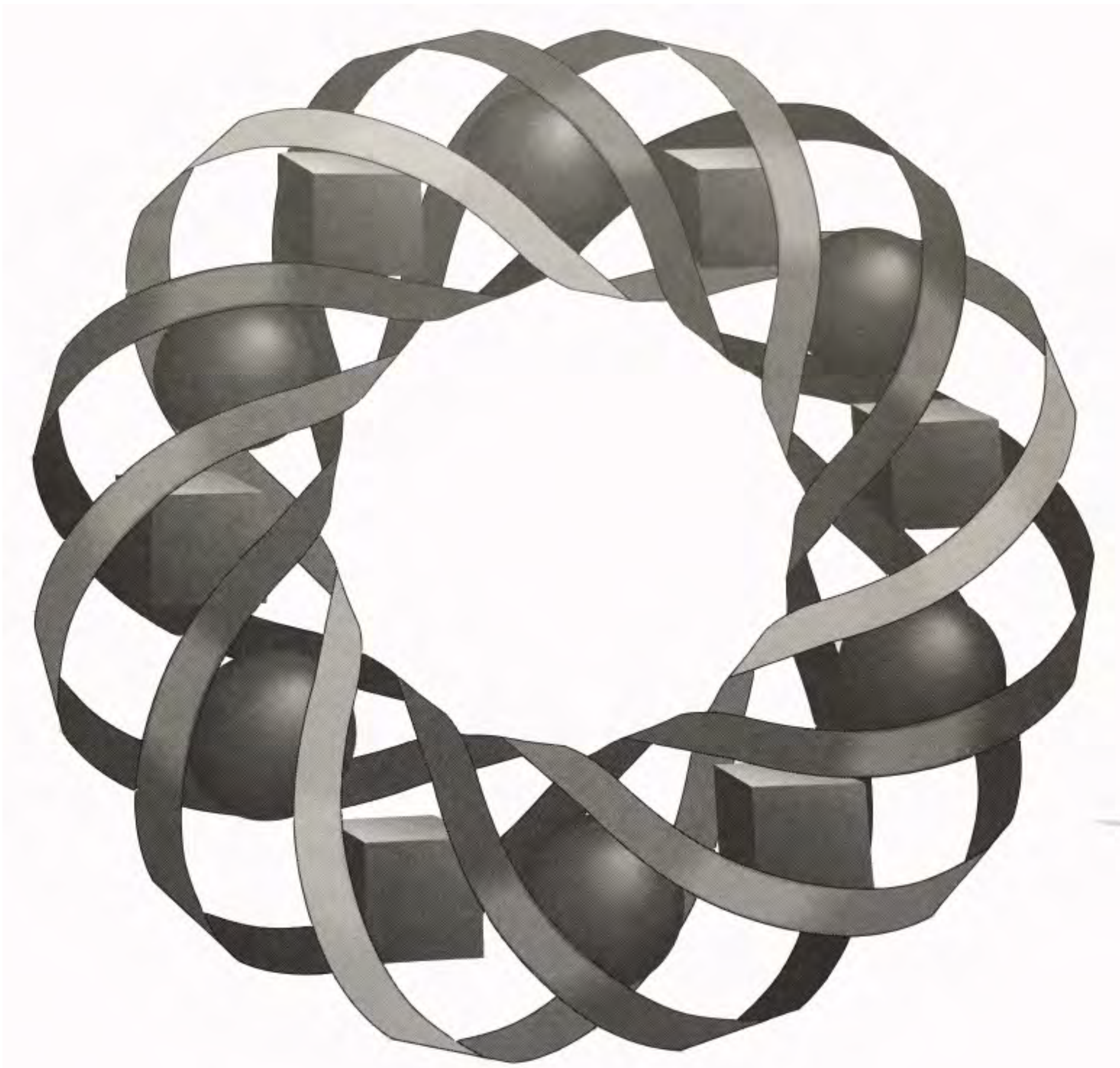
Pixel This



The ability to communicate effectively is an important skill in all walks of life. In this module, you discover how understanding algorithms can enhance your communication skills.

Wendy Driscoll • Margaret Plouvier • Ed Sisolak

Pixel This

Introduction

An image on a video screen is made up of many small discrete elements, or **pixels**. On some types of screens, each pixel can be either on or off. When a pixel is on, you can see it. For example, Figure **1** shows an image of the word *pixel* as it might appear on the screen of a graphing calculator.

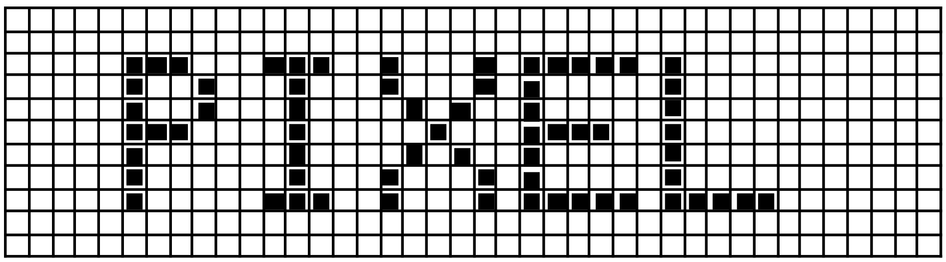


Figure 1: Pixels on a screen

Activity 1

In this activity, you create a picture on grid paper, then write step-by-step directions for recreating the image on a screen.

Exploration

a.

1. Design and construct a simple picture by shading complete squares on a sheet of grid paper. **Note:** Do not allow anyone else to see your picture.

2. Without using diagrams, write a set of instructions to recreate your picture.

3. Exchange instructions with a classmate. Use the instructions that you receive to recreate the classmate's picture.

Mathematics Note

An **algorithm** is a step-by-step process for completing a task. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.

For example, the following steps provide an algorithm to find the distance between two points with coordinates (x_1, y_1) and (x_2, y_2) in the *xy*-plane.

1. Square the difference of x_1 and x_2.
2. Square the difference of y_1 and y_2.
3. Add the values from Steps 1 and 2.
4. Find the square root of the sum from Step 3.
5. Stop. The value for Step 4 is the distance between the two points with coordinates (x_1, y_1) and (x_2, y_2).

b. One way to identify the locations of pixels on an electronic screen is with ordered pairs. In Figure **2**, for example, the location of each pixel can be represented by an ordered pair (x, y), where the *x*-values increase from left to right and the *y*-values increase from bottom to top.

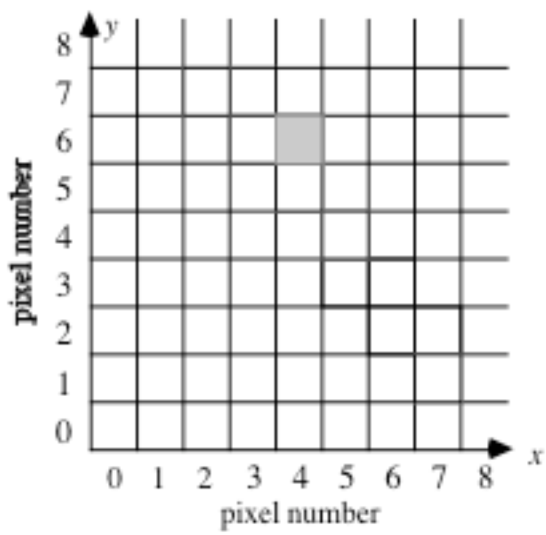


Figure 2: Simulated screen with pixel (4,6) turned on

Note that in this coordinate system, *x* and *y* are used to describe locations of pixels, not points. For example, the coordinates (0,0) describe the pixel in the lower left-hand corner of the screen, not the point at the intersection of the *x*- and *y*-axes.

1. Place an *x*-axis and a *y*-axis on the grid paper you used in Part **a**, and label the grid as shown in Figure **2**.

2. Write another algorithm to recreate your picture by representing each shaded square as an ordered pair.

3. Exchange your algorithm with a different classmate than in Part **a**. Use the algorithm you receive to recreate the classmate's picture.

Discussion

a. **1.** How does the picture you recreated in Part **a** of the exploration compare with its original version?

2. How does the picture you recreated in Part **b** of the exploration compare with its original version?

b. **1.** Is it possible to write more than one algorithm to recreate the same picture? Explain your response.

2. Which algorithm do you think is better, the one from Part **a** or the one from Part **b**? Explain your response.

c. The previous mathematics note describes a five-step algorithm for finding the distance between two points with coordinates (x_1, y_1) and (x_2, y_2). The mathematical formula for this distance is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1. Is this formula an algorithm? Explain your response.

2. In general, do mathematical formulas represent algorithms?

d. **1.** What assumptions did you make about the vocabulary and skills of the classmates with whom you exchanged your algorithms?

2. How might you make your algorithms easier to understand for any potential audience?

3. What characteristics of the audience should you consider when creating an algorithm?

e. Describe some of the characteristics of a "good" algorithm.

Assignment

1.1 **a.** Design and construct another picture using the pixel numbering system shown in Figure **2**.

b. Write an algorithm to recreate your picture.

c. Ask someone who is not in your mathematics class to use the algorithm you wrote in Part **b** to recreate your picture.

d. How does the recreated picture compare with your original?

e. Describe how you could change your algorithm so that the recreated picture would more closely resemble the original.

1.2 In the early part of the 20th century, Russian peasants commonly used the algorithm described below to multiply two positive integers.

1. Create a table with two columns. Write the first integer (x) in the left-hand column and the second integer (y) in the right-hand column.

2. Halve the number in the left-hand column. Disregarding any remainder, and write the result in the next row of the left-hand column. Double the number in the right-hand column, and write that result in the next row of the right-hand column.
3. Repeat Step 2 using the newly created row, until the result in the left-hand column is 1. Calculate the corresponding number for the right-hand column.
4. Find the sum of all the numbers in the right-hand column (including the original integer) that are paired with odd numbers in the left column.
5. Stop. This sum is the product of x and y.

a. Use this algorithm to multiply $x = 19$ and $y = 12$.

b. Use this algorithm to multiply a pair of integers of your choice.

c. Do you think that this a good algorithm? Explain your response.

1.3 Recall that the set of natural numbers is $\{1, 2, 3, 4, \ldots\}$. A **prime number** is any natural number that has exactly two different positive factors. A **composite number** is any natural number that has more than two different positive factors. The number 1 is neither prime nor composite.

For example, 17 is a prime number because it has exactly two positive factors: 1 and 17. The number 4 is composite because it has three positive factors: 1, 2, and 4.

Create an algorithm that determines whether a natural number greater than 1 is prime or composite.

* * * * *

1.4 Imagine that you have created a small business with the help of two friends. Although the company's earnings vary from month to month, you have agreed to share the profits as follows: you receive 50% of each month's profits, while your friends each receive 25%. Write an algorithm that describes a process by which you could divide the monthly profits.

1.5 **a.** Draw a triangle using paper, pencil, a compass and a ruler. Design an algorithm to recreate this triangle. Describe any assumptions you make about your target audience.

b. Draw a triangle using a geometry utility. Design an algorithm to recreate this triangle.

c. Compare the two algorithms you wrote in Parts **a** and **b**. Describe any similarities or differences you observe.

d. Which algorithm do you think is the better one? Explain your response.

1.6 Create an algorithm that generates a random natural number in the interval [1, 10].

1.7 Create an algorithm that rounds any positive real number in decimal form to the nearest tenth.

* * * * * * * * * *

Activity 2

In Activity **1**, you wrote algorithms for your friends and classmates. As precise as you must be when writing algorithms for human audiences, it is even more important to be exact when writing algorithms for technology.

Exploration

In this exploration, you devise an algorithm that a computer could use to draw a line segment between two points.

a.
   1. Use the pixel-numbering system from the exploration in Activity **1** to create a simulated video screen on a sheet of graph paper.
   2. Draw a line segment connecting the pixels labeled (2,8) and (7,1).
   3. Write an algorithm that activates all the pixels required to represent a line segment connecting (2,8) and (7,1).

b. Try your algorithm using two pixels other than (2,8) and (7,1). If it does not correctly represent the line segment, revise the algorithm so that it will work for any line segment that connects pixels (x_1, y_1) and (x_2, y_2).

c. Exchange your algorithm with a classmate. Use your classmate's algorithm to represent the line segment between pixels (2,8) and (7,1).

d. Write down any suggestions you may have for improving your classmate's algorithm, then return it to its author.

e. Revise your own algorithm using the suggestions you received. Verify that the revised version works.

Discussion

a. Describe any problems you encountered when writing the algorithm in Part **a** of the exploration.

b.
   1. Compare your revised algorithm with those of your classmates.
   2. Use the best features of these revisions to write one algorithm that represents a line segment that connects pixels (x_1, y_1) and (x_2, y_2).

c. **1.** What advantages do you see in writing a general algorithm that represents a segment connecting pixels (x_1, y_1) and (x_2, y_2), instead of an algorithm that represents a segment that connects specific pixels like (2,8) and (7,1)?

2. In what types of situations might it be better to write a specific algorithm?

Assignment

2.1 In the exploration, you wrote an algorithm that shaded pixels to represent a line segment between two given pixels. Write an algorithm that could be used to represent the line segment defined by a pixel with coordinates (x_1, y_1), a slope *m*, and a minimum length *d*.

2.2 On most calculators, the square root of any non-negative number *n* may be found by pushing the "$\sqrt{\ }$" button. Write an algorithm that describes how to find the square root, to the nearest hundredth, of a non-negative number without using this button.

2.3 The Pythagorean theorem may be stated as follows: Given any right triangle *ABC*, where *a* and *b* are the measures of the two legs and *c* is the measure of the hypotenuse, $a^2 + b^2 = c^2$.

Write an algorithm that describes how to use this theorem to find the measure of the hypotenuse of any right triangle, given the measures of the two legs. Assume that your potential audience is not familiar with mathematical symbolism, such as exponents.

* * * * *

2.4 Imagine that a friend has called for help with his mathematics homework. He would like to graph the equation:

$$V = \frac{4}{3}\pi r^3$$

where *V* is the volume of a sphere with radius *r*. Unfortunately, he has no idea how to use his new graphing utility. Write an algorithm that you could describe over the phone to help your friend graph this equation.

2.5 Recall that the greatest integer function $y = [x]$ always rounds x to the greatest integer less than or equal to x.

a. For integers n and d, where $d \neq 0$, determine when the following equation is true:

$$\frac{n}{d} = \left[\frac{n}{d}\right]$$

b. The algorithm outlined below can be used to identify leap years:

1. If the year is not divisible by 4, then it is not a leap year. Go to Step 5.
2. If the year is not divisible by 100, then it is a leap year. Go to Step 5.
3. If the year is divisible by 400, then it is a leap year. Go to Step 5.
4. The year is not a leap year.
5. Stop.

Use this algorithm to identify all the leap years between 1895 and 1925.

c. Rewrite the leap-year algorithm using the greatest integer function. Use y to represent the year.

2.6 **a.** Write an algorithm for using paper and pencil to solve an equation of the form $ax + by = c$, where $b \neq 0$, for y.

b. Write an algorithm for using a symbolic manipulator to solve an equation of the form $ax + by = c$, where $b \neq 0$, for y.

2.7 Some calculators allow the user to select the number of digits to be displayed on the screen after a calculation is performed. Some calculators even allow the user to select a floating decimal point or a fixed decimal point.

a. The table below shows how one calculator displays results for two different settings: "Fix 4" and "Float 4." Describe the difference between these two settings.

Calculation	Fix 4	Float 4
5.2894588 • 1	5.2895	5.289
5.2894588 •10,000	52894.5880	5.289E4
5.2894588 • 0.00001	5.2895E–5	5.289E–5

b. Which algorithm would be easier to write: the one for a setting of "Fix 4" or the one for "Float 4"? Explain your response.

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Activity 3

In the first two activities, you examined the characteristics of effective algorithms. In this activity, you focus on creating general algorithms that are useful in many situations.

Exploration

One method of displaying the graph of a function on a screen is the **scan-conversion algorithm**. In this exploration, you investigate how well the scan-conversion algorithm works for graphing several different types of functions.

Technology Note

The **scan-conversion algorithm** calculates ordered pairs of integers, then activates pixels that correspond to these ordered pairs. A general form of the scan-conversion algorithm is outlined below.

1. List a domain for x that consists only of integer values.
2. Use the x-values from Step 1 to determine corresponding values for $f(x)$.
3. Round all the values for $f(x)$ from Step 2 to the nearest integer.
4. Write the corresponding integer values for x and $f(x)$ as ordered pairs.
5. Activate the pixels that correspond to the ordered pairs from Step 4.
6. Stop.

For example, the scan-conversion algorithm can draw a line segment between any two points using the steps below:

1. Determine the equation of the line containing the pixels (x_1, y_1) and (x_2, y_2) using the formula $y = m(x - x_1) + y_1$, where m is the slope of the line and x_1, x_2, y_1, and y_2 are integers.
2. Use the equation from Step 1 to calculate the y-value for each integer x-value between x_1 and x_2, inclusive.
3. Round each y-value to the nearest integer.
4. Write the corresponding x- and y-values as ordered pairs.
5. Activate the pixels that correspond to the ordered pairs from Step 4.
6. Stop.

a. Use the scan-conversion algorithm to create a pixel graph of the function below for integer values of x in the interval [0, 10]:

$$f(x) = \frac{12}{7}x + 2$$

b. Determine whether or not the algorithm generates a good approximation of the graph of the function.

c. Repeat Parts **a** and **b** using each of the following functions:

1. $f(x) = (x-5)^2$

2. $f(x) = 2 + \dfrac{2}{x-5}$

3. $f(x) = (x-5)^3 + 10$

4. $f(x) = 3(2)^{(x-5)}$.

Discussion

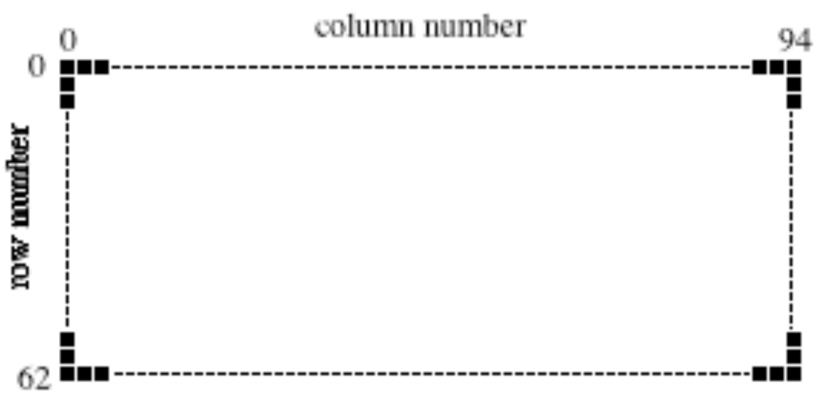
a. Compare the scan-conversion algorithm to the algorithm you developed in the exploration in Activity **2**.

b. Why does the scan-conversion algorithm appear to approximate the graphs of some functions better than others?

c. Do you think that the scan-conversion algorithm is a general algorithm or a specific algorithm? Explain your response.

Assignment

3.1 The screen on one graphing calculator contains 95 columns and 63 rows of pixels. As shown in the diagram below, columns are numbered from left to right, while rows are numbered from top to bottom.



Suppose that the graphing window is set for a domain of $[-10, 10]$ and a range of $[-10, 10]$. In this case, the point (–10,10) corresponds to the pixel in column 0 and row 0.

a. Identify the pixel, by column number and row number, that corresponds to each of the following points:

1. (10,–10)

2. (10,10)

3. (0,0)

4. (3,5)

5. (2,–2)

b. Describe an algorithm for associating a point (x,y), where x and y are integers, with the appropriate pixel on this screen.

3.2 The formula below can be used to calculate the account balance A in a savings account after n years, where r is the annual interest rate and p is the initial principal.

$$A = p(1 + r)^n$$

a. Write an algorithm describing how to calculate A, given values for p, n, and r.

b. What assumptions did you make about the skills and vocabulary of the audience for your algorithm in Part **a**?

c. Use your algorithm to find the value of A when $p = 1000$, $r = 0.06$, and $n = 7$. Describe what this value represents.

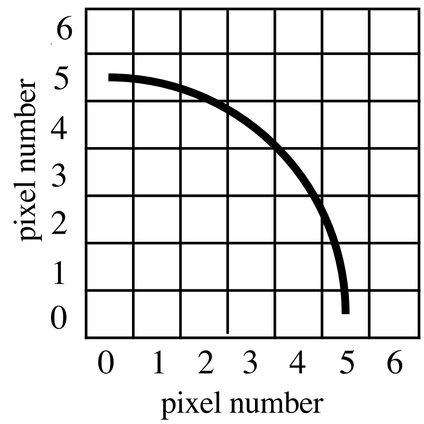
3.3 The diagram below shows the first several rows in the pattern of numbers known as Pascal's triangle. Named in honor of the French mathematician Blaise Pascal (1623–1662), the pattern actually was discovered centuries earlier by Chinese mathematicians.

```
        1          row 0
      1   1        row 1
    1   2   1      row 2
  1   3   3   1    row 3
1   4   6   4   1  row 4
```

a. Determine the numbers in row 5 of Pascal's triangle.

b. Develop a general algorithm for generating any row in Pascal's triangle, assuming that all the previous rows are known.

3.4 The diagram below shows a drawing of a quarter circle on a pixel grid. Do you think that the scan-conversion algorithm will produce a good approximation of this drawing? Explain your response.



* * * * *

3.5 The solution for the system of linear equations below can be found either algebraically or graphically:

$$\begin{cases} y = \dfrac{3}{2}x + 5 \\ y = -2x - 9 \end{cases}$$

a. Write an algorithm for solving the system algebraically.

b. Write an algorithm for solving the system graphically.

c. Use your algorithms from Parts **a** and **b** to solve the following system of equations:

$$\begin{cases} y = \dfrac{3}{4}x + 7 \\ y = \dfrac{9}{12}x - 4 \end{cases}$$

d. Were your algorithms general enough to solve the system in Part **c**? If not, how can you modify them?

3.6 Write a general algorithm to solve any system of linear equations.

3.7 Consider the equations $3x^2 + y^2 = 15$ and $2y^2 - 3x^2 = 21$.

a. Write a specific algorithm to solve each equation for y.

b. Write a general algorithm that will solve either equation for y.

c. What must a person know in order to use your algorithms?

* * * * * * * * * *

Research Project

In 1965, J. E. Bresenham described a method for plotting line segments using technology. This method increased the speed of segment drawing. Find out more about the Bresenham method. Rewrite it as an algorithm that other students in your mathematics class can understand.

Summary Assessment

1. With the discovery of the Rhind papyrus, a mathematical document dating back to 1650 B.C., historians have determined the method by which the ancient Egyptians multiplied any two numbers without using calculators or multiplication tables. By this method, the product of 26 and 33 can be found using an algorithm like the one described below.

1. Make a two-column table.
2. Write 1 in the left-hand column. Write the second factor, 33, in the right-hand column.
3. Double the number in the left-hand column. Write the doubled number in the left-hand column of the next row only if it is smaller than the first factor, 26. Continue this process until the doubled number is greater than or equal to 26.
4. Double the number in the right-hand column and write the result in the right-hand column of the next row. Continue this process until you have written a number in the right-hand column for each number in the left-hand column.

   Identify and mark the numbers in the left-hand column whose sum is the first factor, 26. In this case, the numbers are 2, 8, and 16. Your completed table should resemble the one shown below.

1	33
2	66
4	132
8	264
16	528

5. Add the numbers from the right-hand column $(66 + 264 + 528)$ that correspond to the marked numbers in the left-hand column. The result, 858, is the product of 26 and 33.
6. Stop.

a. Write a general algorithm that uses the method described above to calculate the product of any two numbers.

b. Use your algorithm to find the product of 18 and 65. Show the completed table in your response.

c. Describe why this algorithm works.

2. When a circle is inscribed in a triangle, the circle is tangent to each of the triangle's three sides. Write a general algorithm for using a geometry utility to draw a circle inscribed in a triangle.

Hint: The center of the inscribed circle is located at the point of intersection of the three angle bisectors of the triangle.

Module Summary

- An **algorithm** is a step-by-step process for completing a task. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.
- **Scan-conversion** is a process for determining the pixels that closely represent a function. The **scan-conversion algorithm** calculates ordered pairs of integers, then activates the pixels that correspond to these ordered pairs. A general form of the scan-conversion algorithm is outlined below.

  1. List a domain for x that consists only of integer values.
  2. Use the x-values from Step 1 to determine corresponding values for $f(x)$.
  3. Round all values for $f(x)$ from Step 2 to the nearest integer.
  4. Write the corresponding integer values for x and $f(x)$ as ordered pairs.
  5. Activate the pixels that correspond to the ordered pairs from Step 4.
  6. Stop.

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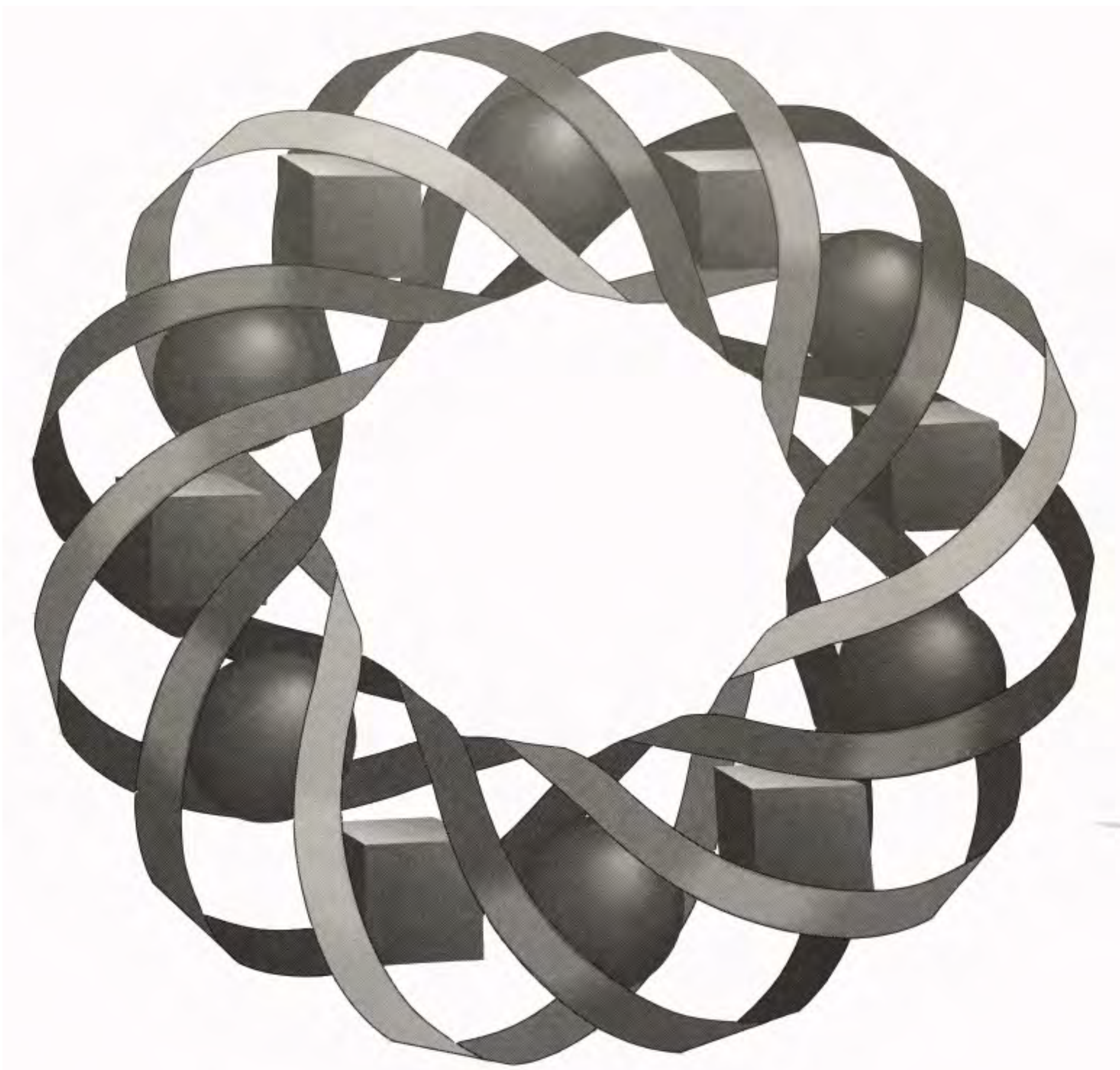
A Fitting Explanation



How can you tell when forecasters have made reasonable predictions for the future? In this module, you use regression to fit curves to data sets and evaluate models.

Terri Dahl • David Thiel • Deanna Turley

A Fitting Explanation

Introduction

In previous modules, you've used various types of equations to model data, understand patterns, and make predictions. To evaluate models, you've used graphs and residuals. In this module, you explore some tools for evaluating linear models.

Activity 1

Before you can analyze how well a linear model describes a data set, you must first suggest the model. In this activity, you create some simple models, then begin to analyze them.

Exploration 1

In this exploration, you use equations to model the relationship between the age and length of fish. Table **1** gives the age and length of a random sample of 20 fish from a population of trout.

Table 1: Ages and lengths of 20 trout

Age (yr)	Length (mm)	Age (yr)	Length (mm)
1	65	3	295
1	72	3	356
1	93	4	355
1	103	4	487
2	209	4	443
2	173	4	371
2	148	5	507
2	181	5	423
3	324	5	398
3	401	5	551

a. Use a graphing utility and the data in Table **1** to create a scatterplot of length versus age.

Mathematics Note

The mean is one of the simplest ways to describe a set of one-variable data. Similarly, the mean line is one of the simplest models for a set of two-variable data.

The equation of the **mean line** for a set of data points in the form (x, y) is $y = \bar{y}$, where $\bar{y}$ represents the mean of the y-values.

For example, consider a data set consisting of the points (1,2), (2,4.1), and (3,5). The mean of the y-values ($\bar{y}$) is 3.7. Therefore, the equation of the mean line for this data set is $y = 3.7$.

b. While the mean line may not always provide an appropriate model for a data set, it can be useful for comparing other potential models.

1. Determine the equation of the mean line for the data in Table **1**.
2. Graph the mean line on the scatterplot from Part **a**.

c. Another possible model for a data set is a linear regression equation.

1. Use technology to determine the linear regression equation for the data in Table **1**. To distinguish this model from the mean line, designate this equation y' and record it in the form $y' = mx + b$.
2. Graph the linear regression equation on the scatterplot from Part **a**.

d. Predict the length of a 5-year-old trout using each of the following models:

1. the mean line
2. the linear regression equation.

Discussion 1

a. Does the data in Table **1** appear to show any kind of association between the age of the fish and the length of the fish? Explain your response.

b. Which appears to be a better model for this data: the mean line or the linear regression equation? Explain your response.

c. If you caught a 5-year-old trout, would you expect its length to be exactly as predicted by either model? Explain your response.

Exploration 2

Figure **1** shows a scatterplot of the data from Table **1**, the mean line for the data, and a graph of the corresponding linear regression equation. Notice that the linear regression equation provides some information about the association between the two variables in the data, while the mean line describes only a central tendency.

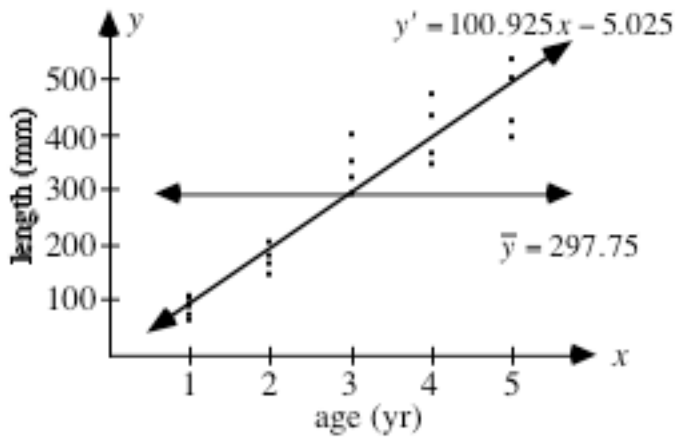


Figure 1: Scatterplot, mean line, and regression equation

a. Figure **2** below shows graphs of the mean line and linear regression equation, along with the data point (5,551). In this case, the deviation of the predicted value $y' = 499.6$ from the mean $\bar{y} = 297.75$ can be attributed to the positive association in the data.

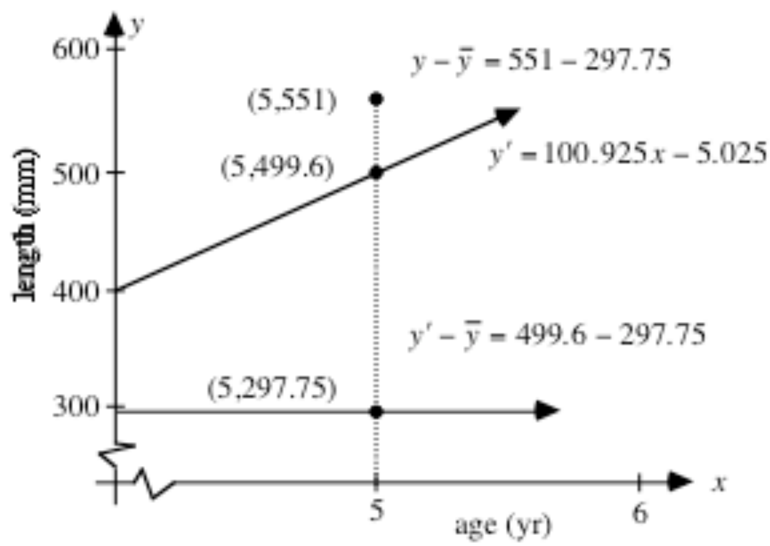


Figure 2: Data point (5,551) with mean line and regression line

1. In Figure **2**, determine the deviation from the mean line of the value predicted by the linear regression model, or $y' - \bar{y}$.
2. Determine the deviation from the mean line of the actual *y*-value of the data point, or $y - \bar{y}$.

3. If a regression model exactly fits a set of data, then there is no deviation of a data point from the regression line. In other words, $y' = y$. Using the mean line as a reference, this can be expressed as:

$$\frac{y' - \bar{y}}{y - \bar{y}} = 1.00$$

This indicates that 100% of the deviation from the mean line is explained by the regression equation.

For the data point (5,551), what percentage of the deviation from the mean line is explained by the linear regression model?

Mathematics Note

The **total variation** for a data set is the sum of the squares of the deviations from the mean line of the data points.

The **explained variation** for a data set is the sum of the squares of the deviations from the mean line of the values predicted by the linear regression equation.

For example, Table **2** shows the distance traveled by a truck for several different amounts of gasoline. In this case, $\bar{y} = 50.65$ and the linear regression model is approximately $y' = 8.33x - 0.09$. As shown in the table, the total variation is 258.37, while the explained variation is 257.78.

Table 2: Total variation and explained variation for a data set

Liters of Gas (x)	Kilometers Traveled (y)	Linear Regression Model (y')	Explained Variation $(y' - \bar{y})^2$	Total Variation $(y - \bar{y})^2$
2.50	20.75	20.74	894.66	894.01
6.25	52.00	51.98	1.78	1.82
6.25	52.50	51.98	1.78	3.42
7.35	59.75	61.15	110.21	82.81
8.10	68.25	67.40	280.45	309.76
		Sum	257.78	258.37

The **coefficient of determination (r^2)** is the percentage of the total variation from the mean line explained by the linear regression equation, or:

$$r^2 = \frac{\text{explained variation}}{\text{total variation}}$$

The value of r^2 is often reported as a decimal. It represents the proportion of the total variation in the y-values that can be explained by the linear relationship in the data set.

For example, the coefficient of determination for the data in Table **2** and the linear regression $y' = 8.33x - 0.09$ is:

$$r^2 = \frac{257.78}{258.37} \approx 0.9989$$

This means that approximately 99.89% of the variation from the mean line is explained by the linear relationship between the liters of gas used and the kilometers traveled by the truck.

b. Calculate the total and explained variation for the data in Table **1**.

c. Calculate r^2, the coefficient of determination, for the data in Table **1**.

Discussion 2

a. Describe what the coefficient of determination calculated in Part **c** of Exploration **2** represents in terms of the ages and lengths of the trout in Table **1**.

b. Why do you think the deviations are squared when determining total variation and explained variation?

Mathematics Note

The statistical association between two variables is referred to as **correlation**.

The **linear correlation coefficient (*r*)** is found by taking the square root of the coefficient of determination r^2. The value of r is in the interval $[-1,1]$, where $|r| = \sqrt{r^2}$. The slope of the linear regression model determines if r is positive, negative, or zero.

The linear correlation coefficient is a measure of how closely the points in a data set can be modeled by a line. The closer $|r|$ is to 1, the stronger the linear relationship between the variables. When $r = 0$, the *y*-values in a data set are said to have no linear correlation to the *x*-values.

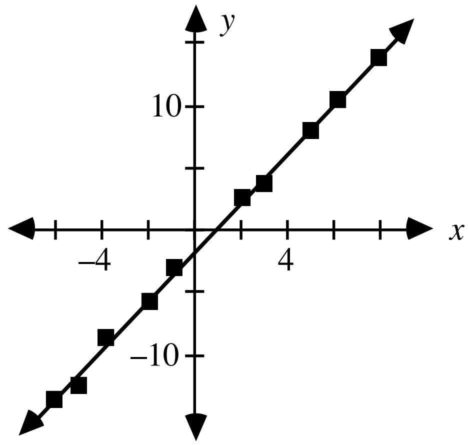
For example, consider the data points (1,2.7), (4,1.2), (7,–0.3), (10,–0.8), and (13,–2.3). The regression equation for this data set is $y = -0.4x + 2.9$. Since the slope of the regression equation is negative, the value of r is negative. Since $r^2 \approx 0.9850$, then $r \approx -0.99$. This indicates a strong negative linear relationship between the variables *x* and *y*.

c.

1. What does it mean when *r* indicates a strong negative relationship between *x* and *y*?
2. What does it mean when *r* indicates a strong positive relationship between *x* and *y*?

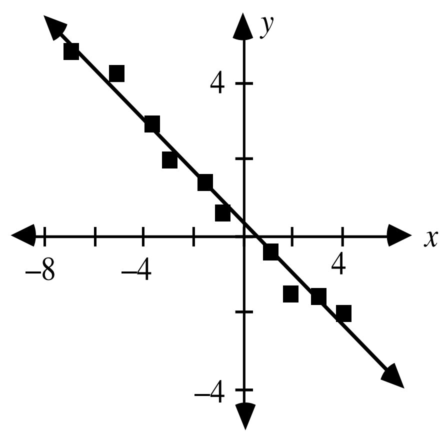
Assignment

1.1 **a.** The graph below shows the linear regression equation for a data set.



1. Explain how the sign of the linear correlation coefficient r can be determined by examining the graph.
2. Given that the coefficient of determination $r^2 \approx 0.996687$, determine r.

b. The following graph shows the linear regression equation for another data set.



1. Explain how the sign of the linear correlation coefficient r can be determined by examining the graph.
2. Given that the coefficient of determination is $r^2 \approx 0.980847$, determine r.

c. Describe how to determine the linear coefficient r, in general, given the coefficient of determination r^2.

1.2 The following table shows the distance traveled by a hiker for several different periods of time.

Time (hr)	Distance (km)
2.50	12.00
6.25	30.00
7.35	35.28
8.10	38.88

a. Determine the linear regression equation that models a scatterplot of distance versus time.

b. Calculate the coefficient of determination, r^2, and describe what this value indicates in terms of the times and distances in the table.

c. Is there a strong linear relationship between time and distance? Explain your response.

1.3 The following table lists the prices of telescopes of comparable quality, along with their lens diameters.

Lens Diameter (in inches)	Price of Telescope
6	\$295.00
8	\$395.00
10	\$565.00
12.5	\$765.00
16	\$995.00

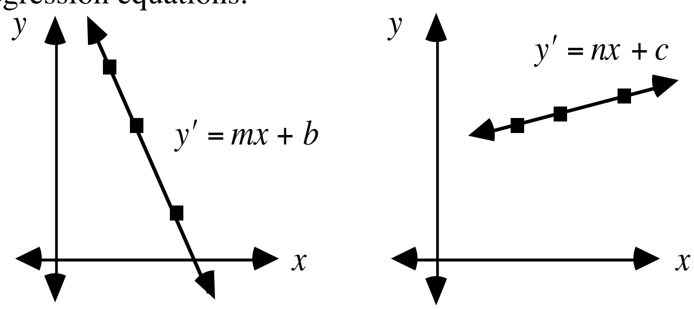
a. Determine the linear regression equation that models a scatterplot of price versus lens diameter.

b. Use your model to predict the price of a 9-inch telescope.

c. Find the linear correlation coefficient, r, for the data set.

d. Describe what the linear correlation coefficient means in terms of the lens diameters and prices given in the table.

1.4 The following graphs show two sets of data modeled by linear regression equations:



a. Assuming that the scales on the graphs are identical, compare the total variations of the two data sets.

b. Given that, in both cases, 100% of the total variation is explained by the linear relationship between the *x*- and *y*-values in the data set, determine the corresponding linear correlation coefficients.

1.5 The table below shows some information on the price of telescope mirrors of comparable quality with respect to their diameters.

Diameter (in inches)	Price of Mirror
1.30	\$26.95
1.52	\$35.95
1.83	\$37.95
2.14	\$45.95
2.60	\$84.95

a. Find the equation of the regression line for this data set.

b. Calculate the coefficient of determination, r^2.

c. What does your response to Part **b** indicate in terms of the prices and diameters of telescope mirrors given in the table?

d. Is there a strong linear relationship between the diameter of a telescope mirror and its price? Explain your response.

* * * * *

1.6 Drew mows lawns in the summer. As part of his weekly accounts, he records both the number of hours worked and the liters of gasoline used. This data is shown in the table below.

Week	Time (hr)	Gasoline (L)
1	32	14.2
2	31	13.3
3	25	12.0
4	35	15.0
5	46	20.2
6	28	12.6
7	34	14.8

a. Determine a linear regression model for a scatterplot of gasoline used versus hours worked.

b. Determine r^2, the coefficient of determination.

c. Describe what r^2 indicates in terms of the hours spent mowing and the amount of gasoline required.

d. Use the model from Part **a** to estimate the amount of gasoline required to mow lawns for 36 hr.

e. Would you expect your estimate from Part **d** to be accurate? Explain your response.

* * * * * * * * * *

Activity 2

A linear correlation coefficient r near either –1 or 1 implies a strong linear relationship. However, when $|r|$ for a linear regression is close to 0, it does not necessarily indicate that there is *no* relationship at all between the variables in the data. It means only that there is no linear relationship.

When analyzing data with regression lines, it is also important to avoid the conclusion that a strong correlation implies a cause-and-effect relationship. For example, the correlation coefficient for a set of data showing average annual salary versus number of years in school is close to 1. This indicates a positive linear relationship between these two quantities. As the number of years in school increases, so does the average salary. However, simply attending school does not *cause* someone to earn a large salary. In this activity, you explore some of the limitations of linear regression models.

Exploration

The data in Table **3** below shows the braking distance required to stop a car moving at various speeds. **Note:** Braking distance varies according to the mass of the car, the condition of the brakes, the road conditions, and other factors. The data below is for one type of car with new brakes on a dry road.

Table 3: Braking distance for various speeds

Speed (km/hr)	Braking Distance (m)
10	0.475
15	1.069
20	1.900
25	2.969
30	4.275
35	5.819

a. Use the data in Table **3** to create a scatterplot of braking distance versus speed and find the equation of the regression line.

b. Determine the coefficient of determination, r^2.

c. Use your graph of the data set, the regression equation, and r^2 to assess how well the equation models the trend in the data.

Mathematics Note

A **residual** is the difference between the *y*-value of a data point and the corresponding *y*-value of the model. In other words, it is the difference between an observed value and the predicted value.

The **average deviation of prediction** is the square root of the average of the squared residuals. It can be found using the following formula, where *n* is the number of data points:

$$\text{average deviation of prediction} = \sqrt{\frac{\text{sum of the squares of the residuals}}{n}}$$

For many data sets, the average deviation of prediction can be used to measure the reliability of a prediction. As a rule of thumb, it is likely that the *y*-values of most data points with *x*-values near the mean ($\bar{x}$) fall within 2 average deviations of the regression line. Thus, it is likely that the predicted *y*-value will be within 2 average deviations of the true *y*-value. The interval of 2 average deviations of prediction on either side of the predicted *y*-value is an **approximation interval**.

For example, the data for trout lengths and ages in Table **1** can be modeled by the regression line $y' = 100.925x - 5.025$. Using this model, the sum of the squares of the residuals is 51,012. Since there are 20 data points, the average deviation of prediction is:

$$\sqrt{51{,}012/20} \approx 50.5$$

The mean of the *x*-values in Table **1** is 3 yr. Using the regression line, the predicted length of a 3-yr-old trout is approximately 298 mm. Thus, it is likely that the actual *y*-value associated with $x = 3$ falls in the interval $[298 - 2(50.5), 298 + 2(50.5)]$ or $[197, 399]$. In other words, it is likely that the length of a 3-yr-old trout is greater than or equal to 197 mm and less than or equal to 399 mm.

d. Use your model for the data in Table **3** to predict a braking distance, along with the corresponding approximation interval, for each of the following speeds:

1. 22 km/hr

2. 34 km/hr

3. 50 km/hr

4. 75 km/hr.

Discussion

a. How well do you think your model from Part **a** of the exploration describes the trend in the data? Explain your response.

b. Describe how the average deviation of prediction resembles the standard deviation of the y-values in a data set.

c. Given the similarities between average deviation of prediction and standard deviation, why would you expect predicted y-values to fall within 2 average deviations of the actual y-values?

d. The actual braking distances for the speeds given in Part **d** of the exploration are shown in Table **4** below.

Table 4: Braking distances for four different speeds

Speed (km/hr)	Braking Distance (m)
22	2.299
34	5.491
50	11.875
75	26.719

For which speeds did the regression line provide a prediction whose approximation interval contained the actual braking distance? Which of these speeds are within the range of the x-values of the original data set in Table **3**?

e. What conclusions can you draw about the reliability of predictions made for values outside the range of a data set?

Assignment

2.1 The following table shows information on motor vehicle registrations and fatalities for eight states.

State	1990 Motor Vehicle Registrations (in thousands)	1990 Motor Vehicle Fatalities
Wyoming	528	125
North Dakota	630	112
South Dakota	650	153
Montana	783	212
Idaho	1054	243
Utah	1206	270
Oregon	2445	578
Washington	4257	825

Source: U.S. Bureau of the Census, 1993.

a. Use this data to create a scatterplot of fatalities versus registrations. Graph the corresponding regression line on the same set of axes.

b. Find the coefficient of determination and describe what it represents in terms of the data.

c. In 1990, Texas had 12,800,000 motor vehicle registrations. Use the regression line to predict this state's number of motor vehicle fatalities.

d. Calculate the average deviation of prediction for your model and determine an approximation interval for your prediction in Part **c**.

e. In 1990, the actual number of motor vehicle fatalities in Texas was 3243. Compare your prediction with this value and suggest some possible reasons for any difference you observe.

2.2 The following table shows the average monthly income in 1993 for various years of education. **Note:** In this case, 12 years of education is equivalent to earning a high school diploma.

Years of Education	Average Monthly Income ($)
12	1380
13	1579
14	1985
16	2625
18	3411
20	4328

Source: U.S. Bureau of the Census, 1995.

a. Use this data to create a scatterplot of average income versus years of education. Graph its corresponding regression line on the same set of axes.

b. Find the coefficient of determination and explain what it represents in terms of the data.

c. Use the regression line to estimate the average monthly income of people with 15 years of education.

d. Determine an approximation interval for your estimate and explain what it represents in terms of the data.

2.3 The following table shows the mean January temperature and mean annual snowfall for 14 U.S. cities.

City	Mean January Temperature (˚C)	Mean Annual Snowfall (cm)
Minneapolis, MN	–11.2	126.5
Mobile, AL	9.9	1.0
Atlantic City, NJ	–0.6	41.9
Omaha, NE	–6.1	76.7
Providence, RI	–2.3	90.7
Raleigh, NC	3.8	18.3
Reno, NV	0.5	62.5
Albuquerque, NM	1.2	27.9
Sacramento, CA	7.3	0.0
Houston, TX	10.2	1.0
Sioux Falls, SD	–10.1	100.1
Spokane, WA	–2.7	127.8
Chicago, IL	–6.1	97.8
Cleveland, OH	–4.1	138.2

Source: U.S. Bureau of the Census, 1993.

a. Create a scatterplot of mean annual snowfall versus mean January temperature.

b. Find the equation of the regression line for the data.

c. Baltimore, Maryland, has a mean January temperature of 0.4 ˚C. Caribou, Maine, has a mean January temperature of –11.8 ˚C. Use approximation intervals to predict a range for the average annual snowfall in each city.

d. Baltimore's average annual snowfall is 55.4 cm, while Caribou's is 287.8 cm. How do these values compare with your predictions?

e. How much of the variation in snowfall appears to be explained by January temperature? Explain your response.

f. Describe some other factors that may influence average annual snowfall other than January temperatures.

2.4 **a.** The following table shows the prices of telescopes of comparable quality and different lens diameters. Graph the data and the corresponding regression line.

Lens Diameter (inches)	Telescope Price
6	$295.00
8	$395.00
10	$565.00
12.5	$765.00
16	$995.00

b. Use an approximation interval to estimate the cost of a telescope with an 11-inch lens.

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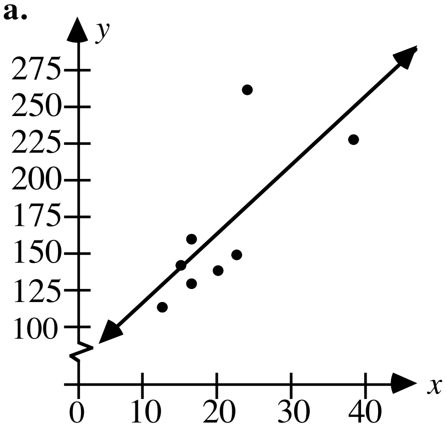
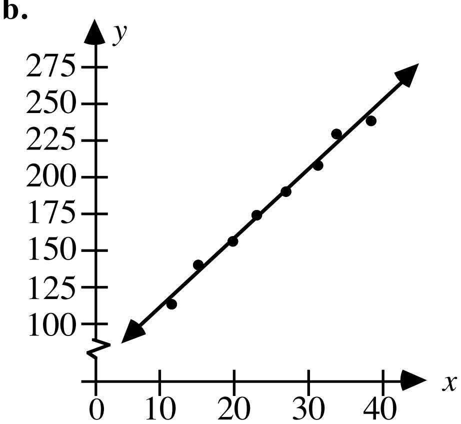
2.5 Each of the two data sets below can be modeled by the same regression equation, $y' = 4.64x - 710$. Compare the approximation intervals for these data sets.

a.



b.



2.6 The magnitude of a celestial body is a measure of its apparent brightness. (As magnitude decreases, apparent brightness increases.) The table below shows the diameter and magnitude of the 10 largest asteroids in the solar system.

Asteroid	Diameter (km)	Magnitude
Ceres	780	7.4
Pallas	489	8.0
Vesta	391	6.5
Hebe	195	8.5
Iris	195	8.4
Juno	190	8.7
Metis	130	8.9
Flora	90	8.9
Astraea	80	9.9
Hygeia	64	9.5

a. Graph a scatterplot of diameter versus magnitude, along with the corresponding regression line.

b. Write a paragraph describing how well the regression equation explains the relationship between diameter and magnitude.

c. **1.** Determine the average deviation of prediction for this model.

2. Calculate an approximation interval for the magnitude of an asteroid whose diameter is 73 km and describe how reliable you think this interval might be.

* * * * * * * * * *

Research Project

Imagine that you are a research statistician. Collect data for two quantities that you believe may have a strong linear correlation. Analyze this data and prepare a presentation of your findings.

Activity 3

As you may recall from Activity **1**, many data sets contain more than one *y*-value for each *x*-value. For example, consider the heights (in centimeters) and ages (to the nearest year) of students in your school. Since there are many different heights for each age, no function can describe every point in this data set.

In the following exploration, you use this type of data to continue your investigation of linear correlation.

Exploration

a. Obtain a stopwatch from your teacher and use it to time several different events. Compare the time you recorded for a specific event with the time recorded by a classmate. How accurate are your measurements?

b.
1. Devise a method for approximating time intervals of 5 sec, 10 sec, 15 sec, and 20 sec without using a clock or stopwatch.
2. Use a stopwatch to measure how accurately your method determines the times you intended. Record four measurements for each interval in a table with headings like those in Table **5**.

Table 5: Time measurements

5 sec	10 sec	15 sec	20 sec

c.
1. Create a scatterplot of measured time versus the intended time and determine the equation of the regression line.
2. Determine the linear correlation coefficient.
3. Determine the average deviation of prediction.

d. Find the mean of the values in each column in Table **5**. Compare these means to the values predicted by the regression line.

Discussion

a. Do you think the regression line is a good model for describing the data in the exploration? Explain your response.

b. In situations where a data set has many y-values for each x-value, statisticians often use regression equations to estimate the mean of the y-values for each x-value.

Does the regression line from Part **c** of the exploration appear to be more useful for predicting the y-values for each x-value or for predicting the mean of the y-values? Explain your response.

c. Given an intended time of 50 sec, do you think that the regression line would provide a good prediction for the measured time? Explain your response.

Assignment

3.1 The table below lists the shoe sizes and heights of 10 different people.

Shoe Size	Height (cm)
8	162
8	178
9	178
9	172
10	180
10	178
13	183
13	188
15	196
15	193

a. Find a linear regression equation that could be used to predict a person's height given that person's shoe size.

b. Using your model from Part **a**, determine an approximation interval for the heights of people with a shoe size of 11.

c. Would you expect your approximation interval to contain the heights of all people with the given shoe size? Explain your response.

3.2 In Activity **1**, you determined a linear regression model for the following data on the age and length of trout.

Age (yr)	Length (mm)	Age (yr)	Length (mm)
1	65	3	295
1	72	3	356
1	93	4	355
1	103	4	487
2	209	4	443
2	173	4	371
2	148	5	507
2	181	5	423
3	324	5	398
3	401	5	551

a. Determine the mean length of trout of each age in the data set.

b. Does the linear regression model for the entire data set provide reasonable predictions for the mean length of trout at a given age? Explain your response.

3.3 The table below shows the engine size in liters and fuel economy in miles per gallon for 10 different cars.

Engine Size (L)	Fuel Economy (mpg)
2.5	22
2.5	24
1.9	27
1.9	29
3.3	19
3.3	20
1.6	29
1.6	29
5.7	17
4.3	15

a. Use this data to create a scatterplot of fuel economy versus engine size and describe any association you observe.

b. Determine the linear regression equation for the data and calculate r^2, the coefficient of determination.

c. Use the model to predict the fuel economy of a car with a 2.8-L engine. Include an approximation interval with your prediction.

d. Explain whether or not you believe the model can accurately predict the fuel economy for a given engine size.

* * * * *

3.4 The following table shows the price and reliability ratings of 10 new cars. In this system, 1 represents a rating of unreliable, while 5 represents a rating of very reliable.

Price ($)	Reliability	Price ($)	Reliability
12,000	1.4	15,500	2.8
12,000	4.2	15,500	3.6
20,500	3.2	51,000	4.1
20,500	3.7	51,000	2.2
17,000	3.1	18,500	3.5

a. Use this data to create a scatterplot of reliability versus price and determine the corresponding regression line.

b. Write a paragraph describing whether or not it is possible to use price to predict the reliability of a car. Use coefficients of determination and graphs to support your opinion.

* * * * * * * * * *

Summary Assessment

1. The following table shows the estimated population for 10 U.S. states, along with the amount of federal funding each state received during fiscal year 1995.

State	Estimated Population (millions)	Federal Funding (billions of dollars)
Alabama	4.3	22.8
Colorado	3.7	19.2
Florida	14.2	75.0
Indiana	5.8	23.0
Maine	1.2	6.6
Mississippi	2.7	14.3
Nebraska	1.6	7.5
North Dakota	0.6	3.8
Tennessee	5.3	26.6
Utah	2.0	8.6

Source: U.S. Bureau of the Census, 1996.

a. Create a scatterplot of federal funding received versus estimated population.

b. Find the equation of the regression line for the data.

c. How much of the variation in federal funding appears to be explained by population? Explain your response.

d. In 1995, Texas' estimated population was 18.7 million, while Minnesota's was 4.6 million. Use approximation intervals to predict a range for the federal funding received by each state.

e. In fiscal year 1995, Texas received $83.9 billion in federal funding, while Minnesota received $19.0 billion. How do these values compare with your predictions?

2. From a sample of at least 5 different people, collect data on the number of heartbeats for each of the intervals given in the following table.

5 sec	10 sec	15 sec	20 sec	25 sec

a. Create a graph of number of heartbeats versus time, along with the corresponding regression equation.

b. Describe how well your model explains any variation in the data.

c. What other factors besides time might influence number of heartbeats?

d. Use your model to predict the number of heartbeats a person might expect in 2.5 hr. Do you think your prediction is valid? Explain your response.

Module Summary

- The equation of the **mean line** for a set of data points in the form (x, y) is $y = \bar{y}$, where $\bar{y}$ represents the mean of the y-values.
- The **total variation** for a data set is the sum of the squares of the deviations from the mean line of the data points.
- The **explained variation** for a data set is the sum of the squares of the deviations from the mean line of the values predicted by the linear regression equation.
- The **coefficient of determination (r^2)** is the percentage of the total variation from the mean line explained by the linear regression equation, or:

  $$r^2 = \frac{\text{explained variation}}{\text{total variation}}$$

  The value of r^2 is often reported as a decimal. It represents the proportion of the total variation in the y-values that can be explained by the linear relationship in the data set.
- The statistical association between two variables is referred to as **correlation**.
- The **linear correlation coefficient (r)** is found by taking the square root of the coefficient of determination r^2. The value of r is in the interval $[-1, 1]$, where $|r| = \sqrt{r^2}$. The slope of the linear regression model determines if r is positive, negative, or zero.

  The closer $|r|$ is to 1, the stronger the linear relationship between the variables. When $r = 0$, the y-values in a data set are said to have no linear correlation to the x-values.
- A **residual** is the difference between the y-value of a data point and the corresponding y-value of the model.
- The **average deviation of prediction** is the square root of the average of the squared residuals. It can be found using the following formula, where n is the number of data points:

  $$\text{average deviation of prediction} = \sqrt{\frac{\text{sum of the squares of the residuals}}{n}}$$
- As a rule of thumb, it is likely that the y-values of most data points with x-values near the mean ($\bar{x}$) fall within 2 average deviations of the regression line. Thus, it is likely that the predicted y-value will be within 2 average deviations of the true y-value. The interval of 2 average deviations of prediction on either side of the predicted y-value is an **approximation interval**.

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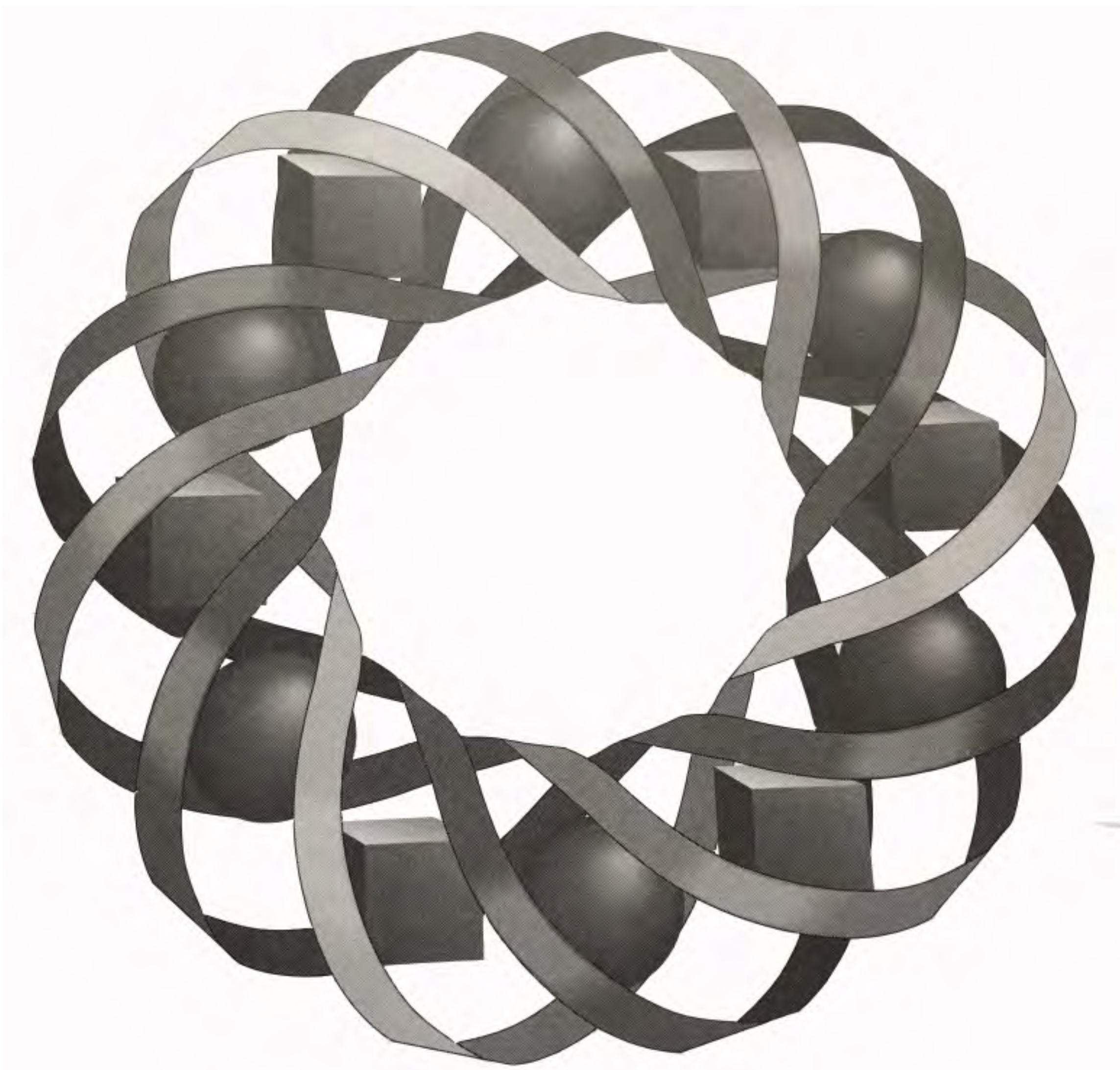
Game of Life



What do two children playing "Rock, Paper, Scissors" have in common with two businesses competing for customers? In both situations, game theory can be used to make decisions and determine strategies.

Masha Albrecht • Verne Schlepp • Steve Yockim

Game of Life

Introduction

While Raoul and Colleen are driving to work on a rainy afternoon, a tire on their car goes flat. Neither one wants to go outside to change it. How should Raoul and Colleen decide who will perform this chore?

The method selected for resolving a conflict often depends on the consequences of the decision. In this case, simply flipping a coin might settle who changes the tire. If the consequences of the decision were more critical, then a more sophisticated method of decision making might be more appropriate.

Game theory is a branch of mathematics used to analyze decision making in situations that involve conflicting interests. While game theory also applies to some recreational games, it was developed during World War II to analyze competitive situations in business, warfare, and society.

In this module, you will analyze the decision-making process using the simplest of all games: the two-person, **zero-sum** game. In a two-person, zero-sum game, one player's loss is the other player's gain. In such situations, it is assumed that each player makes a decision without knowing what the other player will do. In addition, each player is assumed to believe that the other player is an intelligent opponent who will make the best possible move.

Exploration

The game of Odds or Evens can be used to make a decision when two people have conflicting interests. (This is one version of the ancient Italian game Morra.) The rules for Odds or Evens are described below.

- One person chooses odds; the other person chooses evens.
- At the count of three, each person holds out either one or two fingers.
- If the total number of fingers showing is odd, then the person who selected odds wins the game. If the total number of fingers showing is even, then the person who selected evens wins the game.

a. Play Odds or Evens 10 times with a partner. Keep a record of the number of games won by each player, and whether the winner chose odds or chose evens.

b. Play Odds or Evens once more to determine which player will report the results to the class.

c. Report the results to the class.

Discussion

a. How did you decide the number of fingers to show when playing Odds or Evens?

b. Based on the class results, do you think it is better to select odds or evens if you want to win the game?

Activity 1

When two people have conflicting interests, each should develop a plan of action, or **strategy**. In game theory, it is assumed that both parties use strategies that serve their own best interests.

Exploration

In this exploration, you and a partner will investigate three different versions of the game Odds or Evens (Morra). For each version, you will determine if there is a strategy you can use to win every time. The following rules apply to all three versions of the game.

- Randomly decide which person is player A and which is player B. Use this same assignment for all games.
- At the count of three, each player holds out one or two fingers.
- The winner receives a number of points equivalent to the total number of fingers shown by both players, while the loser relinquishes this same number of points. **Note:** It is possible for a player to have a negative number of points.

a. In the game Total, player A wins if the total number of fingers showing is odd. Player B wins if the total number of fingers showing is even.

1. Based on the rules of Total, predict who will win in this game.
2. Play Total 10 times. Record the points won by each player in each game, then determine the total points won by each player in 10 games.
3. Collect the class data.
4. Determine whether or not one player has an advantage in this game.
5. Suppose that players A and B randomly hold out one or two fingers with equal likelihood. Determine the expected value (in points) for each player and compare it to the class data.

b. In the game More, the player showing the most fingers wins. In the case of a tie, player A wins if both players show ones and player B wins if both players show twos.

Repeat Part **a** for the game More.

c. In the game Less, the players showing fewer fingers wins. In the case of a tie, player A wins if both players show ones and player B wins if both players show twos.

Repeat Part **a** for the game Less.

Discussion

a.
1. Is there a strategy for the game Total which ensures that player A always wins? If so, describe this strategy. If not, is there a strategy that minimizes player A's losses?
2. Is there a strategy for Total which ensures that player B always wins? If so, describe this strategy. If not, is there a strategy that minimizes player B's losses?

b. Repeat Part **a** for the game More.

c. Repeat Part **a** for the game Less.

d. For each game, how did the class results compare with the expected value?

e. Once players A and B recognize each other's strategy in a game, what should they do?

Mathematics Note

An **optimal strategy** for a player results in maximizing the winnings or minimizing the losses for that player. In the game More, for example, the optimal strategy for player B is to show two fingers in every game. This will always results in a win for player B. The optimal strategy for player A is to show one finger. This will minimize player A's losses.

In a **pure strategy**, a player makes the same choice each time the game is played. For example, the optimal strategies for both players in More are pure strategies.

In a **mixed strategy**, a player's choice of how to play varies from game to game. For example, the game Total offers no pure strategy that is an optimal strategy for either player. In this case, players should use mixed strategies. **Note:** You will investigate the choice of a mixed strategy later in this module.

f. By showing one finger each time, player A can always win the game of Less. This optimal strategy is a pure strategy. Is player B's optimal strategy also a pure strategy?

g. Why would it be unwise for either player to use a pure strategy in Total?

h. Recommend a mixed strategy for players of Total.

i. **1.** If each player in the game More uses an optimal pure strategy, what is the expected value for each?

2. Compare this expected value to the class data from the exploration.

j. **1.** If each player uses an optimal pure strategy in the game Less, what is the expected value for each?

2. Compare this expected value to the class data from the exploration.

Mathematics Note

In a two-player zero-sum game, the consequences of each player's choices can be summarized in a **payoff matrix**, where a payoff is the amount won or lost by a single player in one game. The rows of the matrix represent the choices of one player (the row player), while the columns represent the choices of the second player (the column player).

In this module, the entries in each cell of a payoff matrix always represent the payoffs for the row player. A positive payoff represents a win for the row player, while a negative payoff represents a loss for the row player. The payoff for the column player is the additive inverse of the payoff for the row player.

For example, Figure **1** shows a payoff matrix for player A in Total. Each player has two choices: show one finger or show two fingers.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} \text{one} & \text{two} \end{array} \\ \text{Player A} & \begin{array}{c} \text{one} \\ \text{two} \end{array} & \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \end{array}$$

Figure 1: Payoff matrix for player A in Total

The entry in row 2, column 2 of the matrix indicates that player A loses 4 points and player B wins 4 points when both show two fingers.

In a **strictly determined** game, the optimal strategy for each player is a pure strategy. Since players make the same choices each time the game is played, the payoffs are always the same.

For example, consider the payoff matrix in Figure **2**. In this case, R_1 and R_2 represent the choices for the row player and C_1 and C_2 represent the choices for the column player. In this game, the optimal strategy for player A is to play row 2. This always ensures a win. Player B's optimal strategy is to play column 1. This strategy minimizes losses. Because both players use pure strategies, the game is strictly determined.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & 5 \\ 2 & 3 \end{bmatrix} \end{array}$$

Figure 2: A payoff matrix for a strictly determined game

When both players use these pure strategies, player A wins 2 points in every game. This constant payoff is the **value of the game** for player A.

k. What does an entry of –2 represent in the payoff matrix in Figure **2**?

l. Which of the games in the exploration are strictly determined? Explain your response.

m. Describe how you could use the payoff matrix for a strictly determined game to find each of the following:

1. the best strategy for the row player
2. the best strategy for the column player.

Assignment

1.1 According to the rules of Total described in the exploration, player B wins when the total number of fingers is even.

a. The payoff matrix for Total when player A is the row player is shown in Figure **1**. Determine a payoff matrix for Total if player B is the row player.

b. Is there a pure strategy for either player? Explain your response.

1.2 **a.** Construct a payoff matrix for the game More if player B is the row player.

b. Explain why the matrix indicates that player B's optimal strategy is a pure strategy.

c. Explain why the matrix indicates that player A's optimal strategy is a pure strategy.

d. Construct a payoff matrix for More if player A is the row player. Does this matrix indicate the same results as the one from Part **a**?

e. What is the value of the game for player B? for player A?

1.3 **a.** Construct a payoff matrix for the game Less if player A is the row player.

b. Explain why the matrix indicates that player A's optimal strategy is a pure strategy.

c. Explain why the matrix indicates that player B's optimal strategy is a pure strategy.

d. What is the value of the game for player A? for player B?

1.4 Consider the following payoff matrix for player A of a two-person, zero-sum game.

$$\text{Player A} \quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} -5 & 6 \\ 2 & -4 \end{bmatrix} \end{array}}$$

a. Determine the payoff matrix if player B is the row player.

b. Describe the relationship between the two payoff matrices.

1.5 For each game described by payoff matrices in Parts **a–c** below, find the optimal pure strategy for player A and for player B. Then determine the value of the game for the winning player.

a.

$$\text{Player A} \quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \end{array}}$$

b.

$$\text{Player A} \quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 62 & 39 \\ 78 & 43 \end{bmatrix} \end{array}}$$

c.

$$\text{Player A} \quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 7 & 3 \\ -5 & 0 \end{bmatrix} \end{array}}$$

Mathematics Note

The value of a strictly determined game can be determined from the payoff matrix by identifying the entry that is less than or equal to all entries in its row and greater than or equal to all entries in its column. This is the **saddle point** of the matrix, and is the value of a strictly determined game.

For example, consider the payoff matrix in Figure **3**. It is the payoff matrix for player B in More. The saddle point of the matrix is 3, since it is less than 4 and greater than –2. This is the value of the game for player B.

$$\begin{array}{ccc} & & \text{Player A} \\ & & \begin{array}{cc} \text{one} & \text{two} \end{array} \\ \text{Player B} & \begin{array}{c} \text{one} \\ \text{two} \end{array} & \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \end{array}$$

Figure 3: Payoff matrix for player B in More

A **strictly determined game** is a fair game if the saddle point is 0.

1.6 **a.** Design an algorithm for determining if a matrix has a saddle point.

b. If a payoff matrix for a game does not have a saddle point, what can you conclude about the game?

c. If a payoff matrix has more than one saddle point, what can you conclude about the game?

1.7 **a.** Design a 2×2 payoff matrix with no saddle point.

b. Design a 3×3 payoff matrix with no saddle point.

1.8 Bill and Ann are running against each other for a single seat on the student council. One of the key issues in the campaign is a proposal to renovate the student lounge. If the proposal is approved, new furniture will be purchased using the extracurricular activities budget. This will make fewer dollars available for other activities.

To predict how his position on this issue will affect undecided voters, Bill conducts a poll. Bill decides to use game theory to find his optimal strategy. The results of the poll are represented in the following payoff matrix, where F represents "favors," N represents "neutral," and O represents "opposes."

$$\begin{array}{ccc} & & \text{Ann} \\ & & \begin{array}{ccc} \text{F} & \text{N} & \text{O} \end{array} \\ \text{Bill} & \begin{array}{c} \text{F} \\ \text{N} \\ \text{O} \end{array} & \begin{bmatrix} -80 & -60 & 60 \\ 90 & 70 & 90 \\ -105 & -50 & 100 \end{bmatrix} \end{array}$$

In this matrix, the entry in row F, column O corresponds with the situation in which Bill is in favor of the new lounge and Ann opposes it. The value of this entry (60) indicates that Bill gains 60 votes and Ann loses 60 votes.

a. Describe the meaning of the matrix entry –80.

b. Which entry in the matrix shows the best outcome for Ann?

c. Bill finds out that a classmate has shown the data to Ann. What is the best outcome for Bill? Explain your response.

d. **1.** What is Bill's optimal strategy?

2. What is Ann's optimal strategy?

e. **1.** What is the saddle point of this payoff matrix?

2. What does this value mean to the two candidates?

* * * * *

1.9 Given the following payoff matrix for a two-person, zero-sum game, determine values of x and y so that the game is not strictly determined:

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}$$

1.10 In the two-person game "Rock, Paper, Scissors," each player holds out one hand to indicate the choice of one of three objects (rock, paper, or scissors). A winner is determined by the following rules: scissors cut paper, rock smashes scissors, and paper covers rock. In the case of a tie, neither player wins.

a. Write a matrix to represent this game.

b. Describe whether or not this game is strictly determined.

* * * * * * * * * *

Activity 2

In games that do not have pure optimal strategies, players should use mixed strategies. In this activity, you investigate this type of game.

Exploration 1

In games that are not strictly determined, the amount won or lost can vary from game to game, depending on the choices made by players. As a result, the optimal strategy for a particular player is one that results in the greatest average winnings per game or the least average losses per game, regardless of the other player's strategy.

a. Consider the following payoff matrix for a game.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \end{array}$$

Since this payoff matrix does not have a saddle point, the game is not strictly determined. In this case, it would not be wise for either player to use a pure strategy. For example, if player A always chose R_1, then player B would eventually recognize this strategy. By always choosing C_1, player B would win 1 point in every remaining game.

1. Devise a mixed strategy that you think will maximize player A's average winnings when the game is repeated a large number of times. Express your strategy in terms of the percentage of games in which player A should choose each row. For example, one strategy for player A would be to play R_1 10% of the time and R_2 90% of the time.
2. Devise a mixed strategy that you believe will maximize player B's average winnings when the game is repeated a large number of times. Express your strategy in terms of the percentage of games in which player B should choose each column.
3. Since the choices made by each player are independent events, the probability of each payoff is the product of the probabilities of the player's respective choices. Create a tree diagram showing the probability of each payoff for player A.

b. Use the tree diagram that you created in Part **a** to complete Table **1**.

Table 1: Determining the expected value for player A

Player A's Choice	Player B's Choice	Payoff	Probability of Payoff	Probability ∞ Payoff
R_1	C_1	-1		
R_2	C_1	1		
R_1	C_2	2		
R_2	C_2	0		

c. Find the sum of the four products in the right-hand column of Table **1**. This is the expected value of the game for player A.

d. Imagine that player B uses a pure strategy and always plays C_1. In this case, the probability that player B chooses C_1 is 1 while the probability that player B chooses C_2 is 0.

Table **2** shows the game that results when player B uses this pure strategy. The probability that player A chooses R_1 is represented by r_1, while the probability that player A chooses R_2 is represented by $1 - r_1$.

Table 2: Game when player B always chooses C_1

Player A's Choice	**Player B's Choice**	**Payoff**	**Probability of Payoff**	**Probability ∞ Payoff**
R_1	C_1	–1	$r_1 \bullet 1$	$-r_1$
R_2	C_1	1	$(1-r_1) \bullet 1$	$1-r_1$
R_1	C_2	2	$r_1 \bullet 0$	0
R_2	C_2	0	$(1-r_1) \bullet 0$	0

Write an expression for the expected value for player A for the game illustrated in Table **2**.

e. Assume that player B changes strategy and always chooses C_2. Write an expression for the expected value of the game for player A. As in Part **d**, represent the probability that player A chooses R_1 as r_1 and the probability that player A chooses R_2 as $1-r_1$.

f. The optimal mixed strategy for the row player does not depend on the strategy employed by the column player. This optimal strategy can be determined by analyzing games in which the column player uses pure strategies.

There are an infinite number of strategies that player A could employ. Table **3** shows 11 of those strategies. Recall that the expected value of the game for player A changes if player A changes the percentages assigned to each choice in a mixed strategy. Use the results from Parts **d** and **e** to complete a table with headings like those in Table **3**.

Table 3: Spreadsheet for various values for r_1

1	**2**	**3**	**4**
Probability that player A selects R_1 (r_1)	Probability that player A selects R_2 ($1-r_1$)	Expected value if player B always selects C_1	Expected value if player B always selects C_2
0.0	1.0		
0.1	0.9		
0.2	0.8		
0.3	0.7		
0.4	0.6		
0.5	0.5		
0.6	0.4		
0.7	0.3		
0.8	0.2		
0.9	0.1		
1.0	0.0		

g. Column 3 of Table **3** contains the expected values that result when player B always chooses C_1 and player A uses 11 different strategies. Column 4 contains the expected values that result when player B always chooses C_2 and player A uses 11 different strategies. Player A's optimal mixed strategy is the strategy that will result in the same expected value in both columns.

Using the entries in Table **3**, estimate the strategy that will result in the same expected value in both columns.

h. Another way to determine the strategy that results in the same expected value in both columns involves analyzing the situation graphically.

1. Write an equation that describes the relationship between the entries in columns 1 and 3 of Table **3**. Use this equation to create a graph of expected value versus the probability that player A selects R_1.
2. Write an equation that describes the relationship between the entries in columns 1 and 4 of Table **3**. Use this equation to create another graph of expected value versus the probability that player A selects R_1. Plot this graph on the same coordinate system as in Step **1**.
3. Use the graphs from Steps **1** and **2** to estimate the optimal strategy for player A.
4. Solve the system of equations from Steps **1** and **2** to determine the optimal strategy for player A.
5. Determine the expected value of the game when player A uses this optimal strategy.

Discussion 1

a. Compare the optimal strategy determined in Part **g** of Exploration **1** with the strategy you identified in Part **h**.

b. How do the coordinates of the point of intersection of the graphs in Part **h** relate to the player's strategy?

c.
1. When player A uses the optimal strategy, what is the expected value of the game?
2. Explain what this value means for player A.
3. If the game was played 1000 times using the optimal strategy, how many points would player A expect to win?

d.
1. If the game was played 1000 times using the optimal strategy, how could you determine player A's average payoff per game?
2. Will the mean payoff per game equal the expected value of the game? Explain your response.

e. **1.** Use Table **3** to describe the range of expected values of the game for player A if different strategies are employed.

2. Why would it be risky for player A to use a strategy other than the optimal one?

f. Assume that the optimal strategy for player A in a game is to select R_1 25% of the time and R_2 75% of the time.

1. Explain why these selections must be made in a random manner.

2. Describe a method for making these selections randomly.

g. Suppose a game is not strictly determined. How can you use algebra to find the optimal strategy for the row player?

Mathematics Note

In a game that is not strictly determined, it can be shown that an optimal mixed strategy exists for each player.

The optimal mixed strategy for a player is the strategy that results in the same expected value regardless of the strategy used by the other player. In other words, the expected value of the game for a player using an optimal strategy is independent of the strategy used by the other player.

For example, consider the payoff matrix in Figure **4**.:

$$\begin{array}{cc} & \text{Column player} \\ & \begin{array}{cc} \quad C_1 & C_2 \end{array} \\ \text{Row player} \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix} \end{array}$$

Figure 4: Payoff matrix

Table **4** shows the expected value of the game for the row player when the column player always chooses C_1, where the probability that the row player chooses R_1 is represented as r_1.

Table 4: Expected value when column player always chooses C_1

Payoff	Probability	Payoff • Probability
4	$r_1 \bullet 1$	$4 \bullet r_1$
–3	$1 \bullet (1-r_1)$	$(-3) \bullet (1-r_1)$
–2	$r_1 \bullet 0$	$(-2) \bullet 0$
5	$0 \bullet (1-r_1)$	$5 \bullet 0$
	Expected Value	$4r_1 + (-3)(1-r_1)$

Table **5** shows the expected value for the row player when the column player always chooses C_2.

Table 5: Expected value when column player always chooses C_2

Payoff	**Probability**	**Payoff • Probability**
4	$r_1 \bullet 0$	$4 \bullet 0$
–3	$0 \bullet (1 - r_1)$	$(-3) \bullet 0$
–2	$r_1 \bullet 1$	$(-2) \bullet r_1$
5	$1 \bullet (1 - r_1)$	$5 \bullet (1 - r_1)$
	Expected Value	$-2r_1 + 5(1 - r_1)$

The optimal strategy for the row player occurs when the two expected values are equal:

$$4r_1 + (-3)(1 - r_1) = -2r_1 + 5(1 - r_1)$$
$$r_1 = \frac{4}{7}$$

In this game, the optimal strategy for the row player is to choose R_1 an average of 4 times in every 7 games. The expected value of the game for the row player when this optimal strategy is used can be determined using either column of the payoff matrix:

$$\frac{4}{7} \bullet 4 + \frac{3}{7} \bullet (-3) = 1 \text{ or } \frac{4}{7} \bullet (-2) + \frac{3}{7} \bullet 5 = 1$$

The value of the game for the row player is the expected value that results when the row player uses the optimal strategy. The value of this game for the row player is 1.

h. What must be true of the value of a game if the game is a fair one?

i. What equation would you solve to determine the expected value for the row player in a game having the following payoff matrix? (Let r_1 represent the probability that the row player chooses R_1.)

$$\begin{array}{cc} & \text{Column player} \\ & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Row player} \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

j. Consider the strictly determined game whose payoffs are represented in the following matrix:

$$\begin{array}{cc} & \text{Column player} \\ \text{Row player} & \begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \end{array} \end{array}$$

In Activity **1**, the value of a strictly determined game was defined as the constant amount won or lost by the row player when both players use pure strategies. Could this value also be defined as the expected value of the game for the row player when both players use pure strategies?

Exploration 2

In this exploration, you investigate the expected values that result when players A and B use a variety of mixed strategies in the game from Exploration **1**. The payoff matrix for player A is shown below:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \end{array} \end{array}$$

a. Based on the results from Exploration **1**, what is the value of the game for player A?

b. Write the payoff matrix for player B.

c. **1.** Determine the optimal strategy for player B.

2. Determine the value of the game for player B.

3. Compare the value of the game for player B with the value of the game for player A.

d. Obtain a simulation of this game from your teacher and complete the following steps.

1. Using the optimal strategies for each player, run the simulation 100 times. Determine the average payoff per game for player A. Compare the average payoff with the value of the game for player A.

2. Run the simulation 1000 times and compare the average payoff per game for player A with the value of the game for player A.

e. **1.** Run the simulation 300 times using the optimal strategy for player A and a strategy for player B that is not the optimal one. Compare the average payoff per game for player A with the value of the game for player A.

2. Repeat the process described in Step **1** using several other non-optimal strategies for player B.

f. Run the simulation 300 times using strategies for both players that are not optimal. Compare the average payoff per game for player A with the value of the game for player A. Repeat this process using several other non-optimal strategies for both players.

Discussion 2

a.
1. Describe the results of the simulation of 1000 games in which both players used their optimal strategies.
2. How did these results compare with the results you obtained by simulating 100 games in which both players used their optimal strategies?

b.
1. Describe the results you obtained when using the optimal strategy for player A and a non-optimal strategy for player B.
2. Explain why these results occurred.

c. What happens when both players fail to use their optimal strategies?

d.
1. Could using a strategy other than the optimal one be advantageous for player A? Explain your response.
2. Could using a strategy other than the optimal one be unfavorable to player A? Explain your response.

Assignment

2.1 Determine the optimal strategy and value of the game for each player in the games represented by the following payoff matrices.

a.

$$\text{Player A}\quad \begin{array}{c c} & \text{Player B} \\ & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \end{array}$$

b.

$$\text{Player A}\quad \begin{array}{c c} & \text{Player B} \\ & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix} \end{array}$$

2.2 In the game Total from Activity **1**, the payoff for the winner is the number of fingers showing.

a. Determine the optimal strategy for player B.

b. What is the value of the game for player B?

c. Is this a fair game? Explain your response.

2.3 When Royal Construction and its chief competitor, Chicago Remodeling, determine their bids for building projects, they have two options. They can emphasize cost or emphasize quality. If they emphasize cost, they use less expensive materials and make lower bids. When they emphasize quality, they use higher quality materials. However, this strategy also raises the amounts of the bids.

Based on Royal's past records, the projects are awarded as shown in the following table.

Strategy	Percentage of Projects Awarded to Royal
Both companies emphasize cost	60%
Royal emphasizes cost, Chicago emphasizes quality	40%
Royal emphasizes quality, Chicago emphasizes cost	45%
Both emphasize quality	55%

a. **1.** Construct a payoff matrix for Royal Construction.

2. Determine if Royal should always emphasize cost, always emphasize quality, or use a mixed strategy.

b. Write an expression to determine the expected value for Royal if the Chicago company always emphasizes cost.

c. Write the expression that will determine the expected value for Royal if Chicago Remodeling always emphasizes quality.

d. Determine the optimal strategy for Royal.

e. If Royal uses this optimal strategy, what percentage of the building projects should the company expect to win?

2.4 Consider a zero-sum game defined by the following payoff matrix:

$$\text{Row player}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array}\overset{\text{Column player}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix} \end{array}}$$

a. Complete a table like the one below for each of the strategies described in Steps **1–3**.

Row Player	Column Player	Payoff	Probability	Expected Payoff
R_1	C_1			
R_1	C_2			
R_2	C_1			
R_2	C_2			
			Total	

1. The row player chooses R_1 40% of the time and the column player chooses C_1 30% of the time.
2. The row player chooses R_1 70% of the time and the column player chooses C_1 50% of the time.
3. The row player uses the optimal strategy for the row player and the column player uses the optimal strategy for the column player.

b. Compare the three total expected payoffs obtained in Part **a**. What are the advantages and disadvantages of the row player using the optimal strategy?

2.5 Consider the following payoff matrix for a game.

$$\text{Player A}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array}\overset{\text{Player B}}{\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 5 & 4 \\ -3 & 7 \end{bmatrix} \end{array}}$$

a. Is this game strictly determined? Explain your response.

b. If player A would like to use a pure strategy, what should this strategy be?

c. Using the pure strategy from Part **b**, what is the expected value of the game for player A?

d. If player A uses a mixed strategy, player A will not win every game. Use expected values to explain why it is still advantageous for player A to use a mixed strategy.

2.6 One row in a payoff matrix **dominates** another row if each of its entries is greater than the corresponding entries in the other row. When making a choice between the two rows, the row player should always select the dominant row in order to maximize winnings.

Similarly, one column **dominates** another column if each of its entries is less than the corresponding entries in the other column. When choosing between the two columns, the column player should always select the dominant column in order to maximize winnings.

Finding dominant rows and columns allows you to identify the rows and columns that should never be chosen. Once these are identified, they can be eliminated to simplify the payoff matrix. Remove any such rows and columns from the following payoff matrices. Using the simplified matrices, find the optimal strategies for each player and the value of the game.

a.

$$\begin{bmatrix} -2 & 8 \\ 2 & -5 \\ 9 & 7 \end{bmatrix}$$

b.

$$\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 0 \end{bmatrix}$$

2.7 A university's decision to change the school's colors has upset some alumni. To urge the administration to reconsider, they have begun a petition drive. Meanwhile, a second group has organized to promote the change. Both sides are campaigning by sending out mailings, making telephone calls, and visiting graduates in their homes.

A consulting firm has estimated the number of signatures that the group opposed to the change can expect to collect with each combination of strategies. This information is shown in the matrix below.

$$\begin{array}{ccc} & & \text{Group for change} \\ & & \begin{array}{ccc} \text{mail} & \text{phone} & \text{visit} \end{array} \\ & \begin{array}{r} \text{mail} \\ \text{phone} \\ \text{visit} \end{array} & \begin{bmatrix} 300 & 125 & 200 \\ 700 & 500 & 400 \\ 900 & 200 & 700 \end{bmatrix} \end{array}$$

Group against change (rows)

a. Use dominance to eliminate a row and a column from the matrix.

b. What are the best strategies for each group?

c. The university has stated that if the opposing group can obtain 500 signatures against the change, it will not change the school colors. Will the university change its colors? Explain your response.

* * * * *

2.8 The doctors at a local hospital are faced with an epidemic of throat infections. Because the results of their laboratory tests are inconclusive, they cannot determine which of two bacterial strains is responsible. Two medicines are available to treat the infections. Medicine 1 is 85% effective against bacterial strain 1 and 70% effective against strain 2. Medicine 2 is 60% effective against strain 1 and 95% effective against strain 2. These medicines cannot be used in combination on any single patient.

a. Use game theory to determine which medicine doctors should use.

b. How effective can doctors expect this treatment to be?

2.9 **a.** Using the information given in Problem **2.3**, construct a payoff matrix for Chicago Remodeling.

b. Write the expression that will determine the expected value for Chicago if Royal always emphasizes cost.

c. Write the expression that will determine the expected value for Chicago if Royal always emphasizes quality.

d. Determine the optimal strategy for Chicago Remodeling.

e. **1.** If Chicago Remodeling uses this optimal strategy, what percentage of the building projects should the company expect to win?

2. How does this percentage compare to the percentage you obtained in Problem **2.3e**?

* * * * * * * * * *

Summary Assessment

1. In baseball, the contest between batter and pitcher can be thought of as a two-person game. Consider a situation in which the pitcher can throw either a fastball or a curve ball. The batter's chances of getting a hit depend not only on the type of pitch thrown, but on whether the batter is anticipating that type of pitch.

This situation is represented in the following matrix, where F represents fastball and C represents curve ball. If the pitcher throws a fastball and the batter guesses fastball, the batter's probability of getting a hit is 0.315.

$$\begin{array}{ccc} & & \text{Pitcher} \\ & & \begin{array}{cc} \text{F} & \text{C} \end{array} \\ \text{Batter} & \begin{array}{c} \text{F} \\ \text{C} \end{array} & \begin{bmatrix} 0.315 & 0.250 \\ 0.100 & 0.565 \end{bmatrix} \end{array}$$

a. Should the players use pure strategies or mixed strategies in this situation?

b. What is the optimal strategy for the batter?

c. Using the strategy described in Part **b**, what is the expected outcome for the batter?

2. Imagine that you are a doctor. You have two medicines available to treat a bacterium with two known strains. One medicine is 55% effective against the first strain and 45% effective against the second. The other medicine is 40% effective against the first strain and 60% effective against the second. Decide which medicine you should use and what kind of results you can expect if no one patient can receive both medicines.

3. The Cougars are playing the Falcons for the conference basketball championship. The coach of the Cougars believes that if her team can control the Falcons' leading scorer, Rhonda Allen, then they have a good chance of winning the game. According to the scouting report, Rhonda has three basic moves: she drives to her left, she drives to her right, or she pulls up and shoots a jump shot.

After comparing Rhonda's strengths with the team's defensive skills, the coach constructs the following matrix for Rhonda's probability of scoring a basket.

In this matrix, L represents either a drive to the left for Rhonda or defense to the left, R represents a drive right or defense to the right, and J represents a jump shot or defense against the jump shot.

$$
\begin{array}{ccc}
 & & \text{Defender} \\
 & & \begin{array}{ccc} \text{L} & \text{R} & \text{J} \end{array} \\
\text{Rhonda} & \begin{array}{c} \text{L} \\ \text{R} \\ \text{J} \end{array} & \begin{bmatrix} 10\% & 60\% & 35\% \\ 85\% & 20\% & 70\% \\ 50\% & 60\% & 40\% \end{bmatrix}
\end{array}
$$

a. Which move will Rhonda be least likely to choose? Explain your response.

b. Considering your response to Part **a**, which defense will the Cougars use least? Explain your response.

c. Delete the column and row indicated by your responses to Parts **a** and **b**. Using the remaining 2×2 matrix, determine the optimal strategy for Rhonda.

d. How should the Cougars defend Rhonda?

e. On average, how often will Rhonda score a basket?

Module Summary

- **Game theory** is a branch of mathematics used to analyze decision making in situations that involve conflicting interests.
- An **optimal strategy** for a player results in maximizing the winnings or minimizing the losses for that player.
- In a **pure strategy**, a player makes the same choice each time the game is played.
- In a **mixed strategy**, a player's choice of how to play varies from game to game.
- A **payoff** is the amount won or lost by a single player in one play of a game.
- The consequences of each player's choices can be summarized in a **payoff matrix**. The rows of the matrix represent the choices of one player (the row player), while the columns represent the choices of the second player (the column player).
- In a **strictly determined** game, the optimal strategy for each player is a pure strategy.
- The **value** of a strictly determined game can be determined from the payoff matrix. It is less than or equal to all entries in its row and greater than or equal to all entries in its column. This value is the **saddle point** of the matrix.
- A strictly determined game is fair if the saddle point is 0.
- In a game that is not strictly determined, the optimal strategies are mixed strategies. When the row player uses an optimal strategy, the expected value of the game for that player is the same, regardless of the choice made by the column player.
- One row in a payoff matrix **dominates** another row if each of its entries is greater than the corresponding entries in the other row. When making a choice between the two rows, the row player should always select the dominant row in order to maximize winnings.

  Similarly, one column **dominates** another column if each of its entries is less than the corresponding entries in the other column. When choosing between the two columns, the column player should always select the dominant column in order to maximize winnings.

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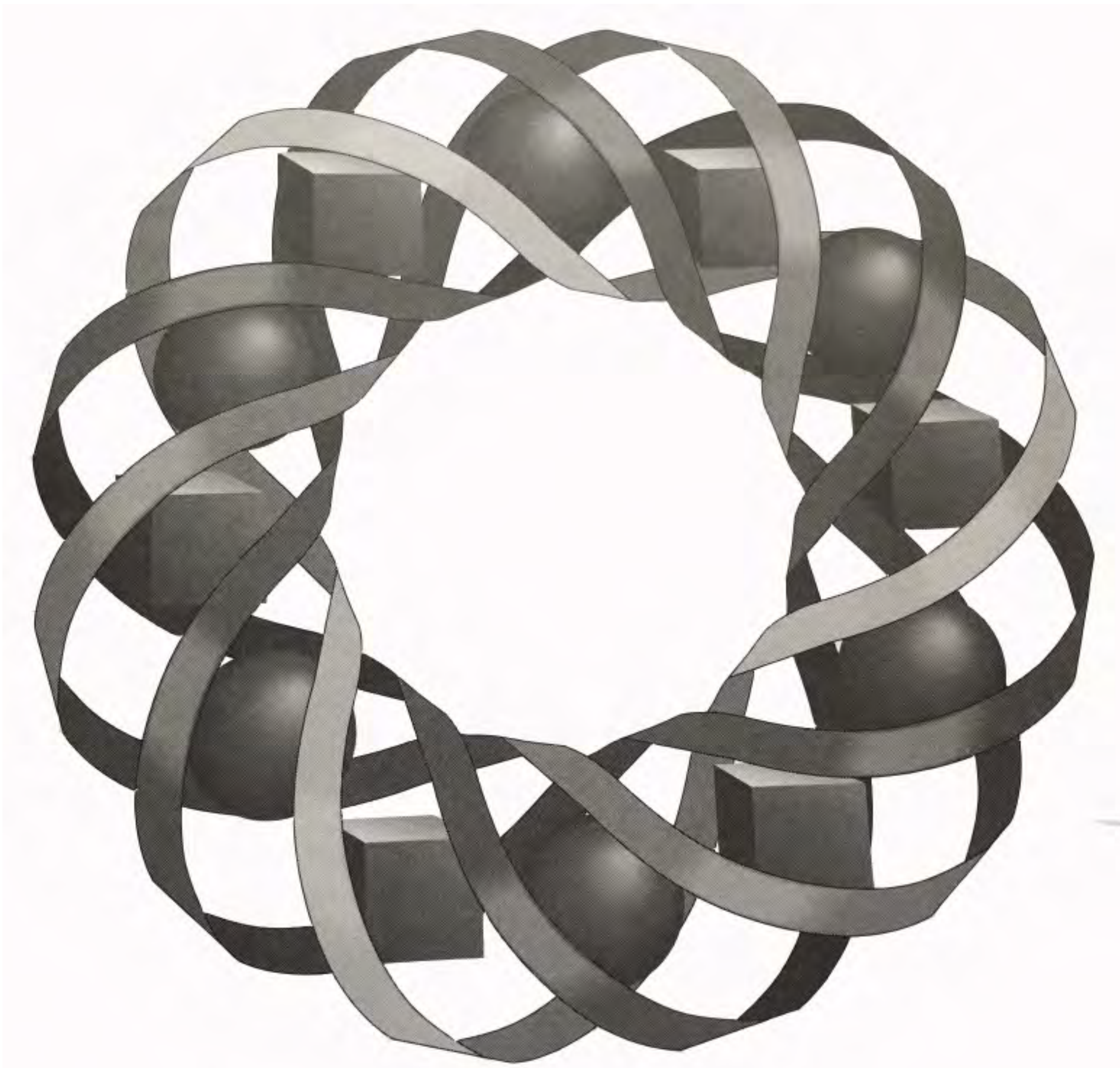
A Walk on the Wild Side



An infestation of whirling disease has had a dramatic effect on the trout populations in Montana's rivers. In this module, you discover how wildlife biologists investigate such problems.

Todd Fife • Pete Stabio

A Walk on the Wild Side

Introduction

In the spring of 1995, an 80-km stretch of Montana's Madison River, one of America's finest trout streams, was closed to fishing. Before the closure, studies by fisheries biologists showed a dramatic decline in the population of rainbow trout in that section of the river—from 2051 rainbows per kilometer in 1991 to 186 per kilometer in 1994. Wildlife managers ordered the closure to protect the trout population during the spawning season.

The rapid decline of the trout population in the Madison has been blamed on whirling disease. This incurable illness causes young fish to spin as if chasing their tails, leaving them vulnerable to predators and unable to feed.

Activity 1

Because it is not feasible to catch and examine every fish in a river, biologists rely on sampling methods when studying trout populations. When monitoring the spread of whirling disease, researchers use statistics to estimate the proportion of the population with the disease.

Exploration

In this exploration, you use a simulation to investigate sample proportions. **Note:** Save your work for use in Activity **2**.

a. Obtain a container of 100 beans from your teacher. In this simulation, the container represents a section of river, while the beans represent the trout population. The marked beans represent diseased fish. To estimate the proportion of diseased fish in your simulated population, complete Steps **1–3** below.

   1. Draw one bean at random from the container and record whether or not it represents a diseased fish.

   2. Return the bean to the container and mix it thoroughly with the others.

   3. Repeat Steps **1** and **2** until you have obtained a sample of 4 beans. (This is an example of sampling with replacement.)

Mathematics Note:

The proportion of a population that displays a certain characteristic can be denoted as p. The value of p is a **population proportion** and can be found as follows:

$$p = \frac{\text{number in population with the characteristic}}{\text{total population}}$$

In the population of all citizens of the United States, for example, the proportion p of females is:

$$p = \frac{\text{number of females in U.S. population}}{\text{total U.S. population}}$$

When a sample is taken from a population, the proportion of the sample that displays the characteristic is denoted as $\hat{p}$. Sample proportions are often used to estimate population proportions. The value of $\hat{p}$ is the **sample proportion** and can be found as shown below:

$$\hat{p} = \frac{\text{number in sample with characteristic}}{\text{total number in sample}}$$

For example, consider a sample of the U.S. population that consists of all secondary school students. The proportion $\hat{p}$ of females in this sample can be used to estimate p, the proportion of females in the U.S population.

b. Determine the proportion $\hat{p}$ of diseased trout from the sample taken in Part **a**.

c. Estimate the proportion p of diseased trout in the population.

d.
1. Repeat Parts **a** and **b** 19 more times to obtain a total of 20 sample proportions.
2. Calculate the mean and standard deviation of the 20 sample proportions.

e. The **relative frequency** of an item is the ratio of its frequency to the total number of observations in the data set. A **relative frequency table** includes columns that describe a data item or interval, its frequency, and its relative frequency.

1. Complete a relative frequency table with headings like those in Table **1** for the 20 sample proportions from Part **d**.

Table 1: Relative frequency table for sample proportions

Sample Proportion	Frequency	Relative Frequency

2. Create a histogram of relative frequency versus sample proportion.

3. Use the graph to revise your estimate of the population proportion from Part **c**.

f. Use technology to repeat Parts **a–e** for a sample size of 30.

g.

1. Combine the class data for the samples of size 4. Create a relative frequency table and histogram of the results. **Note:** Save this data for use in Activity **2**.

2. Combine the class data for the samples of size 30. Create a relative frequency table and histogram of the results. **Note:** Save this data for use in Problem **1.3** and in Activity **2**.

3. Re-evaluate your estimate of p from Part **e**.

Discussion

a. Does the sampling method described in Part **a** of the exploration produce random samples? Explain your response.

b. Compare the results of the sampling process with samples of size 4 with the results for samples of size 30.

c. Imagine that biologists have only enough time and money to collect one sample when estimating the proportion of diseased fish in a population. Which of the sample sizes used in the exploration gave a better chance of obtaining a good estimate? Explain your response.

d.

1. Using the data obtained in the exploration for 20 samples of size 30, estimate the probability of obtaining a sample with exactly 15 diseased fish.

2. Using the combined class data for a sample size of 30, estimate the probability of obtaining a sample with exactly 15 diseased fish.

3. According to the law of large numbers, which set of data is more likely to provide a better estimate of the probability of obtaining a sample with exactly 15 diseased fish?

e. In practice, wildlife biologists often sample without replacement. To sample a fish population, for example, researchers might net fish from one section of a river. While the sample is being gathered, the fish are placed in holding tanks. After the desired data has been collected, the fish are returned to the water.

When is it reasonable to analyze such data as if the sampling had been done with replacement?

Assignment

1.1 In the exploration, you used sample sizes of 4 and 30 to obtain data about a population of fish. For a sample of size 4, the possible proportions of diseased fish are 0, 0.25, 0.5, 0.75, and 1.00. List the different proportions that could result for sample sizes of 30.

1.2 The following relative frequency table shows the data collected from 40 samples of size 4 from a population of fish. As you can see, some entries have been omitted from the table.

Proportion of Diseased Fish	Frequency	Relative Frequency
$0/4 = 0$	1	
$1/4 = 0.25$		0.175
$2/4 = 0.50$	14	
$3/4 = 0.75$		0.375
$4/4 = 1.00$		0.075

a. Determine the missing values in the table.

b. Describe how to use this data to estimate the proportion of diseased fish in the population.

1.3 Use the relative frequency table for the combined class data of samples of size 30 from Part **g** of the exploration to complete Parts **a–c**.

a. Estimate the probability of obtaining a sample with 8 diseased fish.

b. How is the relative frequency of a sample with 8 diseased fish related to the estimated probability?

c. Estimate the probability of obtaining each of the following:

1. a sample with 10 diseased fish
2. a sample with more than 10 diseased fish
3. a sample with 8 or fewer diseased fish.

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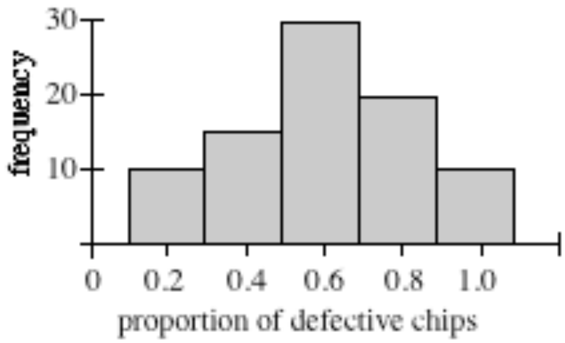
1.4 The following frequency histogram shows the data collected from 85 samples of size 5 from a population of computer chips.



a. Using the histogram, estimate the probability of obtaining a sample with exactly 60% defective chips.

b. Using the histogram, estimate the probability of obtaining a sample with at least 60% defective chips.

c. Sketch a histogram that represents the relative frequency of each proportion in this data set.

d. Estimate the proportion of defective chips in the population.

1.5 The RoboSpace Company makes sprockets. As part of their quality-control process, the company take samples of 5 sprockets each hour and determines the proportion that are defective. The following table shows the data collected for 20 samples of size 5.

Proportion of Defective Sprockets	Frequency	Relative Frequency
$0/5 = 0.0$	5	
$1/5 = 0.20$	4	
$2/5 = 0.40$	5	
$3/5 = 0.60$	5	
$4/5 = 0.80$	1	
$5/5 = 1.00$	0	

a. Determine the relative frequency of each sample proportion.

b. Create a histogram of the relative frequencies.

c. Estimate the probability of obtaining a sample with exactly 2 defective sprockets.

d. Estimate the proportion of defective sprockets in the population.

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Activity 2

Even though the entire population may not be infected, wildlife researchers know that a random sample might contain all diseased fish. Though this is unlikely, it is possible. It is more likely, however, that the proportion of diseased fish in the sample will be close to the actual proportion of diseased fish in the population.

Discussion 1

a. As you may recall from previous modules, a **binomial experiment** has the following characteristics:

- The experiment consists of a fixed number of repetitions of the same action. Each repetition is a trial.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: success or a failure.
- The probability of a success remains constant from trial to trial.
- The total number of successes is observed.

Explain why taking a sample of 4 fish from a lake and determining the number of diseased fish can be modeled by a binomial experiment.

b. In a sample of 4 fish, one possible outcome is HHDH, where H represents a healthy fish and D represents a diseased fish.

Describe how to use combinations to determine the number of different ways in which you could randomly draw 4 fish from a population, one at a time with replacement, and record exactly 1 diseased fish.

c. Recall that the **binomial probability formula** can be used to determine the probability of obtaining r successes in n trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

Given that the probability of catching a diseased fish is 0.6, describe how to determine the probability that a random sample of 4 fish contains exactly 1 diseased fish.

Mathematics Note

The **sampling distribution** of all possible sample proportions $\hat{p}$ for samples of size n is the set of probabilities associated with each possible value of $\hat{p}$.

For example, Table **2** below shows the sampling distribution for $\hat{p}$ given a sample size of 4 and a population proportion of 0.6.

Table 2: Sampling distribution for $n = 4$ and $p = 0.6$

Sample Proportion	Probability
$0/4 = 0$	0.0256
$1/4 = 0.25$	0.1536
$2/4 = 0.5$	0.3456
$3/4 = 0.75$	0.3456
$4/4 = 1$	0.1296

d. What is the sum of the probabilities in any sampling distribution of $\hat{p}$? Explain your response.

Exploration 1

In this exploration, you compare the data you collected in Activity **1** to the sampling distribution of $\hat{p}$ for samples of size 4 and 30.

a. Use the actual population proportion from the exploration in Activity **1** and the binomial probability formula to construct the sampling distribution for samples of size 4.

b. Calculate the mean of the sampling distribution from Part **a**. Compare this value with each of the following:

1. the population proportion
2. the mean of the sample proportions for samples of size 4 from Activity **1**.

Mathematics Note

For samples of size n, the sampling distribution of $\hat{p}$ has a mean μ where $\mu = p$, the population proportion.

The standard deviation σ of the sampling distribution of $\hat{p}$ can be determined using the following formula, where n is the sample size and p is the population proportion:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

For example, consider a population in which the proportion of females is 0.6. The mean of the sampling distribution of $\hat{p}$ for samples of size 30 also is 0.6. The standard deviation of the sampling distribution of $\hat{p}$ for samples of size 30 can be found as follows:

$$\sigma = \sqrt{\frac{0.6(1-0.6)}{30}} \approx 0.089$$

c. Calculate the standard deviation of the sampling distribution of $\hat{p}$ for samples of size 4.

Compare this value with the standard deviation for samples of size 4 determined in Activity **1**.

d. Repeat Parts **a–c** for samples of size 30.

Discussion 2

a. Describe the method you used in Part **b** of Exploration **1** to calculate the mean of the sampling distribution of $\hat{p}$.

b. As the number of samples grows very large, how would you expect a relative frequency table to compare to the corresponding sampling distribution for $\hat{p}$?

c. Consider a population of fish in which 60% are infected with whirling disease. How could you use the sampling distribution for samples of size 30 to show that a sample containing 10 or fewer diseased fish is very unlikely?

Exploration 2

In Exploration **1**, you used sampling distributions to examine the probabilities of obtaining specific sample proportions. In this exploration, you use technology to investigate the probability that a sample proportion will fall within 1, 2, or 3 standard deviations of the population proportion.

a. Consider a population in which the proportion with a certain characteristic is $p = 0.10$. Create connected scatterplots of the sampling distributions when $n = 4$, 10, and 30. Describe any trends you observe.

b. Repeat Part **a** for population proportions of 0.30, 0.50 and 0.80. Describe any trends you observe.

c. Consider a population of trout in which 60% are infected with whirling disease.

1. Determine a sample size n that results in a connected scatterplot which appears bell-shaped and symmetric about the population proportion.
2. Construct a table showing the sampling distribution of $\hat{p}$.
3. Record the mean μ and standard deviation σ of the sampling distribution.

d. Create a connected scatterplot of the sampling distribution in Part **c**. On the *x*-axis of your graph, mark the values that correspond to $\mu + \sigma$, $\mu + 2\sigma$, $\mu + 3\sigma$, μ, $\mu - \sigma$, $\mu - 2\sigma$, and $\mu - 3\sigma$.

Estimate the percentage of sample proportions that fall in each of the following intervals:

1. $[\mu + \sigma,\ \mu - \sigma]$
2. $[\mu + 2\sigma,\ \mu - 2\sigma]$
3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

Discussion 3

a. Which sample sizes in Exploration **2** resulted in connected scatterplots that appear bell-shaped and symmetric about the population proportion?

b. In Part **d** of Exploration **2**, what was the probability that a sample proportion fell within each of the following intervals?

1. $[\mu + \sigma,\ \mu - \sigma]$
2. $[\mu + 2\sigma,\ \mu - 2\sigma]$
3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

c. How do these probabilities compare to those predicted by the 68–95–99.7 rule for normal distributions?

Mathematics Note

The **central limit theorem** states that, regardless of the population, as the sample size increases, the sampling distribution of sample proportions approaches a normal distribution.

As a rule of thumb, the properties implied by the central limit theorem are approximately true when the sample size n satisfies the conditions $np > 5$ and $n(1-p) > 5$. For example, suppose one is reasonably sure that $0.25 \le p \le 0.75$. If $p = 0.25$, then:

$$\begin{aligned} np &> 5 & n(1-p) &> 5 \\ n(0.25) &> 5 & n(1-0.25) &> 5 \\ n &> 20 & n &> 6 \end{aligned}$$

Similarly, if $p = 0.75$, then:

$$\begin{aligned} np &> 5 & n(1-p) &> 5 \\ n(0.75) &> 5 & n(1-0.75) &> 5 \\ n &> 6 & n &> 20 \end{aligned}$$

In this case, the central limit theorem may provide useful interpretations for sample sizes greater than 20.

d. If the population proportion is between 0.10 and 0.90, what sample sizes might you select to ensure that the sample distribution can be approximated by a normal distribution? Explain your reasoning.

Assignment

2.1 Assume that the proportion of females in a trout population is 50%.

a. Construct the sampling distribution of $\hat{p}$ for samples of size 10.

b. What is the probability of obtaining a random sample from this population with less than 3 female trout?

c. What is the probability of obtaining a random sample from this population in which the number of females is more than 3 but less than 8?

d. Determine the probability that the proportion of females in a random sample of 10 trout from this population will be within 1 standard deviation of the population proportion.

2.2 Some individuals in a certain species of bird carry a gene for an enzyme deficiency. The proportion of the bird population in your area that has this gene is 12%.

a. Construct the sampling distribution of $\hat{p}$ for samples of size 50.

b. Use the central limit theorem to determine an interval in which a sample proportion should fall 95% of the time.

2.3 In 1994, 35% of the rainbow trout in the Raynolds Pass area near the western border of Yellowstone National Park were infected with whirling disease.

a. What sample size is required to make the standard deviation of the sampling distribution of $\hat{p}$ less than or equal to 0.05?

b. Calculate each of the following intervals for a sample size of 400:

1. $[\mu + \sigma,\ \mu - \sigma]$

2. $[\mu + 2\sigma,\ \mu - 2\sigma]$

3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

c. Describe the percentage of sample proportions you would expect to fall in each interval in Part **b**.

2.4 A wildlife biologist would like to estimate the proportion of white rabbits in a population with four different color phases: white, brown, black, and mixed. The biologist thinks that at least one-fifth are white and at least one-fifth are not white. What is the minimum sample size the researcher should use to be confident that the sampling distribution will be bell-shaped, symmetrical, and follow the 68–95–99.7 rule?

* * * * *

2.5 One day's production at the RoboSpace Company resulted in a population of sprockets with a 20% defective rate.

a. Determine the probability that a random sample of 4 sprockets will contain each of the following:

1. exactly 1 defective sprocket

2. exactly 0 defective sprockets

3. no more than 2 defective sprockets.

b. For a sample size of 200, in what interval would you expect 95% of the sample proportions to fall?

2.6 A bolt manufacturing company estimates that the probability of manufacturing a defective bolt is 30%.

a. If the company's quality-control process uses samples of 300 bolts, what would you expect the mean and the standard deviation to be for the sampling distribution of $\hat{p}$?

b. In what interval would you expect 95% of the sample proportions to fall for a sample size of 300?

c. What would you conclude if you drew a random sample of 300 bolts in which 40% were defective?

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Activity 3

Consider a population of 30,000 trout. Using the sampling method described in Activity **1** and samples of size 100, the number of possible samples is $30,000^{100}$. Approximately 95% of these sample proportions fall in an interval within 2 standard deviations of the population proportion.

If you knew that the proportion of diseased trout in the population was 56%, this interval would be $[0.56 - 2(0.05), 0.56 + 2(0.05)]$ or $[0.46, 0.66]$. Unfortunately, researchers seldom know the population proportion before studying a population. In this activity, you investigate what a single sample proportion can tell you about an unknown population proportion.

Exploration

In this exploration, you examine the number of sample proportions that fall within a certain interval of the population proportion.

a.
1. On a sheet of graph paper, create a number line from 0 to 1 with increments of 0.1.

2. Mark the proportion p of diseased fish in the fish population from Activity **1** on the number line.

b. Take a random sample of 30 fish from this population and calculate $\hat{p}$, the proportion of diseased fish in the sample.

c. To estimate the standard deviation σ of the sampling distribution of $\hat{p}$ for samples of size 30, substitute $\hat{p}$ for p in the following expression, where n represents the sample size:

$$\sqrt{\frac{p(1-p)}{n}}$$

Call this estimate s.

d. Using your results from Part **c**, determine the interval $\left[\hat{p} - 2s, \hat{p} + 2s\right]$. Draw this interval, to the nearest 0.01, above your number line. This is a **confidence interval** for the population proportion based on your sample.

e. Determine whether or not the actual population proportion falls within the confidence interval.

f. **1.** Use technology to repeat Parts **b–e** 19 more times. Draw each confidence interval separately above your number line. For example, Figure **1** shows three confidence intervals for samples taken from a population where $p = 0.56$.

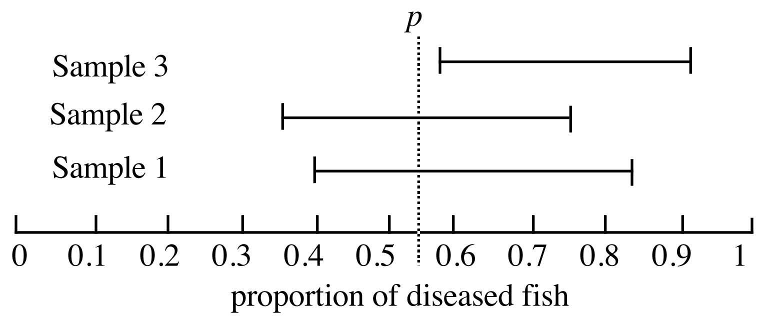

Figure 1: Number line with confidence intervals

2. Record the percentage of confidence intervals that contain the actual population proportion. In the example given in Figure **1**, approximately 67% of the confidence intervals contain p.

g. Repeat Parts **a–f** for samples of size 90.

Mathematics Note

A **confidence interval** for a population proportion is an interval in which the value of the population proportion is expected to be found. Every confidence interval has two aspects: an interval determined by the statistics collected from a random sample and a confidence level that gives the probability that the interval includes the parameter.

For example, a 95% confidence interval is generated by a process that results in an interval in which the probability that the parameter lies in that interval is 95%. In other words, you would expect 95% of the intervals produced by this process to contain the parameter, while 5% of the intervals would not.

Consider a population proportion p and a sample size n such that $np > 5$ and $n(1 - p) > 5$. The mean μ of all the sample proportions of size n equals the population proportion p. Since sample proportions $\hat{p}$ are normally distributed, the 68–95–99.7 rule can be applied.

For example, the proportion $\hat{p}$ of any one sample will fall within 2 standard deviations of p 95% of the time. This fact also indicates a 95% probability that p is within 2 standard deviations of $\hat{p}$. In other words, the 95% confidence interval for p is $[\hat{p} - 2\sigma, \hat{p} + 2\sigma]$, where

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

This assertion can be reworded as a **confidence statement** in the following form: "If $\hat{p}$ is the sample proportion for a random sample of size n, then you can be 95% confident that the actual population proportion falls in the interval $[\hat{p} - 2\sigma, \hat{p} + 2\sigma]$." When p is not known, the value of $\hat{p}$ is used to estimate σ and generate the confidence interval. This estimated standard deviation is denoted by s.

For example, consider a stream that contains several different species of fish. To determine the proportion of rainbow trout in the total fish population, biologists capture a random sample of 150 fish from the stream. Of this sample, 11 are rainbow trout. In this situation, $\hat{p} = 11/150 \approx 0.073$. Since p is not known, $\hat{p}$ is substituted for p to find s, the estimated standard deviation, as follows:

$$s \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.073\,(1-0.073)}{150}} = 0.021$$

Since $\hat{p} - 2s = 0.073 - 0.042 = 0.031$ and $\hat{p} + 2s = 0.073 + 0.042 = 0.115$, the 95% confidence interval for the population proportion is [0.031, 0.115].

Therefore, the researchers can conclude, with 95% confidence, that the proportion of rainbow trout in the total population is in the interval $[0.031, 0.115]$ or, in other words, that rainbow trout represent between 3.1% and 11.5% of the population in the stream.

Discussion

a.
   1. Using a sample size of 30, what percentage of the confidence intervals in the exploration contained the population proportion?

   2. Using a sample size of 90, what percentage of the confidence intervals in the exploration contained the population proportion?

   3. Were these results consistent with the 68–95–99.7 rule?

b. Consider a random sample of 400 trout that contained 160 fish infected with whirling disease. Describe how to construct a confidence interval for the proportion of infected trout in the population using each of the following confidence levels:

   1. 95%

   2. 68%

   3. 99.7%

c. Describe how sample size affects the width of the corresponding confidence intervals.

d. A newspaper reports that 40% of the rainbow trout in a certain section of the Madison River are diseased. To test that claim, a fisheries biologist takes a random sample of 300 rainbow trout from that section of the river.

1. What can the biologist conclude if 60 of the fish are diseased?
2. What can the biologist conclude if 100 of the fish are diseased?

Assignment

3.1 Fisheries managers are trying to determine the proportion of rainbow trout over 40 cm in length in a certain population. As part of their study, biologists collect random samples of 100 trout each day for 16 days. The following table shows the number of trout over 40 cm in each of these samples.

Day 1	6	Day 9	7
Day 2	8	Day 10	6
Day 3	6	Day 11	10
Day 4	10	Day 12	8
Day 5	9	Day 13	7
Day 6	7	Day 14	11
Day 7	8	Day 15	8
Day 8	14	Day 16	8

a. Use the data for one of the days to construct a 95% confidence interval for the proportion of rainbow trout over 40 cm in length and make a confidence statement about this interval.

b. Select a day with a different sample proportion from the one used in Part **a**. Construct a 95% confidence interval using this information and make a confidence statement about this interval.

c. Compare the two confidence intervals. How many different 95% confidence intervals would you expect if you determined one for each day in the table?

d. Would you expect all of the 95% confidence intervals obtained from the data to contain p, the actual proportion of rainbow trout in the river over 40 cm? Explain your response.

e. Describe how the data for all 16 days could be combined to determine a single 95% confidence interval.

f. Determine the 95% confidence interval that results from the technique you described in Part **e**.

3.2 To determine the proportion of rainbow trout in the total fish population, biologists capture a random sample of fish from a stream.

a. Determine a 95% confidence interval for the population proportion and make a confidence statement about this interval for each of the following situations:

1. a random sample of 50 fish contains 7 rainbow trout

2. a random sample of 100 fish contains 14 rainbow trout

3. a random sample of 400 fish contains 56 rainbow trout

b. Compare the sample proportions used to construct each confidence interval in Part **a.**

c. Write a paragraph explaining the significance of the results in Parts **a** and **b**.

3.3 In 1994, 35% of the rainbow trout in the Raynolds Pass area near Yellowstone Park were infected with whirling disease. It is feared that the proportion of diseased fish is increasing rapidly.

Imagine that you are a biologist assigned to determine if the disease rate in the Raynolds Pass area is still 35%. If a sample of 400 fish from the area contains 160 that are diseased, what can you conclude about the disease rate? Explain your response.

3.4 A wildlife biologist is studying the survival rates of calves in a population of elk. As part of her study, she fits a random sample of 100 calves with radio collars. At the end of 1 year, 73 of these calves are still alive.

a. Estimate the proportion of elk calves in the population that survive at least 1 year.

b. Estimate the standard deviation for the sampling distribution.

c. Determine a 95% confidence interval for the proportion of elk calves from this population that survive at least 1 year and make a confidence statement about this interval.

d. Describe how the confidence interval would be affected if the sample size was increased from 100 to 400 and the proportion of calves that survived remained the same.

e. Why is a small standard deviation advantageous to a researcher?

3.5 About 50 species of bats are found in North America. In the 1950s, bats became branded as carriers of rabies. Health departments around the country conducted rabies tests on dead bats found by local residents or their pets. Based on the thousands of bats tested each year, researchers estimated that 10% of the bats in North America carry rabies.

Suppose a group of scientists is studying bats in the southwestern United States. As part of the study, the group collects a random sample of 600 live bats from the region. Of this sample, 6 bats are found to carry rabies.

a. Using a confidence statement, estimate the proportion of rabid bats in the southwestern United States.

b. Based on your results, what conclusions would you draw regarding the claim that 10% of the bats in North America are rabid?

c. If the true proportion of rabid bats in the area you are studying is 10%, what is the probability of obtaining a random sample of 600 bats that contains 6 or fewer with rabies?

d. Describe some circumstances that may have affected the accuracy of the 10% estimate described above.

* * * * *

3.6 **a.** For samples of the same size from the same population, would a 99.7% confidence interval be wider or narrower than a 95% confidence interval? Explain your response.

b. Write an expression that will determine a 99.7% confidence interval given values for $\hat{p}$ and s.

c. Describe some factors that researchers should consider when deciding whether to use a 95% or a 99.7% confidence interval.

3.7 Before an election in Great Britain, a random sample of 100 voters indicated that 55% of them supported the party of the incumbent prime minister.

a. Determine a 95% confidence interval for the proportion of all voters who supported the incumbent's party.

b. Determine a 99.7% confidence interval for the proportion of all voters who supported the incumbent's party.

* * * * * * * * * *

Research Project

Select a population and a characteristic that you are interested in studying. For example, you may wish to estimate the percentage of students at your school who have part-time jobs. Design a sampling experiment that allows you to determine a reasonable estimate of the proportion of the population with this characteristic.

Activity 4

In 1991, biologists estimated that the rainbow trout population in one section of the Madison River was 2051 fish per km. In order to obtain this estimate, they used a sampling technique called **capture-recapture**. In this activity, you use this method to approximate the number of trout in a simulated river.

Exploration

Fishery biologists routinely "tag" or mark captured fish by clipping a small piece from one fin. This identifying mark allows researchers to recognize a recaptured fish. The fin will eventually grow back, but not before biologists have completed their study.

a. Obtain a container with an unknown number of beans. In this exploration, the container represents a river, while each bean represents a trout.

b. Select a random sample of 50 trout, without replacement, and "tag" each one with a marking pen.

c. Return the tagged fish to the river and mix them thoroughly with the rest of the population.

d. Select a random sample of 40 trout from the river and determine the proportion of tagged fish in the sample.

e. Using the estimated population proportion, estimate the number of trout in the river.

f. Determine a 68% confidence interval for the proportion of tagged fish.

g. Use the confidence interval from Part **f** to determine a 68% confidence interval for the number of rainbow trout in the river.

h.
1. Count the actual number of trout in the river.
2. Compare the estimate you made in Part **e** with the actual size of the population.

Discussion

a. What factors might affect the accuracy of a population estimate made using the capture-recapture method?

b. If no tagged fish are recaptured, what conclusions should researchers make about the population?

c. Was the actual size of the population within the confidence interval you determined in Part **g** of the exploration?

d. If you used the process described in the exploration to determine 100 estimates and their corresponding confidence intervals, how many of these intervals would you expect to contain the actual population size? Explain your response.

Assignment

4.1 In one capture-recapture experiment, biologists caught, tagged, and released 350 trout in a lake. Later, they caught 168 trout, 14 of which were tagged. Estimate the number of trout in the lake.

4.2 Imagine that you are a wildlife specialist monitoring the impact of oil-field development on wolves in Alaska.

a. Before any oil drilling occurred, you captured and tagged 50 wolves in one region of the state. Several days later, you captured 45 wolves, of which 15 were tagged. Estimate the wolf population in this region before any drilling took place.

b. One year after the last well was drilled, you captured and tagged 52 wolves with a different set of tags in the same region. One week later, you captured 40 wolves, of which 14 were wearing the new tags. Estimate the wolf population in this region after the oil field was developed.

c. Write a paragraph comparing your results from Parts **a** and **b.** Include statistical evidence to support whether you think that the wolf population is declining in this region of Alaska.

4.3 Seven fish are captured from a small pond. After being tagged, the fish are returned to the pond. A second sample of 6 fish from the pond contains 2 that were tagged earlier. Describe what conclusions, if any, you can make about the pond's fish population.

4.4 Imagine that you are a biologist monitoring the impact of a new sewage disposal site on the trout population downstream.

a. In the year before the site opened, you tagged a sample of 1000 trout in this section of the river. Several days later, you collected another sample of 1000 trout, of which 120 were tagged. Estimate the trout population in this section of the river before the site opened.

b. A year after the site opened, you tagged a sample of 750 trout in this section. Several days later, you captured a sample of 800 trout, of which 105 were tagged. Estimate the trout population in this section after the site opened.

c. Compare your results from Parts **a** and **b**. Do you think that the trout population is declining in this stretch of river? Explain your response.

d. How could you adjust your capture-recapture method to obtain more precise results and allow for more effective monitoring?

* * * * *

4.5 A team of researchers catches 100 bats in a large cave. After tagging the bats, they return them to the cave. A few days later, the scientists catch 500 bats in the same cave and find that 12 are tagged.

a. Determine a 68% confidence interval for the population of bats in the cave.

b. Determine a 95% confidence interval for the population of bats in the cave.

c. Describe any assumptions you made about this sampling method in order to determine your estimates in Parts **a** and **b**.

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Summary Assessment

1. Obtain a model population from your teacher. A certain proportion of this population has a given characteristic.

a. Use sampling to estimate the population proportion.

b. Describe the sampling distribution of $\hat{p}$ for your selected sample size, including the standard deviation of the distribution.

c. Write a report describing your best estimate of the population proportion and the procedure you used to obtain it. Include any data you collected, as well as the resulting statistics and confidence intervals.

2. Obtain a model population from your teacher.

a. Design a capture-recapture experiment to determine a confidence interval for the size of this population.

b. Write a report describing the process you used to determine your confidence interval.

Module Summary

- The proportion of a population that displays a certain characteristic can be denoted as p. The value of p is a **population proportion** and can be found as follows:

$$p = \frac{\text{number in population with the characteristic}}{\text{total population}}$$

- When a sample is taken from a population, the proportion of the sample that displays the characteristic is denoted as $\hat{p}$. The value of $\hat{p}$ is the **sample proportion** and can be found as shown below:

$$\hat{p} = \frac{\text{number in sample with characteristic}}{\text{total number in sample}}$$

- The **sampling distribution** of all possible sample proportions $\hat{p}$ for samples of size n is the set of probabilities associated with each possible value of $\hat{p}$.
- For samples of size n, the sampling distribution of $\hat{p}$ has a mean μ where $\mu = p$, the population proportion.
- The standard deviation σ of the sampling distribution of $\hat{p}$ can be determined using the following formula, where n is the sample size and p is the population proportion:

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

- The **central limit theorem** states that, regardless of the population, as the sample size increases, the sampling distribution of sample proportions approaches a normal distribution. As a rule of thumb, the properties implied by the central limit theorem are approximately true when the sample size n satisfies the conditions $np > 5$ and $n(1-p) > 5$.
- A **confidence interval** for a population proportion is an interval in which the value of the population proportion is expected to be found. Every confidence interval has two aspects: an interval determined by the statistics collected from a random sample and a confidence level that gives the probability that the interval includes the parameter.

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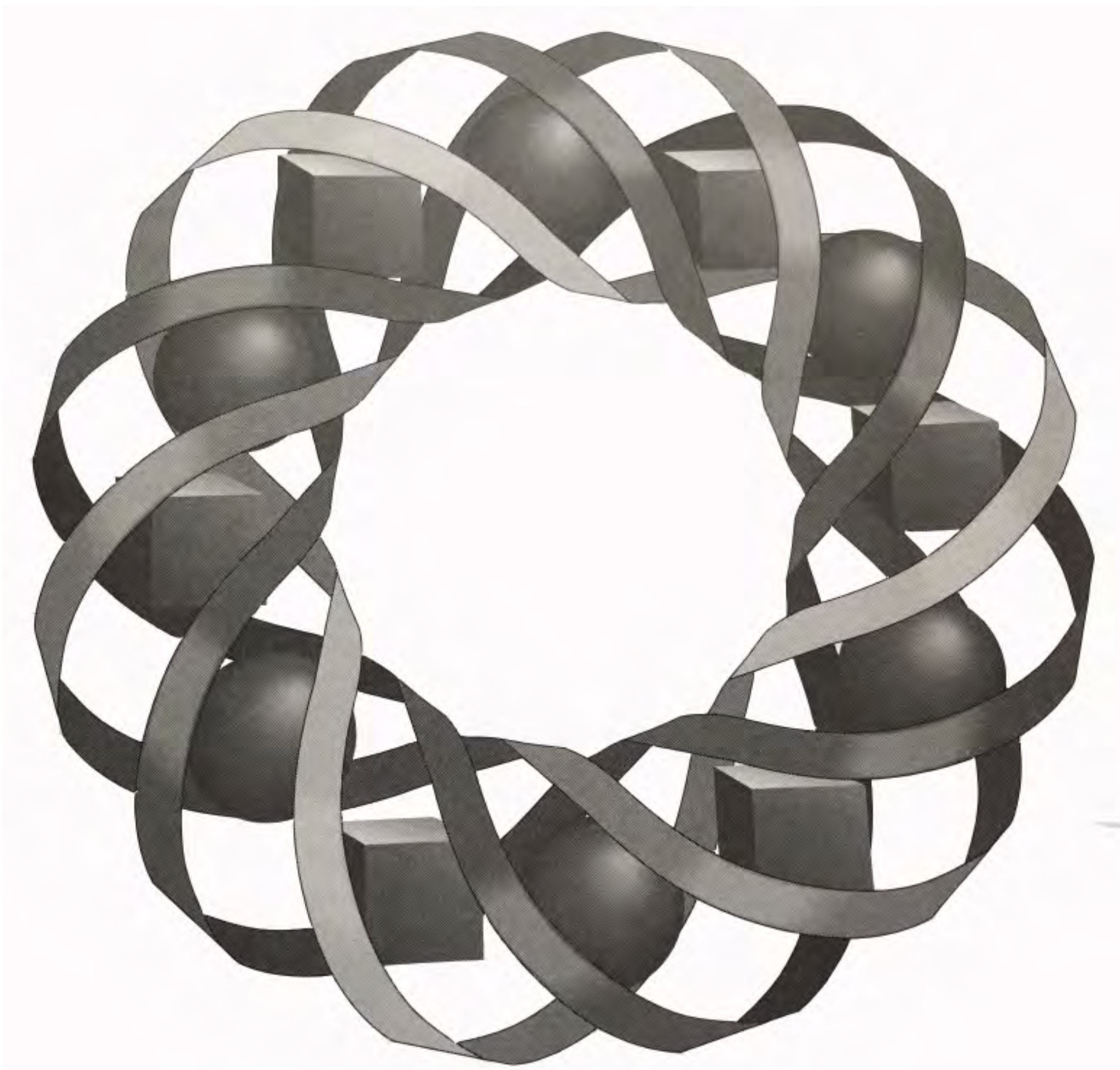
Catch a Wave



Catchin' a wave and listening to tunes—what a glorious summer day. In this module, you explore the mathematics of waves and musical notes using sine and cosine functions.

Ed Sisolak • Danny Jones

Catch a Wave

Introduction

Some events in everyday life occur again and again over time: the phases of the moon, the crash of ocean waves on the beach, the seasons and their temperatures. Other occurrences also display periodic behavior, such as the vibrations of a guitar string or the motion of a ride at an amusement park.

Discussion

a. Cassandra is about to bungee jump off a 40-m platform. The bungee cord is 18 m long. Describe a graph of her height above the ground versus time from the moment she jumps until she stops bouncing.

b. Although Hannah is afraid of heights, her friends have talked her into riding the Ferris wheel. Describe a graph of Hannah's height above the ground versus time as the Ferris wheel makes three complete revolutions.

c. Compare the graphs you described in Parts **a** and **b**.

Activity 1

As the Ferris wheel turns, Hannah's height above the ground changes. A graph of her height above the ground versus time is **periodic**. In other words, the various values for height repeat over time. In this module, you investigate functions that can model periodic events.

Mathematics Note

A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest constant interval of the domain over which the function repeats. For example, Figure **1** shows a graph of a function which repeats itself every 3 units. In this case, $f(x+3) = f(x)$. The period of this function is 3.

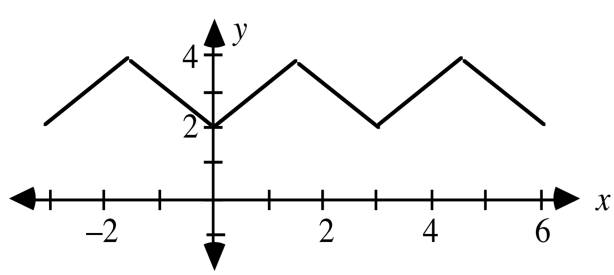


Figure 1: A periodic function

Exploration

Time is measured on a linear scale. When determining the height above ground of a point on a Ferris wheel, however, it can be helpful to associate each unit of time with a point on a circle. In this exploration, you simulate "wrapping" a linear scale onto a circle.

a.

1. Use a can to trace a circle on a sheet of paper.

2. Identify the center of the circle.

3. As shown in Figure **2**, create a coordinate system with its origin at the center of the circle. Let the radius of the circle represent 1 unit. A circle with a radius of 1 unit is a **unit circle**.

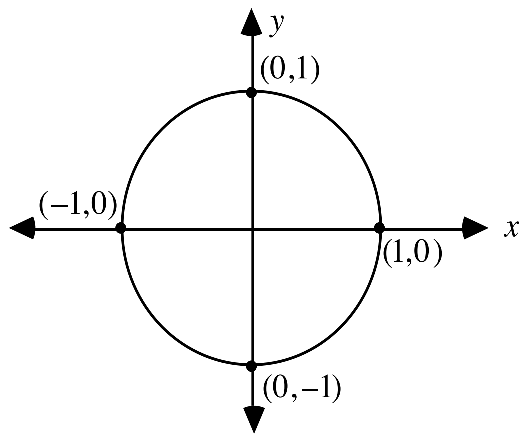


Figure 2: Coordinate system and unit circle

4. Using a paper strip slightly longer than the circumference of the can, create a number line on which 1 unit equals the radius of the circle.

5. Measuring as accurately as possible, label the locations of the points on the number line that correspond to the following real numbers: 0, 1, $\pi/2$, 2, 3, π, 4, $3\pi/2$, 5, 6, 2π, and 7.

6. Tape the beginning of your number line to the lower portion of the can, as shown in Figure **3**.

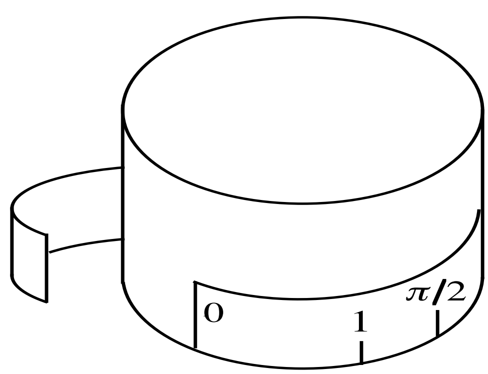


Figure 3: Placement of number line on can

b. Position the can on your drawing of the unit circle from Part **a** so that the 0 on the number line is located at the point (1,0). Use the number line to label every point on the circle that corresponds to a labeled point on the number line. If any point on the circle corresponds to more than one point on the number line, mark it with all corresponding labels.

c. Using your unit circle, find the approximate number of radii that correspond with each of the following lengths:

1. the circumference of the circle
2. three-fourths of the circumference of the circle
3. one-half of the circumference of the circle
4. one-fourth of the circumference of the circle.

d. Approximate the ordered pair that corresponds to each labeled point on the unit circle. Record these values in a table similar to Table **1** below. **Note:** Save this table for use in Activity **2**.

Table 1: Number-line values and ordered pairs

Number	x-coordinate	y-coordinate
0	1.0	0.0
1	0.5	0.8
$\pi/2$	0.0	1.0
$\vdots$	$\vdots$	$\vdots$

e. Draw a ray from the origin through the point that corresponds to 1 unit on the circumference of your unit circle. The angle formed by the non-negative portion of the x-axis and this ray has a measure of 1 **radian**. Find the approximate measure of this angle in degrees. Save your unit circle for use with Problem **1.6**.

Discussion

a.
1. Compare the number of radii that fit around the circumference of your circle with the values obtained by others in your class.
2. How does your answer compare to what you already know about the circumference of a circle?

b. Express each of the following lengths in terms of π:

1. the circumference of your circle
2. one-half the circumference of your circle
3. one-fourth the circumference of your circle

c. In the exploration, you matched each labeled point on your number line with a point on a unit circle.

1. Compare the coordinates of each labeled point on your unit circle with those of others in your class.

2. Describe how you could represent negative numbers from the number line on the unit circle.

3. The point $\pi/2$ on the number line is paired with the point (0,1) on the unit circle. If the number line was continued indefinitely, what other numbers would be paired with (0,1)?

4. Given a point on the unit circle, how many real numbers can be paired with its location?

d. In Figure **4**, arc *AB* has a length of 1 unit, and the two circles have the same center. The smaller circle has a radius of 1 unit and the larger circle has a radius of 2 units.

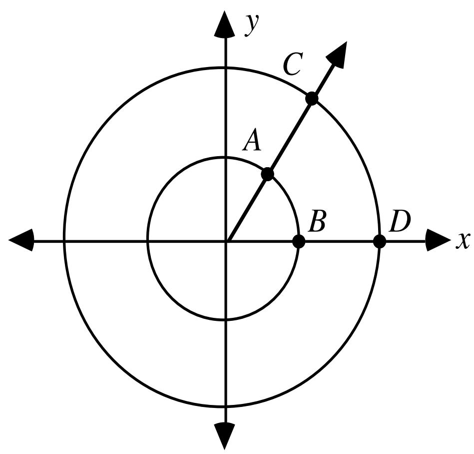


Figure 4: Two concentric circles

1. What is the length of arc *CD*?

2. What appears to be the relationship between the radius of a circle the length of an arc of that circle?

Assignment

1.1 Consider a Ferris wheel with a radius of 12 m.

a. Describe the circumference of the Ferris wheel in terms of its radius.

b. What is the approximate distance around the wheel in meters?

1.2 One revolution of a seat around a Ferris wheel sweeps out an arc with a measure of 2π radians. What radian measure corresponds to each of the following numbers of revolutions?

a. 2

b. 3

c. 8

1.3 Consider a circle with a radius of 12 m. If an arc of the circle has a measure of 4 radians, what is its length in meters?

Mathematics Note

The set of real numbers can be associated with the points on a unit circle by placing a number line so that 0 on the number line is tangent to the circle at the point (1,0), as shown in Figure **5a**. When the positive portion of the number line is wrapped around the unit circle in a counterclockwise direction, as illustrated in Figure **5b**, each positive number is paired with exactly one point, (a,b), on the circle. Positive numbers that differ by 2π units are paired with the same point on the circle.

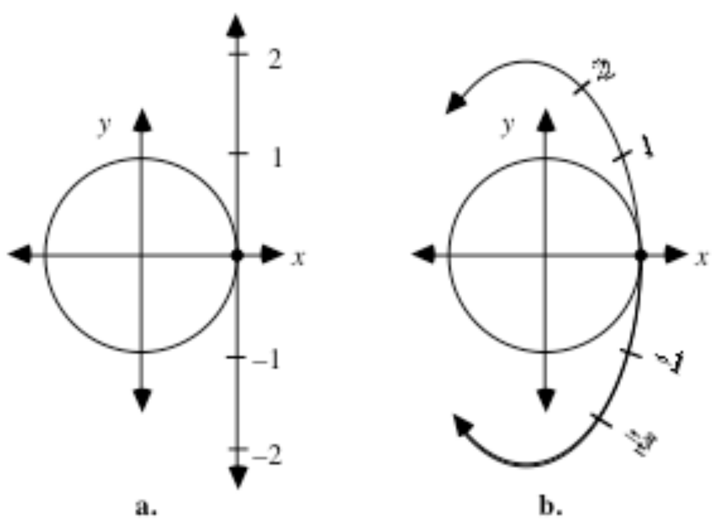


Figure 5: Wrapping function

There is a similar correspondence between the negative real numbers and points on the unit circle, found by wrapping the negative portion of the number line in a clockwise direction. Negative numbers that differ by 2π units are paired with the same point on the circle.

1.4 Consider a wrapping function that pairs each point on the real number line with a location on a unit circle with center at the origin. In this function, 0 on the number line corresponds with the point (1,0).

a. Identify a real number that corresponds with each of the following points on the unit circle:

1. (–1,0)

2. (0,–1)

3. (0,1)

b. Determine the coordinates of the points on the unit circle that are paired with the following real numbers:

1. π

2. $-3\pi/2$

3. $7\pi/2$

4. $-15\pi/2$

5. -11π

1.5 Consider the point on a unit circle with center at the origin that is paired with the real number 3.

a. Identify another positive real number paired with that point.

b. Identify a negative real number paired with that point.

* * * * *

1.6 Use your circle from the exploration to model a Ferris wheel and your number line to model a string of 16 evenly spaced lights placed around the outer edge of the wheel. Mark the position representing the first light at the point corresponding to 0 on the number line. Mark the locations for the remaining 15 lights so that the lights are evenly spaced around the wheel.

a. What is the length of the arc between two consecutive lights?

b. What is the measure of the arc determined by two consecutive lights?

1.7 Consider a sector of a circle whose central angle measures $5\pi/4$ radians. If the circle has a radius of 10 units, what is the area of the sector?

* * * * * * * * * *

Activity 2

Shortly after boarding the Ferris wheel, Hannah reaches the top. A few seconds later, she is at the bottom again. The wheel then begins to carry Hannah back to the top. Hannah's motion on the Ferris wheel is periodic. The process you used in Activity **1** to pair real numbers with points on a circle also is periodic. Many functions pair their domains with their ranges in a similar fashion.

Exploration

In previous modules, you examined the trigonometric ratios *sine* and *cosine* in terms of the sides of a right triangle. In this exploration, you investigate the relationship between these ratios and the **sine** and **cosine functions**.

Recall that a circle is the set of all points in a plane that are the same distance from a given point in the plane. Figure **6** shows a circle with radius r centered at the origin of the xy-plane. The right triangle shows the relationship between the x- and y-coordinates of a point on the circle and the radius of the circle.

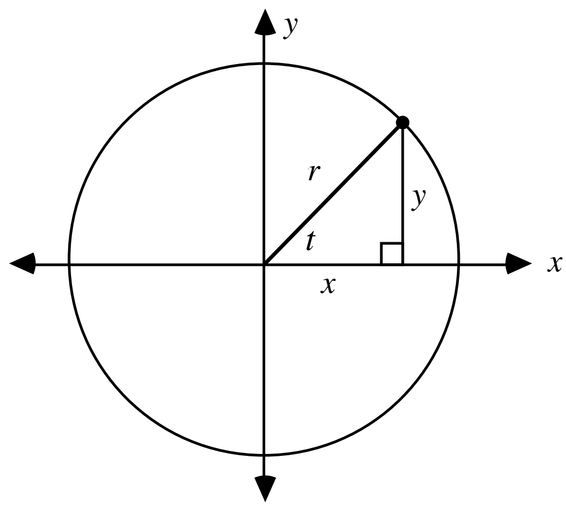


Figure 6: A circle with radius *r*

a. Use the circle in Figure **6** to complete Steps **1–4** below.

1. Write an equation for the sine of the angle t in terms of y and r.
2. Solve this equation for y.
3. Write an equation for the cosine of the angle t in terms of x and r.
4. Solve this equation for x.

b. 1. Use a geometry utility to create a unit circle with its center C located at the origin of a two-dimensional coordinate system. Label the intersection of the positive x-axis and the circle as point D.

2. Construct a point A on the circle so that it moves freely on the circle.

3. From A, construct a segment perpendicular to the x-axis. Label the intersection of the perpendicular and the x-axis as point B.

4. Form right triangle ABC by constructing segments CB, AB, and CA. Your construction should now resemble Figure **7**.

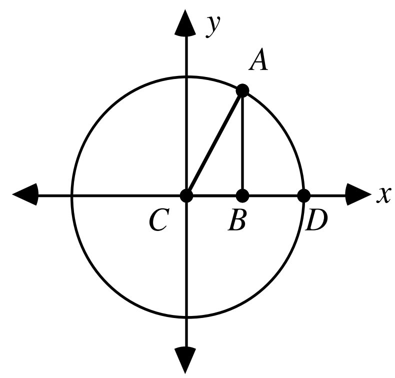


Figure 7: Construction of a triangle in a unit circle

c. 1. Move point A to a location in the first quadrant.

2. Determine the coordinates (x,y) of point A and record them in a table with headings like those in Table **2** below.

Table 2: Measurements on a unit circle for point $A(x,y)$

x	y	$m\angle DCA$	$\sin\angle DCA$	$\cos\angle DCA$

3. Determine the measure of $\angle DCA$ in radians and record it in the table.

4. Use this radian measure, along with the sine and cosine keys on your calculator, to determine the sine and cosine of $\angle DCA$. Compare the sine and cosine with the coordinates of point A.

5. Repeat Steps **1–4** for several more locations of point A in the first quadrant.

d. Make a conjecture about the coordinates of A in terms of $\sin\angle DCA$ and $\cos\angle DCA$.

e. For each location of point A that you used in Part **c**, record the coordinates of A', the reflection of A in the y-axis.

1. Determine the measure of $\angle DCA'$ in radians and record it in the table.

2. Use this radian measure, along with the sine and cosine keys on your calculator, to determine the sine and cosine of $\angle DCA'$. Compare the sine and cosine with the coordinates of point A'.

f. Repeat Part **e** for A'', the reflection of A in the x-axis.

g. Repeat Part **e** for A''', the reflection of A'' in the y-axis.

Mathematics Note

The pairing of each real number with exactly one point on the unit circle is the basis for the **circular functions**. Two of the circular functions, **sine** and **cosine**, can be defined in the following manner:

For any real number x, where x is paired with the point (a,b) on a unit circle with center at the origin, $\sin x = b$ and $\cos x = a$.

For example, the point on the unit circle paired with $\pi/3$ is $\left(1/2, \sqrt{3}/2\right)$. Therefore, $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$.

h. **1.** Graph $y = \sin x$ for the domain $-2\pi \leq x \leq 2\pi$.

2. Describe your graph.

3. Graph $y = 0.5$ on the same coordinate system and approximate the value(s) of x for which $\sin x = 0.5$ over the domain $-2\pi \leq x \leq 2\pi$.

4. What is the range of the values for the sine function?

5. What is the period of the sine function?

Discussion

a. Describe the relationship you observed between the coordinates of a point in the first quadrant and the sine and cosine of $\angle DCA$. Why does this relationship occur?

b. Describe how to find the coordinates of a point on the unit circle in any quadrant.

c. Are your results from Parts **c–g** of the exploration consistent with the values you recorded in Table **1** in Activity **1**?

d. Explain how the range of the sine function is related to the unit circle.

e. Explain how the shape of the graph of the sine function is related to the unit circle.

f. How many solutions are there to the equation $\sin x = 0.5$ over the interval $-2\pi \le x \le 2\pi$? over the real numbers?

g. How many solutions are there to the equation $\sin x = 2$ over the real numbers?

h.

1. Describe the conditions necessary for the cosine of a number to be positive.

2. Describe the conditions necessary for the sine of a number to be positive.

i. The **tangent function** is equivalent to the ratio of the sine function to the cosine function:

$$\tan x = \frac{\sin x}{\cos x}$$

1. Given this fact, how must the domain of the tangent function be restricted?

2. Describe the conditions necessary for the tangent of a number to be positive.

Assignment

2.1 Use a unit circle to determine the sine, cosine, and tangent of each of the following real numbers:

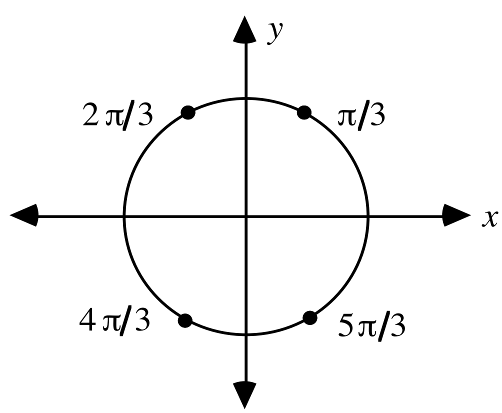
a. 0

b. $\pi/2$

c. π

d. $3\pi/2$

2.2 Consider a real number line wrapped around a unit circle as in Activity **1**. The following diagram shows the locations of points that correspond to four real numbers.



a. Find the x- and y-coordinates of each of the four points.

b. Describe how the coordinates that correspond to the four real numbers are related.

c. Describe how the points that correspond to the following real numbers are related to the coordinates in Part **a**: $7\pi/3$, $8\pi/3$, $10\pi/3$, and $11\pi/3$.

d. Determine four negative real numbers whose corresponding coordinates are the same as the four pairs of coordinates from Part **a**.

2.3 **a.** Graph $y = \cos x$ for $-2\pi \le x \le 2\pi$.

b. Describe your graph.

c. Graph $y = 0.5$ on the same coordinate system and determine the value(s) of x for which $\cos x = 0.5$ over the interval $-2\pi \le x \le 2\pi$.

d. What is the period of the cosine function?

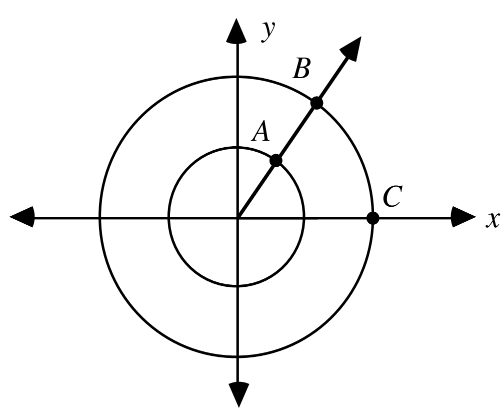
2.4 **a.** Graph $y = \tan x$ for $-2\pi \le x \le 2\pi$.

b. Describe your graph and explain why the graph has these properties.

c. Graph $y = 5$ on the same coordinate system and determine the value(s) of x for which $\tan x = 5$ over the interval $-2\pi \le x \le 2\pi$.

d. Is the tangent function periodic? If it is periodic, what is its period?

2.5 The two circles shown in the diagram below have the same center. The smaller circle has a radius of 1 and the larger circle has a radius of 2. The measure of arc BC is $\pi/3$ radians.



a. Determine the coordinates of point A.

b. Use the coordinates of point A and the properties of similar triangles to determine the coordinates of point B.

c. Describe the coordinates of the point where ray AB intersects a circle with radius r.

2.6 The Ferris wheel that Hannah is riding on has a radius of 12 m. Consider a coordinate system with its origin at the center of the Ferris wheel. The point at which Hannah begins her ride has the coordinates (0,–12).

a. If the wheel completes one revolution every 2.5 min, through how many radians does the wheel rotate in 1 sec?

b. After how many seconds will Hannah's distance above the ground be the greatest?

c. What is the length of the arc that Hannah moves through in 1 sec?

d. What are the coordinates of the point where Hannah will be 1 sec after she begins her ride?

* * * * *

2.7 A watchmaker would like to design the face of a new watch. The watch face is a circle with a radius of 1 cm. The watchmaker locates the center of the face at the origin of a two-dimensional coordinate system and 3 o'clock at the point (1,0).

a. Determine the coordinates of the points where the hours 6, 9, and 12 are located.

b. **1.** Determine the coordinates of the point where 11 o'clock is located.

2. Identify the hours for which the absolute values of the coordinates equal the absolute values of the coordinates for 11 o'clock.

c. The hour hand is currently pointing at 12 o'clock. Identify the coordinates of the location that the hour hand will be pointing to after 33 hr.

2.8 **a.** Graph $y = \cos x$ and $y = \sin x$ for $-2\pi \leq x \leq 2\pi$ on the same coordinate system.

b. Determine the values of x in this interval for which $\cos x = \sin x$.

2.9 **a.** On the same coordinate system, graph the following two equations for $-2\pi \leq x \leq 2\pi$:

$$y = \cos x \text{ and } y = \sin\left(x + \frac{\pi}{2}\right)$$

b. Determine the values of x in this interval for which

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

2.10 **a.** On the same coordinate system, graph the following two equations for $-2\pi \leq x \leq 2\pi$: $y = 2(\sin x)\cos x$ and $y = \sin(2x)$.

b. Determine the values of x in this interval for which $2(\sin x)\cos x = \sin(2x)$.

2.11 Describe the graph of $y = \sin^2 x + \cos^2 x$ for $-2\pi \le x \le 2\pi$. What does this graph tell you about the sum of $\sin^2 x$ and $\cos^2 x$?

* * * * * * * * * *

Activity 3

When any sound is produced, it disturbs nearby air molecules, creating waves of high and low pressure. These variations in pressure over time can be recorded with a microphone and modeled by a sine curve, as shown in Figure **8**.

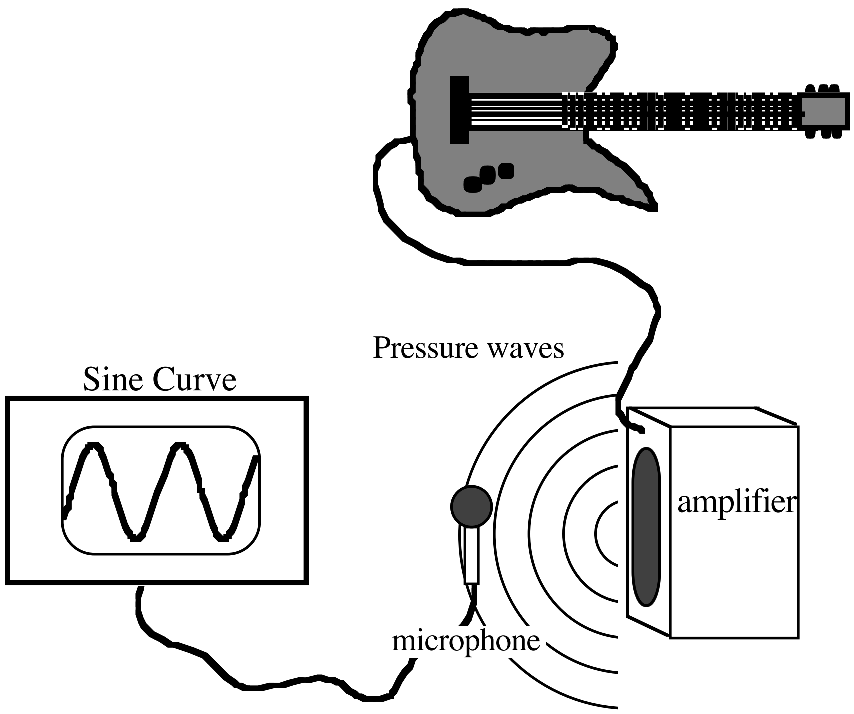


Figure 8: Sine curve modeling pressure waves from a speaker

In this activity, you examine two characteristics of sound waves: amplitude and frequency.

Exploration

a. Graph $y = \sin x$ over the interval $-2\pi \le x \le 4\pi$.

b. Graph $y = r \sin x$ over the domain in Part **a** for several different values of r. Describe how changing the value of r changes the maximum and minimum y-values of the graph.

Mathematics Note

If a periodic function has a maximum M and a minimum m, its **amplitude** is defined as:

$$\frac{M-m}{2}$$

For example, Figure **9** shows a graph of the function $y = 6\sin x$.

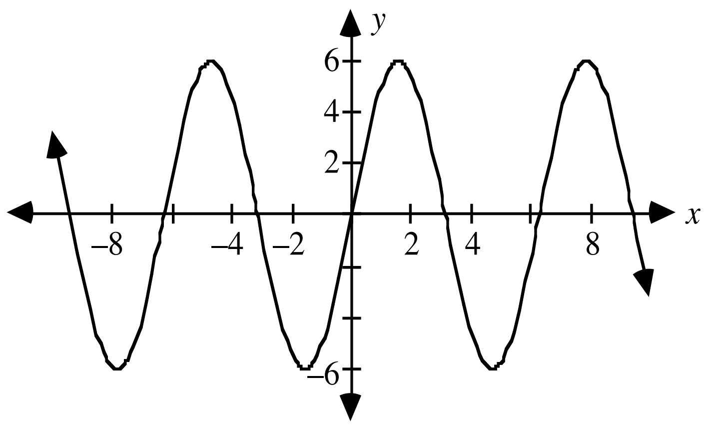


Figure 9: Sine curve with amplitude 6

Since the maximum is 6 and the minimum is –6, the amplitude can be found as follows:

$$\frac{M-m}{2} = \frac{6-(-6)}{2}$$
$$= 6$$

A **cycle** is the portion of a periodic function included in one period. The **frequency** of a periodic function is the number of cycles per unit on the x-axis. The frequency is the reciprocal of the period, as shown below:

$$\text{frequency} = \frac{1}{\text{period}}$$

For example, the sine function in Figure **9** has a period of 2π. In other words, it completes 1 cycle every 2π units. Its frequency is $1/(2\pi)$.

c. The frequency of sound waves is measured in **hertz (Hz)**, where 1 Hz represents 1 cycle per second. If x represents time in seconds, what is the frequency of the graph of $y = \sin x$?

d. Use a graphing utility to complete Table **3**.

Table 2: Cycles, periods, and frequencies

Function	Period	Frequency
$y = \sin x$	2π	$1/(2\pi)$
$y = \sin(2x)$		
$y = \sin(4x)$		
$y = \sin(\pi x)$		
$y = \sin(x/3)$		
$y = \sin(bx)$		
$y = \sin(2\pi x)$		
$y = \cos x$		
$y = \tan x$		

Discussion

a. Compare the graphs of $y = 8\sin x$ and $y = \sin x$.

b. How does changing b in $y = \sin(bx)$ affect the graph of the function?

c.
1. If x represents time in seconds, what function would create a cosine graph that has 50 cycles in 2π sec?
2. What function would create a sine graph that has 12 cycles in π sec?

d.
1. If x represents time in seconds, what function would create a sine graph that has an amplitude of 4 and a frequency of 1 Hz?
2. What function would create a cosine graph with an amplitude of 0.5 and a frequency of 10 Hz?

Assignment

3.1 Write a function that describes the graph of each of the following:

a. a sine curve with an amplitude of 8

b. a cosine curve with an amplitude of 0.4.

3.2 If x represents time in seconds, write a function that describes the graph of a cosine curve with each of the following frequencies:

a. 15 Hz

b. 25 Hz

c. 0.5 Hz

3.3 Write a function that describes the graph of a cosine curve with each of the following periods:

a. 15 units

b. 25 units.

3.4 The human ear can detect sounds with frequencies between 20 Hz and 20,000 Hz.

a. Write a function that describes the graph of a sound wave with each of the following frequencies:

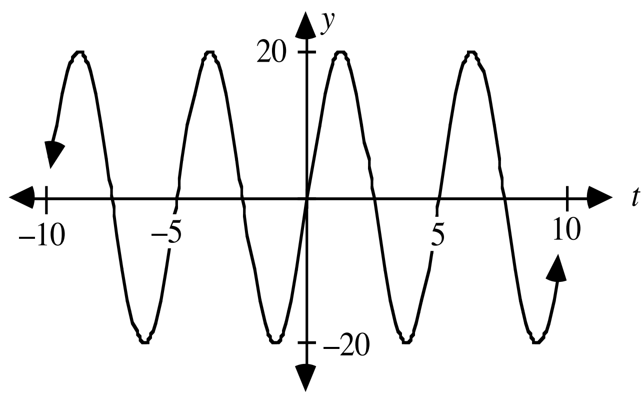
1. 20 Hz

2. 20,000 Hz

b. How would the graphs of all other audible sounds compare to the graphs from Part **a**?

3.5 Write the function that describes the graph of a sine curve with a period of 6 and an amplitude of 7. Graph the function for $0 \leq x \leq 12$.

3.6 Consider the graph below, where *y* represents centimeters and *t* represents seconds.



a. What is the amplitude of the graph?

b. What is the period?

c. What is the frequency?

d. Write an equation that describes this curve.

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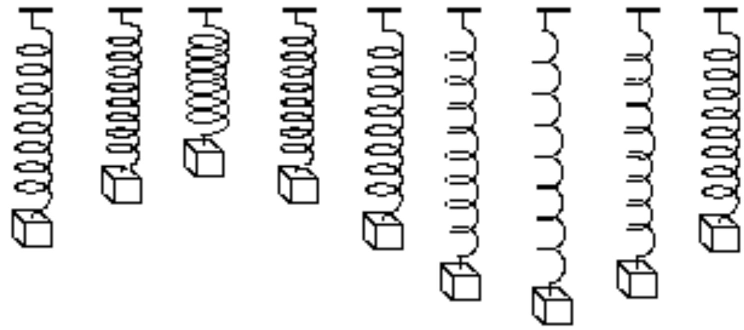
3.7 The diagram below shows the positions of a weight attached to the end of a spring at regular time intervals. The distance between the lowest point and the highest point reached by the weight is 20 cm. It takes the weight 0.4 sec to travel this distance.



a. As the weight bounces up and down, its displacement over time can be modeled with a sine function. Assume that at $t = 0$, the weight is halfway between its highest and lowest positions. If this position corresponds with a displacement of 0 cm, write an equation that models displacement with respect to time.

b. Graph the equation from Part **a** over a domain of 2 sec.

c. Use your equation to describe the position of the weight at each of the following times:

1. 0.1 sec
2. 0.5 sec
3. 0.95 sec
4. 1.75 sec.

d. Indicate the points on the graph from Part **b** that correspond to each of the times in Part **c**.

e. Identify the times when the weight is located 10 cm below its position at $t = 0$.

3.8 Graph $y = \cos x$ and $y = \cos(x + k)$ on the same set of axes for several different values of *k*. Describe the effect that *k* has on the graph of $y = \cos x$.

* * * * * * * * * *

Activity 4

The sounds involved in typical song are made up of millions of pressure waves, each with its own frequency and amplitude. In this activity, you create your own music and model it with sine curves.

Science Note

Each musical note has its own frequency. For example, the note A has a frequency of 440 Hz (440 cycles/sec). Figure **11** shows the changes in pressure produced by the note A over time.

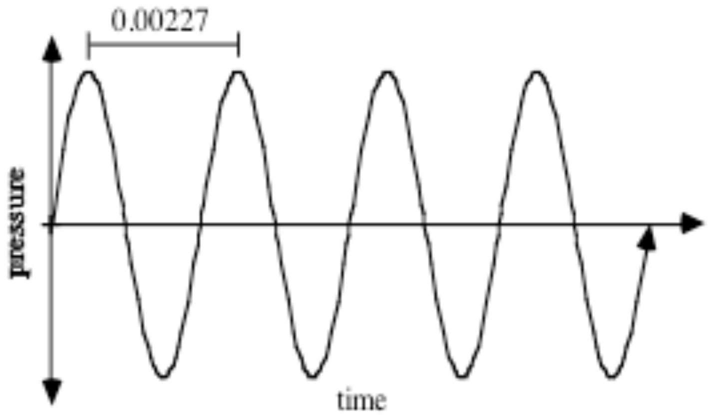


Figure 11: Sine curve of note A

The length of one period is approximately 0.00227 sec. Therefore, the frequency can be determined as follows:

$$\text{frequency} = \frac{1}{\text{period}} \approx \frac{1}{0.00227} \approx 440 \text{ Hz}$$

Table **4** shows some musical notes and their corresponding frequencies. Notice how the frequency of a note doubles when the next octave is reached. The notes on a piano range from low A (27.5 Hz) to high C (4186 Hz).

Table 4: Notes and their frequencies

Note	**Frequency (Hz)**
C	262
C# (C-sharp)	277
D	294
D#	311
E	330
F	349
F#	370
G	392
G#	415
A	440
A#	446
B	494
C (next octave)	524
C#	555
D	588

Exploration

In this exploration, you produce sounds by blowing across the top of a bottle partially filled with water, then model the notes using a graphing utility.

a. Obtain a bottle from your teacher. Fill the bottle about half full of water.

b. Blow across the top of the bottle to produce a whistling sound.

c. Use technology to record the pressure waves produced by this sound.

d. Determine the period, frequency, and amplitude of your graph.

e. Find a function that models the curve formed by the sound.

f. Remove some water from the bottle and repeat Parts **a–e**.

Discussion

a. Describe how you determined the period, frequency, and amplitude in Part **d** of the exploration.

b. Using Table **4**, determine which musical note is closest to the first sound you produced with the bottle.

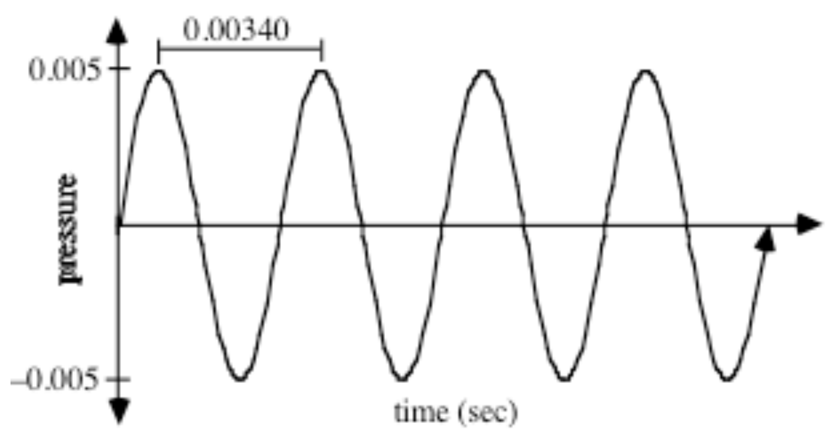
c. Describe the function that models this sound.

d. Compare the function for your note with those of others in the class.

e. How could you create a note with a higher frequency?

Assignment

4.1 The following graph shows the pressure variations produced by a musical note.



a. Determine the frequency of the graph.

b. Determine a function that models the curve.

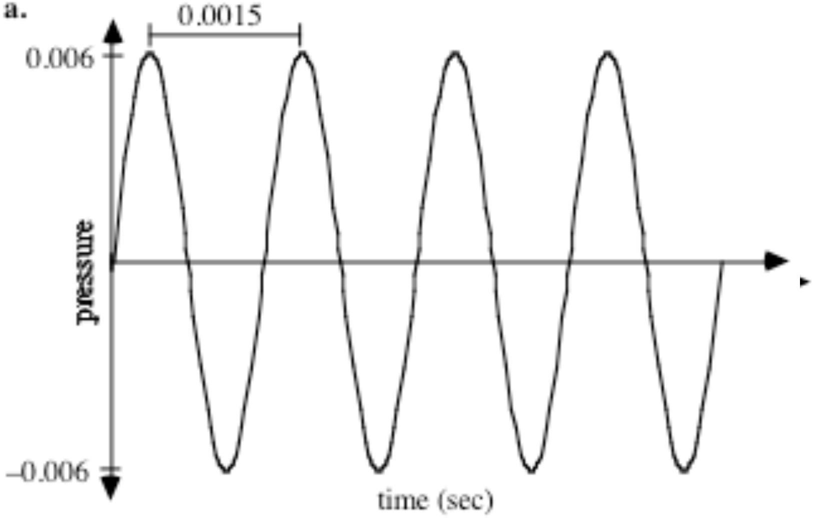
c. Use Table **4** to identify the note.

4.2 Describe the similarities and differences between the graphs of middle C, with a frequency of 262 Hz, and the note C one octave higher.

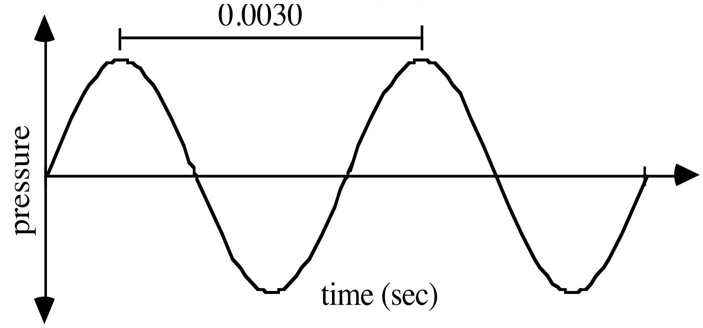
4.3 Write a function that could describe a graph of the note $F^{\#}$.

4.4 For each graph in Parts **a** and **b** below, identify the amplitude, period, frequency, and the name of the corresponding musical note.

a.

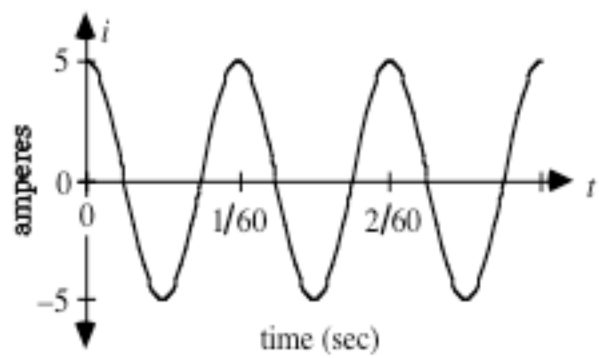


b.



* * * * *

4.5 The electrical power supplied to most households is called "alternating current." The magnitude of alternating current varies over time and can be modeled by a sine or cosine function, as shown in the following graph. The standard unit of current is the **ampere** (amp).



a. Describe the amplitude, period, and frequency of the graph.

b. Write an equation to model the variation in current shown in the graph.

c. Use the equation from Part **b** to determine the current, in amperes, at each of the following times:

1. 0.01 sec

2. 0.15 sec.

d. Identify the times when the current is:

1. 0 amp

2. –5 amp.

* * * * * * * * * *

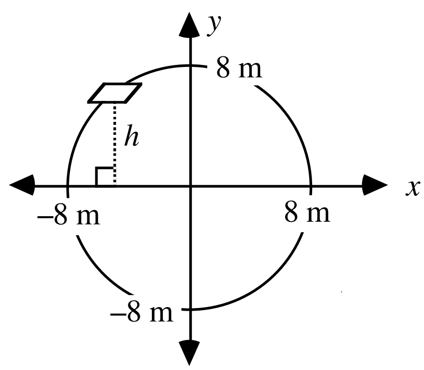
Research Project

Musicians often play musical notes in combinations called **chords**. One chord consists of the notes C, E, and G played simultaneously. Use technology to collect sound-wave data when this chord is played on a musical instrument such as a guitar or piano. Also collect data for the individual notes played on the same instrument.

Report on how the graphs of individual notes compare to that of the chord and describe how the combination of notes affects the graph of the sound.

Summary Assessment

1. Imagine that you are riding the Ferris wheel at a fair. This Ferris wheel has a radius of 8 m and moves at a constant rate of 2 revolutions per minute in a counterclockwise direction. The diagram below shows the path of a chair on the Ferris wheel, with the center of the wheel located at the origin of a two-dimensional coordinate system.



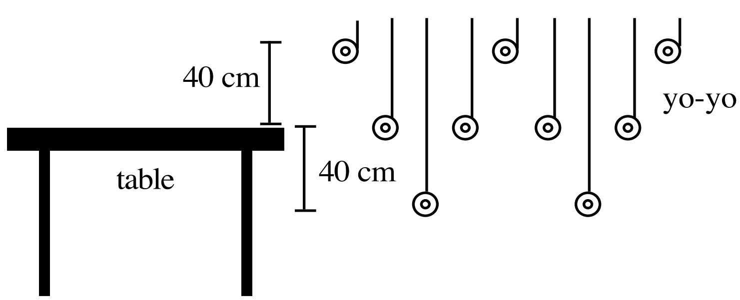
a. Assume that the coordinates of the chair's initial position are (8,0). As the chair travels around the wheel, determine the chair's vertical distance from the *x*-axis at 10 different instants in time.

b. Create a scatterplot of the data from Part **a**.

c. Write the equation of a sine curve that models the scatterplot and graph the curve on the same coordinate system as in Part **b**.

d. Identify the amplitude, period, and frequency of the graph from Part **c**.

2. The diagram below shows the positions of a yo-yo with respect to a table top at regular time intervals.



a. Describe how you could make the vertical distance from a yo-yo to a table top vary periodically with time.

b. Let the x-axis represent the level of the table top in the diagram. Assuming that it takes 2.4 sec for the yo-yo to complete all the cycles illustrated, find a periodic function that models the motion of this yo-yo and graph it over a domain of 0 to 3.6 sec.

c. Identify the amplitude, period, and frequency of the graph from Part **b**.

d. Use your model from Part **b** to describe the position of the yo-yo at each of the following times:

1. 1.5 sec
2. 1.9 sec
3. 3.4 sec.

Module Summary

- A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest constant interval of the domain over which the function repeats.
- A **unit circle** is a circle with a radius of 1.
- On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**.
- The set of real numbers can be associated with the points on a unit circle by placing a number line so that 0 on the number line is tangent to the circle at the point (1,0). When the positive portion of the number line is wrapped around the unit circle in a counterclockwise direction, each positive number is paired with exactly one point, (a,b), on the circle. Positive numbers that differ by 2π units are paired with the same point on the circle.

  There is a similar correspondence between the negative real numbers and points on the unit circle, found by wrapping the negative portion of the number line in a clockwise direction. Negative numbers that differ by 2π units are paired with the same point on the circle.
- The pairing of each real number with exactly one point on the unit circle is the basis for the **circular functions**.

  Two of the circular functions, **sine** and **cosine**, can be defined as follows: For any real number x, where x is paired with the point (a,b) on a unit circle with center at the origin, $\sin x = b$ and $\cos x = a$.
- The **tangent function** is equivalent to the ratio of the sine function to the cosine function:

  $$\tan x = \frac{\sin x}{\cos x}$$
- If a periodic function has a maximum M and a minimum m, its **amplitude** is defined as:

  $$\frac{M - m}{2}$$
- A **cycle** is the portion of a periodic function included in one period.
- The **frequency** of a periodic function is the number of cycles per unit on the x-axis. The frequency is the reciprocal of the period.
- The frequency of waves is typically measured in **hertz (Hz)**, where 1 Hz represents one cycle per second.

Selected References

Gross, E. E., Jr., and A. P. G. Peterson. *Handbook of Noise Measurement*. Concord, MA: General Radio Company, 1972.

Texas Instruments. *CBL™ System Experiment Workbook*. Dallas, TX: Texas Instruments Inc., 1994.

Level 5 Index