SIMMS Integrated Mathematics:

A Modeling Approach Using Technology

Teacher Guide

Level 5   Volumes 1-3

Teacher Guide
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About *Integrated Mathematics: A Modeling Approach Using Technology*

The Need for Change

In recent years, many voices have called for the reform of mathematics education in the United States. Teachers, scholars, and administrators alike have pointed out the symptoms of a flawed system. From the ninth grade onwards, for example, about half of the students in this country's mathematical pipeline are lost each year (National Research Council, 1990, p. 36). Attempts to identify the root causes of this decline have targeted not only the methods used to instruct and assess our students, but the nature of the mathematics they learn and the manner in which they are expected to learn. In its *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics addressed the problem in these terms:

> When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics. (p. 64)

Some Methods for Change

Among the major objectives of the *Integrated Mathematics* curriculum are:

- offering a 9–12 mathematics curriculum using an integrated inter-disciplinary approach for *all* students.
- incorporating the use of technology as a learning tool in all facets and at all levels of mathematics.
- offering a *Standards*-based curriculum for teaching, learning, and assessing mathematics.

The *Integrated Mathematics* Curriculum

An integrated mathematics program "consists of topics chosen from a wide variety of mathematical fields. . . [It] emphasizes the relationships among topics within mathematics as well as between mathematics and other disciplines" (Beal, et al., 1992; Lott, 1991). In order to create innovative, integrated, and accessible materials, *Integrated Mathematics: A Modeling Approach Using Technology* was written, revised, and reviewed by secondary teachers of mathematics and science. It is a complete, *Standards*-based mathematics program designed to replace all currently offered secondary mathematics courses, with the possible exception of advanced placement classes, and builds on middle-school reform curricula.

The *Integrated Mathematics* curriculum is grouped into six levels. All students should take at least the first two levels. In the third and fourth years, *Integrated Mathematics* offers a choice of courses to students and their parents, depending on interests and goals. A flow chart of the curriculum appears in Figure 1.

Each year-long level contains 14–16 modules. Some must be presented in

sequence, while others may be studied in any order. Modules are further divided into several activities, typically including an exploration, a discussion, a set of homework assignments, and a research project.

Entry Point
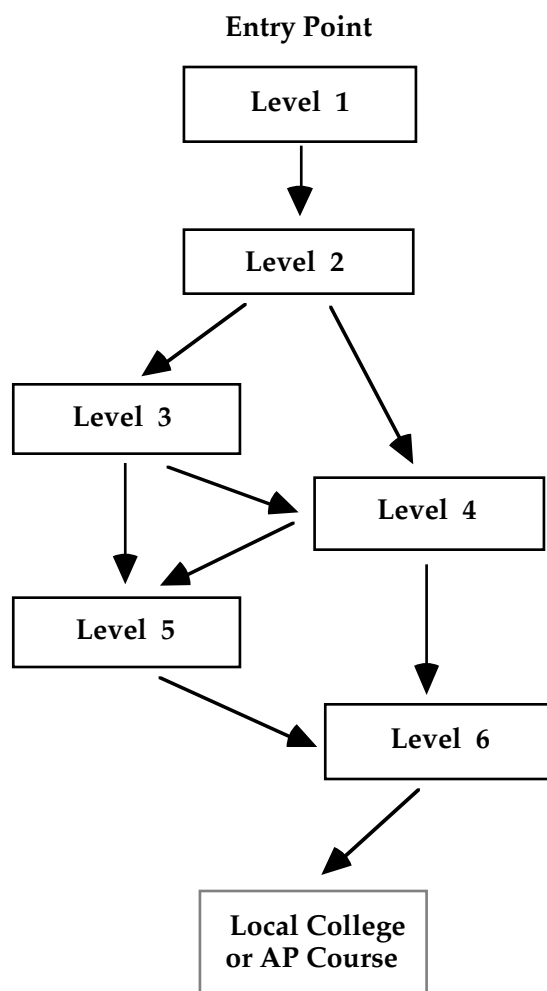
Figure 1: *Integrated Mathematics course sequence*

Assessment materials—including alternative assessments that emphasize writing and logical argument—are an integral part of the curriculum. Suggested assessment items for use with a standard rubric are identified in all teacher editions.

Level 1: a first-year course for ninth graders (or possibly eighth graders)

Level 1 concentrates on the knowledge and understanding that students need to become mathematically literate citizens, while providing the necessary foundation for those who wish to pursue careers involving mathematics and science. Each module in Level 1, as in all levels of the curriculum, presents the relevant mathematics in an applied context. These contexts include the properties of reflected light, population growth, and the manufacture of cardboard containers. Mathematical content includes data collection, presentation, and interpretation; linear, exponential, and step functions; and three-dimensional geometry, including surface area and volume.

Level 2: a second-year course for either ninth or tenth graders

Level 2 continues to build on the mathematics that students need to become mathematically literate citizens. While retaining an emphasis on the presentation and interpretation of data, Level 2 introduces trigonometric ratios and matrices, while also encouraging the development of algebraic skills. Contexts include pyramid construction, small business inventory, genetics, and the allotment of seats in the U.S. House of Representatives.

Levels 3 and 4: options for students in the third year

Both levels build on the mathematics content in Level 2 and provide opportunities for students to expand their mathematical understanding. Most students planning careers in math and science will choose Level 4. While Level 3 also may be suitable for some of these students, it offers a slightly different mixture of context and content.

Contexts in Level 4 include launching a new business, historic rainfall patterns, the pH scale, topology, and scheduling. The mathematical content includes rational, logarithmic, and circular functions, proof, and combinatorics.

In Level 3, contexts include nutrition, surveying, and quality control. Mathematical topics include linear programming, curve-fitting, polynomial functions, and sampling.

Levels 5 and 6: options for students in the fourth year

Level 6 materials continue the presentation of mathematics through applied contexts while embracing a broader mathematical perspective. For example, Level 6 modules explore operations on functions, instantaneous rates of change, complex numbers, and parametric equations.

Level 5 focuses more specifically on applications from business and the social sciences, including hypothesis testing, Markov chains, and game theory.

More About Level 5

In the modules "Sunken Sub," "Risky Business," "Making Cents of Your Income," "A Fitting Explanation," and "A Walk on the Wild Side," students examine various aspects of probability and statistics.

"How Long Is This Going To Take," "Pixel This," and "The Game of Life" involve excursions into the discrete topics of scheduling theory, algorithms, and game theory, respectively.

"Reinvent the Wheel" examines curves of constant width. In "A Ride with Markov," students investigate Markov chains, while in "Catch a Wave," they explore trigonometric functions. The module "Wilderness Wanderings" introduces vectors. In "Let There Be Light," students continue their investigations of conics.

The Teacher Edition

To facilitate use of the curriculum, the teacher edition contains these features:

Overview /Objectives/Prerequisites

Each module begins with a brief overview of its contents. This overview is followed by a list of teaching objectives and a list of prerequisite skills and knowledge.

Time Line/Materials & Technology Required

A time line provides a rough estimate of the classroom periods required to complete each module. The materials required for the entire module are listed by activity. The technology required to complete the module appears in a similar list.

Assignments/Assessment Items/Flashbacks

Assignment problems appear at the end of each activity. These problems are separated into two sections by a series of asterisks. The problems in the first section cover all the essential elements in the activity. The second section provides optional problems for extra practice or additional homework.

Specific assignment problems recommended for assessment are preceded by a single asterisk in the teacher edition. Each module also contains a Summary Assessment in the student edition and a Module Assessment in the teacher edition, for use at the teacher's discretion. In general, Summary Assessments offer more open-ended questions, while Module Assessments take a more traditional approach. To review prerequisite skills, each module includes brief problem sets called "Flashbacks." Like the Module Assessment, they are designed for use at the teacher's discretion.

Technology in the Classroom

The *Integrated Mathematics* curriculum takes full advantage of the appropriate use of technology. In fact, the goals of the curriculum are impossible to achieve without it. Students must have ready access to the functionality of a graphing utility, a spreadsheet, a geometry utility, a statistics program, a symbolic manipulator, and a

word processor. In addition, students should have access to a science interface device that allows for electronic data collection from classroom experiments, as well as a telephone modem.

In the student edition, references to technology provide as much flexibility as possible to the teacher. In the teacher edition, sample responses refer to specific pieces of technology, where applicable.

Professional Development

A program of professional development is recommended for all teachers planning to use the curriculum. The *Integrated Mathematics* curriculum encourages the use of cooperative learning, considers mathematical topics in a different order than in a traditional curriculum, and teaches some mathematical topics not previously encountered at the high-school level.

In addition to incorporating a wide range of context areas, *Integrated Mathematics* invites the use of a variety of instructional formats involving heterogeneous classes. Teachers should learn to use alternative assessments, to integrate writing and communication into the mathematics curriculum, and to help students incorporate technology in their own investigations of mathematical ideas.

Approximately 30 classroom teachers and 5 university professors are available to present inservice workshops for interested school districts. Please contact Kendall Hunt Publishing Company for more information.

Student Performance

During the development of *Integrated Mathematics*, researchers conducted an annual assessment of student performances in pilot schools. Each year, two basic measures—the PSAT and a selection of open-ended tasks—were administered to two groups: students in classes using *Integrated Mathematics* and students in classes using other materials. Students using *Integrated Mathematics* materials typically had access to technology for all class work. During administration of the PSAT, however, no technology was made available to either group. Student scores on the mathematics portion of this test indicated no significant difference in performance.

During the open-ended, end-of-year test, technology was made available to both groups. Analysis of student solutions to these tasks showed that students using *Integrated Mathematics* were more likely to provide justification for their solutions and made more and better use of graphs, charts, and diagrams. They also demonstrated a greater variety of problem-solving strategies and were more willing to attempt difficult problems.

References

Beal, J., D. Dolan, J. Lott, and J. Smith. *Integrated Mathematics: Definitions, Issues, and Implications; Report and Executive Summary*. ERIC Clearinghouse for Science, Mathematics, and Environmental Education. The Ohio State University, Columbus, OH: ED 347071, January 1990, 115 pp.

Lott, J., and A. Reeves. "The Integrated Mathematics Project," *Mathematics Teacher* 84 (April 1991): 334–35.

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National Research Council. *A Challenge of Numbers: People in the Mathematical Sciences*. Washington, DC: National Academy Press, 1990.

The SIMMS Project. *Monograph 1: Philosophies*. Missoula, MT: The Montana Council of Teachers of Mathematics, 1993.

Reinvent the Wheel
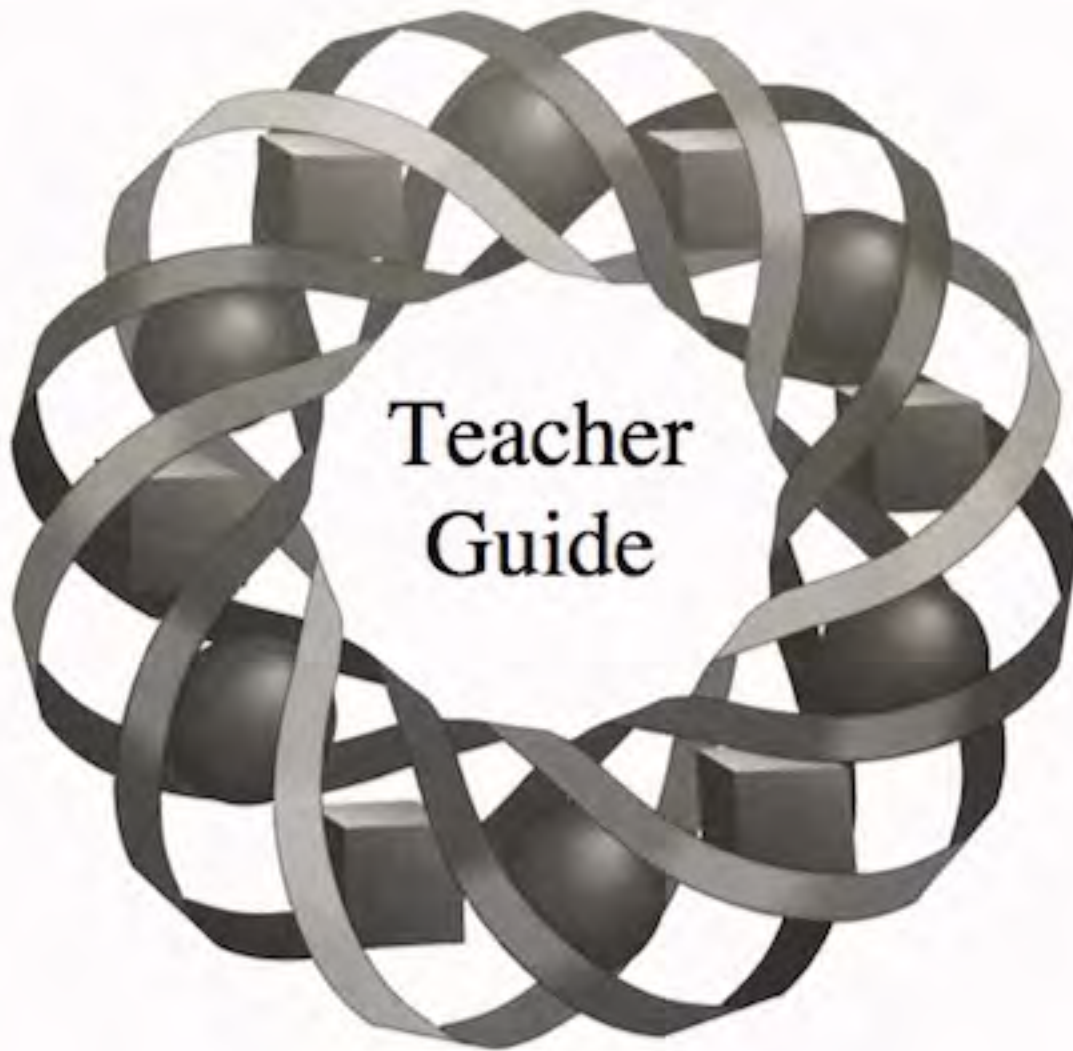
Teacher
Guide

If wheels weren't round, would they still roll smoothly? In this module, you investigate some other types of curves that might supply a smooth ride.

Janet Higgins • Roger Patterson • Mike Trudnowski

Reinvent the Wheel

Overview

In this module, students examine the geometry of curves of constant width: two-dimensional convex figures for which the perpendicular distance between any pair of parallel lines of support is constant. They also investigate perimeters and areas of curves of constant width.

Objectives

In this module, students will:

- explore curves of constant width
- construct and use lines of support for curves
- develop a formula for the perimeter of curves of constant width
- determine the area of curves of constant width
- explore applications of curves of constant width in real-world situations.

Prerequisites

For this module, students should know:

- the geometry of a circle
- the meaning of tangency
- the definition of parallel lines
- the definition of perpendicular lines
- how to identify an axis of symmetry
- the formulas for the area and perimeter of circles
- right-triangle trigonometry
- the formula for the area of a regular polygon
- the concept of a limit.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	2	3	2	10

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
cylinders	X			
flat objects	X			
template A	X			
template B	X	X		
template C	X	X	X	
scissors	X	X		
compass	X	X		
metric ruler	X	X		
cardboard		X		
pencil		X		
tape		X		
template D		X	X	

Teacher Note

Blackline masters for templates A, B, C, and D appear at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
geometry utility	X	X	X	X

Reinvent the Wheel

Introduction

The familiar shape of the wheel—a circle—is only one example of a curve of constant width. In the following activity, students examine the properties of some other curves of constant width.

> ## *Activity 1*

In this activity, students simulate the use of seven different shapes as rollers. They develop an intuitive definition of lines of support and the concept of constant width. In Exploration **2**, students explore Reuleaux polygons: a set of curves of constant width constructed from regular convex *n*-gons, where *n* is odd.

Materials List

- cylinders with the same radius (two per group)
- flat objects (one per group)
- template A (one per student)
- templates B and C (optional; one per group)
- scissors (one pair per student)
- compass (one per group)
- metric ruler (one per group)

Technology

- geometry utility (optional)

Teacher Note

Soft-drink cans work well as cylinders. Books, small wooden boards, or sheets of cardboard may be used as flat objects. Blackline masters of the templates appear at the end of the teacher edition for this module.

You may wish to conduct Parts **a** and **b** of Exploration **1** as a class demonstration.

Exploration 1 (page 3)

a–b. Students should observe that the flat object on top of the rollers remains parallel to the level surface, creating a "level" ride.

c. Predictions will vary.

d. Students should note that shapes A, E, and F provide "level" rides.

e. Students should note that shapes A, E, and F are curves of constant width since both *n* and *m* are lines of support at all times.

Discussion 1 (page 4)

a. A line of support is not always a tangent line. As shown below, a line of support, by definition, need not intersect the curve in only one point, nor must it be the only line in the plane that intersects the curve in a given point.
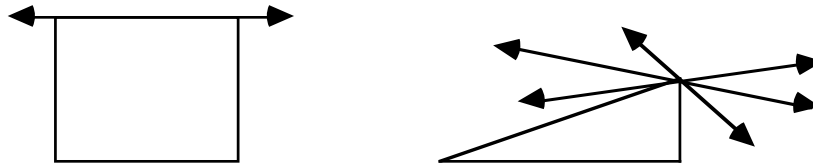
b. Sample response: Lines *m* and *n* are drawn parallel to each other to represent parallel lines of support. The distance between parallel lines is a constant. If they remain lines of support as a shape is rolled through them, then the shape is a curve of constant width.

c. Sample response: Shapes A, E, and F are curves of constant width because they all provide "level" rides.

d. **1.** Shapes A, E, and F are all closed, convex curves that have a constant width. **Note:** The term *diameter* should be used only with respect to circles.

2. Shapes B, C, and G are regular polygons and do not have constant widths. Shape D is an ellipse. Depending on how the lines are drawn, an ellipse either loses contact with one of the lines, or goes outside one of them.

e. Yes. Every circle is a curve of constant width because the distance between every pair of parallel lines of support is the length of a diameter of the circle, a constant.

Teacher Note

In Part **b** of Exploration **2**, students may wish to use a geometry utility to construct the polygons. As an alternative, they may use the square, pentagon, hexagon, and heptagon provided on templates B and C.

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Exploration 2

a. Students construct a Reuleaux triangle.

b. Students should be able to repeat the constructions described in Part **a** for a regular pentagon and a regular heptagon. The constructions are not possible for a square or a regular hexagon. In general, the constructions described in Part **a** are not possible for regular *n*-gons where *n* is even.

c. Students test each of the Reuleaux polygons using the method described in Exploration **1**.

d. Beginning with regular polygons whose sides each measure 5 cm, the constant width of the corresponding Reuleaux triangle is 5 cm, the constant width of the corresponding Reuleaux pentagon is approximately 8.1 cm, and the constant width of the corresponding Reuleaux heptagon is approximately 11.2 cm.

   Students may determine these widths by measuring the distance between any two parallel lines of support, or by measuring the distance from any vertex to the midpoint of the opposite arc.

Discussion 2

a. **1.** Sample response: Regular polygons with an even number of sides do not allow the constructions described in Part **a** of Exploration **2** because for each vertex, there are not two other consecutive vertices that can serve as endpoints for the arc. Therefore, the figure can't be completed.

   2. In general, regular polygons with an odd number of sides can be used to construct Reuleaux polygons.

b. Sample response: Yes. All Reuleaux polygons appear to be curves of constant width because they can be rolled through two parallel lines of support and the distance between any two parallel lines is always constant.

c. There are infinitely many pairs of parallel lines of support for a Reuleaux polygon. For every point on the curve, there is a line of support that contains that point. For each line of support, there is a corresponding parallel line of support.

Assignment

1.1 In Euclidean geometry, the sides of a triangle are line segments, not arcs. Thus, a Reuleaux triangle is not a triangle in Euclidean geometry.

*1.2   The resulting figure is not a curve of constant width, as shown by the distances between the two pairs of parallel lines of support in the sample diagram below.
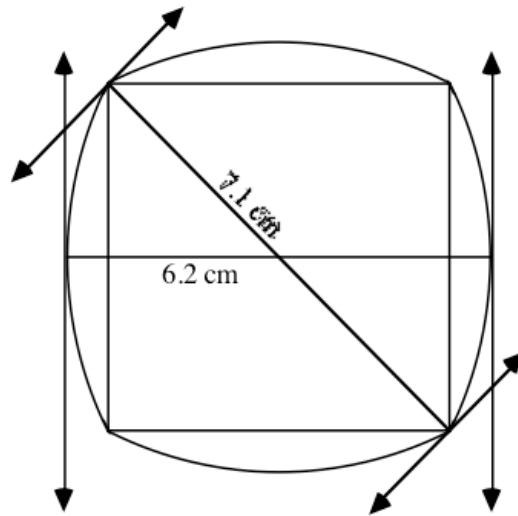

*1.3   **a.** Students may base their predictions on the length of the triangle's longest side. They check their predictions in Part **c**.

b. The sample sketch below shows two pairs of parallel lines of support:
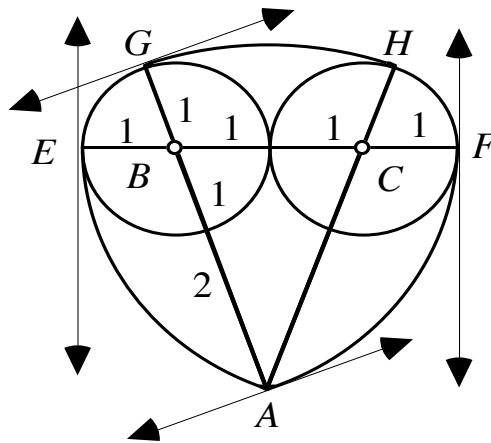
c.   Although not a formal proof that the curve has a constant width, students may note that since $\overline{AG}$ can be rotated about point A until G meets H, the width of that section of the curve is AG, or 4 cm. Since $\overline{AG}$ also can be rotated about point B until G meets E and A meets F, this section of the curve also has a width of 4 cm. Finally, since $\overline{AH}$ can be rotated about point C until A meets E and H meets F, the width of the remaining portion of the curve likewise is 4 cm. Therefore, the entire curve has a width of 4 cm.

   Note: Students should be cautioned that they have not shown that this method can be used to generate a curve of constant width using *any* isosceles triangle, only that a specific triangle can work.

1.4   **a.** A Reuleaux triangle can be created by constructing a perpendicular to $\overline{BC}$ through A and locating point A' on the perpendicular so that $\Delta A'BC$ is equilateral. Arcs then can be drawn from C to A' using B as the center, from B to A' using C as the center, and from B to C using A' as the center, as shown below.
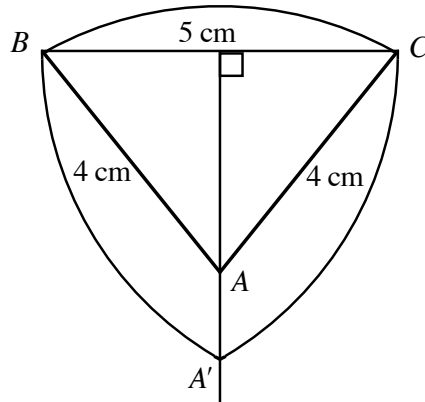
   b. Students should recognize that the result is a curve of constant width, where $w = 5$ cm , since it was constructed using an equilateral triangle whose sides each measure 5 cm.

<div align="center">* * * * *</div>

*1.5   **a.** Sample response: As shown in the following diagram, an equilateral triangle with the same vertices as a Reuleaux triangle that can hold three discs would not have enough room to hold the discs, while the corresponding circle would require more area than necessary.
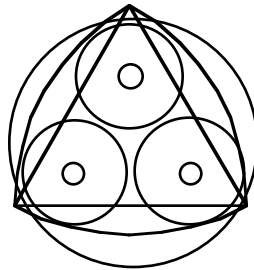
   Note: When the drawer is open, this design also allows access to two discs at once.

   b. Sample response: Although a square could accommodate four discs, when a square is rotated to allow the replacement of discs, the locus of its vertices defines a circle. This would result in wasted space. **Note:** Most CD changers that hold four discs use a stack format.

1.6   Answers will vary. Since the Reuleaux polygon formed from a regular pentagon approaches the shape of a circle, there is little difference in the areas of the two shapes. Changers that hold five discs are typically circular in shape.

<div align="center">* * * * * * * * * *</div>

Activity 2

Students investigate the use of a Reuleaux triangle in a bit that drills square holes.

Materials List
- scissors (one pair per group)
- compass (one per group)
- metric ruler (one per group)
- cardboard (three sheets per group, approximately 14 cm × 14 cm)
- tape
- templates of regular polygons A, B, C, D (optional for use with Problem **2.5**

Technology
- geometry utility

Teacher Note
Blackline masters of templates of regular polygons for optional use with Problem **2.5** appear at the end of the teacher edition for this module.

Exploration(page 9)
a–f.   Students create a model to simulate the path of a drill bit that cuts square holes.

g.   As the Reuleaux triangle rotates inside the square frame, the path of the centroid resembles the shape shown below.
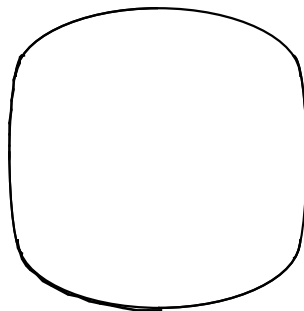
Discussion

a.   A bit shaped like a Reuleaux triangle produces a nearly square hole because, when rotated, it stays in constant contact with the sides of a square. In this case, the sides of the square represent lines of support for the Reuleaux triangle.

b.   See diagram given in Part **g** of the exploration above.

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c. The path of the centroid represents the path that the center of the drill bit must follow to create a square hole. **Note:** When drilling square holes, the drill chuck must be able to assume different positions, or "float." See diagram in Problem **2.1**.

d. If the path of the centroid of a drill bit is a single point, the hole produced is circular.

e. Answers may vary. When circular wheels are connected by an axle, the axle remains a constant distance from the ground. However, when Reuleaux polygons are connected by an axle, this does not occur. The current designs of vehicles do not allow for this kind of axle movement.

Teacher Note

You may wish to encourage students to use a geometry utility in Problems **2.4–2.8**.

Assignment

2.1 **a.** This bit should rotate counterclockwise (as observed from below).

b. In order to drill the hole, only a portion of the Reuleaux triangle need come into contact with the sides. The bit is shaped as shown to form cutting edges and to provide room for shavings to leave the hole.

2.2 A cross section of the bit would have to be shaped like a Reuleaux triangle with a constant width of 4 cm.

*2.3 **a.** As shown in the following diagram, the hole produced by a drill bit shaped like a Reuleaux pentagon would be nearly octagonal, if the centroid follows a path like the one shown in Part **g** of the exploration. It would have shortened pairs of parallel sides and rounded corners.
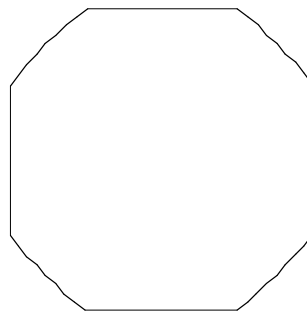
b. The limiting shape of the hole cut by a bit shaped like a Reuleaux *n*-gon is a circle.

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*2.4   **a.** Sample diagram:
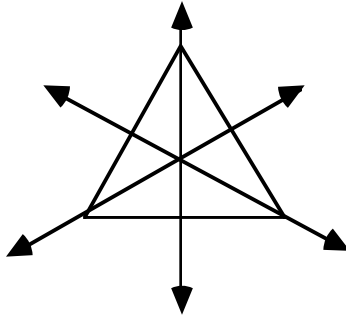
  b. Every median of an equilateral triangle is contained in a line of symmetry. Students may justify their responses by using paper folding to demonstrate reflections.

  c. Sample response: Yes. The medians and lines of symmetry for a Reuleaux triangle are also the medians and lines of symmetry for the inscribed equilateral triangle.

<p style="text-align:center">* * * * *</p>

2.5   **a.** For each regular n-gon, students should draw n lines of symmetry.

  b. Since every regular polygon can be inscribed in a circle, every side of a regular polygon is a chord of that circle. Every line of symmetry of a regular polygon either passes through two opposite vertices or is a perpendicular bisector of one or more sides.

   If the line of symmetry passes through opposite vertices, then it is a diameter of the circumscribed circle and passes through the center. Since the sides of a regular polygon are chords of the circle, the perpendicular bisectors (lines of symmetry) also pass through the center.

   Therefore, since all the lines of symmetry pass through the center of the circumscribed circle, all intersect in one point.

  c. The lines of symmetry for a Reuleaux polygon are the lines of symmetry for the inscribed regular polygon. Since the lines of symmetry for a regular polygon always intersect at a single point, the lines of symmetry for the circumscribed Reuleaux polygon also intersect at a single point.

2.6   **a.** Students draw several triangles, then find the circumcenter of each triangle by constructing the perpendicular bisectors of the sides.

  b. Students construct the circle that circumscribes each triangle created in Part **a**.

  c. Sample response: The circumcenter is the center of the circle that circumscribes a triangle.

d. Sample response: Any three noncollinear points in a plane can define the vertices of a triangle. By finding the triangle's circumcenter, it is possible to draw a circle that circumscribes the triangle. This circle contains each of the vertices of the triangle. Therefore, there is a circle that contains any three points in a plane.

2.7   **a.** Students draw several triangles, then find the incenter of each one by constructing their angle bisectors.

     b. Students construct the inscribed circle for each triangle created in Part **a**.

     c. Sample response: The incenter is the center of the circle that is inscribed within a triangle and is the point of intersection of the angle bisectors.

2.8   **a.** Answers may vary. Students should test their predictions using a geometry utility.

     b. Sample response: Since $\triangle ABC$ is an equilateral triangle, it is also equiangular. By definition, its medians bisect the opposite sides. Given these facts, the following relationships are true:

$AC = AB = BC$, $m\angle ABC = m\angle CAB = m\angle BCA = 60°$, and

$$AE = EC = CD = DB = BF = FA = \frac{1}{2}AC = \frac{1}{2}AB = \frac{1}{2}BC$$

Considering these relationships, $\triangle ADB \cong \triangle ADC$ because corresponding pairs of sides are congruent: $\overline{AD} \cong \overline{AD}$, $\overline{BD} \cong \overline{CD}$, and $\overline{AB} \cong \overline{AC}$. Since $\angle CAD \cong \angle BAD$, this implies that the median $\overline{AD}$ bisects $\angle CAB$, which in turn implies that $m\angle DAB = 30°$. Since $\angle ADC \cong \angle ADB$, this implies that $m\angle ADB = 90°$.

Using a similar argument, it can be shown that $m\angle EBC = 30°$ and $m\angle CFA = 90°$. Therefore, using Angle-Side-Angle Congruence, $\triangle AGF \cong \triangle BGD$ and $FG = GD$.

Using right-triangle trigonometry,

$$\sin \angle GAF = \frac{FG}{AG}$$

$$\sin 30° = \frac{FG}{AG}$$

$$= \frac{GD}{AG} \text{ (since } FG = GD)$$

$$\frac{1}{2} = \frac{GD}{AG}$$

$$\frac{1}{2}AG = GD$$

Since $AD = AG + GD$, the following is also true:

$$AD = AG + \frac{1}{2}AG$$

$$AD = \frac{3}{2}AG$$

$$\frac{2}{3} = \frac{AG}{AD}$$

This property is true for each median.

2.9   The diagram below shows $\triangle ABC$, the equilateral triangle used to generate the Reuleaux triangle ABC.
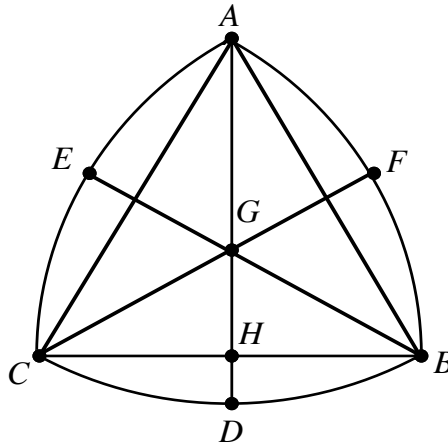
Since $m\angle BAH = 30°$, $m\angle HBA = 60°$, and $m\angle AHB = 90°$, students may use trigonometry to show that

$$HB = \frac{1}{2}AB \quad \text{and} \quad AH = \frac{\sqrt{3}}{2}AB$$

From Problem **2.8**,

$$\frac{AG}{AH} = \frac{2}{3}$$

$$\frac{3}{2}AG = AH$$

Substituting for AH,

$$\frac{3}{2}AG = \frac{\sqrt{3}}{2}AB$$

$$AG = \frac{\sqrt{3}}{3}AB$$

$$\frac{AG}{AB} = \frac{\sqrt{3}}{3}$$

Since $AB = AD$,

$$\frac{AG}{AD} = \frac{\sqrt{3}}{3} \approx 0.5774$$

Students may wish to use a geometry utility to test this ratio for several Reuleaux triangles.

* * * * * * * * * *

Research Project

The Wankel rotary engine is one of many engines with eccentric rotors designed to be more efficient than the popular reciprocating-piston engine invented by August Otto in 1876.

In a Wankel engine, a rotor shaped like a Reuleaux triangle rotates within a trochoidal combustion chamber, as shown in the following diagram. As the rotor turns, a portion of the chamber fills with mixture of air and gas. This mixture is compressed, then ignited by a spark. The exhaust gases are vented from the chamber before the cycle begins again.
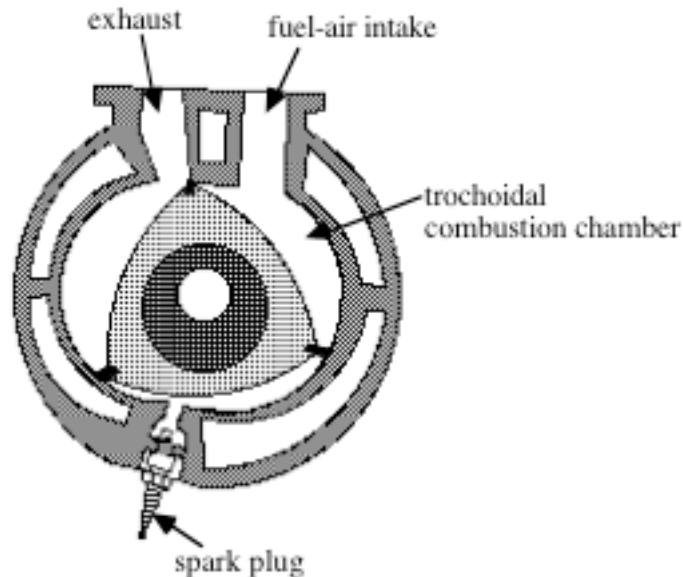
In general, rotary engines have several advantages over reciprocating-piston engines, including higher power output, smoother operation, fewer moving parts, and extremely low vibration and noise levels. Since the pistons of reciprocating-piston engines transfer power from fuel combustion by rotating a connecting rod around a crankshaft, the energy transfer at the beginning and end of the cycle is almost zero. Rotary engines, on the other hand, transfer combustion energy directly, and continuously, to the crankshaft.

In addition, rotary engines do not require valves to let in gasoline and vent exhaust. Instead, the rotor passes over the intake and exhaust ports as it rotates around the combustion chamber. This eliminates problems with valve timing and engine surging.

Wankel engines have been manufactured in a wide variety of sizes and for many different purposes. Curtiss-Wright makes a 20-horsepower engine used primarily in snowmobiles. A British motorcycle company utilizes rotary engines in racing bikes. In 1992, Mazda tested a hydrogen-powered version of the rotary engine.

Activity 3

Students develop a formula for the perimeter of a Reuleaux triangle, then generalize this formula to other curves of constant width. They also explore methods for calculating the areas of Reuleaux polygons.

Materials List

- templates C and D (optional; blackline masters appear at the end of the teacher edition for this module)

Technology

- geometry utility

Exploration 1

a–d. Students should use a geometry utility to create the constructions and measure angles. As an alternative, however, you may wish to provide copies of templates C and D. A completed table is shown below.

Reuleaux Polygon	$m\angle ADB$	$m\angle ADC$	$m\angle ACB$	$\dfrac{m\angle ACB}{360°}$
triangle	120°	120°	60°	1/6
pentagon	72°	144°	36°	1/10
heptagon	$51\frac{3}{7}°$	$154\frac{2}{7}°$	$25\frac{5}{7}°$	1/14
nonagon	40°	160°	20°	1/18
n-gon	$360°/n$	$180°-(180°/n)$	$180°/n$	$1/(2n)$

e. The length of each side of a Reuleaux polygon with constant width w is equal to the product of the fractional part of the circle found in Part **d** and the circumference of a circle with radius w. The perimeter is the product of the number of sides and the arc length.

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A completed table for Reuleaux polygons with a constant width of 5 cm is shown below.

Reuleaux Polygon	Side Length	Perimeter
triangle	$5\pi/3$	5π
pentagon	π	5π
heptagon	$5\pi/7$	5π
nonagon	$5\pi/9$	5π
n-gon	$5\pi/n$	5π

Discussion 1

a. The radius and width are equal.

b. The number of arcs multiplied by the length of each arc equals the perimeter.

c. The perimeter (P) of any Reuleaux polygon of constant width w can be calculated by the formula $P = \pi w$.

d. Sample response: The number of arcs multiplied by the length of each arc equals the perimeter. The fraction of the circle represented by the arc is $1/(2n)$, where n is the number of sides of the Reuleaux polygon. The circumference of the circle is $2\pi w$. The resulting arc length is $\pi w/n$. When this is multiplied by the number of sides, the number of sides factors out of the formula. In other words,

$$n \cdot \frac{1}{2n} \cdot 2\pi w = \pi w$$

e. Sample response: Yes. The constant width of a circle is the diameter. This means that the perimeter should be πd units. This is also the measure of the circumference the circle.

f. **1.** Sample response: The measure of the intercepted arc AC equals the measure of central angle AOC. Because $\angle AOB$ and $\angle AOC$ are supplementary and the sum of the measures of the interior angles of a triangle is 180°, the measure of $\angle AOC$ equals the sum of the measures of $\angle OAB$ and $\angle OBA$.

Since $\overline{OA}$ and $\overline{OB}$ are radii of the same circle, they are congruent, and $\triangle AOB$ is an isosceles triangle. This means that the measures of $\angle OAB$ and $\angle OBA$ are congruent. So,

$$m\angle AOC = m\angle ABC + m\angle BAO$$
$$m\angle AOC = 2 \cdot m\angle ABC$$
$$\frac{m\angle AOC}{2} = m\angle ABC$$

Therefore, the measure of $\angle ABC$ is equal to half the measure of its intercepted arc.

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2. Sample response: Yes. Since $\angle ABX$ and $\angle CBX$ are examples of inscribed angles like the one in Figure **12a**,

$$m\angle ABX + m\angle CBX = \frac{m\angle AOX}{2} + \frac{m\angle XOC}{2}$$
$$= \frac{m\angle AOC}{2}$$

Because $m\angle ABC = m\angle ABX + m\angle CBX$, the measure of $\angle ABC$ is equal to half the measure of its intercepted arc.

3. Sample response: Yes. Since $\angle ABX$ and $\angle CBX$ are examples of inscribed angles like the one in Figure **12a**,

$$m\angle ABX - m\angle CBX = \frac{m\angle AOX}{2} - \frac{m\angle XOC}{2}$$
$$= \frac{m\angle AOC}{2}$$

Because $m\angle ABC = m\angle ABX - m\angle CBX$, the measure of $\angle ABC$ is equal to half the measure of its intercepted arc.

g. Sample response: Both angles intercept the same arc of the circle with center at *D* and radius *DA*. Angle *ADB* is a central angle, while angle *ACB* is an inscribed angle.

Exploration 2

To determine the area of a segment of a circle in Part **c**, students must subtract the area of a triangle from the area of the corresponding sector. For example, consider the sector shown in the diagram below:
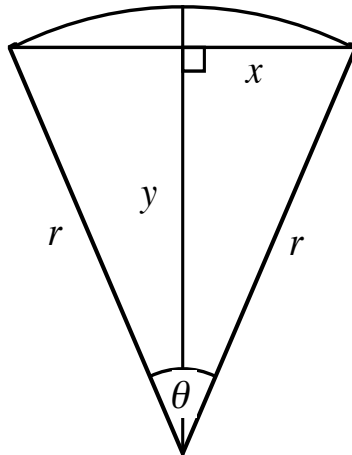
The area of the sector can be found using the following expression:

$$\frac{\theta}{360°} \bullet \pi r^2$$

The area of the triangular region is xy. Students should recall from previous modules that, in this case, $x = r\cos(\theta/2)$ and $y = r\sin(\theta/2)$. Consequently, the area of the triangular region, in terms of the radius r and the measure θ of the arc, can be expressed as follows:

$$r^2 \cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

Given these relationships, the area of the segment is described by the formula below:

$$A = \frac{\theta}{360°} \bullet \pi r^2 - r^2 \cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

a–c.   A completed table is shown below.

Reuleaux Polygon	Area of Sector (cm^2)	Area of Segment (cm^2)
triangle	$\dfrac{25\pi}{6}$	$\dfrac{25\pi}{6} - 25\cos 30°\sin 30° \approx 2.26$
pentagon	$\dfrac{5\pi}{2}$	$\dfrac{5\pi}{2} - 25\cos 18°\sin 18° \approx 0.51$
heptagon	$\dfrac{25\pi}{14}$	$\dfrac{25\pi}{14} - 25\cos 12\tfrac{6°}{7}\sin 12\tfrac{6°}{7} \approx 0.19$
nonagon	$\dfrac{25\pi}{18}$	$\dfrac{25\pi}{18} - 25\cos 10°\sin 10° \approx 0.09$
n-gon	$\dfrac{25\pi}{2n}$	$\dfrac{25\pi}{2n} - 25\cos\left(\dfrac{90°}{n}\right)\sin\left(\dfrac{90°}{n}\right)$

d.   Given a set of Reuleaux polygons with the same constant width w, a Reuleaux triangle has the least area. As the number of sides increases, the area increases, approaching the area of a circle with diameter w.

   Note: The Blaschke–Lebesgue theorem states that the convex set of constant width w (not just the convex polygon) that has the least area is the Reuleaux triangle.

Discussion 2   (page 19)

a.   Sample response: The sector is defined by a side of the Reuleaux triangle and a central angle of $60°$. The area of the sector can be found by multiplying the area of a circle with radius w by $1/6$.

b.   Sample response: The segment is defined by a side of the Reuleaux triangle and a chord. The area of the segment can be found by subtracting the area of the triangle formed by the central angle and the chord from the area of the sector.

The area of the triangle can be found as follows:

$$A = \frac{1}{2}b \bullet h$$

$$= \frac{1}{2}(5)(5\cos 30°)$$

$$\approx 10.8 \text{ cm}^2$$

c. Sample response: The area of a Reuleaux polygon with n sides is the area of the inscribed regular polygon plus the area of n segments of a circle with radius w.

Note: In general, the area of a Reuleaux polygon can be described by the following formula:

$A = n$(area of each segment of circle) + area of regular polygon

This can also be expressed as shown below:

$$A = n\left(\frac{\pi w^2}{2n} - w^2\cos\left(\frac{90°}{n}\right)\sin\left(\frac{90°}{n}\right)\right) + n\frac{\left(w\sin\frac{90°}{n}\right)^2}{\tan\left(\frac{180°}{n}\right)}$$

where n is the number of sides of the Reuleaux polygon and w is the constant width. As shown in the diagram below, each term in this formula can be determined using trigonometry.

$$AD = w\sin(90°/n)$$

$$AB = \frac{w\sin(90°/n)}{\tan(180°/n)}$$

$$AC = w\cos(90°/n)$$
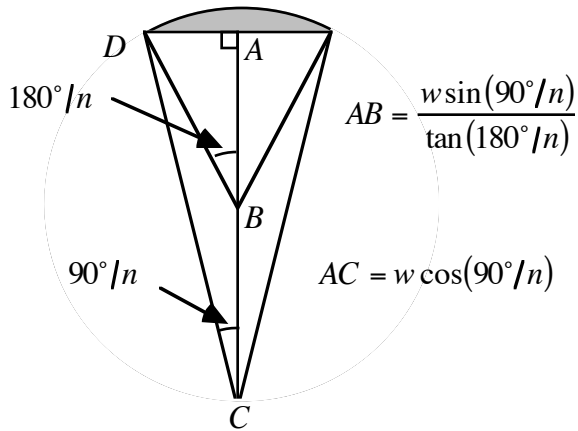
Assignment

3.1 **a.** $11\pi \approx 34.6$ cm

b. $15\pi \approx 47.1$ cm

c. $10\pi \approx 31.4$ cm

3.2 Sample response: The area of each segment of the circle of the Reuleaux triangle is:

$$\frac{\pi(7)^2}{2(3)} - 7^2\cos\left(\frac{90°}{3}\right)\sin\left(\frac{90°}{3}\right) \approx 4.4 \text{ cm}^2$$

The area of the equilateral triangle is:

$$3\frac{\left[7\sin\left(\dfrac{90°}{3}\right)\right]^2}{\tan\left(\dfrac{180°}{3}\right)} \approx 21.2 \text{ cm}^2$$

Therefore, the area of the Reuleaux triangle is approximately:

$$3(4.4) + (21.2) \approx 34.7 \text{ cm}^2$$

*3.3   **a.** The area of the circular tray is $\pi \bullet 13.8^2 \approx 598$ cm^2. The area of the tray shaped like a Reuleaux triangle is:

$$3\left(\frac{1}{6}\bullet\pi\bullet26.1^2\right) - 2\left(\frac{1}{2}\bullet 26.1\bullet 26.1\bullet\sin 60°\right) \approx 480 \text{ cm}^2$$

The tray shaped like a Reuleaux triangle uses approximately 120 cm^2 less material. **Note:** One drawback to this tray is that it will not rotate freely in its own space, as a circular tray will. One CD player that features such a tray uses a mechanism to raise the tray before rotating it.

*3.4   **a.** Students may use a generalized formula or a method of their own design. Sample response: The area of a Reuleaux pentagon is the area of the inscribed regular pentagon plus the area of five segments of a circle. The area of each segment is the area of a sector minus the area of the corresponding isosceles triangle, as shown below.

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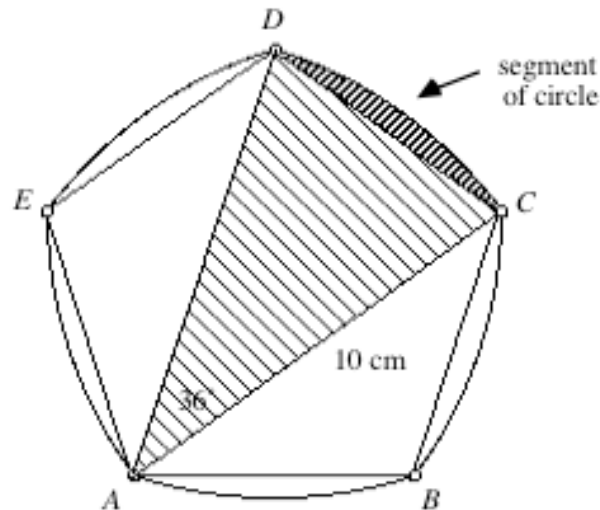
In this case, the area of each segment is:

$$\frac{1}{6} \bullet \pi \bullet 10^2 - \frac{1}{2} \bullet 10 \bullet 10 \bullet \sin 36° \approx 2 \text{ cm}^2$$

One formula for the area of a regular polygon is:

$$A = \frac{1}{2} \bullet \text{apothem} \bullet \text{perimeter}$$

The length of a side of the pentagon can be found using the law of sines and $\triangle ACD$ in the diagram above:

$$2(10 \bullet \sin 18°) \approx 6.2 \text{ cm}$$

Using this approximation, the apothem can be found using as follows: $3.1 \bullet \tan 54° \approx 4.3 \text{ cm}$.
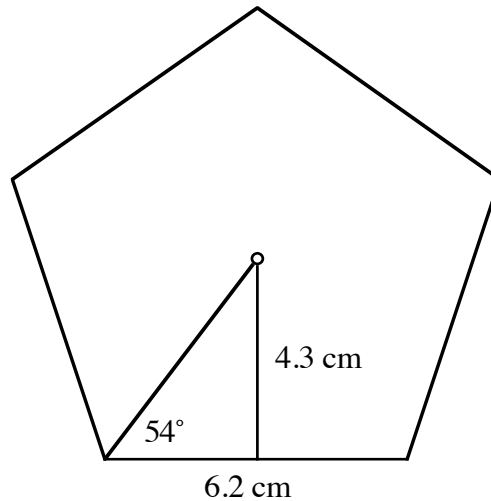
Therefore, the area of the pentagon is approximately:

$$\frac{1}{2} \bullet 4.3 \bullet (5 \bullet 6.2) \approx 67 \text{ cm}^2$$

The area of the Reuleaux pentagon, therefore, is approximately $67 + (5 \bullet 2) = 77 \text{ cm}^2$.

b. The area of the Reuleaux heptagon can be found using a process similar to the one described in Part **a**.

In this case, the area of each segment is:

$$\frac{1}{14} \bullet \pi \bullet 10^2 - \frac{1}{2} \bullet 10 \bullet 10 \sin 25.7° \approx 0.76 \text{ cm}^2$$

The length of each side of the regular heptagon is approximately 4.45 cm. The length of the apothem is approximately 4.6 cm. The area of the heptagon is:

$$\frac{1}{2} \bullet 4.6 \bullet (7 \bullet 4.45) \approx 72 \text{ cm}^2$$

Therefore, the area of the Reuleaux heptagon with constant width 10 cm is approximately $72 + (7 \bullet 0.76) \approx 77.3 \text{ cm}^2$. Students may notice that this is very close to the area of the Reuleaux pentagon in Part **a**.

* * * * *

3.5 Sample response: In general, the area of a Reuleaux polygon can be found by finding the area of the inscribed regular polygon, then adding the areas of the corresponding segments of a circle.

*3.6 Sample response: As determined in Problem **3.4b**, the area of a Reuleaux heptagon with a constant width of 10 cm is approximately 77.3 cm^2. A circle with a diameter of 10 cm has an area of approximately 78.5 cm^2. In this case, the difference is about 1.5%.

3.7 If a Reuleaux triangle is rotated around an axis of symmetry, the surface resembles an inverted top, as shown below.

A solid of this shape may be used as a top, a wheel, or a roller. In fact, some types of furniture use rollers of this shape.

* * * * * * * * * *

1.   **a.** Sample construction:
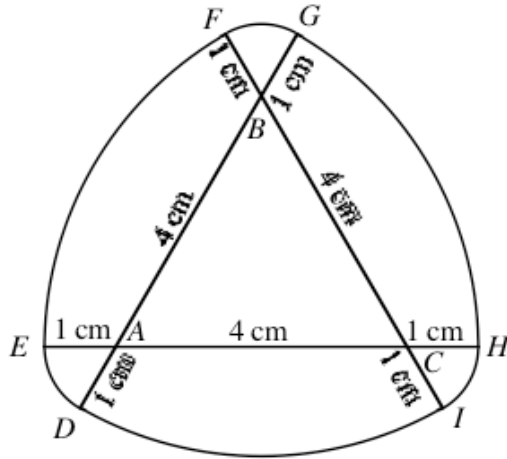
b. This shape is a curve of constant width, where $w = 6$ cm.
Although not a formal proof, students may show that the curve has
a constant width by noting that since $\overline{DG}$ can be rotated about
point B until G meets F, the width of that section of the curve is
the length of $\overline{DG}$, or 6 cm. Since $\overline{DG}$ also can be rotated about
point A until G meets H and D meets E, that section of the curve
also has a width of 6 cm. Finally, since $\overline{FI}$ can be rotated about
point C until F meets E and I meets H, the width of the remaining
portion of the curve is 6 cm. Therefore, the entire curve has a
width of 6 cm.

c. Since this shape is a curve of constant width, its perimeter can be
found using the formula $P = \pi w$. Therefore, the perimeter is
$3 \bullet 6 = 18$ cm.

d. One possible method for finding the area of the shape involves
dividing it into sectors, as shown below.
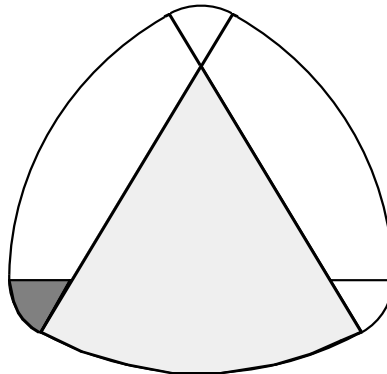
The area of the shape can then be described as three times the area

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of the larger sector, minus two times the area of the equilateral triangle, plus three times the area of the smaller sector.

The area of the larger sector is:

$$\frac{1}{6} \cdot \pi \cdot 5^2 \approx 13.1 \, \text{cm}^2$$

The area of the triangle is:

$$\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} \approx 6.9 \, \text{cm}^2$$

The area of the smaller sector is:

$$\frac{1}{6} \cdot \pi \cdot 1^2 \approx 0.5 \, \text{cm}^2$$

Therefore, the area of the entire shape is:

$$(3 \cdot 13.1) - (2 \cdot 6.9) + (3 \cdot 0.5) \approx 27 \, \text{cm}^2$$

2. Sample response: The best shape for the razor head is a Reuleaux triangle with rounded corners. As shown in the diagram below, this is a smooth shape with a smaller area than the corresponding circle.

$$\boxed{\begin{array}{c} \textit{\textbf{Module}} \\ \textit{\textbf{Assessment}} \end{array}}$$
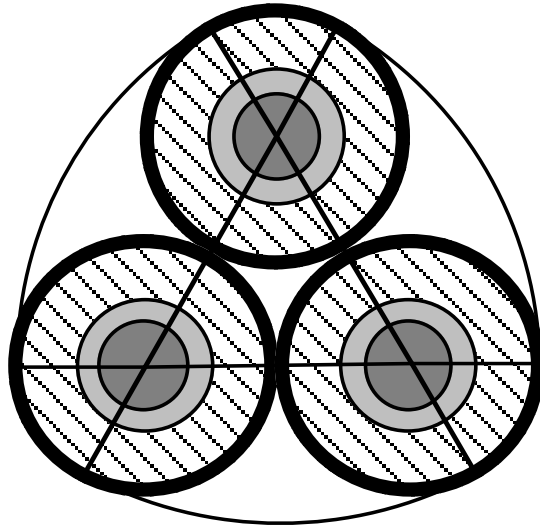
1. What is the measure of the arc that forms a side of a Reuleaux polygon with 11 sides?

2. Why are Reuleaux polygons not Euclidean polygons?

3. Calculate the perimeter of an 11-sided Reuleaux polygon with a constant width of 7.5 cm.

4. **a.** Construct a Reuleaux triangle with a constant width of 4 cm.

   b. Determine the area of the Reuleaux triangle in Part **a**.

5. **a.** What advantages are there to using a curve of constant width as the shape for a sewer cover?

   b. Why is a square not a good shape for a sewer cover?

Answers to Module Assessment

1. The measure of the central angle for a regular 11-gon is $360°/11 \approx 32.7°$. The measure of the arc for the Reuleaux 11-gon is half the measure of the central angle, or approximately $16.4°$.

2. Sample response: The sides of Euclidean polygons must be line segments. Therefore, Reuleaux polygons with arcs for sides do not fit the Euclidean definition.

3. The perimeter of any Reuleaux polygon is πw. In this case, the perimeter is $\pi \bullet 7.5 \approx 23.6$ cm.

4. **a.** Sample sketch:

4 cm
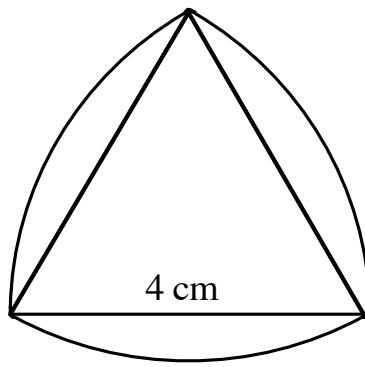
   b. The area of each sector is:

   $$\frac{1}{6} \bullet \pi \bullet 4^2 \approx 8.4 \text{ cm}^2$$

   The area of the equilateral triangle is:

   $$\frac{1}{2} \bullet 4 \bullet 2\sqrt{3} \approx 6.9 \text{ cm}^2$$

   Therefore, the area of the Reuleaux triangle is:

   $$3(8.4) - 2(6.9) \approx 11.3 \text{ cm}^2$$

5. **a.** By using the shape of a curve of constant width, such as a circle, and making the sewer cover a little wider than the hole, the cover will never be able to fall into the sewer.

   b. A square sewer cover could fall into the sewer along the diagonal of the hole, causing property damage, injury, or both.

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Lyusternik, L. A. *Convex Figures and Polyhedra*. Boston: D. C. Heath and Co., 1966.

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Smith, S. "Drilling Square Holes." *Mathematics Teacher* 86 (October 1993): 579–583.

Flashbacks

Activity 1

1.1   In Parts **a–c** below, determine which line, if any, may be tangent to the given figure.

a.

b.

c.
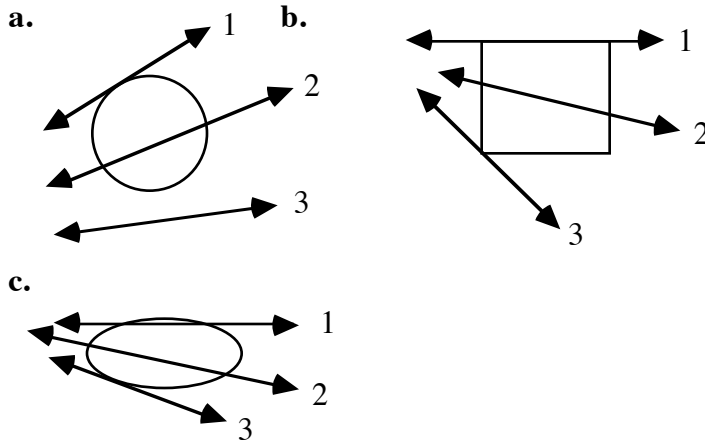
1.2   Which of the following figures are polygons? Explain your responses.

a.

b.

c.

d.
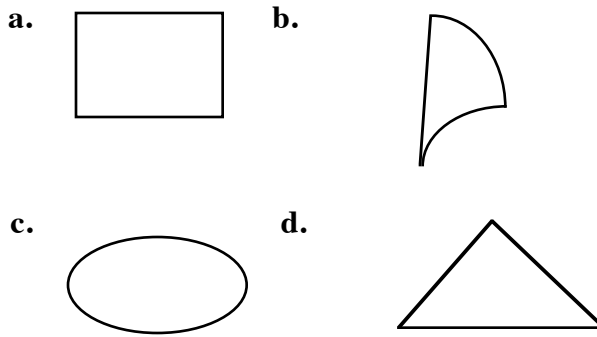
1.3   Which of the following figures appear to be regular polygons?

a.

b.

c.

d.
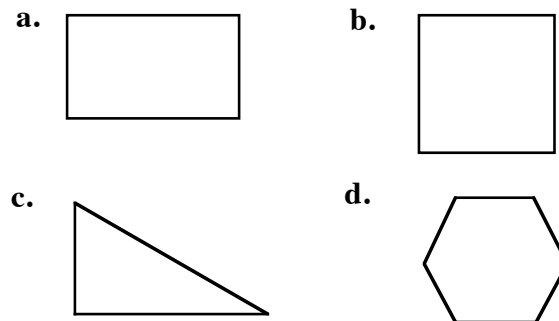
Activity 2

2.1 Determine the perimeter of each of the following polygons:

    a. an equilateral triangle with sides measuring 5 cm

    b. a regular pentagon with sides measuring 8 cm

    c. a regular n-gon with sides measuring l cm

2.2 Calculate the circumference of a circle with each of the following radii:

    a. 5 cm

    b. 8 cm

    c. l cm

2.3 Find the area of each circle in Flashback **2.2**.

Activity 3

3.1 In Parts **a–c** below, determine the fraction of the circle's area represented by the shaded region.

a.

90°

b.

45°

c.

$n°$
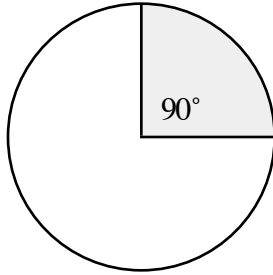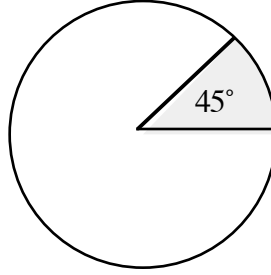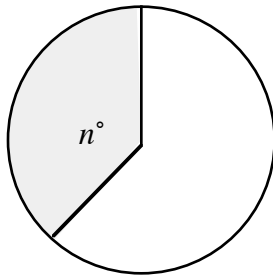
3.2 Determine the height of an equilateral triangle with sides of each of the following lengths:

   a. 4 cm

   b. n cm

3.3 Determine the measure of each interior angle in a regular polygon with the following numbers of sides:

   a. 5

   b. n

3.4 Determine the unknown lengths and angle measures in each of the following regular polygons:

a.

z x

y

8 cm

b.

q

36° p

5 cm

n

$r°$

$s°$ 54°

m
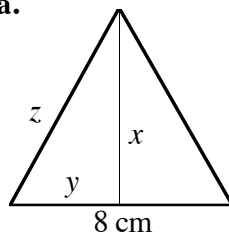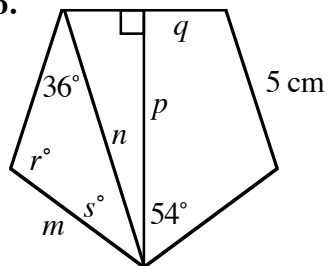
Answers to Flashbacks

Activity 1

1.1   **a.** Line **1** appears to be tangent to the given circle.

   b. None of the lines are tangent to the given rectangle. Line **3** is not a tangent line because it is not the only line in the plane which passes through the point.

   c. Line **3** appears to be tangent to the given ellipse.

1.2   The figures in Parts **a** and **d** are polygons because they are simple closed figures whose sides are segments.

1.3   The figures in Parts **b** and **d** appear to be regular polygons because each has congruent sides and angles of equal measures.

Activity 2

2.1   **a.** 15 cm

   b. 40 cm

   c. $l \bullet n$ cm

2.2   **a.** $2 \bullet \pi \bullet 5 \approx 31.4$ cm

   b. $2 \bullet \pi \bullet 8 \approx 50.3$ cm

   c. $2 \bullet \pi \bullet l$ cm

2.3   **a.** $\pi \bullet 5^2 \approx 78.5$ cm^2

   b. $\pi \bullet 8^2 \approx 201.1$ cm^2

   c. $\pi \bullet l^2$ cm^2

Activity 3

3.1   **a.** 90/360 = 1/4

   b. 45/360 = 1/8

   c. n/360

3.2   **a.** $2\sqrt{3} \approx 3.46$ cm

   b. $(n/2) \bullet \sqrt{3}$ cm

3.3   **a.** $108°$

　　　b. $180° - (360°/n)$

3.4   **a.** $x = 4\sqrt{3}$ cm; $y = 4$ cm; and $z = 8$ cm

　　　b. $m = 5$ cm; $n = 2.5/\sin 18° \approx 8.1$ cm; $p = 2.5/\tan 18° \approx 7.7$ cm; $q = 2.5$ cm; $r = 108$; $s = 36$

Template A

A

B

C

D

E

F

G

34
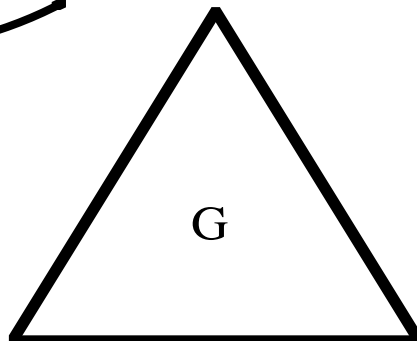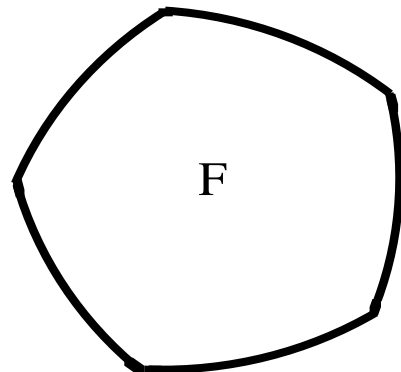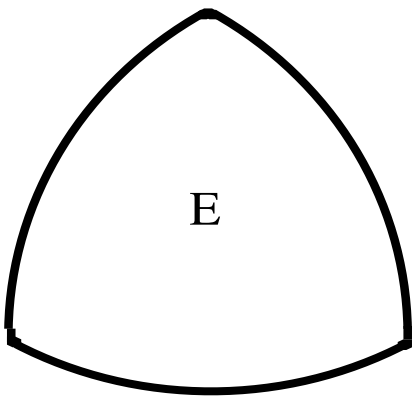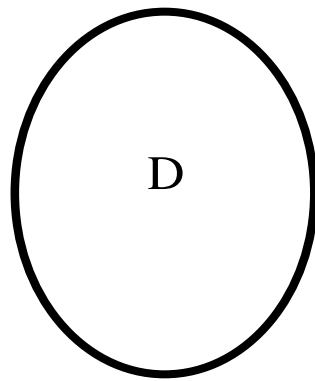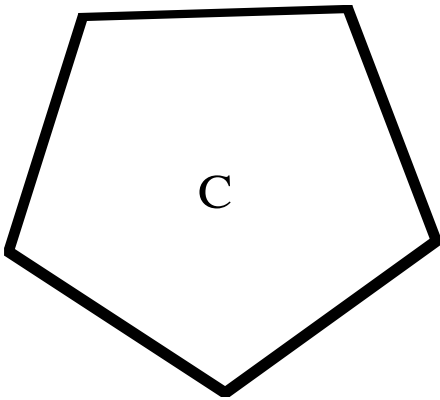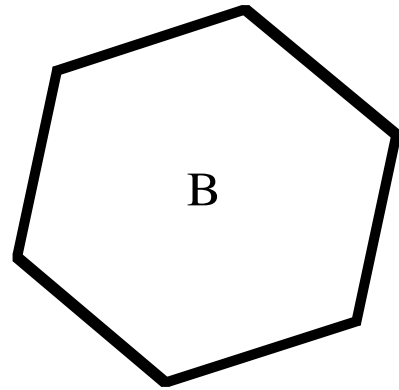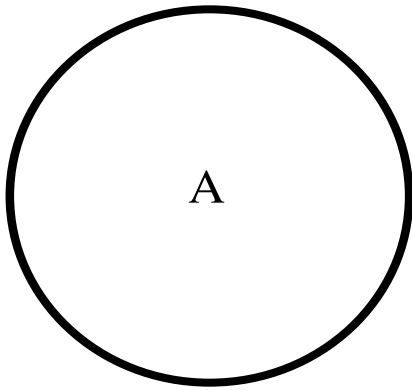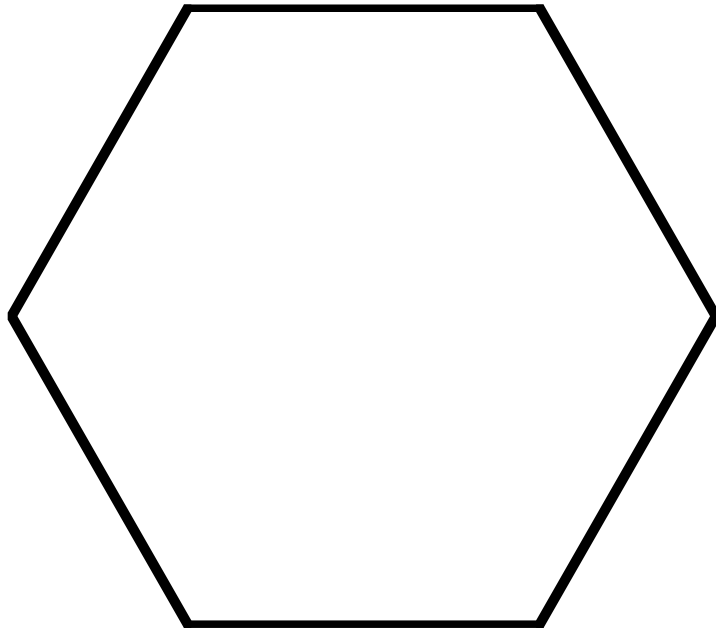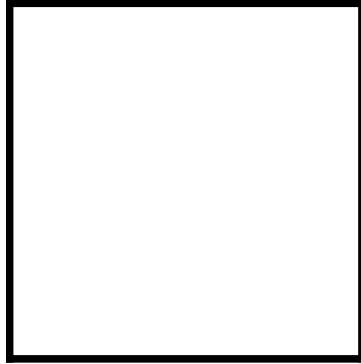
Template B

Template C
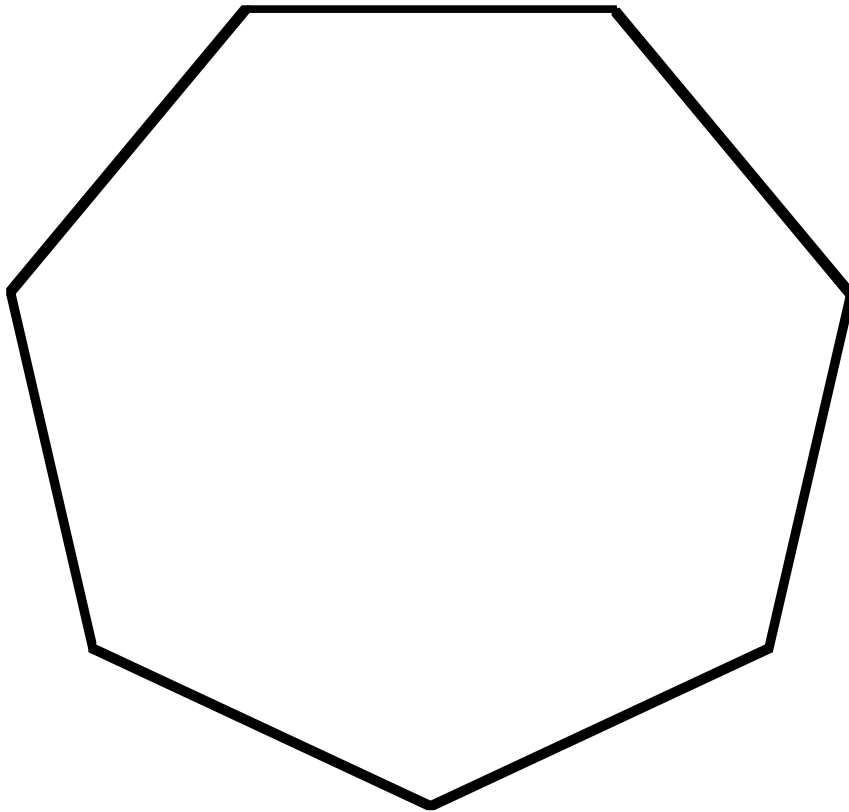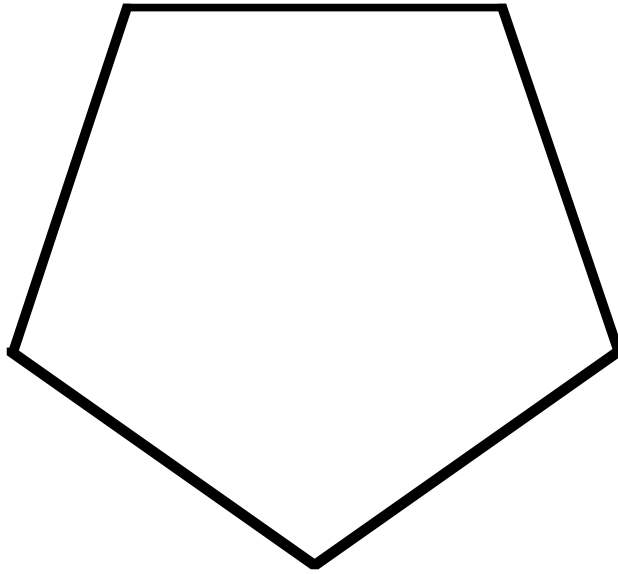
Template D
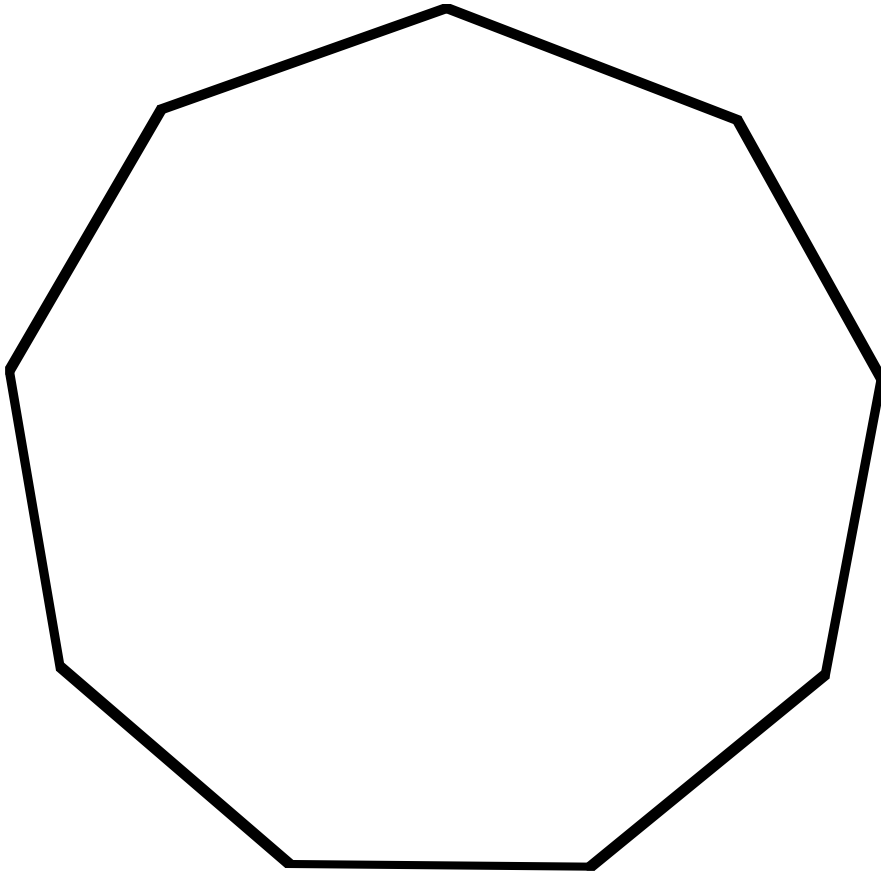
Sunken Sub
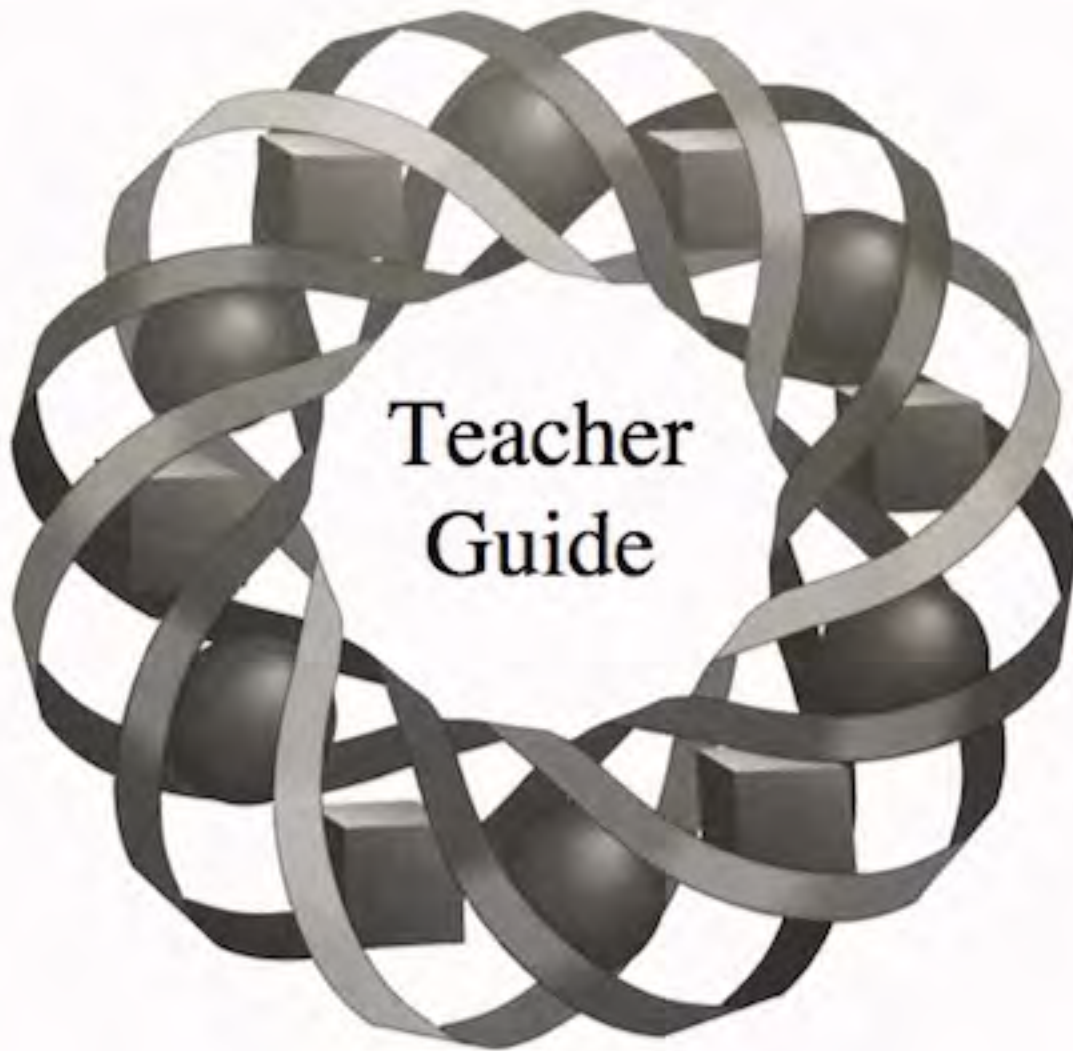
"This is the USS *Livingston*. We've lost ballast and are going down! Our coordinates are . . . " A submarine is sinking and you're in charge of the rescue. Can you save the crew?

John Knudson-Martin • Laurie Paladichuk

Sunken Sub

Overview

In this module, students investigate the use of geometric and conditional probability in search-and-rescue operations.

Objectives

In this module, students will:

- use geometric models to describe events and sample spaces
- examine geometric probability in two dimensions
- combine geometric probability and conditional probability in multistage settings.

Prerequisites

For this module, students should know:

- how to find the area of geometric figures
- the definitions of event, sample space, and theoretical probability
- the definition of conditional probability
- how to use tree diagrams
- how to determine expected value.

Time Line

Activity	1	2	Summary Assessment	Total
Days	2	2	1	5

Materials Required

- none

Technology

Software	1	2	Summary Assessment
geometry utility	X		

Sunken Sub

Introduction

In this module, students explore geometric probability through the search for a lost submarine. The history note describes two well-known submarine disasters.

Activity 1

Students apply geometric and conditional probability to determine the chances of finding a lost submarine.

Materials List

- none

Technology

- geometry utility (optional)

Teacher Note

The description of geometric probability given in the mathematics notes assumes that the probability distribution is uniform.

Exploration

a–b.   Sample sketch:

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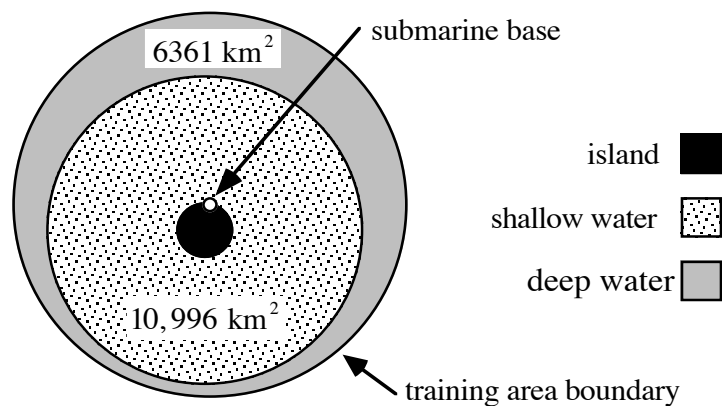
Since submarines can travel no farther than 75 km of the base and the island is roughly circular with radius 10 km, the surface area of the training region is:

$$\left(\pi \bullet 75^2\right) - \left(\pi \bullet 10^2\right) \approx 17,357 \text{ km}^2$$

The surface area of the region of shallow water is:

$$\left(\pi \bullet 60^2\right) - \left(\pi \bullet 10^2\right) \approx 10,996 \text{ km}^2$$

The surface area of the region of deep water is:

$$17,357 - 10,996 = 6361 \text{ km}^2$$

c.   **1.**   The probability that the *Livingston* went down in shallow water is:

$$\frac{\text{surface area of shallow water}}{\text{total surface area}} = \frac{10996}{17357} \approx 0.63 \approx 63\%$$

2.   The probability that the *Livingston* went down in deep water is $1 - 0.63 = 0.37$ or 37%. It may also be calculated using areas:

$$\frac{\text{surface area of deep water}}{\text{total surface area}} = \frac{17357 - 10996}{17357} \approx 0.37 \approx 37\%$$

d.   The probability that the *Livingston* will not be found in one complete search, given that it went down in deep water, is $1 - 0.1 = 0.9$ or 90%.

Discussion

a.   Sample response: The probability that the *Livingston* went down in shallow water is the ratio of the area of the shallow region to the area of the region which includes all the submarine's possible locations.

b.   Sample response: The probability that the Livingston went down in deep water can be found using a ratio of areas or by subtracting the probability that the submarine went down in shallow water from 1.

c.   **1.**   Sample response: The probability of finding the *Livingston*, given that it went down in deep water, is a conditional probability. If you know it went down in deep water, the probability of finding it in one complete search is 0.1.

2.   Sample response: The probability that the *Livingston* went down in deep water and will be found is $0.37 \bullet 0.1 = 0.037$.

d.   Sample response: The rescue crew should begin their search in shallow water since there is a higher probability the submarine is there. Also, if the submarine did go down in shallow water, the probability of finding it is much higher than if it went down in deep water.

1.1   **a.** Sample tree diagram:

probabilities

0.1	found	$0.037 \approx 4\%$
0.9	not found	$0.333 \approx 33\%$
0.7	found	$0.441 \approx 44\%$
0.3	not found	$0.189 \approx 19\%$
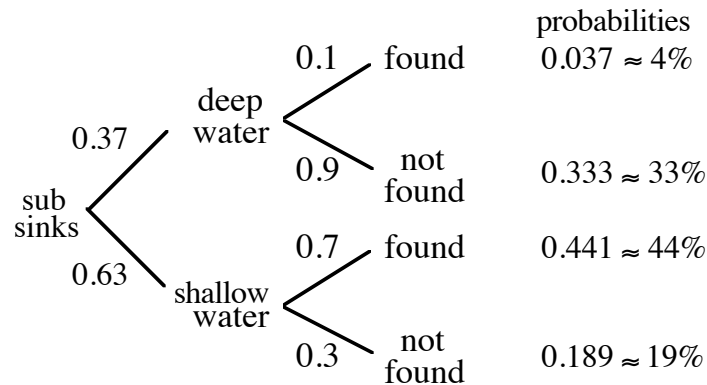
     b. The probability that the *Livingston* went down in deep water and will be found can be calculated by multiplying the probabilities on the appropriate branches of the tree diagram: $0.37 \bullet 0.1 = 0.037$.

1.2   **a.** The probability that the *Livingston* went down in shallow water and will be found is $0.63 \bullet 0.7 \approx 0.441$, or about 44%.

     b. The probability that the *Livingston* went down in deep water and will not be found is $0.37 \bullet 0.9 \approx 0.333$, or about 33%.

     c. Assuming that the entire area is searched, the total probability that the *Livingston* will be found is the sum of the probability that it went down in shallow water and will be found and the probability that it went down in deep water and will be found, or $0.037 + 0.441 = 0.478 \approx 48\%$.

*1.3   **a.** Answers will vary, depending on student estimates of Laurie's body size. The probability of finding her on the first probe may be calculated by the following ratio:

$$\frac{\text{area of back of Laurie's body}}{\text{total area of avalanche}}$$

For example, if a student estimates that Laurie's area is 6800 cm^2, then the probability of finding her body on the first probe is:

$$\frac{6800 \text{ cm}^2}{10 \text{ m} \bullet 12 \text{ m}} = \frac{0.68 \text{ m}^2}{120 \text{ m}^2} \approx 0.006 = 0.6\%$$

     b. It will take Jamal approximately 20 min to search the entire area:

$$\frac{120 \text{ m}^2}{6 \text{ m}^2/\text{min}} = 20 \text{ min}$$

c. Since Jamal can search the entire area in 20 min, the probability of finding her in the first 10 min is 0.5, as is the probability of finding her in the second 10 min.

From the tree diagram below, the probability that she is found in the first 10 min and survives is $0.5 \cdot 0.8 = 0.4$. The probability that she is found in the second 10 min and survives is $0.5 \cdot 0.55 = 0.275$. Her total probability of survival is $0.4 + 0.125 = 0.675$.

```
                              0.8   survives      probabilities
                  found                           0.4 = 40%
                  in first
         0.5      10 min
                          0.2   does not
                                survive          0.1 = 10%
  buried
                          0.55  survives         0.275 = 27.5%
         0.5     found
                 in second
                 10 min    0.45  does not
                                 survive         0.225 = 22.5%
```
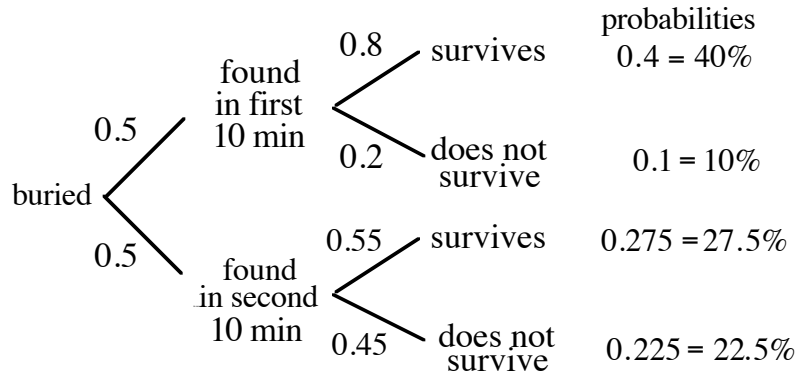
1.4 **a.** Sample response: The total number of pixels on the screen is $800 \cdot 800 = 640,000$. The "area" of the target is $\pi \cdot 280^2 \approx 246,300 \text{ pixels}$. Assuming that each pixel is equally likely to be illuminated, the probability that the lighted pixel is inside the target area is $246,300/640,000 = 0.38$.

b. Sample response: If all pixels have an equal chance of being illuminated, moving the target will not increase the probability of winning. The target area still contains the same number of pixels.

* * * * *

1.5 **a.** Sample response: The area of the circle is $\pi r^2 \approx 3.14 r^2$. The area of the square is $2r \cdot 2r = 4r^2$. The probability that a dart falls in the circle, given that it lands in the square, is:

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{3.14 r^2}{4r^2} = 0.785$$

b. Answers may vary. Sample response: Since the perimeter of a circle has no area, the probability that the dart lands exactly on the circle is 0.

c. The probability of winning, given that the dart lands inside the square, is $1 - 0.785 = 0.215$.

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1.6 **a.** The probability is the ratio of the areas of the shaded regions to the total area of the board:

$$\frac{2(25 \bullet 50)}{(50 + 25)^2} = \frac{2500}{5625} = \frac{4}{9}$$

b. The probability is $2/4 = 1/2$.

c. Students may use a tree diagram to determine the probability of winning the game. A sample diagram is shown below.
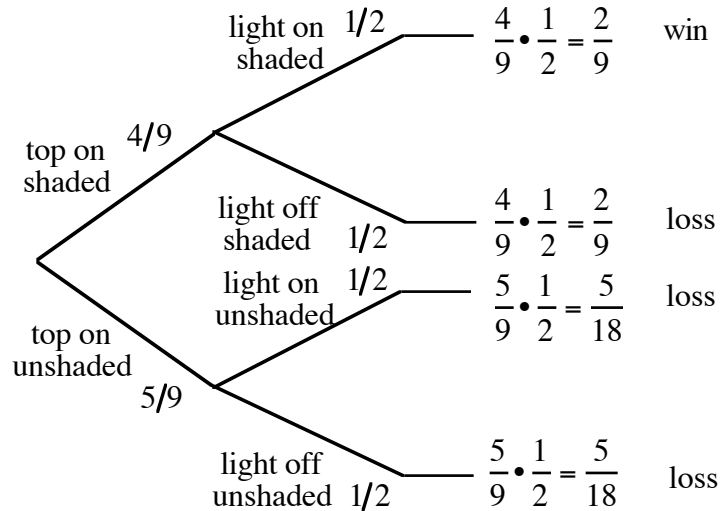
d. **1.** Sample response: Yes, these events are independent. If event A is that the top lands in a shaded region and event B is that a light is on in a shaded region, then $P(B|A) = 1/2$. In this case, $P(B|A) = P(B)$, and

$$P(A \text{ and } B) = P(A) \bullet P(B)$$

$$\frac{2}{9} = \frac{4}{9} \bullet \frac{1}{2}$$

2. Sample response: No, these events are not independent. If event A is that the top lands in a shaded region and event B is that a player wins, then $P(B|A) = 1/2$. The probability of winning, however, or $P(B)$, is only $2/9$. Since $P(B|A) \neq P(B)$, the events are not independent.

* * * * * * * * * *

> ## *Activity 2*

In this activity, students add a time constraint to the search for the *Livingston* and consider the design of an optimal search strategy.

Materials List

*   none

Exploration

a.   In five days, each submarine has $5 \bullet 24 = 120$ hr in which to search. At a speed of 10 km/hr, each submarine can travel 1200 km. As shown in the diagram below, the maximum area covered can be thought of as a rectangular region with a semicircle at each end.
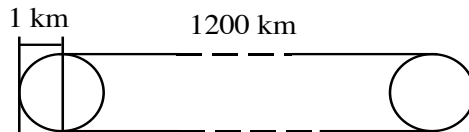
The area of this region is:

$$(1200 \bullet 2) + 2\left(\frac{\pi \bullet 1^2}{2}\right) \approx 2403 \text{ km}^2$$

Therefore, the two submarines can search a maximum of approximately 4806 km^2 in five days.

b.   **1.** The surface area of the shallow region around the island is about 10,996 km^2. If the *Livingston* went down in shallow water, the probability that the rescue submarines pass over it is no greater than:

$$\frac{4806}{10,996} \approx 0.44 = 44\%$$

2. The surface area of the region of deep water is about 6361 km^2. If the *Livingston* went down in deep water, the probability that the rescue submarines pass over it is no greater than:

$$\frac{4806}{6361} \approx 0.76 = 76\%$$

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c. **1.** Sample tree diagram:

probabilities

```
                                                    0.7
                                          pass over  ───── detect        0.19
                                  0.44               0.3
                                          ─────          ───── don't detect  0.08
                      shallow                         0
                                  0.56    don't     ───── detect        0.00
          0.63                            pass over   1
                                                        ───── don't detect  0.35
                                                    0.1
                                  0          pass over  ───── detect        0
          0.37                                        0.9
                          deep            ─────          ───── don't detect  0
                                  1       don't       0
                                          pass over ───── detect        0
                                                      1
                                                        ───── don't detect  0.37
```

2. Sample tree diagram:

probabilities

```
                                                    0.7
                                  0          pass over  ───── detect        0
                                                      0.3
                                          ─────          ───── don't detect  0
                      shallow                         0
                                  1       don't     ───── detect        0
          0.63                            pass over   1
                                                        ───── don't detect  0.63
                                                    0.1
                                  0.76       pass over  ───── detect        0.03
          0.37                                        0.9
                          deep            ─────          ───── don't detect  0.25
                                  0.24    don't       0.0
                                          pass over ───── detect        0.00
                                                      1.0
                                                        ───── don't detect  0.09
```

Discussion

(page 38)

a. Sample response: No. The officer was incorrect. The submarines can only cover 4806 km^2 in five days.

b. The surface area of the entire training region is 17,357 km^2. Given that two submarines can search 4806 km^2 in 1 five-day period, it would take $17,357/4806 \approx 3.6$ five-day periods to search the entire training region. This corresponds to more than 18 days.

c. **1.** Since each submarine can search no more than 2403 km^2 in five days, the rescue team would need at least 8 submarines ($17357/2403 \approx 7.2$).

2. Sample response: This would not guarantee that the missing sub will be located, it would only guarantee that most of the training area would be searched.

d. Sample response: The probability of finding the sub after one complete search of both the shallow water and the deep water is $0.19 + 0.03 = 0.22$. The probability of finding the sub after two complete searches is shown in the following tree diagram:
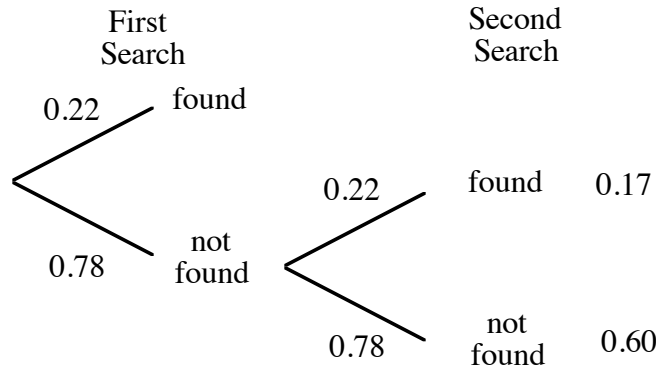
After two complete searches, the probability of not finding the sub is about 0.60. Therefore, the probability of finding it is approximately $1 - 0.60 = 0.4$.

Assignment (page 35)

2.1 **a.** $0.63 \cdot 0.44 \cdot 0.70 \approx 0.19 = 19\%$

   b. $0.08 + 0.35 + 0.37 = 0.8 = 80\%$

   c. $0.63 + 0.25 + 0.09 = 0.97 = 97\%$

2.2 **a.** In three days, the submarines each have $3 \cdot 24 = 72$ hr in which to search. At 10 km/hr, each submarine can travel 720 km. The maximum area that can be scanned by each submarine in 72 hr is the sum of the areas of a rectangular region ($720 \cdot 2 = 1440 \text{ km}^2$) and two semicircular regions (about 3.14 km^2) for a total of about 1443 km^2.

   Therefore, the three submarines together can search no more than 4329 km^2. Since the area of the region of deep water is about 6361 km^2, the percentage is $4329/6361 \approx 0.68$ or 68%.

b. Sample tree diagram:
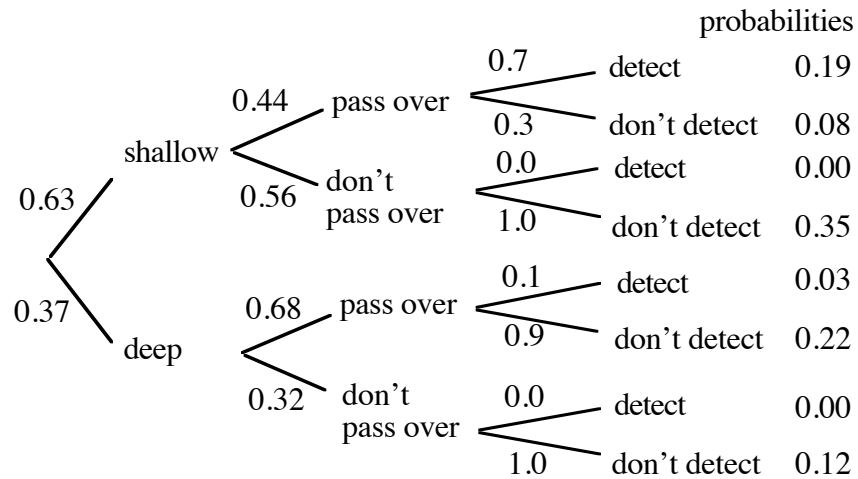
c. The probability that the three new submarines will find the *Livingston* is the probability that the *Livingston* is in the searched area and it is found. From the tree diagram above, this equals:

$$0.37 \bullet 0.68 \bullet 0.1 \approx 0.03$$

d. The total probability that the *Livingston* will be found is the sum of the probability that the *Livingston* is found by the two submarines searching the shallow water and the probability that it is found by the three submarines searching the deep water: $0.19 + 0.03 = 0.22$.

*2.3 The maximum area that the five submarines can scan is $4806 + 4329 = 9135 \text{ km}^2$. The probability that these submarines will scan the area where the *Livingston* went down is:

$$\frac{\text{searched area}}{\text{total area}} = \frac{9135}{17,357} \approx 0.53 \approx 53\%$$

The probability that the *Livingston* is not in the searched area is:

$$1 - 0.53 = 0.47 \approx 47\%$$

2.4 Using the tree diagram created in Problem **2.2b**, the probability that the search submarines will pass over the area where the *Livingston* went down, but not detect it, is $0.08 + 0.22 = 0.30$.

2.5 The probability that three submarines searching in shallow water for three days will find the *Livingston* is:

$$P(\text{shallow}) \bullet P(\text{pass over}) \bullet P(\text{detect}) = \frac{10{,}996}{17{,}357} \bullet \frac{4329}{10{,}996} \bullet 0.7 \approx 0.17$$

The probability that two submarines searching in deep water for five days will find the *Livingston* is:

$$P(\text{deep}) \bullet P(\text{pass over}) \bullet P(\text{detect}) = \frac{6362}{17{,}357} \bullet \frac{4806}{6362} \bullet 0.1 \approx 0.03$$

The sum of these probabilities is $0.17 + 0.03 = 0.20$

$$* \ * \ * \ * \ *$$

2.6 Sample response: After 100 games, you would expect the ratio of wins to games to be close to the theoretical probability of winning. Therefore, solving the equation below for *A* results in an approximate area for the ellipse:

$$P(\text{winning}) = \frac{A}{\text{area of square}}$$

$$\frac{12}{100} = \frac{A}{30 \text{ cm} \bullet 30 \text{ cm}}$$

$$A = 108 \text{ cm}^2$$

2.7 **a.** After two rolls, the following scores are possible: 40, 30, 20, 10 and 0 points.

b. The table below shows the ways in which each of the scores in Part **a** can be earned.

Total Score	Ways to Earn
40	(20,20)
30	(20,10), (10,20)
20	(20,0), (0,20), (10,10)
10	(10,0), (0,10)
0	(0,0)

c. The total area in which the ball can land is $75 \bullet 90 = 6750 \text{ cm}^2$. The area of each zone can be calculated as follows:

$$\text{Area}_{20 \text{ pts}} = \pi \bullet 10^2 \approx 314 \text{ cm}^2$$

$$\text{Area}_{10 \text{ pts}} = (\pi \bullet 30^2) - (\pi \bullet 10^2) \approx 2827 - 314 = 2513 \text{ cm}^2$$

$$\text{Area}_{0 \text{ pts}} = 75^2 - (\pi \bullet 30^2) + (15 \bullet 75)$$

$$\approx 5625 - 2827 + 1125 = 3923 \text{ cm}^2$$

The table below shows the probability of obtaining each possible score.

Total Score	Probability
40	$\left(\dfrac{314}{6750}\right)^2 \approx 0.002$
30	$2 \cdot \left(\dfrac{314}{6750}\right) \cdot \left(\dfrac{2513}{6750}\right) \approx 0.03$
20	$2 \cdot \left(\dfrac{314}{6750}\right) \cdot \left(\dfrac{3923}{6750}\right) + \left(\dfrac{2513}{6750}\right)^2 \approx 0.19$
10	$2 \cdot \left(\dfrac{2513}{6750}\right) \cdot \left(\dfrac{3923}{6750}\right) \approx 0.43$
0	$\left(\dfrac{3923}{6750}\right)^2 \approx 0.34$

    d. The following table shows the probability of winning each prize.

Total points	Prize	Probability
40	large stuffed animal	0.002
20–30	poster	0.22
10 or less	piece of candy	0.77

2.8 The table below shows the ways in which each possible score can be earned.

Total Score	Ways to Earn
200	(100,100)
150	(100,50), (50,100)
100	(100,0), (0,100), (50,50)
50	(0,50), (50,0)
0	(0,0)

    The board is divided into 11 slots of equal size. The probability of earning 100 points on one ball is $1/11 \approx 0.09$. The probability of earning 50 points on one ball is $4/11 \approx 0.36$. The probability of earning 0 points on one ball is $6/11 \approx 0.55$.

The table below shows the probability of obtaining each possible score.

Total Score	Probability
200	$\left(\dfrac{1}{11}\right)^{2} \approx 0.008$
150	$2 \bullet \left(\dfrac{1}{11}\right)\left(\dfrac{4}{11}\right) \approx 0.066$
100	$2 \bullet \left(\dfrac{1}{11}\right)\left(\dfrac{6}{11}\right) + \left(\dfrac{4}{11}\right)^{2} \approx 0.231$
50	$2 \bullet \left(\dfrac{4}{11}\right)\left(\dfrac{6}{11}\right) \approx 0.397$
0	$\left(\dfrac{6}{11}\right)^{2} \approx 0.298$

    a.   0.231

    b. $0.066 + 0.008 = 0.074$

    c. $0.298 + 0.397 = 0.695$

2.9   Answers will vary. Sample response: Rafael should search the desk first. The probability of finding the keys there is $0.7 \bullet 0.7 = 0.49$. This is greater than the probability of finding them either in his coat pocket (0.2) or the glove compartment (0.1). The probability of finding them on a second search of the desk, given that he didn't find them on the first search, is $0.7 \bullet 0.3 \bullet 0.7 = 0.147$. Since this is less than the probability of finding them in his coat pocket, he should search the coat.

If he still doesn't find the keys, then he should search the desk again, followed by the glove compartment. If they are still not found, he should keep searching the desk, since one search is enough to eliminate both his coat pocket and the glove compartment.

<p style="text-align:center">* * * * * * * * * *</p>

Answers to Summary Assessment (page 39)

Sample response: Each submarine traveling 10 km/hr for five days can cover 2403 km^2. This means that eight submarines can search $8 \cdot 2403 = 19,224$ km^2. This is 1867 km^2 more than the area of the entire training region.

If the entire training area is searched once, the following equations show the probabilities of each possible outcome.

$$P(\text{shallow}) \cdot P(\text{detected}) = 0.63 \cdot 0.7 = 0.441$$

$$P(\text{shallow}) \cdot P(\text{not detected}) = 0.63 \cdot 0.3 = 0.189$$

$$P(\text{deep}) \cdot P(\text{detected}) = 0.37 \cdot 0.1 = 0.037$$

$$P(\text{deep}) \cdot P(\text{not detected}) = 0.37 \cdot 0.9 = 0.333$$

The chances of finding the submarine after one complete search are $0.441 + 0.037 = 0.478$

In the time remaining after one complete search, the submarines can cover 1867 km^2. After one complete search, the probability that the sub is in shallow water but not found is 0.19. Given that the sub is not found on the first search, the chances of finding the lost sub in shallow water are:

$$\frac{1867}{10,996} \cdot 0.7 \approx 0.12$$

Therefore, the chances of finding it by searching a portion of the shallow water again are $0.19 \cdot 0.12 \approx 0.02$.

Similarly, after one complete search, the probability that the sub is in deep water but not found is 0.33. Given that it was not found on the first search, the chances of finding the sub in deep water are:

$$\frac{1867}{6361} \cdot 0.1 \approx 0.03$$

Therefore, the chances of finding it by searching a portion of the deep water again are $0.33 \cdot 0.03 \approx 0.01$.

Since the chances of finding the sub are better in shallow water, the team should concentrate their efforts there.

1. By tagging a bear with a radio collar, wildlife biologists can track the movements of the animal during the summer and locate its den during the winter. Imagine that one bear's radio collar emits a signal that is monitored by a hand-held receiver within a 2-km radius. From a map of the bear's summer range, the biologist knows that its den is somewhere within a square region measuring roughly 16 km on each side.

   a. Once a bear enters its den in the fall, it typically remains there all winter, in a state of hibernation. If the biologist begins the search for the bear in the middle of the square region, what is the probability of detecting the collar's radio signal the moment the receiver is turned on?

   b. After turning on the receiver, the biologist hikes due north from the middle of the region for 4 km. What is the probability of detecting the collar's radio signal during this phase of the search?

   c. Suppose that the square region were surrounded by a fence. Using a geometric model, determine if it would be reasonable for the biologist to search for the bear by riding along the outside of the fence.

2. Lin Yu is a guard for the Blackhall Jets basketball team. When she shoots free throws in a "one-and-one" situation, she makes the first shot 60% of the time. (In a one-and-one situation, a player who misses the first shot does not receive a second shot.) Given that Lin Yu makes her first shot, she makes the second shot 52% of the time.

   a. Draw a tree diagram to describe this situation.

   b. Draw a geometric region that models this situation.

   c. If Lin Yu is at the foul line in a one-and-one situation, what is the probability that she will make two shots?

3. At the local school fair, each class uses a carnival game as a charity fund-raiser. One class has developed a game involving a spinner, as shown in the diagram below. The green region represents 43% of the area of the spinner, the blue region represents 38% of the area, the yellow 14%, and the red 5%.

blue

green

55°

105°

yellow

65°

135°

red
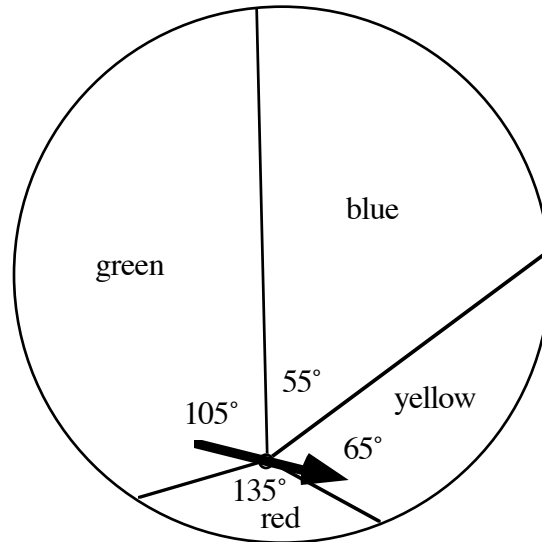
If the arrow lands on blue or yellow, the player wins a prize. If the arrow lands on red, the player loses. If the arrow lands on green, the player may spin a second time. The game ends after the second spin, regardless of the outcome.

a. What is the probability that a player will win a prize on the first spin?

b. What is the probability that a player will lose on the first spin?

c. What is the probability that a player will receive a second spin?

d. What is the probability that a player will receive a second spin and win a prize on that spin?

e. What is the probability that a player will receive a second spin but fail to win a prize on that spin?

Answers to Module Assessment

1.   **a**. The probability of detecting a radio signal the moment the receiver is turned on is about 4.9%. This can be calculated as follows:

$$\frac{\text{area of radio reception}}{\text{area of habitat}} = \frac{\pi \bullet 2^2}{16^2} \approx \frac{12.56}{256} \approx 0.049 \approx 4.9\%$$

   b. The probability of detecting a radio signal is about 11.2%. The region covered by the radio receiver is a square 4 km on each side with a semicircle of radius 2 km at each end, as shown below:
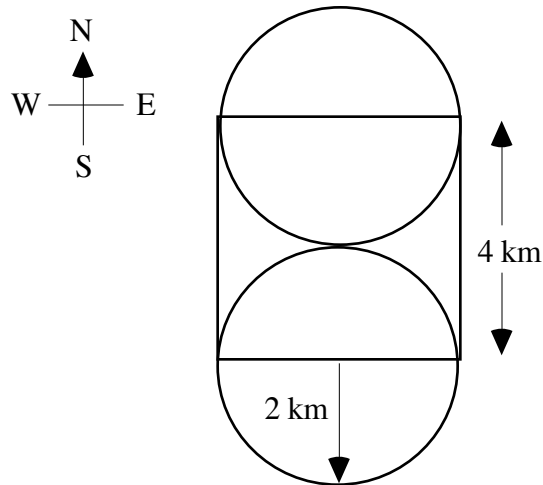
The probability can be calculated as follows:

$$\frac{\text{area of radio reception}}{\text{area of habitat}} = \frac{4^2 + \left(\pi \bullet 2^2\right)}{16^2} \approx \frac{28.56}{256} \approx 0.112 \approx 11.2\%$$

c. If the biologist rides along the fence line, the region covered by the radio receiver resembles a picture frame, as shown below:

fence
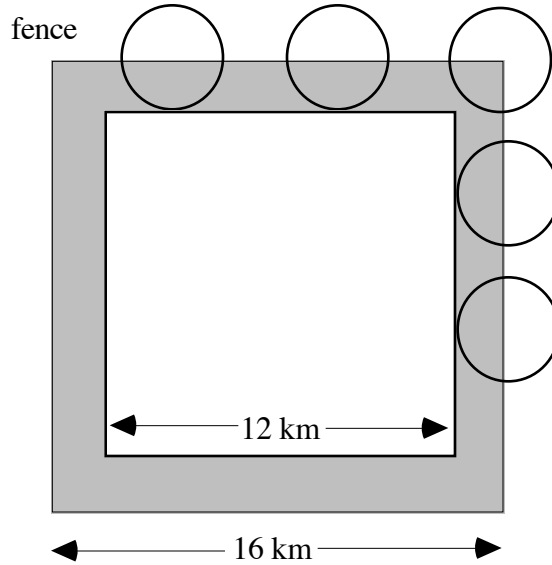
The probability of detecting a radio signal can be calculated by:

$$\frac{\text{area of radio reception}}{\text{area of habitat}} \approx \frac{16^2 - 12^2}{16^2} = \frac{112}{256} \approx 0.438 \approx 43.8\%$$

Since this means that the biologist has less than a 50% chance of finding the bear, this does not seem like a reasonable way to conduct the search.

2. **a.** Sample tree diagram:
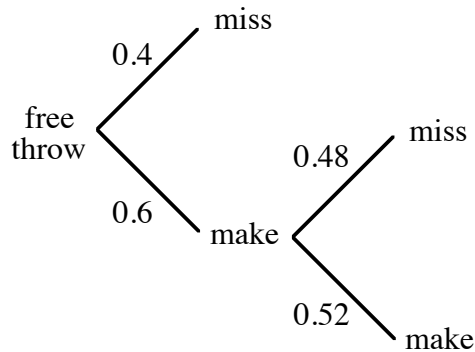
b. Answers will vary. Students should draw a geometric region divided into two smaller parts: one which represents 60% of the total area and another that represents 40% of the total area. The larger region should be further divided into two parts: one which represents 52% of the region and another which represents 48% of the region. Sample drawing:
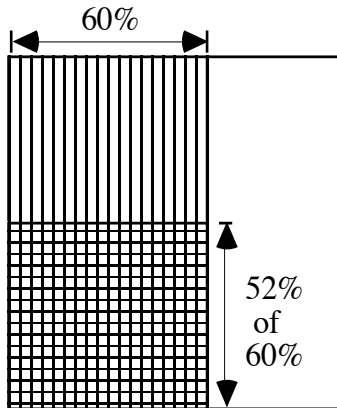
c. Lin Yu makes both shots of a one-and-one 31.2% of the time. This solution can be found by multiplying the individual probabilities: $0.6 \cdot 0.52 = 0.312$.

3. **a.** The probability that the spinner will land in either the blue or the yellow region is $(55 + 65)/360 \approx 0.333$.

b. The probability that the spinner will land in the red region is $135/360 = 0.375$.

c. The probability that the spinner will land in the green region is $105/360 \approx 0.292$.

d. The probability that a player will win a prize on the second spin is the probability of landing in the green region on the first spin and in either the blue or the yellow region on the second spin, or $0.292 \cdot 0.333 \approx 0.097$.

e. The probability of not winning a prize on the second spin is $1 - 0.333 = 0.67$. The probability of receiving a second spin but failing to win a prize is $0.292 \cdot 0.667 \approx 0.195$.

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Flashbacks

Activity 1

1.1 There are 25 students in a mathematics class. Eight of the students are female and sophomores, six of the students are male and sophomores, five are female and juniors, and six are male and juniors.

    a. What is the probability that a student selected at random from the class will be male?

    b. What is the probability that a student selected at random from the class will be a sophomore?

    c. What is the probability that a student selected at random from the class is a junior, given that the student is a female?

    d. What is the probability that a student selected at random from the class will be a female or a junior?

1.2 Andrea purchased tickets to two different raffles: one in which the prize is a calculator, and one in which the prize is a $200 gift certificate. There were 200 tickets sold in the first raffle and 500 tickets sold in the second raffle. What is the probability that Andrea will win both raffles?

1.3 Tonya is playing a dice game at a charity bazaar. The winnings for each roll of the fair, six-sided die are shown in the table below. How much can Tonya expect to win if she plays the game 100 times?

Roll	Winnings ($)
1	0.05
2	0.10
3	0.15
4	0.20
5	0.25
6	0.50

Activity 2

2.1 Solve each of the following proportions for x.

a. $\dfrac{x}{18} = \dfrac{37}{56}$

b. $\dfrac{52}{x} = \dfrac{96}{125}$

2.2 Consider an experiment that involves generating two random numbers (x,y) in the interval $[0, 10]$. The following graph shows the sample space for this experiment. The shaded region represents the event that $x + y \le 4$.
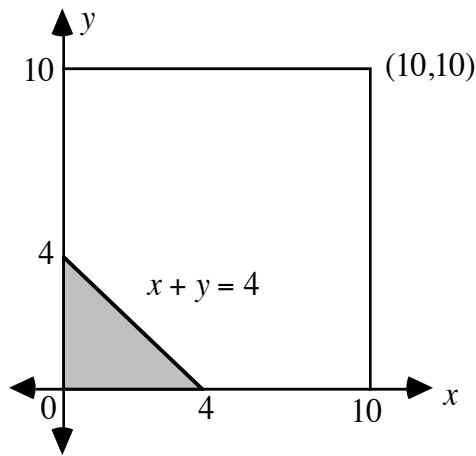
a. Determine the probability of the event that $x + y \le 4$.

b. 1. Draw a graph like the one above to illustrate the event that $x + y \le 16$.

   2. Determine the probability of this event.

c. Given that the value for x is 3, what is the probability that the point will fall in the shaded region?

Answers to Flashbacks

Activity 1

1.1   **a.** $P(\text{male}) = \dfrac{6+6}{25} = \dfrac{12}{25} = 0.48$

   b. $P(\text{sophomore}) = \dfrac{8+6}{25} = \dfrac{14}{25} = 0.56$

   c. $P(\text{junior}|\text{female}) = \dfrac{5}{13} \approx 0.38$

   d. $P(\text{female or junior}) = \dfrac{13+11}{25} - \dfrac{5}{25} = \dfrac{19}{25} = 0.76$

1.2   Since the two events are independent, the probability that Andrea wins both is:

$$\frac{1}{200} \cdot \frac{1}{500} = \frac{1}{100,\,000} = 0.00001$$

1.3   The expected value (in dollars) for one roll is:

$$\frac{1}{6}(0.05 + 0.10 + 0.15 + 0.20 + 0.25 + 0.50) \approx 0.21$$

After 100 rolls, therefore, Tonya's expected winnings would be $21.00.

Activity 2

    2.1   **a.** $x \approx 11.9$

            b. $x \approx 67.7$

    2.2   **a.** $P(x + y \leq 4) = \dfrac{0.5 \cdot 4 \cdot 4}{10 \cdot 10} = \dfrac{8}{100} = 0.08$

            b. **1.** Sample graph:
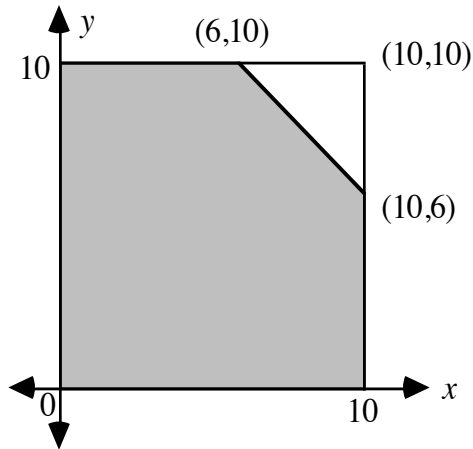
                 2. $P(x + y \leq 16) = \dfrac{(10 \cdot 10) - (0.5 \cdot 4 \cdot 4)}{10 \cdot 10} = \dfrac{92}{100} = 0.92$

            c. If x is 3, then y must be less than or equal to 1 for the point to fall in the shaded region. The probability that $y \leq 1$, where y is in the interval [0, 10], is 0.1. Therefore, the probability that $x + y \leq 4$, given that $x = 3$, is 0.1.

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How Long Is This Going to Take?

Teacher Guide

When building a house, how does a contractor estimate the time required to complete the job? In this module, you explore how to design a schedule for a complicated project.

Russ Killingsworth • Verne Schlepp • Dan West

How Long Is This Going to Take?

Overview

In this module, students investigate how various aspects of scheduling theory and bin packing can be used to plan complicated projects.

Objectives

In this module, students will:

- apply the nearest neighbor and cheapest link algorithms

- use network diagrams to analyze scheduling problems

- revise schedules given specific constraints

- use bin packing to analyze problems.

Prerequisites

For this module, students should know:

- the definitions of a graph, a path, and a circuit

- how to identify Hamiltonian circuits

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	2	1	1	7

Materials Required

Materials	1	2	3	Summary Assessment
construction paper		X	X	X
ruler		X	X	X
scissors		X	X	X
graph paper		X	X	X

How Long Is This Going to Take?

Introduction

In this module, students explore algorithms that can be used to make a construction project more efficient. **Note:** Problem situations that involve the evaluation of many possibilities are typically analyzed using computers. In this module, the numbers of possibilities are limited to allow students to familiarize themselves with the appropriate algorithms.

Activity 1

Students examine strategies for finding a reasonably efficient Hamiltonian circuit for a weighted graph. As alternatives to evaluating all possible Hamiltonian circuits, they investigate the nearest neighbor and cheapest link algorithms.

Materials List

• none

Exploration 1

a. Answers will vary. One possible route starts at the building site, proceeds to United Drywall, Empire Plumbing, and J&C Hardware, then returns to the site.

b. Using the fundamental counting principle, the number of possible Hamiltonian circuits for a complete graph with four vertices is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

c. If students distinguish between routes that differ only in the direction the truck must travel, they should identify six circuits, as shown below.

B–J–E–U–B
or
B–U–E–J–B

B–U–J–E–B
or
B–E–J–U–B

B–J–U–E–B
or
B–E–U–J–B

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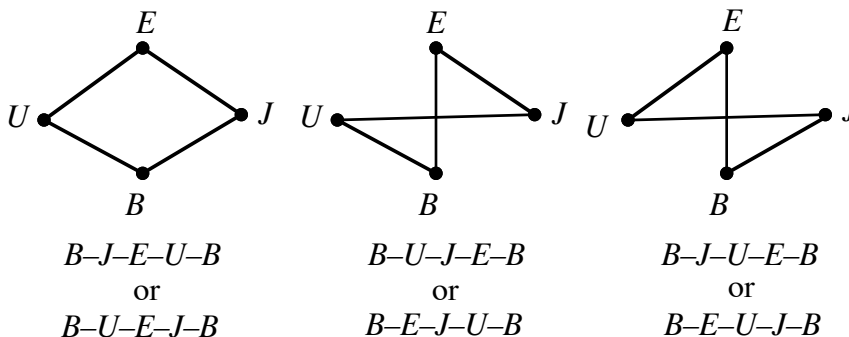
d. The shortest routes are *B–J–E–U–B* or *B–U–E–J–B* and *B–U–J–E–B* or *B–E–J–U–B*. These all have a total distance of 32.8 km.

e. The minimum cost is $0.18 \cdot 32.8 \approx \$5.90$.

Discussion 1 (page 46)

a. The graph in Figure **1** is a complete graph since each pair of vertices is connected by exactly one edge.

b. Sample response: One-way streets and heavy traffic patterns could make route direction important. The order in which materials should be loaded may also be a consideration.

Exploration 2 (page 46)

a. Using the fundamental counting principle, the number of possible Hamiltonian circuits for a complete graph with five vertices is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

b. The nearest neighbor algorithm results in the following circuit: *B–E–A–U–J–B*. This route is 41.2 km long.

c. Using the route determined in Part **b**, the cost is approximately $7.42.

d. Beginning the nearest neighbor algorithm with each vertex produces the following routes and approximate costs:

 1. *U–B–E–A–J–U*: 45.6 km ($8.21)

 2. *A–U–B–E–J–A*: 38 km ($6.84)

 3. *E–B–U–A–J–E*: 38 km ($6.84)

 4. *J–E–B–U–A–J*: 38 km ($6.84)

Discussion 2 (page 46)

a. Sample response: The nearest neighbor algorithm does not always produce a circuit that uses the same edges. In Exploration **2**, starting at different vertices generated circuits that use different sets of edges.

b. Sample response: Yes. It is possible to use this route when beginning at any vertex since it is a Hamiltonian circuit.

c. The nearest neighbor algorithm is quick and easy to use. However, it does not always identify the shortest possible route.

d. 1. Sample response: There are 5 times as many circuits possible in a complete graph with 5 vertices than in a complete graph with 4 vertices.

 2. The number of possible routes for visiting 20 businesses is extremely large: $(20 - 1)! \approx 1.2 \cdot 10^{17}$. Determining the shortest route by examining every one would be virtually impossible.

e. **1.** The figure below shows the construction of a route using the cheapest link algorithm. This route is 38 km long.
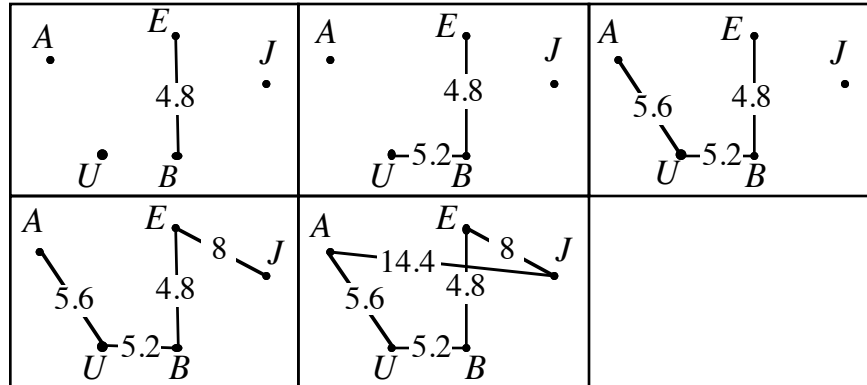
2. The route found using the cheapest link algorithm has the same length as three of the routes found using the nearest neighbor algorithm.

Teacher Note

It is important to emphasize that neither the nearest neighbor nor the cheapest link algorithm guarantees the shortest circuit. These algorithms simply provide a means to generate reasonably short circuits without checking every possible circuit.

Assignment   (page 50)

*1.1 **a.** Answers will vary, depending on the algorithm chosen. The cheapest link algorithm produces the circuit: *B–E–J–A–U–I–B*. This route is 43.2 km long.

b. Answers will vary, depending on the route. Using the sample response above, the cost is approximately $7.78.

1.2 **a.** The nearest neighbor algorithm produces the route *W–S–H–E–W*, which is 458 km long.

b. Using the cheapest link algorithm results in the route *W–H–E–S–W*, which is 434 km long.

1.3 **a.** The following sample response begins and ends in Seattle: *S–GF–SLC–D–P–S* (5184 km).

b. Beginning at each of the five cities, the nearest neighbor algorithm produces the following routes: *S–P–GF–SLC–D–S* (5411 km); *P–S–GF–SLC–D–P* (5184 km); *SLC–D–GF–S–P–SLC*(4691 km); *D–SLC–GF–S–P–D*(5184km); and *GF–SLC–D–P–S–GF* (5184 km).

c. Answers may vary. Sample response: In this case, the shortest of the routes identified in Parts **a** and **b** is *SLC–D–GF–S–P–SLC*, which is 4691 km long. This route was identified by the nearest neighbor algorithm. However, this may or may not be the shortest possible route. To make this determination, you would have to check all possible circuits.

*1.4 a–b. Using the nearest neighbor algorithm, the circuits vary in length, depending on the beginning vertex. Beginning at Charleston or Charlotte results in a route 1252 km long. Beginning at Chesterfield produces a route 1177 km long. Beginning at either Columbia or Raleigh, the route is 1140 km long. The cheapest link algorithm produces a circuit 1140 km long.

<center>* * * * *</center>

1.5 a. Beginning at vertex *A*, the nearest neighbor algorithm produces the circuit *A–B–E–C–D–A*, which is 25 units long.

b. The cheapest link algorithm produces the circuit *A–C–E–B–D–A*, which is 25 units long.

1.6 a. Sample graph:
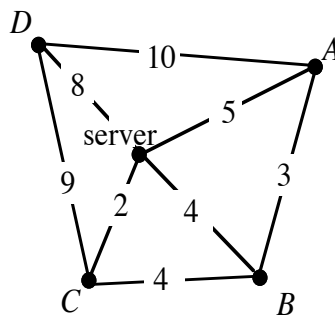
b. Sample response: Using the cheapest link algorithm, one possible circuit is *S–C–B–A–D–S*. This would require at least 27 m of cable.

1.7 a. Sample response: No. The map does not represent a complete graph because each vertex is not connected to every other vertex. For example, there is no edge connecting Boise and Great Falls.

b. Sample response: One route that begins and ends in Seattle is *S–P–B–SLC–D–GF–S* (4691 km).

c. Beginning at Seattle, the nearest neighbor algorithm produces the following route: *S–P–B–SLC–D–GF–S* (4691 km).

<center>* * * * * * * * * *</center>

Activity 2

In this activity, students begin planning a schedule for the construction project.

Materials List

- construction paper (several sheets per student)
- graph paper (several sheets per student)
- scissors (one pair per student)
- ruler (one per student)

Exploration

Students use a paper model to analyze a scheduling problem. **Note:** The sample responses given below are not drawn to scale.

a–b. The maximum completion time is 148 hr.

c–d. The minimum completion time is 104 hr. Sample sketch:
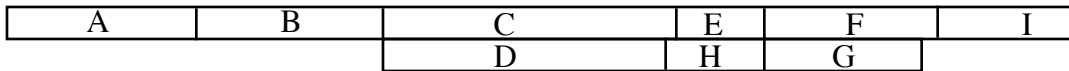
A	B	C	E	F	I
		D	H	G	

d.

e. **1.** Sample network diagram:
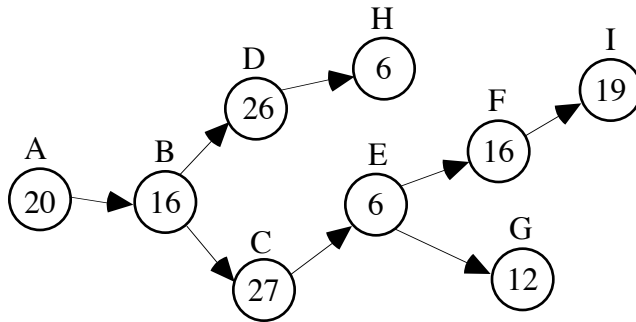
2. The critical path is A–B–C–E–F–I (104 hr).

Discussion

a. Sample response: After I lined up all the strips of paper with the appropriate prerequisites, I added up the time it took to complete the longest row of tasks.

b. Students should observe that different schedules may produce the same completion time.

c. Students should note that the paper model and the network diagram are two ways of representing the same situation.

d. Sample response: The minimum time and the length of the critical path are both 104 hr. This occurs because the critical path describes the time required to complete the longest set of tasks linked by prerequisites. Until these tasks are done, the job cannot be finished.

e. The lack of arrows indicates that Q is not a prerequisite task for T and that T is not a prerequisite for Q.

Assignment

*2.1 **a.** Sample diagram:
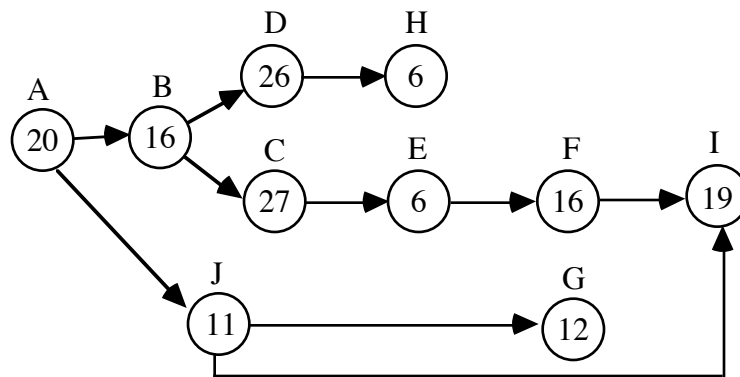
b. The minimum completion time is still 104 hr.

*2.2 The critical path is C–H–J–K. It is 20 units long.

2.3 Answers will vary, depending on how students assign prerequisites to each task. The critical path should include (at the least) unloading passengers, cleaning the plane interior, and loading passengers. This results in a minimum time of 59 min. Sample schedule:

2.4 The following schedule requires 67 hr to complete.

2.5　**a.** Shortening each task by 1 hr reduces the length of the critical path by 4 hr. The project can be completed in 63 hr.

　　b. Sample response: Removing J as a prerequisite for G does not change the schedule since the schedule outlined in Problem **2.4** has task J finished 2 hr before G starts. The only thing that would change is the order in which Crew 2 might do the tasks. Since adding one person to each crew allows the tasks to be completed in 63 hr, the schedule in Part **a** would save more time.

2.6　**a.** The minimum time required for three crews is 52 hr. This can be accomplished using the following schedule:

Crew 1	A		D	F	H	
Crew 2	C		E		G	
Crew 3	B		J		I	

　　b. Sample response: There is no way to do the job in 45 hr. Even if another crew does task G, the critical path is 48 hr long and cannot be reduced. (Task H must occur after tasks A, D, E, and F and task I must occur after task F.)

Crew 1	A		D	F	H	
Crew 2	C		E		I	
Crew 3	B		J			
Crew 4	G					

＊ ＊ ＊ ＊ ＊

2.7　**a.** Sample response: In the picture below, I1 represents sewing the inside of the first coat, O1 represents sewing the outside of the first coat and T1 represents sewing the first coat together. The critical path is O1–T1–T2–T3–T4–T5. The total time to make 5 coats is 230 min.
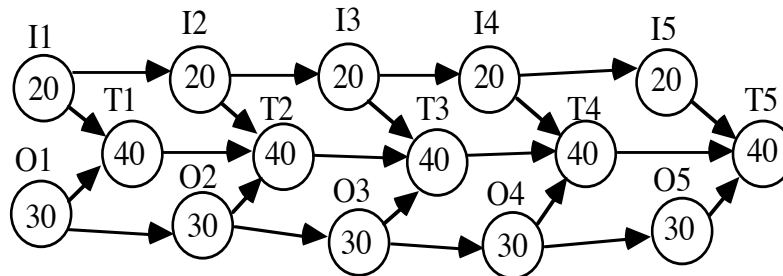
I: sew inside of coat
O: sew outside of coat
T: sew coat together

　　b. If the time to sew the two parts of a coat together is reduced to 20 min, the critical path becomes O1–O2–O3–O4–O5–T5. The total time to finish 5 coats is 170 min.

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*2.8   **a.** The mail cannot be processed on time because Kodjo's assignments will take 125 min.

b. Using the following schedule, the three workers can finish the orders in 109 min. With a 10 min break, the workers would still finish the orders within the 2 hours required.

A	B	C	E	G
K		F	D	M
L		J	H	I

2.9   **a.** There are 27,230 copies ordered. If the copy service works without a break, they will finish in approximately 7 hr and 34 min.

b. Sample response: Assign orders A, E, F, G, I, and K to the first copier, for a total of 13,600 copies. Assign orders B, C, D, H, and J to the second copier, for a total of 13,630 copies. If the copy service works without a break, they will finish in approximately 3 hr, 47 min.

c. Sample response: Assuming that each machine is ready for its first job, it will take a minimum of 4 hr, 32 min to complete the copying. This can be accomplished by assigning orders A, B, C, D, and J to one copier and orders E, F, G, H, I, and K to the other copier.

* * * * * * * * * *

(page 60)

Activity 3

In this activity, students explore bin packing in various settings.

Materials List

- graph paper
- construction paper (optional)
- scissors (optional)
- ruler (optional)

Teacher Note

Students can complete the exploration by diagramming the timbers on graph paper. Alternately, you may wish to allow students to use the paper-strip method as in previous activities.

Exploration

a. **1.** The project requires a total of 130 ft of timbers.

2. Sample response: No. There are two 9-ft timbers specified in Table **2**, but no 1-ft timbers.

b. The job requires 14 timbers. This results in 10 ft of waste.

c. Sample diagram:

10		
10		
10		
9		
9		
8	2	
7	3	
7	3	
7	2	
6	4	
6	4	
5	4	
4	4	2
4		

Discussion

a. Answers may vary. A minimum of 14 timbers are required.

b. Sample response: The locations of the cuts on each timber are different.

c. Sample response: Placing the longest timbers into "bins" first, then fitting the shorter timbers into the available spaces, seems to work efficiently.

d. The total length of fourteen 10-ft timbers is 140 ft. This is 10 ft more than the total length of all the timbers in Table **2**.

Assignment (page 61)

3.1 **a.** A minimum of nine 16-ft timbers are required. Sample diagram:

10		6	
10		6	
10		4	2
9		7	
9		7	
8		5	3
7		4	3	2
4	4	4	4
2			

b. By cutting the 9 timbers as shown above, there is 14 ft of waste.

3.2 **a.** One possible combination consists of six 10-ft timbers and six 12-ft timbers. Sample diagram:

10		
10		
10		
9	3	
9	3	
8	2	
7	4	
7	4	
7	5	
6	4	2
6	4	
2	4	4

b. By cutting the timbers as shown above, there is 2 ft of waste.

c. The total length of the order is $6 \cdot 10 \text{ ft} + 6 \cdot 12 \text{ ft} = 132 \text{ ft}$. This results in a cost of $132 \cdot \$0.60 = \$79.20$.

*3.3   A minimum of eighteen 12-ft lengths are needed for the deck. This results in 12 ft of waste. Using any of the other lengths results in more waste. (Using 16-ft lengths, there is 20 ft of waste. For 20-ft lengths, there is 16 ft of waste. Using 8-ft lengths, there is 60 ft of waste.) Sample diagram:

7	3
7	3
7	3
7	3
7	3
7	3

6	6
6	6
6	6
6	6
6	6
6	6

3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3
3	3	3	3

* * * * *

3.4   **a.** This can be analyzed as a bin-packing problem. In this case, the bin is the 6-hr workday in which Jake can print shirts. The items are the hours required to print each shirt color. Sample schedule:

black: 5 hr	
gray: 5 hr	
navy: 4.17 hr	purple: 1.67 hr
red: 3.33 hr	yellow: 2.5 hr
orange: 2.5 hr	green: 1.67 hr

b. Yes, Jake can finish in time. The pink shirts will take 0.83 hr to print, while the teal shirts will take 1.25 hr. They can be fit into the sample schedule as shown below:

black	pink
gray	
navy	purple
red	yellow
orange	green	teal

3.5   A minimum of seven 75-sec commercial breaks are needed. One possible schedule is shown below. **Note:** To find the minimum number of commercials breaks, students may start by trying to pack the ads into 90-sec bins. However, these ads do not fit evenly in 90-sec intervals.

60				15
60				15
45			30	
45			30	
45			30	
45			30	
15	15	15	15	15

3.6   **a.** Using the next-fit algorithm, it will take twenty 10-ft timbers to provide the materials in the table.

10				
10				
9				
9				
9				
8				
8				
7				
7				
7				
7				
6				
6				
6				
6				
5		5		
4		4		
4		4		
3		3	2	2
2	2	2	2	2

b. Using the first-fit algorithm, it will take seventeen 10-ft timbers to provide the materials in the table.

10			
10			
9			▓
9			▓
9			▓
8		2	
8		2	
7		3	
7		3	
7		2	▓
7		2	▓
6		4	
6		4	
6		4	
6		4	
5		5	
2	2	2	▓

c. Sample response: The next-fit algorithm appears to result in more waste, since bins may be closed before they are close to full. However, since it is very simple to apply, it may save time.

* * * * * * * * * *

Research Project (page 63)

In the spanning tree algorithm, the "cheapest" edges on a graph are chosen until it is possible to get from every vertex to another along at least one path. If any circuits have been created during the selection of edges, the most costly edge in the circuit is eliminated. The resulting graph is a minimum spanning tree and represents the least-cost network that connects all of the vertices.

This algorithm is used by communications companies to determine how to connect various cities via telephone lines. It also may be used by transportation companies to determine the best way to connect cities using rail delivery, or by computer system operators to minimize the cost of constructing a network.

Answers to Summary Assessment

1.   **a.** Sample graph:
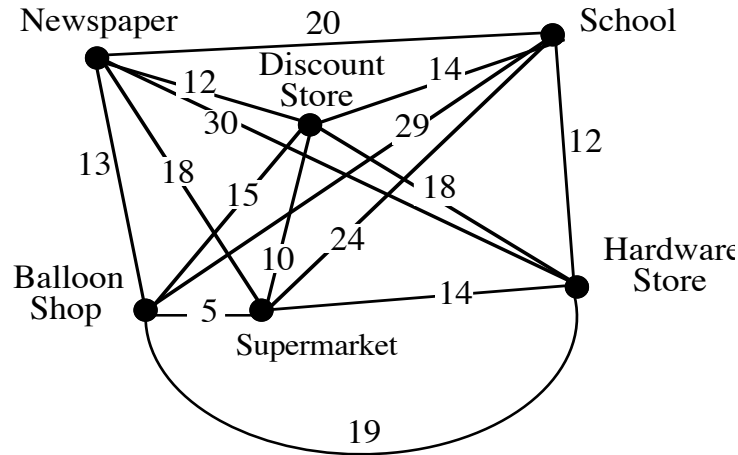
      b. The cheapest link algorithm produces the following route:
School–Newspaper–Discount Store–Supermarket–Balloon Shop–
Hardware Store–School.

2.   **a.** Answers will vary, depending on how students interpret the
necessary order of tasks. Sample response:

	Task	Prerequisites
A	inflate 300 balloons	none
B	build frames for pyramids	none
C	decorate pyramids with crepe paper	B
D	build large cardboard sphinx	none
E	apply gold foil to sphinx	D
F	decorate ceiling with balloons and streamers	A
G	drape plastic sheeting at end of gym	none
H	cover bleachers with plastic sheeting	none
I	decorate stage with balloons and install sphinx	E, A
J	draw and cut out designs for walls	none
K	build palm trees	none
L	set up platform for disc jockey	none
M	decorate walls and plastic sheeting	J, G, H
N	set up palm trees on floor	K
O	set up snack bar	I
P	set up tables and chairs	I

      b. Sample response: It is not possible to finish the decorations in
time using two crews. Only 22 hr are available for decorating the
gym. Two crews, working every available hour, can only
complete 44 hr of tasks. The tasks listed in the table require a total
of 47 hr.

c. Answers will vary. However, students should schedule task J prior to Wednesday, assign each crew less than 22 hr of tasks, and follow their list of prerequisites. The following sample schedule assumes that both J and K are completed before Wednesday:

A	D	E	G	H	I	M	O
B	C		F	L	N		P

3.   The sum of the lengths of the songs is 3630 sec. Since 3630/300 = 12.1, the disc jockey will need at least 13 time slots. The songs can be arranged in 13 slots as shown below.

270	
240	60
240	60
240	60
210	90
210	90
210	90
180	120
180	120
180	120
180	
180	
150	150

1. Two students are planning a barbecue for their 20-member club. The menu features potato salad, baked beans, barbecued ribs, home-made apple pie, and iced tea. The students plan to prepare the entire meal the same day so that everything will be fresh.

   The apples will take 30 min to peel. The pies must bake for 45 min. The pies must be finished before the potato salad is mixed or the beans placed in the oven. It will take 0.75 hr to peel the potatoes and 0.5 hr to boil them. Mixing the potato salad will take another hour. The beans will take about 1.5 hr to bake. The ribs take 1 hr to barbecue and must be started after the pies, beans, and potato salad are ready. The iced tea takes 4 hr to brew and cool.

   a. Draw a network diagram that illustrates the prerequisites for each task.

   b. If the students plan to serve the meal at 6:00 P.M., when should they start their preparations?

2. Imagine that you are restoring an antique car. The seven tasks remaining, along with their prerequisites, are listed in the table below.

	Task	Days	Prerequisites
A	rebuild engine and drive train	5	none
B	sand and paint body	6	none
C	reassemble body	8	B
D	install glass	1	C
E	install engine and drive train	3	C
F	mount tires and wheels	1	E
G	upholster interior	5	C

   a. Assuming that you can work on only one task at a time, design a schedule that allows you to complete this project in the minimum number of days.

   b. Suppose that six friends offer to help you work on the car. Although the additional help does not shorten the time required for each task, it does allow more than one task to be done at a time. Determine the minimum number of days required to complete the project with your friends' help.

3.   Jillian's list of errands requires her to drive to school, the post office, the grocery store, and the mall before returning home again. The following table describes each stop's position in relation to her home.

Destination	Distance East or West	Distance North or South
Post office	19 blocks east	8 blocks north
School	14 blocks east	3 blocks north
Grocery store	13 blocks east	1 block north
Mall	18 blocks east	1 block south

a. The streets in Jillian's town form a rectangular grid. Draw a map of this situation, including Jillian's home and each destination shown in the table.

b. Draw a complete graph to model your map. Label each edge with the shortest distance between the two locations. **Note:** When determining distances, assume that it is not possible to drive diagonally across a block.

c. Jillian's gas tank is almost empty and she cannot afford to buy more until next week. Determine a short route that visits all the destinations before returning home. Describe how you selected this route.

4.   Joaquin is remodeling his house. The lumber required for his remodeling plan is shown in the following table.

Length (feet)	Quantity
8	7
5	5
4	2
3	8
2	5

When Joaquin arrives at the lumber yard, he finds that they only have 14-ft boards in stock. Determine the number of boards he should purchase, how he should cut each board, and the amount of waste that will result.

Answers to Module Assessment

1. **a.** Sample diagram:
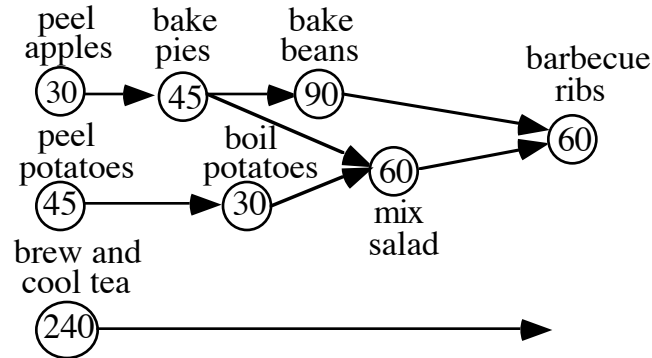
b. The students should begin no later than 2:00 P.M.

2. **a.** Working on each task in sequence, the restoration requires 29 days.

b. With the help of friends, the restoration requires only 19 days. Sample schedule:
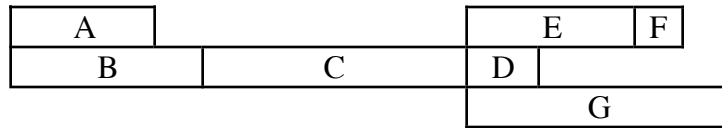
3. **a.** Sample map:

b. Sample graph:

85
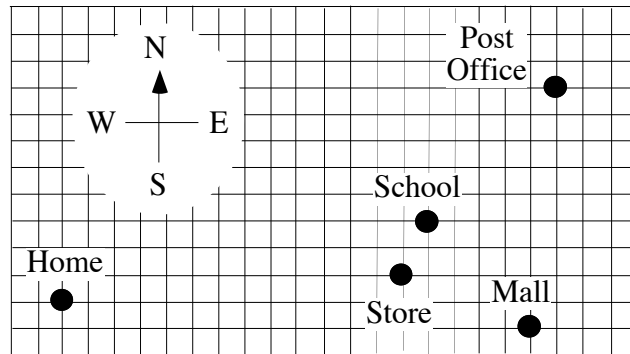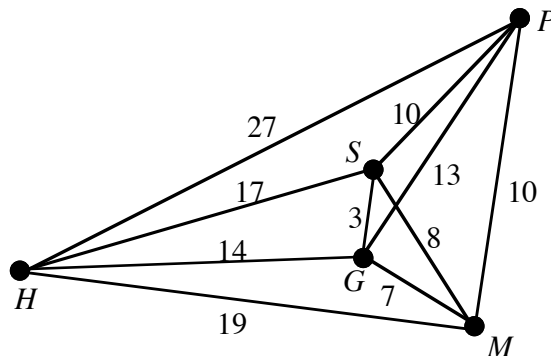
c. Answers will vary, depending on the technique used to find a route. The cheapest link algorithm produces the following route: *H–S–G–M–P–H* (64 blocks).

The shortest route is *H–G–S–P–M–H* (56 blocks).

4. Joaquin needs nine 14-ft boards. This results in 3 ft of waste, as shown in the diagram below.

8		4	2
8		4	2
8		3	3
8		3	3
8		3	3
8		3	3
8		5	
5	5	2	2
5	5	2	

Selected References

Consortium for Mathematics and its Applications (COMAP). *For All Practical Purposes: Introduction to Contemporary Mathematics*. New York: W. H. Freeman and Co., 1991.

Graham, R. L. "The Combinatorial Mathematics of Scheduling." *Scientific American* 238 (March 1978): 124–133.

Steen, L. A., ed. *Mathematics Today*. New York: Springer-Verlag, 1978.

Flashbacks

Activity 1

1.1 Draw a graph with five vertices in which each vertex is connected by edges to two other vertices.

1.2 In the following weighted graph, the value associated with each edge represents a distance in kilometers.
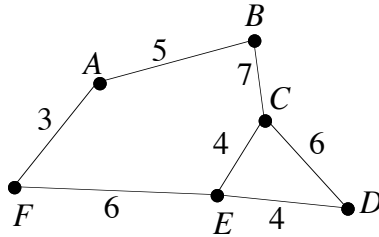
a. Find the shortest path from A to C.

b. Which path is shorter, A–B–C–E–D or A–F–E–D?

Activity 2

2.1 Identify a reasonably efficient Hamiltonian circuit in the following weighted graph that begins at vertex G:
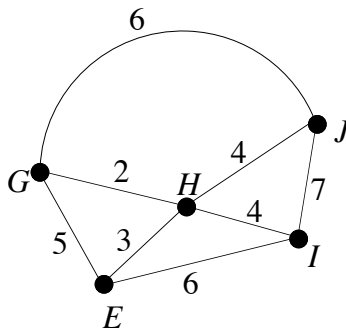
2.2 Given the graph in Flashback **2.1**, find the longest path from G to J that does not visit any vertex more than once.

Activity 3

3.1 If 1 bus can carry 42 passengers, how many buses are needed to carry 100 passengers?

3.2 Describe the possible combinations of 2-m and 3-m lengths of wire that can be cut from 14 m of wire.

Answers to Flashbacks

Activity 1

1.1   Sample graph:
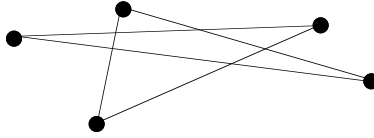
1.2   **a.** *A–B–C*

      b. *A–F–E–D*

Activity 2

2.1   Answers may vary. Sample responses: *G–J–I–E–H–G* (24 units) or *G–H–J–I–E–G* (24 units).

2.2   *G–E–I–H–J*

Activity 3

3.1   3 buses

3.2   The following table shows the possible combinations of lengths.

Two Meters	Three Meters
7	0
5	1
4	2
2	3
1	4

A Ride with Markov

Teacher
Guide
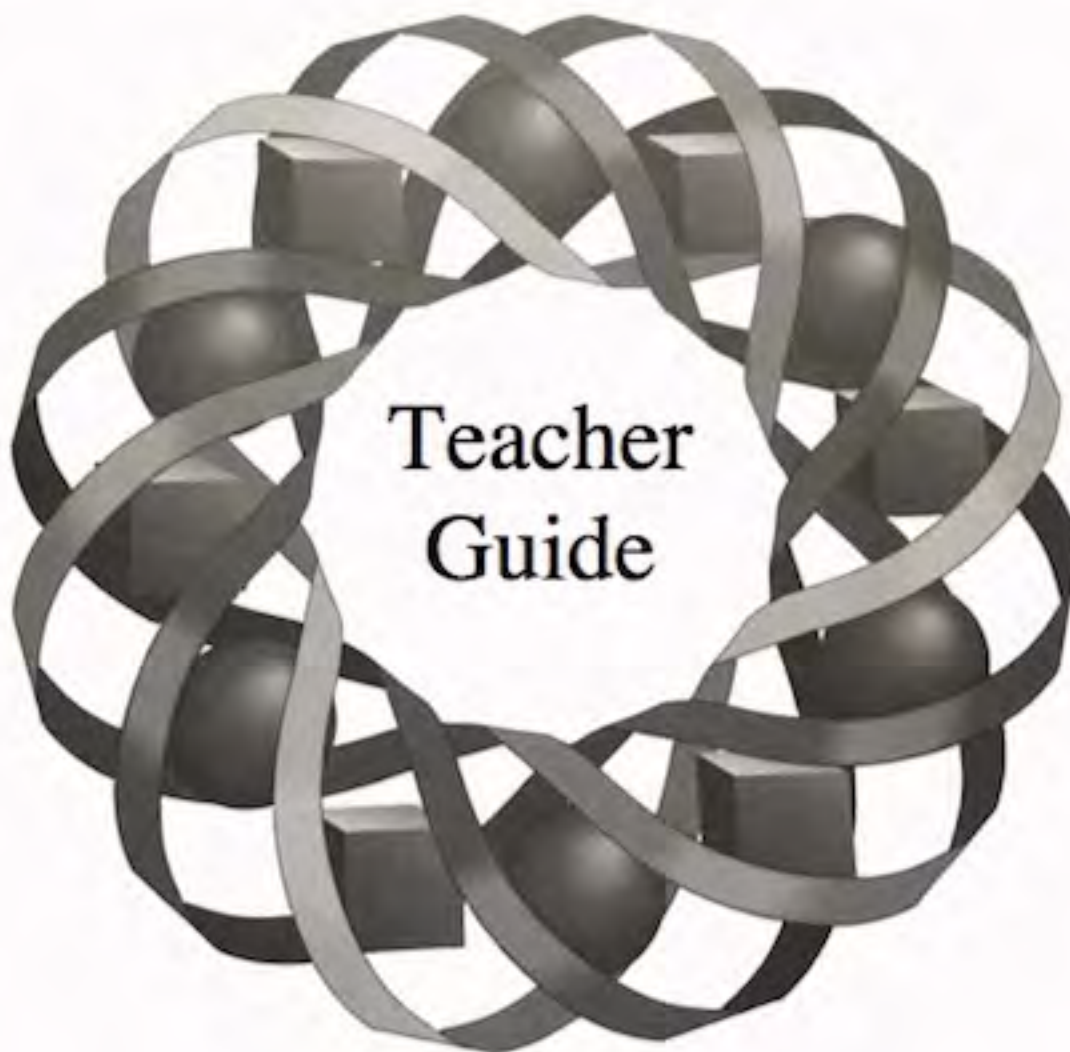
At Boards Incorporated, the first priority is giving customers a quality ride—on their skateboards, of course. In this module, you examine how a process developed by Russian mathematician Andrei Markov can help the company control quality.

Douglas H. Mack • Laurie Paladichuk • Arthur R. Perleberg

A Ride with Markov

Overview

In this module, students use quality control in a manufacturing process to investigate Markov chains. They then use the Markov process to predict trends.

Objectives

In this module, students will:

- determine the mean and standard deviation of a binomial experiment
- use the results of a binomial experiment to approximate a normal curve
- use the 68–95–99.7 rule of a normal distribution
- create and interpret transition diagrams
- calculate the probabilities of transitions
- identify Markov chains and the states of Markov chains
- create and interpret transition matrices and state vectors
- identify and use stable state matrices and vectors.

Prerequisites

For this module, students should know:

- how to create histograms
- how to use tree diagrams to determine probabilities
- the characteristics of a binomial experiment
- how to calculate conditional probabilities
- how to use the binomial probability formula
- how to create a probability distribution for a random variable
- the concepts of directed graphs, vertices, and edges
- how to create and multiply matrices.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	3	1	10

Materials Required

Activity	1	2	3	4	Summary Assessment
dry beans	X				
markers	X				

Technology

Software	Activity				
	1	2	3	4	Summary Assessment
matrix manipulator			X	X	X

A Ride with Markov

Introduction

The introduction describes a fictitious company that uses sampling in its quality-control process. This context is used throughout the module. You may wish to invite a quality-control professional to discuss the types of sampling used and the rate of defective items that companies are willing to accept in a manufacturing process.

Activity 1

In this activity, students use a binomial experiment to model a sampling technique. They find the mean and standard deviation of this experiment and use the results to gain insights into the normal curve. They are introduced to the 68–95–99.7 rule for a normal distribution.

Materials List

- dry beans (one population of approximately 150 per group)
- marker (one per group)

Exploration

a. Students create a population of 150 beans in which 20% are marked to represent defective items.

b. The expected number of defective items in a sample of 30 is $30 \bullet 0.20 = 6$.

c–d. Students take 50 samples of size 30 and create a frequency histogram of the results. Sample histogram:
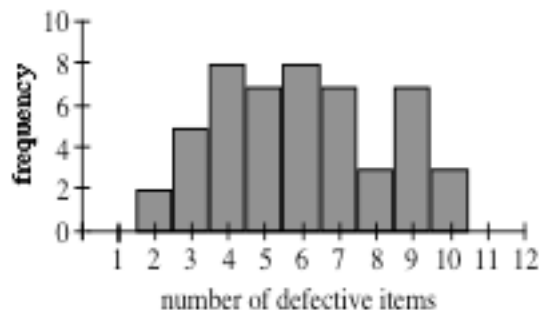
e. Answers will vary. The mean for the 50 samples displayed in the histogram given in Part **d** is approximately 6.

f.   The mean for the sample classroom data, shown in the histogram below, is approximately 5.97.
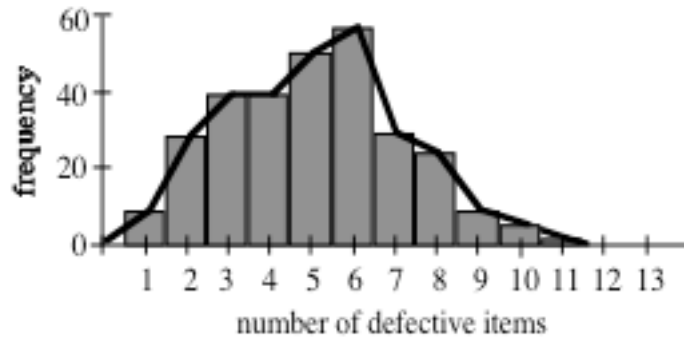
g.   The mean of the binomial distribution is $30 \cdot 0.2 = 6$. The standard deviation is $\sigma = \sqrt{30 \cdot 0.2 \cdot 0.8} \approx 2.19$.

h.   Sample graph:
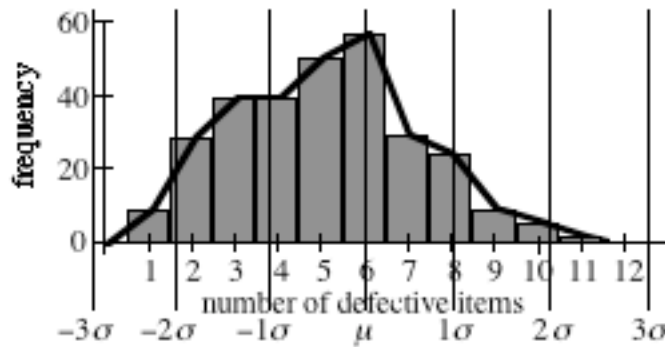
i.   Students may wish to use grid paper to determine the approximate area in each region of the frequency polygon. The responses below were determined using the sample data given in Part **f**.

1. 58%

2. 82%

3. 100%

Teacher Note

As the number of samples increases, the shape of the frequency polygon should more closely approximate a normal curve. The table below shows data collected for 300 samples of size 30 from a population with a 20% defective rate. You may wish to ask students to add these frequencies to the class data.

No. of Defectives	Frequency	No. of Defectives	Frequency
0	1	8	25
1	9	9	9
2	29	10	6
3	40	11	2
4	40	12	0
5	51	13	0
6	58	⋮	⋮
7	30	30	0

Discussion

(page 74)

a. Sample response: When a relatively small number of items is removed from the population, the probability of a success on each trial remains approximately the same.

b. **1.** Sample response: The mean of the 50 samples was close to the mean of the binomial distribution for 30 trials and a 20% probability of success.

    2. Sample response: It also is close to the mean of the distribution.

    3. Sample response: The more samples that are collected, the nearer the mean of the samples should get to the mean of the distribution.

c. Sample response: The data tended to group near the mean. The bars nearer the mean are taller than the bars far from the mean.

d. **1.** Sample response: If the defective rate had been 40%, the mean would have been 12 defective items. I would expect the histogram to be centered about 12 rather than 6.

    2. Sample response: It would look more symmetrical about the mean and the frequency polygon would be smoother.

e. Sample response: The shape of the histogram roughly resembles the shape of the normal curve, but it is not symmetric.

f. Answers will vary, depending on classroom data. The following sample responses are based on the data given in the exploration.

  1. Approximately 58% of the samples fell in this interval. This is less than predicted by the 68–95–99.7 rule.

  2. Approximately 82% of the samples fell in this interval. This also is less than predicted by the 68–95–99.7 rule.

  3. Approximately 100% of the samples fell in this interval. This is the same as predicted by the 68–95–99.7 rule.

g.
  1. 50%

  2. 15.85%

  3. 15.85%

  4. 99.85%

  5. 84%

Assignment (page 76)

1.1 **a.** This is a binomial experiment.

  b. This is not a binomial experiment because it involves sampling without replacement. A binomial experiment would not be a reasonable model because the sample size is large relative to the population size. The change in probabilities between trials would be significant.

  c. This is not a binomial experiment because there are more than two possible outcomes.

  d. If sampling is done with replacement, this is a binomial experiment. If sampling is done without replacement, this is not a binomial experiment. In this case, a binomial experiment would be a reasonable model because the sample size is very small when compared to the population size.

  e. This is a binomial experiment.

1.2 **a.** $[\mu + \sigma, \ \mu - \sigma]$ or $[6 - 2.19, 6 + 2.19] = [3.81, 8.19] \approx [4, 8]$

  b. 3

  c. 9

  d. The quality-control process would stay in the ordinary state if 4 to 8 defectives are found in the sample. It moves to the heightened state if the number of defectives is 9 or greater. It moves to the relaxed state if the number of defectives is 3 or fewer.

1.3   **a.** 20

    b. 24

    c. 16%

    d. The quality-control process would stay in the heightened state if the number of defectives is 25 or greater. It moves back to the ordinary state if the number of defectives is 24 or fewer.

1.4   **a.** 1

    b. 2

    c. 84%

    d. The quality-control process would stay in the relaxed state if the number of defectives is 0 or 1. It moves back to the ordinary state if the number of defectives is 2 or more.

＊ ＊ ＊ ＊ ＊

1.5   **a.** Sample response: Yes. These experiments represent binomial experiments where n is the number of tosses, the probability of a success is 0.4, and the probability of a failure is $(1 - 0.4) = 0.6$.

    b. 1. $\mu = 8$; $\sigma \approx 2.19$

        2. $\mu = 16$; $\sigma \approx 3.10$

        3. $\mu = 24$; $\sigma \approx 3.79$

    c. When the probability of obtaining a head is 0.2, the means are 4, 8, and 12, respectively. The standard deviations are approximately 1.79, 2.53, and 3.10, respectively.

1.6   Since the top 2.5% of the grades in a normal distribution are those that are more than 2 standard deviations above the mean, the lowest score that would result in a scholarship is $72 + 2 \bullet 9 = 72 + 18 = 90$.

＊ ＊ ＊ ＊ ＊ ＊ ＊ ＊ ＊ ＊

Activity 2

In this activity, students revisit the binomial probability distributions. They then create and interpret transition diagrams. Students are introduced to Markov chains and Markov processes, asked to identify the existing states of Markov chains, and determine the probability of moving from one state to another.

Materials List

- none

Technology

- spreadsheet (optional)

Teacher Note

The probabilities in the following sample responses have been rounded so that the sum of the probabilities for an event is 1.

Exploration

a. Probability distribution for a sample size of 30:

n	*P(n)*	*n*	*P(n)*
0	0.001	10	0.036
1	0.009	11	0.016
2	0.034	12	0.006
3	0.079	13	0.002
4	0.133	14	0.001
5	0.172	15	0.000
6	0.180	16	0.000
7	0.154	17	0.000
8	0.111	$\vdots$	$\vdots$
9	0.068	30	0.000

b. In the following equations, H represents the heightened state, O represents the ordinary state, and R represents the relaxed state.

    1. $P(\text{H}|\text{O}) \approx 0.128$

    2. $P(\text{R}|\text{O}) \approx 0.123$

    3. $P(\text{O}|\text{O}) \approx 0.128$

c.   **1.** Probability distribution for a sample size of 5:

No. of Defectives (*n*)	*P(n)*
0	0.328
1	0.410
2	0.205
3	0.051
4	0.006
5	0.000

    2. $P(\text{O}|\text{R}) \approx 0.263$

    3. Since the quality-control process as described in the student edition does not allow this transition, $P(\text{H}|\text{R}) = 0$.

    4. $P(\text{R}|\text{R}) \approx 0.737$

d.   **1.** $P(O|H) \approx 0.869$

      2. Since the quality-control process as described in the student edition does not allow this transition, $P(R|H) = 0$.

      3. $P(H|H) \approx 0.131$

e.   Sample table:

Present State	Next State			Total
	relaxed	ordinary	heightened	
relaxed	0.737	0.263	0.000	1
ordinary	0.123	0.749	0.128	1
heightened	0.000	0.869	0.131	1

f.   Sample diagram:
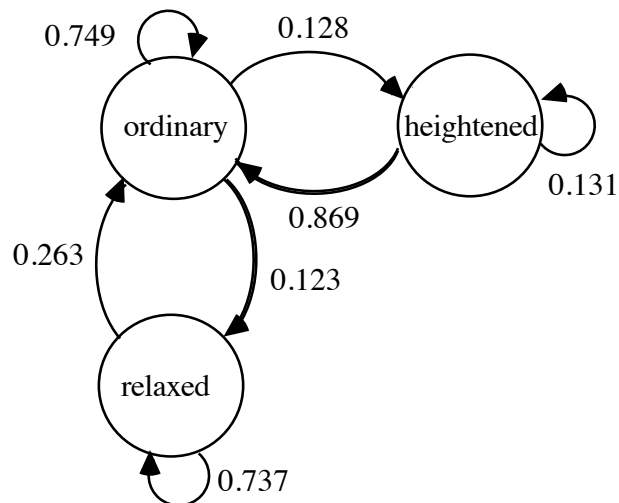
Discussion

(page 82)

a.   **1.** Sample response: Each entry can be interpreted as the probability of moving from the present state to another state (or the same state).

      2. Sample response: The probability in each cell is represented as the label on the directed edge leaving the vertex that corresponds to the cell's row heading.

b.   **1.** Sample response: An entry of 0 in a cell means that there is no possibility of a transition between the row state and the column state.

      2. An entry of 0 in a cell may be represented in a transition diagram with a directed edge labeled 0, or with no edge.

c.　**1.** The total for each row is 1, the sum of the probabilities for all the outcomes in an event.

　　　2. Each entry in each row is represented by an edge leaving a vertex. The sum of the probabilities of all edges leaving any one vertex is 1.

Assignment　　　　　　　　　　　　　　　　　　　(page 83)

2.1　**a.** Probability distribution for sample size of 25:

No. Attending College	Probability	No. Attending College	Probability
0	0.000	13	0.065
1	0.000	14	0.103
2	0.000	15	0.141
3	0.000	16	0.164
4	0.000	17	0.161
5	0.000	18	0.133
6	0.000	19	0.091
7	0.000	20	0.051
8	0.001	21	0.022
9	0.002	22	0.008
10	0.006	23	0.002
11	0.016	24	0.000
12	0.035	25	0.000

　　b. $P(n \geq 13) \approx 0.94$

2.2　**a.** Students need not include branches that have a probability of 0. In the following sample diagrams, O represents ordinary, R represents relaxed, and H represents heightened.

Present State	Next State	Present State	Next State	Present State	Next State
	0.737　R		0.123　R		0　R
R	0.263　O	O	0.749　O	H	0.869　O
	0　H		0.128　H		0.131　H
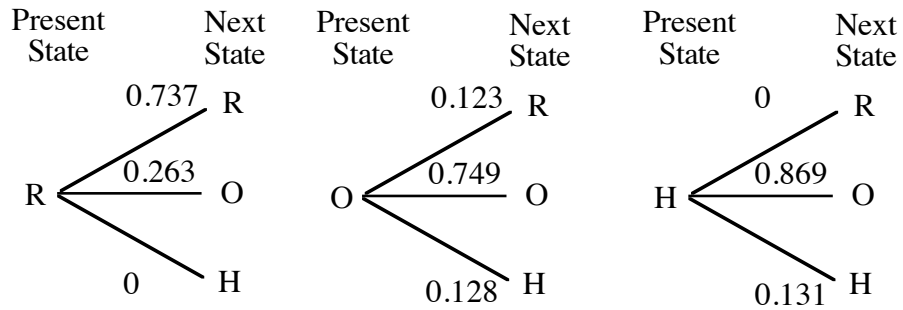
　　b. Sample response: Yes. These could be transition diagrams because they display the transitions from a single state to other states by edges labeled with corresponding probabilities.

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2.3   **a.** Sample transition diagram:
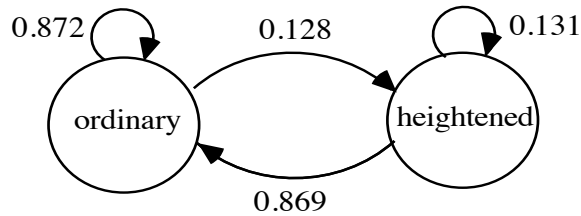
b. Sample response: No. The probability of transition from the ordinary state to the heightened state is the same in both plans. **Note:** Students revisit this question Problem **4.3**.

c. Sample response: I would recommend using three states. The only thing that changes when the relaxed state is eliminated is the transition from ordinary state to the less-expensive relaxed state. In either plan, the probabilities of moving from ordinary to heightened and back again remain the same. Therefore, it would cost the company less to use three states.

* * * * *

*2.4   **a.** The entry in this cell represents the probability that a person who lived in the Midwest moved to the South in 1991.

b. The entries in the principal diagonal represent the proportion of people who did not move to another region in 1991 or who moved within the same region.

c. In the following sample diagram, N represents Northeast, M represents Midwest, S represents South, and W represents West:
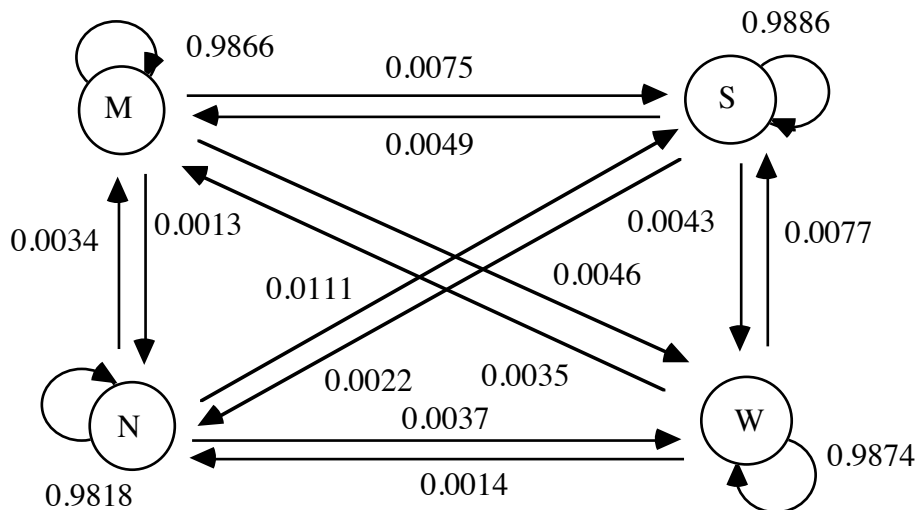
*2.5   **a.** The ordered pairs for the data—where U represents up, D represents down, and N represents no change—are shown in the table below.

Day	Move	Day	Move	Day	Move
(1,2)	(U,U)	(2,3)	(U,N)	(3,4)	(N,D)
(4,5)	(D,N)	(5,6)	(N,N)	(6,7)	(N,U)
(7,8)	(U,U)	(8,9)	(U,D)	(9,10)	(D,D)
(10,11)	(D,D)	(11,12)	(D,U)	(12,13)	(U,U)
(13,14)	(U,D)	(14,15)	(D,N)	(15,16)	(N,N)
(16,17)	(N,U)	(17,18)	(U,U)	(18,19)	(U,D)
(19,20)	(D,D)	(20,21)	(D,U)	(21,22)	(U,U)
(22,23)	(U,D)	(23,24)	(D,D)	(24,25)	(D,N)
(25,26)	(N,N)	(26,27)	(N,U)	(27,28)	(U,U)
(28,29)	(U,N)	(29,30)	(N,D)		

b. Sample table:

Given Day	Following Day		
	up	no change	down
up	0.500	0.167	0.333
no change	0.375	0.375	0.250
down	0.222	0.333	0.444

c. Sample diagram:
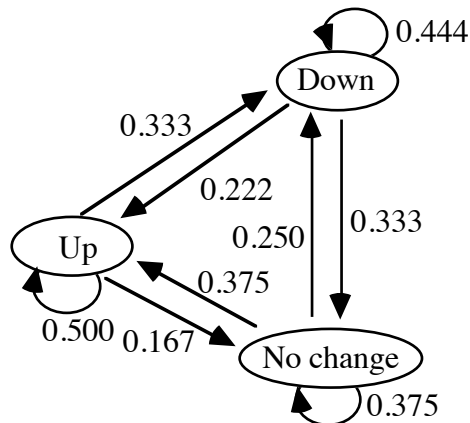
d. Sample response: Given that the stock price is down on day 31, the probability of going up on day 32 is 0.222, the probability of remaining unchanged is 0.333, and the probability of going down is 0.444. **Note:** You may wish to point out that many other factors, besides the previous day's behavior, influence stock prices.

* * * * * * * * * *

Activity 3

In this activity, students predict probabilities using transition matrices.

Materials List

- none

Technology

- matrix manipulator

Exploration

a. Sample tree diagram.

```
                    0.123   R
                   /
        O ─────────── 0.749  O
                   \
                    0.128   H
```

b. Sample tree diagram:

```
                              0.737   R
                             /
                      R ─────── 0.263  O
              0.123  /        \
                    /          0    H
                   /
                  /                    0.123   R
         0.749   /                    /
        O ─────────────────── O ─────── 0.749  O
                  \                    \
                   \                    0.128   H
                    \
            0.128    \          0     R
                      \        /
                      H ─────── 0.869  O
                             \
                              0.131   H
```

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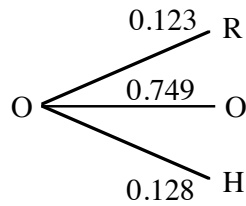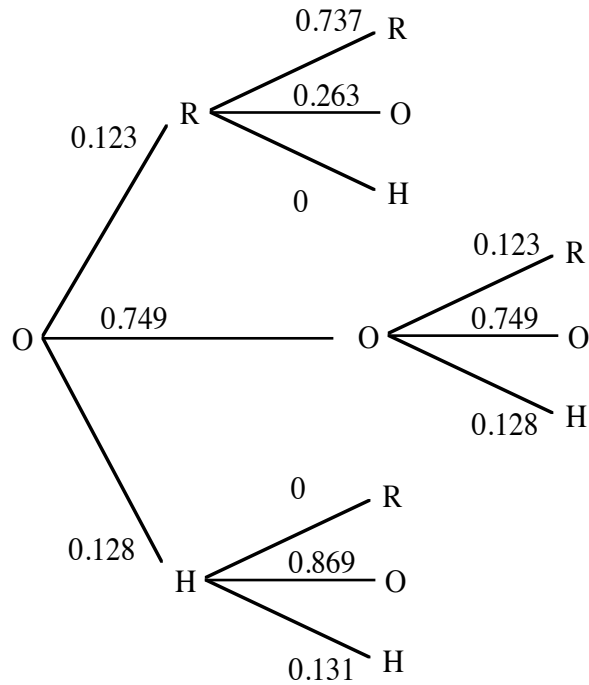
c. **1.** Students should multiply the two probabilities along each of the three paths that begin and end with O, then add the three products: $(0.749)(0.749) + (0.123)(0.263) + (0.128)(0.869) \approx 0.704$.

2. Students should multiply the two probabilities along each of the three paths that begin with O and end with R, then add the three products: $(0.123)(0.737) + (0.749)(0.123) + (0.128)(0) \approx 0.183$.

3. Students should multiply the two probabilities along each of the three paths that begin with O and end with T, then add the three products: $(0.123)(0) + (0.749)(0.128) + (0.128)(0.131) \approx 0.113$.

d. Sample matrix:

$$\mathbf{T} = \begin{array}{cccc} & R & O & H \\ R & \begin{bmatrix} 0.737 & 0.263 & 0.000 \\ O & 0.123 & 0.749 & 0.128 \\ H & 0.000 & 0.869 & 0.131 \end{bmatrix} \end{array}$$

e. Using matrix **T** from Part **d**,

$$\mathbf{T}^2 \approx \begin{array}{cccc} & R & O & H \\ R & \begin{bmatrix} 0.575 & 0.391 & 0.034 \\ O & 0.183 & 0.704 & 0.113 \\ H & 0.107 & 0.765 & 0.128 \end{bmatrix} \end{array}$$

f. Using matrix **T** from Part **d**,

$$\mathbf{T}^3 \approx \begin{array}{cccc} & R & O & H \\ R & \begin{bmatrix} 0.472 & 0.473 & 0.054 \\ O & 0.221 & 0.674 & 0.105 \\ H & 0.173 & 0.712 & 0.115 \end{bmatrix} \end{array}$$

At about $\mathbf{T}^{12}$ the values in each column appear to be nearing the same value. By $\mathbf{T}^{25}$, the values in each column are identical when rounded to the nearest thousandth.

$$\mathbf{T}^{25} \approx \begin{array}{cccc} & R & O & H \\ R & \begin{bmatrix} 0.290 & 0.619 & 0.091 \\ O & 0.290 & 0.619 & 0.091 \\ H & 0.290 & 0.619 & 0.091 \end{bmatrix} \end{array}$$

Discussion

a. Sample response: The rows represent the initial states and the columns represent the successive states. Since there are the same number of states each time, the matrix is square. The fact that it is square allows the matrix to be raised to powers greater than 1.

b. Sample response: Each element represents the probability of moving from one state to another. Probability values must be between 0 and 1, inclusive.

c. Since the elements in each row consist of the probabilities of moving from a given state to every other state, the sum of the probabilities is 1.

d. Sample response: Matrix **1** is not square. Matrix **2** contains elements that are not between 0 and 1. In matrix **3**, the sum of the second row is not 1.

e. **1.** Sample response: The probabilities determined in Part **c** of the exploration are the same as the elements in the second row of $\mathbf{T}^2$. This is because multiplying the probabilities along the branches of the tree diagram is analogous to multiplying a row of one matrix by a column of another.

2. Sample response: Each element in the first row of $\mathbf{T}^2$ represents the probability of moving from the relaxed state to another state after two transitions. Likewise, each element in the third row of $\mathbf{T}^2$ represents the probability of moving from the heightened state to another state after two transitions.

f. **1.** Each element in $\mathbf{T}^3$ represents the probabilities of moving among the various states after three transitions.

2. Each element in $\mathbf{T}^n$ represents the probabilities of moving among the various states after n transitions.

g. Sample response: Yes. The elements of $\mathbf{T}^n$ appear to be greater than 0 for many values of n.

h. Sample response: The values for each column in the transition matrix became closer and closer to the same value.

i. The stable state matrix indicates the probabilities of being in each state over the long run.

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3.1   **a.** Sample matrix:

$$\begin{array}{cc} & \begin{array}{cc} O & H \end{array} \\ \mathbf{T} = \begin{array}{c} O \\ H \end{array} & \begin{bmatrix} 0.872 & 0.128 \\ 0.869 & 0.131 \end{bmatrix} \end{array}$$

b. Sample response:

$$\begin{array}{cc} & \begin{array}{cc} O & H \end{array} \\ \mathbf{T}^3 \approx \begin{array}{c} O \\ H \end{array} & \begin{bmatrix} 0.872 & 0.128 \\ 0.872 & 0.128 \end{bmatrix} \end{array}$$

c. 1. Sample response: Yes. This is a square matrix whose entries are between 0 and 1, inclusive. The entries represent the probabilities of changing from one state to another.

   2. Sample response: Yes. The matrix is regular since all of its elements are positive. Therefore, the Markov chain also is regular.

*3.2   **a.** In the following sample matrix, N represents Northeast, M represents Midwest, S represents South, and W represents West:

$$\begin{array}{cc} & \begin{array}{cccc} N & M & S & W \end{array} \\ \mathbf{T} = \begin{array}{c} N \\ M \\ S \\ W \end{array} & \begin{bmatrix} 0.9818 & 0.0034 & 0.0111 & 0.0037 \\ 0.0013 & 0.9866 & 0.0075 & 0.0046 \\ 0.0022 & 0.0049 & 0.9886 & 0.0043 \\ 0.0014 & 0.0035 & 0.0077 & 0.9874 \end{bmatrix} \end{array}$$

b. The transition matrix **T** represents a Markov chain because it is square; all of its elements are between 0 and 1, inclusive; all of its rows have a sum of 1; and $\mathbf{T}^n$ represents the probability of moving from one state to another after n transitions.

   Because all of its elements are positive, $\mathbf{T}^1$ is a regular transition matrix. Therefore, the data represents a regular Markov chain.

c. Students should raise matrix **T** to the power n where n is the number of transitions, then identify the appropriate element.

   1. The probability of living in the West after two moves, given that a person originally lived in the Northeast, is about 0.0073.

   2. The probability of living in the Northeast after five moves, given that a person originally lived in the Midwest, is approximately 0.0063.

   3. The probability of living in the South after three moves, given that a person originally lived in the West, is about 0.0227.

d. Sample response: No. The probabilities of moving from one region to another depend on more than just present residence. For example, job availability, educational goals, and family relationships all affect the probability of moving.

    Note: You may wish to point out that the analysis of the probabilities of future states using Markov chains is valid only as long as the transition matrix remains a good model of the situation. This means that predictions made for a large number of transitions must be taken as approximations at best. For example, a model of the movement of the U.S. population can be affected by regional variations in economic growth, as well as changes in the perceived desirability of particular regions. The development of large water storage reservoirs, for instance, dramatically encouraged movement to the arid Southwest.

3.3   **a.** Sample matrix:

$$\mathbf{T} \approx \begin{array}{cc} & \begin{array}{ccc} R & O & H \end{array} \\ \begin{array}{c} R \\ O \\ H \end{array} & \begin{bmatrix} 0.879 & 0.121 & 0.000 \\ 0.206 & 0.598 & 0.196 \\ 0.000 & 0.129 & 0.871 \end{bmatrix} \end{array}$$

    b. Sample response: Yes. It appears that the entries in each column are approaching the same values. For $\mathbf{T}^{100}$, the column entries are equal when rounded to the nearest 0.00001.

*3.4   A transition matrix for this information is shown below. To complete Parts **a–c**, students should raise matrix $\mathbf{T}$ to the power n where n is the number of transitions, then identify the appropriate element.

$$\mathbf{T} = \begin{array}{cc} & \begin{array}{ccc} CBA & CBN & SBC \end{array} \\ \begin{array}{c} CBA \\ CBN \\ SBC \end{array} & \begin{bmatrix} 0.50 & 0.20 & 0.30 \\ 0.60 & 0.30 & 0.10 \\ 0.60 & 0.30 & 0.10 \end{bmatrix} \end{array}$$

    a. The probability that viewers will be watching CBA after 4 hr, given that they originally were watching SBC, is approximately 0.55.

    b. The probability that viewers will be watching SBC after 5 hr, given that they originally were watching CBN, is approximately 0.21.

    c. The probability that viewers will be watching CBA after 10 hr, given that they originally were watching CBA, is approximately 0.55.

* * * * *

3.5   **a.** Sample transition diagram

$$\begin{array}{c} \quad\quad 80\%\ \ 60\% \\ \mathbf{T} = \begin{array}{c} 80\% \\ 60\% \end{array}\!\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

b. Sample response: Over the long run, the 80% shooter would shoot about two-thirds of the shots, or 100. The 60% shooter would take the remaining 50 shots. This is illustrated in the transition matrix below.

$$\begin{array}{c} \quad\quad\ \ 80\%\quad\ \ 60\% \\ \mathbf{T}^{150} \approx \begin{array}{c} 80\% \\ 60\% \end{array}\!\begin{bmatrix} 0.667 & 0.333 \\ 0.667 & 0.333 \end{bmatrix} \end{array}$$

c. Sample response: Since the 80% shooter makes 80% of her free throws, she should make about 80 shots. The 60% shooter should make about 30 shots.

3.6   Since this is a transition matrix and the element in row 2, column 1 is 0, $c = 1$. When $\mathbf{A}$ is raised to the power n, the resulting element in row 1, column 1 is a^n. If a is not 0 or 1, then it's value must be between 0 and 1. For large values of n, therefore, a^n approaches 0. This means that for large values of n,

$$\mathbf{A}^n = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Note: This is an absorbing state matrix. Students are introduced to this term in Problem **4.5**.

* * * * * * * * * *

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Activity 4

In this activity, students predict future population distributions using transition matrices, state vectors, and stable state matrices and vectors.

Materials List

• none

Technology

• matrix manipulator

Exploration

a. The initial state vector is shown below:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 = [0 \quad 1 \quad 0]$$

b. **1.** After 1 transition, the predicted distribution can be found as follows:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1 = \begin{bmatrix} 0.123 & 0.749 & 0.128 \end{bmatrix}$$

2. After 2 transitions, the predicted distribution is:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_1 \bullet \mathbf{T} = \mathbf{X}_2 = \begin{bmatrix} 0.183 & 0.704 & 0.113 \end{bmatrix}$$

3. The predicted distributions for 3–30 transitions are almost identical, and will continue to be so if transitions are continued. The approximate stable state vector is shown below.

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{P} = \begin{bmatrix} 0.290 & 0.619 & 0.091 \end{bmatrix}$$

c. **1.** The first transition vector, second transition vector, and stable state vector for the initial state vector $\mathbf{X}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ are given below:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1 = \begin{bmatrix} 0.737 & 0.263 & 0.000 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_1 \bullet \mathbf{T} = \mathbf{X}_2 = \begin{bmatrix} 0.575 & 0.391 & 0.034 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{P} = \begin{bmatrix} 0.290 & 0.619 & 0.091 \end{bmatrix}$$

2. The first transition vector, second transition vector, and stable state vector for the initial state vector $\mathbf{X}_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ are given below:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1 = \begin{bmatrix} 0.000 & 0.869 & 0.131 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_1 \bullet \mathbf{T} = \mathbf{X}_2 = \begin{bmatrix} 0.107 & 0.765 & 0.128 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{P} = \begin{bmatrix} 0.290 & 0.619 & 0.091 \end{bmatrix}$$

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d.   Responses will vary. The first transition vector, second transition vector, and stable state vector for the initial state vector $\mathbf{X}_0 = \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix}$ are given below:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1 = \begin{bmatrix} 0.000 & 0.869 & 0.131 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_1 \bullet \mathbf{T} = \mathbf{X}_2 = \begin{bmatrix} 0.107 & 0.765 & 0.128 \end{bmatrix}$$
$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{P} = \begin{bmatrix} 0.290 & 0.619 & 0.091 \end{bmatrix}$$

e.   **1.** Using the initial state vector given in Part **d** yields the following:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{X}_0 \bullet \mathbf{T}^{30} = \mathbf{X}_{30} = \begin{bmatrix} 0.290 & 0.619 & 0.091 \end{bmatrix}$$

   2. This vector is the same as the one obtained in Part **e**.

f.   The general formula is $\mathbf{X}_n = \mathbf{X}_0 \bullet \mathbf{T}^n$.

Discussion

a.   Sample response: This occurs because each element represents the probability of a state, and the sum of the probabilities of all the states for a population is 1.

b.   The distributions eventually become identical, or stable.

c.   You may wish to use recursion to demonstrate that $\mathbf{X}_{n-1} \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^n$:

$$\mathbf{X}_1 = \mathbf{X}_0 \bullet \mathbf{T}$$
$$\mathbf{X}_2 = \mathbf{X}_1 \bullet \mathbf{T} = (\mathbf{X}_0 \bullet \mathbf{T}) \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^2$$
$$\mathbf{X}_3 = \mathbf{X}_2 \bullet \mathbf{T} = (\mathbf{X}_0 \bullet \mathbf{T}^2) \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^3$$
$$\vdots$$
$$\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T} = (\mathbf{X}_0 \bullet \mathbf{T}^{n-1}) \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^n$$

d.   **1.** After a large number of transitions, the resulting state vector is the same, regardless of the initial population distribution.

   2. Each row of $\mathbf{T}^{30}$ has the same elements as $\mathbf{X}_{30}$.

Assignment

4.1   The stable state vectors for **a–c** are shown below:

a. $\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$ **b.** $\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.1 & 0.5 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

d. This transition matrix does not stabilize, but oscillates between the following two matrices:

$$\begin{bmatrix} 0.4 & 0.0 & 0.6 \\ 0.0 & 1.0 & 0.0 \\ 0.4 & 0.0 & 0.6 \end{bmatrix} \text{ and } \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.4 & 0.0 & 0.6 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}$$

4.2   In the following transition matrix, C represents cloudy, R represents rainy, and S represents sunny:

$$\begin{array}{c c} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{T} = \begin{array}{c} C \\ R \\ S \end{array} & \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

The corresponding stable state matrix is shown below:

$$\begin{array}{c c} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{S} = \begin{array}{c} C \\ R \\ S \end{array} & \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

This produces the following stable state vector:

$$\begin{array}{c c} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{P} = & \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

Therefore, this model predicts about 146 cloudy days (40% of 365), 73 rainy days (20% of 365), and 146 sunny days (40% of 365) in a year.

*4.3   **a.** A stable state matrix exists for this data because it is a regular Markov chain. Sample matrix:

$$\begin{array}{c c} & \begin{array}{cc} O & H \end{array} \\ \mathbf{T}^{30} \approx \begin{array}{c} O \\ H \end{array} & \begin{bmatrix} 0.872 & 0.128 \\ 0.872 & 0.128 \end{bmatrix} \end{array}$$

b. The corresponding stable state vector is shown below:

$$\begin{array}{c c} & \begin{array}{cc} O & H \end{array} \\ \mathbf{P} = & \begin{bmatrix} 0.872 & 0.128 \end{bmatrix} \end{array}$$

c. Sample response: Assuming that the model remains a good one, the distribution will eventually consist of about 87.2% ordinary state, and 12.8% heightened state.

4.4   **a.**   **1.** The probability of being in the ordinary state after two weeks, given that the initial state was ordinary, is approximately 0.408. This is the second entry in the state vector found by the product below:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{T}^2$$

2. The probability of being in the ordinary state after five weeks, given that the initial state was relaxed, is approximately 0.227. This is the second entry in the state vector found by the product:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \mathbf{T}^5$$

3. The probability of being in the relaxed state after three weeks, given that the initial state was ordinary, is approximately 0.351. This is the first entry in the state vector found by the product:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{T}^3$$

b. The transition matrix representing an entire year of weekly sampling is $\mathbf{T}^{52}$. At this point, the transition matrix is stable. Regardless of the initial state vector, the probabilities will be the same:

$$\begin{array}{ccc} R & O & H \end{array}$$
$$\mathbf{P} = \begin{bmatrix} 0.403 & 0.237 & 0.360 \end{bmatrix}$$

c. Sample response: For small powers of the transition matrix, the initial state vector affects the probability of being in any given state. However, since this is a regular transition matrix (one which becomes stable), the initial state vector makes no difference for larger powers.

4.5 **a.** Sample transition matrix:

$$\text{make}\quad\text{miss}\quad\text{quit}$$

$$\mathbf{T} = \begin{array}{c}\text{make}\\\text{miss}\\\text{quit}\end{array}\begin{bmatrix} 0.25 & 0.65 & 0.1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

b. The sixth toss occurs 5 tosses after the success on the first toss. The probability of making a successful sixth toss is the first entry in the following product, or 0.001:

$$\mathbf{X_0} \bullet \mathbf{T}^5 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0.25 & 0.65 & 0.10 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.001 & 0.003 & 0.996 \end{bmatrix}$$

c. The probabilities over the long run stabilize to the steady state vector: $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

d. Sample response: This is called an absorbing-state Markov chain because, over the long run, there is one state whose probability is 1. The other states' probabilities are all 0.

* * * * *

4.6 **a.** **1.** After 1 yr, the resort can expect about 36% of the skiers to use the mass transit system.

2. After 2 yr, the resort can expect about 39% of the skiers to use the mass transit system.

3. After 5 yr, the resort can expect about 42% of the skiers to use the mass transit system.

b. If the transition matrix remains a good model, the resort can expect about 42% of skiers to use the mass transit system in the long run.

* * * * * * * * * *

Research Project

Students should determine that the probability that the two rolls will match is $6/36 \approx 0.1667$. The probability that the two rolls will differ by 1 is $10/36 \approx 0.2778$. The probability of all other outcomes is $20/36 \approx 0.5556$. Using these probabilities, they may construct a table similar to the following, in which Start is the initial position and End is the position after one turn:

Start	End						
	LOSE	**1**	**2**	**3**	**4**	**5**	**WIN**
LOSE	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.5556	0.0000	0.2778	0.1667	0.0000	0.0000	0.0000
2	0.0000	0.5556	0.0000	0.2778	0.1667	0.0000	0.0000
3	0.0000	0.0000	0.5556	0.0000	0.2778	0.1667	0.0000
4	0.0000	0.0000	0.0000	0.5556	0.0000	0.2778	0.1667
5	0.0000	0.0000	0.0000	0.0000	0.5556	0.0000	0.4444
WIN	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

The corresponding transition matrix, where L represents Lose and W represents Win, is shown below:

$$
\mathbf{A} = \begin{array}{c} \\ L \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ W \end{array}
\begin{array}{ccccccc}
L & 1 & 2 & 3 & 4 & 5 & W \\
\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.5556 & 0.0000 & 0.2778 & 0.1667 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.5556 & 0.0000 & 0.2778 & 0.1667 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.5556 & 0.0000 & 0.2778 & 0.1667 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.5556 & 0.0000 & 0.2778 & 0.1667 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.5556 & 0.0000 & 0.4444 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}
\end{array}
$$

Raising **A** to a large power, say 100, results in the matrix that represents the probabilities of winning and losing when starting at each location. Using the following matrix, for example, the probability of a win if starting at position 1 is 0.1943. **Note:** Although the values in the columns that correspond with positions 1–5 are not precisely 0, they are very near 0.

$$
\mathbf{A}^{100} = \begin{array}{c} \\ L \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ W \end{array}
\begin{array}{ccccccc}
L & 1 & 2 & 3 & 4 & 5 & W \\
\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.8063 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1943 \\
0.6265 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3748 \\
0.4601 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.5411 \\
0.3023 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6986 \\
0.1679 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.8326 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}
\end{array}
$$

In one variation on Tug O'Spot, if the dice match when the marker is at position 5, the player who placed the marker loses. This changes the row that corresponds with position 5 in transition matrix **A** to the following:

$$
\begin{bmatrix} 0.1667 & 0.0000 & 0.0000 & 0.0000 & 0.5556 & 0.0000 & 0.2778 \end{bmatrix}
$$

Another way to analyze Tug O'Spot is to create a simulation. The following program, for example, is written for the TI-92 calculator. **Note:** Students should seed the random number generator with a positive integer. If all students use the same integer, they will obtain identical results.

```
: markov()
: Prgm
: ClrIO
: {2,1,–1,–1,–1,–1} → list1
: 0 → w
: Disp "Enter integer for seed."
: Input rs
: randseed(rs)
: Lbl start
: Disp "Enter starting position(sp). (1≤sp≤5)"
: Input sp
: If sp-int(sp)>0
: Goto start
: If sp > 5 or sp < 1
: Goto start
: sp → j
: Disp "How many games?"
: Input n
: For c,1,n,1
: Lbl play
: int(rand(6)) → a
: int(rand(6)) → b
: 1+abs (a–b) → d
: j+list1[D] → j
: If j>0 and j<6
: Goto play
: w+int (J/6) → w
: sp → j
: EndFor
: Disp "No. of wins",w
: Disp "No. of losses",n–w
: EndPrgm
```

Using this program to simulate 1000 plays starting at position 3 resulted in 565 wins and 435 losses. Using this same position, the matrix $\mathbf{A}^{100}$ predicts 54.1% wins.

Answers to Summary Assessment

1. Sample response: Using the transition diagram and the probability distributions for samples of size 5, 30, and 100 with a defect rate of 20%, you can develop the following transition matrix:

$$T = \begin{array}{c} \\ R \\ O \\ H \\ S \end{array}\begin{array}{cccc} R & O & H & S \\ \begin{bmatrix} 0.737 & 0.263 & 0.000 & 0.000 \\ 0.123 & 0.749 & 0.128 & 0.000 \\ 0.000 & 0.869 & 0.125 & 0.006 \\ 0.000 & 1.000 & 0.000 & 0.000 \end{bmatrix} \end{array}$$

To find the expected probabilities after 52 weeks, the transition matrix is raised to the 52nd power. The resulting matrix appears stable when the probabilities are rounded to three places. **Note:** Students should realize that this is not, however, a stable state matrix.

$$T^{52} = \begin{array}{c} \\ R \\ O \\ H \\ S \end{array}\begin{array}{cccc} R & O & H & S \\ \begin{bmatrix} 0.290 & 0.619 & 0.090 & 0.001 \\ 0.290 & 0.619 & 0.090 & 0.001 \\ 0.290 & 0.619 & 0.090 & 0.001 \\ 0.000 & 1.000 & 0.000 & 0.000 \end{bmatrix} \end{array}$$

The expected number of weeks at each state can be determined by multiplying 52 weeks by each of the probabilities in the following state vector:

$$\begin{array}{cccc} R & O & H & S \end{array}$$
$$X_0 \bullet T^{52} = \begin{bmatrix} 0.290 & 0.619 & 0.090 & 0.001 \end{bmatrix}$$

The approximate number of weeks at each state is shown in the table below.

State	Predicted No. of Weeks
relaxed	14
ordinary	32
heightened	5
shut down	1

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The annual cost of this plan can be estimated by multiplying the projected cost for each sample, the number of samples in each state, the number of weeks at that state, adding these three products, then adding the number of weeks shut down times $2000, as shown below:

$$\text{cost} = (\$15 \bullet 5 \bullet 14) + (\$15 \bullet 30 \bullet 32) + (\$15 \bullet 100 \bullet 5) + \$2000$$
$$= \$24,950$$

2.   Sample response: If the actual rate is greater than the assumed rate of 20%, the probability distribution of defective items for each sample size would shift to the right. This means that in the ordinary state, for example, the probability of obtaining more than 8 defectives would increase. I would expect the company to spend more weeks in the heightened and shut-down states, and fewer weeks in the ordinary and relaxed states.

1. Which of the following could represent transition matrices? Explain your responses.

   a. $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 2/5 & 3/5 \\ 4/7 & 3/7 \end{bmatrix}$

2. Which of the following could represent regular Markov chains? Explain your responses.

   a. $\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$ **b.** $\begin{bmatrix} 4 & 6 \\ 1 & 9 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0.1 & 0.9 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0 & 0.9 \end{bmatrix}$

3. In the transition diagram for the weather shown below, C represents cloudy, R represents rainy, and S represents sunny.
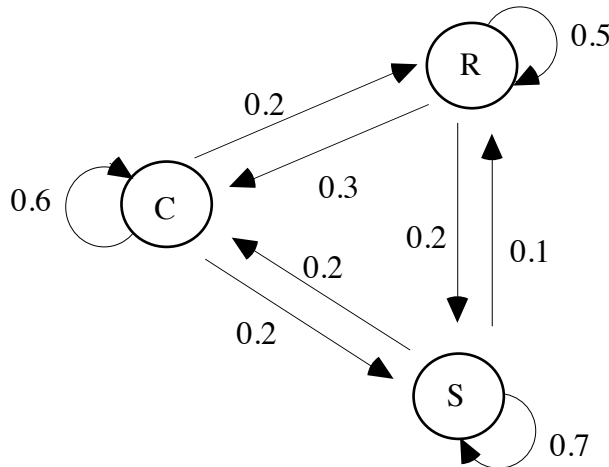
   Use this information to determine the probability that the weather will be rainy 10 days from now, given that today is rainy.

4. To represent the percentages of radio listeners who change music preferences each year, the Singers Songs Statistics Survey developed the following tree diagrams. In each diagram, C represents country, R represents rock, and O represents other kinds of music.

a. Create a transition matrix for this data.
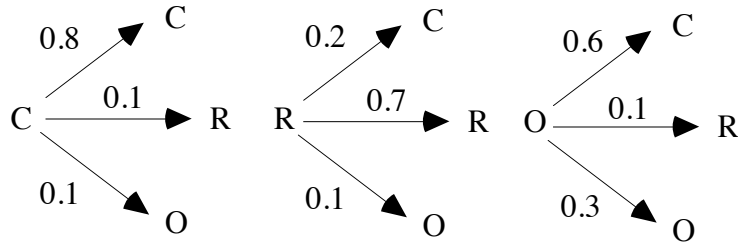
b. Does your transition matrix represent a regular Markov chain? Justify your response.

c. The lone radio station in Cowpoke County, ROCK 93, plays only rock music. A country station, KKOW, is considering a move to the area. The station's manager has asked you to determine whether or not KKOW has a promising future in Cowpoke County. Using the results from the survey given above, do you think KKOW will ever have more listeners than ROCK 93? If so, how many years do you predict it will take?

d. In 20 years, Cowpoke County expects to have a radio audience of about 23,700. By that time, how would you expect the audience to be distributed among rock, country, and other kinds of music?

Answers to Module Assessment

1. **a.** This is not a transition matrix because it is not square.

    b. This is a transition matrix because it meets all the required criteria.

    c. This is a transition matrix because it meets all the required criteria.

2. **a.** This is a regular Markov chain because it is a transition matrix with all positive elements.

    b. This is not a regular Markov chain because this is not a transition matrix (its entries are not between 0 and 1).

    c. This is a regular Markov chain because this transition matrix, raised to the second power, has all positive elements.

3. In the following sample transition matrix, C represents cloudy, R represents rainy, and S represents sunny:

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{ccc} C & R & S \\ \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

By raising the transition matrix to the 10th power, the probability that the weather will be rainy 10 days from now, given today is rainy, is approximately 0.2.

$$\mathbf{T}^{10} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{ccc} C & R & S \\ \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

4. **a.** A sample transition matrix is shown below:

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ O \end{array} \begin{array}{ccc} C & R & O \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \end{array}$$

    b. Sample response: This transition matrix is a regular Markov chain because all of its elements are positive.

    c. Sample response: KKOW should have a promising future in Cowpoke County because, after only 4 years, the majority of the audience should be listening to country music. This can be shown by multiplying the initial state vector by $\mathbf{T}^4$.

$$\mathbf{X}_0 \cdot \mathbf{T}^4 = \mathbf{X}_4$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix}^4 = \begin{bmatrix} 0.53 & 0.35 & 0.12 \end{bmatrix}$$

d. In 20 years (assuming that the transition matrix remains a good model), approximately 14,220 people will be listening to country, 7110 people will be listening to rock, and 2370 people will be listening to other types of music. This can be calculated by multiplying 23,700 by the stable state vector:

$$\begin{matrix} C & R & O \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix} \end{matrix}$$

Selected References

Consortium for Mathematics and Its Applications (COMAP). *Principles and Practice of Mathematics*. *Volume 1: Integration of Vectors and Matrices*. Arlington, MA: COMAP, 1993.

Hill, D. R. *Experiments in Computational Matrix Algebra*. New York: Random House, 1988.

Hunkins, D. R., and L. R. Mugridge. *Applied Finite Mathematics*. Boston, MA: Prindle, Weber, and Schmidt, 1985.

Isaacson, D. L., and R. W. Madsen. *Markov Chains: Theory and Applications*. New York: John Wiley & Sons, 1976.

National Council of Teachers of Mathematics (NCTM). *New Topics for Secondary School Mathematics: Matrices*. Reston, VA: NCTM, 1993.

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Flashbacks

Activity 1

1.1   If the probability of an event occurring is 0.7, what is the probability of the event not occurring?

1.2   If a standard die is tossed 10 times, what is the probability of obtaining a 4 on all 10 tosses?

1.3   Consider an experiment that involves drawing two cards from a standard deck, one at a time, with replacement. After each draw, the color of the card is recorded: red or black. Are these two draws independent events? Explain your response.

Activity 2

2.1   Use the following Venn diagram to determine each of the probabilities listed in Parts **a–c** below.
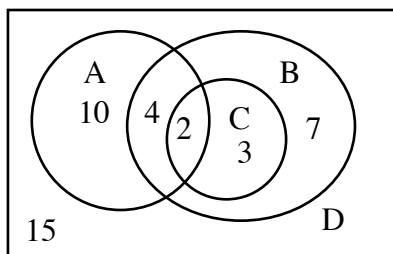
a. $P(A)$

b. $P(A \cap B)$

c. $P(C|B)$

2.2   Consider an experiment that involves drawing two cards from a standard deck of 52 playing cards, one at a time and without replacement.

a.   What is the probability of obtaining a spade on the second draw, given that one spade has already been drawn?

b.   What is the probability of obtaining a 10 on the second draw, given that a 10 has already been drawn?

2.3   Consider another experiment that involves drawing two cards from a standard deck of 52 playing cards, one at a time and without replacement.

Draw a tree diagram that shows the probabilities of drawing each of the following suits, given that a spade has already been drawn: spades, diamonds, hearts, clubs.

Activity 3

3.1 Perform each of the following matrix multiplications.

a. $\begin{bmatrix} 6 & 1 \\ -9 & 5 \end{bmatrix} \bullet \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \bullet \begin{bmatrix} -5 & 4 & 5 \\ 0 & 0 & 1 \\ -3 & 3 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3.2 Explain why the following operation cannot be performed.

$$\begin{bmatrix} 12 & 34 & -9 & 6 \end{bmatrix} \bullet \begin{bmatrix} -7 & 4 & -1 \\ 0 & 0 & 1 \\ -3 & 5 & 0 \end{bmatrix}$$

3.3 Use technology to perform the following matrix multiplications.

a. $\begin{bmatrix} 3 & 0 & 3 \\ 0.78 & 1 & 0 \\ 45 & 1 & 6 \end{bmatrix} \bullet \begin{bmatrix} 0.9 & 0.1 & 0 \\ 2 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0.23 & -0.76 & 5 & -1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 23 & 4 & -8 & 7 \\ 0 & 0 & 1 & 12 & 0 \\ 2 & 2 & 4 & -5 & 5 \\ 6 & 15 & 9 & 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0.25 & 0.25 & 0.1 & 0.4 \\ 0.3 & 0.7 & 0 & 0 \\ 0.4 & 0 & 0.3 & 0.3 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 24 & 65 & 90 & 121 \\ 9 & 45 & 41 & 32 \\ 6 & 8 & 4 & 67 \\ 0 & 0 & 15 & 17 \end{bmatrix}$

Activity 4

4.1   **a.** What is 15% of 600?

       b. What is 22% of 450?

       c. What is 53% of 725?

4.2   **a.** What percentage of 210 is 16?

       b. What percentage of 20,500 is 800?

4.3   Use the properties of exponents to simplify each of the following expressions.

    a. $4^2 \bullet 4^3$

    b. $17^4 \bullet 17^{108}$

    c. $5^{40}/5^{18}$

    d. $8^4/8^6$

    e. $x^{19} \bullet x^{20}$
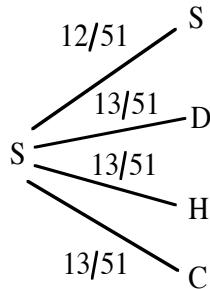
    f. x^{14}/x^7

Answers to Flashbacks

Activity 1

1.1   0.3

1.2 $(1/6)^{10} \approx 1.7 \cdot 10^{-8}$

1.3   Sample response: These events are independent; the result of the first has no effect on the probabilities in the second.

Activity 2

2.1   **a.** $P(A) = 16/41 \approx 0.39$

          b. $P(A \cap B) = 6/41 \approx 0.15$

          c. $P(C|B) = 5/16 \approx 0.31$

2.2   **a.** $P(\text{spade}|\text{spade}) = 12/51 \approx 0.24$

          b. $P(10|10) = 3/51 \approx 0.06$

2.3   Sample tree diagram:

Activity 3

3.1   **a.** $\begin{bmatrix} 23 & 1 \\ -2 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} -14 & 13 & 5 \\ -4 & 2 & 9 \end{bmatrix}$ **c.** $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

3.2   The operation cannot be performed because the number of columns in the first matrix does not equal the number of rows in the second matrix.

3.3   **a.** $\begin{bmatrix} 2.7 & 0.3 & 0 \\ 2.702 & 6.078 & -2 \\ 42.5 & 10.5 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 4.23 & 0.29 & 11.16 & -36.96 & 26.61 \end{bmatrix}$

c. $\begin{bmatrix} 8.85 & 28.3 & 39.15 & 51.75 \\ 13.5 & 51 & 55.7 & 58.7 \\ 11.4 & 28.4 & 41.7 & 73.6 \\ 24 & 65 & 90 & 121 \end{bmatrix}$

Activity 4

4.1   **a.** 90

b. 99

c. 384

4.2   **a.** 7.6%

b. 3.9%

4.3   **a.** 4^5

b. 17^{112}

c. 5^{22}

d. $1/8^2$

e. x^{39}

f. x^7

Let There Be Light

Teacher
Guide

Why are circles, ellipses, parabolas, and hyperbolas so important in our technological world? In this module, you answer this question by studying a group of shapes called conics.

Gary Bauer • Sherry Horyna • John Knudson-Martin

Let There Be Light

Overview

Students investigate properties of the four conic sections: the circle, the ellipse, the parabola, and the hyperbola. To illustrate some applications of these shapes, students examine the reflective properties of conics. Students also study the equations of the ellipse, hyperbola, and parabola.

Objectives

In this module, students will:

- investigate the reflective properties of the four conic sections
- write equations for conics in standard form
- use conics to design curves for practical applications.

Prerequisites

For this module, students should know:

- the definition of a circle
- the definitions of angle of incidence and angle of reflection
- the Pythagorean theorem
- how to solve equations for a given variable
- how to determine the distance between two points in a coordinate plane.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	3	3	2	1	11

Materials Required

Materials	1	2	3	4	Summary Assessment
flashlight	X				
laser (optional)	X				
cellophane tape	X				
electrical tape	X				
scissors	X				
cardboard	X				
reflective material	X				
conic templates	X				
conic models (optional)	X				
rubber bands (optional)	X				
graph paper		X	X		
ruler		X	X		
string		X			
template for Problem **3.5**			X		

Teacher Note

Blackline masters for the templates appear at the end of the teacher edition FOR THIS MODULE. The reflective material must be flexible, and wide enough to extend 2–3 cm above a cardboard model. Strips of Mylar or the thick film used for overhead transparencies both work well.

You also may wish to create reusable models of the conics for use in Activity **1**. Such models can be cut from wood, fiberboard, or polystyrene foam. To permit repeated use, cover each model with the corresponding paper template before beginning the exploration.

Technology

Software	1	2	3	4	Summary Assessment
graphing utility		X	X	X	
3-D graphing utility				X	
symbolic manipulator		X	X	X	X

Teacher Note

Although many graphing calculators can be used to graph conic sections, a symbolic manipulator or other computer graphing utility may provide more flexibility (and include asymptotes).

With some graphing calculators, it may be necessary to graph conic sections as two separate functions, or by using parametric equations.

Let There Be Light

Introduction

The conics can be produced by a plane slicing through a double-napped cone.
Note: To simulate slicing a cone with a plane, you may wish to point a flashlight towards the floor, then move a sheet of cardboard through the cone of light.

Activity 1

Students investigate the reflective properties of the conic sections.

Materials List

- flashlight (one per group)
- electrical tape
- laser (optional)
- sheets of cardboard (three per group)
- scissors (one pair per group)
- reflective material (one strip per group)
- cellophane tape
- conic templates (ellipse, circle, and parabola; one set per group)
- conic models (optional)
- rubber bands (optional)
- hyperbolic mirror template (for Problem **1.2**; two per student)

Teacher Note

Blackline masters of the templates appear at the end of the teacher edition for this module. Although a flashlight will serve as an adequate and economical light source, a pocket laser, such as the type used for pointers, provides much better results.

You may wish to create demonstration models by cutting each of the conic sections from a piece of plywood or heavy plastic foam, using the templates supplied at the end of this teacher edition. A rubber band can be used to secure the reflective material to the model's edge, as shown below.

reflective material   template

rubber band   model
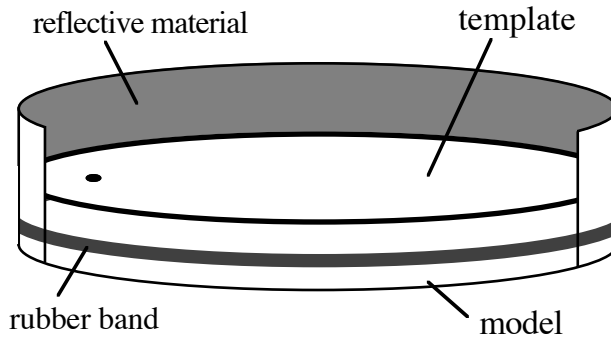
Exploration

a–c.   Students explore the reflective properties of an ellipse. As shown in the diagram below, they should discover that light passing through one focus reflects off the ellipse and passes through the second focus.

area of
second focus
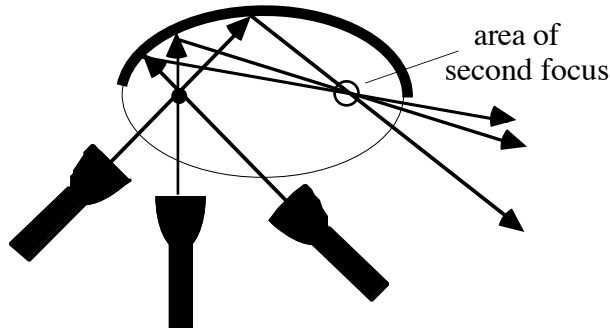
When the light does not pass through one of the foci, it reflects off the ellipse and does not pass through the other foci either, as shown below.

d.   Students repeat Parts **b** and **c** for a circle. As shown in the diagram below, they should discover that light passing through the center of the circle reflects off the circle and back to the center. Light that does not pass through the center is not reflected back through the center.
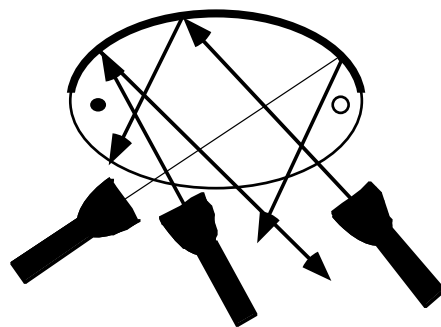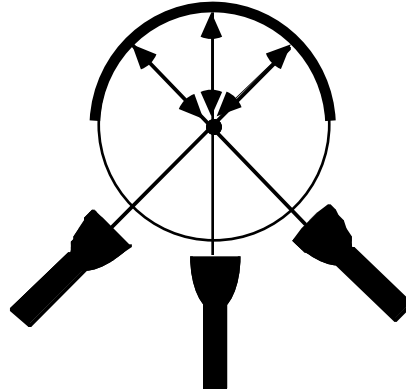

e.   Students should discover that light passing through the focus before striking the parabola reflects parallel to the parabola's axis of symmetry. If the incoming light is parallel to the parabola's axis of symmetry, it will reflect off the surface and pass through the focus. If the incoming light is not parallel to the parabola's axis of symmetry, it will not reflect through the focus.
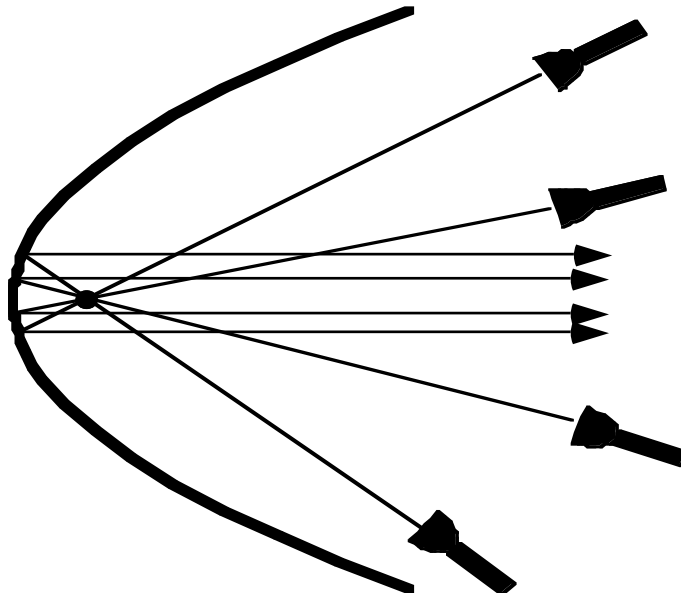
Discussion   (page 106)

a.   For a circle, any line that contains a diameter is an axis of symmetry. An ellipse has two axes of symmetry: one that contains the foci and one that contains the perpendicular bisector of the segment that joins the foci. The same is true for a hyperbola. The line of symmetry for a parabola contains the focus and the vertex.

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b.   **1.** Light passing through one focus before striking the ellipse reflects through the other focus.

   2. Light passing through the center of the circle reflects back through the center.

   3. Light passing through the focus of a parabola reflects parallel to the axis of symmetry. Light directed into the parabola, parallel to the axis of symmetry, reflects through the focus.

c.   Sample response: Light entering the paraboloid parallel to the axis of symmetry would be reflected towards the focus.

Teacher Note

To complete Problem **1.2**, each student will need two copies of the hyperbolic mirror template. A blackline master appears at the end of the teacher edition for this module.

Assignment   (page 107)

1.1   **a.** In the following sample drawing, $\theta_1 = \theta_2$.
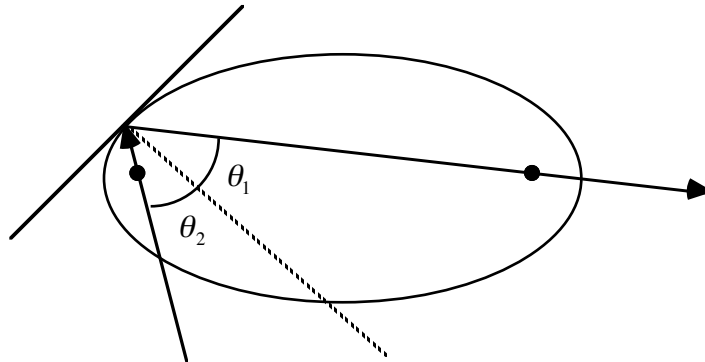
b. Students may note that the reflected ray on the exterior is collinear with the reflected ray in the interior. Sample drawing:

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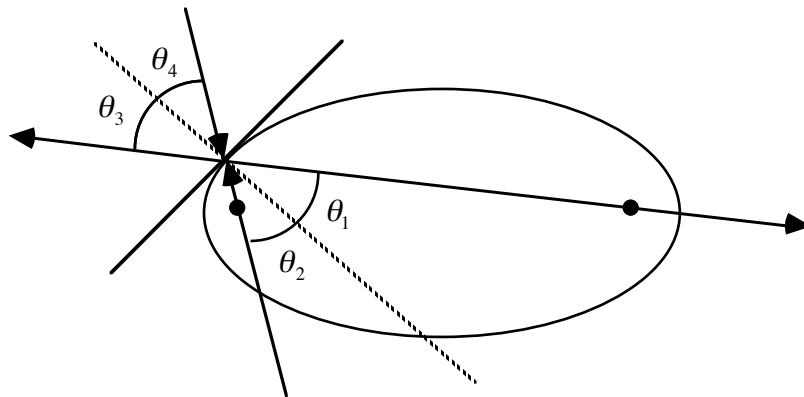
c. Sample response: For a circle, light heading toward the center from the exterior will reflect in the opposite direction, directly away from the center.
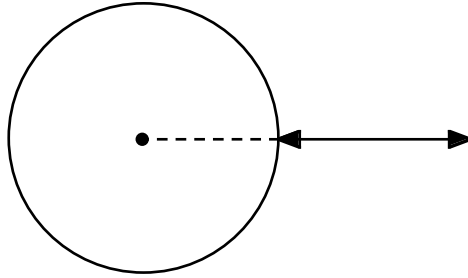
d. Sample response: For a parabola, light heading towards the focus from the exterior will reflect in a direction parallel to the axis of symmetry.
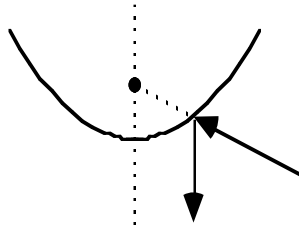
1.2   **a.** To construct a line tangent to the hyperbola, students may bisect the angle formed by the incoming light ray and its reflected ray, then construct a line perpendicular to the angle bisector.

b. Sample sketch:
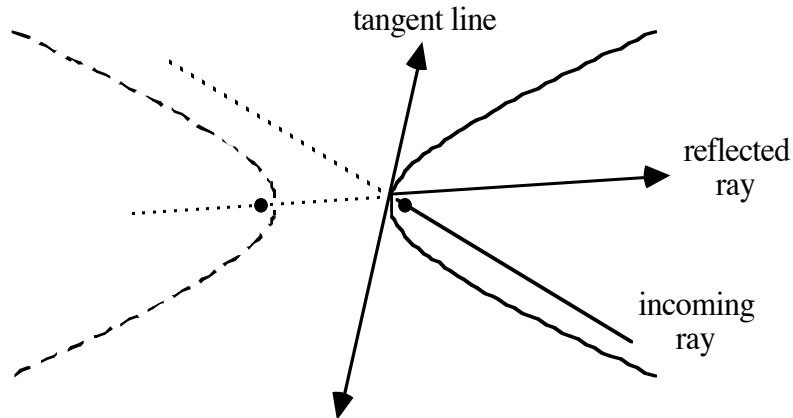
c–d. Sample response: If the line containing the incoming light ray passes through one focus of the hyperbola, then the line containing the reflected ray passes through the other focus.

*1.3   **a.** Sample sketch:
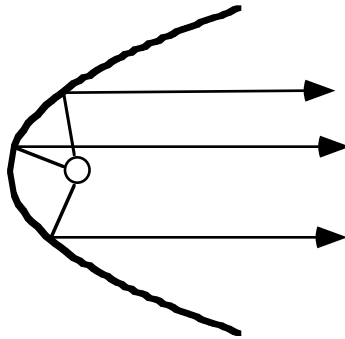
       b. Sample response: By placing the bulb at the focus, light rays reflecting off the mirror will be parallel to its axis of symmetry. This creates a concentrated beam of light.

       c. Sample response: Signals entering a parabolic reflector parallel to the axis of symmetry reflect towards the focus. By placing the receiver at the focus, incoming signals parallel to the antenna's line of symmetry are reflected to the receiver.

<p align="center">* * * * *</p>

1.4   Sample response: A hand cupped around the ear acts roughly like a conic section with the ear canal at a focus. Incoming sounds are reflected towards the ear canal, making them easier to hear.

1.5   Since Statuary Hall is shaped like an ellipse, the sound waves coming from a person whispering at one focus are reflected off the walls and towards the other focus.

1.6   Light rays emanating from the bulb and reflecting off the ellipsoidal reflector pass through the other focus of the ellipse.

1.7   Sample response: As the light rays enter the telescope parallel to the axis of symmetry for the parabola, they will be reflected off the parabolic mirror. The rays then pass through the focus of the paraboloid, which is also one focus of the ellipse. The reflected light rays will then pass through the eyepiece since it is at the other focus of the ellipse.

<p align="center">* * * * * * * * * *</p>

Activity 2

Students investigate the standard form of the equation of an ellipse.

Materials List

- ruler (one per group)
- string (one 25-cm length per group)
- graph paper (several sheets per group)

Technology

- graphing utility
- symbolic manipulator

Exploration   (page 110)

a–g.   The following sample graph shows the three ellipses to be constructed in the exploration.
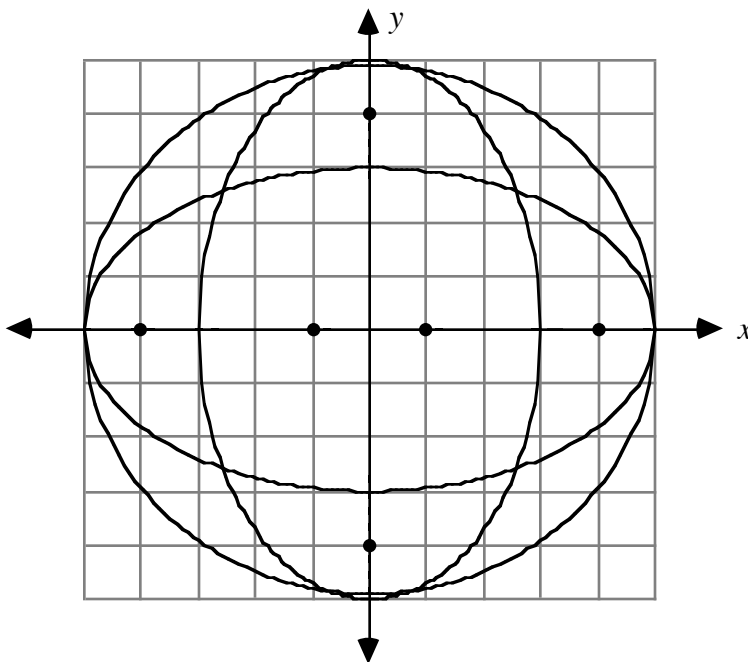
Since all these ellipses were created using a segment of string 10 units long, $PF_1 + PF_2 = 10$ in each case. Similarly, the length of the segment that contains the foci and whose endpoints are on the ellipse (the major axis) is also 10 units in each case.

Sample tables:

	Foci: $F_1(-4,0)$ and $F_2(4,0)$											
x	0	0	5	–5	1	1	2	2	–1	–1	–2	–2
y	3	–3	0	0	2.9	–2.9	2.7	–2.7	2.9	–2.9	2.7	–2.7
PF_1	5.0	5.0	9.0	1.0	5.8	5.8	6.6	6.6	4.2	4.2	3.4	3.4
PF_2	5.0	5.0	1.0	9.0	4.2	4.2	3.4	3.4	5.8	5.8	6.6	6.6
$PF_1 + PF_2$	10	10	10	10	10	10	10	10	10	10	10	10

	Foci: $F_1(-1,0)$ and $F_2(1,0)$											
x	0	0	5	–5	1	1	2	2	–1	–1	–2	–2
y	4.9	–4.9	0	0	4.8	–4.8	4.5	–4.5	4.8	–4.8	4.5	–4.5
PF_1	5.0	5.0	6.0	4.0	5.2	5.2	5.1	5.1	4.8	4.8	4.9	4.9
PF_2	5.0	5.0	4.0	6.0	4.8	4.8	4.9	4.9	5.2	5.2	5.1	5.1
$PF_1 + PF_2$	10	10	10	10	10	10	10	10	10	10	10	10

	Foci: $F_1(0,-4)$ and $F_2(0,4)$											
x	0	0	3	–3	1	1	2	2	–1	–1	–2	–2
y	5	–5	0	0	4.7	–4.7	3.7	–3.7	4.7	–4.7	3.7	–3.7
PF_1	9.0	1.0	5.0	5.0	8.8	1.2	8.0	2.0	8.8	1.2	8.0	2.0
PF_2	1.0	9.0	5.0	5.0	1.2	8.8	2.0	8.0	1.2	8.8	2.0	8.0
$PF_1 + PF_2$	10	10	10	10	10	10	10	10	10	10	10	10

Discussion (page 111)

a. Sample response: Each ellipse has two axes of symmetry: the x-axis and the y-axis. There is also a rotational symmetry of 180° centered at the origin.

b. 1. Sample response: The shape becomes more circular.

2. Sample response: If both foci were located at the same point, the shape would be a circle with a radius half the length of the string.

c. Sample response: The shape becomes longer and narrower.

d. Sample response: The shapes are similar. However, when the foci are on the y-axis, the ellipse is taller than it is wide. When the foci are on the x-axis, the ellipse is wider than it is tall.

e. If the length of the segment had been 12 units, then $PF_1 + PF_2$ also would have been 12 units.

f. The sum of the distances from each point on the ellipse to the two foci equals the length of the major axis.

g. The vertices of the ellipse with foci $F_1(-4,0)$ and $F_2(4,0)$ are at $(5,0)$, $(0,3)$, $(-5,0)$ and $(0,-3)$.

The vertices of the ellipse with foci $F_1(-1,0)$ and $F_2(1,0)$ are at $(5,0)$, $(0,4.9)$, $(-5,0)$ and $(0,-4.9)$.

The vertices of the ellipse with foci $F_1(0,-4)$ and $F_2(0,4)$ are at $(3,0)$, $(0,5)$, $(-3,0)$ and $(0,-5)$.

h. **1.** Sample response: The sum of the distances from a point on the ellipse to the two foci is always the length s of the segment of string. So $s = c + a + (a - c) = 2a$. This means that no matter where the foci are located, the vertices on the major axis will be half the length of the string from the center of the ellipse.

2. Sample response: Since the length of the string is $2a$, the distance from a focus to a vertex on the minor axis is a. The distance from the center to the same vertex is b, while the distance from the center to the focus is c. By the Pythagorean theorem, $a^2 = b^2 + c^2$ or $b = \sqrt{a^2 - c^2}$.

i. The equation of the ellipse with foci $F_1(-4,0)$ and $F_2(4,0)$ is:

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

The equation of the ellipse with foci $F_1(-1,0)$ and $F_2(1,0)$ is:

$$\frac{x^2}{5^2} + \frac{y^2}{4.9^2} = 1$$

The equation of the ellipse with foci $F_1(0,-4)$ and $F_2(0,4)$ is:

$$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$$

j. **1.** Using the Pythagorean theorem, $b^2 + c^2 = a^2$.

2. Using the relationship given above, $c = \sqrt{a^2 - b^2}$ or $-\sqrt{a^2 - b^2}$. The coordinates of the foci are $(-c,0)$ and $(c,0)$.

k. If $a = b$, then the ellipse is a circle.

Assignment (page 113)

2.1 Sample response: The sum of the distances from each point on the ellipse to the foci is the length of the string segment. The length of the string segment is the constant mentioned in the definition.

2.2 Sample response: The y-intercepts are b and $-b$. The x-intercepts are a and $-a$.

2.3　　**a.** $P(0,8)$, $R(0,-8)$, $S(-10,0)$, $Q(10,0)$, $F_1(-6,0)$, $F_2(6,0)$

　　　　b. $PR = 16$, $SQ = 20$, $PO = 8$, $PF_2 = 10$, $OQ = 10$

　　　　c. Sample response: The square of the length of the major axis minus the square of the length of the minor axis is the square of the distance between the foci, or $(2a)^2 - (2b)^2 = (2c)^2$.

*2.4　　**a.** The equation of the ellipse, in standard form, is:

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

　　　　The coordinates of points in the table, though only estimates, should roughly satisfy this equation.

　　　　b. Since $a = 5$ and $b = 3$, $c = 4$. Students should substitute as follows: $\sqrt{(x-4)^2 + (y-0)^2} + \sqrt{(x+4)^2 + (y-0)^2} = 2 \bullet 5$.

　　　　Note: You may wish to demonstrate to students that this equation can be transformed into standard form as shown below.

$$\sqrt{(x-4)^2 + y^2} + \sqrt{(x+4)^2 + y^2} = 10$$
$$\sqrt{(x-4)^2 + y^2} = 10 - \sqrt{(x+4)^2 + y^2}$$
$$(x-4)^2 + y^2 = 100 - 20\sqrt{(x+4)^2 + y^2} + (x+4)^2 + y^2$$
$$-8x = 100 - 20\sqrt{(x+4)^2 + y^2} + 8x$$
$$20\sqrt{(x+4)^2 + y^2} = 100 + 16x$$
$$400\left[(x+4)^2 + y^2\right] = 10,000 + 3200x + 256x^2$$
$$144x^2 + 400y^2 = 3600$$
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

2.5 **a.** $x^2 + y^2 = 4^2$ or $x^2 + y^2 = 16$

b. $y = \sqrt{16 - x^2}$ or $y = -\sqrt{16 - x^2}$

c. Sample graph:
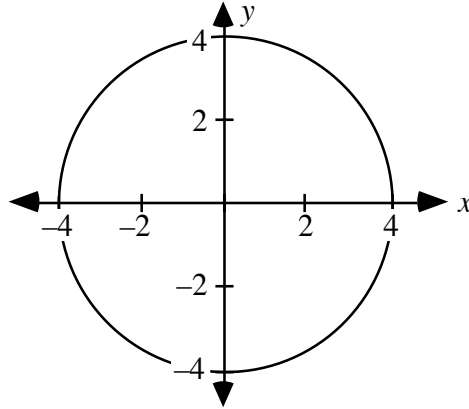
d. Sample response:

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} = 1$$

$$\left(4^2\right) \bullet \left(\frac{x^2}{4^2} + \frac{y^2}{4^2}\right) = 1 \bullet \left(4^2\right)$$

$$x^2 + y^2 = 4^2$$

*2.6 **a.** Sample response: By using a reflector shaped like an ellipsoid and placing the light source at one focus, all the light emitted will be reflected towards the other focus. Placing the end of the strand at the other focus would allow it to pick up the concentrated light.

b. Sample drawing:
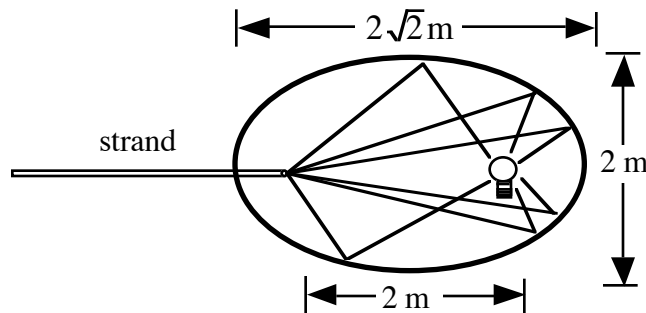
c. $\dfrac{x^2}{2} + \dfrac{y^2}{1} = 1$

* * * * *

2.7   **a.**   Students should solve the equation for y, then graph the resulting two functions. Sample graph:
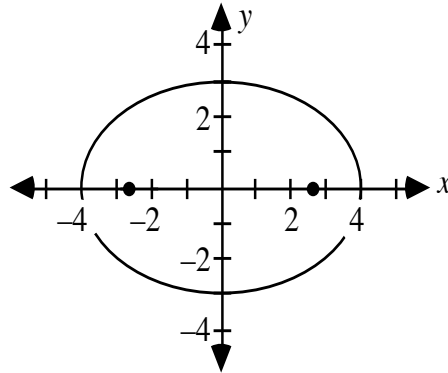
b.   The x-intercepts are 4 and –4; the y-intercepts are 3 and –3.

c.   The coordinates of the foci are $\left(\sqrt{7},0\right)$ and $\left(-\sqrt{7},0\right)$.

2.8   **a.**   The equation of the ellipse, in standard form, is:

$$\frac{x^2}{8000^2} + \frac{y^2}{8000^2 - 500^2} = 1$$

$$\frac{x^2}{64,000,000} + \frac{y^2}{63,750,000} = 1$$

b.   Sample response: Since the denominators are very close in size, the ellipse is almost circular in shape.

c.   **1.**   Sample response: Half of the major axis is 8000 km. It is 500 km from the center of the ellipse to the focus at which earth is located. The distance from the center of the ellipse to the surface of the earth nearer to the satellite is $6400 + 500 = 6900$ km. Therefore, the satellite is $8000 - 6900 = 1100$ km from earth at its nearest point.

2.   Sample response: When the satellite is at its farthest point, it is $8000 + 500 = 8500$ km from earth's center. Therefore, the distance from the satellite to the earth's surface is $8500 - 6400 = 2100$ km.

2.9   **a.** Sample graph:
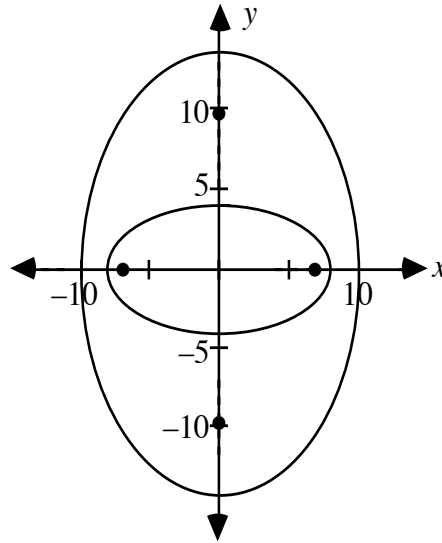
b. For the smaller ellipse, $a = 8$ and $b = 4$. Its area is $\pi \cdot 8 \cdot 4 = 32\pi$ units^2.

For the larger ellipse, $a = 10$ and $b = 14$. Its area is $\pi \cdot 10 \cdot 14 = 140\pi$ units^2.

* * * * * * * * * *

Activity 3

Students investigate the standard form of the equation of a hyperbola.

Materials List

• wall heater template (for Problem **3.5**; one per student)

Technology

• graphing utility
• symbolic manipulator

Exploration   (page 118)

a. Answers will vary. For the following sample responses, $a = 4$ and $b = 3$.

b. **1.** Sample response: The coordinates of the vertices are (–4,0) and (4,0).

  2. Sample response: Since $c = \sqrt{4^2 + 3^2}$, the coordinates of the foci (–5,0) and (5,0).

  3. Sample response: The distance between each focus and the center is 5 units.

c. Sample response: Solving the equation for y yields $y = 0.75\sqrt{x^2 - 16}$ or $y = -0.75\sqrt{x^2 - 16}$.
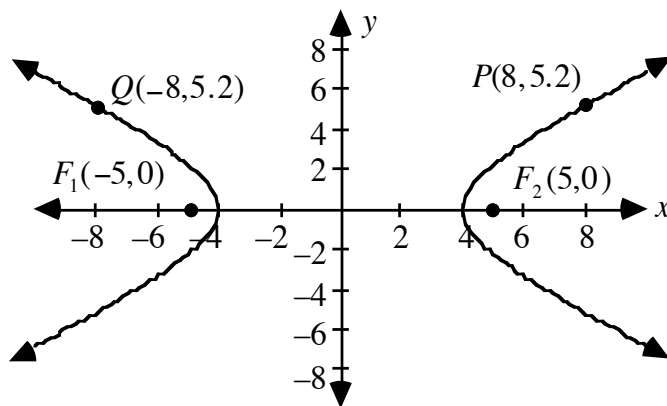
d. **1.** Students may select any point on the hyperbola. Sample response: For point P with coordinates (8,5.2), the constant difference is:

$$\sqrt{(8+5)^2 + (5.2-0)^2} - \sqrt{(8-5)^2 + (5.2-0)^2} \approx 8$$

  2. Sample response: For point Q with coordinates (–8,5.2), the constant difference is:

$$\sqrt{(-8-5)^2 + (5.2-0)^2} - \sqrt{(-8+5)^2 + (5.2-0)^2} \approx 8$$

  3. Sample response: The distance between the vertices is 8.

  4. The distance between the vertices equals the constant difference.

e. Sample table:

x	10	10	100	100	1000	1000	10,000	10,000
y	6.87	–6.87	74.9	–74.9	749.9	–749.9	7499.9	–7499.9
y/x	0.687	–0.687	0.749	–0.749	0.7499	–0.7499	0.74999	–0.74999

f. Sample response: For this hyperbola, $b/a = 3/4$ and $-b/a = -3/4$. As x increases, the values of y/x in the table approach $3/4$ and $-3/4$.

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g–h. Students should observe that the graphs of the lines $y = (b/a)x$ and $y = -(b/a)x$ contain the diagonals of the rectangle with center at the origin, a horizontal length of $2a$, and a vertical length of $2b$. Sample graph:
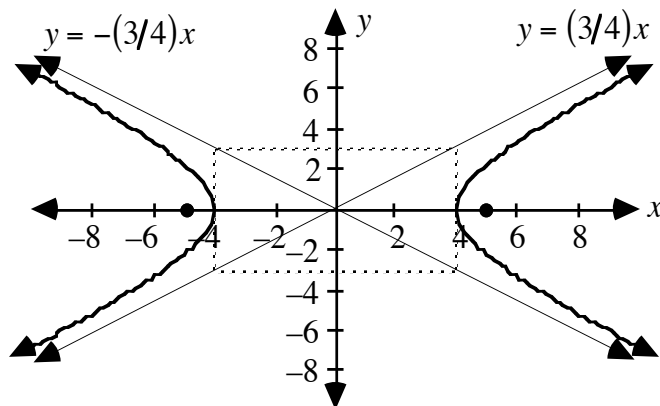
i. Students should observe that as x increases, the difference between the y-value of a point on the hyperbola and the corresponding point on one of the lines approaches 0.

j. Students should observe that this equation describes an hyperbola with foci on the y-axis. Sample response: The coordinates of the vertices are $(0,4)$ and $(0,-4)$. Since $c = \sqrt{4^2 + 3^2}$, the coordinates of the foci $(0,5)$ and $(0,-5)$. The distance between each focus and the center is 5 units.

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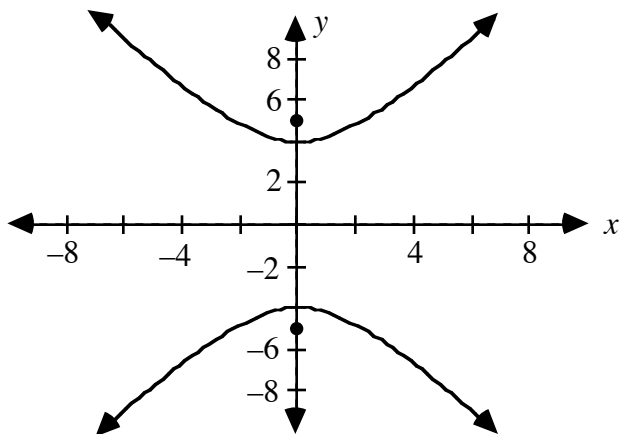
Discussion

a.   The constant difference of the hyperbola is $2a$. This is also the distance between the vertices. As a increases for a given value of b, the hyperbola gets narrower and the vertices move apart.

$$\frac{x^2}{(a+1)^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(a+3)^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(a+2)^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
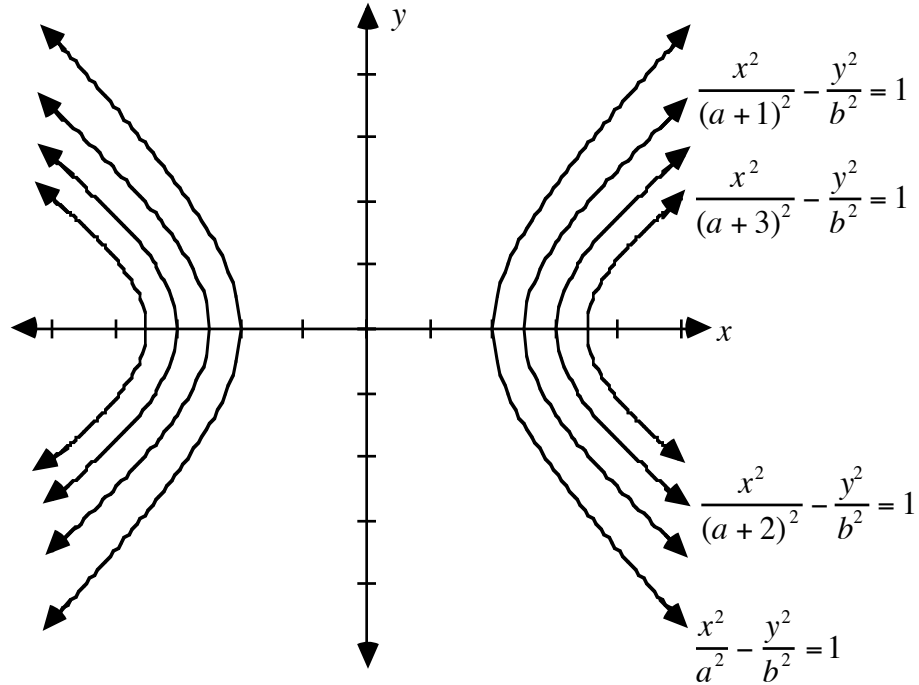
As b increases for a given value of a,, the hyperbola continues to have the same vertices but opens up more quickly.

$$\frac{x^2}{a^2} - \frac{y^2}{(b+3)^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{(b+1)^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{(b)^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{(b+2)^2} = 1$$

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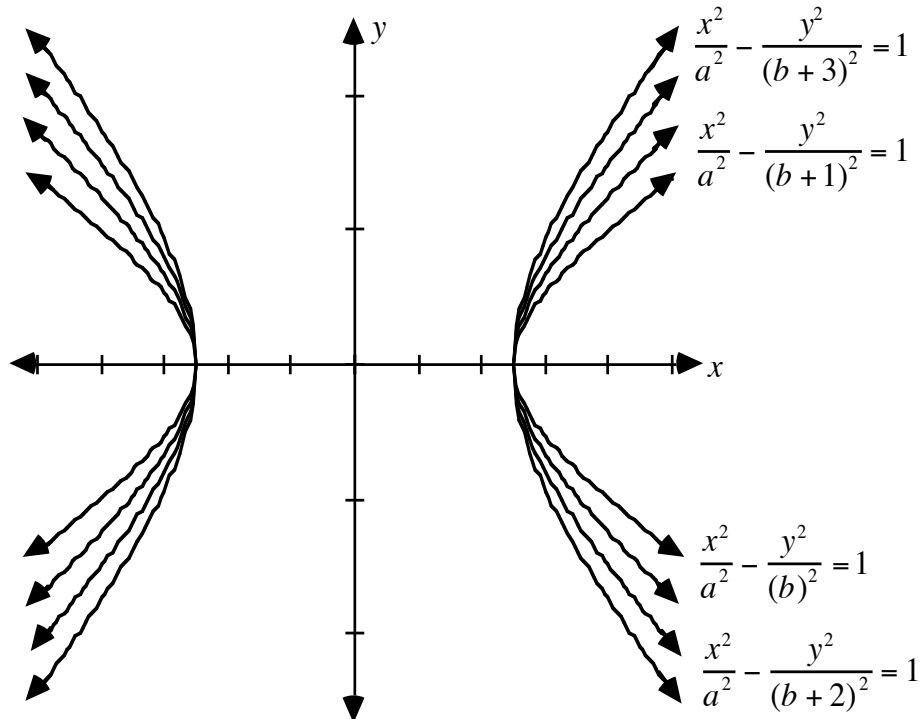
b. The distance between each focus and the center is c, where $a^2 + b^2 = c^2$. **Note:** Students should discover this by analogy with the ellipse, where the three quantities are also related using the Pythagorean theorem.

c. The coordinates of the foci are $(-c, 0)$ and $(c, 0)$, where $c = \sqrt{a^2 + b^2}$.

d. Sample response: The vertices of the hyperbola are located at the points where the major axis intersects the rectangle. The asymptotes are the lines containing the diagonals. You can sketch the hyperbola by starting at each vertex and drawing a curve that approaches the asymptotes.

e. Sample response: One hyperbola has foci on the x-axis, while the other has foci on the y-axis. All of these foci are the same distance from the origin (the center of both hyperbolas). The two hyperbolas share the same asymptotes.

f. Sample response: When the major axis of a hyperbola is the y-axis, its equation is of the form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

g. The asymptotes for a hyperbola with an equation of the form given above are the lines $y = (b/a)x$ and $y = -(b/a)x$.

Teacher Note

Each student will need a copy of the wall heater template to complete Problem **3.5**. A blackline master appears at the end of the teacher edition for this module.

Assignment (page 121)

3.1   **a.**   **1.**   The equation, in standard form, is:

$$\frac{x^2}{100} - \frac{y^2}{25} = 1$$

    2.   The foci are located at $\left(-\sqrt{125}, 0\right)$ and $\left(\sqrt{125}, 0\right)$.

    3.   Sample graph:

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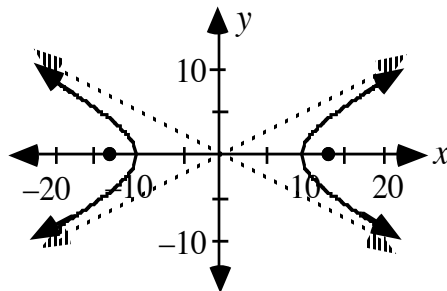
b. **1.** The equation, in standard form, is:

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

2. The foci are located at $\left(-\sqrt{41},0\right)$ and $\left(\sqrt{41},0\right)$.

3. Sample graph:
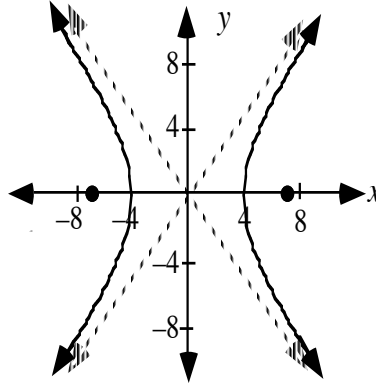
*3.2 **a.** Sample graph:
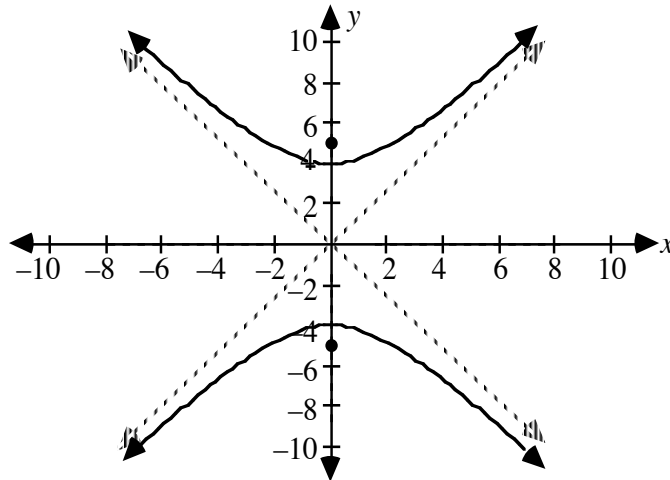
b. The coordinates of the vertices are (0,4) and (0,–4). The coordinates of the foci are (0,5) and (0,–5). The equations of the asymptotes are $y = \left(4/3\right)x$ and $y = -\left(4/3\right)x$.

3.3 **a.** $\dfrac{x^2}{3^2} - \dfrac{y^2}{4^2} = 1$

b. Sample response: The ray is reflected away from F_2 along the line that contains P and F_2.

3.4 The following equations describe the curve of the band shell using a hyperbola with center at the origin:

$$\frac{x^2}{20^2} - \frac{y^2}{15^2} = 1 \text{ or } \frac{x^2}{400} - \frac{y^2}{225} = 1$$

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3.5   **a.** Sample sketch:
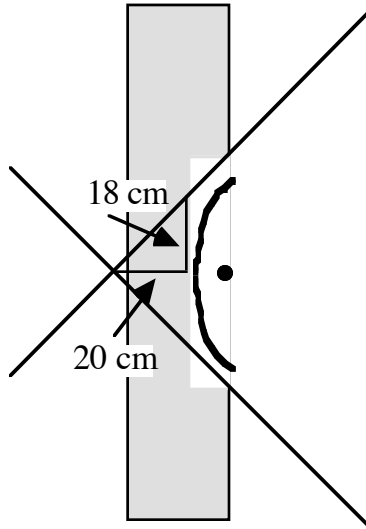
b. Answers may vary. Students can determine a and b by constructing the intersection of the asymptotes on their diagram. The value of a is the distance from the intersection (the center of the hyperbola) to the vertex of the hyperbola. The value of b is the distance from the vertex to a point on the asymptote directly above the vertex.

c. Sample response:

$$\frac{x^2}{20^2} - \frac{y^2}{18^2} = 1 \text{ or } \frac{x^2}{400} - \frac{y^2}{324} = 1$$

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3.6   Answers will vary. Some students may estimate the slopes of the asymptotes, then use the corresponding values for a and b to write an equation for the hyperbola. Others may estimate the coordinates of two points on the hyperbola, then substitute these coordinates into the general form of the equation for a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A symbolic manipulator can then be used to solve the resulting system of equations for a and b.

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3.7 **a.** Sample graph:

$$\frac{y^2}{144} - \frac{x^2}{25} = 1$$

$$\frac{x^2}{25} - \frac{y^2}{144} = 1$$
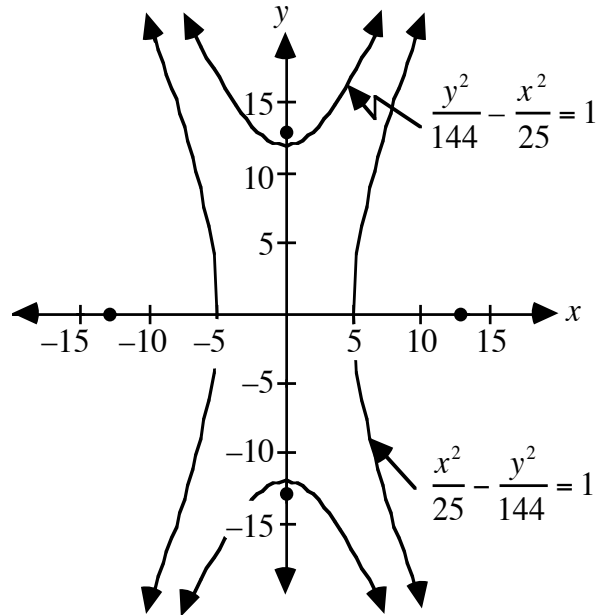
b. Sample response: The foci of both hyperbolas are located 13 units from the origin—one set on the x-axis and one set on the y-axis.

3.8 **a.** Sample graph:
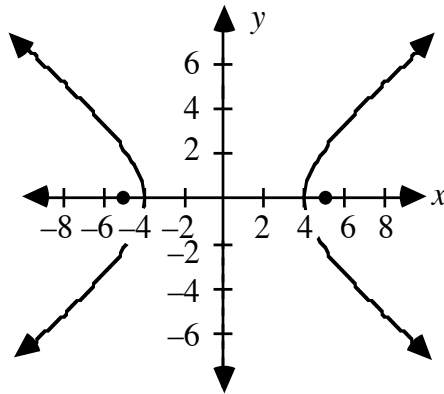
b. The x-intercepts are 4 and –4; there are no y-intercepts.

c. The coordinates of the foci are (5,0) and (–5,0).

d. Sample response:

$$9x^2 - 16y^2 = 144$$

$$\frac{9x^2 - 16y^2}{144} = \frac{144}{144}$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

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Activity 4

Students investigate the standard form of the equation of a parabola and its applications.

Materials List

* none

Technology

* graphing utility
* symbolic manipulator

Exploration

a. **1.** Since $\overline{FC}$ and $\overline{CD}$ must be equal by the definition of a parabola, *CDGF* must be a square. Since the length of $\overline{FG}$ is 4 units, each side of the square is also 4 units long. Therefore, the coordinates of point *C* are (4,2).

2. Students should substitute as follows:

$$y = ax^2$$
$$2 = a(4)^2$$
$$1/8 = a$$

3. $y = (1/8)x^2$

b. In this case, the coordinates of point *C* are $(2p, p)$. Students should substitute $2p$ and p for x and y, respectively, then solve for a.

$$y = ax^2$$
$$p = a(2p)^2$$
$$\frac{1}{4p} = a$$

The standard form of the equation of a parabola with vertex at the origin and focus on the *y*-axis is:

$$y = \frac{1}{4p}x^2$$

Discussion

a. The coordinates of the focus are $(0, p)$, where

$$p = \frac{1}{4a}$$

b. Sample response: The values of a and p are inversely proportional. As a gets larger, p gets smaller, and vice versa.

c. Sample response: Since p is a measure of the distance from the focus to the vertex, p increases as the focus and vertex move apart.

d. Sample response: As p increases, the shape of the parabola widens.

e. This is a second-degree polynomial (or quadratic) function.

Assignment

4.1. **a.** Student graphs will vary.

b. Sample response: Changing the value of the coefficient a in the equation $y = ax^2$ "stretches" the parabola vertically. As $|a|$ decreases, the parabola gets wider; as $|a|$ increases, the parabola gets narrower. When a is negative, the parabola opens downward.

4.2 **a.** $(0, 1/8)$

b. $(0, 1/2)$

c. $(0, -1/40)$

4.3 **a.** Sample sketch:
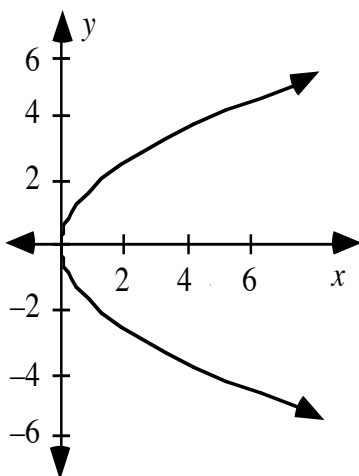
b. The focus is located at $(1/12, 0)$.

c. The equation of the directrix is $x = -1/12$.

d. Sample response: No. For all x-values in the domain, other than the vertex of the parabola, there are two corresponding y-values.

4.4   **a.** **1.** In the following sample sketch, 1 unit represents 1 m.
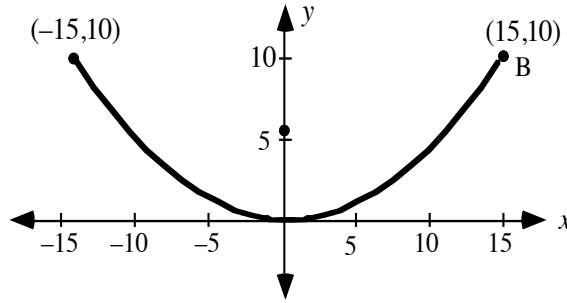
   2–3. Sample response: Substituting 15 and 10 for x and y, respectively, $p = 45/8$.

   4. Sample response:

$$y = \frac{1}{4(45/8)}x^2 = \frac{2}{45}x^2$$

   5. The coordinates of the focus are $(0, 45/8)$. This means that the receiver should be located $5\frac{5}{8}$ m (about 5.63 m) above the vertex.

b. Students should substitute 15 and 5 for x and y, respectively, in the general equation and solve for p. Since $p = 45/4$, the equation that describes the cross section of the dish is:

$$y = \frac{1}{4(45/4)}x^2 = \frac{1}{45}x^2$$

The coordinates of the focus are $(0, 45/4)$. This means that the receiver should be located $11\frac{1}{4}$ m (11.25 m) above the vertex.

*4.5   **a–b.** Students should give reasonable dimensions for a hand-held microphone. Sample sketch:

parabolic reflector

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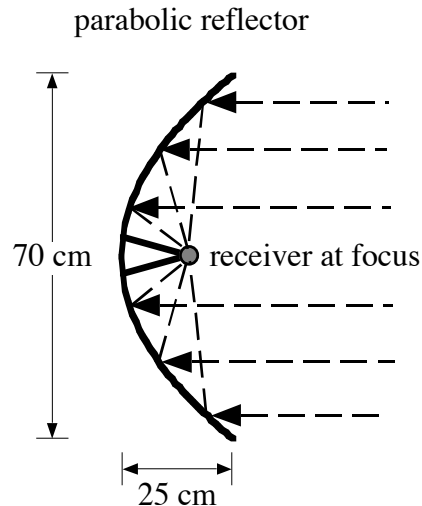
c. To calculate the distance from the vertex to the focus, students may graph a cross section of the dish with its vertex at the origin and focus on the y-axis. By substituting the coordinates of one edge of the dish into the standard form of the equation of a parabola, they can then solve for p. Using the sample dimensions given above:

$$y = \frac{1}{4p}x^2$$

$$25 = \frac{1}{4p} \cdot 35^2$$

$$p = 12.25$$

This represents a distance from the vertex to the focus of 12.25 cm.

4.6   **a. 1.**   Substituting (8,8) into the equation $y = ax^2$ yields $8 = a64$ or $a = 1/8$.

   2.   The coordinates of the focus are (0,2).

   b.   The coordinates of the vertex are (0,0,0). The coordinates of the focus are (0,0,2).

   c.   The radius of the rim is 8 units.

   d.   The set of points defined by the path of Q is a circle in a plane perpendicular to the z-axis with radius x.

   e.   The following sample graphs were created using two different graphing utilities. Students should recognize that these are graphs of the paraboloid described in Part **b**.

* * * * *
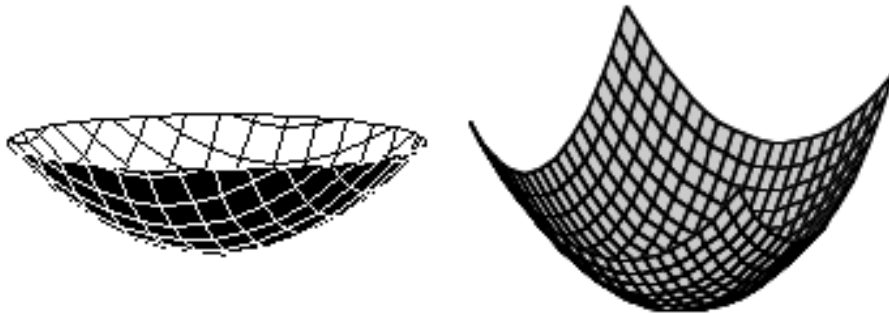
4.7 In the following sample responses, the vertex of the parabola that represents a cross section of the reflector is located at (0,0).

    a. Sample response: By substituting 7.5 and 5 for y and x respectively, in the standard form equation of a parabola, $p = 2.8$. This means that if the bulb is mounted as planned, the filament will be 0.8 cm from the focus of the parabola. In this case, all the light rays leaving the bulb and reflecting off the mirror will not be reflected parallel to the axis of symmetry.

    b. Sample response: The three parameters of the problem are the width of the beam, the depth of the headlight, and the location of the bulb. The first two parameters cannot be changed because of the manufacturer's design requirements. The solution is to change the location of the bulb. Mounting the base of the bulb 0.8 cm from the surface of the reflector will place the filament at the focus of the parabola.

4.8   **a.** Sample response: When the parabola's vertex is located at the origin, the point (40,25) is located on its edge. Substituting these values for x and y in the standard form equation, one can solve for p, the distance from the focus to the vertex.

$$y = \frac{1}{4p}x^2$$

$$25 = \frac{1}{4p}(40)^2$$

$$p = 16$$

The radio receiver should be located 16 m from the vertex.

    b. $y = \frac{1}{64}x^2$

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Answers to Summary Assessment

1.   **a.** Sample response: Beta III and Nimeus are farthest apart when Nimeus is on the major axis across the center from Beta III.

        b. Sample response: The length of the major axis $(2a)$ is $1 \bullet 10^7$ km. The length of the minor axis $(2b)$ is $8 \bullet 10^6$ km. The distance from the center of the ellipse to either focus (c) can be calculated using the equation $b^2 + c^2 = a^2$.

$$b^2 + c^2 = a^2$$
$$\left(4 \bullet 10^6\right)^2 + c^2 = \left(5 \bullet 10^6\right)^2$$
$$c = 3 \bullet 10^6 \text{ km}$$

           The maximum distance between Beta III and Nimeus is $a + c$, or $8 \bullet 10^6$ km. The minimum distance between Beta III and Nimeus is $a - c$, or $2 \bullet 10^6$ km.

2.   **a.** Sample response: Signals generated at the focus of the paraboloid will be reflected parallel to the axis of symmetry.

        b. Sample response: One disadvantage of using a paraboloid reflector is that it produces a beam only about as wide as the reflector. This would make it difficult to broadcast a signal to large areas of the earth.

        c. Answers will vary. Sample response: One branch of a hyperbola would be a more appropriate choice. If the transmitter is placed at a focus, this would scatter reflected rays in a region defined by the asymptotes. Otherwise, you could use a parabolic reflector and place the transmitter nearer the vertex than the focus.

        d. The diagram below (not drawn to scale) shows a satellite in the Clarke Belt, about 36,000 km from the equator.

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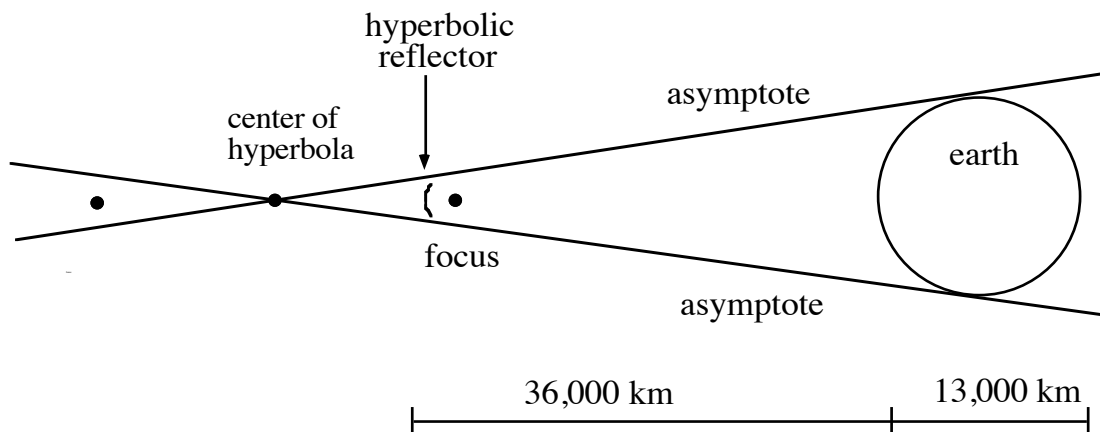
To determine the equation that describes the curve of the hyperbolic reflector, students must determine the values of a, b, and c in the following diagram (also not drawn to scale):
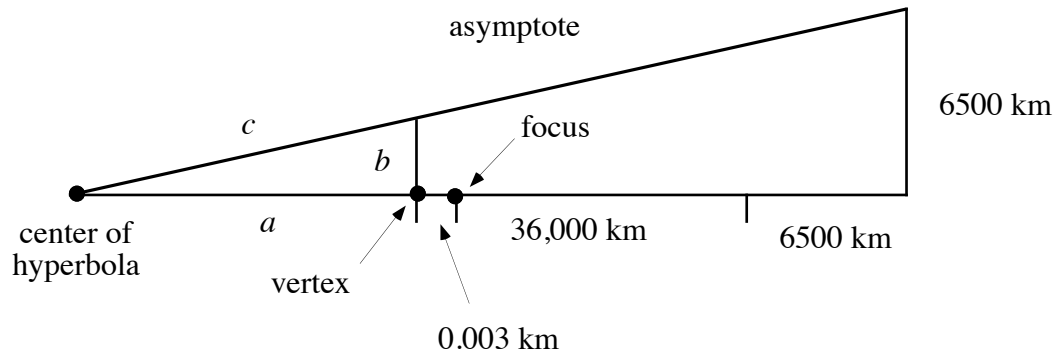
The right triangle with sides a, b, and c is similar to the large right triangle. From this fact—and the Pythagorean theorem—the following two relations are true: $c^2 = a^2 + b^2$ and

$$\frac{a}{b} = \frac{42500 + a}{6500}$$

Since the focus is 0.003 km from the dish, the distance from the center to the focus, or c, is $a + 0.003$.

Using a symbolic manipulator, $a = 0.258$ km, $b = 0.0395$ km, and $c = 0.261$ km. The equation of the hyperbola, therefore, is

$$\frac{x^2}{(0.258)^2} - \frac{y^2}{(0.0395)^2} = 1$$

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1. Write an equation in standard form for each of the following conics:

   a. a circle with center at the origin and a radius of 6 units

   b. an ellipse in which the endpoints of the major axis are at (5,0) and (–5,0) and the foci are at (2,0) and (–2,0)

   c. an ellipse with center at the origin, a major axis of 8 units, and a minor axis of 3 units.

2. Write an equation in standard form for each of the following:

   a. a parabola with vertex at the origin and focus at (0,3.8)

   b. a parabola with vertex at the origin and a distance from the vertex to the focus of 3 units.

3. Write an equation in standard form for each of the following graphs:

   a.

   b.

   c.

   d.
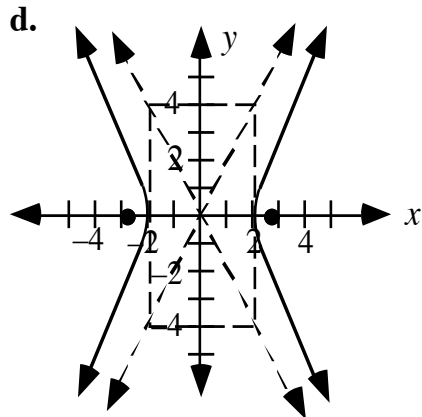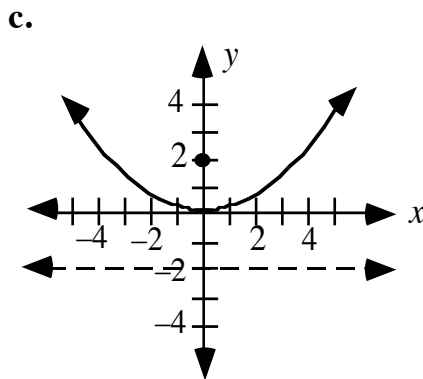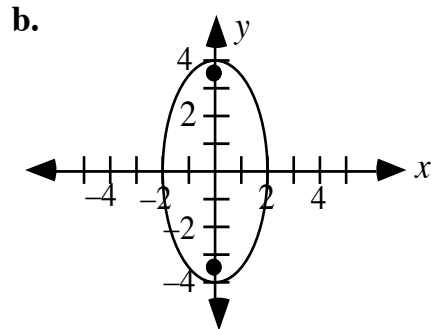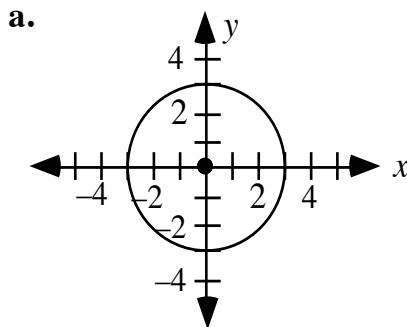
4. Compare the geometric and reflective properties of a circle and an ellipse.

5. Compare the geometric and reflective properties of a parabola and a hyperbola.

Answers to Module Assessment

1.
    a. $x^2 + y^2 = 36$

    b. $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$

    c. $\dfrac{x^2}{4^2} + \dfrac{y^2}{(1.5)^2} = 1$ or $\dfrac{y^2}{4^2} + \dfrac{x^2}{(1.5)^2} = 1$

2.
    a. $y = \dfrac{1}{15.2}x^2$ or $y = \dfrac{1}{15.2}x^2$

    b. $y = \dfrac{1}{12}x^2$ or $x = \dfrac{1}{12}y^2$

3.
    a. $x^2 + y^2 = 9$

    b. $\dfrac{x^2}{2^2} + \dfrac{y^2}{4^2} = 1$ or $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

    c. $y = \dfrac{1}{4 \bullet 2}x^2$ or $y = \dfrac{1}{8}x^2$

    d. $\dfrac{x^2}{2^2} - \dfrac{y^2}{4^2} = 1$ or $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

4. Sample response: A circle is a special form of an ellipse in which the major and minor axes have the same length. The sum of the distances from any point on an ellipse to the two foci is a constant value. The distance from any point on the circle to the center of a circle is a constant value.

   In an ellipse, light starting at one focus reflects off the ellipse to the other focus. Light starting at the center of a circle reflects off the circle back to the center.

5. Sample response: While a branch of a hyperbola looks somewhat like a parabola, the two curves have different characteristics. For any point on a parabola, the distances from the focus and the directrix are equal. For any point on a hyperbola, the difference between the distances to the two foci is a constant. The farther a point on the hyperbola is from the center, the closer it is to a line known as an asymptote.

   Light emanating from the focus of a parabola is reflected parallel to its line of symmetry. Light emanating from a focus of a hyperbola is scattered. Light directed towards one of the foci will be reflected along the line containing the point of reflection and the other focus.

Selected References

Clarke, A. C. *2001: A Space Odyssey*. New York: New American Library, 1968.

Downs, J. W. *Practical Conic Sections*. Palo Alto, CA: Dale Seymour Publications, 1993.

Lyng, M. J. *Dancing Curves*. Reston, VA: National Council of Teachers of Mathematics (NCTM), 1978.

Parker, W. O. *Scene Design and Stage Lighting*. New York: CBS College Printing, 1985.

Flashbacks

Activity 1

1.1 The following diagram shows a light ray approaching a flat mirror.
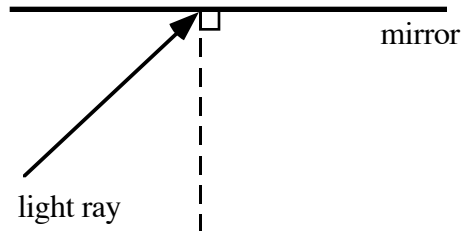
    a. On a copy of this diagram, sketch the reflected ray and label the angle of incidence and angle of reflection.

    b. Describe the relationship between the angle of incidence and the angle of reflection.

1.2 Describe how to use a radius of a circle to construct a line tangent to the circle. Make a sketch of a circle and a tangent line.

1.3 The diagram below shows the graph of a line l on a two-dimensional coordinate system. Make a sketch of the three-dimensional figure that results when the line is rotated about the y-axis.
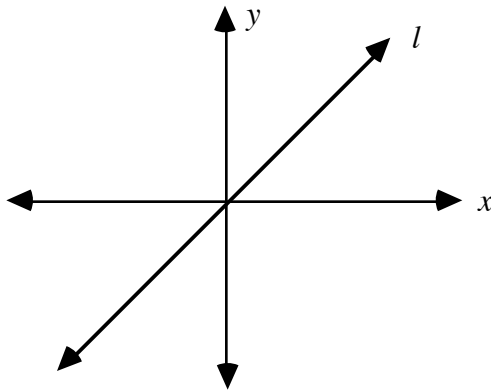
Activity 2

2.1 Consider the points $P(x,y)$, $A(4,0)$, and $B(-4,0)$. Given these coordinates, determine each of the following distances:

    a. PA

    b. PB

    c. $PA + PB$

2.2 The equation $8x^2 + 8y^2 = 24$ describes a circle. Determine the x- and y-intercepts of the graph of this equation.

Activity 3

3.1 The following diagram shows the graph of a rectangle on a two-dimensional coordinate system. Determine the equations of the lines that contain the diagonals of this rectangle.
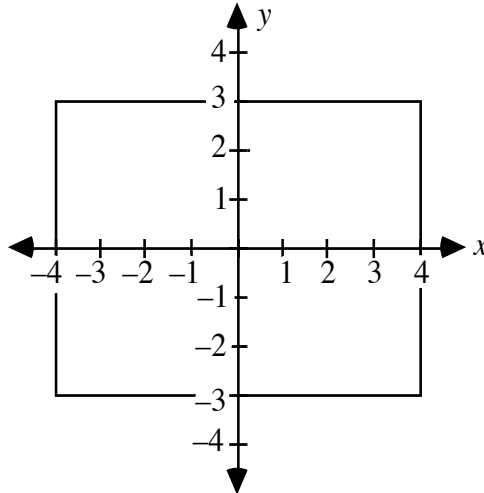
Activity 4

4.1 Solve each of the following equations for x.

    a. $16 = \dfrac{1}{4x}$

    b. $25 = 16x^2$

4.2 Sketch the path of the reflected ray on each of the following parabolas.

a.

light ray

b.

light ray
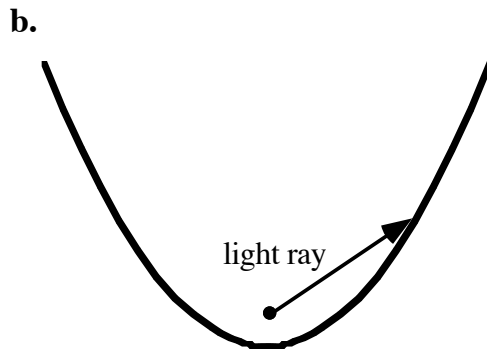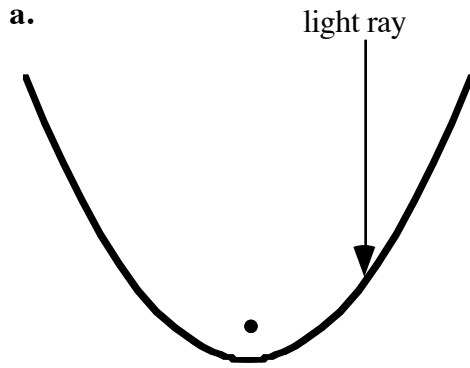
Answers to Flashbacks

Activity 1

1.1   **a.** Sample sketch:
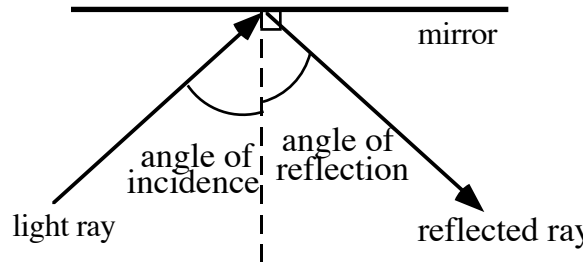
b. The angles are congruent.

1.2   Sample response: To draw a line tangent to a circle, construct a line perpendicular to the radius at the point of intersection with the circle, as shown in the diagram below.
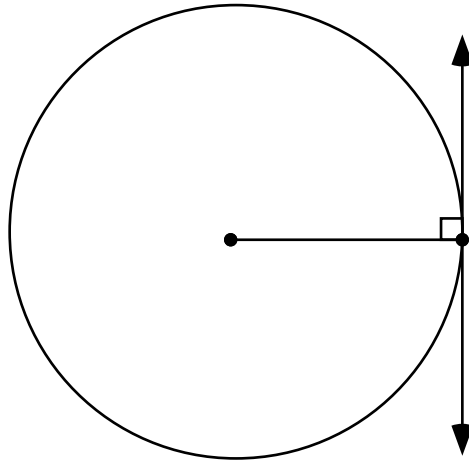
1.3   As shown in the following sketch, rotating line *l* about the *y*-axis results in a double-napped cone with its vertex at the origin.
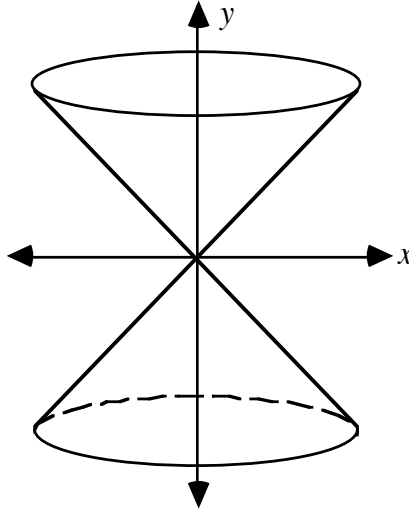
Activity 2

2.1   **a.** $PA = \sqrt{(x-4)^2 + (y-0)^2} = \sqrt{(x-4)^2 + y^2}$

        b. $PB = \sqrt{(x+4)^2 + (y-0)^2} = \sqrt{(x+4)^2 + y^2}$

        c. $PA + PB = \sqrt{(x-4)^2 + y^2} + \sqrt{(x+4)^2 + y^2}$

2.2   The *x*-intercepts are $-\sqrt{3}$ and $\sqrt{3}$. The *y*-intercepts are $-\sqrt{3}$ and $\sqrt{3}$.

Activity 3

3.1   The equations of the lines are $y = (3/4)x$ and $y = -(3/4)x$.

Activity 4

    4.1   **a.** $x = 1/64$

            b. $x = 5/4$ or $x = -5/4$

    4.2   **a.** Sample sketch:

light ray

reflected
ray

           b. Sample sketch:

reflected
ray

light ray

Ellipse Template

A ●

●

Circle Template

A •
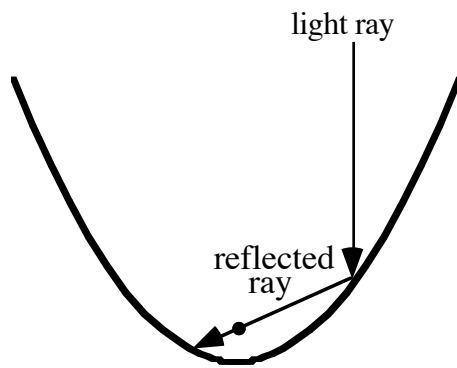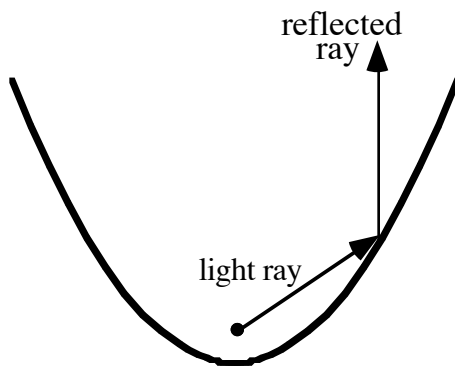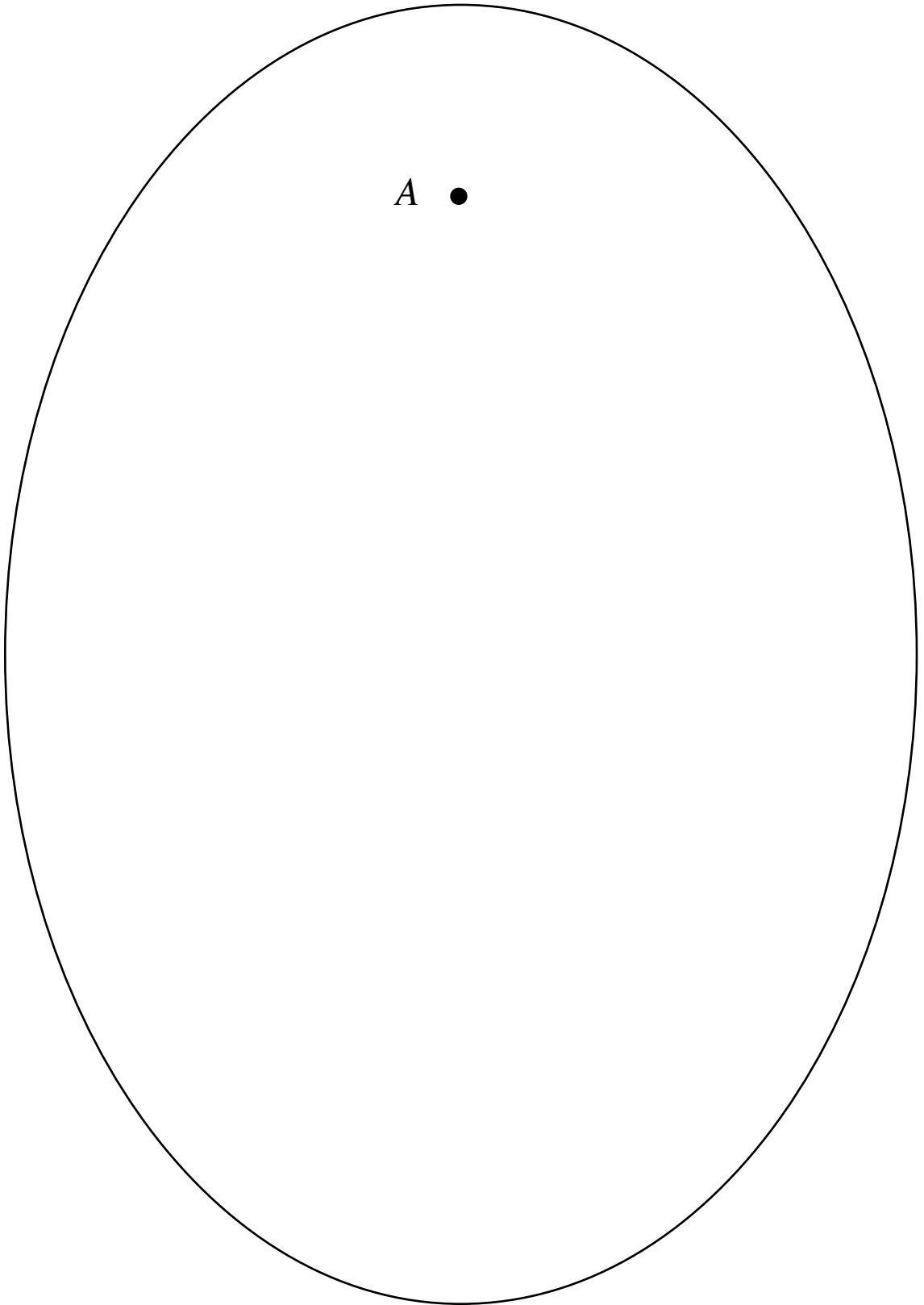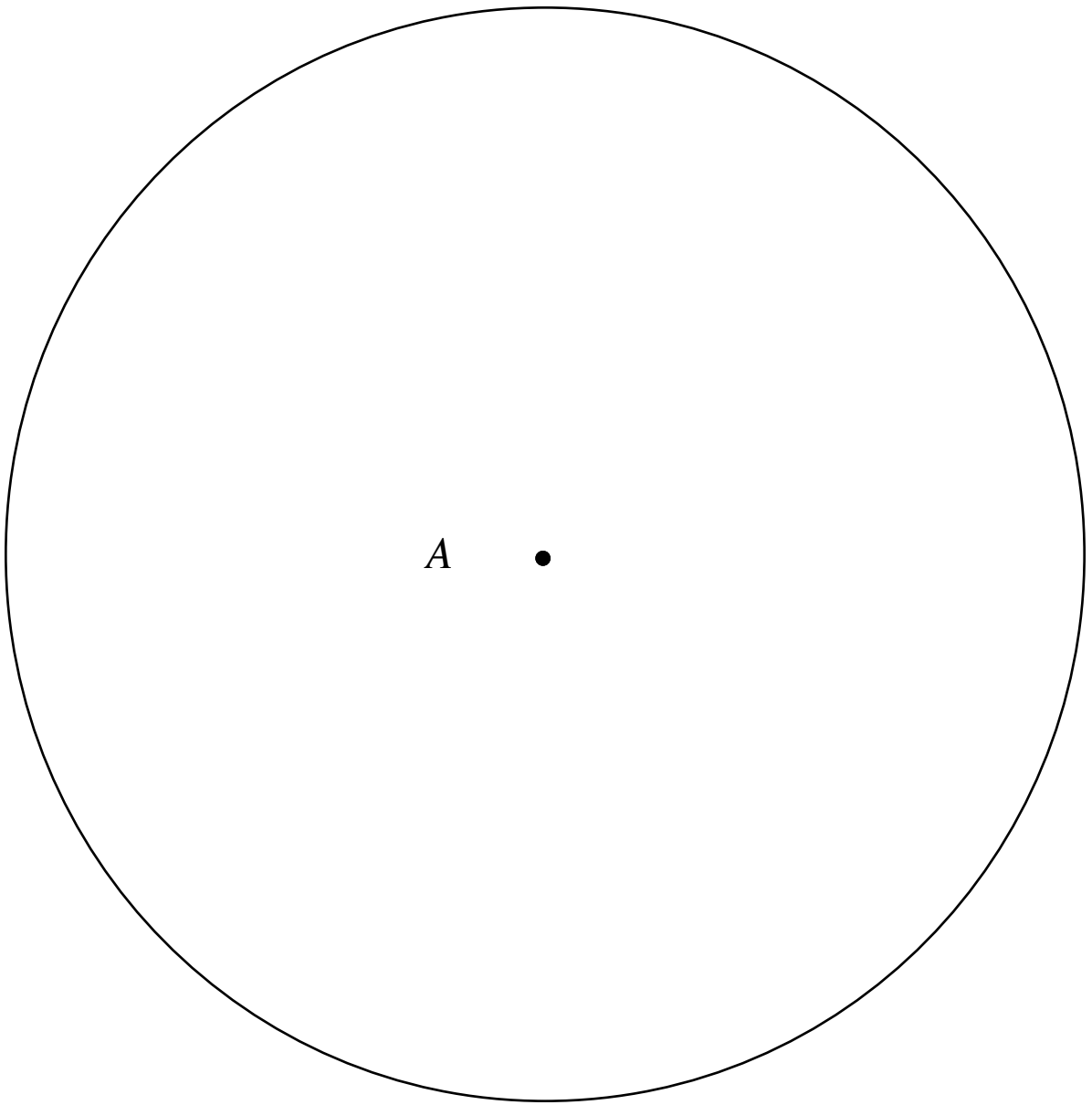
Parabola Template
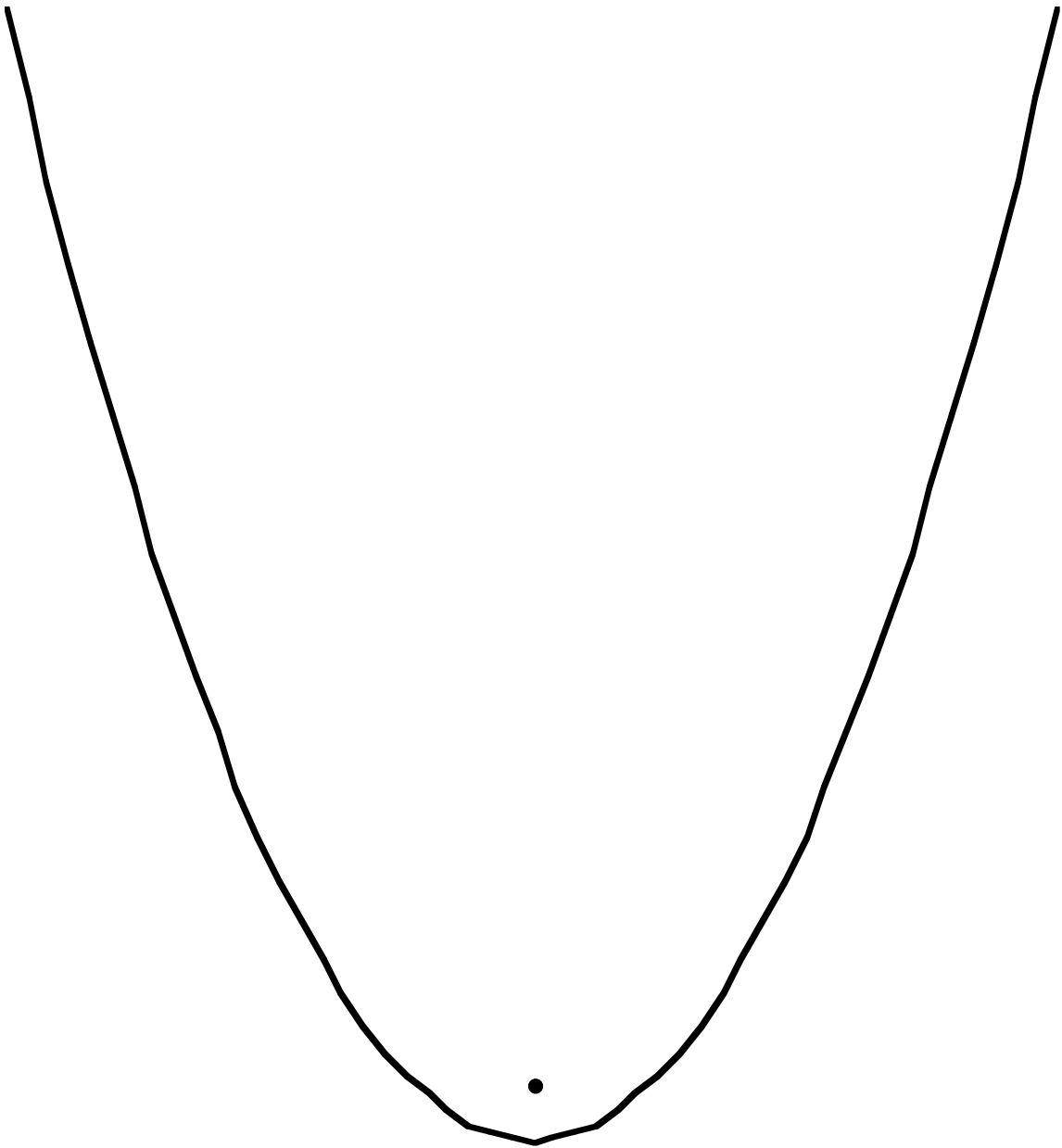
Hyperbolic Mirror Template
for Problem 1.2
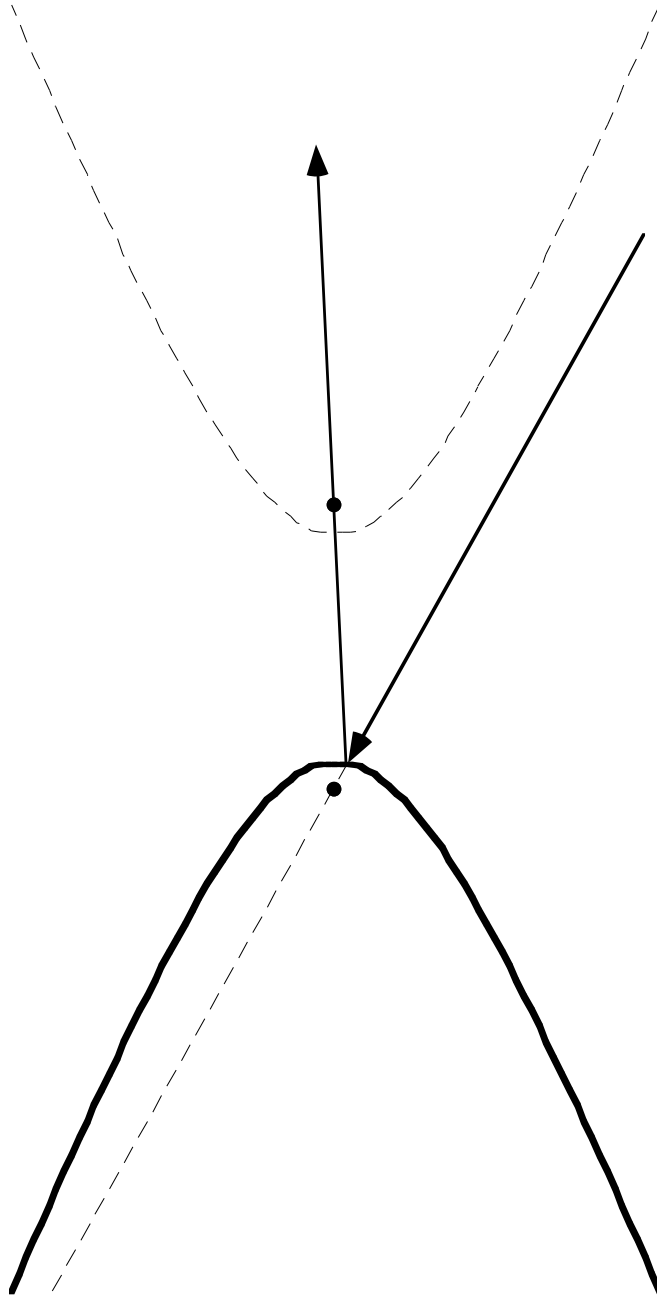
Wall Heater Template
for Problem 3.5
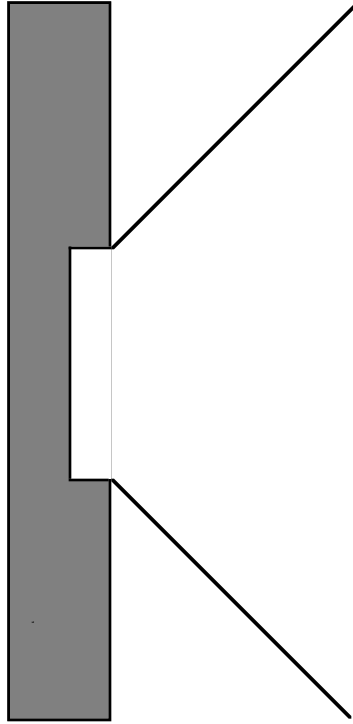
Scale: 1 mm = 2 cm

Risky Business

Teacher Guide
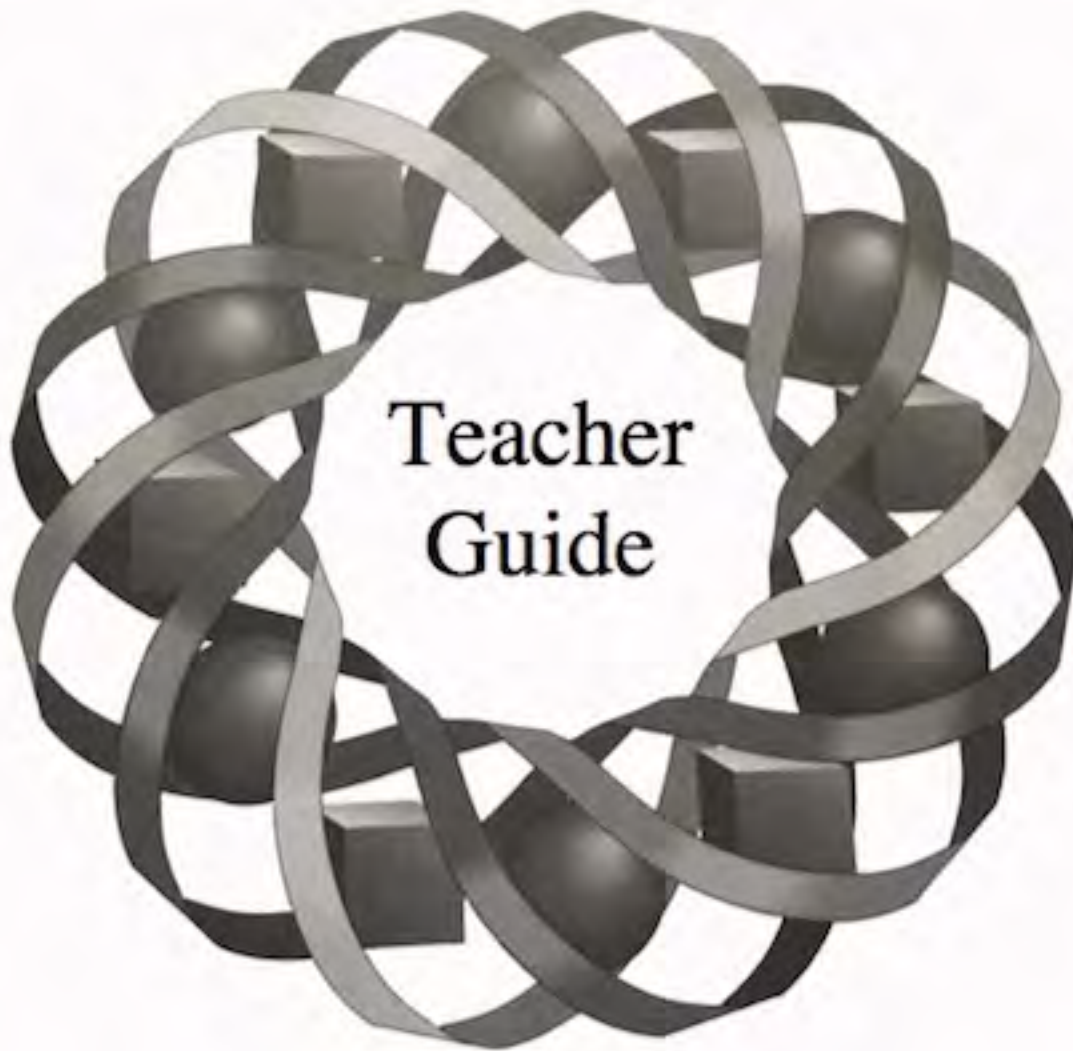
Why buy automobile insurance? The rising costs of accidents—and an examination of their probabilities—can help explain why people choose to share the economic risks of driving their cars.

Randy Carspecken • *Todd Fife*

Risky Business

Overview

In this module, students investigate random variables and probability distributions in the context of automobile insurance. They use expected value to suggest insurance premiums and the law of large numbers to predict claim values.

Objectives

In this module, students will:

- investigate the law of large numbers
- examine random variables and their probability distributions
- determine the expected value of random variables.

Prerequisites

For this module, students should know:

- how to calculate experimental and theoretical probabilities
- how to calculate expected value
- how to model exponential growth.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	2	1	9

Materials Required

- none

Technology

Software	Activity			
	1	2	3	Summary Assessment
statistics package	X	X	X	X
graphing utility	X	X		
spreadsheet		X	X	X

Risky Business

Introduction

The sample premium renewal notice in Figure **1** introduces students to a typical automobile insurance policy.

Discussion

a. The effective dates of coverage are noted in the upper left-hand portion of the figure. Since coverage is effective for 6 months, the listed premium must be paid twice per year. The annual premium is $\$316.50 \cdot 2 = \$633.00$.

b. Answers may vary. This was the average premium in 1995 for a person in the 16–20 age category driving a family automobile.

c. The premium for an individual in the 35–44 age category, driving the same type of vehicle, would normally be lower.

d. Sample response: It depends on the cost of the accident. If the policyholder is at fault, the insurance company is likely to place the driver in a higher risk category. This change may be reflected in a higher premium on the next renewal notice. After a period in which the policyholder is accident-free, the driver may be moved to a lower risk category with a reduced premium.

e. Answers will vary. Most companies consider age, driving record of the principal driver, type of automobile, number of miles driven annually, and the type of use (for example, whether or not the car is used to commute to work). Geographic location also is important. Premiums for urban drivers are typically higher than those for rural residents.

f. Sample responses:

- Bodily injury liability insurance—an individual crossing a busy street is hit by the insured automobile.

- Collision insurance—the driver of the insured automobile slides off an icy road and into a telephone pole, damaging the car.

- Comprehensive insurance—the insured automobile is dented during a severe hail storm.

- Property damage liability insurance—the driver of the insured automobile misjudges a turn, damaging the fender of another car.

g. Students may notice some of the following:

- Classification—the type of automobile (for example: sports car, passenger car, luxury car, or pickup truck).

- Term—the number of months that the policy covers.

- Accident/Violation points—reflects the number of accidents involving the driver and the number of violation points accumulated against the driver's license.

- Territory—the area of the state in which the driver lives.

- Deductible—the amount the driver must pay for damages incurred before the insurance company will pay a claim.

- Uninsured motor vehicle and bodily injury—covers the driver and passengers of the insured automobile if it is involved in an accident with an automobile that is not insured.

(page 136)

Activity 1

Students calculate the probabilities of having an accident using historical statistics. These probabilities are then used to develop a simulation of the annual driving record for an individual in the 16–20 age category. The law of large numbers is introduced in this activity.

Materials List

- none

Technology

- statistics package
- graphing utility

Exploration   (page 136)

a. Sample response:

$$P(\text{wreck}) = \frac{15,775}{100,000} \approx 0.16$$

b. Simulations will vary. For example, students may use technology to generate a random number between 0 and 1. If the number is less than or equal to 0.16, the driver was involved in an accident. If the number is greater than 0.16, the driver was not involved in an accident.

c. Sample table:

Number of Policyholders	Number of Accidents	Accident Rate
5	2	0.40
25	5	0.20
50	11	0.22
100	14	0.14
250	40	0.16
500	78	0.16
1000	157	0.16

d. The table below shows the difference in accident rates obtained by one class.

Number of Policyholders	Difference in Accident Rates
5	0.20
25	0.20
50	0.10
100	0.06
250	0.08
500	0.06
1000	0.02

Discussion (page 138)

a. Sample response: It is reasonable because the table is a summary of accident statistics for the entire United States in 1992. Therefore, it contains data for a large number of people.

b. Answers will vary. Up to age 69, the probability that drivers will be involved in an accident decreases as age increases. The statistics in Table **1** yield the following probabilities for 1992:

Age	Probability of an Accident
16–20	0.16
21–24	0.10
25–34	0.07
35–44	0.06
45–54	0.05
55–64	0.04
65–69	0.03
69 or older	0.04

c. Answers will vary. One drawback is that the simulation does not account for a person having more than one accident per year.

d. Students should observe that, when the number of policyholders is small, the accident rates may vary by a large amount from simulation to simulation (year to year). As the number of policyholders increases, the accident rates usually vary by a lesser amount.

e. The difference between the maximum and minimum accident rates for should become smaller as the number of policyholders increases.

f. When the number of policyholders is small, the accident rate may differ by a relatively large amount from the theoretical probability of having an accident. As the number of policyholders increases, the accident rates should be closer to this probability.

g. Sample response: If the simulation were continued with larger groups, the accident rate should approach 0.16.

h. Sample response: A group of 100 drivers may be too small to generate reliable statistics. The difference in the accident rates from year to year could be large.

i. Sample response: Yes. Since the sample size is very large, the law of large numbers indicates that there is high probability that the sample proportion is very close to the population proportion.

Assignment

(page 139)

*1.1 **a.** Sample response: Use a random number generator to generate a number between 0 and 1. If the number is less than or equal to 0.5, the baby is a girl.

b. Sample data:

Number of Births	Proportion of Girls
10	0.30
20	0.60
40	0.38
80	0.64
100	0.48
200	0.52
300	0.48
400	0.49
500	0.50

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c. The following graph shows a scatterplot of the sample data in Part **b**.
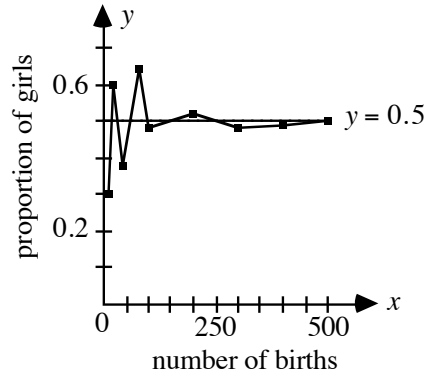
number of births

d. Sample response: Yes. The graph illustrates the law of large numbers because, as the number of births increases, the proportion of babies that are girls approaches the theoretical probability of a child being a girl.

1.2 **a.** Sample response: The accident rate allows the insurance company to predict the number of accidents in which a certain group might be involved. They can use this to predict the cost of claims, which in turn can be used to determine an appropriate premium.

   b. Sample response: In general, the actuary can be more confident when making predictions as the size of the group gets larger.

1.3 A number of different factors can influence the accident rate. Changes in these factors from year to year limit the accuracy of predictions. In the past two decades, for example, the enforcement of strict drunken driving laws and the development of new safety features (such as anti-lock brakes) have made driving safer. These and other factors might produce an accident rate lower than the expected one. On the other hand, a severe winter might contribute to a slightly higher accident rate.

1.4 **a.** The probability of a bodily-injury accident can be calculated as follows:

$$\frac{5753}{100,000} = 0.05753 \approx 0.06$$

The probability of a property-damage-only accident is:

$$\frac{10,022}{100,000} = 0.10022 \approx 0.10$$

The probability of no accident is $1 - (0.06 + 0.10) = 0.84$.

   b. Sample response: The sum of the probabilities of all the events in a sample space is 1. In this case, 1 represents the sum of the probabilities for a bodily-injury accident, a property-damage-only accident, and no accident.

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c. Sample table:

		Accident Probabilities		
Age	**Bodily Injury**	**Property Damage Only**	**No Accident**	**Total**
16-20	0.06	0.10	0.84	1.00
21–24	0.04	0.07	0.89	1.00
25–34	0.03	0.04	0.93	1.00
35–44	0.02	0.04	0.94	1.00
45–54	0.02	0.03	0.95	1.00
55–64	0.01	0.03	0.96	1.00
65–69	0.01	0.02	0.97	1.00
69 or older	0.01	0.02	0.97	1.00

d. Answers will vary. Students should observe that the probabilities decrease in both accident columns as age increases. This trend may occur partly because more experienced drivers are safer drivers, and because elderly individuals tend to drive less frequently.

*1.5   **a.** The statistic representing accidents per 10,000 drivers favors drivers in the "75 and older" category (3 fatal accidents compared to 7).

b. The statistic that represents accidents per 100 million miles favors teenage drivers (9 fatal accidents compared to 11).

c. **1.** Sample response: For the statistics that represent fatal accidents per 100 million miles driven, the 40–44 age category has the best record. Using the statistics that represent fatal accidents per 10,000 drivers, the 60–64 age category has the best record.

   This is possible because drivers in different age groups tend to drive different numbers of miles.

2. Answers will vary. Sample response: Drivers in the 40–44 age category have the best driving skills because they have the fewest fatal accidents per 100 million miles driven. The other statistic may favor those who drive very little.

d. Responses will vary. Students may observe that beginning drivers have a high fatal accident rate as a result of inexperience and immaturity, which can lead to speeding, tailgating, rapid accelerations, and other overly aggressive driving practices. As drivers become more experienced, they tend to take fewer risks and have a lower accident rate. The rate of fatal accidents increases as age increases past 50, perhaps due to slower reaction times and diminishing eyesight.

* * * * *

1.6 **a.** In each case, the sample proportion can be used to estimate the population proportion.

1. $813/1084 = 0.75$

2. $41{,}600/64{,}000 = 0.65$

3. $41/90 \approx 0.46$

4. $25/750 \approx 0.03$

b. **1.** The survey of vehicle accidents should produce the most reliable estimate because it involves the largest sample.

2. The survey of 90 hospital patients is likely to produce the least reliable estimate since it is based on the smallest sample.

1.7 **a.** Sample response: It is not uncommon to flip a fair coin 5 times and obtain 4 heads. You should not be suspicious based on these results.

b. Answers will vary. The following simulation was created for the TI-92 calculator. (It took approximately 5 minutes to collect the required data.)

```
coin()
Prgm
ClrIO
For t,1,10
0→h
For f,1,1000
If rand() ≤ 0.5
h+1→h
EndFor
Disp "Trial number"
Disp t
Disp "Number of heads"
Disp h
EndFor
EndPrgm
```

The results for 10 runs of this simulation are shown below. None of the trials produced 800 or more heads.

Trial	No. of Heads	Trial	No. of Heads
1	512	6	527
2	500	7	495
3	469	8	519
4	505	9	505
5	483	10	493

c. Sample response: The coin is probably not fair. In Part **b**, none of the trials resulted in a proportion of heads greater than 0.6. If the coin were fair, the law of large numbers indicates that the proportion of heads would be close to 0.5 with very high probability.

* * * * * * * * * *

Activity 2

Students calculate the expected costs of accidents using claim statistics. Random variables and probability distributions are used to examine claim costs.

Materials List

* none

Technology

* graphing utility
* spreadsheet
* statistics package (optional)

Exploration

a. The rows in the following table show claim values in each decile sorted from left to right.

Decile	Sorted Claim Values ($)			
1	52	89	94	111
2	175	250	290	388
3	410	562	570	700
4	888	995	1131	1244
5	1402	1452	1802	2252
6	3250	3300	3563	4001
7	3900	4200	4300	4589
8	4602	5300	7200	8550
9	12,889	13,202	15,236	15,461
10	20,336	20,500	23,555	23,898

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b. A completed table is shown below.

Decile	Mean Claim Value ($)
1	86.50
2	275.75
3	560.50
4	1064.50
5	1727.00
6	3528.50
7	4247.25
8	6413.00
9	14,197.00
10	22,072.25

c. Answers may vary. Sample response: Since consumer prices appear to be rising at 4.1% annually, this situation can be modeled by an exponential equation of the form $y = a \cdot b^x$. In this case, a represents the mean claim value in 1990, $b = 1 + r$, where r is the annual rate of increase, and x represents years after 1990. The table below shows the predicted values for 1992 using the equation $y = a(1.041)^2$.

Decile	Predicted Mean ($)
1	93.74
2	298.83
3	607.40
4	1153.58
5	1871.52
6	3823.77
7	4602.66
8	6949.65
9	15,385.02
10	23,919.28

Discussion 1

a. Sample response: The first step in finding the mean decile values is to sort the claims from least to greatest. Next, divide the claims into 10 sets of 2.5 million claims each. The mean of each set of 2.5 million claims is the mean decile value.

b. Since the claims are divided into 10 deciles, the total number of claims is $10 \cdot 1,000,000 = 10,000,000$.

c. Sample response: Since consumer prices increased by an average of 4.1% annually, an exponential model of the form $y = a \cdot 1.041^x$ seemed appropriate. In this case, a represents the mean claim value in 1990 and x represents years after 1990.

    Note: This is a very simplified model for an extremely complex situation. Actual increases (or decreases) in average claim costs are affected by changes in medical costs, numbers of weather-related accidents, and the use of deductibles, along with many other factors.

Exploration 2   (page 144)

a. **1.** Since each decile represents 10% of the total number of claims, the number of claims per decile is $0.1(120,000,000) = 12,000,000$.

     2. The probability that a claim value for a given category will fall in any one decile is 0.1.

b. The table below shows the probability distribution for C in 1992:

Decile	Bodily Injury		Property Damage	
	Claim ($)	Probability	Claim ($)	Probability
1	125	0.006	47	0.01
2	378	0.006	150	0.01
3	753	0.006	266	0.01
4	1400	0.006	389	0.01
5	2300	0.006	527	0.01
6	3550	0.006	699	0.01
7	5400	0.006	936	0.01
8	8000	0.006	1313	0.01
9	13,250	0.006	2061	0.01
10	24,380	0.006	3462	0.01
	No Accident			
	0	0.84		

c. The expected value of C is approximately $456. This value is calculated by multiplying the mean claim value for each outcome by the corresponding probability, then adding these products.

Discussion 2   (page 146)

a. Sample response: The expected value of C represents the average amount the insurance company would expect to pay in claims per policyholder.

b. Sample response: There are 21 possible values for C: 10 bodily-injury claim values, 10 property-damage claim values, and 1 no-accident claim value.

c. Sample response: The expected value of C gives the company an idea of what they should expect to pay in claims. However, since the company also has to cover the costs of doing business, as well as earn a profit, they must charge more than the expected mean claim per policyholder.

d. Sample response: In a weighted mean, each value is assigned a certain relative weight. When calculating expected value, each possible value of the random variable is given the "weight" of its corresponding probability.

Teacher Note

The sample responses given below show predicted values for 1999. Students should give predictions for the current year.

Assignment (page 146)

2.1 **a.** One way to estimate the total value is to multiply each mean decile value by the corresponding number of claims in the decile, then find the sum of these products. Since the numbers in each decile for a given category are the same, this is the same as multiplying the sum of the mean decile values by the number of claims in a decile, then finding the sum for the two categories:

$$(\$59,536 \cdot 2,500,000) + (\$9850 \cdot 12,000,000) \approx \$267,400,000,000$$

b. The cost per person is approximately \$1045.

2.2 **a.** One possible model is $y = a \cdot 1.026^x$, where a represents the mean claim value in 1992 and x represents number of years after 1992.

b. The following sample table shows predicted values for 1999.

Decile	Bodily Injury ($)	Property Damage ($)
1	150	56
2	452	180
3	901	318
4	1676	466
5	2753	631
6	4249	837
7	6463	1120
8	9575	1571
9	15,858	2467
10	29,179	4143

c. The following table shows the probability distribution for C in 1999. The expected value is approximately $367.54.

	Bodily Injury		Property Damage	
Decile	Claim ($)	Probability	Claim ($)	Probability
1	150	0.004	56	0.007
2	452	0.004	180	0.007
3	901	0.004	318	0.007
4	1676	0.004	466	0.007
5	2753	0.004	631	0.007
6	4249	0.004	836	0.007
7	6463	0.004	1120	0.007
8	9575	0.004	1571	0.007
9	15,858	0.004	2467	0.007
10	29,179	0.004	4143	0.007

No Accident	
0	0.89

2.3 **a.** Sample response: The mean claim values can be found by dividing the sum of the predicted decile values by 10. The predicted mean bodily-injury claim in 1999 is about $7125.60. The predicted mean property-damage-only claim in 1999 is about $1178.80.

b. Sample table:

Outcome	Bodily Injury	Property Damage	No Accident
Value of C	$7125.60	$1178.80	$0
Probability	0.04	0.07	0.89

c. The expected value of C can be found as follows:

$$E(C) = \$7125.60(0.04) + \$1178.80(0.07) + \$0(0.89) = \$367.54$$

This indicates that the insurance company can expect to pay $367.54 in claims per policyholder in the 21–24 age group.

2.4 Students should show algebraically that the two equations for expected value given in the student edition are equivalent. **Note:** You may wish to emphasize this relationship, as it gives students another method of calculating expected value.

2.5 **a.** Sample table:

Outcome	Bodily Injury	Property Damage	No Accident
Value of C	$7125.60	$1178.80	$0
Probability	0.01	0.02	0.97

b. In this case,
$E(C) = \$7125.60(0.01) + \$1178.80(0.02) + \$0(0.97) \approx \$94.83$.
This indicates that the insurance company can expect to pay $94.83 in claims per policyholder in the 65–69 age group.

c. Sample response: The expected claims per policyholder in the 21–24 age category are about $273 more than the expected claims per policyholder in the 65–69 age category. This difference results from the difference in the probabilities of being involved in an accident for the two age groups. To compensate for the higher expected claims, insurance companies charge higher premiums to insure the younger drivers.

* * * * *

2.6   **a.** $P(N) = 1 - P(H) = 0.685$

b. The table below shows the probability of each outcome.

Outcome	Probability	Outcome	Probability
HHHH	0.010	NNHH	0.047
HHHN	0.021	NHHN	0.047
HHNH	0.021	HNNH	0.047
HNHH	0.021	NNNH	0.101
NHHH	0.021	NNHN	0.101
HHNN	0.047	NHNN	0.101
HNHN	0.047	HNNN	0.101
NHNH	0.047	NNNN	0.220

c. The following table shows the completed probability distribution.

Hits	Probability
0	0.220
1	0.404
2	0.282
3	0.084
4	0.010

d. Sample response: Since it is the most likely of the five outcomes, I predict the player will get 1 hit in the next 4 at-bats.

2.7   **a.** Sample response: The mean of the group-work scores is 84.4. The mean of the project scores is 82.5. The mean of the test scores is 88.5. The final grade can be found as follows:
$$0.20(84.4) + 0.30(82.5) + 0.50(88.5) \approx 86$$

b. Sample response: This method is like the calculation of expected value in that each score is multiplied by a percentage, and the sum of the percentages is 1. In this case, however, the percentages represent relative weights instead of probabilities.

Activity 3

Students create a simulation that models the operation of an insurance company. The business terms *pure premium*, *overhead*, and *loss ratio* are introduced and applied in the assignment problems.

Materials List

*   none

Technology

*   spreadsheet
*   statistics package

Teacher Note

There are many ways to conduct the simulation for the exploration, including the use of statistics packages, calculator programs, and spreadsheets. One possible simulation is outlined below; another is described in the sample response to Part **b**.

The following method involves the use of a spreadsheet.

*   The probability of a claim falling in the no-accident category is 0.84 or $420/500$. In one column, enter 420 zeros.

    Note: On some spreadsheets, students may avoid the need to enter the zeros by hand using a built-in function. On the TI-92, for example, the following command fills a column with 420 zeros: seq(0,x,1,420,1). Once the zeros have been entered, the formula is removed to unlock the column. The claim values then can be entered as described below.

*   The probability of a claim falling in each decile for bodily-injury accidents is 0.006 or $3/500$. In the same column as above, enter each claim value in the bodily-injury category three times.

*   The probability of a claim falling in each decile for property-damage-only accidents is 0.01 or $5/500$. In the same column as above, enter each claim value in the property-damage-only category five times.

*   In the adjacent column, enter 500 random numbers.

*   Sort the columns together, using the one that contains the random numbers as the reference. This sort will randomize the 500 entries in the first column. The claim values for 20 policyholders, for example, can be represented by the values in the first 20 rows of the first column.

*   To determine the mean claim value for n policyholders, use the mean of the claim values in n rows.

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Exploration

a. $E(C) = 0.006(70,005) + 0.01(9193) \approx \$512$

b.   The following sample simulation can be completed using the random number generator on a calculator. Students may program their calculators to give all three random numbers or enter the appropriate generator at each stage of the simulation.

   1. Sample response: Generate a random number between 0 and 1. If the number is 0.16 or less, the driver has an accident during the year. If not, the driver is accident-free for the year. (If no accident occurs, skip Steps **2** and **3**.)

   2. Sample response: The probability of being involved in a bodily-injury accident is 0.06 or 6%. The probability of having a property-damage-only accident is 0.10 or 10%. Since the probability of being involved in an accident of either type is 16%, the proportion of accidents that involve bodily injury is 6/16 or 0.375.

      To decide if a claim is for bodily injury or property damage only, generate a random number between 0 and 1. If the number is less than or equal to 0.375, the claim is for bodily injury. If the number is greater than 0.375, the claim is for property damage only.

   3. Sample response: Generate a number between 0 and 1. If the number is less than or equal to 0.1, the claim is in the first decile. If the number is greater than 0.1 and less than or equal to 0.2, the claim is in the second decile. This pattern can be continued for the remaining deciles.

   4. Students record the cost of the claim, if any.

c.   Each student should complete this step. Sample response: The first random number generated was 0.04. Since $0.04 \leq 0.16$, the driver had an accident. The second random number generated was 0.56. Since $0.56 > 0.375$, the accident was for property damage only. The third random number generated was 0.34. Since $0.3 < 0.34 < 0.4$, the claim value falls in the fourth decile. From Table **9**, the claim value is \$319 (fourth decile of "Property Damage Only" category).

d.   1. To find the total for the class, students should add the claim amounts for those policyholders involved in accidents.

   2. Students divide the total from Step **1** by the number of students in the class.

   3. The difference between the mean claim and the individual premium (\$512) is the profit or loss per policyholder. A negative number indicates a loss.

   4. The total profit or loss is found by multiplying the profit or loss per policyholder by the number of students.

e. Students should collect class data for each trial. The following sample data is based on 20 policyholders.

Simulation Number	Premium ($)	Mean Claim Value ($)	Profit or Loss Per Person ($)	Total Profit or Loss ($)
1	512	47	465	9300
2	512	0	512	10,240
3	512	1413	–901	–18,020
4	512	80	432	8640
5	512	141	371	7420
6	512	598	–86	–1720
7	512	320	192	3840
8	512	0	512	10,240
9	512	155	357	7140
10	512	2064	–1,552	–31,040

1. Using the sample data, the mean claim per policyholder is about $482.

2. Using the sample data, the mean profit per policyholder is about $30.

3. Using the sample data, the mean annual profit for the insurance company is about $600.

f. **Note:** To reduce the time required to collect data for 100 policyholders, you may wish to divide the 10 trials among class members. Sample data:

Simulation Number	Premium ($)	Mean Claim Value ($)	Profit or Loss Per Person
1	512	1053	–541
2	512	628	–116
3	512	314	198
4	512	533	–21
5	512	328	184
6	512	350	162
7	512	533	–21
8	512	283	229
9	512	482	30
10	512	301	211

1. Using the sample data, the mean claim per policyholder is about $480.

2. Using the sample data, the mean profit per policyholder is about $31.50.

3. Using the sample data, the mean annual profit for the insurance company is about $3150.

Discussion

a. Sample response: In the simulation, the probability of having an accident is the same for all policyholders. In real life, some people drive more safely than others, decreasing the risk of an accident. Other people have bad driving habits and are more likely to file a claim. The probabilities and claim amounts are also likely to change over a 10-year period. In addition, the simulation does not allow a person to have more than one accident per year.

b. **1.** Sample response: There are other costs to the insurance company besides paying claims, such as wages for employees and administrative expenses. The insurance companies try to make a profit, so the actual premium would have to be greater than the expected value of claims per policyholder.

  2. Sample response: No. If premiums are set too high, then consumers will buy insurance elsewhere. To stay competitive, companies must set a reasonable premium, given their expected costs.

c. Answers will vary. In the sample data given in the exploration, the company lost money 3 of the 10 years for a class of 20. Using 100 policy holders, the company averaged a larger profit, but still lost money 4 of the 10 years.

d. Sample response: For small groups, one large claim could cost each member a great deal, while the same claim divided among a large group would be less costly for each member. On the other hand, there is a greater likelihood for more large claims in a large group.

e. As the number of policyholders in the simulation increases, the amount of variability is likely to decrease. The experimental means should approach the theoretical means.

f. Sample response: As the number of policyholders increases, the number of claims should approach the theoretical number of claims. Therefore the resulting claim value per policyholder should approach the expected value.

g. Answers will vary. In the sample data for a class of 20, the largest claims cost was in year 10, when the loss per policyholder was $31,040/20 = \$1552$. To ensure that the company would not lose money in such a year, the premium would have to be $\$512 + \$1552 = \$2064$.

  For 100 policyholders, the greatest loss per policyholder was \$541. To guard against such a loss, the annual premium would have to be $\$512 + \$541 = \$1053$. These premiums do not guarantee that the company will avoid losses in future years. Some students may argue that the company could lose money in some years, as long as the average profit was positive.

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Assignment

*3.1   Students should focus on the law of large numbers in this situation: the larger the number of policyholders, the closer the mean number of claims is likely to be to the theoretical mean and the closer the mean claim value is likely to be to the theoretical mean. This makes it easier to accurately predict the number of claims the company will have to pay.

3.2   **a.** Sample response: The premium used for the simulation was the expected cost of claims per policyholder. This does not take into account the amount needed for overhead and profit.

     b. Sample response: The full premium would be $788, since $512 is 65% of $788.

     c. Based on a full premium of $788, the mean annual profit would be 5% of $788 or $39 per policyholder.

3.3   **a.** Sample response: To keep losses low and profits high, an insurance company would want the loss ratio to be low.

     b. Sample response: Once the insurance company has determined the expected claims cost per policyholder, these costs are 65% of the premium. Therefore, the theoretical loss ratio is:

$$\frac{\text{Total claim costs}}{\text{Total premiums}} = \frac{65\%}{100\%} = 0.65 = 65\%$$

     c. Answers will vary. The following responses are based on the sample data given in the exploration and a full premium of $788.

       1. The largest loss ratio occurs when the greatest claims cost occurs. For the simulation with 20 policyholders, this is $2064/$788 ≈ 262%; the smallest loss ratio is $0/$788 = 0%.

       2. For the simulation with 100 policyholders, the largest loss ratio is $1053/$788 ≈ 134%; the smallest is $283/$788 ≈ 36%.

       3. Since the law of large numbers indicates that the experimental probability will approach the theoretical probability, the largest and the smallest should be very close to the same value: $0.65 = 65\%$.

     d. Sample response: Insurance companies would want to insure a large enough group of people to enable a reasonably accurate prediction of claim costs. The answers given in Part **c** reveal that insuring only 100 drivers is a risky business.

3.4 Sample response: The average number of claims is 9% of 3500 policyholders or 315. With an average claim value of $707, this amounts to a cost of $222,705. The pure premium for each policyholder would be:

$$\frac{\$222,705}{3500} = \$63.63$$

The full premium would be $\$63.63/0.65$, or $\$97.89$.

*3.5 a. Sample response: From the tables, the average bodily-injury claim is about $6384. The probability that a 17-year-old will be involved in a bodily-injury accident is about 0.06. The pure premium for bodily-injury liability is therefore $0.06(\$6384) = \$383.04$.

The average property-damage-only claim is about $1084. The probability that a 17-year-old will be involved in a property-damage-only accident is about 0.10. Therefore, the pure premium for property-damage-only liability is $0.10(\$1084) = \$108.40$.

The full premium should be $(383.04 + 108.40)/0.65 \approx \$756.06$.

b. The probability that a 67-year-old will have a bodily-injury accident is approximately 0.01. The pure premium for bodily injury liability is $0.01(\$6384) = \$63.84$.

The probability that a 67-year-old will be involved in a property-damage-only accident is about 0.02. The pure premium for property-damage-only is $0.02(\$1084) = \$21.68$.

The full premium should be $(63.84 + 21.68)/0.65 = \$131.57$.

c. The annual premium for the 17-year-old is almost 6 times more than the premium for the 67-year-old. The difference is $624.49.

* * * * *

3.6 a. $p_6 = 1 - (p_1 + p_2 + p_3 + p_4 + p_5) = 0.9908$

b. Sample response:

$$E(X) = 0.00175(-\$39,800) + 0.00181(-\$39,600) +$$
$$0.00184(-\$39,400) + 0.00189(-\$39,200) +$$
$$0.00191(-\$39,000) + 0.9908(\$1000)$$
$$= \$628.40$$

c. 1. Sample response: If the insured does not die, the company will have an income of $1000 over the 5-year period. However, if the insured dies, the company will lose at least $39,000. The potential gain is very small compared to the potential loss. Even though there is a very small chance that the insured will die in the 5-year period, the financial risk is too great.

2. Sample response: The law of large numbers indicates that the proportion of insured that die is very likely to be close to the probabilities given in the table. Because of this, the income per policyholder will be close to the expected value.

3. The company profit should be close to
$10,000(\$628.40) = \$6,284,000$.

* * * * * * * * * *

Research Project

Students may contact an insurance agent to obtain a list of applicable discounts. (Those who currently have insurance already may have this information.) The most common discounts are for maintaining a certain grade-point average and for completion of a driver's education course. These discounts reflect the fact that such teenagers have lower probabilities for filing claims.

Answers to Summary Assessment (page 155)

1. Student simulations will vary. Some students may wish to use random numbers, as described in the sample response to Part **b** of the exploration in Activity **3**. A completed probability distribution table is shown below.

Random Variable C				
Value of C	$0	$75	$125	$300
Probability	0.90	0.01	0.06	0.03

The expected value for C can be found as follows:

$$E(C) = 0.9(\$0) + 0.01(\$75) + 0.06(\$125) + 0.03(\$300) = \$17.25$$

2. The results of a sample simulation are shown in the following table:

Simulation Number	Claim Value per Policyholder ($)	
	20 Policyholders	100 Policyholders
1	0.00	15.00
2	21.25	17.00
3	6.25	29.00
4	12.50	11.00
5	6.25	17.00
6	10.00	16.00
7	27.50	15.00
8	15.00	25.00
9	0.00	19.00
10	12.50	16.00
Difference of High and Low Values	27.50	18.00
Mean Claim Value	11.13	18.00

In this sample data, the claim values for 20 policyholders vary widely on either side of the expected value. They vary less widely for 100 policyholders.

3. The pure premium should be the expected value for C, $17.25. Using the percentages described in Problem **3.2**, the full premium should be $17.25/0.65 = \$26.54$.

4.   Sample response: The variability for claims cost per policyholder with 100 policyholders is less than that for 20 policyholders. This implies more reliable predictions for a larger group of policyholders. The law of large numbers tells us that as the number of policyholders increases, the experimental probabilities of the events in the simulation are likely to approach the theoretical probabilities. For a large number of policyholders, the claim value per policyholder should approach the expected value of $17.25. With a large enough group of policyholders, I am confident that both claims cost per policyholder and profit can be predicted accurately.

1.	Since diamond rings are very expensive, most owners insure them against theft, loss, or damage. A company called Diamond Ring Insurance (DRI) pays an average of 756 claims per year for every 10,000 policyholders.

	What is the probability that a DRI policyholder will make a claim during a year?

2.	The following table summarizes the claim values (in dollars) for DRI in five selected years. Using this data, develop a model to predict claim values for this year.

Average Claim Value ($)					
Decile	1988	1990	1992	1994	1996
1	30	32	34	37	39
2	39	41	44	47	50
3	49	53	58	64	70
4	75	82	89	98	107
5	141	152	165	179	194
6	193	206	220	236	252
7	248	272	300	330	363
8	345	370	398	428	460
9	725	793	868	950	1039
10	11,800	12,337	12,898	13,485	14,099

3.	Let the random variable C represent predicted claim costs for this year. Create a probability distribution table for C and determine its expected value.

4.	a.	If DRI uses the expected value for C as the pure premium, what should they charge for the full premium? Explain your response.

	b.	How much annual profit should the company expect to earn?

Answers to Module Assessment

1. The probability of a claim is $756/10,000 = 0.0756$.

2. Answers will vary. The predicted claims in the table below were obtained using an exponential model, $y = m(1.039)^x$, where m is the decile value for 1988, x is the number of years since 1988, and y is the decile value for the year $(1988 + x)$. The growth rate was determined by finding the mean growth rate for all 10 deciles.

Decile	Predicted Claim Value ($) in 1998
1	44
2	57
3	72
4	110
5	207
6	283
7	364
8	506
9	1063
10	17,300

3. Answers will vary. The claim values shown in the table below are for the year 1998, using the model $y = m(1.039)^x$.

Claim Value ($)	Probability
0	0.92440
44	0.00756
57	0.00756
72	0.00756
110	0.00756
207	0.00756
283	0.00756
364	0.00756
506	0.00756
1063	0.00756
17,300	0.00756

The expected value can be calculated as follows:

$$E(C) = 0.9244(\$0) + 0.00756(\$44 + \$57 + \$72 + \$110$$
$$+ \$207 + \$283 + \$364 + \$506 + \$1063 + \$17,300)$$
$$\approx \$151$$

4.   **a.** Sample response: Assuming operating costs of 30% and a profit of 5%, the full premium should be $\$151/0.65 \approx \$232$.

b. Sample response: A typical insurance company would expect a profit of $\$232 \bullet 0.05 = \$11.60$ per policyholder.

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U.S. Department of State. *Addressing the Safety Issues Related to Younger and Older Drivers: A Report to Congress*. Washington, DC: U.S. Government Printing Office, 1993.

*For the latest statistics on highway accidents, contact The Highway Loss Data Institute, Washington, DC.

Flashbacks

Activity 1

1.1 Consider an experiment that involves rolling an ordinary six-sided die and recording the number that appears. In 10 trials, the following rolls are obtained: 3, 2, 1, 4, 5, 3, 2, 1, 6, 3.

Given this data, what is the experimental probability of each of the following?

   a. rolling a 2

   b. rolling an odd number

   c. rolling a number greater than 2

1.2 Given one roll of an ordinary six-sided die, what is the theoretical probability of each of the following?

   a. rolling a 2

   b. rolling an odd number

   c. rolling a number greater than 2

Activity 2

2.1 List the sample space for flipping three coins.

2.2 A carnival game involves tossing three fair coins. A player wins $2 if exactly two heads are showing, $4 if all three coins show the same side, and $0 for all other outcomes. What is the expected value of this game?

2.3 The wholesale price of a jacket is $95, while its retail price is $130. What is the percent increase in the price?

2.4 The equation $y = \$100 \bullet 1.07^x$ models the annual growth in the balance of a savings account. In this model, describe what is represented by each of the following:

   a. y

   b. $100

   c. 1.07

   d. x

Activity 3

3.1 In one card game, a player draws a card from a standard deck of 52 playing cards. The player wins $1.00 for drawing an ace, $0.50 for drawing a face card, and nothing for drawing any other card. What is the expected value of this game?

3.2 To attract more customers, a clothing store offers the following promotion: each customer draws a card from a deck of 100 cards. Each card indicates one of five discounts: 10%, 20%, 30%, 40%, or 50%. The deck contains 30 cards marked 10%, 40 cards marked 20%, 15 cards marked 30%, 10 cards marked 40%, and 5 cards marked 50%. What is the expected value, in terms of the percent discount, for this promotion?

Answers to Flashbacks

Activity 1

1.1 **a.** $2/10 = 0.2$

　　　b. $6/10 = 0.6$

　　　c. $6/10 = 0.6$

1.2 **a.** $1/6 = 0.1\overline{6}$

　　　b. 0.5

　　　c. $4/6 = 0.6\overline{6}$

Activity 2

2.1 $\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

2.2 $0.375(\$2) + 0.25(\$4) + 0.375(\$0) = \$1.75$

2.3 $(130 - 95)/95 \approx 0.37 = 37\%$

2.4 **a.** The variable y represents the account balance after x years.

　　　b. The quantity $100 represents the initial investment.

　　　c. The quantity 1.07 represents $(1 + r)$, where r is the annual interest rate.

　　　d. The variable x represents time in years.

Activity 3

3.1 $\dfrac{4}{52} \cdot \$1 + \dfrac{12}{52} \cdot \$0.50 + \dfrac{36}{52} \cdot \$0 \approx \$0.19$

3.2 The expected value for the discount is 22%.

Wilderness Wanderings

Teacher Guide
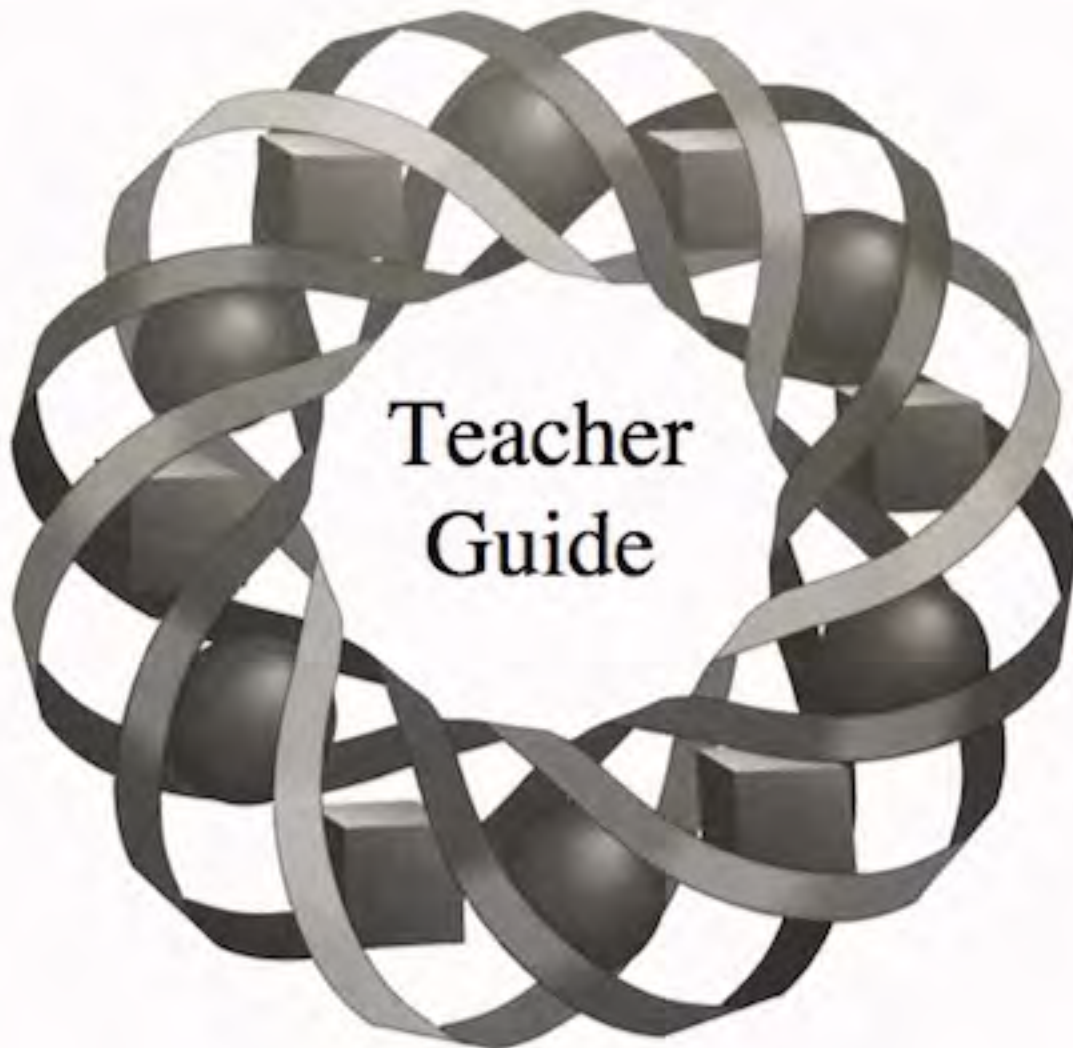
What do a hike through the forest, a ride down a country road, and the thrust of a jet engine have in common? They all can be modeled by vectors. In this module, you learn how to use vectors to represent displacement, velocity, and force.

Wendy Driscoll • Darlene Pugh • Peter Rasmussen

Wilderness Wanderings

Overview

Students explore vectors and vector addition in the context of a wilderness expedition.

Objectives

In this module, students will:

- model displacement, velocity, and force with vectors
- represent vectors graphically and as ordered pairs
- find resultant vectors both graphically and algebraically
- determine the horizontal and vertical components of vectors both graphically and algebraically.

Prerequisites

For this module, students should know:

- the Pythagorean theorem
- the trigonometric ratios of sine, cosine, and tangent
- the law of sines and the law of cosines
- how to find an angle measure given one of its trigonometric ratios.

Time Line

Activity	1	2	Summary Assessment	Total
Days	3	3	1	7

Materials Required

Materials	Activity		
	1	**2**	**Summary Assessment**
centimeter grid paper	X	X	X
protractor	X	X	X
metric ruler	X	X	X
template A	X		
map of United States	X		
transparent tape		X	
spring scales		X	
metal ring		X	
template B			X

Teacher Note

Blackline masters of templates A and B appear at the end of the teacher edition FOR THIS MODULE. The spring scales for use in Activity **2** should measure force in newtons (N).

Technology Required

Software	Activity		
	1	**2**	**Summary Assessment**
geometry utility	X	X	X
science interface device		X	

Wilderness Wanderings

Introduction

Students are introduced to displacement vectors using the azimuth as a measure of direction.

Materials List

* none

Teacher Note

A compass rose on a map shows true north. The needle on a compass, however, points to magnetic north.

Discussion

a. Answers will vary. If the teacher's desk is located 5 m southeast of the student's desk, for example, the route could be described by the vector (5,135), where the magnitude is given in meters and the azimuth in degrees.

b. Any angle measurement that unambiguously describes the direction of the vector will work—for instance, an angle measured counterclockwise from the positive *x*-axis. Other possibilities include measuring counterclockwise from north or using magnetic north as a reference instead of true north.

    Students also may suggest that an ordered pair on a coordinate plane can be used to describe a vector.

Activity 1

Students add vectors graphically by drawing tip-to-tail diagrams.

Materials List

* centimeter grid paper (several sheets per student)
* rulers (one per student)
* protractors (one per student)
* template A (for Problem **1.3**; a blackline master appears at the end of the teacher edition for this module)
* map of United States (for Problem **1.4**)

Technology

- geometry utility (optional)

Exploration

The sample responses given below were approximated using scale drawings. Students are expected to measure magnitudes and azimuths in Activity **1**. In Activity **2**, vectors will be resolved into vertical and horizontal components and right-triangle trigonometry will be used to determine the resultant.

a–d. Sample drawing:
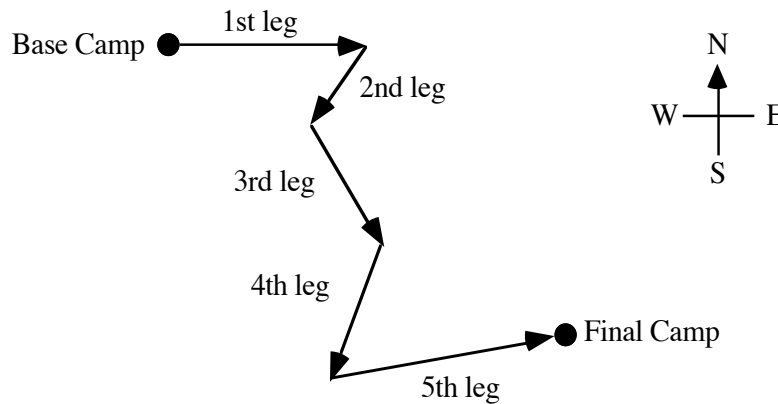
e. Students should measure the length and the azimuth of the resultant vector, represented by the bold arrow in the drawing below. Sample response: The resultant vector is approximately (10.3,127).
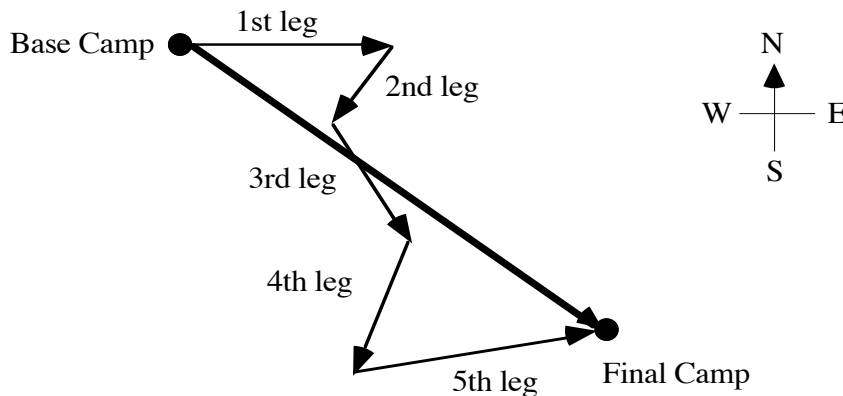
f. Students should discover that the order in which the vectors are added does not affect the resultant vector.

g. **1.** Students draw two opposite vectors. Answers will vary, depending on where students place the spring.

2. Students should observe that the magnitudes of the two vectors are the same, while their azimuths differ by 180°.

3. The resultant is a vector with 0 magnitude.

Discussion

a. **1.** Sample response: The two resultant vectors are the same.

2. Sample response: Yes. The resultant vector appears to have the same magnitude and direction, regardless of the order in which the vectors are added.

b. Vector addition is both commutative and associative. The resultant vector always has the same magnitude and direction, regardless of the order in which the vectors are added.

   Note: You may wish to illustrate the commutativity of vector addition using a parallelogram, as shown below. If two adjacent sides represent a vector sum, the two remaining sides represent the commuted sum. Both have the same resultant—a diagonal of the parallelogram.
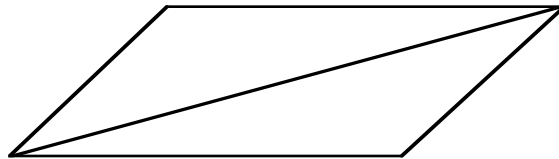
   The associativity of vector addition can be illustrated by considering a walk from a point A to a point B, followed by a walk from B to C to D. If one walks from A to B to C, then walks from C to D, the resultant displacement is the same.

c. **1.** Sample response: The magnitudes in the ordered pairs are equal. Their azimuths differ by 180°.

2. The resultant vector has a magnitude of 0.

Teacher Note

Each student will require a copy of template A to complete Problem **1.3** and a map of the United States to complete Problem **1.4**.

Assignment

1.1 **a.** The kayaker is paddling at an azimuth of 90°; the current is flowing at an azimuth of 270°.

b. The resultant vector has a magnitude of 8 km/hr with an azimuth of 90°.

c. The resultant vector has a magnitude of 14 km/hr with an azimuth of 270°.

d. Sample response: Using the tip-to-tail method, the resultant velocity has a magnitude of approximately 13.6 km/hr and an azimuth of approximately 242°.

(13.6,242)

(11,235)

(3,270)
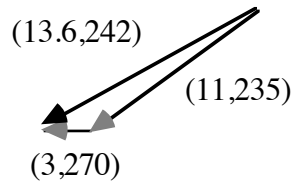
e. Sample response: Using the tip-to-tail method, the resultant velocity has a magnitude of approximately 3.9 km/hr and an azimuth of approximately 262°.

(3.9,262)

(1,235)

(3,270)
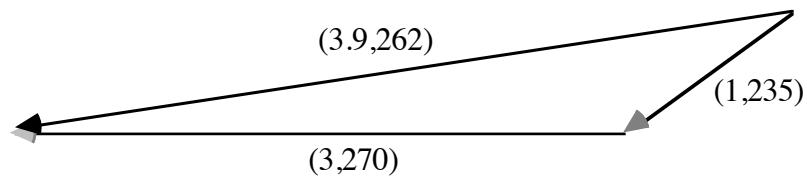
1.2   **a.** There are $3 \cdot 2 \cdot 1 = 6$ different orders for jogging these distances. Using E, N, and W to designate the directions, they can be listed as ENW, EWN, NEW, NWE, WEN, and WNE.

b. Sample diagram:

ENW   EWN   NEW

NWE   WEN   WNE

○   designates starting point
●   designates ending point
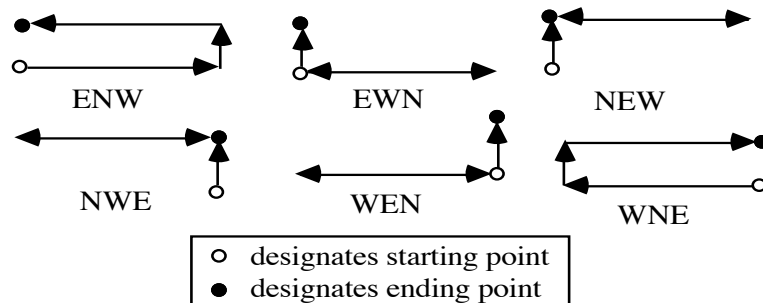
c. Because the jogger always ends up 100 m north of the starting point, as demonstrated by the diagrams in Part **b**, the vector is (100,0).

d. Sample response: Part **c** demonstrates that vector addition is both associative and commutative.

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*1.3   **a.**   Sample drawing:

Parents' cabin
fourth tack (2.0,308)
Resultant of first three tack
(6.3,355)
Beach
Pelican Island
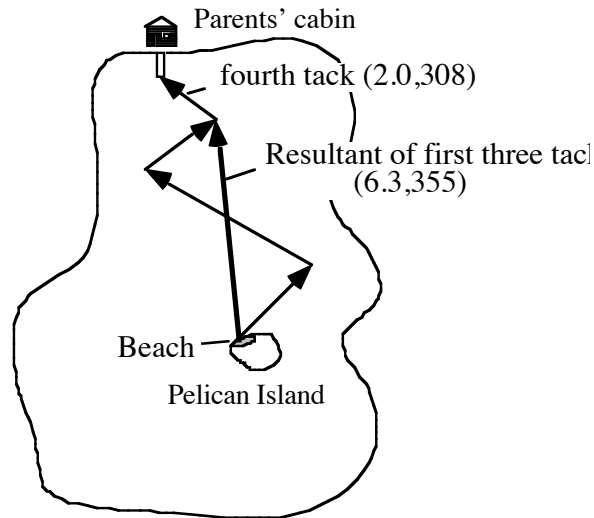
b.   The resultant should have a magnitude (in kilometers) and azimuth (in degrees) of approximately (6.3,355), as shown in Part **a** above.

c.   The vector of the fourth tack should have a magnitude and azimuth of approximately (2.0,308), as shown in Part **a**.

*1.4   Answers will vary. Although some students may recognize that the paths of direct flights follow great circles, they should estimate distances in two dimensions. The vector that describes a direct flight from Seattle to San Francisco with magnitude in kilometers and azimuth in degrees is approximately (1100,198).

* * * * *

1.5   **a.**   In the following sample drawing, the resultant is shown in bold:

(2.4,300)
scale:
1 km
(4,205)
(1.5,0)
(2,65)
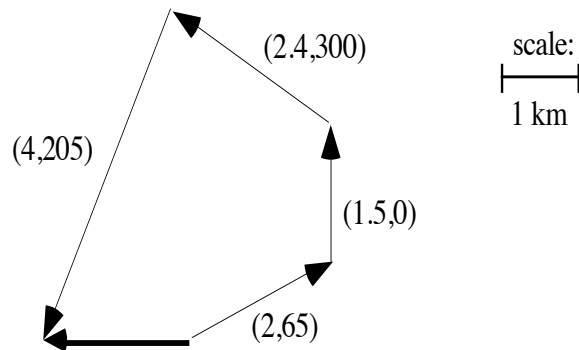
b.   The resultant vector that describes the straight-line path from the starting point to the ending point is approximately (2,269).

c.   The displacement vector that describes the straight-line path needed to return to the starting point is approximately (2,89).

d.   Sample response: They are opposite vectors. Their magnitudes are equal and their azimuths differ by 180˚.

1.6   **a.** Sample response: Manuel could find the angle between the first two sections of his route by drawing vertical lines as shown in the diagram below. Using the fact that alternate interior angles are congruent, he could determine that the measure of angle **1** is 45°. Using supplementary angles, the measure of angle **2** is 112°. The sum of these two measures is 157°.
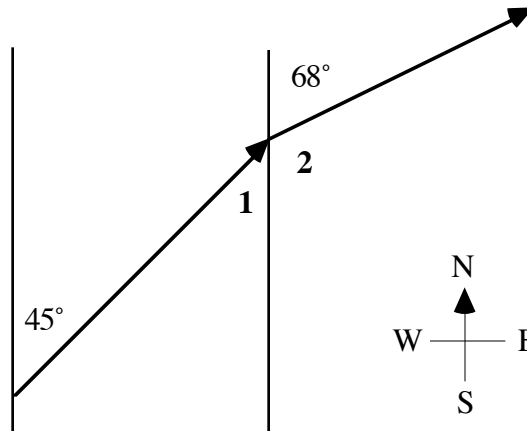
b.   1. The appropriate two sections of the original route, along with the resultant, form the following triangle:
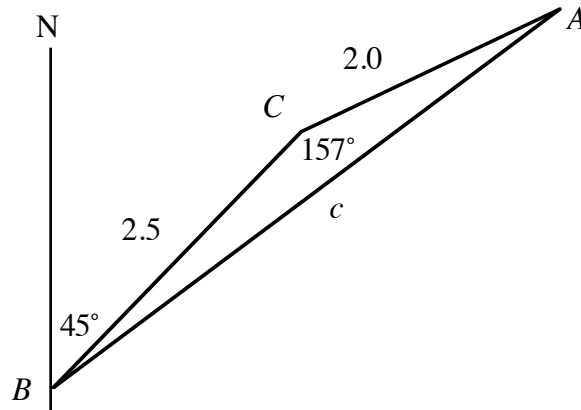
Using the law of cosines, the magnitude of the resultant can be found as shown below:

$$c^2 = 2.5^2 + 2.0^2 - 2(2.5)2.0 \bullet \cos 157° \approx 19.46$$

$$c \approx 4.4 \text{ km}$$

Using the law of sines, the angle B, formed by the resultant vector and the vector representing the first section of the route, can be found as follows:

$$\frac{\sin 157°}{4.4} = \frac{\sin B}{2.0}$$

$$\sin B = \frac{2.0(\sin 157°)}{4.4}$$

$$\approx 0.1776$$

$$B \approx 10.2°$$

Since the first section of the hike had an azimuth of $45°$, the resultant vector is $(4.4, 55.2)$.

2. The appropriate two sections of the original route, along with the resultant, form the following triangle:
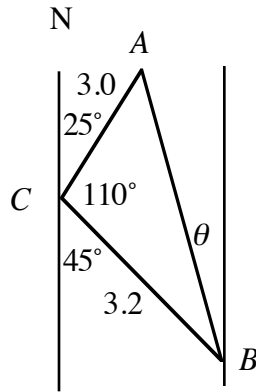
Using the law of cosines, the magnitude of the resultant can be found as shown below:

$$c^2 = 3.2^2 + 3.0^2 - 2(3.2)3.0 \cdot \cos 110° \approx 25.81$$

$$c \approx 5.1 \text{ km}$$

Using the law of sines, the angle B can be found as follows:

$$\frac{\sin 110°}{5.1} = \frac{\sin B}{3.0}$$

$$\sin B = \frac{3.0(\sin 110°)}{5.1}$$

$$\approx 0.5528$$

$$B \approx 33.6°$$

The azimuth for the resultant vector is $360° - \theta$. Since $B + \theta = 45°$, the azimuth is $360° - (45° - 33.6°) = 348.6°$. Therefore, the resultant vector is $(5.1, 348.6)$.

3. The appropriate two sections of the original route, along with the resultant, form the following triangle:
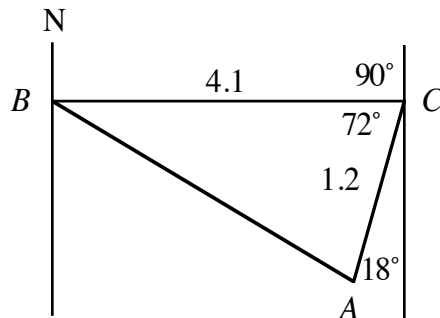
Using the law of cosines, the magnitude of the resultant can be found as shown below:

$$c^2 = 4.1^2 + 1.2^2 - 2(4.1)1.2 \bullet \cos 72° \approx 15.21$$

$$c \approx 3.9 \text{ km}$$

Using the law of sines, the angle B can be found as follows:

$$\frac{\sin 72°}{3.9} = \frac{\sin B}{1.2}$$

$$\sin B = \frac{1.2(\sin 72°)}{3.9}$$

$$\approx 0.2926$$

$$B \approx 17.0°$$

The azimuth for the resultant vector is $90 + 17 = 107°$. Therefore, the resultant vector is (3.9,107).

* * * * * * * * * *

(page 167)

Activity 2

In this activity, students determine the vertical and horizontal components of vectors.

Materials List

- spring scales (three per group)
- metal ring (one per group)
- transparent tape (several pieces per group)
- centimeter grid paper (several sheets per group)
- rulers (one per group)
- protractors (one per group)

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Technology

- geometry utility (optional)
- science interface device (optional)

Teacher Note

In Exploration **1**, you may wish to use a science interface device along with the corresponding tool for measuring force (for example, Texas Instruments' CBL).

Exploration 1

a–g.   Each group's graph paper should resemble the following diagram. **Note:** Using the Pythagorean theorem, the sum of the squares of the readings on the two scales positioned over the axes should equal the square of the reading on the third scale. Observed values may vary by as much as 20%.
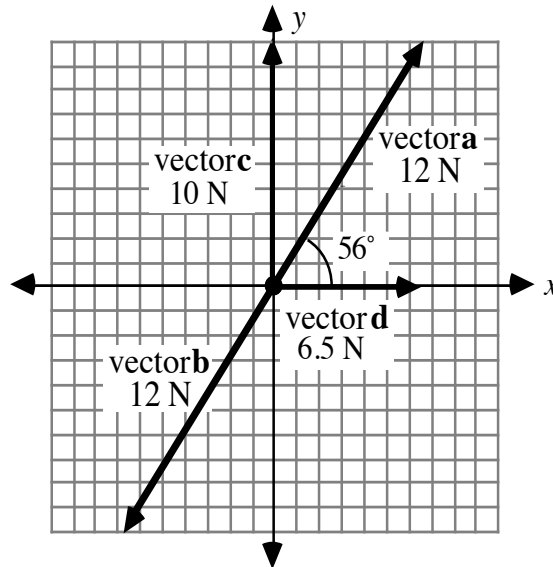
h.   The following diagram shows one method for adding vectors **c** and **d**. The resultant is equal to vector **a**. **Note:** In this module, the vertical component is added to the horizontal component so that one side of the reference angle coincides with the *x*-axis.
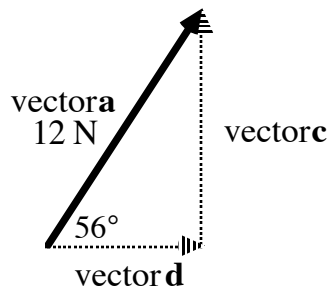
Teacher Note

Since students who enter Level 5 directly from Level 3 may have not yet been introduced to trigonometric functions, the magnitudes of component vectors are determined using a reference angle whose measure is between $0°$ and $90°$.

    i.   Sample response: Using the angle and magnitude of vector **a**, the magnitude of vector **d** is $12 \bullet \cos 56° \approx 6.7$ N.

           Similarly, the magnitude of the vector **c** is $12 \bullet \sin 56° \approx 9.9$ N.

    j.   Students repeat Parts **a–i** using a different reference angle.

Discussion 1 (page 171)

    a.   Since the ring is not moving, the forces on the scales are at equilibrium. The total magnitude of the force on the two scales positioned along the axes must be the same as the magnitude of the force on the third scale. The azimuths of the vectors describing these forces must differ by $180°$.

    b.   The magnitude can be found using the Pythagorean theorem. If x is the magnitude of the horizontal component, y is the magnitude of the vertical component, and a is the magnitude of the resultant, then:

$$\sqrt{x^2 + y^2} = a$$

    c.   The reference angle θ can be found using the inverse tangent relationship:

$$\tan^{-1}\left|\frac{\mathbf{r}_y}{\mathbf{r}_x}\right| = \theta$$

Exploration 2 (page 171)

    a.   Sample diagram:

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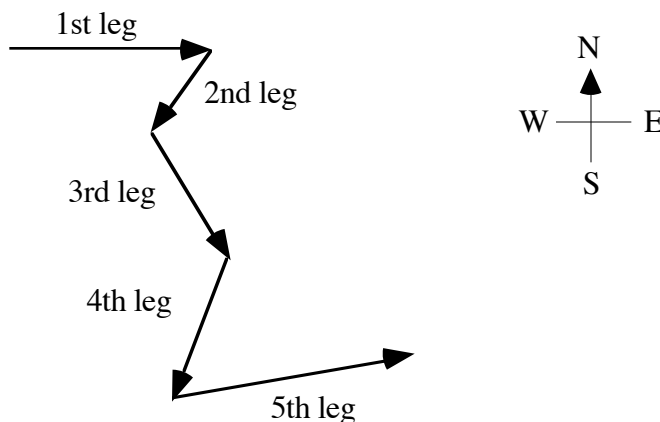
b. **1.** The magnitudes of the horizontal and vertical components for each vector can be calculated as follows:

1st section: $4\cos 0° = 4.00$; $4\sin 0° = 0.00$

2nd section: $2\cos 55° \approx 1.15$; $2\sin 55° \approx 1.64$

3rd section: $3\cos 60° \approx 1.50$; $3\sin 60° \approx 2.60$

4th section: $3\cos 70° \approx 1.03$; $3\sin 70° \approx 2.82$

5th section: $5\cos 10° \approx 4.92$; $5\sin 10° \approx 0.87$

2. Students should use the direction of each component to determine its sign, as shown in the diagram below.
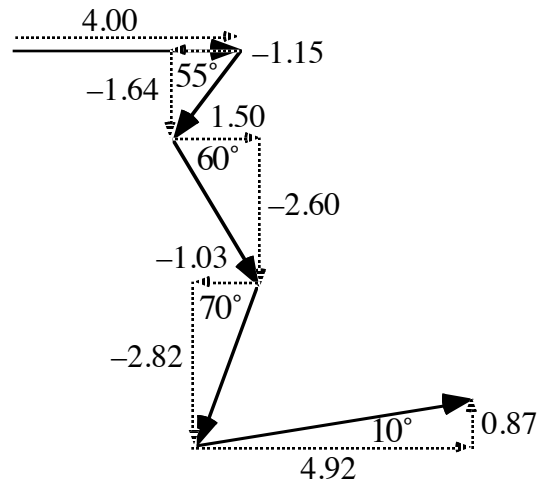
The horizontal and vertical components for each section of the route are shown in the following table:

Section of Route	Horizontal Component	Vertical Component
1	4.00	0.00
2	−1.15	−1.64
3	1.50	−2.60
4	−1.03	−2.82
5	4.92	0.87
Sum	8.24	−6.19

c. The horizontal and vertical components for the resultant are approximately 8.23 and −6.20, respectively.

d. **1.** The horizontal component of the resultant should be the same as the sum of the horizontal components of the five individual vectors. Any differences are due to rounding.

2. The vertical component of the resultant should be the same as the sum of the vertical components of the five individual vectors. Any differences are due to rounding.

Discussion 2

a. **1.** Sample response: Vertical components are negative in the third and fourth quadrants, where azimuths are between 90° and 270°. Horizontal components are positive in the first and fourth quadrants, where azimuths are between 0° and 180°. Therefore, if the vertical component is negative and the horizontal component is positive, the azimuth must be between 90° and 180°.

   2. Sample response: Vertical components are positive in the first and second quadrants, where azimuths are between 0° and 90° or 270° and 360°. Horizontal components are negative in the second and third quadrants, when the azimuths are between 180° and 360°. Therefore, if the vertical component is positive and the horizontal component is negative, the azimuth is between 270° and 360°.

b. Sample response: The magnitude of the horizontal component was calculated by multiplying the length of the vector by the cosine of the reference angle. The magnitude of the vertical component was calculated by multiplying the length of the vector by the sine of the reference angle.

c. Sample response: Yes. The magnitudes of the components can be determined as described previously in Part **b**. The sign of each component can be determined by considering the azimuth.

d. Sample response: The horizontal component of a resultant can be determined by adding the horizontal components of all the vectors in the sum. The vertical component of a resultant can be determined by adding the vertical components of all the vectors in the sum.

   This method works because by determining the total horizontal displacement, then the total vertical displacement, the actual displacement can be found using trigonometry and the Pythagorean theorem.

e. Sample response: If the resultant is positioned so that its tail is at the origin, its horizontal component coincides with the *x*-axis, and the *y*-axis represents north, then the following is true. For a resultant positioned in the first quadrant, the azimuth is equal to 90° minus the measure of the reference angle. In the second quadrant, the azimuth is equal to 270° plus the measure of the reference angle. In the third quadrant, the azimuth is equal to 270° minus the measure of the reference angle. In the fourth quadrant, the azimuth is equal to 90° plus the measure of the reference angle.

f. Sample response: One advantage of component form is that vectors can quickly be added or subtracted by simply adding or subtracting horizontal and vertical components.

*2.1 **a.** Sample sketch:
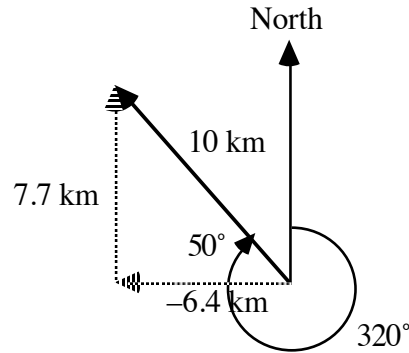
b. The reference angle is 50°.

c. The magnitude of the horizontal component can be found as follows:

$$\frac{x}{10} = \cos 50°$$

$$x \approx 6.4 \text{ km}$$

Since this component vector is in the direction of the negative x-axis, its sign is negative. So, the horizontal component is −6.4 km .

The magnitude of vertical component is:

$$\frac{x}{10} = \sin 50°$$

$$x \approx 7.7 \text{ km}$$

Since this component vector is in the direction of the positive y-axis, its sign is positive.

2.2 **a.** Sample response: The vector back to camp is the opposite vector of the sum of the first two vectors, as shown in the following diagram. The vector back to camp is approximately (8.7,258).

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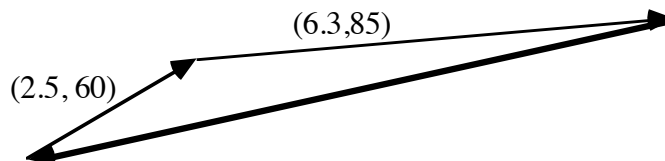
b. Sample response: The horizontal and vertical components of the vector $(2.5,60)$ are $2.5 \bullet \cos 30°$ and $2.5 \bullet \sin 30°$, respectively. Likewise, the horizontal and vertical components of the vector $(6.3,85)$ are $6.3 \bullet \cos 5°$ and $6.3 \bullet \sin 5°$, respectively.

Therefore, the horizontal and vertical components of the resultant are $2.5 \bullet \cos 30° + 6.3 \bullet \cos 5° \approx 8.44$ and $2.5 \bullet \sin 30° + 6.3 \bullet \sin 5° \approx 1.80$, respectively. Using trigonometry and the Pythagorean theorem, the resultant vector can be represented as $(8.63,258)$.

c. Sample response: No. After 2 hours, you will still have about 0.63 km to walk.

*2.3 Sample response: As shown in the diagram below, the airspeed and azimuth can be found by subtracting **w** from **v**.
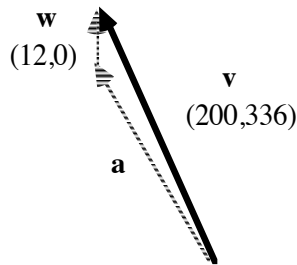
Using reference angles, the horizontal component of **a** is: $200 \cos(66°) - 0 \approx 81$. Since it is in the direction of the negative *x*-axis, its sign is negative. As a result, $\mathbf{a}_x \approx -81$.

The vertical component of **a** is: $200 \sin(66°) - 12 \approx 171$.

Using these components, the magnitude of **a** is:

$$\sqrt{81^2 + 171^2} \approx 189 \text{ km/hr}$$

The azimuth is:

$$270° + \tan^{-1}\left(\frac{171}{81}\right) \approx 334°$$

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2.4   **a.**   Sample response: As shown in the diagram below, the distance back to camp is the magnitude of the resultant vector.

second hike
(320,90)

first hike
(400,315)

vector back
to camp

camp
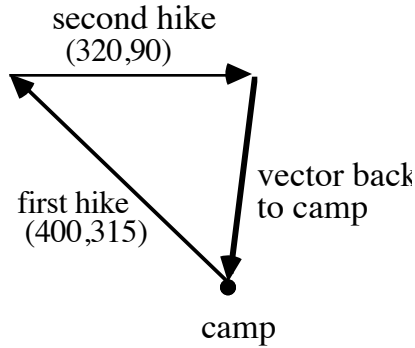
The horizontal component of the resultant is:
$-400\cos 45° + 320\cos 0° \approx 37.2$.

The vertical component is: $400\sin 45° + 320\sin 0° \approx 282.8$.

Using these components, the magnitude of the resultant is:

$$\sqrt{(37.2)^2 + (282.8)^2} \approx 285.2 \text{ m}$$

b.   The azimuth of the resultant is:

$$270° - \tan^{-1}\left(\frac{282.8}{37.2}\right) \approx 187.5°$$

2.5   **a.**   Sample diagram:

67°

800 km/hr
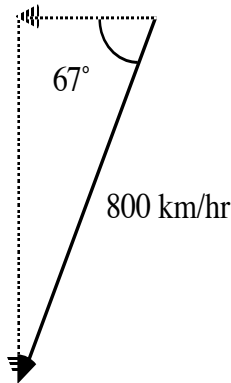
b.   The horizontal component is $-800(\cos 67°) \approx -312.6$. The vertical component is $-800(\sin 67°) \approx -736.4$.

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2.6   **a.** Students may draw a diagram of the situation, as shown below:
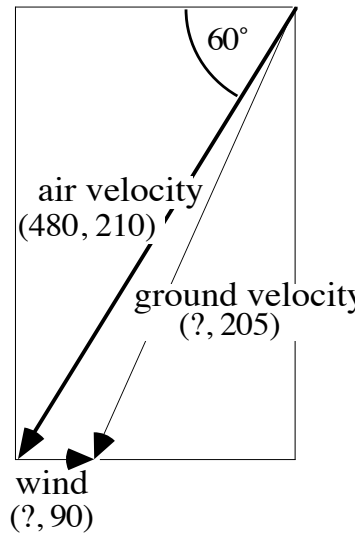
The horizontal component of the air velocity is: $-480\cos 60° \approx -240$. Its vertical component is: $-480\sin 60° \approx -416$.

The reference angle for the ground velocity vector is 65°. Because the vertical component of the ground velocity is equal to the vertical component of the air velocity, the magnitude of the ground velocity vector is:

$$\frac{416}{\sin 65°} \approx 459 \ \text{km/hr}$$

b. The speed of the wind is the difference between the horizontal component of the air velocity and the horizontal component of the ground velocity. The magnitude of the horizontal component of the ground velocity is:

$$\frac{416}{\tan 65°} \approx 194 \ \text{km/hr}$$

Therefore, the wind speed is $240 - 194 = 46 \ \text{km/hr}$.

* * * * *

2.7   **a.** Sample sketch:
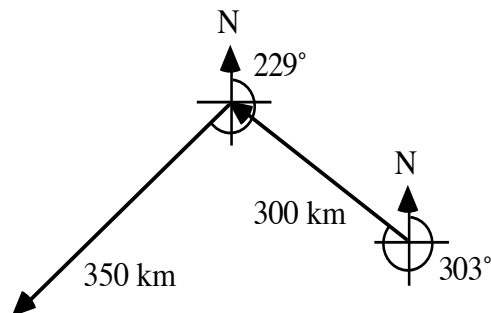
b. The horizontal component of the resultant vector is:
$-(300\cos 33° + 350\cos 41°) \approx -516$. The vertical component of the resultant is: $300\sin 33° - 350\sin 41° \approx -66$.

c. The magnitude of the resultant vector is:

$$\sqrt{(-516)^2 + (-66)^2} \approx 520 \text{ km}$$

The azimuth of the resultant vector is:

$$270° - \tan^{-1}\left(\frac{66}{516}\right) \approx 263°$$

2.8 Sample response: The vectors in this situation can be represented as shown in the following diagram.
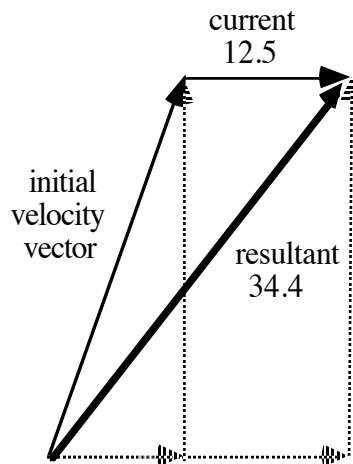
The reference angle for the ferry's initial velocity vector is 70°. The horizontal and vertical components are $28\cos 70° \approx 9.58$ and $28\sin 70° \approx 26.31$.

The reference angle for the resultant is 50°. The vertical component of the resultant is the same as the vertical component for the ferry, or about 26.31. This means that the horizontal component of the resultant is

$$\frac{26.31}{\tan 50°} \approx 22.08$$

The speed of the current is the difference between the horizontal components of the ferry and the resultant, or about 12.5 km/hr.

Using the Pythagorean theorem, the actual speed of the ferry is $\sqrt{22.08^2 + 26.31^2} \approx 34.35$ km/hr.

2.9   Sample response: The vectors in this situation can be represented as shown in the following diagram.
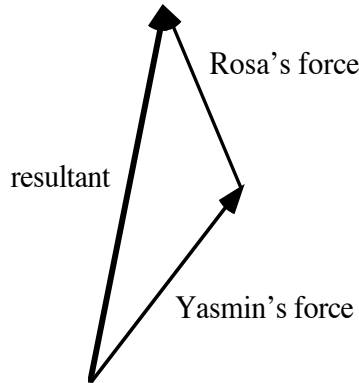
The reference angle for the force vector created by Yasmin is $50°$. The horizontal and vertical components are $90\cos 50° \approx 57.85$ and $90\sin 50° \approx 68.94$, respectively. The reference angle for the force vector created by Rosa is $65°$, so its horizontal and vertical components are $-80\cos 65° \approx -33.81$ and $80\sin 65° \approx 72.50$.

The horizontal component of the resultant is the sum of the horizontal components of Yasmin and Rosa, or about 24 N. The vertical component of the resultant is the sum of the vertical components of Yasmin and Rosa, or about 141 N.

Using the Pythagorean theorem, the magnitude of the resultant is $\sqrt{24^2 + 141^2} \approx 143$ N. The reference angle for the resultant is:

$$\tan^{-1}\left(\frac{141}{24}\right) \approx 80°$$

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Teacher Note

To complete the summary assessment, each student will require a copy of template B. A blackline master appears at the end of the teacher edition for this module.

Answers to Summary Assessment   (page 175)

1–2. The following diagram shows a set of displacement vectors that describe one possible route from the starting position to the destination.
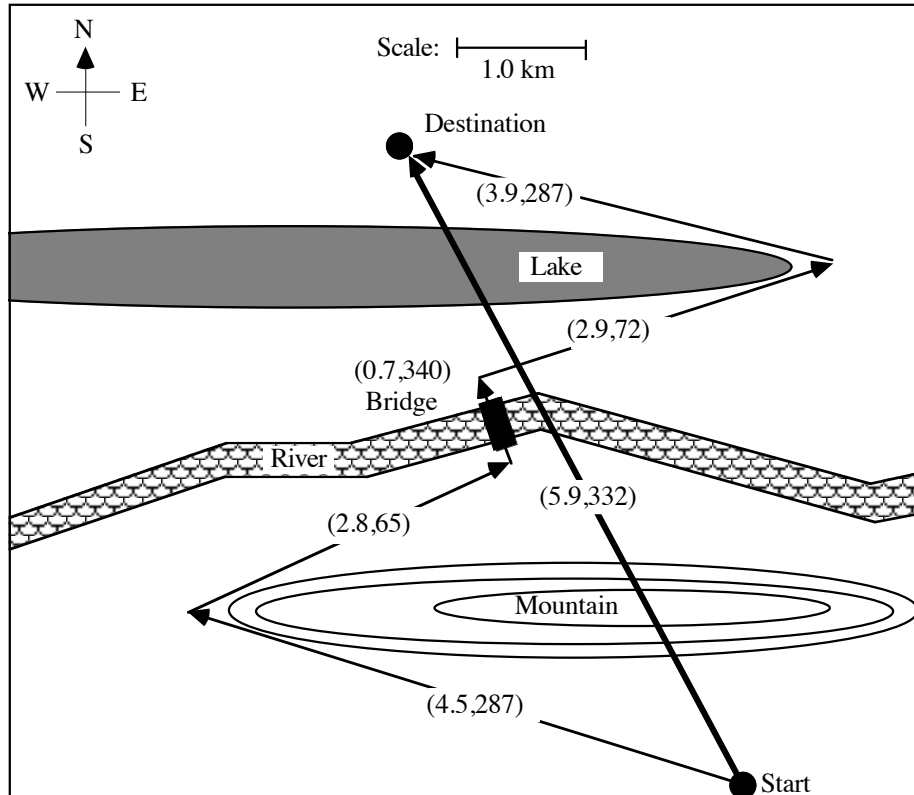
3. Sample response: The following diagram shows the reference angle for each vector in the route.
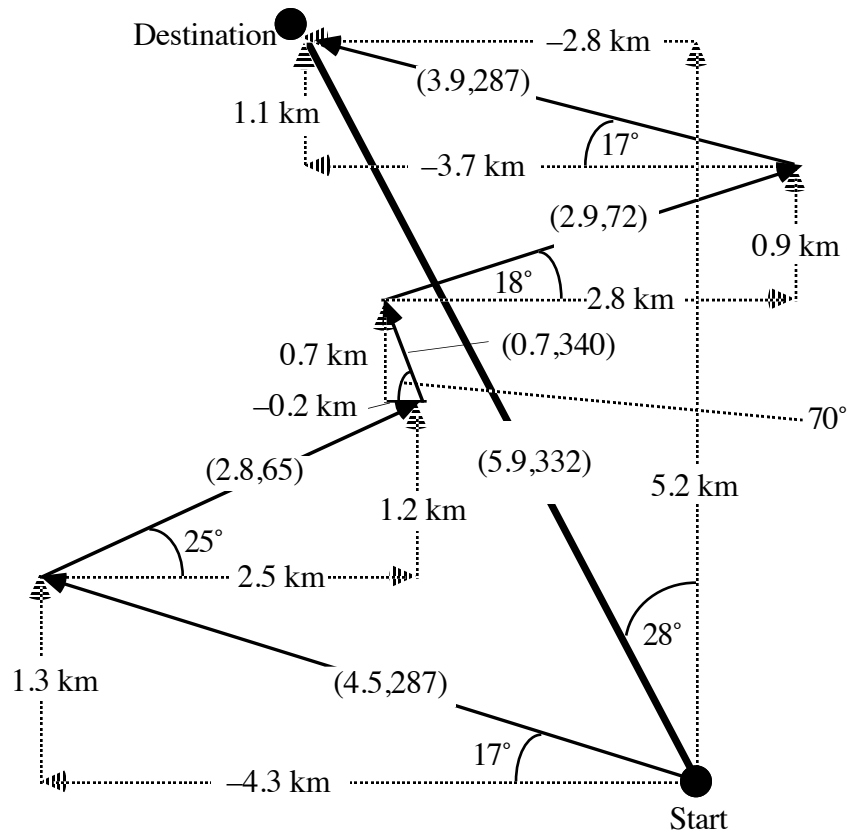
The horizontal and vertical components of each vector can be found using the sine and cosine ratios.

Vector	Horizontal Component	Vertical Component
(4.5, 287)	–4.3	1.3
(2.8, 65)	2.5	1.2
(0.7, 340)	–0.2	0.7
(2.9, 72)	2.8	0.9
(3.9, 287)	–3.7	1.1
Sum	–2.9	5.2

The sum of the horizontal components is –2.9 km, which is close to –2.8 km, the horizontal component of the vector from the start to the destination. This is reasonably close considering rounding and possible errors in measurement. The sum of the vertical components is 5.2 km, which equals the vertical component of the vector from the start to the destination.

By the Pythagorean theorem, $\sqrt{2.9^2 + 5.2^2} \approx 5.9$, the magnitude of the resultant. Using the inverse tangent,

$$\tan^{-1}\left(\frac{2.8}{5.2}\right) \approx 28°$$

The azimuth of the resultant is $360 - 28 = 332°$.

4.   Sample response: The sum of the magnitudes of the five vectors in the route is $4.5 + 2.8 + 0.7 + 2.9 + 3.9 = 14.8$ km. Hiking at a speed of 3 km/hr, it should take about 4.9 hr to reach the destination point.

1. Two tugboats are pushing a barge out of Chesapeake Bay. One tugboat pushes with a force of 4445 N at an azimuth of 105°, while the second pushes with a force of 8889 N at an azimuth of 10°.

    a. Draw a scale diagram of this situation, using vectors to represent the forces involved. Label the force generated by the first tugboat vector **f**, the force generated by the second tugboat vector **s**, and the resultant vector **r**.

    b. Describe the direction and magnitude of the resultant force.

2. A motorboat leaves a marina on the Missouri River at an azimuth of 30° and a speed of 32 km/hr. The main current is flowing directly to the east at a rate of about 10 km/hr.

    a. Draw a scale diagram of this situation using vectors to represent the velocities involved. Label the vector that represents the motorboat's velocity **m**, the vector that represents the river's velocity **w**, and the resultant vector **r**.

    b. Describe the actual velocity of the motorboat, including its speed and azimuth.

3. A plane is flying with an airspeed of 153 km/hr at an azimuth of 52°. As a result of a wind blowing directly to the south, the plane's ground speed is 135.4 km/hr at an azimuth of 63°.

    a. Determine the wind's velocity.

    b. Write the vector that represents the wind's velocity as an ordered pair.

4. As shown in the diagram below, a pilot flies from point A to point B by following an azimuth of 34° for 360 km, then following an azimuth of 71° for 310 km.
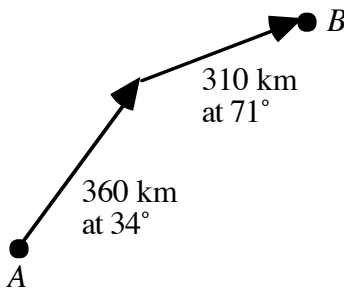
    Given an average speed of 200 km/hr, how much time could the pilot save by flying directly from A to B?

Answers to Module Assessment
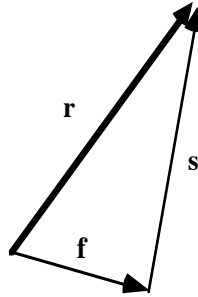
1.   **a.**   Sample diagram:

   b.   The resultant force has a magnitude of about 9630 N and an azimuth of about 37˚.

2.   **a.**   Sample diagram:
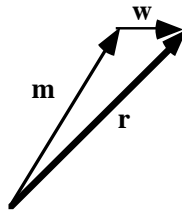
   b.   The motorboat is moving at 36 km/hr at an azimuth of about 46˚.

3.   **a.**   Sample response: The following diagram shows the vectors for the velocities of the plane and the wind.
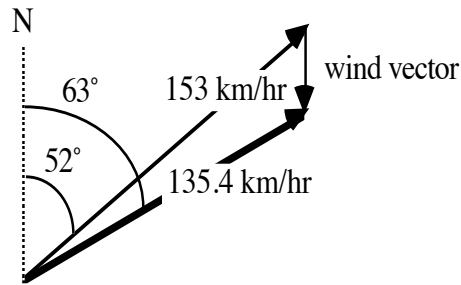
The diagrams below show the vertical and horizontal components of the plane's vectors. The wind speed is approximately $94.2 - 61.5 = 32.7$ km/hr at an azimuth of 180˚.
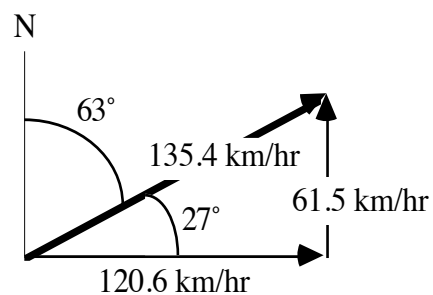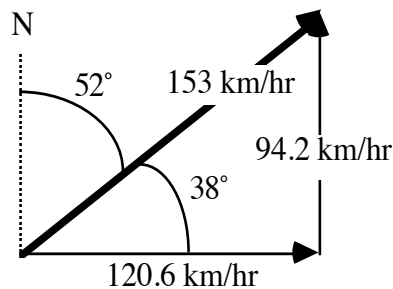
   b.   The wind velocity is (32.7,180).

4.   Sample response: As shown in the following diagram, the resultant vector has a magnitude of 635.6 km.
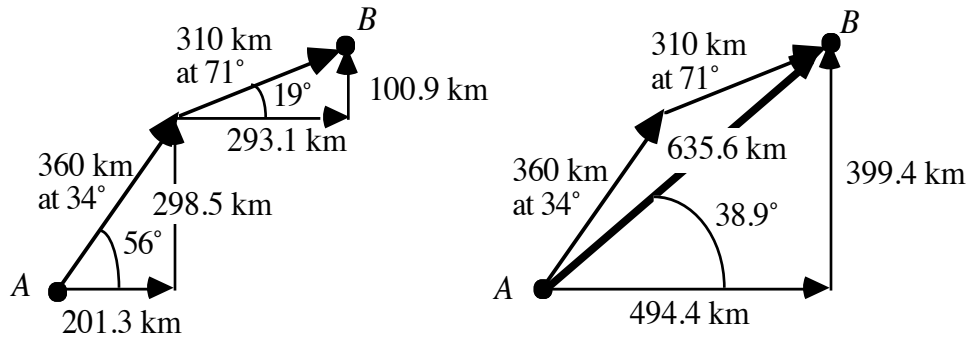
The time required for the direct route from *A* to *B* can be found as follows:

$$\frac{635.6 \text{ km}}{200 \text{ km/hr}} \approx 3.2 \text{ hr}$$

The time required for the original route is;

$$\frac{670 \text{ km}}{200 \text{ km/hr}} \approx 3.35 \text{ hr}$$

The difference is about 0.15 hr or 9 min.

Selected References

de Lange, J. *Flying through Math*. Scotts Valley, CA: WINGS for Learning, 1991.

Dolan, S., ed. *Newton's Laws of Motion*. New York: Cambridge University Press, 1989.

Edge, R. D. "Physics and the New Games." *The Physics Teacher* 20 (December 1982): 596–598.

Escobar, C. "Amusement Park Physics." *The Physics Teacher* 28 (October 1990): 446–452.

Minnix, R. B., and D. R. Carpenter, Jr. "Apparatus for Teaching Physics." *The Physics Teacher* 22 (January 1984): 54–55.

Ritter, T. "The Great American Vector Game." *The Physics Teacher* 20 (May 1982): 310–311.

Wilson, J. E. *Physics: A Practical and Conceptual Approach*. Philadelphia: Saunders College Publishing, 1989.

Flashbacks

Activity 1

1.1   Use an example to illustrate each of the following:

        a. the commutative property of addition

        b. the associative property of addition.

1.2   Estimate the measure of each angle indicated in Parts **a–d** below.

        a.   **b.**

        c.   **d.**
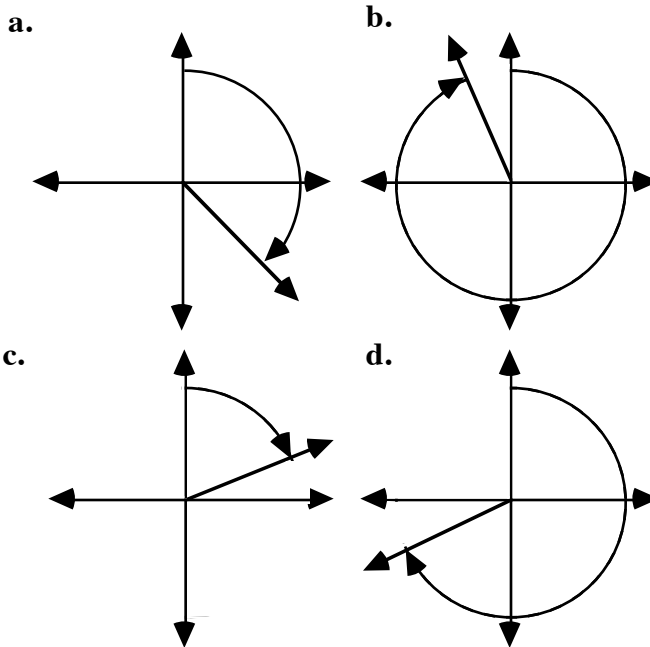
1.3   Use the law of cosines to find the unknown measure in each triangle below.

        a. x **b.** x 11

                           14   8

        12   15

            87°
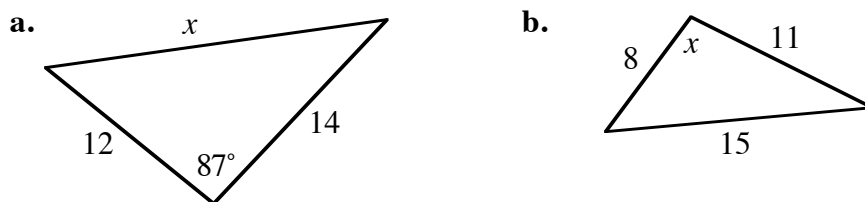
1.4   Use the law of sines to find the unknown measure in each triangle below.

        a.   16   **b.**   98°   15

                    38°   11

                                54°

        10   x

            x

238
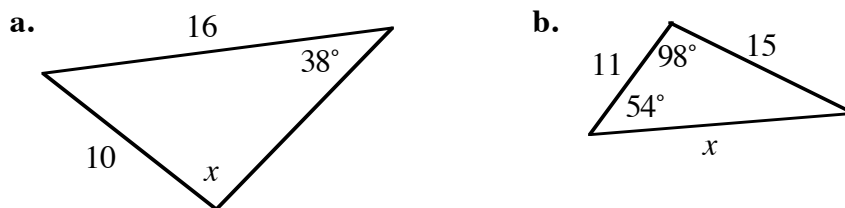
Activity 2

2.1 Use the Pythagorean theorem to determine the value of x in each of the following triangles.

a.

b.
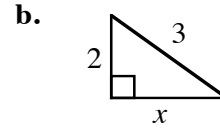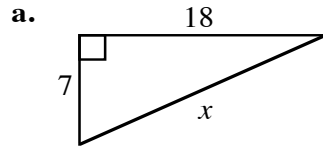
2.2 Use right-triangle trigonometry to determine the unknown values in each of the triangles below.

a.

b.

c.
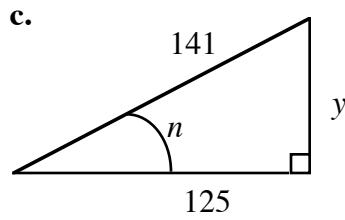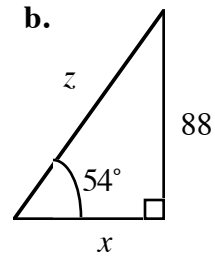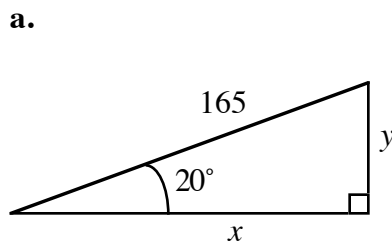
Answers to Flashbacks

Activity 1

1.1   **a.** Sample response: $2 + 3 = 3 + 2 = 5$.

      b. Sample response: $4 + (2 + 9) = (4 + 2) + 9 = 15$.

1.2   **a.** $135°$

      b. $330°$

      c. $60°$

      d. $240°$

1.3   **a.** Sample response:

$$x^2 = 12^2 + 14^2 - 2(12)14 \cdot \cos 87°$$
$$\approx 322.42$$
$$x \approx 17.96$$

      b. Sample response:

$$15^2 = 11^2 + 8^2 - 2(11)8 \cdot \cos x$$
$$\cos x = -\frac{5}{22}$$
$$x \approx 103.14°$$

1.4   **a.** Sample response:

$$\frac{\sin x}{16} = \frac{\sin 38°}{10}$$
$$\sin x = \frac{16 \cdot \sin 38°}{10}$$
$$x \approx 80.1°$$

      b. Sample response:

$$\frac{\sin 98°}{x} = \frac{\sin 54°}{15}$$
$$x = \frac{15 \cdot \sin 98°}{\sin 54°}$$
$$x \approx 18.4$$

Activity 2

2.1   **a.** $x = \sqrt{18^2 + 7^2} \approx 19.313$

         b. $x = \sqrt{3^2 - 2^2} \approx 2.236$

2.2   **a.** $x = 165 \bullet \cos 20° \approx 155$, $y = 165 \bullet \sin 20° \approx 56$

         b. $z = \dfrac{88}{\sin 54°} \approx 109$, $x = \dfrac{88}{\tan 54°} \approx 64$

         c. $y = \sqrt{141^2 - 125^2} \approx 65$, $n = \cos^{-1}\left(\dfrac{125}{141}\right) \approx 28°$

Template A

Crystal Lake

Template B

N

W — E

S

Scale: 1.0 km

Lake

Bridge

River

Mountain

● Start
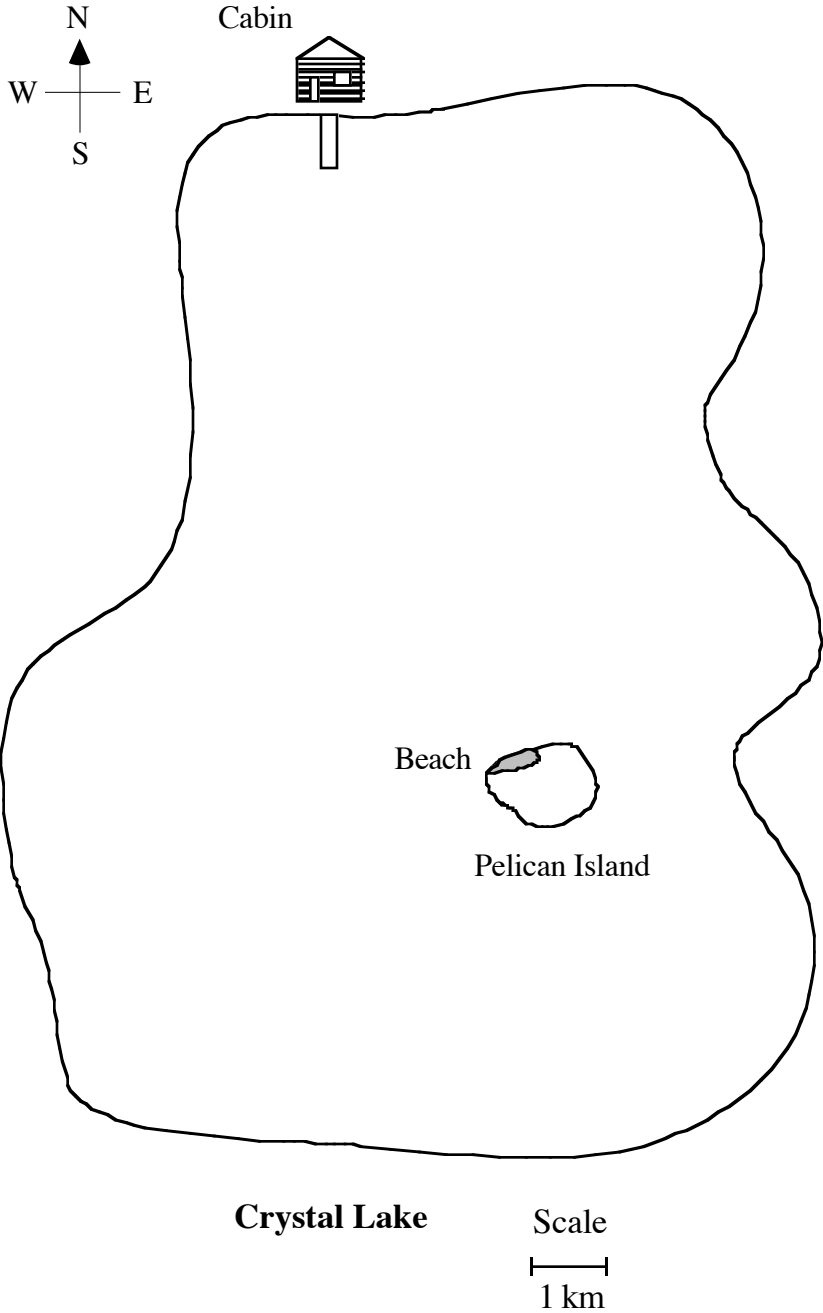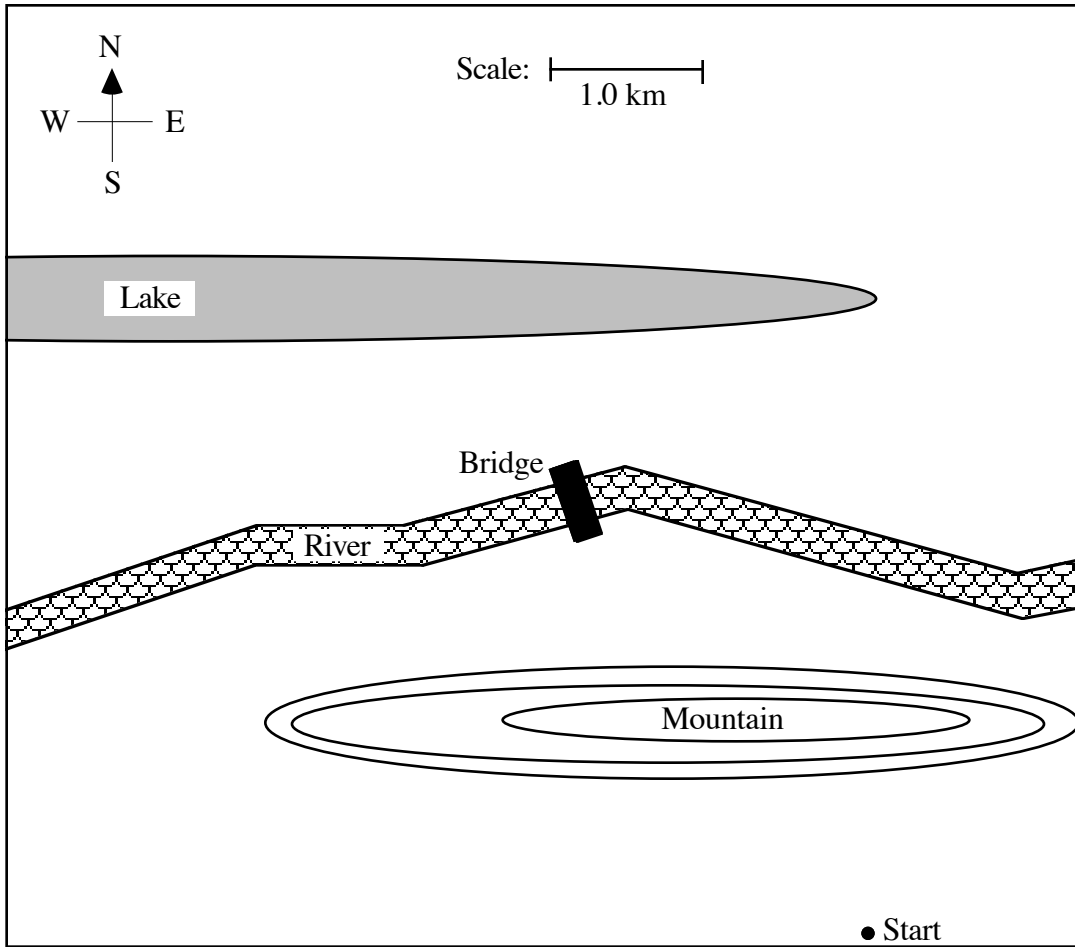
Making Cents
of Your Income

Teacher
Guide
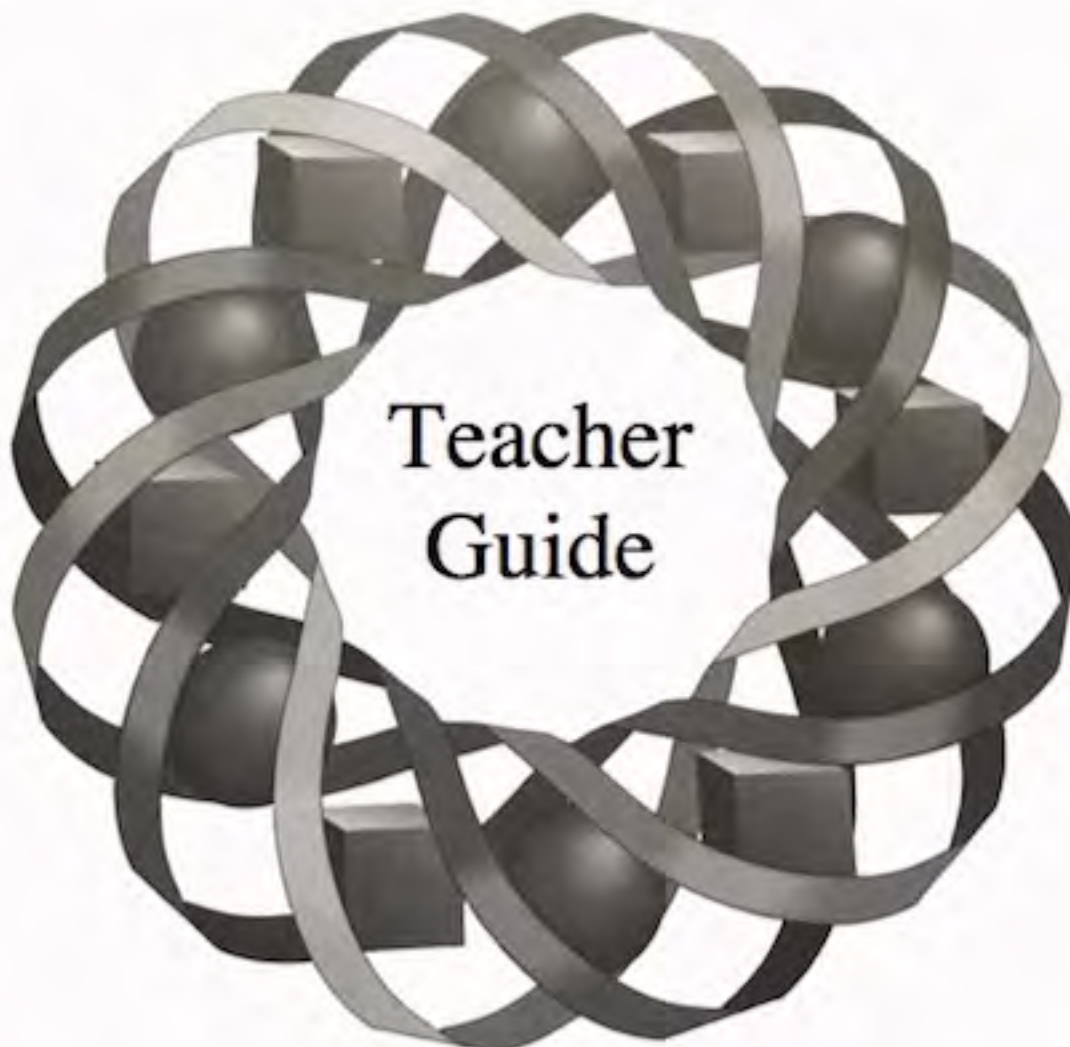
What's the average income in your state? If a researcher claimed that the mean is $30,000 after surveying 100 residents, would you find this claim believable? In this module, you investigate how to use sampling to test claims about populations.

Doug Mack • David Thiel • Deanna Turley
Danny Jones • John Knudson-Martin

Making Cents of Your Income

Overview

In this module, students determine the means and standard deviations of populations and samples, use sample means and standard deviations to estimate population means and standard deviations, construct confidence intervals, and test claims about the population mean.

Objectives

In this module, students will:

- estimate a population mean using sample means

- estimate the standard deviation of a population using the standard deviation of a sample

- use the standard deviation of a sample to construct confidence intervals

- use the 68-95-99.7 rule to determine the probability that the population mean lies within certain confidence intervals

- use confidence intervals to estimate a population mean

- formulate null and alternative hypotheses

- use confidence intervals to test a hypothesis.

Prerequisites

For this module, students should know:

- how to determine the mean and standard deviation of a set of numbers

- how to create frequency histograms

- the definitions of population, sample, census, parameter, statistic, and bias

- how to construct line plots

- the characteristics of a normal distribution.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	3	2	11

Materials Required

Materials	Activity			
	1	2	3	**Summary Assessment**
graph paper	X	X	X	
containers	X			
scissors	X			
population template	X			
pennies (optional)	X			
plastic tokens or cardboard strips				X

Teacher Note

A blackline master of the population template appears at the end of this teacher edition.

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
statistics package	X	X	X	X
programmable graphing utility (optional)	X			

Making Cents of Your Income

Introduction (page 181)

The context of determining average income for high school graduates recurs throughout the module.

(page 181)

Activity 1

In this activity, students investigate the effects of sample size on the variability of sample means and sample standard deviations. They also estimate population parameters using sample statistics.

Materials List

- graph paper (several sheets per group)
- containers (one per group)
- scissors (one pair per group)
- population template (one copy per group; a blackline master appears at the end of this teacher edition)

Technology

- statistics package
- programmable graphing utility (optional)

Teacher Note

Some technology reports both the standard deviation of a sample (s) and the standard deviation of a population (σ), while others report only s. (In the formula for s, the average squared deviation is found by dividing by $n - 1$, since only $n - 1$ of the squared deviations can vary freely.)

Exploration 1 (page 181)

Students sample from a simulated population of pennies (which may not be representative of all pennies in circulation). As an alternative to this simulation, you may wish to use actual coins. **Note:** In this case, the age of each coin in the population should be entered in a statistics package or calculator program for use in Exploration **2**.

a. To model a population of pennies, each group of students should cut the population template into 99 slips of paper and place the slips in a container.

They should then sample 5 "pennies" from the container, one at a time with replacement, and record the age of each penny in the sample. One possible sample is {12, 2, 1, 3, 28}.

b. Students determine the mean age $\bar{x}$ and standard deviation s and record these values in Table **1**. For example, the mean age of the sample {12, 2, 1, 3, 28} is 9.2, with a standard deviation of approximately 11.4.

c. Sample data:

Sample Number	$\bar{x}$	s
1	2.6	1.82
2	9	9.70
3	7.6	5.41
4	6.4	8.53
5	6.6	10.43
6	4.8	5.22
7	5.6	8.65
8	14.6	9.07
9	7.4	4.56
10	8.4	6.58

d. Sample histogram:
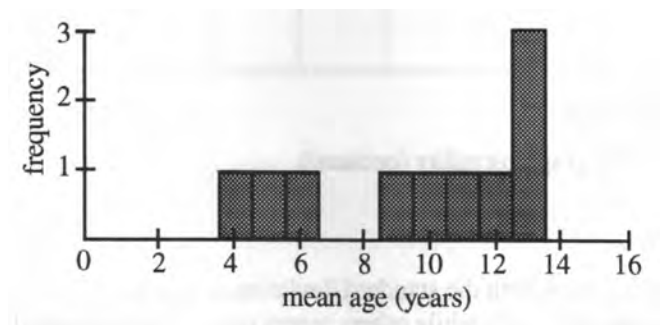
e. Students create a frequency histogram using the class data. A frequency histogram for the means of 50 samples of size 5 is shown below.

> **Note:** You may wish to point out that compiling data is reasonable in this case since all data was collected from random samples of the same population.
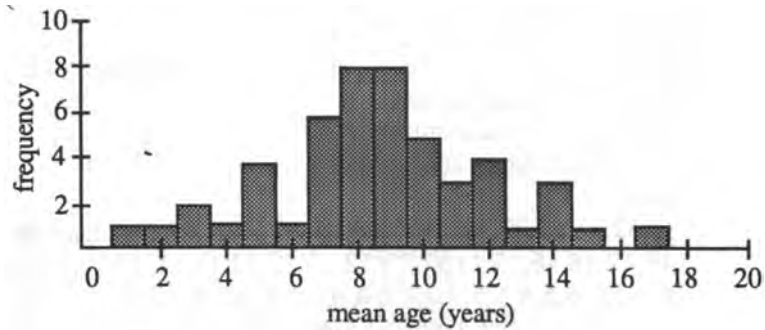
Discussion 1 (page 183)

a. Sample response: Since the calculated standard deviation is from a sample rather than the entire population s is appropriate.

b. In the frequency histogram for the class data, the sample means should tend to cluster more visibly about a single value.

c. Sample response: One estimate of the mean age would be the sample mean that appears most frequently in the histogram.

d. Sample response: Since the histogram for the class data shows the results of many more samples, it is more likely to give a better estimate of the population mean.

Teacher Note

In Exploration **2**, students simulate samples of various sizes using a statistics package or calculator program. The following program was written for the TI-92 calculator and may be adapted for other programmable utilities. This program records the sample means in a list named "meanli" and the standard deviations in "stddli." Both meanli and stddli are combined into one data file called "samdat." Students can quickly create a histogram of the sample means using samdat. "Penlis" contains the ages of the population of 99 pennies given in the template, while "samlis" contains the data values for the last sample. Those lines marked with an asterisk (*) could be eliminated for faster processing.

The mean age for the population given in this program is 9.60 yr, with a standard deviation of 9.28 yr. Since the program samples one item at a time, with replacement, sample sizes may be as large as desired. The program asks students to seed the random number generator by entering a positive integer. Each group

should use a different integer. To save time, students may use one calculator for each of the sample sizes.

```
  :cents()
  :Prgm
  :getMode("ALL")→modset
  :setMode("Exact/Approx","Approximate")
  :setMode("Display Digits","Float 4")
  :ClrIO
  :DelVar samdat,meanli,stddli
  :Local n,r,c,q,penlis,samlis
  :Disp "Input random number seed."
  :Input q
  :{34,32,31,31,29,28,28,27,26,25,24,23,23,22,21,21,20,19,19,18,18,17,
    17,17,16,16,15,14,14,13,13,13,12,12,11,11,11,10,10,10,9,9,8,8,7,7,6,
    6,6,6,6,5,5,5,5,4,4,4,4,4,4,3,3,3,3,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,
    1,1,1,1,1,1,1,1,1,1,1,1,1,1,1}→penlis
  :Disp "Sample size   :"
  :Input n
  :Disp "Number of samples   :"
  :Input r
* :ClrIO
  :For k,1,r
  :For c,1,n
  :penlis[rand(99)]→samlis[c]
  :EndFor
* :Disp {"SAMPLE",k}
* :Disp :Disp left(samlis,8):Disp mid(samlis,9,8):Disp mid(samlis,17,8):Disp
   mid(samlis,25,8):Disp mid(samlis,33,8)
  :mean(samlis)→meanli[k]
* :Disp {"Sample Mean:",meanli[k]}
  :stdDev(samlis)→stddli[k]
* :Disp {"Sample standard deviation:", stddli[k]}: Pause
* :ClrIO
  :EndFor
  :ClrIO
  :NewData samdat,meanli,stddli
  :Disp "In summary the means and " : Disp "standard deviations are:"
  :Disp : Disp {"Sample","Mean","Sample std dev"}
  :For k,1,r
  :Disp {k, (samdat[1])[k],(samdat[2])[k]}
  :If k≠r Then
  :If int(k/5)=k/5 Then
  :Pause
  :ClrIO:Disp {"Sample","Mean","Sample std dev"}
```

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```
:EndIf
:EndIf
:EndFor
:SetMode(modset)
:EndPrgm
```

Exploration 2   (page 183)

a.   The following graph shows a frequency histogram for 50 samples of size 20:
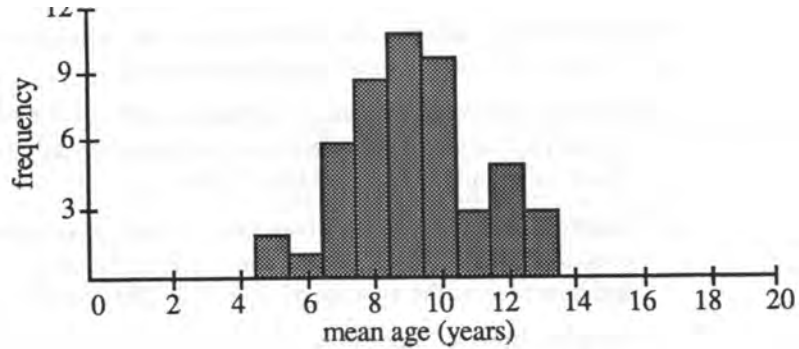
b.   The following graph shows a frequency histogram for 50 samples of size 40:
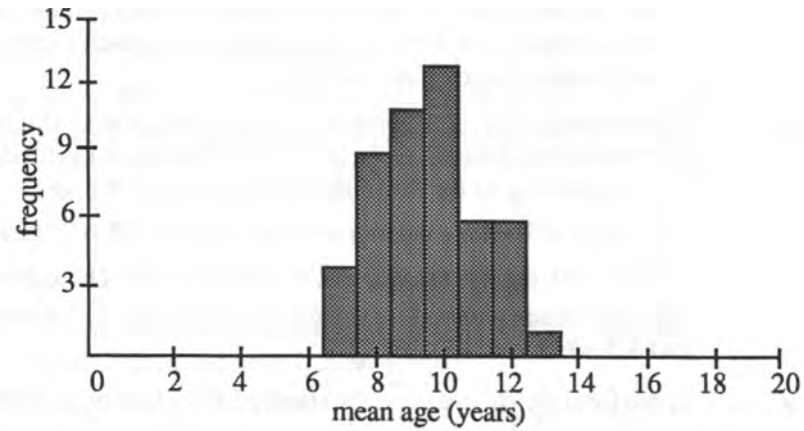
c–d.   Students determine the mean and the standard deviation for the sample means, then record this information in Table **2**. The data in the following table corresponds with 50 samples of each size:

Sample Size (*n*)	Mean of Sample Means	Standard Deviation of Sample Means
5	9.0	3.9
20	9.0	2.0
40	9.6	1.5

e. Using the sample data given previously, one estimate for the population mean is 9.6 yr.

Discussion 2 (page 184)

a. As the sample size increases, the histograms should become narrower, taller, and more symmetrical.

b. Sample response: The standard deviations become smaller as the sample size increases. The mean appears to stay about the same.

c. By the law of large numbers, as the sample size increases, the sample means tend to get closer to the population mean. In other words, the variability of the sample means tends to decrease.

d. 1. Answers may vary. Students could estimate that the mean age μ for the population of pennies should be approximately equal to the mean for samples of size 40 from Table **2**.

   2. Answers may vary. Sample response: The estimate should be very good since the mean in Table **2** for $n = 40$ is based on 50 samples, and it represents the average of $40 \bullet 50 = 2000$ observations.

e. 1. Sample response: Since both are based on a large number of observations, both should be close to the actual population mean.

   2. The larger the sample, the more the sample should resemble the entire population. Therefore, the larger the sample, the smaller the difference between the sample standard deviation s and the population standard deviation σ.

f. Answers will vary. The population used to obtain the sample responses given in the exploration had a mean of 9.60 and a standard deviation of 9.28. According to the formulas, therefore, $\mu_{\bar{x}} = 9.6$ and $\sigma_{\bar{x}} = 9.28/\sqrt{5} \approx 4.15$ when $n = 5$, $\sigma_{\bar{x}} = 9.28/\sqrt{20} \approx 2.08$ when $n = 20$, and $\sigma_{\bar{x}} = 9.28/\sqrt{40} \approx 1.47$ when $n = 40$. These compare well with the values reported in Table **2**: 3.9 when $n = 5$, 2.0 when $n = 20$, and 1.5 when $n = 40$.

g. In the formula $\sigma_{\bar{x}} = \sigma/\sqrt{n}$, the standard deviation of all possible sample means decreases as n increases. This indicates that as the sample size increases, the spread of sample means decreases.

h. When the results of the sampling are very critical and the margin of error needs to be very small, it might be worth the cost of taking a large sample. For example, studies that involve health issues, such as the risk of disease or chemical toxicity, often use large samples.

Assignment

1.1   **a.** $\bar{x} = 6$

       b. $s = 5$

       c.   Students should use $\bar{x}$ to estimate μ.

*1.2   **a.** $\mu_{\bar{x}} \approx \$23{,}000$

       b. $\sigma_{\bar{x}} \approx \$2000$

       c.   An estimate of the mean μ for the population from which this sample came is $\$23{,}000$. Since $\sigma/\sqrt{40} = \sigma_{\bar{x}} \approx 2000$, then $\sigma \approx 2000\sqrt{40} \approx \$12{,}650$.

       d.   Sample response: Since 1000 samples of size 40 were collected, the estimate should be very accurate.

       e.   Students should use relative frequencies to estimate each probability.

          1.   0.002

          2.   0.057

          3.   0.941

1.3   Sample response: Given a population standard deviation of σ, then for $n = 120$, $\sigma_{\bar{x}} = \sigma/\sqrt{120} \approx 0.0913\sigma$. For $n = 240$, $\sigma_{\bar{x}} = \sigma/\sqrt{240} \approx 0.0645\sigma$. Similarly, for $n = 480$, $\sigma_{\bar{x}} = \sigma/\sqrt{480} \approx 0.0456\sigma$. Since 0.0456σ is not half of 0.0645σ, and 0.0645σ is not half of 0.0913σ, Andreas has made a mistake.

<div align="center">

* * * * *

</div>

1.4   **a.** $\bar{x} \approx 529$

       b. $s \approx 125$

       c.   One estimate of the mean SAT score for the population of Lincoln High School students is 529.

       d.   Sample response: The sample could be biased if students in this class are not representative of all students who take the SAT at Lincoln High School.

*1.5   a. $\bar{x} \approx 16$ yrs.

b. $s \approx 2$ yrs.

c.   One estimate of the mean for the population from which this sample came is 16 yr.

d.   1.   about 0.39

  2.   about 0.61

  3.   about 0.49.

e.   Sample response: Since the sample size is relatively large, the estimates should be fairly accurate, unless the sample is biased.

1.6   a. $\bar{x} \approx 3$

b. $s \approx 1.4$

c.   One estimate of the mean number of persons in a household for the city from which this sample came is 3.

d.   1.   0.17

  2.   0.63

e.   Sample response: I don't have much confidence in the accuracy of the estimate because the sample size is not very large.

* * * * * * * * * *

(page 188)

Activity 2

In this activity, students investigate confidence intervals for sample means.

Materials List
• graph paper

Technology
• statistics package

Exploration   (page 189)

In this exploration, students use the sampling data from Activity **1** to examine how confident they can be with the results of one sample. Students should use $s/\sqrt{n}$ to estimate $\sigma_{\bar{x}}$. **Note:** In order to save time, you may want to assign a different sample size to each group of students.

a–b. Student graphs should resemble Figure **1**. Sample data:

Sample No.	$\bar{x}$	s	Estimate of $\sigma_{\bar{x}}$	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$
1	4	3.8	1.7	[2.3, 5.7]
2	19	12.5	5.6	[13.4, 24.6]
3	4	3.5	1.6	[2.4, 5.6]
4	6	8.5	3.8	[2.2, 9.8]
5	9	12.8	5.7	[3.3, 14.7]
6	12	8.1	3.6	[8.4, 15.6]
7	10	6.6	3.0	[7.0, 13.0]
8	13	11.9	5.3	[7.7, 18.3]
9	8	6.2	2.8	[5.2, 10.8]
10	11	12.7	5.7	[5.3, 16.7]

c. 1. For the sample data given in Activity **1**, the mean age is 9.6 yrs.

  2. Student graphs should resemble Figure **2**.

d–g. The percentages can be calculated by counting the number of intervals that are intersected by the line representing the population mean. In Figure **2**, for example, 2 of 3 intervals (67%) include the population mean. Sample data:

Sample Size (n)	Interval	Percentage that Contained μ
5	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	70
20	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	70
40	$[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$	70
5	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	80
20	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	90
40	$[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$	90
5	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	90
20	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	100
40	$[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$	100

Discussion (page 190)

a. Sample response: As the size of the interval increases, the percentage of intervals that contain the population mean also increases. **Note:** By comparing their results with other groups, students also may notice that the percentage of means contained in the interval $[\bar{x} - \sigma_{\bar{x}}, \bar{x} + \sigma_{\bar{x}}]$ is approximately 70%, the percentage of means contained in the interval $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ is approximately 95%, and the percentage of means contained in the interval $[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$ is approximately 100%.

b. Sample response: Our results were 70%, 90%, and 100%. These are pretty close to the percentages in a normal distribution.

c. Sample response: The size of the interval decreases as the sample size increases. This occurs because the standard deviation $\sigma_{\bar{x}}$ decreases as the sample size n increases.

d. Sample response: A 95% confidence interval means that if you construct an interval of $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$, you would expect it to contain the population mean 95% of the time.

e. Sample response: Since 95% of the intervals $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ can be expected to contain the population mean, 5% of these intervals would not contain the population mean.

f. Sample response: The researcher would need to know the standard deviation of the sample. A confidence interval could then be found to determine the range of probable values for the population mean.

Assignment (page 192)

2.1 Sample response: You should anticipate about 19 of the intervals to include the population mean, because 95% of 20 is 19. However, it would not be surprising to obtain 18 or 20 intervals that include the population mean, since the sample means will vary.

2.2   **a.** $\sigma_{\bar{x}} \approx 45/\sqrt{100} = 4.5$

   b. $[67 - 4.5, 67 + 4.5] = [62.5, 71.5]$

   c. Sample response: Since $n \geq 30$, this is a 68% confidence interval. Therefore, you can be 68% confident that μ is in the interval.

*2.3   **a.** The estimated standard deviation of all sample means is:

$$\sigma_{\bar{x}} \approx 79{,}678/\sqrt{810} \approx 2817$$

 b. **1.** One can be 68% confident that the actual mean life expectancy of the bulbs is between 813.6 hr and 840.4 hr.

 2. One can be 95% confident that the actual mean life expectancy of the bulbs is between 800.2 hr and 853.8 hr.

 3. One can be 99.7% confident that the actual mean life expectancy of the bulbs is between 786.8 hr and 867.2 hr.

*2.4   **a.** Students may use any of the three confidence levels. The estimated standard deviation of all sample means is $\sigma_{\bar{x}} \approx 4/\sqrt{20} \approx 0.9$ g. Sample response: One can be 95% confident that the mean mass of the candy bags is between 50.2 g and 53.8 g.

 b. Since the sample size is less than 30, students should not be confident in the estimate of $\sigma_{\bar{x}}$, or in using the 68–95–99.7 rule.

 c. Sample response: The results of this experiment would be more conclusive if the sample size was larger. This would make the distribution of sample means closer to a normal distribution and also make the sizes of the confidence intervals smaller.

2.5   The following statistics can be calculated from data in the table: $\bar{x} = 1.97$ L, $s = 0.038$ L, and $\sigma_{\bar{x}} \approx 0.038/\sqrt{30} \approx 0.0069$ L. Sample response: One can be 95% confident that the mean volume is between 1.96 L and 1.98 L.

2.6   **a.** $\sigma_{\bar{x}} \approx 217/\sqrt{500} \approx \$9.70$

 b. $[720 - 2(9.7), 720 + 2(9.7)] \approx [\$700.60, \ \$739.40]$

 c. Sample response: Since this is a 95% confidence interval, you can be 95% confident that the interval contains μ.

* * * * *

2.7   The estimated standard deviation of all sample means is:

$$\sigma_{\bar{x}} \approx 79{,}678/\sqrt{800} \approx 2817 \text{ km}$$

 a. One can be 68% confident that the actual mean life expectancy of tires is between 194,307 km and 199,941 km.

 b. One can be 95% confident that the actual mean life expectancy of tires is between 191,490 km and 202,758 km.

 c. One can be 99.7% confident that the actual mean life expectancy of tires is between 188,673 km and 205,575 km.

2.8 The following statistics can be calculated from the data in the table: $\bar{x} = 25.1$ months, $s = 13.8$ months, and $\sigma_{\bar{x}} \approx 13.8/\sqrt{32} \approx 2.4$ months.

  a. One can be 68% confident that the actual mean life of tennis shoes is between 22.7 months and 27.5 months.

  b. One can be 95% confident that the actual mean life of tennis shoes is between 20.3 months and 29.9 months.

  c. One can be 99.7% confident that the actual mean life of tennis shoes is between 17.9 months and 32.3 months.

2.9 Sample response: In general, a 95% confidence interval is $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$. Given the stated interval of $[1250, 1420]$, the following two equations result:

$$\bar{x} - 2\sigma_{\bar{x}} = 1250$$
$$\bar{x} + 2\sigma_{\bar{x}} = 1420$$

Solving this system of equations, $\bar{x} \approx 1335$ and $\sigma_{\bar{x}} \approx 42.5$.

* * * * * * * * * *

Activity 3

In this activity, students investigate the use of confidence intervals to test hypotheses.

Materials List
  • graph paper

Technology
  • statistics package

Exploration 1

In this exploration, students estimate possible values of an unknown population mean using confidence intervals.

  a. The mean of the sample is $23,863 and the sample standard deviation is $23,400. The estimated standard deviation of all sample means is:

$$23,400/\sqrt{30} \approx \$4272$$

Using these values for $\bar{x}$ and $\sigma_{\bar{x}}$, the 68% confidence interval is $[\$19,591, \$28,135]$, the 95% confidence interval is $[\$15,319, \$32,407]$, and the 99.7% confidence interval is $[\$11,047, \$36,679]$.

b. Sample graph:
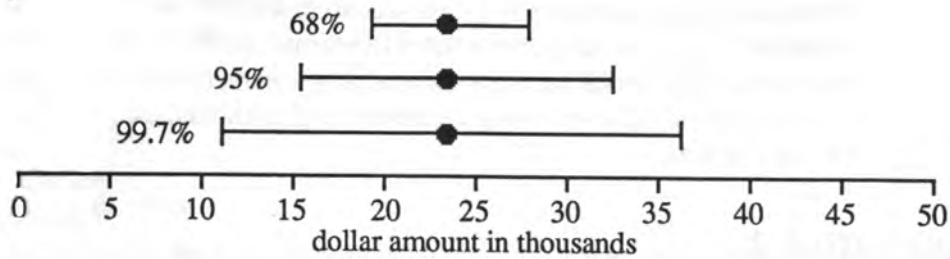
c. 1. One can be 68% confident that the average income of high school graduates in the state is in the interval [$19,591, $28,135].

   2. One can be 95% confident that the average income of high school graduates in the state is in the interval [$15,319, $32,407].

   3. One can be 99.7% confident that the average income of high school graduates in the state is in the interval [$11,047, $36,679].

Discussion 1 (page 195)

a. Sample response: The 99.7% confidence interval is most likely to include the actual average income because it includes all values 3 standard deviations on either side of the sample mean.

b. Since the 68% confidence interval is the narrowest, it has the least range of values.

c. Sample response: The greater the size of the interval, the less chance there is of making an error.

d. Sample response: The confidence intervals can be narrowed by increasing the sample size. The width of the interval is determined by $\sigma/\sqrt{n}$. As n increases, $\sigma/\sqrt{n}$ decreases.

e. 1. The advantage of the 99.7% confidence interval is that the probability of including the population mean is high. The disadvantage is that the interval may be so wide that it does not provide a useful estimate.

   2. The 95% confidence interval has some of the advantages of both the 99.7% and 68% confidence intervals. Students may suggest that this interval is the most useful.

   3. The disadvantage of the 68% confidence interval is that the probability of including the population mean is relatively low. The advantage is that the interval is narrow and has a small range of possible values for the mean.

f.   Sample response: The only way to get the true mean and standard deviation of the population is to take a census, which may not be feasible. When a sample mean is used to estimate the population mean, the confidence interval provides a probability statement about the accuracy of the estimate. Using a random sample eliminates possible sources of bias.

Exploration 2 (page 196)

In this exploration, students are introduced to the logic and structure of hypothesis testing.

a.   The null hypothesis for Roberto's claim is $H_0: \mu = \$36,000$. The alternative hypothesis for Roberto's claim is $H_a: \mu \neq \$36,000$.

b.   Sample graph:
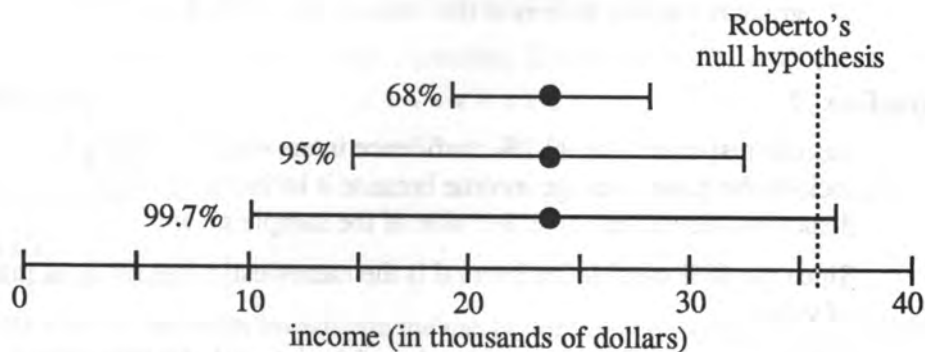
c.   Roberto's predicted mean is included only in the 99.7% confidence interval.

d.   Since the value for μ in the null hypothesis is outside the 68% and 95% confidence intervals, H_0 can be rejected at those levels. However, it cannot be rejected at the 99.7% level, since it falls within the 99.7% confidence interval.

e.   The null hypothesis for the editor's claim is $H_0: \mu = \$16,000$. The alternative hypothesis for the editor's claim is $H_a: \mu \neq \$16,000$.
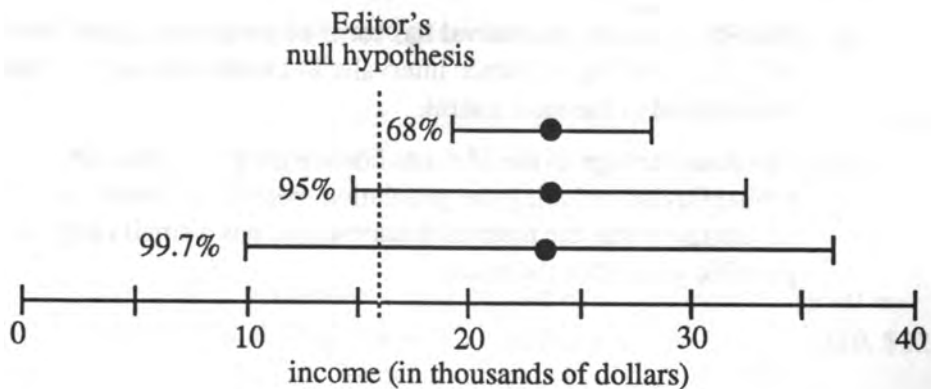
The editor's predicted mean is included in the 95% and 99.7% confidence intervals. Since the value for μ in the null hypothesis is outside the 68% confidence interval, H_0 can be rejected at this level. It cannot be rejected at the 95% and 99.7% levels, because it falls within these confidence intervals.

Teacher Note

Although hypothesis testing can be used to either reject a null hypothesis or to fail to reject it, it cannot be used to prove that a null hypothesis is true. This is an often misunderstood concept. Intuitively, this idea can be illustrated using a courtroom scenario. Although a jury may find a defendant not guilty, this does not necessarily prove that the person is innocent.

The same concept can be demonstrated mathematically with the idea of proof. When rejecting a null hypothesis based on a statistic, you have found a counterexample, realizing that there is the possibility for error. If you fail to reject, you have found a single example that supports the conjecture. A single example does not constitute a proof.

For example, consider a sample taken from a set of numbers with an unknown mean. The sample has a mean $\bar{x}$ of 50; the estimated value of $\sigma_{\bar{x}}$ is 10. This information can be used to construct a 95% confidence interval $[30, 70]$. Now consider the following two hypotheses: $H_1: \mu = 78$ and $H_2: \mu = 66$.

Approximately 95% of intervals $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ contain the mean of the set. Since 78 is not contained in the 95% confidence interval, H_1 can be rejected at the 95% confidence level.

The number 66 does fall in the 95% confidence interval. Therefore, H_2 cannot be rejected at the 95% confidence level. This hypothesis test does not prove that 66 is the mean of the set. There are many possible values for the mean between 30 and 70; 66 is only one of them.

Discussion 2 (page 197)

a. Sample response: Since Roberto's null hypothesis can be rejected with 68% and 95% confidence, while his editor's null hypothesis can be rejected only with 68% confidence, the editor's claim about the actual mean income is probably more reasonable.

b. Sample response: "Failing to reject" does not mean the null hypothesis is true. It only means you have not found evidence to reject it. "Accepting" the null hypothesis implies that it is likely to be true.

c. The alternative hypothesis is the statement that must be true if the null hypothesis is false. Therefore, rejecting the null hypothesis results in the acceptance of the alternative hypothesis.

d. Sample response: No. This only means that you have not found enough evidence to reject it.

e. Sample response: No. There is a chance that the population mean falls outside the confidence interval. For example, 5% of intervals $[\bar{x} - 2\sigma_{\bar{x}}, \bar{x} + 2\sigma_{\bar{x}}]$ do not contain the mean.

f. Sample response: Because the editor's hypothesized value is included in the 95% confidence interval, you cannot reject her hypothesis with 95% confidence. Because Roberto's hypothesized value is not included in the 95% confidence interval, you can reject his hypothesis with 95% confidence. Therefore, you can be more confident that Roberto's hypothesis is wrong than that the editor's hypothesis is wrong.

g. Sample response: Roberto's statement is wrong. The 99.7% applies to how confident one can be that the population mean lies in the interval, not to the probability that his hypothesis is right.

Assignment (page 198)

3.1 **a.** For this data $\bar{x} \approx 16{,}797$ and $s \approx 8864$. The 68% confidence interval is [$15,179, $18,415], the 95% confidence interval is [$13,560, $20,033], and the 99.7% confidence interval is [$11,942, $21,652].

The editor's claim that $\mu = \$16{,}000$ is included in each of these intervals. Therefore, the null hypothesis $H_0\!: \mu = \$16{,}000$ cannot be rejected at any of the three confidence levels.

b. Sample response: Since $16,000 falls in the 95% confidence interval, I fail to reject the null hypothesis at the 95% confidence level.

c. Sample response: No. A hypothesis test cannot prove that a null hypothesis is true. Such a test can only reject or fail to reject a null hypothesis with a certain level of confidence.

d. Sample response: The editor's sample consists of recent graduates from the same high school. Unless this sample is representative of graduates statewide, it may contain biases.

*3.2 **a.** The 95% confidence interval for this data is [$26,076, $41,044]. Since $36,000 is included in this interval, Roberto's null hypothesis $H_0\!: \mu = \$36{,}000$ cannot be rejected at the 95% confidence level.

b. Sample response: Since there is not enough evidence to reject the null hypothesis at the 95% level, you can only conclude that H_0 may (or may not) be true.

c. Sample response: Students who show leadership ability in school may be more likely to earn higher salaries. In addition, a sample of 30 class presidents from one school is not likely to be representative of graduates statewide.

3.3 Students should formulate a null hypothesis where the population mean equals some value M. In this situation, $H_0: \mu = \$M$ and $H_a: \mu \neq \$M$. They should propose sampling methods that avoid bias and include their justification for selecting a particular confidence level.

3.4 Sample response: They are confusing the standard deviation s of their sample with the standard deviation $\sigma_{\bar{x}}$ of all sample means. There is a big difference, since $\sigma_{\bar{x}}$ is estimated by $s/\sqrt{n}$.

Based on their sample, one can be 99.7% confident that the mean life expectancy is in the interval:

$$\left[827 - 3 \cdot \frac{424}{\sqrt{1000}}, 827 + 3 \cdot \frac{424}{\sqrt{1000}}\right] \approx [787, 867] \,,$$

Since 1250 is not in this interval, the null hypothesis is rejected. One can be 99.7% confident that the mean life expectancy of the bulbs is not 1250 hr.

*3.5 For the Shiny Bright Company, the null hypothesis is $H_0: \mu = 2000$ and the alternative hypothesis is $H_a: \mu \neq 2000$.

The 95% confidence interval is:

$$\left[2040 - 2 \cdot \frac{470}{\sqrt{100}}, 827 + 2 \cdot \frac{470}{\sqrt{100}}\right] = [1946, 2134] \,,$$

The value of μ falls in the 95% confidence interval. Therefore, students should fail to reject H_0.

For the Hi-Glow company, the null hypothesis is $H_0: \mu = 2000$ and the alternative hypothesis is $H_a: \mu \neq 2000$.

The 95% confidence interval is:

$$\left[2010 - 2 \cdot \frac{400}{\sqrt{10,000}}, 2010 + 2 \cdot \frac{400}{\sqrt{10,000}}\right] \approx [2002, 2018]$$

The value of μ does not fall within the 95% confidence interval. Therefore, students should reject H_0 and accept the alternative hypothesis.

Students should realize that although there appears to be a difference in the mean life spans found in the two studies, this difference might not be significant. Since the confidence intervals for the two studies overlap, the mean life spans may be equal.

* * * * *

3.6   **a.** For this situation, $H_0: \mu = 397$ and $H_a: \mu \neq 397$.

b. For the sample data, $\bar{x} \approx 394$ and $s = 9$. The 95% confidence interval is:

$$\left[394 - 2 \cdot \frac{9}{\sqrt{40}}, 394 + 2 \cdot \frac{9}{\sqrt{40}} \right] \approx [391, 397]$$

c. Since 397 falls in the confidence interval, there is not enough evidence to reject the null hypothesis.

d. Sample response: Since the class cannot reject the null hypothesis, they cannot conclude that the average net mass is not 397 g. They can only conclude that the average net mass may be 397 g.

3.7   **a.** In this case, $\bar{x} \approx 166$ and $s = 23$. The 68% confidence interval is:

$$\left[166 - \frac{23}{\sqrt{36}}, 166 + \frac{23}{\sqrt{36}} \right] \approx [162, 170]$$

b. Sample response: Loretta's average of 152 from last season is not included in this interval. Therefore, the null hypothesis should be rejected at the 68% confidence level.

c. Sample response: One can be 68% confident that Loretta's average score has changed significantly since she started using her new ball.

$$* * * * * * * * * *$$

Research Project (page 200)

Students should formulate a null hypothesis where the population mean equals some value A. In this case, $H_0: \mu = A$ and $H_a: \mu \neq A$. They also should propose sampling methods that avoid bias and include their justification for selecting a particular confidence level.

Teacher Note

The summary assessment is designed to allow students to proceed through all steps of a hypothesis test. To model a population, students will need 100–200 identical objects—such as plastic tokens or cardboard strips—marked with various lengths, as well as a suitable container. The objects simulate fish, while the container simulates a lake.

Answers to Summary Assessment (page 201)

1. **a.** Answers will vary. Students may hypothesize about the mean using various rationales. Sample hypotheses: $H_0: \mu = 10$ and $H_a: \mu \ne 10$.

   b. Answers may vary. The following sample responses use a 95% confidence interval.

   c. Students should use a sample size of at least 30.

   Note: As an alternative to requiring students to sample the population, you may wish to distribute the following data set, which was randomly drawn from a population with a mean of 13 and a standard deviation of 3.5. The mean of this sample is 13, with a standard deviation of 3.9.

11	12	10	19	14	20
7	17	10	13	12	8
7	13	20	18	11	15
12	12	8	7	15	14
16	14	18	16	9	16

2. **a.** For the data given above, the estimated standard deviation of all sample means is $3.9/\sqrt{30} \approx 0.7$. Using 13 and 0.7 for $\bar{x}$ and $\sigma_{\bar{x}}$, respectively, the 95% confidence interval is [11.6, 14.4].

   b. Using the sample data given in Part **c** and the null hypothesis $H_0: \mu = 10$, the value is not in the 95% confidence interval. In this case, the null hypothesis is rejected.

3. Sample response: Our samples may have been biased in favor of objects that were on the top of the population. When resampling the lake, we would increase the sample size and use a sampling technique that ensures each object is equally likely to be drawn.

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1.   A random sample of size 126 has $\bar{x} = 18$ and $s = 2$.

   a. Estimate the standard deviation of all sample means $\sigma_{\bar{x}}$.

   b. Construct the interval $[\bar{x} - 3\sigma_{\bar{x}}, \bar{x} + 3\sigma_{\bar{x}}]$.

   c. How confident are you that the population mean is within the interval in Part **b**?

2.   The fish and game commission is studying the trout population in a local lake. A sample of 30 trout had a mean length of 18.4 cm and a standard deviation of 6.2 cm.

   a. Estimate the standard deviation of all sample means in this situation.

   b. Write a statement about the mean length of the trout population at:

      1. a 68% confidence level

      2. a 95% confidence level

      3. a 99.7% confidence level.

3.   The Shiny Bright Company is studying the mean life expectancy of their 100-watt bulbs. A sample of 850 bulbs had a mean life expectancy of 524 hr, with a standard deviation of 250 hr. Use this information to write a confidence statement about the mean life expectancy of the company's 100-watt bulbs.

4.   A local bottling company claims that the volume of product in their 2-L soft drink bottles averages 2 L. You have been hired by a consumer group to test this claim. The volumes (in liters) of a sample of 30 bottles are shown in the table below.

1.90	1.95	1.97	1.98	2.00	1.94
1.91	2.04	2.02	1.99	1.97	1.96
1.92	2.00	1.97	1.96	1.94	1.94
1.93	2.05	2.04	2.00	1.98	1.97
2.00	1.98	1.97	1.94	2.01	1.99

   a. Use the sample data to construct a 95% confidence interval, then test the company's claim that the average volume is 2 L.

   b. Write a report explaining the results of your hypothesis test.

Answers to Module Assessment

1.   **a.** $\sigma_{\bar{x}} \approx 2/\sqrt{126} \approx 0.2$

       b.   [17.4, 18.6]

       c.   This is a 99.7% confidence interval since it represents values within 3 standard deviations of all sample means from $\bar{x}$.

2.   **a.**   The estimated standard deviation of all sample means is:

$$\sigma_{\bar{x}} \approx 6.2/\sqrt{30} \approx 1.1 \text{ cm}$$

       b.   **1.**   One can be 68% confident that the mean length of fish is between 17.3 cm and 19.5 cm.

             2.   One can be 95% confident that the mean length of fish is between 16.2 cm and 20.6 cm.

             3.   One can be 99.7% confident that the mean length of fish is between 15.1 cm and 21.7 cm.

3.   The estimated standard deviation of all sample means is:

$$\sigma_{\bar{x}} \approx 250/\sqrt{850} \approx 8.6 \text{ hrs.}$$

Students may use any of the three confidence levels. For example, one can be 99.7% certain that the mean life expectancy of 100-watt bulbs is between 498.2 hr and 549.8 hr.

4.   **a.**   The 95% confidence interval is [1.95 L, 1.99L]. The claimed volume of 2 L is not included in this interval. Therefore, the hypothesis that the mean volume is 2 L can be rejected at a 95% confidence level.

       b.   Based on this sample of 30 bottles, one can be 95% confident that the true mean volume is between 1.95 L and 1.99 L. Since 2 L is not in that interval, the null hypothesis can be rejected at the 95% confident level.

Selected References

Neter, J., W. Wasserman, and G. A. Whitmore. *Applied Statistics*. Boston: Allyn and Bacon, 1993.

Billingsley, P., and D. V. Huntsberger. *Elements of Statistical Inference*. Boston: Allyn and Bacon, 1981.

Flashbacks

Activity 1

1.1 Determine the mean (to the nearest whole number) for each set of data below.

    a. {1, 13, 11, 1, 1, 1, 24, 2, 4, 15, 1, 26, 1, 23, 1}

    b. {31, 8, 1, 1, 3, 18, 1, 2, 10, 31, 5, 21, 1, 10, 13}

    c. {28, 1, 5, 1, 14, 32, 1, 23, 24, 21, 1, 2, 3, 19, 26}

1.2   **a.** Create a frequency histogram for the following data set:

      {1, 13, 11, 1, 1, 1, 14, 2, 4, 15, 1, 6, 1, 13, 1}

    b. Determine the probability that a number chosen at random from the data set in Part **a** is:

      1. less than 10

      2. greater than 4

      3. between 4 and 10, inclusive.

Activity 2

2.1 Plot each of the following intervals on a number line.

    a. $[2, 6]$

    b. $[6, 13)$

    c. $(4, 10)$

2.2 Express each of the following statements as an inequality.

    a. x is in the interval $[3, 9]$

    b. x is in the interval $[-13, 8)$

    c. x is in the interval $(5, 19)$

Activity 3

    3.1   Write the negation of each of the following statements:

        a. $a = b$

        b. $a < b$

        c. $a \geq b$

    3.2   Determine the standard deviation for the set of data below.

$$\{1, 13, 11, 1, 1, 1, 24, 2, 4, 15, 1, 26, 1, 23, 1\}$$

Answers to Flashbacks

Activity 1

1.1   **a.** The mean is 8.

        b. The mean is 10.

        c. The mean is 13.

1.2   **a.** Sample graph:
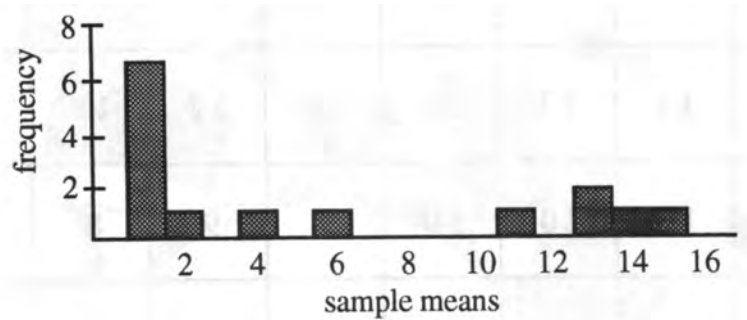
        b. **1.** $10/15 = 2/3 \approx 0.7$

            2. $6/15 = 2/5 = 0.4$

            3. $2/15 \approx 01$

Activity 2

2.1   Sample graph:
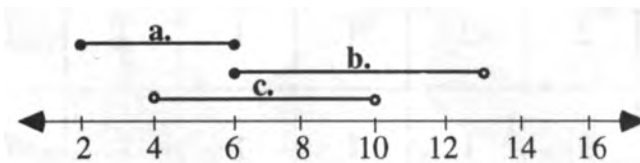
2.2   **a.** $3 \le x \le 9$

        b. $-13 \le x < 8$

        c. $5 < x < 19$

Activity 3

3.1   **a.** $a \ne b$

        b. $a \ge b$

        c. $a < b$

3.2   The standard deviation is 9.6.

Population Template

34	32	31	31	29	28	28	27	26
25	24	23	23	22	21	21	20	19
19	18	18	17	17	17	16	16	15
14	14	13	13	13	12	12	11	11
11	10	10	10	9	9	8	8	7
7	6	6	6	6	6	5	5	5
5	4	4	4	4	4	4	3	3
3	3	2	2	2	2	2	2	2
2	2	2	2	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

Is It Really True?

Teacher Guide

In this module, you continue your investigations of logical reasoning.

Terri Dahl • Laurie Paladichuk • Peter Rasmussen

Is It Really True?

Overview

In this module, students examine the uses of connectives and quantifiers in mathematical statements. They also investigate truth values in conditional statements of the form "if p, then q."

Objectives

In this module, students will:

- define the negation of statements using the quantifiers *some*, *all*, and *none*

- use logically equivalent forms to rewrite statements

- write negations of statements

- determine the difference between the inclusive *or* and the exclusive *or* through truth tables and Venn diagrams

- investigate the negations of compound statements through truth tables, Venn diagrams, and De Morgan's laws

- examine the truth values of a conditional statement

- investigate the converse, the inverse, and the contrapositive of the conditional.

Prerequisites

For this module, students should know:

- the definition of a mathematical statement and its truth value

- how to disprove a statement through the use of a counterexample

- how to draw and use Venn diagrams

- the use of the mathematical language *and*, *or*, and *not*

- the truth tables for compound statements using the connectives *and* and *or*

- the definition and notation of a conditional statement.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	3	2	2	2	1	10

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
two sets of colored chips	X				

Is It Really True?

Introduction

In the introduction, students are asked to discuss the implications of an if-then statement.

Discussion

a. Sample response: Colter should expect a reprimand. Since the foundation is not complete, he should not have the weekend off.

b. Sample response: Colter should not expect a reprimand. Since the foundation is complete, it is acceptable to take the weekend off.

Teacher Note

This module focuses on mathematical logic and reasoning. In order to create unambiguous examples, some of the sentences given may violate rules of English usage. In Problem **1.1a**, for example, "All buildings have elevators" is rewritten as "No buildings do not have elevators."

Activity 1

This activity introduces logically equivalent statements that include the quantifiers *all*, *some*, and *none*. The negation of each statement is developed through the use of counterexamples.

Teacher Note

It is important for students to understand that logical equivalence refers to the form, not the content, of a pair of statements. For example, the conditional $p \rightarrow q$ and the contrapositive $\sim q \rightarrow \sim p$ have identical truth tables, no matter what selections are made for p and q. Therefore, the conditional and the contrapositive are logically equivalent.

While it is true that a conditional and its converse may sometimes have the same truth values, this is not always the case. Therefore, the conditional and the converse are not logically equivalent.

Materials List

• colored chips (at least 4 each of two different colors per group)

Exploration (page 207)

a. In the following sample responses, a blue chip (B) represents materials purchased locally and a red chip (R) represents materials not purchased locally.

  1. Sample response: There are 5 possible combinations: 4R, 1R3B, 2R2B, 3R1B, 4B.

  2. Sample response: Only one combination correctly represents this statement, four blue chips (4B).

b. Sample response: The following four combinations of chips represent counterexamples—1R3B, 2R2B, 3R1B, and 4R.

c. Sample response: Not all materials used by Mykelti Construction will be purchased locally. **Note:** Statements such as, "All materials used by Mykelti Construction will not be purchased locally" or "No materials used by Mykelti Construction will be purchased locally" do not correctly describe all the counterexamples.

d. Sample responses: Some materials used by Mykelti Construction will not be purchased locally. At least one of the materials used by Mykelti Construction will not be purchased locally.

e. In the following sample responses, a blue chip (B) represents oak and a red chip (R) represents pine.

  There are four possible combinations for the statement "Some of the wood used by Mykelti Construction is oak": 1R3B, 2R2B, 3R1B, 4B. **Note:** Some students may question the use of four blue chips to correctly describe the given statement. You may wish to emphasize that if a statement is true for the quantifier *at least one*, then it is true for the quantifier *some*.

  Only one combination represents a counterexample to the statement: 4R. The following statement describes this counterexample: "None of the wood used by Mykelti Construction is oak." **Note:** Statements such as "Some of the wood used by Mykelti Construction is not oak" do not correctly describe this counterexample.

  Another way to rewrite this statement is "All of the wood used by Mykelti Construction is not oak" or "It is not the case that some of the wood used by Mykelti Construction is oak."

f. In the following sample responses, a blue chip (B) represents an employee who does not smoke on the job site and a red chip (R) represents an employee who does smoke on the job site.

  Only one combination correctly represents the statement "No employees will smoke on the job site": 4B.

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The following four combinations represent counterexamples: 1R3B, 2R2B, 3R1B, and 4R. The following statement describes these counterexamples: "Some employees will smoke on the job site." **Note:** Statements such as "Some employees will not smoke on the job site," "All employees will smoke on the job site," or "No employees will not smoke on the job site" do not correctly describe all counterexamples.

Another way to rewrite this statement is: "At least one employee will smoke on the job site" or "It is not the case that no employees will smoke on the job site."

g. **1.** Sample responses:

- Not all materials used by Mykelti Construction will be purchased locally.

- It is not the case that all material used by Mykelti Construction will be purchased locally.

- Some materials used by Mykelti Construction will not be purchased locally.

- At least one of the materials used by Mykelti Construction will not be purchased locally.

2. Sample responses:

- It is not the case that some of the wood used by Mykelti Construction is oak.

- No wood used by Mykelti Construction is oak.

- All wood used by Mykelti is not oak.

3. Sample responses:

- It is not the case that no employees will smoke on the job site.

- Some of the employees will smoke on the job site.

- At least one employee will smoke on the job site.

Discussion

a. Sample response: Yes, since it has the opposite truth value of the given statement. **Note:** You may wish to point out that negating words in a statement (such as changing *all* to *none* or smoker to nonsmoker) does not necessarily result in the negation of the statement.

b. **1.** The negation is "At least one house is not painted white."

2. Sample response: The statement "Some houses are painted white" is not the negation of the original because it is possible for both statements to have the same truth value.

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c. **1.** The minimum requirement needed to negate the given statement is to find at least one Saint Bernard that is not a dog. Sample responses: "Not all Saint Bernards are dogs," "Some Saint Bernards are not dogs," or "At least one Saint Bernard is not a dog."

   2. Sample responses: "No presidents of the United States have been women" and "It is not the case that some presidents of the United States have been women."

   3. Sample responses: "Some of my friends have been to Los Angeles" and "At least one of my friends has been to Los Angeles."

   4. Sample responses: "It is not the case that some puppies are canines" and "No puppies are canines."

d. Sample response: Sentences that are not statements have no truth value. Therefore, they cannot have a negation. For example, the command "Write a letter" has no negation.

Assignment (page 211)

*1.1 **a.** **1.** One statement logically equivalent to "some A are B" is "at least one A is B." Therefore, "Some people are old" can be rewritten as "At least one person is old."

   2. One statement logically equivalent to "no A are B" is "all A are not B." Therefore, "No one can dance" can be rewritten as "All can not dance."

   3. One statement logically equivalent to "all A are B" is "no A are not B." Therefore, "All buildings have elevators" can be rewritten as "No buildings do not have elevators."

   b. Answers may vary. Sample responses are given below.

   1. The negation of "Some people are old" is "No people are old."

   2. The negation of "No one can dance" is "Someone can dance."

   3. The negation of "All buildings have elevators" is "Some buildings have no elevators."

1.2 **a.** The negation can be written as "It is not the case that all horses are not black," or "Some horses are black," or "There is at least one horse that is black."

   b. One statement logically equivalent to "all A are not B" is "no A are B." Therefore, the statement can be rewritten as "No horses are black."

1.3 **a.** This is not the negation because it does not correctly describe all counterexamples to "All police officers are single." The negation need only state that "At least one police officer is not single."

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b. This is the negation because it includes the minimum requirement needed to negate "All police officers are single."

c. This is not the negation. See response to Part **a**.

d. This is not a negation because there are situations in which both statements have the same truth value.

e. This is not the negation. See response to Part **d**.

f. This is the negation because it includes the minimum requirement needed to negate "All police officers are single."

1.4 **a.** One statement logically equivalent to "all A are B" is "no A are not B." Therefore, the statement can be rewritten as "No Mykelti construction workers are not men" or "No Mykelti construction workers are women."

b. One statement logically equivalent to "some A are B" is "at least one A is B." Therefore, the statement can be rewritten as "At least one college student majors in engineering."

c. One statement logically equivalent to "no A are B" is "all A are not B." Therefore, the statement can be rewritten as "All professional football teams include only male players."

1.5 The statement "Some houses are painted white" includes the possibility that all houses are painted white. The statement "Some houses are not painted white" includes the possibility that no houses are painted white.

The following diagram shows all the true cases when considering three houses and the statement "some houses are painted white."
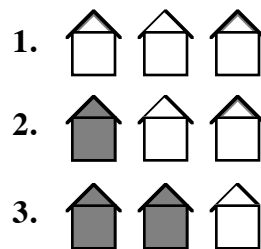
The diagram below shows all the true cases when considering three houses and the statement "some house are not painted white."

* * * * *
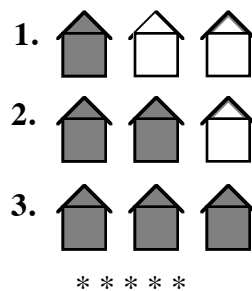
1.6   **a.** Sample response: To determine if this statement is true, you need to find one person in your math class that plays football. To determine if this statement is false, you need to show that all people in your class do not play football.

         b. Sample response: To determine if this statement is true, you need to show that all people in the math class are taking a science class. To determine if this statement is false, you need to find one person in the class that is not taking a science class.

         c. Sample response: To determine if this statement is true, you need to show that all students in your math class do not take the bus. To determine if this statement is false, you need to find one student in the class that rides the bus.

1.7   Sample response: No. This statement is false, because the purchase was a counterexample. It was 20% off.

1.8   **a.** Since this statement is true, its negation is false. Sample responses: "Not all integers are real numbers," "Some integers are not real numbers," or "At least one integer is not a real number."

         b. Since this statement is true, its negation is false. Sample responses: "No integers are whole numbers" or "All integers are not whole numbers."

         c. Since this statement is true, its negation is false. Sample responses: "Some irrational numbers are rational numbers" or "At least one irrational number is a rational number."

* * * * * * * * * *

(page 212)

Activity 2

In this activity, students investigate the negation of compound statements using De Morgan's laws.

Materials List

- none

Teacher Note

Some symbolic manipulators can implement logical rules for user-defined statements. You may wish to ask students to compare statements using such technology. On a TI-92 calculator, for example, students might compare the statements "not $(x > 10$ or $x < 10)$" and "$x \le 10$ and $x \ge -10$" by following these steps:

- In the y= editor, define y1(x) = When(not($x{>}10$ or $x{<}{-}10$),–5, 5) and y2(x)= When($x{\le}10$ and $x{\ge}{-}10$),–4, 4)

- Select an appropriate viewing window, graph the two functions, and compare the graphs.

In this case, the two statements have the same truth values. The graph of y1(x) is a horizontal line at –5 for x in the interval $[-10, \ 10]$ and a horizontal line at 5 for all other values of x. The graph of y2(x) is a horizontal line at –4 for x in the interval $[-10, \ 10]$ and a horizontal line at 4 for all other values of x. Students could also compare the tables associated with the functions.

Exploration   (page 212)

a.   Sample Venn diagram:
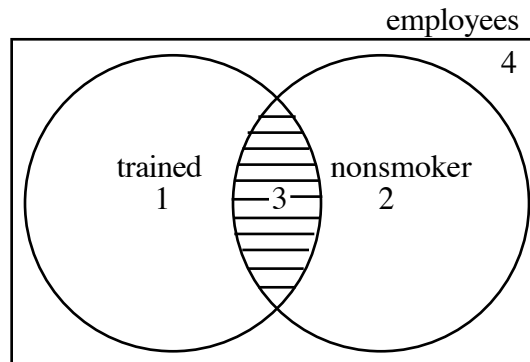
b.   **1.** The employees who have been trained for supervisory positions and are nonsmokers are located in region 3 of the Venn diagram in Part **a**.

       2. The employees who do not satisfy the director's recommendation are those in regions 1, 2, or 4 of the Venn diagram in Part **a**.

       3. The employees who do not satisfy this recommendation have received training and are smokers (region 1), have not received training and are nonsmokers (region 2), or have not received training and are smokers (region 4).

4. Sample table:

Nonsmoker	Trained	Possible Candidate
T	T	T
T	F	F
F	T	F
F	F	F

5–6. Sample table:

p	q	p and q	~(p and q)	~p	~q	~p or ~q
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

c.   **1.** Sample Venn diagram:
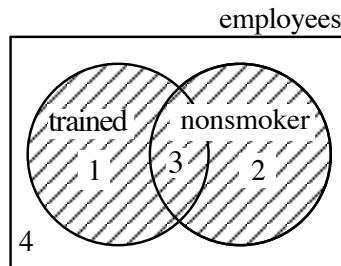
2. The employees who have been trained for supervisory positions or are nonsmokers are located in regions 1, 2, or 3 of the Venn diagram. Therefore, the employees who do not satisfy the assistant's recommendation are those in region 4.

3. The employees who do not satisfy this recommendation have not been trained and are smokers (region 4).

4. Sample table:

Nonsmoker	Trained	Possible Candidate
T	T	T
T	F	T
F	T	T
F	F	F

5–6. Sample table:

p	q	p or q	~(p or q)	~p	~q	~p and ~q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Discussion (page 215)

a. Two general statements are logically equivalent when all of their corresponding truth values are the same.

b. "It is not the case that *p* and *q*" or "Not *p* or not *q*."

c. "It is not the case that *p* or *q*" or "Not *p* and not *q*."

d. Sample response: To use De Morgan's laws, negate the statement before the connective *and* and the statement after the connective *and*, then change the connective *and* to a connective *or*. The result is "They are not overworked or not underpaid."

Assignment (page 216)

2.1 a. The truth table for the exclusive *or* (*xor*) and the inclusive *or* is shown below

p	*q*	*p* xor *q*	*p* or *q*
T	T	F	T
T	F	T	T
F	T	T	T
F	F	F	F

b. Only the first row of this truth table differs from the truth table for the inclusive *or*. The combination TT results in a truth value of true using the inclusive *or*, while the same combination results in a truth value of false using the exclusive *or*.

2.2 a. Sample response: Typically, this means that you can select either coffee or tea, but not both, which is the exclusive *or*. However, if it is possible to have both tea and coffee, the inclusive *or* would apply.

b. Sample response: Because you may have cream, sugar, or both cream and sugar, the inclusive *or* is implied.

c–d. The use of the exclusive or inclusive *or* is typically determined by context. Sample responses are provided in the table below.

Context	Inclusive *or*	Exclusive *or*
ordering a burger	ketchup or mustard	cheese or plain
class enrollment	English or Art	
gender		male or female

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2.3 **a.** Sample truth table:

driver's door open	passenger's door open	light on
T	T	T
T	F	T
F	T	T
F	F	F

b. This situation implies the inclusive *or* because the light is on when either door is open and when both doors are open.

*2.4 **a.** Sample response: $\sim (x < 5 \text{ or } x = 5)$.

b. Using De Morgan's laws, the statement can be rewritten as " $\sim (x < 5)$ and $x \neq 5$ "

2.5 **a.** Let p represent "Eileen is too short" and q represent "Eileen is too tall." The sentence may be represented as follows: $\sim (p \text{ or } q)$.

b. Using De Morgan's laws, an equivalent statement is: $\sim p$ and $\sim q$. The sentence is written as follows: "Eileen is not too short and she is not too tall."

c. Student's answers may vary. However, the sentence in Part **b** may be easier for most to interpret since one does not have to guess which parts are being negated.

* * * * *

2.6 **a.** The statement is false. Its negation is "The number 3 is not an even integer or the number 6 is not an odd integer."

b. The statement is true. Its negation is "The Pacific is not a river and the Pacific is not an ocean."

c. The statement is true. Its negation is "$3 \neq 5$ and $4 = 5$."

2.7 **a.** This circuit depicts the connective *or* because electricity can flow from C to D if one or both of the switches are closed.

b. This circuit depicts the connective *and* because electricity can flow from C to D only if both switches are closed.

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Research Project (page 218)

George Boole (1815–1864) came from a poor British family. He was a brilliant, self-taught scholar, whose most profound work was *An Investigation of the Laws of Thought*, published in 1854. In this book, he defined the algebraic properties of *and*, *or*, and *not*, clarified the ideas of Boolean algebra, and established the algebra of sets and logic. Boolean algebra is applied today in the fields of engineering, electronics, and computer technology.

(page 218)

Activity 3

Students examine the truth values of a conditional statement and rewrite the conditional in a logically equivalent form using the connective *or*.

Materials List

- none

Exploration (page 218)

a. The hypothesis is "the dog is smart"; the conclusion is "the dog is a Labrador."

b. The dogs in region 1 can be described by the statement "All smart dogs are Labradors." Those in region 2 can be described by "Some Labradors are not smart." Those in region 3 can be described by "No dogs that are not Labradors are smart."

c. A completed table is shown below:

Hypothesis	Conclusion	Conflict?	Statement
T	T	no	T
T	F	yes	F
F	T	no	T
F	F	no	T

d. The statements "$p \rightarrow q$" and "$\sim p$ or q" are logically equivalent, as shown by the truth table below:

p	q	$p \rightarrow q$	$\sim p$	$\sim p$ or q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Discussion

a. **1.** No. Since the hypothesis is false, then the statement is true.

   2. Yes. This provides a counterexample to the statement. The hypothesis is true and the conclusion is false. This results in a false statement.

b. A conditional statement is false when it can be shown that the hypothesis is true and the conclusion is false.

c. The conditional "$p \rightarrow q$" is logically equivalent to "$\sim p$ or q." To rewrite the statement "If the dog is smart, then it is a Labrador," negate the hypothesis, then insert the connective *or*, followed by the conclusion. This results in the compound statement, "The dog is not smart or it is a Labrador."

d. Since the hypothesis "The moon is made of green cheese" is false, the truth table indicates that this if-then statement is true. **Note:** Many would argue that the statement is neither true nor false. "Counterfactual conditionals" such as the one above have stirred much debate among philosophers and logicians.

Assignment

3.1 **a.** Sample Venn diagram:
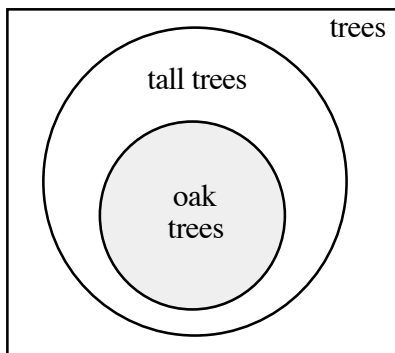
b. Sample response: All oak trees are tall trees. Some tall trees are oaks. No trees that are not tall are oaks.

3.2   **a.** The hypothesis is "the house was built by Mykelti"; the conclusion is "its materials were purchased locally."

        b. The only time that this conditional statement is false is when a house built by Mykelti is not built with materials purchased locally.

        c. Sample responses: All houses built by Mykelti use materials that are purchased locally. No houses built with materials that are not purchased locally are built by Mykelti.

        d. The house was not built by Mykelti or its materials were purchased locally.

3.3   **a.** The hypothesis is "you finish the foundation"; the conclusion is "you're off for the weekend."

        b. Sample response: Colter finishes the foundation and takes the weekend off. Colter finishes the foundation and doesn't take the weekend off. Colter doesn't finish the foundation and takes the weekend off. Colter doesn't finish the foundation and doesn't take the weekend off.

        c. Some students may argue that the supervisor implied that Colter could take the weekend off *only* if he finished the foundation. In this case, Colter cannot justify taking the weekend off.

           Other students may follow the rules of logic for a conditional. In this case, the only situation in which Colter's actions result in a false statement is when he finishes the foundation and does not take the weekend off.

3.4   **a.** It is not fall or the leaves are falling.

        b. The worker is a carpenter or she is an electrician.

        c. The person is not happy or he is not young.

        d. It is snowing or it is not winter.

3.5   **a.** If a prime number is greater than 2, then the prime number is odd.

        b. Sample response: This statement is true for all prime numbers greater than 2 because all even numbers are of the form $2n$ and are therefore divisible by 2. Prime numbers are not divisible by any numbers other than 1 and itself, so only odd numbers greater than 2 could be prime.

3.6   Students should look for a counterexample: a situation in which the three angles of one triangle are congruent to the three angles of another triangle and the triangles are not congruent. Since this is possible in Euclidean geometry when two triangles are similar but not congruent, the original statement is false.

<div align="center">* * * * *</div>

3.7   **a.** Sample Venn diagram:
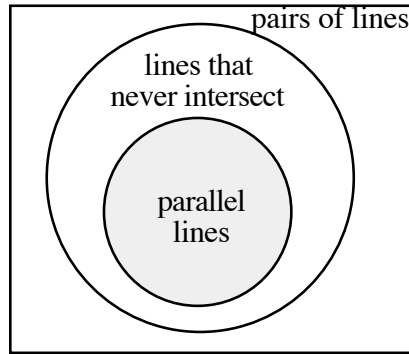
b. Sample response: All parallel lines never intersect. Some lines that never intersect are parallel. No lines that intersect are parallel.

c. **1.** Sample response: They belong in the set of lines that never intersect but outside the set of parallel lines.

2. Sample response: No. To be a counterexample, skew lines would have to illustrate a situation in which the hypothesis is true and the conclusion is false. Since skew lines are not parallel lines, the hypothesis is false.

*3.8   **a.** The hypothesis is "the polygon is a triangle"; the conclusion is " $a^2 + b^2 = c^2$, where a, b, and c are the lengths of the sides of the triangle and c is the length of the longest side."

b. The only time a conditional, "if p, then q," is false is when p is true and q is false. Therefore, students should look for a counterexample: a triangle in which $a^2 + b^2 \neq c^2$. For instance, when $a = 3$, $b = 5$, and $c = 7$, $3^2 + 5^2 \neq 7^2$. Therefore, the original statement is false.

3.9   **a.** Sample response: If x is a root of the polynomial, then $10 > x > -10$.

b. Sample response: There is a root x of the polynomial such that $x > 10$ or $x < -10$.

* * * * * * * * * *

Activity 4

In this activity, students are introduced to the contrapositive, converse, and inverse of a conditional statement.

Materials List

* none

Exploration

a. Students should create a truth table similar to the one shown below.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

b. The conditional statement and its contrapositive are logically equivalent. The converse and inverse are logically equivalent.

Teacher Note

When a conditional and its converse are both true, they can be written as a single statement using the words *if and only if*. Symbolically, this can be represented as $p \leftrightarrow q$, or *p* iff *q* (read "*p* if and only if *q*"). All definitions may be written as statements using *if and only if*.

Discussion

a. Sample response: Not necessarily. The truth tables show that a statement and its converse are not logically equivalent. The converse can be—but is not always—true if the original statement is true.

For example, consider the statement "If a figure is a square, then it is a polygon." Its converse is "If a figure is a polygon, then it is a square." This is not true. On the other hand, consider the statement "If a figure is a three-sided polygon, then it is a triangle." Its converse is true: "If a figure is a triangle, then it is a three-sided polygon."

b. Sample response: Not always. The truth tables show that a statement and its inverse are not logically equivalent. The inverse can be—but is not always—true if the original statement is true.

For example, consider the statement "If a figure is a square, then it is a polygon." Its inverse is "If a figure is not a square, then it is not a polygon." This is not true. On the other hand, consider the statement "If a figure is a three-sided polygon, then it is a triangle." Its inverse is true: "If a figure is not a three-sided polygon, then it is not a triangle."

c. Sample response: Yes. The truth tables show that a statement and its contrapositive are logically equivalent. The contrapositive of a statement is always true if the original statement is true. For example, consider the statement "If a figure is a square, then it is a polygon." Its contrapositive also is true: "If a figure is not a polygon, then it is not a square." Similarly, consider the statement "If a figure is a three-sided polygon, then it is a triangle." Its contrapositive is likewise true: "If a figure is not a triangle, then it is not a three-sided polygon."

Note: The truth tables also show that the inverse and the converse of the same conditional are logically equivalent.

d. Sample response: Since it is the contrapositive of the original conditional, the third worker's statement is best—"If a dog's not a Labrador, then it's not smart." These two statements represent logically equivalent forms.

Assignment (page 224)

4.1 **a.** The converse is "If a polygon is a quadrilateral, then it is a square." This is not true because there are quadrilaterals that are not squares, such as some rectangles.

  b. The inverse is "If a polygon is not a square, then it is not a quadrilateral." This is not true for the same reason give in Part **a**.

  c. The contrapositive is "If a polygon is not a quadrilateral, then it is not a square." This is true because all squares are quadrilaterals.

*4.2 **a.** Sample response: "If the sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent."

  b. Sample response: "If one triangle is congruent to another triangle, then the corresponding angles are congruent."

4.3 **a.** The contrapositive of the first sentence is "If its materials were not purchased locally, then the house was not built by Mykelti Construction." In this case, students can conclude that the house was not built by Mykelti Construction.

b. No valid conclusion can be reached, other than that the house was not built by Mykelti Construction. You may wish to remind students that it would not be valid to conclude that the materials were not purchased locally.

4.4   **a.** Sample response: You can conclude that the sidewalk is wet. Since $p \to q$ is true and p is true, then q is true. This is verified by the truth table for $p \to q$.

b. Sample response: You can't conclude that it is raining. When $p \to q$ is true and q is true, p is not always true. The converse of a statement is not necessarily true if the original statement is true.

c. Sample response: You can conclude that it isn't raining. Since $p \to q$ is true and q is false, then p is false. This is verified by the truth table for $p \to q$. The contrapositive and the conditional are logically equivalent.

d. Sample response: You can't conclude that the sidewalk is not wet. When $p \to q$ is true and p is false, q is not always false. This is verified by the truth table for $p \to q$.

4.5   **a.** Sample response: The converse is "If a quadrilateral is a square, then its diagonals are congruent." This is true because every square is a rectangle and the diagonals of all rectangles are congruent.

b. Sample response: The inverse is "If the diagonals of a quadrilateral are not congruent, then it is not a square." This is true because every square is a rectangle and the diagonals of all rectangles are congruent.

c. Sample response: The contrapositive is "If a quadrilateral is not a square, then its diagonals are not congruent." This is false because there exists a rectangle that is not a square and the diagonals of all rectangles are congruent. Thus, there is a quadrilateral that is not a square but has congruent diagonals.

* * * * *

4.6   **a.** Sample response: You can conclude that the sum of the measures of the exterior angles is 360°. Since $p \to q$ is true and p is true, then q is true. This is verified by the truth table for $p \to q$.

b. Sample response: You can't conclude that the polygon is a triangle. When $p \to q$ is true and q is true, p is not always true. This is verified by the truth table for $p \to q$.

c. Sample response: You can't conclude that the sum of the measures of the exterior angles is not 360°. When $p \to q$ is true and p is false, q is not always false. This is verified by the truth table for $p \to q$.

d. Sample response: You can conclude that the polygon is not a triangle. Since $p \to q$ is true and q is false, then p is always false. This is verified by the truth table for $p \to q$. The contrapositive is always true if the original statement is true.

4.7 **a.** This statement is a tautology.

p	q	$p \to q$	$(p \to q)$ and p	$\left[(p \to q)\text{ and }p\right] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

b. This statement is a tautology.

p	$\sim p$	p or $\sim p$
T	F	T
F	T	T

c. This statement is not a tautology.

p	q	$\sim p$	$\sim q$	p and $\sim q$	$\sim p$ or q	$(p$ and $\sim q)$ and $(\sim p$ or $q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

d. This statement is not a tautology.

p	q	$\sim p$	p and q	$(p$ and $q)$ or $(\sim p)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

4.8 **a.** Sample response: You can't conclude that it is sunny. When $p \to q$ is true and q is true, p is not always true.

b. Sample response: You can conclude that it is not sunny. Since $p \to q$ is true and q is false, then p is always false.

c. Sample response: You can conclude that I'll go for a bike ride. Since $p \to q$ is true and p is true, then q is always true.

d. Sample response: You can't conclude anything about going for a bike ride. When $p \to q$ is true and p is false, q is not always false.

* * * * * * * * * *

Answers to Summary Assessment (page 226)

1. By drawing the correct conclusions from each of the false statements, students can gain the additional information needed to solve the crime.

   a. Since the original statement is false, its negation must be true. Using De Morgan's laws, the negation of "The niece and not the butler committed the crime" is the statement "Not the niece or the butler committed the crime." In other words, "The niece did not commit the crime or the butler did commit the crime."

   b. Since the original statement is false, its negation must be true. The negation of "No silver was taken" is the statement "Some silver was taken."

   c. This statement is true.

   d. Since the original statement is false, its negation must be true. The negation of "Some of the fingerprints at the crime are those of the woman's sister" is the statement "None of the fingerprints at the crime are those of the woman's sister."

   e. This statement is true.

   f. Since the original statement is false, its negation must be true. Using De Morgan's laws, the negation of "The thief took silver and no jewelry" is the statement "The thief took no silver or took some jewelry."

   g. This statement is true.

   h. The only situation in which the conditional statement "If the crime happened on Monday, then the woman's sister committed the crime" can be false is if the crime happened on Monday and the sister did not commit the crime. Therefore, the crime must have happened on Monday and the sister did not commit the crime.

   i. This statement is true.

   Because statement **h** is false, students can conclude that the crime happened on Monday and the sister didn't do it. This, along with statement **c**, implies that the criminal was the banker or the lawyer. If the criminal is not the lawyer, then the husband is guilty (by the contrapositive of statement **i**. This violates statement **c**. Therefore, the lawyer committed the crime on Monday.

2. Sample response: The only cards that need to be checked are those that could be counterexamples to Hood Wink's claim. The only cards that can make Hood Wink's statement false are cards which have an even number on one side and a consonant on the other side. Thus, the only cards that need to be checked are the 8, D, P, and 10.

1.   Consider the following true statements:

  • All squares are rectangles.

  • No circles are squares.

  a. Rewrite each statement as a conditional in the form "if p, then q."

  b. Write the negation of each statement using one of the quantifiers *all*, *some*, *none*, or *at least one*.

  c. Rewrite each statement in a logically equivalent form using one of the quantifiers *all*, *some*, *none*, or *at least one*.

2.   Consider the conditional statement, "If a polygon is a rectangle, then its opposite sides are parallel."

  a. Construct a Venn diagram that models this statement.

  b. Using the quantifiers *all*, *some*, and *no*, write statements that describe the sets of polygons in the Venn diagram. Do not use the words *if* or *then* in your three statements.

3.   **a.** Write an example of a conditional statement in mathematics for which the converse is true.

  b. Write an example of a conditional statement in mathematics for which the converse is false.

4.   Assuming that each of the following statements is true, describe any conclusions you can draw.

  • All babies are illogical.

  • Nobody is despised who can manage a crocodile.

  • All illogical persons are despised.

5.   Use truth tables to determine which of the following statements are tautologies. (A tautology is a statement that is always true.)

  a. $\left[(q \rightarrow p) \text{ and } p\right] \rightarrow q$

  b. $(p \text{ and } q) \text{ or } (\sim p \text{ or } q)$

  c. $\sim (p \text{ and } q) \text{ or } (\sim p \text{ or } q)$

Answers to Module Assessment

1. **a.** Answers may vary. Sample responses:

    - If the figure is a square, then it is a rectangle.

    - If the figure is a circle, then it is not a square.

    b. Sample responses:

    - Some squares are not rectangles.

    - Some circles are squares.

    c. Sample responses:

    - No squares are not rectangles.

    - All circles are not squares.

2. **a.** Sample Venn diagram:
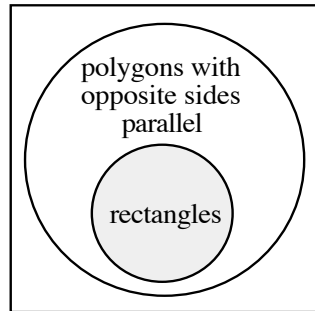
    b. Sample response: All rectangles are polygons with opposite sides parallel. Some polygons in which opposite sides are parallel are rectangles. No polygon in which opposite sides are not parallel is a rectangle.

3. **a.** Answers will vary. Sample response: If the sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

    b. Answers will vary. Sample response: "If one triangle is congruent to another triangle, then the corresponding angles are congruent."

4. Sample response: No baby can manage a crocodile.

5.   **a.** This statement is a tautology.

p	q	$q \to p$	$(q \to p)$ and p	$[(p \to q)$ and $p] \to q$
T	T	T	T	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	T

b. This statement is not a tautology.

p	q	$\sim p$	p and q	$\sim p$ or q	(p and q) or ($\sim p$ or q)
T	T	F	T	T	T
T	F	F	F	F	F
F	T	T	F	T	T
F	F	T	F	T	T

c. This statement is a tautology.

p	q	$\sim p$	p and q	$\sim(p$ and $q)$	$\sim p$ or q	$\sim(p$ and $q)$ or ($\sim p$ or q)
T	T	F	T	F	T	T
T	F	F	F	T	F	T
F	T	T	F	T	T	T
F	F	T	F	T	F	T

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Consortium for Mathematics and Its Applications (COMAP). *Principles and Practice of Mathematics, Volume 2*. Arlington, MA: COMAP, 1993.

O'Regan, P. J. "Intuition and Logic." *Mathematics Teacher* 81(November 1988): 664–668.

Flashbacks

Activity 1

1.1 Determine if each of the following statements is true or false.

    a. All integers are real numbers.

    b. Some real numbers are integers.

    c. No irrational numbers are rational numbers.

1.2 Give one example that disproves each of the following statements.

    a. All real numbers are integers.

    b. No whole numbers are rational numbers.

Activity 2

2.1 For each of the following statements, create a Venn diagram similar to the one shown below and shade the appropriate region.
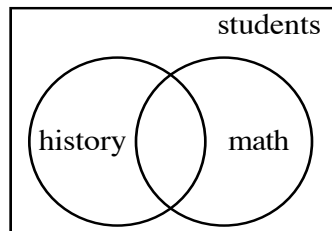
    a. All students are in a math class and a history class.

    b. All students are in a math class or a history class, or both.

    c. All students are in a math class or a history class, but not both.

2.2 Complete the following truth table for statements p and q.

p	q	p and q	p or q
True	True		
True	False		
False	True		
False	False		

Activity 3

3.1 Identify the hypothesis and conclusion in the conditional statement, "If you finish your work, then you can play basketball."

3.2 Negate each of the following statements.

    a. All roads are paved or some roads are gravel.

    b. Some airplanes have jet engines and no airplanes have wings.

Activity 4

4.1 Rewrite each of the following conditional statements using the word *or*.

    a. If all trucks have four-wheel drive, then you are a farmer.

    b. If some people are doctors, then all people are patients.

4.2 Given that the conditional statement "if p, then q" is false, determine the truth value of each of the following statements.

    a. p

    b. q

Answers to Flashbacks

Activity 1

1.1   **a.**   true

   b.   true

   c.   true

1.2   **a.**   Sample response: The number –1.8 is a real number that is not an integer.

   b.   Sample response: The number 8 is whole number that is a rational number.

Activity 2

2.1   **a.**   **b.**

   c.
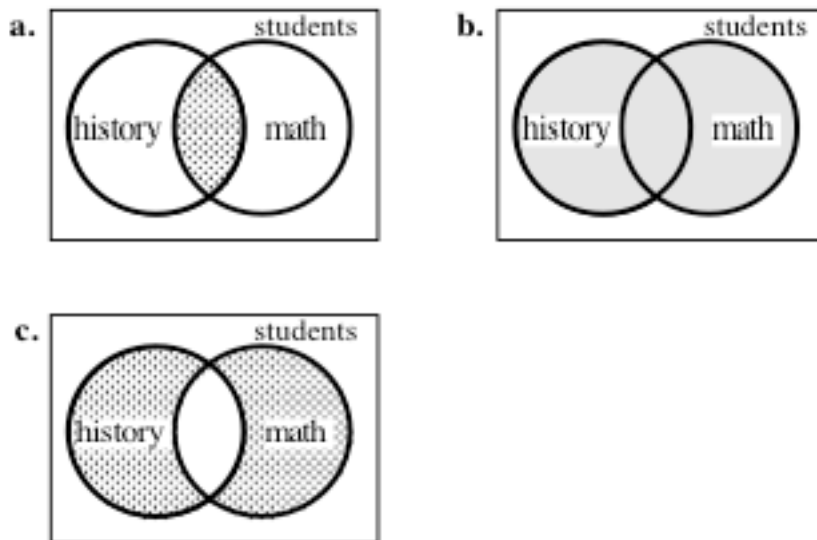
2.2   A completed table is shown below.

p	*q*	*p* and *q*	*p* or *q*
True	True	True	True
True	False	False	True
False	True	False	True
False	False	False	False

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Activity 3

3.1 The hypothesis is "you finish your work"; the conclusion is "you can play basketball."

3.2 **a.** Some roads are not paved and all roads are not gravel.

b. All airplanes do not have jet engines or some airplanes have wings.

Activity 4

4.1 **a.** Some trucks do not have four-wheel drive or you are a farmer.

b. Not all people are doctors or all people are patients.

4.2 **a.** true

b. false

Pixel This

Teacher Guide

The ability to communicate effectively is an important skill in all walks of life. In this module, you discover how understanding algorithms can enhance your communication skills.

Wendy Driscoll • Margaret Plouvier • Ed Sisolak

Pixel This

Overview

In this module, students investigate algorithms. While drafting and revising algorithms, they consider the desirability of making algorithms usable for more than one application.

Objectives

In this module, students will:

- study the characteristics of algorithms
- write algorithms for general and specific applications.

Prerequisites

For this module, students should know:

- how to distinguish a function from a relation
- how to solve linear and quadratic equations
- how to solve one-variable and two-variable equations for a given variable.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	2	2	1	7

Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
straightedge	X	X		
grid paper	X	X	X	

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
spreadsheet			X	
symbolic manipulator		X	X	
geometry utility	X			X
matrix manipulator			X	
graphing utility		X	X	

Pixel This

Introduction

The introduction familiarizes students with the term *pixel*.

Activity 1

In this activity, students investigate the characteristics of an effective algorithm.

Materials List

- grid paper
- straightedge (for Problem **1.5**)

Technology

- geometry utility (for Problem **1.5**)

Teacher Note

Pixels are usually located by numbering rows and columns from the upper left-hand corner of a screen (see Problem **3.1**). In order to simply the algorithms in this module, students are asked to number rows and columns from the lower left-hand corner. This provides for a numbering system analogous to a two-dimensional coordinate system.

Exploration

a. **1.** Students create a simple picture by shading squares on a sheet of grid paper. Sample response:
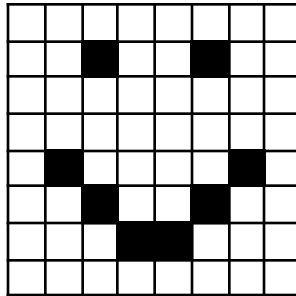
2. Since students cannot use diagrams, they must concentrate on written instructions. The following instructions correspond to the sample picture given in Step **1**.

   1. Label each row of the grid paper as follows: the top row is row 1, the next row down is row 2, the next is row 3, and so on.

   2. Label each column of the grid paper as follows: The left-hand column is column 1, the next column to the right is column 2, the next is column 3, and so on.

   3. Shade the squares that are at the intersections of row 2 and column 3, row 2 and column 6, row 5 and column 2, row 5 and column 7, row 6 and column 3, row 6 and column 6, row 7 and column 4, and row 7 and column 5.

   4. Stop.

3. Students exchange instructions with classmates and try to reproduce the original pictures.

b. **1.** Students place *x*- and *y*-axes on the grid paper they used in Part **a**, then number the squares. The bottom row is row 0, the next row up is row 1, the next is row 2, and so on. The left-hand column is column 0, the next column to the right is column 1, the next is column 2, and so on.

2. Students write another algorithm to recreate their pictures by representing each shaded square as an ordered pair. The following algorithm corresponds to the sample picture given in Part **a**.

   1. Shade the squares that are represented by the coordinates (1,3), (2,2), (3,1), (4,1), (5,2), (6,3), (2,6), and (5,6).

   2. Stop.

3. Students exchange algorithms with a different classmate from the one in Part **a** and try to reproduce the original pictures.

Discussion (page 233)

a. **1.** If students wrote good algorithms and the algorithms were followed accurately, then the pictures should be very similar.

2. If students numbered the grids correctly and identified the shaded squares using the correct coordinates, then the pictures should be very similar.

b. **1.** Sample response: As demonstrated in the exploration, it is possible to have more than one algorithm to create the same picture. There may be more than one way to describe a particular task, or there may be more than one order in which steps can be followed to produce the final product.

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2. Answers may vary. Some students may think that the algorithm in Part **b** is better only if the reader of the algorithm is familiar with two-dimensional coordinate systems.

c. **1.** Sample response: Since the distance formula is not a process, it does not appear to fit the definition of an algorithm. However, the formula could be used as a blueprint for writing the algorithm. If a person understands the mathematical symbols and the implied process, they could create an algorithm to find the distance between two points on the coordinate plane.

2. Answers may vary. Sample response: Any formula could serve as a blueprint for an algorithm, but since no formula is actually a process, it does not really represent an algorithm.

d. **1.** Answers may vary. The algorithm written in Part **b** assumed that the potential audience was familiar with two-dimensional coordinate systems.

2. Answers may vary. Students may recommend writing the algorithms without using mathematical vocabulary or concepts.

3. To make sure that the reader can understand and use the algorithm, students should consider their familiarity with the context and their understanding of any specialized terms or procedures, as well as any potential conflicts caused by physical disabilities or differences in native language. In most cases, the more familiar the reader is with the subject of the algorithm, the shorter the algorithm can be.

e. Answers may vary. In general, a good algorithm gives a step-by-step process for completing a task. It is efficient, containing only the steps necessary to complete the task. It is precise, producing a result that successfully completes the task or determines that the task cannot be done. And it is appropriate for the intended audience.

Assignment (page 233)

1.1 **a–e.** Answers will vary. Students design and construct pictures, then write algorithms to recreate them. They may find that their algorithms are not appropriate for someone who is not in their math class, and that the vocabulary must be changed to accommodate different skills.

*1.2 **a.** Students should create tables similar to the one shown below. The result (228) is the sum of the numbers in the right-hand column that are paired with odd numbers in the left-hand column (the shaded cells).

19	12
9	24
4	48
2	96
1	192
Sum	228

Note: This algorithm indirectly represents the given x-value in base 2. For example, 19 is represented as 10011 in base 2, which expands to $1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$. Multiplying this expansion by 12 produces:

$$12\left(1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0\right) =$$

$$12 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 12 \cdot 2^1 + 12 \cdot 2^0 =$$

$$192 + 24 + 12 = 228$$

Each of the numbers in the sum corresponds to a shaded cell in the right-hand column of the table.

b. Answers will vary. Students should include tables in their responses.

c. Although the algorithm produces the correct result, students may argue that it is not an efficient way to multiply two numbers.

*1.3 Answers will vary. A possible algorithm is given below.

1. To determine if the number n (greater than 1) is a composite or prime number, begin testing possible factors. Let the first test factor f be 2.

2. Divide the number n by the test factor f. If there is no remainder, go to Step 3. If there is a remainder, go to Step 4.

3. If the number n is the same as the test factor f, then go to Step 5. If the number n is not the same as the test factor f, go to Step 6.

4. Increase the test factor f by 1. Go to Step 2.

5. The number is prime. Go to Step 7.

6. The number is composite.

7. Stop.

* * * * *

*1.4   Answers may vary. Some students may write algorithms for use on a spreadsheet. A sample algorithm is given below.

1. Determine the monthly profit.

2. Divide the profit into fourths. If necessary, round so that the value of each portion does not differ by more than 1 cent from any of the others.

3. Give two of the portions to yourself.

4. Give one of the remaining portions to one of your two partners. Give the last portion to the other partner.

5. Stop.

1.5   **a.** The sample algorithm below uses a compass, a ruler, and the mathematical terms *horizontal*, *point*, and *segment* to construct the following triangle.
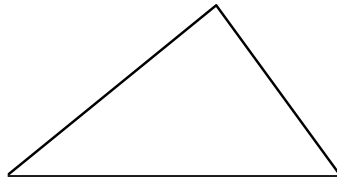
1. Draw a horizontal line segment 6.7 cm long.

2. Use a compass to draw a circle with its center at the left-hand endpoint of the segment and a radius of 5.4 cm.

3. Use a compass to draw a circle with its center at the right-hand endpoint of the segment and a radius of 4.2 cm.

4. Use the point of intersection of the two circles that is above the line segment as the third vertex of the triangle. Connect this point with the two endpoints of the line segment to form the triangle.

5. Stop.

b. The following sample algorithm uses a geometry utility to create the triangle in Part **a**.

1. Draw a horizontal line segment 6.7 cm long.

2. Construct a circle with its center at the left-hand endpoint of the segment and a radius of 5.4 cm.

3. Construct a circle with its center at the right-hand endpoint of the segment and a radius of 4.2 cm.

4. Use the point of intersection of the two circles that is above the line segment as the third vertex of the triangle. Connect this point with the two endpoints of the line segment to form the triangle.

5. Stop.

    c. Answers will vary. The steps in the two sample algorithms are very similar. The vocabulary may differ depending on the tools used.

    d. Answers will vary. In their explanations, students should consider the relative efficiency of the algorithms.

1.6 Students may devise algorithms that use dice, playing cards, or some type of spinner. They may also incorporate technology in their algorithms. On some calculators, for example, typing "int(rand()*10+1)" produces a random natural number from 1 to 10, inclusive.

1.7 Sample response: To round a positive real number in decimal form to the nearest tenth, follow the steps below.

1. If the number is a rational number exact at the units or tenths place, go to step 7.
2. Examine the digit in the hundredths place. If it is five or greater, go to step 4.
3. If the hundredths digit is less than 5, go to step 6.
4. If the tenths digit is not 9, add 1 to the tenths digit and drop any digits to the right of the tenths place. Go to step 7.
5. If the tenths digit is 9, add 1 to the units place, place a 0 in the tenths place, and drop any digits to the right of the tenths place. Go to step 7.
6. Drop any digits to the right of the tenths digit.
7. Stop.

* * * * * * * * * *

(page 235)

Activity 2

In this activity, students explore an algorithm for identifying the pixels necessary to represent a line segment. They also discuss whether an algorithm is specific (usable only for one application) or general (usable for more than one application).

Materials List
- grid paper
- straightedge

Technology
- graphing utility
- symbolic manipulator (for Problem **2.6**)

Exploration

a. Answers will vary. The following sample response gives instructions that would not be appropriate for technology. **Note:** You may wish to point out that technology cannot use an algorithm that draws a segment first, then activates the pixels that touch the segment.

1. Draw the line segment connecting the centers of the two pixels that correspond to the endpoints.

2. Shade all the pixels that touch this line.

3. Stop.

In the following graph, this sample algorithm was used to draw the segment connecting pixels (2,8) and (7,1).
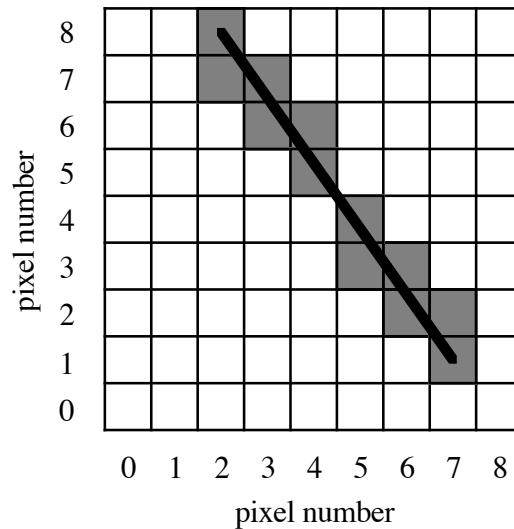
b. Students may or may not decide to revise their algorithms. The following sample algorithm uses a more appropriate approach than the one given in Part **a**.

1. Use the ordered pairs that identify the two pixels to determine the slope m of the line containing them. In this case, the slope is:

$$m = \frac{8-1}{2-7} = -\frac{7}{5}$$

2. Determine the equation of the line passing through the two points in the form $y = m(x - x_1) + y_1$. In this case, the equation is:

$$y = -\frac{7}{5}(x - 2) + 8$$

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3. For all integer values between x_1 and x_2, inclusive, determine y-values using the equation found in Step 2. Create a table of these values. The table for the points in Part **a** is shown below.

x-values	$y = -1.4(x-2) + 8$
2	8
3	6.6
4	5.2
5	3.8
6	2.4
7	1

4. Record ordered pairs of x- and y-values using the following rules:

   - If the corresponding y-value in the table is an integer, then record the ordered pair (x,y).

   - If the corresponding y-value is not an integer, then record the ordered pairs $(x,[y])$ and $(x,[y]+1)$, where $[y]$ is the greatest integer function.

5. Shade the pixels represented by the ordered pairs recorded in Step 4.

6. Stop

   In the following graph, this sample algorithm was used to draw the segment connecting pixels (2,8) and (7,1).
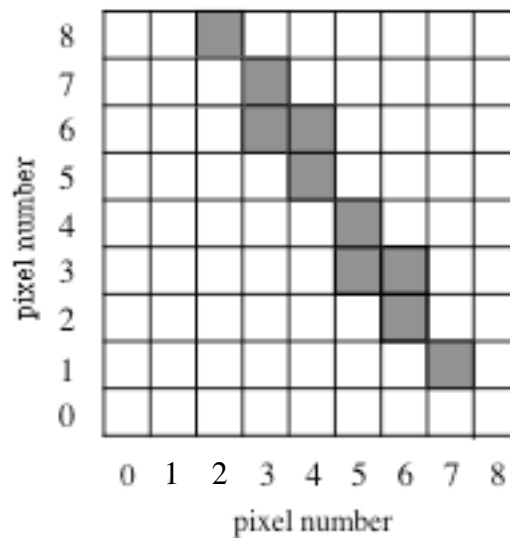
c–e. Students test each others' algorithms using pixels (2,8) and (7,1). They then revise their own algorithms using peer feedback.

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Discussion

a. Sample response: It was hard to write an appropriate algorithm without knowing what mathematics a graphics program would recognize or what instructions it would need to activate pixels.

b. 1. Students compare algorithms.

    2. As a class, students create one general algorithm.

c. 1. Sample response: A general algorithm can be reused. For example, a general algorithm for representing a line segment can be used to represent any line segment. A specific algorithm could only be used to represent a particular line segment.

    2. Sample response: If it is only necessary to draw one line segment, it may be easier to write an algorithm for that specific line segment. Also, a specific algorithm may be shorter and faster than a general algorithm.

Assignment

2.1 Answers may vary. One possible algorithm is given below. In this case, the length of the segment is at least d.

1. Let $x_2 = x_1 + d$.

2. Let $y_2 = m(x_2 - x_1) + y_1$.

3. Using the algorithm from Part **b** of the exploration, draw the line segment connecting pixels (x_1, y_1) and (x_2, y_2).

4. Stop.

*2.2 Answers may vary. One possible algorithm, using repeated division and averaging, is given below.

1. Let $r = 1$ be the first approximation of the square root of some non-negative number n.

2. Let q equal the quotient when n is divided by r.

3. Calculate the mean a of the quotient q and the approximate square root r.

4. If $|a - r| \leq 0.01$, go to Step 7.

5. Let a equal the new approximate square root.

6. Go to Step 2.

7. Round a to the nearest hundredth.

8. Stop. The number a is the square root of n to the nearest hundredth.

2.3 Answers may vary. Some students may describe how to use a calculator to complete this task, as follows:

1. Using a calculator, push the "$\sqrt{}$" button.

2. Type in "(" followed by the value of a.

3. Type in "^2+".

4. Type in the value of b followed by "^2)".

5. Stop. The result from Step 4 is the approximate value of c, which is the measure of the hypotenuse of the given right triangle.

Others may describe how to complete the task without using technology, as shown below.

1. Multiply a by a.

2. Multiply b by b.

3. Add the result from Step 1 to the result from Step 2.

4. Use the algorithm from Problem **2.2** to approximate the "square root" of the result from Step 3.

5. Stop. The result from Step 4 is the approximate value of c, which is the measure of the hypotenuse of the given right triangle.

* * * * *

2.4 The following sample response was written for the TI-92 calculator.

1. Press "ON."

2. Press "MODE."

3. "FUNCTION" should be highlighted. If "FUNCTION" is not highlighted, use the right arrow key to highlight "FUNCTION."

4. Press "ENTER."

5. Press "APPS" and select "Y=Editor."

6. Press "F1."

7. Press "8."

8. Press "ENTER."

9. Move cursor to "y1 = ."

10. Enter "$4\pi x^\wedge 3 \div 3$".

11. Press "F2" to set the window.

12. Select "6" to graph in the standard window.

13. Stop.

*2.5 **a.** The equation is true when d is a factor of n.

  b. The leap years between 1895 and 1925 were 1896, 1904, 1908, 1912, 1916, 1920, and 1924. The year 1900 was not a leap year because 1900 is divisible by 100 but not 400.

  c. Answers may vary. A sample algorithm is given below.

   1. If $y/4 \neq [y/4]$, then y is not a leap year. Go to Step 5.

   2. If $y/100 \neq [y/100]$, then y is a leap year. Go to Step 5.

   3. If $y/400 = [y/400]$, then y is a leap year. Go to Step 5.

   4. The year y is not a leap year.

   5. Stop.

*2.6 **a.** Sample response:

   1. Add the additive inverse of ax, which is $-ax$, to both sides of the equation.

   2. Multiply both sides of the equation in Step **1** by the multiplicative inverse of b, which is $1/b$.

   3. Stop. The result is the value of y.

  b. The following sample algorithm uses the symbolic manipulator on a TI-92 calculator:

   1. Type in "SOLVE $(A * X + B * Y = C, Y)$".

   2. Press "ENTER."

   3. Stop.

2.7 **a.** Sample response: When the calculator is set for "Fix 4," it appears to round the number to the nearest 10,000th. When the calculator is set for "Float 4," it displays four significant digits with the decimal point in the correct location. In both cases, the calculator automatically switches to scientific notation when the number gets very large or very small.

  b. Sample response: While it appears that both algorithms should be fairly simple, the algorithm for the "Fix 4" setting would be easier to write for all numbers greater than 0.0001. In this case, all that would be required is to round the number to the nearest 10,000th. If the number is less than 0.0001, the calculator would first need to express the number in scientific notation, then keep four significant digits after the decimal point. Identifying these significant digits would require another algorithm. In the "Float 4" setting, the calculator must determine four significant digits and the correct location of the decimal point in every case.

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Activity 3

In this activity, students explore a general algorithm for activating the pixels on a screen to represent graphs of functions and relations.

Materials List

- grid paper

Teacher Note

You may wish to use grid paper with smaller squares than ordinary graph paper. If so, a blackline master appears at the end of the teacher edition FOR THIS MODULE.

Technology

- spreadsheet
- matrix manipulator
- symbolic manipulator
- graphing utility (optional)

Exploration

Students apply the scan-conversion algorithm to determine how well it approximates the graphs of five different functions. **Note:** To save time, you may wish to divide the functions listed in Part **c** among several groups.

   a. Sample graph:

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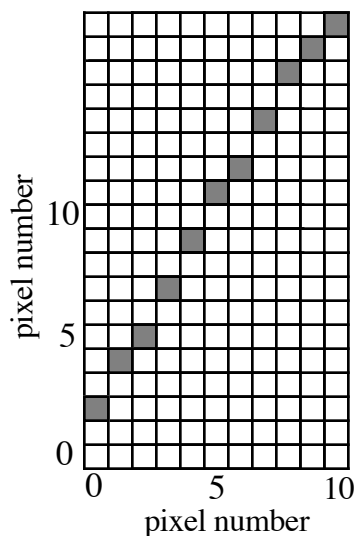
b. Sample response: The scan-conversion algorithm represents the line fairly well, although there are gaps in the graph.

c. Students should consider the domain of each function first. In some cases, all integer x-values in the interval $[0, 10]$ will not have a corresponding real y-value.

  1. As shown below, the pixel graph generated by the scan-conversion algorithm leaves many gaps in the graph of $f(x) = (x-5)^2$.
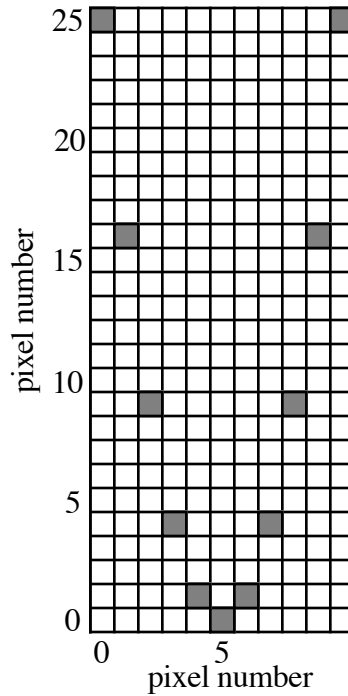
  2. The pixel graph generated by the scan-conversion algorithm is a good approximation of the graph of

$$f(x) = 2 + \frac{2}{x-5}$$

  for $x \le 4$ and $x > 5$. However, it does not show the vertical asymptote as $f(x)$ approaches 5.
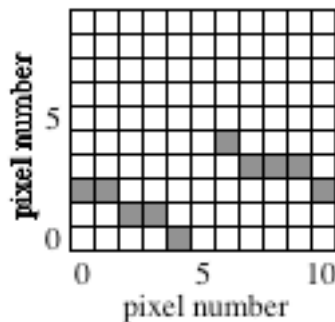
3. The pixel graph generated by the scan-conversion algorithm leaves many gaps in the graph of $f(x) = (x-5)^3 + 10$.
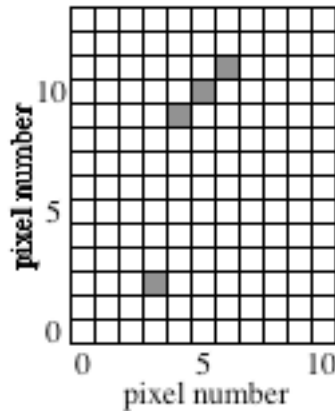
4. The pixel graph generated by the scan-conversion algorithm is a somewhat accurate representation of $f(x) = 3(2)^{(x-5)}$ for $x < 5$, but leaves many gaps for $x > 5$.
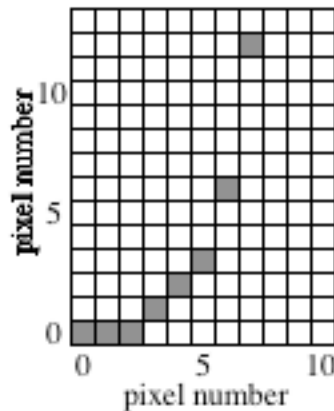
Discussion

a. Answers will vary. The scan-conversion algorithm requires fewer calculations than the sample algorithm given in Activity **2**.

b. Since it rounds to the nearest integer, the scan-conversion algorithm more closely approximates the graph when a small increase in the value of x does not produce a large change in the value of $f(x)$. Hence, the graph of a line may be more accurate than the graph of a higher-order polynomial function.

c. The scan-conversion algorithm is a general algorithm since it can be used to represent the graphs of many different functions, not just linear functions.

Assignment   (page 239)

3.1   **a.** **1.** The point (10,–10) corresponds to the pixel in column 94, row 62.

         2. The point (10,10) corresponds to the pixel in column 94, row 0.

         3. The point (0,0) corresponds to the pixel in column 48, row 32.

         4. The point (3,5) corresponds to the pixel in column 62, row 16.

         5. The point (2,–2) corresponds to the pixel in column 57, row 38.

   b. Sample response:

Steps	Example Find the pixel for the point (2,5) on a calculator with 63 rows and 95 columns and viewing window set for a domain of $\left[-10, 10\right]$ and a range of $\left[-10, 10\right]$
1. Determine the width of the viewing window by finding the difference between the maximum and minimum x-values.	1. In this case, the width of the viewing window is $(10 - (-10)) = 20$.
2. Determine the height of the viewing window by finding the difference between the maximum and minimum y-values.	2. In this case, the height of the window is $(10 - (-10)) = 20$.
3. Determine the horizontal scale factor by dividing the number of pixel columns by the width of the viewing window	3. The horizontal scale factor in this case is: $\dfrac{95}{20} = 4.75$
4. Determine the vertical scale factor by dividing the number of pixel rows by the height of the window.	4. The vertical scale factor is: $\dfrac{63}{20} = 3.15$
5. To find the column number of the pixel representing the x-coordinate, add half the width of the window to x and multiply the sum by the horizontal scale factor found in Step 3.	5. In this case: $(2 + 0.5 \bullet 20) \bullet 4.75 = 57$

                                   325

6. To find the row number of the pixel representing the *y*-coordinate, add half the height of the window to *y* and multiply the sum by the vertical scale factor found in Step 4.	6. In this case: $(5 + 0.5 \cdot 20) \cdot 3.15 = 47.25$
7. If the numbers generated in Steps 5 and 6 are not integers, round each to the nearest integer.	7. The column number is an integer but the row number found in Step 6 needs to be rounded to 47.
8. The integers from Step 7 represent the column and row numbers, respectively, of the appropriate pixel.	8. The pixel representing the point (2,5) is located in column 57 and row 47.
9. Stop.	9. Stop

3.2   **a.** The following sample algorithm was written for a graphing calculator.

1. Turn the calculator on.

2. Enter the value for *p*.

3. Type "(1+".

4. Enter the value for *r*.

5. Type ")^".

6. Enter the value for *n*.

7. Press "ENTER."

8. Stop. The number on the screen is the value of *A*.

b. The audience must know the order of operations, how to use exponents, how to add, and how to multiply. They must also be able to identify variables and substitute values for the variables into the formula. In addition, the audience must be able to determine whether the formula is appropriate for the situation.

c. The following sample response corresponds with the algorithm given in Part **a**.

1. Turn the calculator on.

2. Enter the value "1000".

3. Type "(1+".

4. Enter "0.06".

5. Type ")^".

6. Enter "7".

7. Press "ENTER."

8. Stop. The number "1503.63" on the screen is the value of A.

This is the account balance, in dollars, after 7 years, given an initial principal of $1000 and an annual interest rate of 6%.

*3.3   **Note:** The numbers in Pascal's triangle can be used as the coefficients when expanding powers of binomials. For example, to expand $(x + y)^3$, the numbers in the third line of Pascal's triangle can be used as follows:

$$(x + y)^3 = 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

a. The numbers in row 5 of Pascal's triangle are $1, 5, 10, 10, 5, 1$.

b. Answers will vary. Two possible algorithms are shown below.

Sample algorithm 1:

1. The first and last number of each row is always 1.

2. To get the other numbers, add the entries to the upper left and upper right in the previous row as shown in the following diagram.
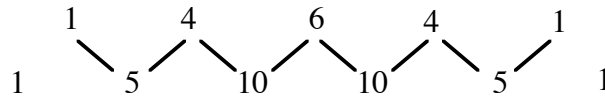
3. Stop.

Sample algorithm 2:

1. The first and last number of each row is always 1.

2. To get the other numbers, add the numbers in the diagonal either to the left or to the right, as shown in the diagram below.
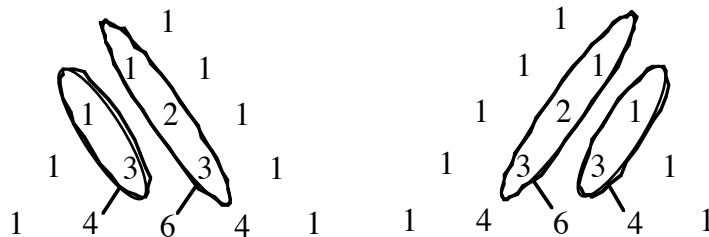
3. Stop.

*3.4 The equation of the quarter circle is $y = \sqrt{25 - x^2}$, where $0 \le x \le 5$. The y-values for this equation, and the corresponding values for the scan-conversion algorithm, are shown in the table below.

x	y-value	scan-conversion
0	5	5
1	4.899	5
2	4.583	5
3	4	4
4	3	3
5	0	0

The pattern of pixels represents the graph fairly well, as shown below.

* * * * *
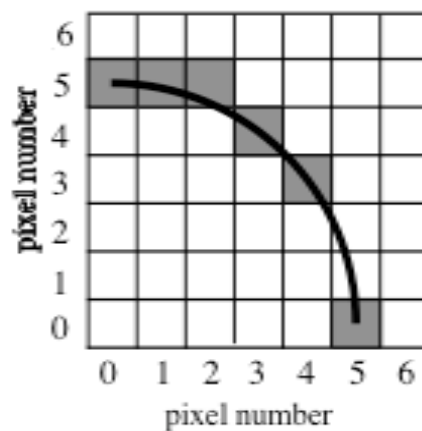
3.5 **a.** The following sample algorithm was written for the symbolic manipulator on a TI-92 calculator.

1. Turn the calculator on.

2. Type "SOLVE (".

3. Enter the first equation.

4. Type ",x)".

5. Press "ENTER."

6. Type "SOLVE (".

7. Enter the second equation.

8. Type ",y)|".

9. Press the "up" arrow once.

10. Press "ENTER."

11. Press "ENTER" again.

12. Record the value of y for use in Step 16.

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13. Type "SOLVE (".

14. Enter the first equation.

15. Type ",x)ly=".

16. Enter the value from Step 12.

17. Press "ENTER."

18. Stop. The *x*-value from Step 17 and the *y*-value from Step 12 are the solutions of the system.

The sample algorithm below was written for pencil-and-paper manipulation:

1. Substitute the first equation into the second equation.

2. Solve the equation from Step 1 for *x*.

3. Substitute the *x*-value found in Step 2 into one of the original equations and find *y*.

4. Stop. The values of *x* and *y* are the solutions of the system.

b. Sample algorithm:

1. Graph both equations on a coordinate grid.

2. Find the point where the lines intersect.

3. Determine the coordinates of the point found in Step 2.

4. Stop. The *x*- and *y*-values of the coordinates are the solutions of the system.

c. Using the sample calculator algorithm given in Part **a**, the result of Step 11 is the word "false," not a *y*-value, so the algorithm cannot be completed.

Using the paper-and-pencil algorithm, the result of Step 2 is the false statement "$7 = -4$," so the algorithm cannot be completed.

Using the graphical algorithm, the result of Step 1 is two parallel lines, so the algorithm cannot be completed.

d. The algorithms are not general enough to solve all possible systems of linear equations. They do not account for parallel or coincidental lines. Possible modifications to handle these systems are shown in the response to Problem **3.6**.

3.6   Each of the sample algorithms shown below is a modification of an algorithm given in Problem **3.5**.

Modify Step 12 of the calculator algorithm in Problem **3.5** to the following:

12. If the result of Step 11 is a y-value, then record the value of y and proceed to Step 13. If the result of Step 11 is the word "false," then the system is a system of parallel lines and has no solutions, so stop. If the result of Step 11 is the word "true," then the system is a system of coincident lines and has infinite solutions, so stop.

Modify the paper-and-pencil algorithm in Problem **3.5** by inserting the following step between Steps 2 and 3:

2a. If the result of Step 2 is a y-value, then proceed to Step 3. If the result of Step 2 is an inequality of two real numbers, then the system is a system of parallel lines and has no solutions, so stop. If the result of Step 2 is an equality of two real numbers, then the system is a system of coincident lines and has infinite solutions, so stop.

Modify the graphical algorithm in Problem **3.5** by inserting the following step between Steps 1 and 2:

1a. If the result of Step 1 is a set of two intersecting lines, then proceed to Step 2. If the result of Step 1 is a set of two parallel lines, then the system has no solutions, so stop. If the result of Step 1 is one line, then the system is a system of coincident lines and has infinite solutions, so stop.

3.7   **a.** The following sample algorithm for solving the equation $3x^2 + y^2 = 15$ for y was written for the TI-92 calculator:

1. Type "SOLVE(3x^2+y^2=15,y)".

2. Press "ENTER."

3. Stop.

A sample paper-and-pencil algorithm for solving the equation $3x^2 + y^2 = 15$ for y is outlined below:

1. Subtract the x-term, $3x^2$, from both sides of the equation.

2. Take the square root of both sides of the equation. Remember to include both the positive and negative square roots.

3. Stop.

b. The following sample algorithm was written for a TI-92 calculator:

1. Type "SOLVE(".

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2. Enter the equation.

3. Type ",y)".

4. Press "ENTER."

5. Stop.

A sample paper-and-pencil algorithm is outlined below:

1. Add the additive inverse of the x^2-term to both sides of the equation.

2. Divide each term of the equation by the coefficient of the y^2-term.

3. Take the square root of both sides of the equation. Remember to include both the positive and negative square roots.

4. Stop.

   c. To use the calculator algorithms, a person must know the correct buttons to push. To use the paper-and-pencil algorithms, a person must know how to perform the required algebraic operations.

<p align="center">* * * * * * * * * *</p>

Research Project   (page 242)

The general Bresenham method and a sample adaptation are given below.

1.   Assign the endpoints of the segment to (x_1, y_1) and (x_2, y_2).

2.   Create a table with headings like the one below to organize your data.

x	x_1	y_1	x_2	y_2	d_1	d_2	d	y
	x_1	y_1	x_2	y_2	$2\big((y_2 - y_1) - (x_2 - x_1)\big)$	$2(y_2 - y_1)$	$x_1 - x_2$	y_1
x_1								

3.   a. Calculate the values for d_1, d_2, d, and y using the formulas given.

         b. Replace the formulas in the row below the headings with their corresponding values. (Leave the cell under x blank.)

4.   a. Move to the next row.

         b. Assign the value of x_1 to x and write this value in the appropriate row in the x column.

5.   The values for y_1, x_2, y_2, d_1, and d_2 remain the same for this row. Write these values in the appropriate cells.

<p align="center">331</p>

6. The values for d and y may change for this row. Calculate these values according to the following rules:

    a. If d is less than 0, then add d to d_2 and assign this value to d. Assign y the same value as before. Write the values for d and y in the appropriate cells. Go to Step 7.

    b. If d is greater than or equal to 0, then add d to d_1 and assign this value to d. Add 1 to y and assign this value to y. Write the values for d and y in the appropriate cells.

7. If $x + 1 > x_2$, then go to Step 8. If not, complete the following.

    a. Move to the next row.

    b. Add 1 to x, then write this value in the appropriate cell in the x column.

    c. Go to Step 5.

8. Form an ordered pair (x,y) for the x-value and y-value in each row (except the first).

9. Using grid paper and the pixel numbering system shown below, shade the grid that corresponds to each ordered pair (x,y).

10. Stop.

For example, to create a pixel graph of the segment with endpoints (0,1) and (5,4), the Bresenham method can be used to create the table below:

x	x_1	y_1	x_2	y_2	d_1	d_2	d	y
	0	1	5	4	$2(3-5) = -4$	$2(3) = 6$	5	1
0		1	5	4	−4	6	1	2
1		1	5	4	−4	6	−3	3
2		1	5	4	−4	6	3	3
3		1	5	4	−4	6	−1	4
4		1	5	4	−4	6	5	4
5		1	5	4	−4	6	1	5

The appropriate ordered pairs—(0,2), (1,3), (2,3), (3,4), (4,4), and (5,5)—are plotted below.
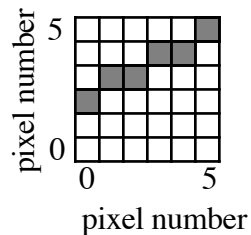
A pixel graph made with the scan-conversion algorithm is shown below.
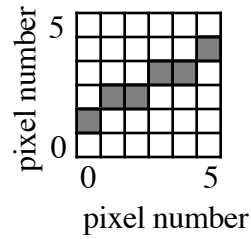
In this case, the Bresenham graph is the same as the scan-conversion graph, except it is translated up by one pixel. For this reason, the scan-conversion graph may appear to be a more accurate representation of the segment. However, Bresenham pixel graphs of segments never contain "holes," regardless of the slope of the line. As demonstrated in Activity **3**, scan-conversion pixel graphs of steep segments may have gaps. For this reason, the Bresenham method is preferred in some cases.

Answers to Summary Assessment

1.   **a.** Answers may vary. The following sample response may be used to multiply any two numbers $m \bullet n$.

    1. Make a two-column table.

    2. Write 1 in the left-hand column. Write the second factor n in the right-hand column.

    3. Double the number in the left-hand column. Write the doubled number in the next row only if it is smaller than the first factor, m. Continue this process until the doubled number is greater than or equal to m.

    4. Double the number in the right-hand column and write it in the next row. Continue this process until there is a number in the right-hand column for each number in the left-hand column.

    5. Mark the numbers in the left-hand column whose sum is the first factor n. Add the numbers in the right-hand column that are paired with the marked numbers in the left-hand column. The result is the product of m and n.

    6. Stop.

b. The product of 18 and 65 is 1170. A completed table is shown below.

1	65
2	130
4	260
8	520
16	1040

c. Sample response: The algorithm works because one of the original factors is written as a sum of powers of 2. These powers of 2 are then multiplied by the other factor and summed. For example, if you multiply 18 and 65, this algorithm expresses 18 as $0 \bullet 2^0 + 1 \bullet 2^1 + 0 \bullet 2^2 + 0 \bullet 2^3 + 1 \bullet 2^4$. This sum is then multiplied by 65, which yields the following:

$$\left(0 \bullet 2^0 + 1 \bullet 2^1 + 0 \bullet 2^2 + 0 \bullet 2^3 + 1 \bullet 2^4\right) \bullet 65 =$$

$$1 \bullet 2^1 \bullet 65 + 1 \bullet 2^4 \bullet 65 =$$

$$130 + 1040 = 1170$$

    Note: You may wish to point that every integer has a unique representation in base 2.

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2. Sample response:

1. Use the point tool to construct three noncollinear points.

2. Use the segment tool to connect consecutive points to form a triangle.

3. Construct the angle bisector of one interior angle of the triangle from Step 2.

4. Repeat Step 3 for another interior angle of the triangle.

5. Construct a point at the intersection of the two angle bisectors.

6. Select a side of the triangle and the intersection point from Step 5.

7. Construct a line perpendicular to the side of the triangle and the intersection point.

8. Construct a point at the intersection of the perpendicular line and the side of the triangle.

9. Use the circle tool to construct a circle with the point found in Step 5 at the center and the point found in Step 8 on the circle.

10. Stop. The circle constructed in Step 9 is the inscribed circle.

1. Write an algorithm that explains how to use a geometry utility to draw a circle with radius r.

2.   **a.** Write an algorithm for using paper and pencil to solve an equation of the form $ax + c = bx + d$ for x.

      b. Write an algorithm for using a symbolic manipulator to solve an equation of the form $ax + c = bx + d$ for x.

3. Write an algorithm for using a symbolic manipulator to solve an equation of the form $ax^2 + bx + c = 0$ for x.

4.   **a.** Write an algorithm to determine the area of a trapezoid.

      b. Revise your algorithm so that it can also be used to calculate the area of a triangle and a square.

5. Write an algorithm for using a geometry utility to draw a triangle and a circle that circumscribes it.

6. The three-step algorithm described below can be used to find the area of a circle.

    1. Measure the radius of the circle.

    2. Square the radius.

    3. Multiply the result from Step 2 by π.

    4. Stop. This is the area of the circle.

Explain whether or not you think this is a good algorithm.

Answers to Module Assessment

1.   Sample response:

      1. Draw a circle.

      2. Measure the distance from the center to a point on the circle.

      3. Enlarge or shrink this distance until you have the desired radius.

      4. Stop.

2.   **a.** Sample algorithm:

      1. Add the additive inverse of c to both sides and simplify expressions by combining like terms.

      2. Add the additive inverse of bx to both sides and simplify expressions by combining like terms.

      3. Multiply both sides of the equation by the multiplicative inverse of the coefficient of x.

      4. Stop.

      b. The following sample algorithm was written for the symbolic manipulator on a TI-92 calculator:

      1. Type "SOLVE $(A * X + C = B * X + D, X)$".

      2. Press "ENTER."

      3. Stop.

3.   The following sample algorithm was written for the symbolic manipulator on a TI-92 calculator:

      1. Type "SOLVE $(A * X^2 + B * X + C = 0, X)$".

      2. Press "ENTER."

      3. Stop.

4.   **a.** The following sample algorithm refers to the diagram below.
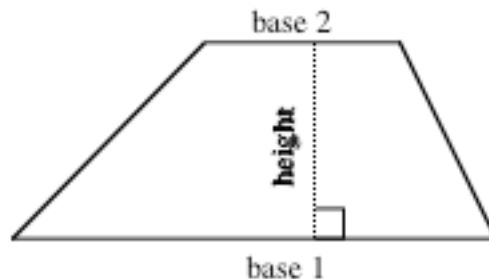
      1. Add the lengths of base 1 and base 2.

      2. Divide the result from Step 1 by 2.

      3. Multiply the result from Step 2 by the height.

      4. Stop.

b. Answers may vary. The sample algorithm given in Part **a** can be used without modification for squares and triangles. Since the measure of "base 2" in a triangle is 0, the formula simplifies to the familiar:

$$A = \frac{1}{2} b \bullet h$$

In a square, the measures of the two bases are equal. The formula therefore simplifies to:

$$A = \frac{1}{2}(b)(b + b) = b^2$$

5. Sample algorithm:

1. Use the point tool to construct three noncollinear points.

2. Use the segment tool to connect consecutive points into a triangle.

3. Construct the perpendicular bisector of one side of the triangle from Step 2.

4. Repeat Step 3 for another side of the triangle.

5. Construct a point at the intersection of the two perpendicular bisectors.

6. Use the circle tool to construct a circle with the point found in Step 5 at the center and one of the points constructed in Step 1 on the circle.

7. Stop. The circle constructed in Step 6 is the circumscribed circle.

6. Sample response: This algorithm is satisfactory if the center of the circle is known. If the center of the circle is not known, there could be some difficulty determining the measure of the radius.

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Flashbacks

Activity 1

1.1 Perform the instructions given below.

    a. Plot the ordered pairs A (0,0), B (2,7), C (4,3), and D (7,5) on a two-dimensional coordinate system.

    b. Connect the points represented by these ordered pairs.

    c. Stop.

1.2 Write a definition for each of the following:

    a. the set of natural numbers

    b. a composite number

    c. a prime number.

Activity 2

2.1 Write an equation in the form $ax + by = c$ for the line that contains each of the following pairs of points:

    a. (2,4) and (–3,–4)

    b. (2,4) and (8,4)

    c. (2,4) and (2,6)

2.2 Write an equation in the form $ax + by = c$ for each of the following:

    a. the line with a slope of –4 that contains the point (2,4)

    b. the line with a slope of $2/3$ that contains the point (–8,–4)

2.3 In each of the following expressions, $[x]$ represents the greatest integer function. Evaluate each expression.

    a. [3.7]

    b. [1.2]

    c. [–3.7]

    d. [–1.2]

2.4 Solve each of the following equations for y.

    a. $2x - 3y = 7$

    b. $(2/3)x + (3/4)y = -7$

Activity 3

3.1 Classify each of the following functions into one of the following categories: linear, quadratic, cubic, exponential, trigonometric or hyperbolic.

    a. $f(x) = \sin(x)$

    b. $f(x) = 4/x$

    c. $f(x) = 3x^3$

    d. $f(x) = 4(3^x)$

3.2 Which of the following categories best describes the graph of each function shown below: linear, quadratic, cubic, exponential, trigonometric or hyperbolic?

a.
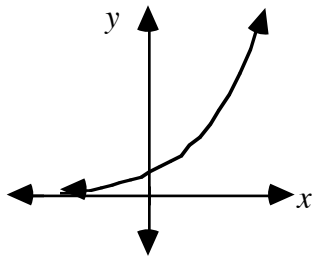
b.
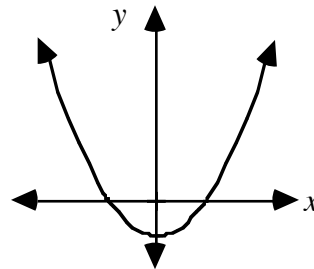
c.
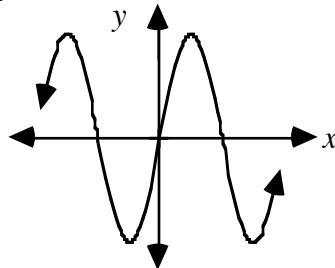
d.
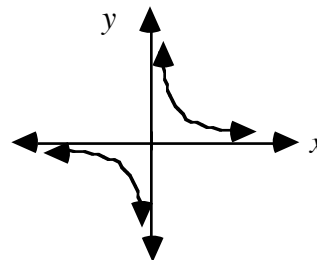
Answers to Flashbacks

Activity 1

1.1 **a.** Sample graph:
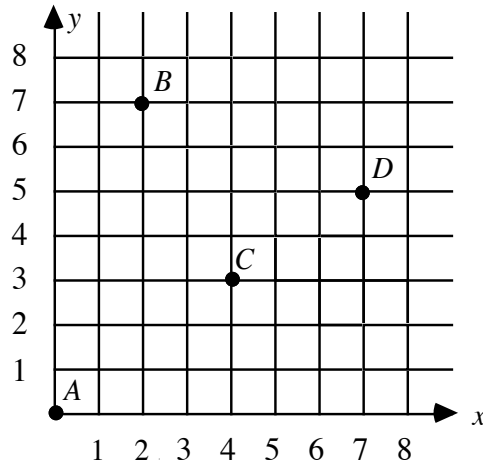
b. Answers may vary, depending on how the students connect the four points.

1.2 **a.** The set of natural numbers is the positive integers, or $\{1, 2, 3, 4, \ldots\}$.

b. A composite number is any natural number that is not prime or 1. In other words, any natural number that has factors other than 1 and itself is a composite number.

c. A prime number is any natural number, other than 1, that only has positive factors of 1 and itself.

Activity 2

2.1 **a.** $8x - 5y = -4$

b. $0x + y = 4$ or $y = 4$

c. $x + 0y = 2$ or $x = 2$

2.2 **a.** $4x + y = 12$

b. $2x - 3y = -4$

2.3   **a.** $[3.7] = 3$

      b. $[1.2] = 1$

      c. $[-3.7] = -4$

      d. $[-1.2] = -2$

2.4   **a.** $y = \dfrac{2x - 7}{3} = \dfrac{2}{3}x - \dfrac{7}{3}$

      b. $y = \dfrac{-8x - 84}{9} = -\dfrac{8}{9}x - \dfrac{28}{3}$

Activity 3

3.1   **a.** This function is trigonometric.

      b. This function is hyperbolic.

      c. This function is cubic.

      d. This function is exponential.

3.2   **a.** This function appears to be exponential.

      b. This function appears to be quadratic.

      c. This function appears to be trigonometric.

      d. This function appears to be hyperbolic.

Grid Paper Template

A Fitting Explanation

Teacher Guide
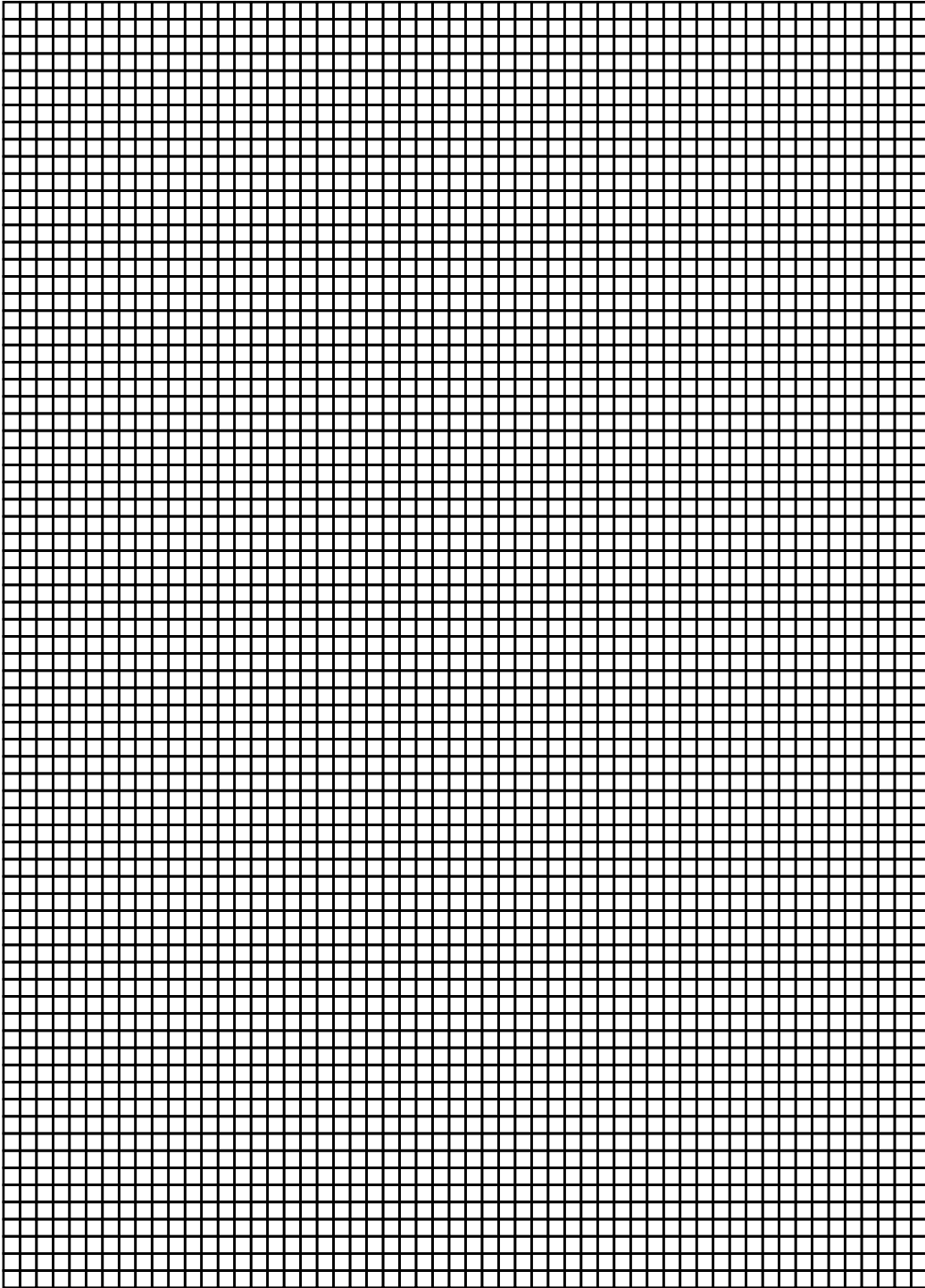
How can you tell when forecasters have made reasonable predictions for the future? In this module, you use regression to fit curves to data sets and evaluate models.

Terri Dahl • *David Thiel* • *Deanna Turley*

A Fitting Explanation

Overview

Students investigate and interpret linear regression equations, the coefficient of determination r^2, and the correlation coefficient r. They also work with the average deviation of prediction and approximation intervals, and consider the proper uses—and potential misuses—of linear regression models.

Objectives

In this module, students will:

- calculate the coefficient of determination r^2 as a percentage of variation
- use r^2 to explain the linear association of the variables in a data set
- calculate the linear correlation coefficient r of a regression line
- use regression equations to make predictions and determine approximation intervals
- analyze appropriate and inappropriate uses of regression equations.

Prerequisites

For this module, students should know:

- how to calculate the mean and standard deviation of a data set
- how to use technology to find regression equations.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	2	2	1	7

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
stopwatch			X	X

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
spreadsheet	X	X	X	X
graphing utility	X	X	X	X

A Fitting Explanation

Introduction

Students are reminded that they have been creating and using mathematical models throughout the curriculum. In this module, they investigate some tools for analyzing linear models.

Activity 1

Students determine a regression line and a mean line for a data set. They investigate the coefficient of determination as the percentage of total variation from the mean line that is explained by the regression line. They then examine the linear correlation coefficient as an indicator of the strength of a linear relationship in the data.

Materials List

- none

Technology

- spreadsheet
- graphing utility

Exploration 1

Students compare the linear regression line to the mean line for a set of data.

a–c. The equation of the mean line is $\bar{y} = 297.75$. The linear regression equation is $y' = 100.925x - 5.025$. Sample graph:

349
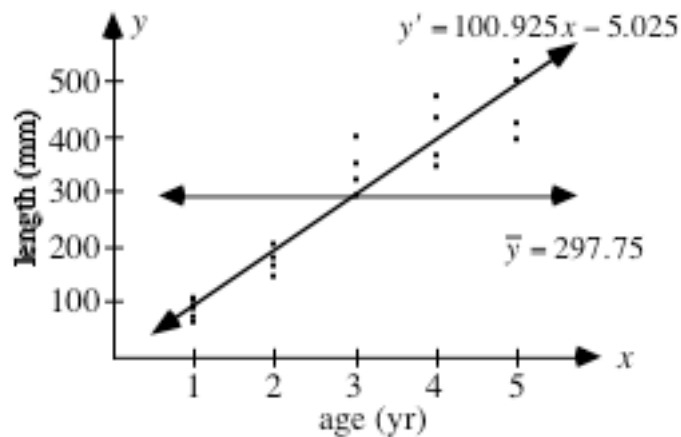
d.   **1.** Using the mean line, the predicted value is 297.75 mm.

         2. Using the linear regression equation, the predicted value is 499.6 mm.

Discussion 1   (page 250)

    a. The data in Table **1** shows a positive association between the age of the fish and the length of the fish.

    b. Sample response: The regression equation seems to be a better model than the mean line. The mean line does not reflect a positive association. The slope of the linear regression model is positive, reflecting the positive association in the data.

    c. Using the mean line, students should not expect a close prediction. Using the linear regression equation, students may expect the predicted value to fall within the range for the points whose x-values are 5. **Note:** Students explore approximation intervals in Activity **2**.

Exploration 2   (page 251)

Students calculate the coefficient of determination r^2 and the linear correlation coefficient r for a data set.

    a.   **1.** $499.6 - 297.75 = 201.85$

         2. $551 - 297.75 = 253.25$

         3. $201.85/253.25 \approx 79.7\%$

b.   In this case, $\bar{y} = 297.75$. As shown in the following spreadsheet, the total variation is 458,445 and the explained variation is 407,434.

Age (yr)	Length (mm)	Linear Model (y')

Age (yr)	Length (mm)	Linear Model (y')	$(y' - \bar{y})^2$	$(y - \bar{y})^2$
1	65	95.90	40,743	54,173
1	72	95.90	40,743	50,963
1	93	95.90	40,743	41,923
1	103	95.90	40,743	37,928
2	209	196.83	10,186	7,877
2	173	196.83	10,186	15,563
2	148	196.83	10,186	22,425
2	181	196.83	10,186	13,631
3	324	297.75	0	689
3	401	297.75	0	1,0661
3	295	297.75	0	8
3	356	297.75	0	3,393
4	355	398.68	10,186	3278
4	487	398.68	10,186	35,816
4	443	398.68	10,186	21,098
4	371	398.68	10,186	5,366
5	507	499.60	40,743	43,786
5	423	499.60	40,743	15,688
5	398	499.60	40,743	10,050
5	551	499.60	40,743	64,136
		Sum	407,434	458,446

c.   Sample response:

$$r^2 \approx \frac{407,434}{458,445} \approx 0.8887$$

Teacher Note

You may wish to remind students that a linear regression minimizes the sum of the squares of the residuals. In one sense, it is the line that best fits the data. The coefficient of determination, r^2, is a measure of the linear relationship in the data.

Discussion 2 (page 253)

a. Since $r^2 \approx 0.8887$, approximately 88.87% of the total variation in the data can be explained by the linear relationship between fish age and fish length. The relationship can be modeled by $y' = 100.925x - 5.025$.

b. The deviations are squared when determining the total variation and explained variation because the sum of the unsquared deviations is 0.

c. **1.** Sample response: If r indicates a strong negative relationship between x and y, then as the values of x increase, the values of y decrease. In other words, the slope of the linear regression equation is negative.

    2. Sample response: If r indicates a strong positive relationship between x and y, then as the values of x increase, the values of y also increase. In other words, the slope of the linear regression equation is positive.

Assignment (page 254)

*1.1   **a.** **1.** The data has a positive correlation coefficient since the regression line has a positive slope.

    2. $|r| \approx \sqrt{0.996687} \approx 0.998342$. Since the regression line has a positive slope, $r \approx 0.998342$.

   b. **1.** The data has a negative correlation coefficient since the regression line has a negative slope.

    2. $|r| \approx \sqrt{0.980847} \approx 0.990377$. Since the regression line has a negative slope, $r \approx -0.990377$.

   c. Sample response: To determine $|r|$, the absolute value of the linear correlation coefficient, take the positive square root of r^2, the coefficient of determination. To determine whether r is positive or negative, find the linear regression equation and observe whether its slope is positive or negative. The sign of r may also be determined visually in some scatterplots where there is a distinct positive or negative association.

1.2   **a.** $y' = 4.8x$

b. The following table shows the explained and total variation:

Time	Distance	$(y' - \bar{y})^2$	$(y - \bar{y})^2$
2.5	12.00	290.36	290.36
6.25	30.00	0.9216	0.9216
7.35	35.28	38.938	38.938
8.10	38.88	96.826	96.826
	Sum	427.0464	427.0464

In this case,

$$r^2 \approx \frac{427.0464}{427.0464} = 1.0 = 100\%$$

Since $r^2 = 1$, 100% of the total variation in the distance can be explained by the model $y' = 4.8x$, where x is time. The regression line models all the data points exactly.

c. Sample response: Since r is exactly 1, there is a perfect positive linear relationship between time and distance.

*1.3   **a.** $y' = 72.2131x - 155.2377$

b. For a 9-inch telescope, the predicted price is $494.68.

c. The following table shows the explained and total variation:

Lens Diameter	Price of Telescope	$(y' - \bar{y})^2$	$(y - \bar{y})^2$
6	295	105,598	94,864
8	395	32,592	43,264
10	565	1,304	1,444
12.5	765	20,859	26,244
16	995	157,745	153,664
	Sum	318,098	319,480

In this case,

$$r^2 = \frac{318,098}{319,480} \approx 0.9957$$

$$r \approx 0.9978$$

d. Since $r^2 \approx 0.9957$, approximately 99.57% of the total variation in the data can be explained by the regression model $y' = 72.2131x - 155.2377$. There is a strong positive linear relationship between lens diameter and price.

1.4   **a.** Sample response: Since the data deviates farther from the mean line in the left-hand graph, the total variation for this data set is greater than the total variation for the other data set.

   b. If 100% of the variation is explained by the linear relationship between the x-values and the y-values in the data set, then $r^2 = 1$ for both graphs.

   If 100% of the variation is explained by the linear relationship between the x-values and the y-values in the data set, then $|r| = \sqrt{1} = 1$ for both graphs. Since the left-hand graph shows a negative association, $r = -1$. Since the right-hand graph shows a positive association, $r = 1$.

1.5   **a.** The regression line is $y' = 40.877x - 30.417$.

   b. $r^2 \approx 0.862$

   c. Sample response: Since $r^2 \approx 0.862$, this indicates that approximately 86% of the total variation in price is explained by the linear relationship between diameter of the mirror and the price.

   d. Sample response: Because the linear association is positive, $r \approx 0.93$. This indicates a fairly strong positive linear relation since the value is close to 1.

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1.6   **a.** $y' = 0.40x + 1.34$

   b. $r^2 \approx 0.978$

   c. Sample response: Since $r^2 \approx 0.978$, 97.8% of the total variation is accounted for by the linear relationship between the hours spent mowing and the amount of gasoline used.

   d. The predicted amount of gasoline required to mow for 36 hr is 15.7 L.

   e. Answers may vary. Sample response: Since there is a strong positive linear relationship between hours mowed and gasoline used, the estimate is probably accurate.

* * * * * * * * * *

Activity 2

In this activity, students are introduced to the average deviation of prediction and approximation intervals. They also consider the uses and misuses of linear regression models.

Materials Required

- none

Technology

- spreadsheet
- graphing utility

Exploration

a. **Note:** Students may realize that a quadratic equation provides a more appropriate model. In this case, a linear model is used to point out some possible misuses of linear regressions.

The regression line is $y' = 0.214x - 2.058$. Sample graph:
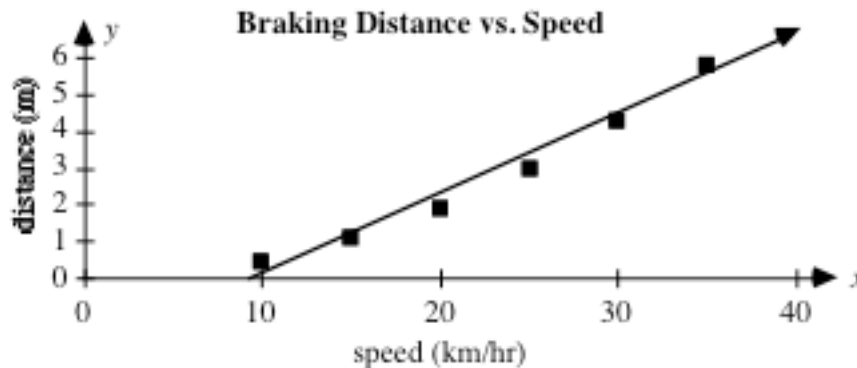
b. In this case, $\bar{y} = 2.751$ and $r^2 \approx 20.0356/20.516 \approx 0.977$.

x	y	y'	$(y' - \bar{y})^2$	$(y - \bar{y})^2$
10	0.475	0.082	7.124	5.181
15	1.069	1.152	2.557	2.830
20	1.900	2.222	0.280	0.724
25	2.969	3.292	0.293	0.047
30	4.275	4.362	2.595	2.322
35	5.819	5.432	7.187	9.412
		Sum	20.036	20.516

c. Sample response: Since $r^2 \approx 98\%$, the line appears to describe the trend in this data quite well. However, the scatterplot appears to curve upward slightly.

d. The average deviation of prediction is:

$$\sqrt{\frac{0.527}{6}} \approx 0.296$$

The approximation interval for each prediction is listed in the table below.

x (km/hr)	Predicted y (m)	Approximation Interval
22	2.650	[2.058, 3.242]
34	5.218	[4.626, 5.810]
50	8.642	[8.050, 9.234]
75	13.992	[13.400, 14.584]

Discussion   (page 259)

a. Sample response: The model seems to fit the data well for the speeds given. Since $r^2 \approx 98\%$, approximately 98% of the total variation in the distance can be explained by the speed.

   Note: Due to the small number of data points, students should not have a great deal of confidence in this model for a larger range of speeds. Other students may argue that since braking distance varies by the square of the speed, a line is not an appropriate model.

b. Sample response: The average deviation is the square root of the average squared residual from the model. Similarly, standard deviation is the square root of the average squared distance from the mean.

   Note: Average deviation is a more general measure than standard deviation. In fact, using the mean line as the model, the average deviation of prediction is the standard deviation.

c. Sample response: Since the average deviation is similar to the standard deviation, it makes sense that predicted y-values should fall within 2 average deviations of the actual y-value for x-values near the mean. In a normal distribution, 95% of the data will lie within 2 standard deviations of the mean.

d. The approximation intervals contain the actual distances for 22 km/hr and 34 km/hr. These speeds are contained in the range of the data, and 22 km/hr is near the mean of the x-values ($\bar{x} = 22.5$). The actual braking distances for speeds of 50 km/hr and 75 km/hr do not fall in the corresponding approximation intervals. Both speeds are beyond the range of the given data.

e. Sample response: Even if a line appears to be a reasonable model, you should be cautious when using it to make predictions beyond the range of the data set.

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Assignment

*2.1   **a.** The regression line is $y' = 0.1944x + 34.0602$. Sample graph:
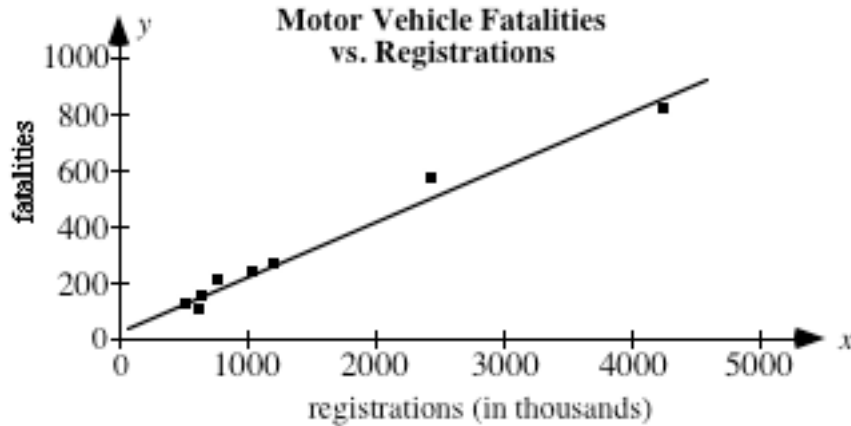
b. For this data, $\bar{y} = 314.75$, $r^2 \approx 0.981$, and $r \approx 0.990$. The value of r^2 indicates that about 98% of the total variation in the number of motor vehicle fatalities can be attributed to the number of motor vehicle registrations.

c. Sample response: Since registrations are given in thousands, 12.8 million registrations corresponds with $x = 12,800$. The regression model predicts approximately 2522 fatalities.

d. The average deviation of prediction is approximately 33. The corresponding approximation interval is $[2456, 2589]$.

e. Sample response: The actual value does not fall in the approximation interval. The number of motor vehicle registrations for Texas was far greater than the number in Washington, the next most populous state in the data (and therefore far greater than the mean of the eight states used to create the model), so this prediction was risky. Other factors such as population density and number of large cities may explain some of the variation.

*2.2   **a.** The regression line is $y' = 369.71x - 3179.10$. Sample graph:

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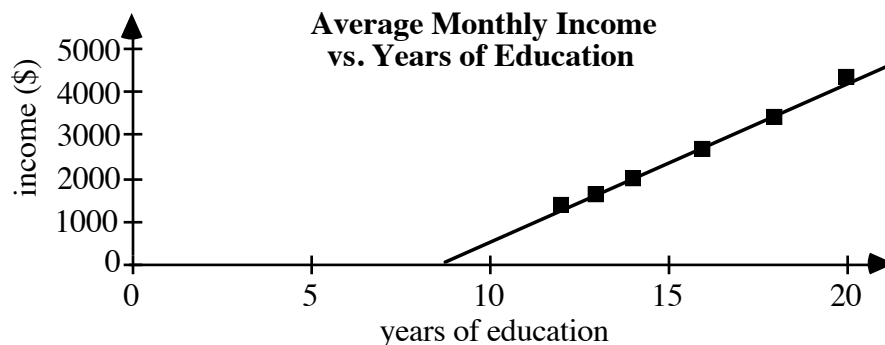
b. In this case, $r \approx 0.996$ and $r^2 \approx 0.993$. The value of r^2 indicates that approximately 99.3% of the variation in monthly income can be attributed to a linear relationship with the number of years of education.

c. For 15 years of education, the predicted monthly salary is $2367.

d. Sample response: Since the average deviation is approximately $88.31, the approximation interval is [$2190, $2544]. Since 15 yr is near the mean ($\bar{x} = 15.5$), it is very likely that the actual average monthly salary for 15 yr of education is in the approximation interval.

*2.3 **a.** Sample graph:

snowfall (cm)

mean January temperature (°C)
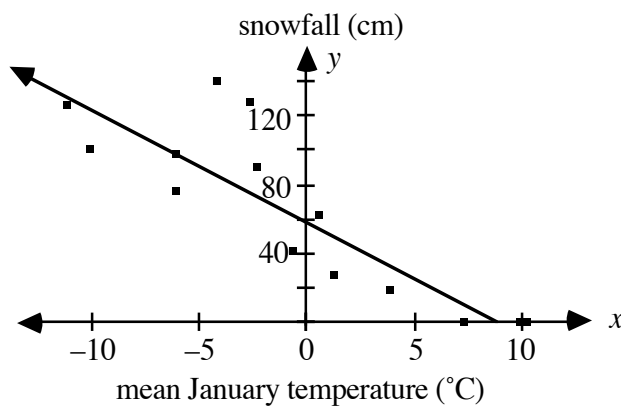
b. The regression line is $y' = -6.4x + 60.3$.

c. The average deviation of prediction is approximately 24.3 cm. The approximation interval for snowfall in Baltimore is [9.1, 106.3], while the approximation interval for Caribou is [87.2, 184.4].

d. Sample response: The interval for Baltimore includes the actual average snowfall. The interval for Caribou does not. Since the January temperature for Baltimore is near the mean of the data ($\bar{x} = -0.7$), while the January temperature for Caribou is not, this appears to agree with the rule of thumb for approximation intervals.

e. Sample response: Since $r^2 \approx 0.746$, approximately 74.6% of the variation in snowfall is explained by temperature.

f. Answers will vary. Some factors that influence local climate are proximity to large bodies of water, nearby mountain ranges, and location relative to the jet stream.

2.4 **a.** This same data set was given in Problem **1.3**. The regression line is: $y' = 72.2x - 155$.
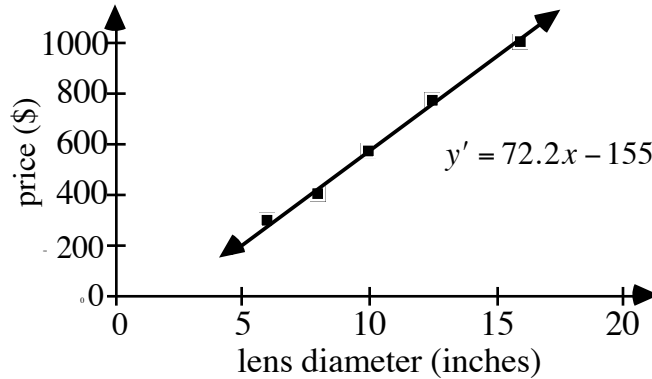
b. The average deviation of prediction is $\sqrt{1382/5} \approx 16.6$. The linear regression model predicts a cost of $639.20 for an 11-inch telescope. The approximation interval for this estimate is $[606.00, 672.40]$.

* * * * *

2.5 Sample response: You would expect the approximation intervals to be smaller for graph **b**, since the residuals are smaller than in graph **a**.

2.6 **Note:** Magnitude is determined by a logarithmic scale in which an increase of 1 unit corresponds with a reduction in brightness by a factor of 2.512.

a. The regression line is $y' = -0.003x + 9.31$. Sample graph:
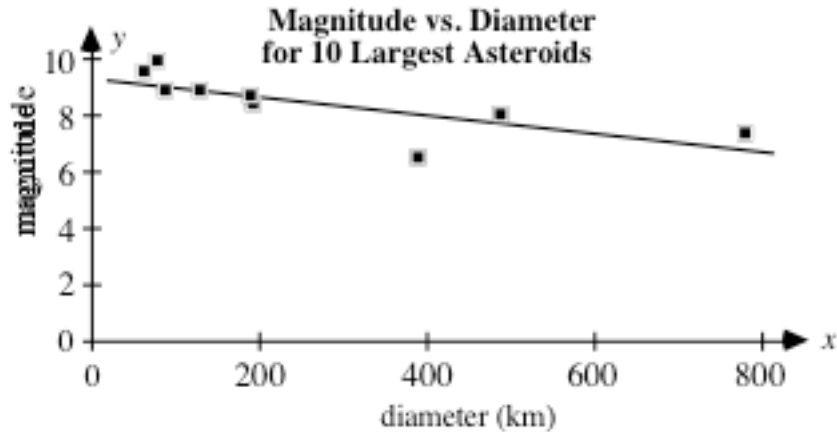
b. Sample response: The value of r is approximately –0.750, which suggests a negative linear correlation between the diameter of an asteroid and its magnitude. The value of r^2 indicates that 56.3% of the variation in the magnitude is explained by the diameter. This means that there is much variation in the magnitude unaccounted for by a linear relationship with diameter.

c. **1.** The average deviation of prediction is approximately 0.619.

   2. Sample response: Using the linear regression model, the approximation interval for the magnitude of a 73-km asteroid is [7.8, 10.3]. Since a diameter of 73 km is far from the mean diameter ($\bar{x} = 260.4$), you should hesitate to apply the rule of thumb for an approximation interval. Considering the percentage of variation unexplained by the regression line, this is probably not a good estimate.

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Research Project (page 263)

This project allows students to collect and analyze a data set that appeals to their own interests.

(page 264)

Activity 3

In this activity, students use regression equations to describe trends in data sets that have many *y*-values for each *x*-value.

Materials List
- stopwatch (one per group)

Technology
- spreadsheet
- graphing utility

Exploration (page 264))

a. Students will probably find that their measurements for a given event are not exactly the same.

b. **1.** Some students may use their pulses, others may count aloud ("one-one-thousand, two-one-thousand").

2. Sample data:

5 sec	10 sec	15 sec	20 sec
4.71	10.06	14.97	20.00
4.80	9.98	14.99	20.00
4.98	9.95	14.98	19.84
4.83	9.93	14.98	19.96

c. **1.** The regression line for the sample data is $y' = 1.01x - 0.16$.

2. The linear correlation coefficient is approximately 1.

3. The average deviation is approximately 0.079.

d. Using the sample data and the regression line, the predicted y-values are 4.88, 9.92, 14.95, and 19.99. The average y-values are 4.83, 9.98, 14.98, and 19.95, respectively. These averages are very close to the corresponding y-values on the regression line.

Discussion   (page 265)

a. Sample response: Yes. It is very good for describing the trend within the range of the data set. The sum of the squares of the residuals is very small and the linear correlation coefficient is approximately 1.

b. Answers will vary. For the sample data given in the exploration, the model appears to be more useful in predicting the means. The average deviation of the means from the model is 0.046, compared to an average deviation of 0.079 for the data points.

c. Sample response: No. This model may not be good for predicting far outside the range of the data. You would expect greater differences for longer time intervals than for shorter time intervals.

Assignment   (page 265)

*3.1 **a.** The regression line is $y' = 3.235x + 145.2$. The coefficient of determination is $r^2 = 0.798$.

b. The average deviation for the regression line is 4.24. The approximation interval for a shoe size of 11 is [172.3, 189.3].

c. Sample response: Approximately 80% of the variability in height can be explained by shoe size. Since $\bar{x} = 11$, the rule of thumb for approximation intervals should apply. I would not anticipate the heights of all people with a shoe size of 11 to fall between 172 cm and 189 cm, but most should be in this approximation interval.

3.2 For the linear regression equation $y' = 100.925x - 5.025$, $r^2 \approx 0.889$. The average deviation is 50.5.

a–b. Sample response: Based on the calculations shown in the table below, the model does not seem to be a particularly good predictor of the mean length of trout based on age. Since the average deviation of prediction is 27.5, however, it is a better predictor of the mean length than of individual lengths.

Age (yr)	Mean Length (mm)	Predicted Length (mm)
1	83	96
2	178	197
3	344	298
4	414	399
5	470	500

*3.3 a. Based on their graphs, students should observe that fuel economy tends to decrease as engine size increases for the given cars. Sample graph:
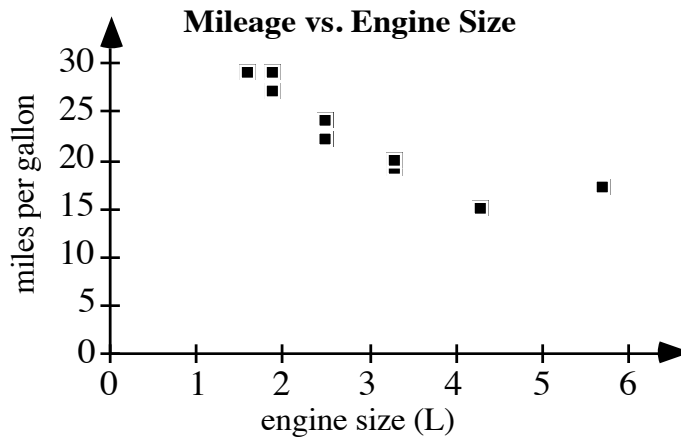
b. The linear regression equation is $y' = -3.57x + 33.3$. In this case, $r^2 \approx 0.801$.

c. Sample response: The linear model predicts that a car with a 2.8-L engine will get about 23.3 miles per gallon. The approximation interval is [18.8, 27.8].

d. Sample response: Since $r^2 \approx 0.80$, a line describes 80% of the variability in the data. This model should predict fuel economy reasonably well within the range of the data. However, it should not be used outside this range. For example, $y' = -3.57x + 33.3$ has a y-intercept of 33.3. This means that fuel economy of 33.3 mpg corresponds with an engine size of 0 L!

* * * * *

3.4   **a.** The linear regression equation is $y = 0.00002x + 3.13$. Sample graph:

Reliability vs. Price of Car
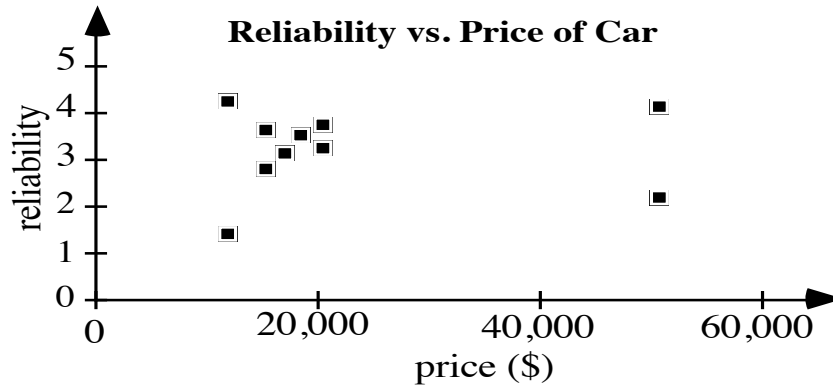
b. Sample response: Based on the information provided in this problem, there is no reason to believe that there is a relationship between price and reliability in new cars. The coefficient of determination for the linear regression model is 0.4%, which shows that very little of the total variation can be explained by a linear relationship in the data set. Consequently, a linear model should not be used to predict the reliability of a car using price.

* * * * * * * * * *

1.   **a.** Sample graph:

Federal Funds vs. Population
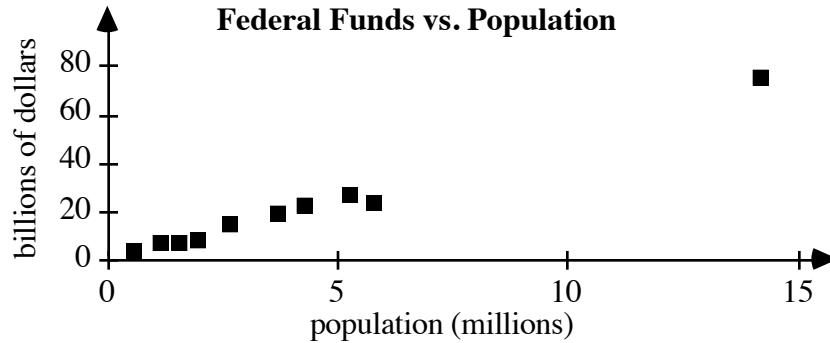
     b. The equation of the regression line is $y' = 5.206x - 0.813$.

     c. Sample response: Considering the coefficient of determination, about 98.6% of the variation in the federal funding for the given states can be explained by the population.

     d. The approximation interval for Texas, in billions of dollars, is $[95.10, 97.98]$. The approximation interval for Minnesota, in billions of dollars, is $[21.69, 24.58]$.

     e. Sample response: In both cases, the actual amounts fall outside the approximation intervals found in Part **d**. The population for Texas was outside the range of the data used for the model, while the population for Minnesota was near the mean ($\bar{x} = 4.14$). Other factors, such as the presence or absence of large federal projects, may explain why neither fell within the predicted interval.

2.   **Note:** You may wish to collect a single set of data for the entire class, or use the sample data given below.

5 sec	10 sec	15 sec	20 sec	25 sec
5	12	20	26	31
6	11	17	23	28
6	11	17	21	27
5	11	16	21	29
6	12	17	20	26
6	10	15	20	29
7	13	19	25	24
5	10	16	20	25

a. The linear regression equation for the sample data is $y' = 1.08x + 0.5$. Sample graph:
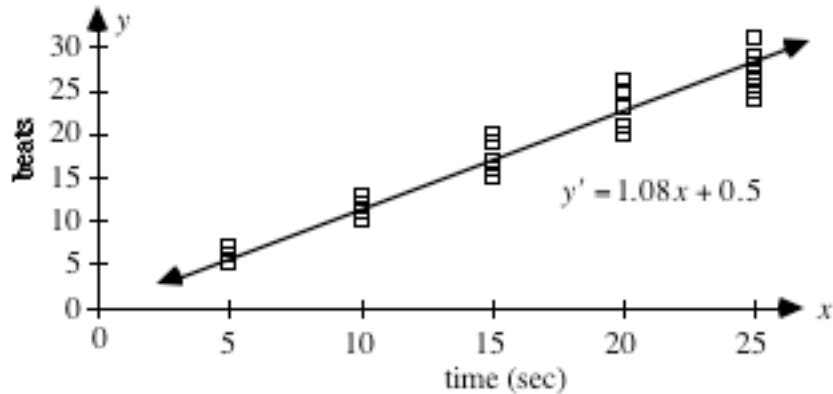
b. Since $r^2 \approx 0.955$, 95.5% of the variation in the sample data can be explained by the model $y' = 1.08x + 0.5$, which relates the number of heartbeats to the amount of time.

c. Sample response: Age, physical condition, and the amount of recent physical activity all influence heart rate.

d. Using the model for the sample data, students might calculate an interval of $[9717, 9723]$. Sample response: This prediction should be suspect, since 2.5 hr is far beyond the range of the given x-values. Unless the conditions under which data was collected remain the same for 2.5 hr, I would not expect this regression model to give a reliable estimate.

1. The following table lists some typical airfares for flights from Los Angeles International Airport (LAX) to selected cities, along with their respective distances in miles.

City	Distance from LAX (km)	Airfare ($)
Las Vegas, NV	407	258
Seattle, WA	1543	830
Rapid City, SD	1749	986
Minneapolis, MN	2453	1196
Atlanta, GA	3146	1302
Toledo, OH	3188	1258
Orlando, FL	3597	1344
New York, NY	3945	1486
Tokyo, Japan	8744	2038
Paris, France	8993	2660

   a. Create a scatterplot of airfare versus distance.

   b. Find the linear regression equation for this data and describe how well it explains the variation in the data.

   c. What other factors besides distance might influence airfares?

2. The following table shows the mean distance from the sun and the orbital period for the nine planets in the solar system.

Planet	Mean Distance from Sun (millions of km)	Orbital Period (in Earth days)
Mercury	57.9	88.00
Venus	108.2	224.70
Earth	149.6	365.26
Mars	227.9	687.00
Jupiter	778.3	4331.98
Saturn	1427.0	10,760.56
Uranus	2869.6	30,685.49
Neptune	4496.7	59,939.17
Pluto	5900.0	90,474.90

   a. Create a scatterplot of orbital period versus distance from the sun and find the corresponding linear regression equation.

b. The mean distance from the sun for the asteroid Ceres is $412.4 \bullet 10^6$ km. Use your model to predict the asteroid's orbital period. Include an approximation interval with your prediction.

c. The actual orbital period of Ceres is 1675 days. How does this compare with your prediction in Part **b**?

d. Write a paragraph describing your model, how well it explains the variation in the data, and its validity for predicting other orbital periods given the mean orbital distance.

Answers to Module Assessment

1. **a.** Sample graph:

Airfare vs. Distance
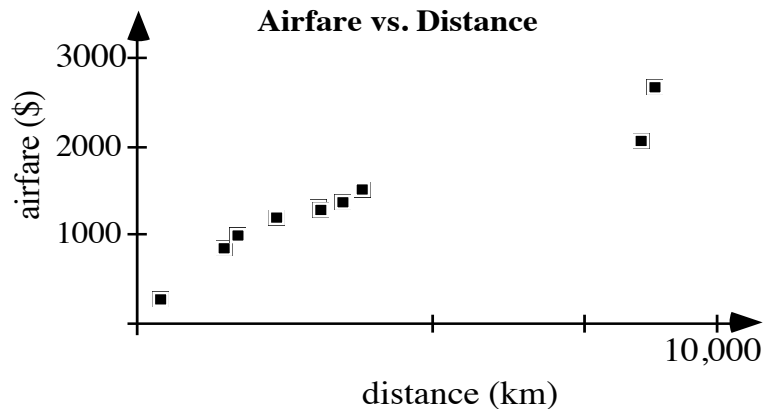
b. The linear regression equation is $y' = 0.216x + 521.86$. For this model, $r = 0.95$, $r^2 = 0.91$, and the average deviation is $188.87. The value of r indicates that there is a strong linear correlation between airfare and distance. The value of r^2 indicates that approximately 91% of the variation in airfares is explained by the linear relationship between distance and price.

c. Answers will vary. Students may mention airport taxes and fees, overseas tariffs, and market competition.

2. **a.** The linear regression is $y' = 14.9x - 4567.5$. Sample graph:

Orbital Period vs. Mean Distance from Sun
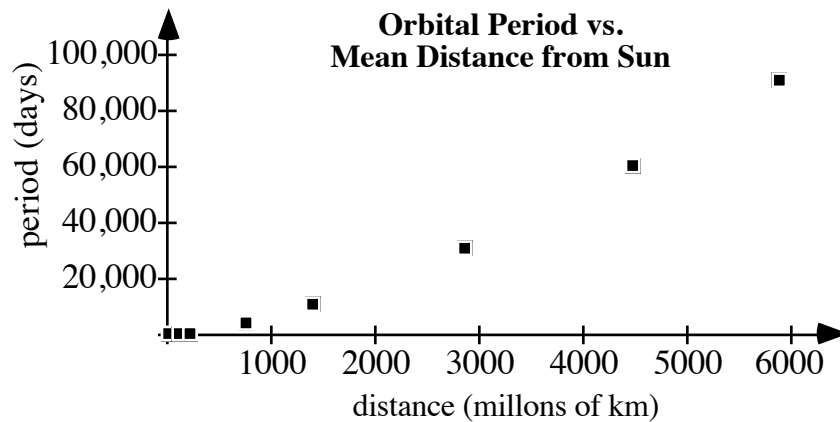
b. Using $y' = 14.9x - 4567.5$, the predicted orbital period for Ceres is 1577 days. The average deviation from the model is approximately 4609 days. The corresponding approximation interval is [0, 10794].

c. Answers may vary. The approximation interval described in Part **b** contains the actual period of 1675 days.

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d. Sample response: Using the linear regression, the coefficient of determination is 0.977. This indicates that 97.7% of the variation in orbital periods is due to the linear relation between mean distance from the sun and period. Therefore, this linear regression appears to be a reasonable model for predicting within the original data set.

Since the scatterplot appears to be curved, however, there may be a better model for this data than a line.

Selected References

Consortium for Mathematics and Its Applications (COMAP). *Against All Odds: Inside Statistics*, program 9. "Correlation." 30 min. 1989. Distributed by The Annenberg/CPB Collection, South Burlington, VT. Videocassette.

Meyers, R. A., ed. *Encyclopedia of Astronomy and Astrophysics*. San Diego, CA: Academic Press, 1989.

Swan, T. "Comparison Test: Luxury Sports Sedans." *Popular Mechanics*. (February 1993): 38–42.

U.S. Bureau of the Census. *Statistical Abstract of the United States 1993*. Washington, DC: U.S. Government Printing Office, 1993.

Flashbacks

Activity 1

1.1   Find a linear regression equation to model the following data points: (120,118), (136,122), (160,143), (98,105), (115,98), and (110,98).

1.2   Calculate the mean of the following numbers: $60.8, 63.8, 64.8, 64.3$, and 68.7.

Activity 2

2.1   Consider the following set of numbers: $\{2, 3, 5, 6, 9, 11\}$.

        a. Calculate the mean and standard deviation.

        b. Determine the interval that represents all values within 2 standard deviations of the mean.

Activity 3

3.1   What type of equation would you use to model the data in the following scatterplot? Explain your response.
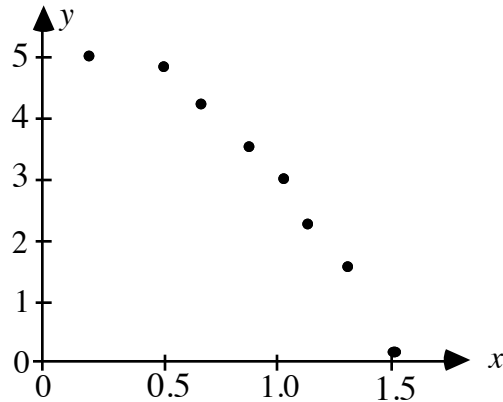
3.2   Consider a data set containing 36 points. If the sum of the squares of the residuals is 18.5, calculate the average deviation of prediction.

Answers to Flashbacks

Activity 1

1.1 $y = 0.718x + 25.6$

1.2   64.48

Activity 2

2.1   **a.** The mean is 6.0 and the standard deviation is 3.46.

            b. $[-0.92, \ 12.92]$

Activity 3

3.1   Sample response: A quadratic model would be appropriate for this data because the pattern resembles the shape of a parabola.

3.2 $\sqrt{18.5/36} = 0.717$

Game of Life

Teacher
Guide
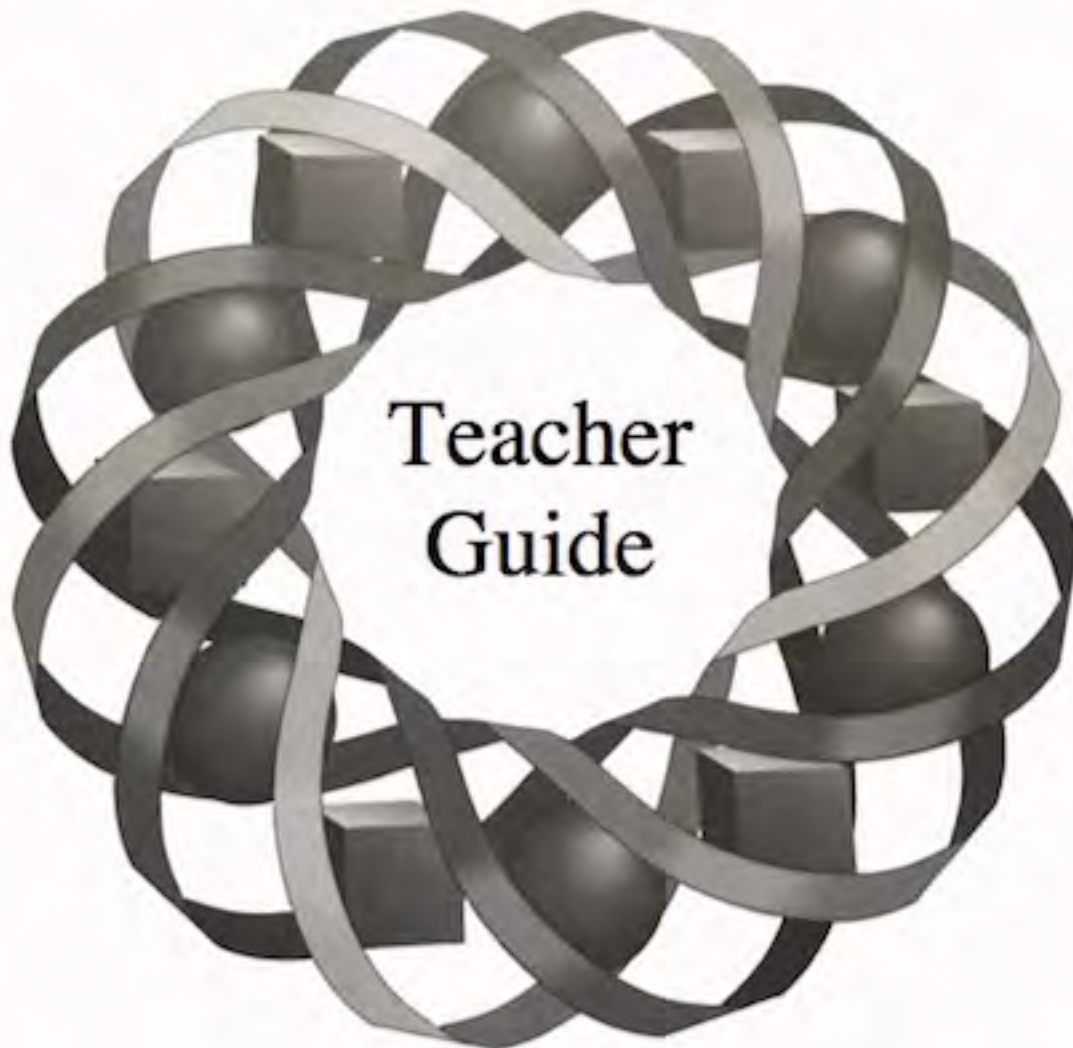
What do two children playing "Rock, Paper, Scissors" have in common with two businesses competing for customers? In both situations, game theory can be used to make decisions and determine strategies.

Masha Albrecht • Verne Schlepp • Steve Yockim

Teacher Edition

Game of Life

Overview

In this module, students investigate situations in which two people have conflicting interests. They use game theory to develop strategies for both players.

Objectives

In this module, students will:

- investigate the properties of strictly determined games
- construct and interpret payoff matrices
- find and interpret the saddle points of payoff matrices
- find the expected values of games
- identify when pure strategies or mixed strategies should be used
- find an optimal strategy for each player in a game.

Prerequisites

For this module, students should know:

- how to determine probabilities of independent events
- how to calculate expected value
- the definition of a fair game.

Time Line

Activity	1	2	Summary Assessment	Total
Days	2	3	1	6

Materials Required

- none

Technology

Software	Activity		
	1	**2**	**Summary Assessment**
spreadsheet		X	
symbolic manipulator (optional)		X	X

Game of Life

Introduction

Students investigate the need to develop strategies in situations where two people have conflicting interests. **Note**: In this module, the term *game* is used to describe any situation in which two individuals have conflicting interests.

Exploration

a–b. Students play Odds or Evens, recording the number of wins for each player.

c. Students report the results of their games to the class. Sample response: The "even" player won 6 times and the "odd" player won 4 times.

Discussion

a. Sample response: I tried to anticipate what the other player would do and varied my number to try to fool my opponent.

b. There should be no advantage to choosing odds or evens. There are four possible outcomes for the number of fingers showing: two of them are odd and two are even.

Activity 1

This activity introduces optimal strategies, pure strategies, and mixed strategies. Students examine payoff matrices and use them to identify strictly determined games. In strictly determined games, the optimal strategy for each player is a pure strategy. **Note:** You may wish to remind students that these strategies are designed to maximize winnings or minimize losses, assuming an intelligent opponent (in other words, one who is aware of the rules of the game and knows how to determine effective strategies for both players).

Teacher Note

Students should collect class data in each part of the exploration. Depending on the amount of data collected, students should be able to identify trends in each game. If the amount of data is too small to identify trends, ask students to play the game more than 10 times each.

Exploration

Students look for optimal strategies in three different versions of the game Morra. They should randomly select the roles they play in each version of the game. This can be done by flipping a coin. Each student should play the same role in all repetitions of each version.

a. **1.** Answers will vary. In Total, neither player appears to have an advantage. In Activity **2**, however, students discover that there is mixed strategy which gives player A an advantage.

2–3. Results will vary, but the point totals for each player should be close to 0.

4. Sample response: It seems that neither player has an advantage.

5. If each player randomly shows one or two fingers, the expected value for player A is: $0.25(-2) + 0.25(3) + 0.25(3) + 0.25(-4) = 0$.

The expected value for player B is:
$0.25(2) + 0.25(-3) + 0.25(-3) + 0.25(4) = 0$.

b. **1.** Answers will vary. In More, player B can win every game.

2–3. Results will vary. If each player recognizes the other's strategy, the point total for player B should be more than the point total for player A. If the players randomly show one or two fingers without any strategy, the results will be similar to the expected values given in Step **5** below.

4. Sample response: It seems player B has an advantage over player A.

5. If both players randomly show one or two fingers, the expected value for player A is:
$0.25(2) + 0.25(-3) + 0.25(3) + 0.25(-4) = -0.5$.

The expected value for player B is:
$0.25(-2) + 0.25(3) + 0.25(-3) + 0.25(4) = 0.5$.

c. **1.** Answers will vary. In Less, player A can win every game.

2–3. Results will vary. If each player recognizes the other's strategy, the point total for player A should be more than the point total for player B. If the players randomly show one or two fingers without any strategy, the results will be similar to the expected values given in Step **5**.

4. Sample response: It seems player A has an advantage over player B.

5. If both players randomly show one or two fingers, the expected value for player A is: $0.25(2) + 0.25(-3) + 0.25(3) + 0.25(-4) = -0.5$.

The expected value for player B is:
$0.25(-2) + 0.25(3) + 0.25(-3) + 0.25(4) = 0.5$.

a. **1.** There is no strategy that allows player A to win every game of Total. Player A should display one or two fingers at random so player B cannot predict the next play.

2. There is no strategy that allows player B to win every game of Total. Player B should display one or two fingers at random so player A cannot predict the next play.

b. **1.** There is no strategy that allows player A to win every game of More. To minimize losses, player A should display one finger in every game so that player B wins 3 points rather than 4 points.

2. Player B can win every game of More by showing two fingers.

c. **1.** Player A can win every game of Less by showing one finger.

2. Player B can minimize losses by always showing one finger. There is no strategy that allows player B to win every game of Less.

d. Answers will vary. Students should observe that the class data should reflect the expected values only if each player randomly shows one or two fingers.

e. Sample response: Each player should try to minimize losses or maximize winnings.

f. In Less, player B's optimal strategy is also a pure strategy because the choice of showing one finger each time minimizes losses.

g. Sample response: If one player uses a pure strategy, the other player will recognize the strategy after a few games and use that information to win the remaining games.

h. Sample response: Each player should display one finger 50% of the time and two fingers 50% of the time. This should be done randomly so the opponent cannot detect a pattern.

i. **1.** Player A should always show one finger to minimize losses and player B should always show two fingers to maximize winnings. In this case, the expected value for player A is $1(-3) = -3$ and the expected value for player B is $1(3) = 3$.

2. If the players used pure strategies, the class data may be close to the expected values in Step **1**.

j. **1.** Player A should always show one finger to maximize winnings and player B should always show one finger to minimize losses. In this case, the expected value for player A is $1(2) = 2$ and the expected value for player B is $1(-2) = -2$.

2. If the players used pure strategies, the class data may be close to the expected values in Step **1**.

k. The entry of –2 indicates that if player A chooses R_1 and player B chooses C_1, then player A loses 2 points and player B wins 2 points.

l. The games More and Less are strictly determined. In both games, the optimal strategy for each player is a pure strategy.

m. **1.** Sample response: Look for the highest positive value that is not affected by the choice of the column player.

    2. Sample response: Look for the least possible value in the row which the row player will choose.

Assignment (page 279)

1.1 **a.** If player B is the row player, the payoff matrix is:

Player A

$$\text{Player B}\begin{array}{c} \\ \text{one} \\ \text{two} \end{array}\begin{array}{cc} \text{one} & \text{two} \\ \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \end{array}$$

    b. Sample response: Neither player benefits by using a pure strategy. With any choice, a player can either win or lose depending on the opponent's choice.

1.2 **a.** If player B is the row player, the payoff matrix is:

Player A

$$\text{Player B}\begin{array}{c} \\ \text{one} \\ \text{two} \end{array}\begin{array}{cc} \text{one} & \text{two} \\ \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \end{array}$$

    b. The second row of the matrix contains two positive entries. Player B will win every time by showing two fingers.

    c. Since player B should always show two fingers, player A should show one finger to minimize losses.

    d. Sample response: If player A is the row player, the payoff matrix is as shown below.

Player B

$$\text{Player A}\begin{array}{c} \\ \text{one} \\ \text{two} \end{array}\begin{array}{cc} \text{one} & \text{two} \\ \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \end{array}$$

    It indicates the same strategies as the matrix in Part **a**. Player B should always show two fingers to guarantee a win. This is shown by negative numbers, which mean a win for player B. Player A would show one finger to guarantee losing 3 instead of 4. This is indicated by the –3 in the matrix.

e. The value of the game for player B is 3. The value of the game for player A is –3.

1.3 **a.** If player A is the row player, the payoff matrix is:

$$\text{Player B}$$

$$\begin{array}{cc} & \text{one} \quad \text{two} \end{array}$$

$$\text{Player A} \quad \begin{array}{c} \text{one} \\ \text{two} \end{array} \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

b. Since the first row contains only positive payoffs, player A can win every time by displaying one finger.

c. Player B will lose either 2 points or 3 points in every game. To minimize losses, player B should always show one finger.

d. The value of the game for player A is 2. The value of the game for Player B is –2.

1.4 **a.** If player B is the row player, the payoff matrix is:

$$\text{Player A}$$

$$\begin{array}{cc} & \text{one} \quad \text{two} \end{array}$$

$$\text{Player B} \quad \begin{array}{c} \text{one} \\ \text{two} \end{array} \begin{bmatrix} 5 & -2 \\ -6 & 4 \end{bmatrix}$$

b. Sample response: The payoff matrix for player B can be obtained from the payoff matrix for player A by changing the signs of all the payoffs, then interchanging the rows and columns such that row 1 becomes column 1 and row 2 becomes column 2.

  Note: The matrix for player B is the opposite of the transpose of the matrix for player A.

1.5 **a.** Player B can win every time by choosing C_1. Player A should always choose R_1 to minimize losses. The value of the game for player B is 2; the value of the game for player A is –2.

b. Player A will win every time. Since player B will always choose C_2 to minimize losses, player A should always choose R_2. The value of the game for player A is 43; the value of the game for player B is –43.

c. Player A can always win by choosing R_1. Player B should choose C_2 to minimize losses. The value of the game for player A is 3; the value of the game for player B is –3.

1.6   **a.** Sample response: Circle the least number or numbers in each row and place a square around the greatest number or numbers in each column. If any payoffs have both a circle and a square around them, they are saddle points.

      b. If the payoff matrix does not have a saddle point, the game is not strictly determined.

      c. If the payoff matrix has more than one saddle point, there is more than one pure strategy for the game.

1.7   **a.** Answers will vary. Sample response:

$$\begin{bmatrix} 5 & -2 \\ 4 & 6 \end{bmatrix}$$

      b. Answers will vary. Sample response:

$$\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$$

*1.8   **a.** If both candidates favor the new lounge, Bill will lose 80 votes to Ann.

      b. The entry -105 means a gain of 105 votes for Ann.

      c. The best outcome for Bill occurs if both he and Ann oppose the lounge, since his payoff is 100 votes. Since Ann has seen the matrix, she will not oppose the lounge. In that case, Bill's best option is to remain neutral on the proposal and gain either 90 or 70 votes from Ann.

      d.   **1.** Bill will gain votes by remaining neutral on the proposal, no matter what Ann does.

          2. Ann can minimize her loss of votes by also remaining neutral.

      e.   **1.** The saddle point is 70.

          2. If both Bill and Ann plan their strategies and remain neutral, Bill will gain 70 votes and Ann will lose 70 votes.

<p style="text-align:center">* * * * *</p>

1.9   Sample response: There will always be a saddle point. Therefore, this game will be strictly determined no matter what the values of x and y are. There are four possibilities: x and y both less than 1, x and y both greater than 1, x less than 1 and y greater than 1, or x greater than 1 and y less than 1. If both x and y are less than 1, then the saddle point value is 1. If both x and y are greater than 1, then the saddle point value is the lesser of x and y. If x is less than 1 and y is greater than 1, the saddle point value is 1. If x is greater than 1 and y is less than 1, then the saddle point value is 1.

1.10   **a.** In the following sample matrix, the circles represent the least value in each row and the squares represent the greatest value in each column.

$$
\begin{array}{c c c c}
 & R & S & P \\
R & 0 & \boxed{1} & \left(\!-1\!\right) \\
S & \left(\!-1\!\right) & 0 & \boxed{1} \\
P & \boxed{1} & \left(\!-1\!\right) & 0
\end{array}
$$

b. The game is not strictly determined because there is no saddle point.

* * * * * * * * * *

(page 282)

Activity 2

In this activity, students use expected value to determine optimal strategies in games that are not strictly determined.

Materials List

- none

Technology

- spreadsheet

Teacher Note

When a game is not strictly determined, players should use mixed strategies. To determine the optimal mixed strategy for the row player, consider the games that result from the column player using a pure strategy. There are two possibilities: the column player always chooses C_1 or always chooses C_2. The two lines in the following graph represent the expected values of these two games for the row player when the row player uses mixed strategies.

column player always selects column 2

$(0.7, \text{max})$ —— maximum payoff to row player

············ minimum payoff to row player

$(0.7, \text{min})$

r_1 (probability of playing row 1)

0.7   1

column player always selects column 1

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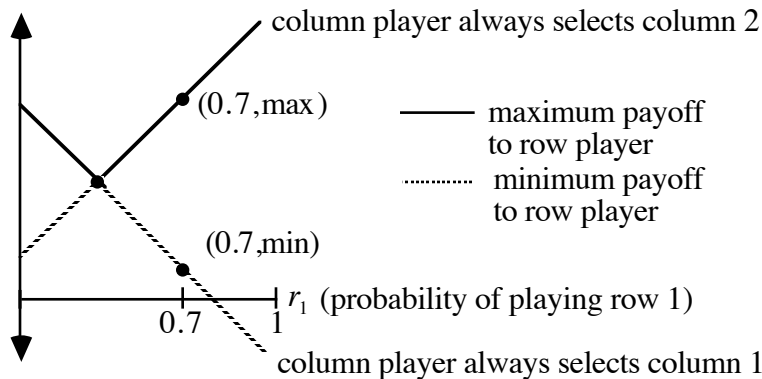
The solid portion of each line represents the maximum payoffs possible for the row players' mixed strategies. The dotted portion represents the minimum payoffs possible. For example, if the row player's strategy is to choose R_1 70% of the time, the y-coordinate of the point $(0.7, \text{min.})$ is the minimum expected payoff and the y-coordinate of the point $(0.7, \text{max.})$ is the maximum expected payoff. Since the column player will not use a pure strategy, the actual expected payoff will be somewhere between these two values. The row player's optimum strategy occurs when the minimum expected payoff is as large as possible. This happens at the point where the two lines intersect. This is also the point where the expected payoff for the row player does not depend on the column player's strategy.

Exploration 1 (page 282)

Since the amount won or lost for each game varies when a game is not strictly determined, expected values are used to determine the optimal strategies. The optimal strategy for the row player is defined as the strategy that results in an expected value that is independent of the choice made by the column player.

    a.   Students guess what strategies will result in maximum earnings for the players. These strategies should be described in terms of the percentage of times each row or column is selected.

        1. Sample response: Player A should choose R_1 70% of the time and R_2 30% of the time.

        2. Sample response: Player B should choose C_1 50% of the time and C_2 50% of the time.

        3. Sample response using percentages from Steps **1** and **2**:

			payoff	probability
column player	0.5	C_1	−1	0.35
R_1	0.5	C_2	2	0.35
R_2	0.5	C_1	1	0.15
	0.5	C_2	0	0.15

row player   0.7   0.3
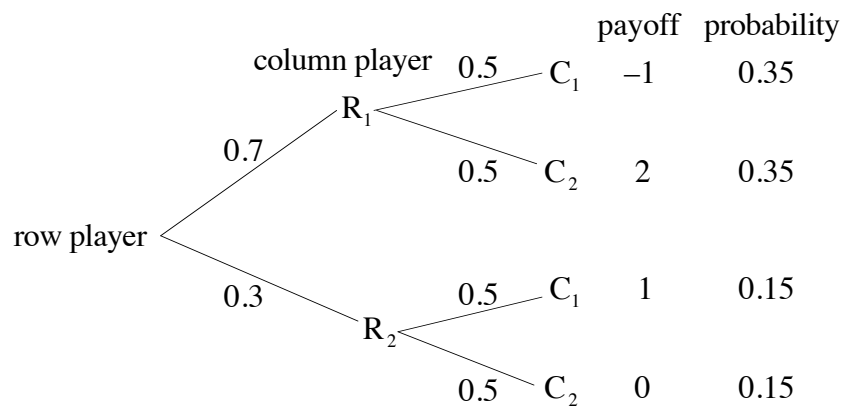
b. The entries in the following table are based on the sample responses given in Part **a**.

Player A's Choice	Player B's Choice	Payoff	Probability of Payoff	Probability ∞ Payoff
R_1	C_1	−1	0.35	−0.35
R_2	C_1	1	0.15	0.15
R_1	C_2	2	0.35	0.70
R_2	C_2	0	0.15	0

c. The expected value for the sample strategies is 0.5.

d. The expected value is $1 - 2r_1$.

e. The expected value is $2r_1$.

f. Sample spreadsheet:

1	2	3	4
Probability that player A selects R_1 (r_1)	Probability that player A selects R_2 $(1-r_1)$	Expected value if player B selects C_1 $(1-2r_1)$	Expected value if player B selects C_2 $(2r_1)$
0.0	1.0	1	0
0.1	0.9	0.8	0.2
0.2	0.8	0.6	0.4
0.3	0.7	0.4	0.6
0.4	0.6	0.2	0.8
0.5	0.5	0	1
0.6	0.4	−0.2	1.2
0.7	0.3	−0.4	1.4
0.8	0.2	−0.6	1.6
0.9	0.1	−0.8	1.8
1.0	0.0	−1	2

g. Sample response: The row player should choose R_1 25% of the time.

h. **1.** The equation is $e(r_1) = 1 - 2r_1$. (See sample graph given in Step **2**.)

2. The equation is $f(r_1) = 2r_1$. Sample graph:
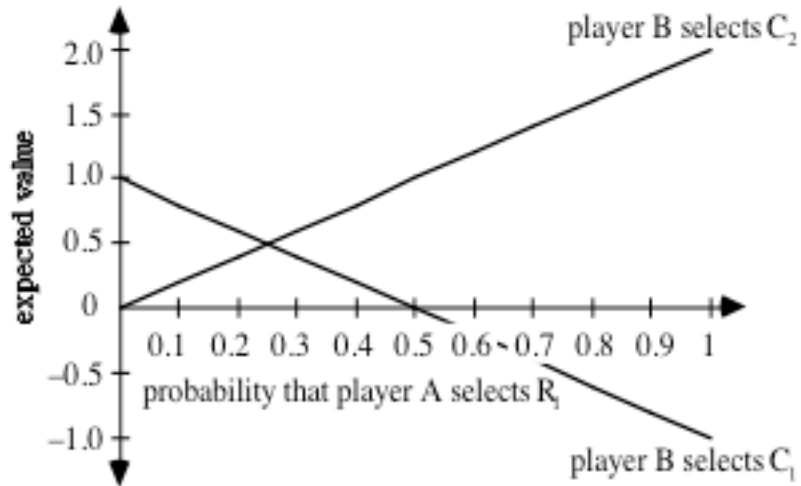
3. Sample response: The two expected values are equal when player A's probability of choosing R_1 is approximately 0.25.

4. Sample response:

$$1 - 2r_1 = 2r_1$$
$$1 = 4r_1$$
$$0.25 = r_1$$

5. Using this strategy, the expected value for player A is 0.5. (Students can determine this value from the graph.)

Discussion 1 (page 285)

a. Answers will depend on the strategies selected in Part **g**.

b. The coordinates of the point of intersection are (0.25,0.5). The first coordinate represents the probability that player A chooses R_1. The second coordinate represents the expected value for player A when using that strategy.

c. **1.** The expected value is 0.5.

   2. This means that over a large number of games, player A could expect to win an average of 0.5 points per game.

   3. Player A would expect to win $1000 \cdot 0.5 = 500$ points.

d. **1.** The average payoff can be found by dividing the sum of the 1000 payoffs by 1000.

   2. Sample response: Not necessarily. The expected value is based on probabilities. When the game is played, the actual payoffs may differ from the theoretical payoffs.

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e.   **1.** The expected values range from –1 to 2. The least value occurs when player A always chooses R_1 and player B always chooses C_1. The greatest value occurs when player A always chooses R_1 and player B always chooses C_2. These values are reflected in the payoff matrix.

         2. Sample response: If player A uses the optimal strategy, the expected value is 0.5. This indicates that player A should end up winning in the long run. If a different strategy is used, the game could have an expected value for player A as low as –1. This would result in player A losing in the long run rather than winning.

f.   **1.** Sample response: If player A establishes a pattern when selecting the row to play, player B could determine the next selection in that pattern and use that knowledge to win games.

         2. Sample response: A random generator on a calculator could be used to generate a number between 0 and 1. If the number is less than or equal to 0.25, select R_1. If the number is between 0.25 and 1.0, select R_2.

g.   Sample response: Determine the expression for the expected value for the row player when the column player always plays C_1. Find the related expression when the column player always plays C_2. Set these two expressions equal to each other and solve the resulting equation.

h.   Sample response: If the game is fair, the value of the game must be 0.

i.   The equation to find the optimal strategy is:

$$ar_1 + c(1 - r_1) = br_1 + d(1 - r_1)$$

Solving this equation for r_1 yields

$$r_1 = \frac{-(c - d)}{a - b - c + d}$$

So, the equation to determine expected value is either

$$E = a\left(\frac{-(c - d)}{a - b - c + d}\right) + c\left(1 - \frac{-(c - d)}{a - b - c + d}\right)$$

or

$$E = b\left(\frac{-(c - d)}{a - b - c + d}\right) + d\left(1 - \frac{-(c - d)}{a - b - c + d}\right)$$

j.   Sample response: Yes. The saddle point of the matrix is 1. Therefore, the value of the game for the row player is 1. Both players use a pure strategy with the row player choosing R_2 with a probability of 1 and the column player choosing C_1 with a probability of 1. The expected value is:

$$-2 \bullet 0 + 3 \bullet 0 + 1 \bullet 1 + 5 \bullet 0 = 1$$

The expected value of the game when both players use their optimal strategy is the same as the number defined to be the value of the game.

Teacher Note

Part **d** of Exploration **2** calls for a simulation of the game. The following program was written for the TI-92 calculator and may be adapted for other utilities. Because of the manner in which the TI-92 names rows and columns in its spreadsheet, "r1" and "c1" cannot be used to define values. In this program, "a1" represents r_1 and "a2" represents r_2, while "b1" represents c_1 and "b2" represents c_2.

```
: game()
: Prgm
: ClrIO
: Input "Input 4 digits", RandSeed r
: 0 → won
: Input "Probability row 1 is selected",a1
: 1 – a1 → a2
: Input "Probability column 1 is selected",b1
: 1 – b1 → b2
: Input "Number of games",n
: For s,1,n
: rand()→ x
: rand()→ y
: If x < a1 and y < b1
: won – 1 → won
: If x < a1 and y > b1
: won + 2 → won
: If x > a1 and y < b1
: won + 1 → won
: If x > a1 and y > b1
: won + 0 → won
: EndFor
: Disp "Average payoff per game for player A"
: Disp won / n
: EndPrgm
```

Exploration 2 (page 288)

a. From Part **h** of Exploration **1**, the value of the game for player A is 0.5.

b. The payoff matrix for player B is:

Player A

$$\begin{array}{c} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player B} \quad \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} \end{array}$$

c. **1.** The optimal strategy for player B is determined by solving for the value of r_1 that results in the same expected value, regardless of player A's strategy:

$$1 \bullet r_1 - 2(1 - r_1) = -r_1 + 0(1 - r_1)$$

$$r_1 = 0.5$$

Player B should choose R_1 50% of the time and R_2 50% of the time.

2. The value of the game for player B is –0.5.

3. The value of the game for player B is the additive inverse of the value of the game for player A.

d. Students require a simulation for the game. See previous teacher note for one example.

1. The average payoff per game for player A should be reasonably close to the value of the game, 0.5. Sample response: The average payoff per game of 0.56 is close to the expected value of 0.5.

2. The average payoff per game should be close to 0.5.

e. When player A uses the optimal strategy, the average payoff per game should still be close to the value of the game for player A. This is because the optimal strategy is determined in such a manner that the expected value is the same, regardless of player B's strategy.

f. When non-optimal strategies are used, the average payoff may vary greatly. For example, using $r_1 = 0.8$ and $c_1 = 0.2$, one simulation resulted in an average payoff per game for player A of 1.092. Using $r_1 = 0.95$ and $c_1 = 0.85$, the simulation resulted in an average payoff of – 0.523.

Discussion 2 (page 289)

a. **1.** The average payoff per game should be close to the value of the game.

2. In most cases, the average payoff per game for 1000 games should be closer to the value of the game than the average payoff for 100 games.

b.   **1.** Sample response: The average payoff per game was still close to the value of the game.

2. Sample response: This occurs because the value of the game is the expected value obtained using a strategy for player A that is independent of the strategy used by player B.

c.   Sample response: The average payoff per game takes on a wide range of values, which are often different from the value of the game.

d.   **1.** Sample response: Yes. The average payoff per game could be greater than the value of the game.

2. Sample response: Yes. The average payoff per game could be less than the value of the game.

Assignment (page 289)

2.1   **a.** Students can find the optimal strategy for player A by solving the following equation for r_1:

$$r_1 \bullet 1 + (1 - r_1) \bullet 4 = r_1 \bullet 2 + (1 - r_1) \bullet 1$$

The optimal strategy for player A is $r_1 = 0.75$ and $r_2 = 0.25$.

Either column can be used to find the value of the game for player A. Using C_1, the value of the game for player A is:

$$0.75 \bullet 1 + (1 - 0.75) \bullet 4 = 1.75$$

The payoff matrix for player B is:

Player A

$$
\begin{array}{c}
 \\
\text{Player B}
\end{array}
\begin{array}{cc}
 & C_1 \quad C_2 \\
R_1 & \begin{bmatrix} -1 & -4 \\ -2 & -1 \end{bmatrix} \\
R_2 &
\end{array}
$$

To find the optimal strategy for player B, solve the following equation for r_1:

$$r_1 \bullet (-1) + (1 - r_1) \bullet (-2) = r_1 \bullet (-4) + (1 - r_1) \bullet (-1)$$

The optimal strategy for player B is $r_1 = 0.25$. (This means that when the payoff matrix is written for player A, player B should use the strategy $c_1 = 0.25$ and $c_2 = 0.75$.)

To find the value of the game for player B, either column of player B's payoff matrix can be used. Using C_1, the value of the game for player B is:

$$0.25 \bullet (-1) + (1 - 0.25) \bullet (-2) = -1.75$$

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b. To find the optimal strategy for player A, solve the following equation for r_1:

$$r_1 \bullet (2) + (1 - r_1) \bullet (3) = r_1 \bullet 4 + (1 - r_1) \bullet (-2)$$

The optimal strategy for player A is $r_1 = 5/7$ and $r_2 = 2/7$.

Either column can be used to find the value of the game for player A. Using C_1, the value of the game for player A is:

$$\frac{5}{7} \bullet (2) + (1 - \frac{5}{7}) \bullet (3) = \frac{16}{7}$$

The payoff matrix for player B is:

Player A

$$\begin{array}{c} \\ \text{Player B} \end{array} \begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & -3 \\ -4 & 2 \end{bmatrix} \end{array}$$

To find the optimal strategy for player B, solve the following equation for r_1:

$$r_1 \bullet (-2) + (1 - r_1) \bullet (-4) = r_1 \bullet (-3) + (1 - r_1) \bullet 2$$

The optimal strategy for player B is $r_1 = 6/7$ and $r_2 = 1/7$. (This means that when the payoff matrix is written for player A, player B should use the strategy $c_1 = 6/7$ and $c_2 = 1/7$.)

To find the value of the game for player B, either column of player B's payoff matrix can be used. Using C_1, the value of the game for player B is:

$$\frac{6}{7} \bullet (-2) + (1 - \frac{6}{7}) \bullet (-4) = -\frac{16}{7}$$

2.2 **a.** The payoff matrix for player B is:

Player A

$$\begin{array}{c} \\ \text{Player B} \end{array} \begin{array}{cc} & \begin{array}{cc} \text{one} & \text{two} \end{array} \\ \begin{array}{c} \text{one} \\ \text{two} \end{array} & \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \end{array}$$

The optimal strategy for player B is $r_1 = 7/12$ and $r_2 = 5/12$.

b. Either column of the payoff matrix can be used to determine the value of the game. Using C_1, the value of the game for player B is $(7/12) \bullet 2 + (5/12) \bullet (-3) = -1/12$.

c. This is not a fair game because the value of the game is not 0.

2.3   **a.** **1.** The payoff matrix for Royal is:

Chicago

cost   quality

Royal   cost $\begin{bmatrix} 0.60 & 0.40 \\ 0.45 & 0.55 \end{bmatrix}$
quality

2. Since the payoff matrix has no saddle point, a mixed strategy should be used.

b. $r_1(0.6) + (1 - r_1)(0.45)$

c. $r_1(0.40) + (1 - r_1)(0.55)$

d. To find the optimal strategy for Royal, solve the following equation for r_1: $r_1(0.6) + (1 - r_1)(0.45) = r_1(0.40) + (1 - r_1)(0.55)$.

The optimal strategy for the company is $r_1 = 1/3$ and $r_2 = 2/3$.

e. Royal should receive approximately 50% of the bids, since the expected value is: $(1/3)(0.6) + (2/3)(0.45) = 0.2 + 0.3 = 0.5$.

*2.4   **a.** The three different strategies result in three different expected payoffs.

1.

Row Player	Column Player	Payoff	Probability	Expected Payoff
R_1	C_1	–2	0.12	–0.24
R_1	C_2	5	0.28	1.4
R_2	C_1	4	0.18	0.72
R_2	C_2	–3	0.42	–1.26
			Total	0.62

2.

Row Player	Column Player	Payoff	Probability	Expected Payoff
R_1	C_1	–2	0.35	–0.7
R_1	C_2	5	0.35	1.75
R_2	C_1	4	0.15	0.6
R_2	C_2	–3	0.15	–0.45
			Total	1.2

3. The optimal strategy for the row player is to choose R_1 50% of the time. The optimal strategy for the column player is to choose C_1 an average of 4 times out of every 7 games.

Row Player	Column Player	Payoff	Probability	Expected Payoff
R_1	C_1	–2	0.29	–0.58
R_1	C_2	5	0.21	1.05
R_2	C_1	4	0.29	1.16
R_2	C_2	–3	0.21	–0.63
			Total	1

b. Sample response: The expected payoff obtained when both players use optimal strategies is between the two expected payoffs obtained using the other strategies. You could win more by using a strategy that was not the optimal strategy but you could also win less. Using the optimal strategy, the row player knows that the average payoff will not depend on the column player's strategy.

*2.5 **a.** The game is not strictly determined since the payoff matrix has no saddle point.

b. Player A's pure strategy should be to choose R_1 in order to win every game.

c. If player A always chooses R_1, player B should always choose C_2 to minimize losses. The expected value for player A would be 4.

d. The optimal mixed strategy for player A is $r_1 = 10/11$ and $r_2 = 1/11$. This results in an expected value of approximately 4.3. This is better than the value of 4 for the pure strategy described in Part **b**.

2.6 **a.** Row 3 dominates row 2. Eliminating row 2, the matrix becomes:

$$\begin{bmatrix} -2 & 8 \\ 9 & 7 \end{bmatrix}$$

The row player should play row 1 an average of 1 time out of every 6 games and the column player should play column 1 an average of 1 time out of every 12 games. The value of the game for the row player is $7\frac{1}{3}$.

b. Column 1 and column 3 dominate column 2. Eliminating column 2, the matrix becomes:

$$\begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}$$

The row player should play row 1 an average of 1 time out of every 3 games and the column player should play column 1 an average of 1 time out of every 3 games. The value of the game for the row player is $-2/3$.

2.7 **a.** Eliminating column 1 and row 1 results in the following matrix:

Group for change

		phone	visit
Group against change	phone	500	400
	visit	200	700

b. There is no pure strategy so both groups should use a mixed strategy. Students should solve the equation below for r_1:

$$500 \bullet r_1 + 200(1 - r_1) = 400 \bullet r_1 + 700(1 - r_1)$$

Since $r_1 = 5/6$, the group against the change should phone people $5/6$ of the time and visit people $1/6$ of the time.

c. The expected value of the game is $500 \bullet (5/6) + 200 \bullet (1/6) = 450$. Since the university requires 500 signatures to reverse its decision, the colors will probably be changed.

* * * * *

2.8 **a.** The payoff matrix in this situation is:

bacterial strain

		C_1	C_2
medicine	R_1	85	70
	R_2	60	95

The optimal strategy would be to select R_1 70% of the time and R_2 30% of the time.

b. The doctor could expect this strategy to be effective about 77.5% of the time.

2.9   **a.**   The payoff matrix for Chicago Remodeling is:

$$\text{Chicago} \quad \begin{array}{cc} & \text{Royal} \\ & \begin{array}{cc} \text{cost} & \text{quality} \end{array} \\ \begin{array}{c} \text{cost} \\ \text{quality} \end{array} & \begin{bmatrix} 0.40 & 0.55 \\ 0.60 & 0.45 \end{bmatrix} \end{array}$$

b. $r_1(0.4) + (1 - r_1)(0.6)$

c. $r_1(0.55) + (1 - r_1)(0.45)$

d. $r_1 = 1/2$ and $r_2 = 1/2$

e.   **1.**   Chicago Remodeling should receive approximately 50% of the bids, since the expected value is:
$(1/2)(0.4) + (1/2)(0.6) = 0.2 + 0.3 = 0.5$.

   2.   This agrees with the percentage of bids Royal should expect to receive.

$$* * * * * * * * * *$$

Answers to Summary Assessment (page 294)

1. **a.** Since the payoff matrix has no saddle point, the players should use mixed strategies.

    b. The optimal strategy for the batter is the solution to the equation:

    $$0.315r_1 + 0.1(1 - r_1) = 0.25r_1 + 0.565(1 - r_1)$$

    The optimal strategy is to guess fastball about 88% of the time and to guess curve ball the remaining 12% of the time.

    c. The expected value is about 0.29, which means that by using this strategy, the batter should attain a batting average of .290 against this pitcher.

2. The optimal strategy is the solution to the equation below:

    $$0.55m_1 + 0.4(1 - m_1) = 0.45m_1 + 0.6(1 - m_1)$$

    Medicine 1 should be used about 67% of the time and medicine 2 the remaining 33% of the time. The total effectiveness of this strategy is 0.5, which means that this strategy is effective half the time.

3. **a.** Using the concept of dominance, Rhonda should not drive left, because shooting a jump shot is always a better choice no matter how the Cougars defend against her. This leaves the following matrix:

    Defender

    $$\begin{array}{c c} & \begin{array}{ccc} L & R & J \end{array} \\ \text{Rhonda} \begin{array}{c} R \\ J \end{array} & \begin{bmatrix} 85\% & 20\% & 70\% \\ 50\% & 60\% & 40\% \end{bmatrix} \end{array}$$

    b. Using the concept of dominance, the Cougars should never defend left, because Rhonda scores well in this case, no matter which move she chooses.

    c. Deleting row L and column L leaves the following 2×2 matrix:

    Defender

    $$\begin{array}{c c} & \begin{array}{cc} R & J \end{array} \\ \text{Rhonda} \begin{array}{c} R \\ J \end{array} & \begin{bmatrix} 20\% & 70\% \\ 60\% & 40\% \end{bmatrix} \end{array}$$

    The optimal strategy for Rhonda is to drive right $3/7$ of the time, and shoot a jump shot $4/7$ of the time.

    d. The Cougars should defend right $3/7$ of the time and defend a jump shot $4/7$ of the time.

    e. Rhonda should expect to score 49% of the time.

1.   Design payoff matrices with the following dimensions for games in which both players should use a pure strategy and the value of the game is 5.

   a. 2×2

   b. 3×3

2.   Write the payoff matrix for player A in each game described below.

   a. Each player in a two-person card game holds a 5 and a 6. The players simultaneously show one card. If the sum of the numbers on the cards is even, player A pays player B an amount equal to that sum. If the sum of the numbers on the cards is odd, player B pays player A an amount equal to that sum.

   b. Two players, player A and player B, simultaneously show one coin. Each player has a choice of showing either a nickel, a dime, or a quarter. If the total amount (in cents) is even, player A wins player B's coin. If the total amount is odd, player B wins player A's coin.

3.   In a game with the following payoff matrix, the optimal strategy for player A is to choose R_1 $3/5$ of the time. Determine the value of x if the game is fair.

$$
\begin{array}{cc}
 & \text{Player B} \\
 & \begin{array}{cc} C_1 & C_2 \end{array} \\
\text{Player A}\begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -4 & x \\ x & -9 \end{bmatrix}
\end{array}
$$

4.   After studying market trends for a number of years, an automobile maker has determined that its profits are affected by both the rate of inflation and the company's own level of production. The table below shows annual profit depending on production and inflation.

	Normal Inflation	High Inflation
Reduced Production	$3.7 million	$3.9 million
Normal Production	$4.0 million	$3.0 million

   a. What is the automobile maker's optimal mixed production strategy?

   b. Using the production strategy from Part a, what is the company's expected annual profit?

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Answers to Module Assessment

1. **a.** Answers will vary. Sample payoff matrix:

$$\begin{bmatrix} 5 & 8 \\ -3 & 2 \end{bmatrix}$$

b. Answers will vary. Sample payoff matrix:

$$\begin{bmatrix} 8 & 5 & 7 \\ 3 & 1 & 0 \\ 2 & -3 & -4 \end{bmatrix}$$

2. **a.** The payoff matrix for player A is:

player B

$$\begin{array}{cc} & \begin{array}{cc} \text{five} & \text{six} \end{array} \\ \text{player A} \quad \begin{array}{c} \text{five} \\ \text{six} \end{array} & \begin{bmatrix} -10 & 11 \\ 11 & -12 \end{bmatrix} \end{array}$$

b. The payoff matrix for player A is:

player B

$$\begin{array}{cc} & \begin{array}{ccc} 5 & 10 & 25 \end{array} \\ \text{player A} \quad \begin{array}{c} 5 \\ 10 \\ 25 \end{array} & \begin{bmatrix} 5 & -5 & 25 \\ -10 & 10 & -10 \\ 5 & -25 & 25 \end{bmatrix} \end{array}$$

3. The game is a fair one if the value of the game is 0. The value of the game is the expected value when the row player uses the optimal strategy. Using C_1, the expected value is:

$$-4 \cdot \frac{3}{5} + x \cdot \frac{2}{5} = 0$$

Solving this equation, $x = 6$.

4. **a.** The automobile maker should use reduced production about 86.7% of the time and normal production about 13.3% of the time. This can be determined by solving the following equation for p_1:

$$3.7p_1 + 4(1 - p_1) = 3.9p_1 + 3(1 - p_1).$$

b. Using this strategy, the expected annual profit is about $3.74 million.

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Flashbacks

Activity 1

1.1   Consider an experiment that involves flipping two fair coins, one at a time.

      a. Identify the sample space for this experiment.

      b. What is the probability of obtaining two heads?

      c. What is the probability of obtaining a head and a tail?

1.2   In a coin-tossing game, a player flips two fair coins. If both are heads, the player wins $1.00. If both are tails, the player wins $2.00. If the two coins do not match, the player wins nothing.

      a. What is the expected value for this game?

      b. Is this a fair game? Explain your response.

Activity 2

2.1   Consider a game that involves flipping three fair coins, one at a time. For each head that appears, players receive $1.00

      a. Identify the sample space for this game.

      b. What is the expected value of the game?

2.2   The following matrix represents a payoff matrix for a two-person, zero-sum game.

$$\begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$$

      a. Determine the value of the game.

      b. Is this a strictly determined game? Explain your response.

Answers to Flashbacks

Activity 1

1.1   **a.** HH, HT, TH, TT

          b. 0.25

          c. 0.5

1.2   **a.** $0.25(\$1.00) + 0.25(\$2.00) + 0.5(\$0) = \$0.75$

          b. Sample response: Assuming that the cost to play is 0, this game is not fair. It would only be a fair game if the cost to play was $0.75.
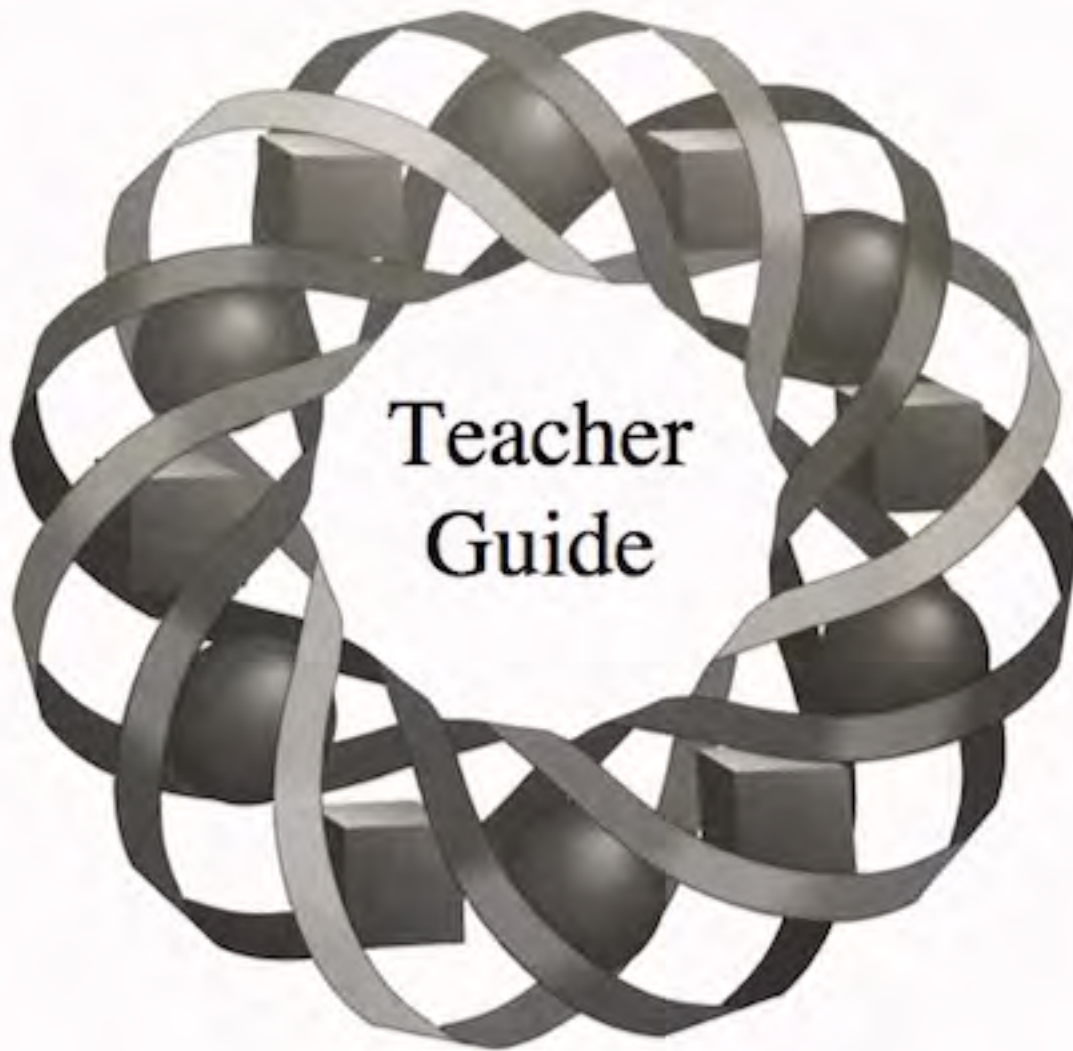
Activity 2

2.1   **a.** HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

          b. The expected value is:

$$\$0 \bullet \frac{1}{8} + \$1 \bullet \frac{3}{8} + \$2 \bullet \frac{3}{8} + \$3 \bullet \frac{1}{8} = \$1.50$$

2.2   **a.** The value of the game is 1.

          b. Sample response: The matrix represents a strictly determined game because it has a saddle point which is 1.

A Walk on the Wild Side

Teacher Guide

An infestation of whirling disease has had a dramatic effect on the trout populations in Montana's rivers. In this module, you discover how wildlife biologists investigate such problems.

Todd Fife • Pete Stabio

<div align="center">

Teacher Edition

A Walk on the Wild Side

</div>

Overview

In this module, students use statistics from samples to determine confidence intervals for population parameters. This topic is addressed within the context of wildlife management. The capture-recapture method of sampling is introduced to estimate the number of animals in a population.

Objectives

In this module, students will:

- take simple random samples from populations
- use technology to simulate sampling procedures
- develop sampling distributions
- apply the central limit theorem and the 68–95–99.7 rule
- determine confidence intervals from random samples
- use the capture-recapture method to estimate population size.

Prerequisites

For this module, students should know:

- how to calculate the mean and standard deviation of a data set
- how to create frequency histograms
- how to use the binomial probability formula
- the definitions of population, sample, census, parameter, statistic, and bias
- how to construct line plots
- the basic characteristics of a normal distribution.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	3	3	2	1	11

Teacher Note

Activity **3** in this module is similar to Activity **2** in the module "Making Cents of Your Income." Although it allows students to strengthen their understanding of confidence intervals, you may wish to omit the exploration in order to save time.

Materials Required

Materials	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
beans	X		X	X	
permanent markers	X			X	
shoe boxes	X		X	X	
graph paper	X	X	X		

Technology

Software	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
graphing utility	X	X	X		X
statistics package	X	X	X	X	X

A Walk on the Wild Side

Introduction

Sampling techniques are important tools for wildlife managers. You may wish to ask students to identify wildlife species in your state or region and discuss the complications involved with managing specific populations.

Activity 1

Students collect random samples, determine sample proportions, and construct relative frequency tables.

Materials List

- beans (100 per group; with the same number marked in each population)
- shoe box or other container of similar size (one per group)
- graph paper
- permanent markers

Teacher Note

Before the exploration, you should mark a certain percentage of beans in each container with a permanent marker (to represent diseased fish). *The percentage marked should be the same in every population*. For best results, the percentage of marked beans should be between 25% and 75%. Do not tell students the proportion of marked beans.

Technology

- statistics package

Teacher Note

After students take a few samples by hand, they should simulate the process using technology. If a statistics package that provides sampling distributions is not available, it is also possible to use a programmable calculator to create an appropriate simulation.

For example, the following program was written for the TI-92 calculator and simulates a population in which 40% show a certain characteristic. Since it uses the calculator's built-in random number generator, students are first requested to enter a seed value. The sample data is stored in the variable "freqtab." The sample proportions are listed in the first column, the frequencies in the second column, and the relative frequencies in the third column. This program requires approximately 8 sec to collect 20 samples of size 4. For 20 samples of size 30, the program takes approximately 40 sec.

Note: The percentage of diseased fish used in the simulation must be the same as the percentage used in the bean populations.

```
:probsim()
:Prgm
:Local a,c,h,k,n,r,s,x
:PlotsOff:FnOff:ClrIO
:Input "Seed Value", s
:RandSeed s
:Disp " "
:Input "Population proportion: (enter as a decimal):",p
:Input "Sample Size?",n
:Input "Number of samples?",r
:ClrIO
:seq(x,x,0,n,1)→list1
:seq(0,x,0,n,1)→list2
:For c,1,r
:0→h
:   For k,1,n
:   rand()→x
:   If x<p Then
:        h+1→h
:   EndIf
:   EndFor
:list2[h+1]+1→list2[h+1]
:EndFor
:list2/r→list4
:list1/n→list3
:sum(list3*list4)→m
```
$$:\sqrt{\left(\text{sum}\big((\text{list}3-m)\hat{}2*\text{list}2\big)/(r-1)\right)}\to a$$
```
:Disp "Mean of sample proportions is",m
:Disp "Standard deviation of the sample"
:Disp "proportions is",a
:NewData freqtab,list3,list2,list4
:EndPrgm
```

Exploration

Students take samples from a simulated population of fish and investigate the distribution of the resulting sample proportions. **Note:** Some students may wonder why they are searching for a population proportion that the teacher already knows or that they could quickly count. The point of this exploration is not to guess the true population proportion but to observe the variability in the results of the sampling process, as well as the patterns in that variability. You may wish to tell the students the true population proportion before beginning the discussion.

a–b. Students determine the proportion of diseased fish for one sample. Responses will vary. The sample responses given below correspond with a population proportion of 0.4, or 40% diseased fish.

c. Students may not make good estimates of the population proportion because the sample size is small. The law of large numbers states that the larger the sample size, the more likely the sample proportion will be close to the population proportion.

d–e. The following relative frequency table was generated using a population proportion of 0.4. The 20 samples of size 4 had a mean of 0.413 and a standard deviation of 0.260.

| Sample Proportion	Frequency	Relative Frequency
0/4 = 0	2	0.10
1/4 = 0.25	8	0.40
2/4 = 0.50	6	0.30
3/4 = 0.75	3	0.15
4/4 = 1.00	1	0.05

Sample histogram:
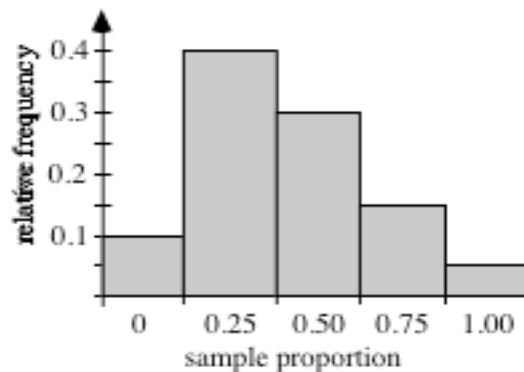
Students should change their predictions to reflect the mean of their 20 sample proportions, since this value is equivalent to the mean of a much larger sample.

f.   Students should use technology to simulate this sampling. (See previous teacher note for a description of a simulation written for a TI-92 calculator.) The larger the sample size, the less variation there should be in the sample proportions.

The following relative frequency table was generated using a population proportion of 0.4. These 20 samples of size 30 had a mean of 0.403, with a standard deviation of 0.070.

| Sample Proportion	Frequency	Relative Frequency
0.267	2	0.10
0.300	1	0.05
0.333	1	0.05
0.367	3	0.15
0.400	2	0.10
0.433	6	0.30
0.467	3	0.15
0.500	2	0.10

Sample histogram:
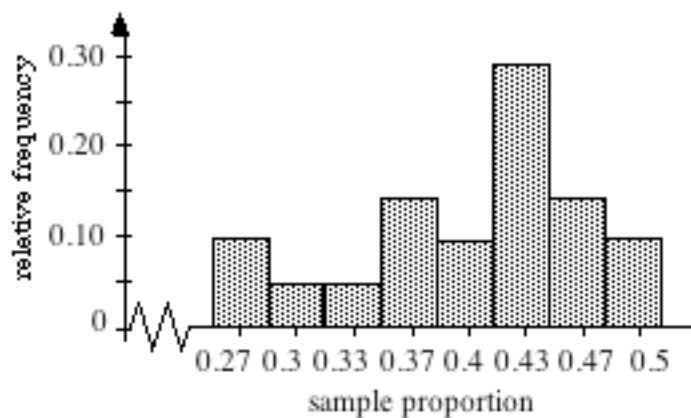
Students may change their predictions to reflect the mean of the 20 samples of size 30.

g.   **Note:** Students should be reminded that all groups were working with identical populations. Without this condition, it makes no sense to combine the data.

1. The shapes of the histograms should more closely resemble the shape of a normal curve than the graphs in Parts **e** and **f**. The following relative frequency table and histogram represent 500 samples of size 4. These samples had a mean of 0.408 and a standard deviation of 0.2458.

| $\hat{p}$	Frequency	Relative Frequency
0/4 = 0	59	0.118
1/4 = 0.25	178	0.356
2/4 = 0.50	163	0.326
3/4 = 0.75	88	0.176
4/4 = 1.00	12	0.024

Sample histogram:
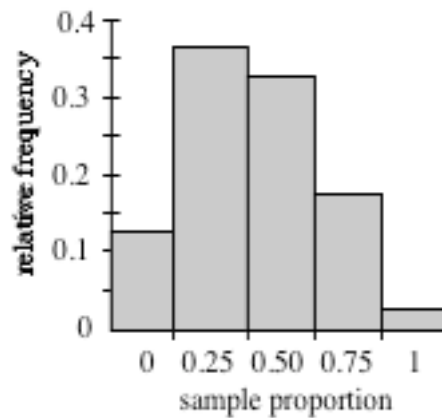
2. The following relative frequency table and histogram represent 500 samples of size 30. These samples had a mean of 0.4039, with a standard deviation of 0.0945.

$\hat{p}$	Freq.	Rel. Freq.	$\hat{p}$	Freq.	Rel. Freq.
0.100	1	0.002	0.433	64	0.128
0.133	3	0.006	0.467	57	0.114
0.167	1	0.002	0.500	42	0.084
0.200	3	0.006	0.533	28	0.056
0.233	17	0.034	0.567	15	0.030
0.267	23	0.046	0.600	8	0.016
0.300	41	0.082	0.633	4	0.008
0.333	48	0.096	0.667	0	0.000
0.367	72	0.144	0.700	1	0.002
0.400	71	0.142	0.733	1	0.002

Sample histogram:
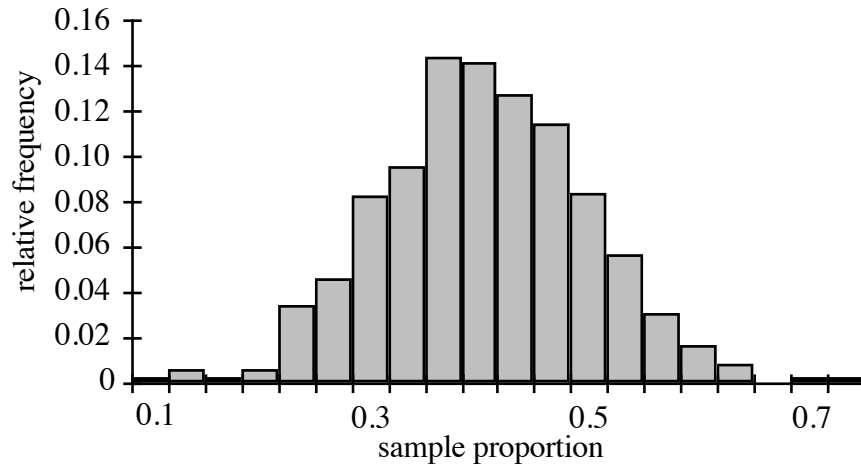
3. Students may change their predictions to reflect the mean of the combined data.

Discussion

a. Sample response: Yes, mixing the beans back with the others gives each member of the population an equal chance of being selected, therefore yielding a random sample.

b. Sample response: The samples of size 30 provided histograms that were more bell-shaped, means that were closer to the actual proportions, and smaller standard deviations.

c. Sample response: The larger sample gave a better estimate of the sample proportion. This makes sense, since as n increases, there is more information about the population.

d. Answers will vary, depending on the given population proportion and student data.

    1. Using the sample data from Part **e** of the exploration, the experimental probability of obtaining a sample proportion of 0.5 is 0.1.

    2. Using the sample data from Part **f** of the exploration, the experimental probability of obtaining a sample proportion of 0.5 is 0.084.

    3. Since they include a much larger number of observations, the samples of size 30 should provide a better estimate. **Note:** In this case, the theoretical probability of obtaining a sample proportion of 0.5 is $C(30,15) \bullet (0.4)^{15} \bullet (0.6)^{15} \approx 0.078$.

e. When the total population is large and the sample size relatively small, the results of sampling without replacement closely approximate the results of sampling with replacement. Consequently, biologists often analyze their data as if the sampling had been done with replacement.

1.1 The possible proportions for samples of size 30 are:

$$\frac{1}{30}, \frac{2}{30}, \frac{3}{30}, ..., \frac{29}{30}, \frac{30}{30}$$

*1.2 **a.** A completed table appears below.

Proportion of Diseased Fish	Frequency	Relative Frequency
0/4 = 0	1	0.025
1/4 = 0.25	7	0.175
2/4 = 0.50	14	0.350
3/4 = 0.75	15	0.375
4/4 = 1.00	3	0.075

b. The best estimate can be found by determining the mean of these sample proportions, which is calculated as the sum of the products of each sample proportion and its relative frequency:

$$0(0.025) + 0.25(0.175) + 0.5(0.35) + 0.75(0.375) + 1(0.075) = 0.575$$

*1.3 Answers will vary, depending on the given population proportion and student data.

a. Using the sample data given in the exploration for a population with 40% diseased fish, the experimental probability of obtaining a sample proportion of $8/30$ is 0.046.

b. The relative frequency of the sample is the same as its experimental probability.

c. **1.** Using the sample data given in the exploration, the experimental probability of obtaining a sample proportion of $10/30$ is 0.096.

2. Using the sample data, the experimental probability of obtaining a sample proportion greater than $10/30$ is:

$$\frac{72 + 71 + 64 + 57 + 42 + 28 + 15 + 8 + 4 + 1 + 1}{500} = \frac{363}{500} \approx 0.73$$

3. Using the sample data, the experimental probability of obtaining a sample proportion less than or equal to $8/30$ is:

$$\frac{23 + 17 + 3 + 1 + 3 + 1}{500} \approx 0.096$$

* * * * *

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1.4   **a.**   Since 30 of the 85 samples had 60% defective chips, then the estimated probability of getting exactly 60% defective chips in random samples of size 5 is $30/85 \approx 35\%$.

b.   Since 60 samples had at least 60% defective chips, then the estimated probability is $60/85 \approx 71\%$.
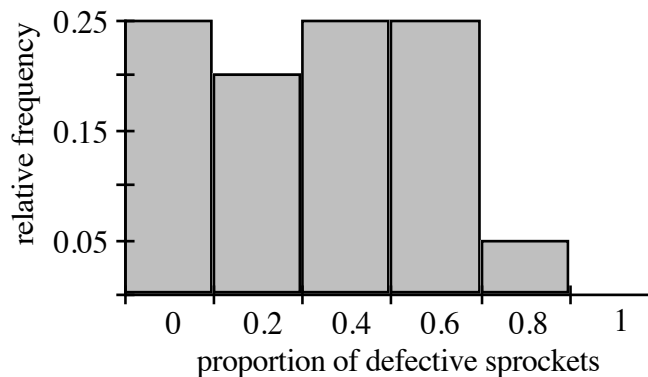
c.   Sample histogram:
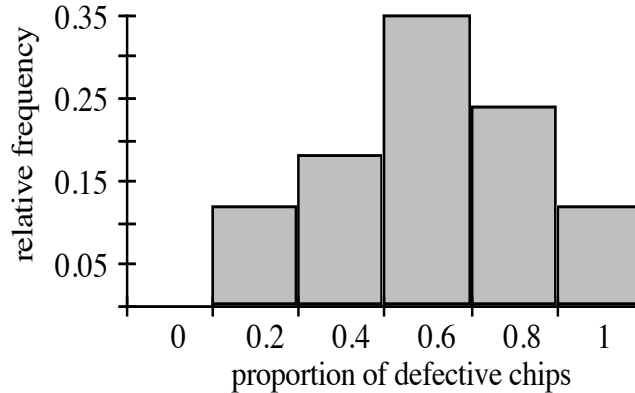
d.   Using the mean of the sample proportions, the estimated population proportion is:

$$\frac{0.2(10) + 0.4(15) + 0.6(30) + 0.8(20) + 1(10)}{85} \approx 0.61$$

1.5   **a.**   A completed table is shown below.

Proportion	Frequency	Relative Frequency
$0/5 = 0.0$	5	0.25
$1/5 = 0.20$	4	0.20
$2/5 = 0.40$	5	0.25
$3/5 = 0.60$	5	0.25
$4/5 = 0.80$	1	0.05
$5/5 = 1.00$	0	0.00

b.   Sample histogram:

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    c. Sample response: Because $2/5 = 0.4$, this proportion of defective sprockets has a probability of $5/20 = 0.25$.

    d. Using the mean of the sample proportions, the estimated population proportion is:

$$0 \bullet 0.25 + 0.2 \bullet 0.2 + 0.4 \bullet 0.25 + 0.6 \bullet 0.25 + 0.8 \bullet 0.05 = 0.33$$

* * * * * * * * * *

Activity 2

In this activity, students use the binomial probability formula to investigate sampling distributions, then apply the central limit theorem and the 68–95–99.7 rule.

Materials List
* graph paper

Technology
* statistics package

Teacher Note

To allow students to generate binomial sampling distributions more quickly in Exploration **2**, you may wish to provide them with a calculator program such as the following, written for the TI-92. In this program, the sample size (or number of trials) is n. The possible number of successes (0 through n) are stored in list1[1] through list1[n+1]. The probabilities of each possible number of successes are stored in list2[1] through list2[n+1].

This program reports the mean and standard deviation of the data and draws a connected scatterplot of the relative frequencies with a vertical line at the true population proportion.

```
:binomial()
:Prgm
:PlotsOff:FnOff:ClrIO:DelVar list1, list2
:Input "Population Proportion?",p
:Input "Sample Size?",n
:ClrIO
:seq(c/n,c,0,n)–>list1
:seq(nCr(n,c)*p^c*(1-p)^(n-c),c,0,n)–>list2
:Disp "Mean",p, "Standard Deviation", √(p*(1 – p) / n)
:Pause
:0–>xmin
:1–>xmax
:1/n–>xscl
:0–>ymin
:1.1(max(list2))–>ymax
:0.1–>yscl
:NewPlot 1,2,list1,list2,,,,4
:LineVert p
:PtText string(p),p,ymax/10
:EndPrgm
```

Discussion 1 (page 306)

a. Sample response: It is a binomial experiment because there are 4 trials of catching a fish and replacing it, each trial is independent of the others, each fish is either diseased or not, and the probability of catching a diseased fish remains constant from trial to trial.

b. In a sample of size 4, there are $C(4,1)$ ways that a sample with exactly 1 diseased fish could occur: HHHD, HHDH, HDHH, and DHHH, where H represents healthy and D represents diseased.

c. If the probability of obtaining a diseased fish is 0.6, then the probability of obtaining a healthy fish is $1 - 0.6 = 0.4$. The number of ways that you could obtain exactly 1 diseased fish in a sample of 4 fish is $C(4,1) = 4$. The probability of obtaining any one of these samples is $(0.6)^1(0.4)^3$. The total probability, therefore, is:

$$C(4,1) \bullet (0.6) \bullet (0.4)^3 = 0.1536 \approx 15\%$$

d. Since the sampling distribution represents the probabilities of all possible outcomes in the experiment, the sum of the probabilities is 1.

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Exploration 1

a. Answers will vary, depending on the value of p. The following table shows the sampling distribution of sample proportions for $p = 0.4$ and $n = 4$.

Sample Proportion	Probability
$0/4 = 0$	$0.1296 \approx 13\%$
$1/4 = 0.25$	$0.3456 \approx 35\%$
$2/4 = 0.50$	$0.3456 \approx 35\%$
$3/4 = 0.75$	$0.1536 \approx 15\%$
$4/4 = 1.00$	$0.0256 \approx 3\%$

b. The mean of the sampling distribution of sample proportions for $p = 0.4$ and $n = 4$ can be found as follows:

$$0(0.1296) + 0.25(0.3456) + 0.50(0.3456) +$$
$$0.75(0.1536) + 1(0.0256) = 0.40$$

1. The mean of the sampling distribution equals the population proportion.

2. The mean of the sample data given in Activity **1** was 0.408.

c. The standard deviation of the sampling distribution for $p = 0.4$ and $n = 4$ is:

$$\sigma = \sqrt{\frac{0.4(1 - 0.4)}{4}} \approx 0.24$$

The standard deviation of the sample data given in Activity **1** was 0.25.

d. The following table shows the sampling distribution of $\hat{p}$ for $p = 0.4$ and $n = 30$.

$\hat{p}$	Probability	$\hat{p}$	Probability
0/30 = 0	0.0000	16/30 ≈ 0.53	0.0489
1/30 ≈ 0.03	0.0000	17/30 ≈ 0.57	0.0269
2/30 ≈ 0.07	0.0000	18/30 = 0.6	0.0129
3/30 = 0.1	0.0003	19/30 ≈ 0.63	0.0054
4/30 ≈ 0.13	0.0012	20/30 ≈ 0.67	0.0020
5/30 ≈ 0.17	0.0041	21/30 = 0.7	0.0006
6/30 = 0.2	0.0115	22/30 ≈ 0.73	0.0002
7/30 ≈ 0.23	0.0263	23/30 ≈ 0.77	0.0000
8/30 ≈ 0.27	0.0505	24/30 = 0.8	0.0000
9/30 = 0.3	0.0823	25/30 ≈ 0.83	0.0000
10/30 ≈ 0.33	0.1152	26/30 ≈ 0.87	0.0000
11/30 ≈ 0.37	0.1396	27/30 = 0.9	0.0000
12/30 = 0.4	0.1474	28/30 ≈ 0.93	0.0000
13/30 ≈ 0.43	0.1360	29/30 ≈ 0.97	0.0000
14/30 ≈ 0.47	0.1101	30/30 = 1	0.0000
15/30 = 0.5	0.0783		

The mean of the sampling distribution equals the population proportion, 0.4. The mean of the sample data given in Activity **1** was 0.404.

The standard deviation of all possible sample proportions where $p = 0.4$ and $n = 30$ is:

$$\sqrt{\frac{0.4(1-0.4)}{30}} \approx 0.089$$

The standard deviation of the sample data given in Activity **1** was 0.095.

Discussion 2 (page 308)

a. Sample response: The mean is the sum of the products of each sample proportion and its corresponding probability. For $p = 0.4$ and $n = 4$,

$$0(0.1296) + 0.25(0.3456) + 0.5(0.3456) + 0.75(0.1536) + 1(0.0256) = 0.4$$

b. Sample response: In the long run, as one takes more and more samples of size n from a population with population proportion p, the relative frequency table gets closer and closer to the sampling distribution.

c. The sum of the probabilities in the sampling distribution that correspond to 10 or fewer diseased fish is:

$$C(30,10) \cdot (0.6)^{10}(0.4)^{20} + C(30,9) \cdot (0.6)^9(0.4)^{21} + \cdots +$$

$$C(30,0) \cdot (0.6)^0(0.4)^{30} \approx 0.003$$

Thus, from such a population, one would expect to obtain a sample with 10 or fewer diseased fish only about 0.3% of the time or 3 out of 1000 samples.

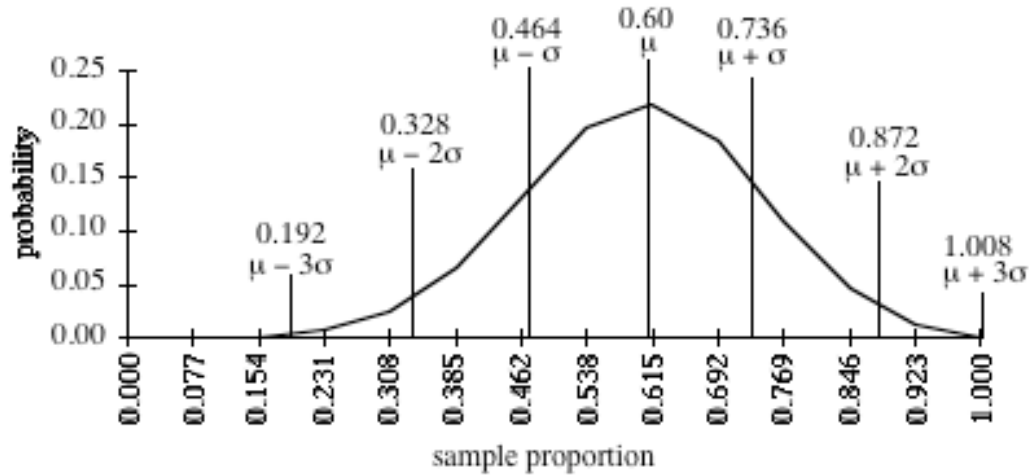## Exploration 2 (page 309)

a–b. Students should observe that the graphs become more bell-shaped and more symmetric around the population proportion as n gets large.

c.   **1.** Sample response: The sampling distribution appears to be bell-shaped and symmetric for $n = 13$. **Note:** Some students may report values for n as small as 10.

   2. Using the TI-92 program described in the previous teacher note, students can estimate the probabilities for each sample proportion with the trace feature.

   The following table shows the sampling distribution for $n = 13$.

$\hat{p}$	Probability	$\hat{p}$	Probability
$0/13 = 0$	0.0000	$7/13 \approx 0.54$	0.1968
$1/13 \approx 0.08$	0.0001	$8/13 \approx 0.62$	0.2214
$2/13 \approx 0.15$	0.0012	$9/13 \approx 0.69$	0.1845
$3/13 \approx 0.23$	0.0065	$10/13 \approx 0.77$	0.1107
$4/13 \approx 0.31$	0.0243	$11/13 \approx 0.85$	0.0453
$5/13 \approx 0.38$	0.0656	$12/13 \approx 0.92$	0.0113
$6/13 \approx 0.46$	0.1312	$13/13 = 1$	0.0013

   3. The mean of the distribution given in Step **2** is 0.6; the standard deviation is 0.136.

d. Sample graph:

probability (y-axis): 0.25, 0.20, 0.15, 0.10, 0.05, 0.00

0.464 $\mu - \sigma$ 0.60 μ 0.736 $\mu + \sigma$
0.328 $\mu - 2\sigma$ 0.872 $\mu + 2\sigma$
0.192 $\mu - 3\sigma$ 1.008 $\mu + 3\sigma$

sample proportion (x-axis): 0.000, 0.077, 0.154, 0.231, 0.308, 0.385, 0.462, 0.538, 0.615, 0.692, 0.769, 0.846, 0.923, 1.000

1. For $n = 13$, the interval of values within 1 standard deviation of the mean is [0.464, 0.736]. Students should determine the probability that the sample proportion falls in this interval by finding the sum of the appropriate probabilities in their table from Part **c**. The probability that the sample proportion falls in this interval is:

$$0.1968 + 0.2214 + 0.1845 = 0.6027$$

2. For $n = 13$, the interval of values within 2 standard deviations of the mean is [0.328, 0.872]. The probability that the sample proportion falls in this interval is 0.9555.

3. For $n = 13$, the interval of values within 3 standard deviations of the mean is [0.192, 1]. The probability that the sample proportion falls in this interval is 0.998. **Note:** The upper bound of the interval is 1 even though $\mu + 3\sigma > 1$.

Discussion 3

a. Sample response: For $n \geq 13$, the sampling distribution appears to be bell-shaped and symmetric. **Note:** Some students may report values for n as small as 10.

b. Answers will vary. The following sample responses correspond with $n = 13$.

1. 0.621

2. 0.956

3. 0.998

c. Sample response: These values are reasonably close to those predicted by a normal distribution.

d. For $0.10 \le p \le 0.90$, the sample size should be at least 51.

$$np > 5 \qquad n(1-p) > 5$$
$$n(0.10) > 5 \qquad n(1-0.10) > 5$$
$$n > 50 \qquad\qquad n > 6$$

Assignment

*2.1 **a.** The following table shows the sampling distribution for $p = 0.5$ and $n = 10$.

$\hat{p}$	Probability	$\hat{p}$	Probability
0.0	0.0010	0.6	0.2051
0.1	0.0098	0.7	0.1172
0.2	0.0439	0.8	0.0439
0.3	0.1172	0.9	0.0098
0.4	0.2051	1.0	0.0010
0.5	0.2461		

b. $P(0 \text{ trout}) + P(1 \text{ trout}) + P(2 \text{ trout}) = 0.0010 + 0.0098 + 0.0439 = 0.055$

c. The probability can be found as follows:

$P(4 \text{ trout}) + P(5 \text{ trout}) + P(6 \text{ trout}) + P(7 \text{ trout})$

$= 0.2051 + 0.2461 + 0.2051 + 0.1172 = 0.7735$

d. The interval for 1 standard deviation is $[0.342, 0.658]$. The sample proportions that fall in this interval are $0.4, 0.5$, and 0.6. The sum of their corresponding probabilities is:

$$0.2051 + 0.2461 + 0.2051 = 0.6563$$

2.2 **a.** The sampling distribution is shown in the table below.

$\hat{p}$	Probability	$\hat{p}$	Probability
$0/50 = 0.00$	0.002	$10/50 = 0.20$	0.038
$1/50 = 0.02$	0.011	$11/50 = 0.22$	0.019
$2/50 = 0.04$	0.038	$12/50 = 0.24$	0.008
$3/50 = 0.06$	0.083	$13/50 = 0.26$	0.003
$4/50 = 0.08$	0.133	$14/50 = 0.28$	0.001
$5/50 = 0.10$	0.167	$15/50 = 0.30$	0.000
$6/50 = 0.12$	0.171	$16/50 = 0.32$	0.000
$7/50 = 0.14$	0.147	$17/50 = 0.34$	0.000
$8/50 = 0.16$	0.108	$\vdots$	$\vdots$
$9/50 = 0.18$	0.068	$50/50 = 1.00$	0.000

b. Since $\sigma = \sqrt{((0.12)(0.88))/50} \approx 0.046$, the interval is $[0.03, \ 0.21]$.

*2.3   **a.**   Using the formula for standard deviation of a sampling distribution:

$$\sqrt{\frac{(0.35)(0.65)}{n}} < 0.05$$

$$91 < n$$

b.   The standard deviation of the sampling distribution is:

$$\sigma = \sqrt{\frac{(0.35)(0.65)}{400}} = 0.0238$$

   1. $[0.3262, 0.3738]$

   2. $[0.3024, 0.3976]$

   3. $[0.2786, 0.4214]$

c.   Sample response: Since n is sufficiently large, the sampling distribution can be approximated by a normal distribution. Therefore, the 68–95–99.7 rule can be applied. You would expect approximately 68% of all sample proportions to fall within 1 standard deviation of the mean, 95% within 2 standard deviations, and 99.7% within 3 standard deviations.

2.4   As mentioned in the mathematics note, the properties implied by the central limit theorem are approximately true when $np > 5$ and $n(1 - p) > 5$. In this situation, $0.2 \le p \le 0.8$. Assuming $p = 0.2$, then $n(0.2) > 5$ or $n > 25$.

* * * * *

2.5   **a.**   **1.** $P(1) = C(4,1) \bullet (0.2)^1 \bullet (0.8)^3 = 0.4096$

   2. $P(2) = C(4,0) \bullet (0.2)^0 \bullet (0.8)^4 = 0.4096$

   3. $P(\text{at most } 2) = P(0) + P(1) + P(2) = 0.973$

b.   The standard deviation of the sampling distribution is:

$$\sqrt{\frac{(0.2)(0.8)}{200}} \approx 0.0283$$

The approximate interval in which 95% of the sample proportions will fall is $[0.143, 0.257]$.

2.6   **a.**   The mean is $\mu = p = 0.3$. The standard deviation is:

$$\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(0.3)(0.7)}{300}} \approx 0.0265$$

b. $[0.247, \ 0.353]$

c.   Sample response: Such a result would be extremely unlikely given the assumption that only 30% are defective. Therefore, one might consider testing another sample. If similar results are obtained, one might conclude that the assumption is wrong.

* * * * * * * * * *

Activity 3

Students continue their investigations of confidence intervals.

Materials List

- graph paper
- bean populations from Activity **1**

Teacher Note

As an alternative to using the original bean populations, you may wish to create a single new population for the entire class.

Technology

- statistics package

Teacher Note

The following calculator program, written for the TI-92, reports the mean and estimated standard deviation for a random sample of size n from a population with proportion p.

```
:confiden()
:Prgm
:ClrIO
:Input "Seed Value",s
:RandSeed s
:Input "Sample Size?",n
:Input "Population Proportion?",p
:ClrIO
:0→h
:For c,1,n
:   If rand()<p Then
:       h+1→h
:   EndIf
:EndFor
:h/n–>m
:√(m*(1 – m)/n)→s
:Disp "Sample Proportion", m
:Disp "Estimated Standard Deviation",s
:EndPrgm
```

Teacher Note

When estimating the standard deviation of the sampling distribution of $\hat{p}$, the unbiased estimator of its square is:

$$s^2 = \frac{\hat{p}(1-\hat{p})}{n-1}$$

where n is the sample size. Therefore,

$$s = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}$$

When n is relatively large, as in the remainder of this module, the following formula provides a reasonable approximation:
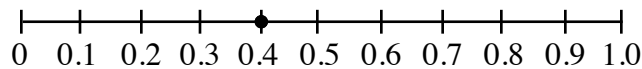
$$s = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Exploration (page 312)

a. Answers will vary. The following sample responses are based on a population proportion of 0.4.

Sample number line:

b. Sample response: In one sample, 9 out of 30 were diseased. The sample proportion is $9/30 = 0.3$.

c. Using $\hat{p}$ to approximate p, $p = 0.3$ and $1 - p = 0.7$. In this case,

$$s = \sqrt{\frac{0.3(0.7)}{30}} = 0.084$$

d. Using $\hat{p} = 0.3$ and $s = 0.084$, $\hat{p} - 2s = 0.3 - 2(0.084) = 0.132$ and $\hat{p} + 2s = 0.3 + 2(0.084) = 0.468$. Sample graph:

proportion of diseased fish
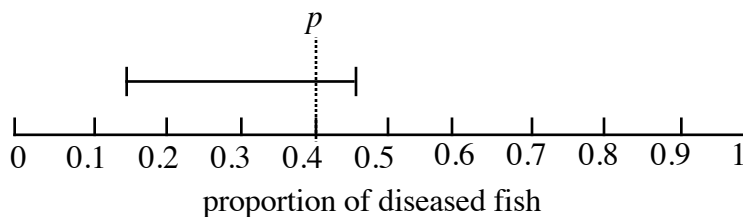
e. In the example shown above, the population proportion falls within the confidence interval.

f. Answers will vary. In one simulation, 100% of the 20 confidence intervals generated contained p.

g.   Answers will vary. In one simulation, 95% of the 20 confidence intervals generated contained p.

Discussion

right(page 314)

a.   **1.** Answers will vary. Sample response: For $n = 30$, 100% of the confidence intervals contained $p = 0.4$.

    2. Answers will vary. Sample response: For $n = 90$, 95% of the confidence intervals contained $p = 0.4$.

    3. Sample response: Yes. According to the 68–95–99.7 rule, you would expect about 95% of the intervals $\left[\hat{p} - 2s, \hat{p} + 2s\right]$ to contain the population proportion.

b.   **1.** Sample response: First calculate the sample proportion $\hat{p} = 160/400 = 0.4$. Then, use $\hat{p}$ to calculate s as follows:

$$s = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{(0.4)(0.6)}{400}} \approx 0.0245$$

The values of $\hat{p}$ and s are used to determine the width of the interval: $\hat{p} - 2s = 0.4 - 2(0.0245) = 0.351$ and $\hat{p} + 2s = 0.4 + 2(0.0245) = 0.449$. Thus, the 95% confidence interval is $[0.351, 0.449]$.

    2. To find the 68% confidence interval, use the values for $\hat{p}$ and s to calculate $\hat{p} - s$ and $\hat{p} + s$. The 68% confidence interval is $[0.3755, 0.4245]$.

    3. To find the 99.7% confidence interval, use the values for $\hat{p}$ and s to calculate $\hat{p} - 3s$ and $\hat{p} + 3s$. The 99.7% confidence interval is $[0.3265, 0.4735]$.

c.   As sample size increases, the width of the confidence interval decreases. This occurs because s becomes smaller as n increases.

d.   **1.** Sample response: If 60 of the fish were diseased, the biologist can conclude with 95% confidence that the newspaper overestimated the percentage of diseased fish.

    2. Sample response: If 100 of the fish were diseased, the biologist can conclude that the newspaper reported a reasonable estimate, although it may be a little high.

Assignment

3.1   **a.** Sample response using day 1: Since $\hat{p} = 6/100 = 0.06$,

$$s = \sqrt{\frac{\hat{p}(1-\hat{p})}{100}} = \sqrt{\frac{(0.06)(0.94)}{100}} = 0.024$$

Therefore, the 95% confidence interval is:
$[0.06 - 2(0.024),\ 0.06 + 2(0.024)]$ or $[0.013, 0.108]$.

Since 0.06 is the sample proportion of trout over 40 cm in length for a random sample of 100 trout, then you can be 95% confident that the actual proportion of trout over 40 cm in length is in the interval $[0.013, 0.108]$.

b. Sample response using day 8: Since $\hat{p} = 14/100 = 0.14$,

$$s = \sqrt{\frac{0.14(0.86)}{100}} = 0.035$$

Therefore, the 95% confidence interval is: $[0.07, 0.21]$.

Since 0.14 is the sample proportion of trout over 40 cm in length for a random sample of 100 trout, then you can be 95% confident that the actual proportion of trout over 40 cm in length is in the interval $[0.07, 0.21]$.

c. Sample response: Although the two intervals overlap, they are quite different. If you determined a confidence interval for each day in the table, you would get a different interval for each different sample proportion. Since there are seven different values for $\hat{p}$, there would be seven different confidence intervals.

d. Sample response: No. According to the 68–95–99.7 rule, you would expect about 95% of all similarly constructed confidence intervals to contain p.

e. Sample response: Since these samples came from the same population, it seems reasonable to combine the data, assuming that the results of one day's samples does not affect the probability of catching fish on other days. By adding the samples together, the total number of fish over 40 cm sampled during the 16 days is 133 out of 1600.

f. In this case, $\hat{p} = 133/1600 = 0.083$ and

$$s = \sqrt{\frac{(0.083)(0.917)}{1600}} = 0.007$$

Therefore, the 95% confidence interval is $[0.069, 0.097]$.

*3.2   **a.**   **1.**   Since $\hat{p} = 7/50 = 0.14$ and $s = 0.049$, the 95% confidence interval is $[0.042, 0.238]$. If 0.14 is the sample proportion of rainbow trout in a random sample of 50 fish, then you can be 95% confident that the actual proportion of rainbow trout is in the interval [0.042, 0.238].

  2.   Since $\hat{p} = 0.14$ and $s = 0.035$, the 95% confidence interval is [0.070, 0.210]. If 0.14 is the sample proportion of rainbow trout in a random sample of 100 fish, then you can be 95% confident that the actual proportion of rainbow trout is in the interval [0.070, 0.210].

  3.   Since $\hat{p} = 0.14$ and $s = 0.017$, the 95% confidence interval is [0.106, 0.174]. If 0.14 is the sample proportion of rainbow trout in a random sample of 400 fish, then you can be 95% confident that the actual proportion of rainbow trout is in the interval [0.106, 0.174].

b.   Sample response: The sample proportions are all 0.14. The 95% confidence intervals are all different. As the sample size increases, the interval becomes narrower.

c.   Sample response: Even though the proportion of rainbow trout in each sample was the same (0.14), the confidence intervals are different. As n increases, the estimate of s decreases. This translates into narrower 95% confidence intervals.

3.3   Sample response: One could conclude that the disease rate is $160/400 = 0.4$. Since this is not within 2 standard deviations of 0.35, it is an unlikely occurrence if the disease rate is 0.35. Thus, you could reject the assumption that the disease rate has not changed since 1994. You can be 95% confident that the assumption is false. However, you can't be 99.7% confident since 0.4 does fall within 3 standard deviations of 0.35.

3.4   **a.** $\hat{p} = 73/100 = 0.73$

b.   Since $\hat{p} = 0.73$:

$$s = \sqrt{\frac{(0.73)(0.27)}{100}} = 0.044$$

c.   Sample response: If 0.73 is the sample proportion of elk calves that survive for at least one year in a random sample of 100, then you can be 95% confident that the actual proportion of surviving calves is in the interval [0.642, 0.818].

d.   In this case, the width of the 95% confidence interval would be half that of the confidence interval described in Part **c**: [0.686, 0.774].

e.   Sample response: For a given level of confidence, a smaller standard deviation will produce a narrower confidence interval.

*3.5   **a.** Sample response: Since $\hat{p} = 6/600 = 0.01$ and $s = 0.004$, you can be 95% confident that the true proportion is in the interval $[0.002, 0.018]$.

        b. Sample response: If you assume the bat population in the Southwest is representative of the North American population, the claim that $p = 0.10$ most likely is not true.

        c. Sample response: The probability for exactly 6 bats is extremely small, since $C(600, 6) \bullet (0.1)^6 \bullet (0.9)^{594} \approx 4 \bullet 10^{-20} \approx 0$. As the probabilities for 5, 4, 3, 2, 1, or 0 rabid bats are even smaller, the probability of 6 or less is nearly 0.

        d. Sample response: In this case, the estimate of 10% was based on bats that were found dead or dying. This is not a random sample of the bat population. (In fact, rabies kills bats after a short period of time. Thus, instead of a random selection, many bats were selected precisely because they had rabies.)

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3.6   **a.** Sample response: The 99.7% confidence interval would be wider, since it is constructed using 3 standard deviations instead of 2.

        b. $[\hat{p} - 3s, \; \hat{p} + 3s]$

        c. Sample response: In some situations, it is critical to be more confident of one's estimates. If a life depended on it, you would want 99.7% confidence instead of 95% confidence. At the same time, the researcher must decide how precise the estimate needs to be. A wide 99.7% confidence interval might be of no use. In such cases, the researcher would have to increase the sample size to create a sufficiently narrow interval.

3.7   **a.** The 95% confidence interval $[0.45, 0.65]$ can be found as follows:

$$0.55 + 2\sqrt{\frac{(0.55)(0.45)}{100}} \approx 0.65$$

$$0.55 - 2\sqrt{\frac{(0.55)(0.45)}{100}} \approx 0.45$$

        b. The 99.7% confidence interval $[0.40, 0.70]$ can be found as shown below:

$$0.55 + 3\sqrt{\frac{(0.55)(0.45)}{100}} \approx 0.70$$

$$0.55 - 3\sqrt{\frac{(0.55)(0.45)}{100}} \approx 0.40$$

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Students should select well-defined populations, such as the members of the senior class, and easily determined characteristics. Sampling techniques should provide reasonably random samples of sufficient size.

Reports should include the number surveyed, the resulting proportion and standard deviation, and at least one confidence interval.

(page 318)

Activity 4

Students explore the capture-recapture method of sampling used to estimate the size of animal populations. They then extend the concept of confidence intervals for a population proportion to confidence intervals for population size.

Materials List

- beans (between 180 and 220 per group)
- shoe box or other container (one per group)
- marking pen (one per group)

Technology

- statistics package

Teacher Note

When sampling is done without replacement (as is usually the case), then the unbiased estimator of the square of the standard deviation of the sampling distribution is:

$$\left(\frac{N-n}{N-1}\right)\left(\frac{\hat{p}(1-\hat{p})}{n-1}\right)$$

where N is the population and n is the sample size. Consequently,

$$s = \sqrt{\left(\frac{N-n}{N-1}\right)}\sqrt{\left(\frac{\hat{p}(1-\hat{p})}{n-1}\right)}$$

When the population is large relative to the sample size, then

$$\sqrt{\left(\dfrac{N-n}{N-1}\right)}$$

is close to 1. However, if $n \approx N/3$ for example, then this factor is approximately 0.8, which narrows the 95% confidence interval significantly. In fact, the interval stated in Parts **f** and **g** are conservative estimates, and students can be more than 68% confident of their predictions.

Exploration (page 318)

a. Each group should receive a container with between 180 and 220 unmarked beans.

b. Students should clearly mark each bean in the sample.

c. Mixing the beans ensures that each bean (marked or unmarked) has an equal chance of being selected in the next sample.

d. Answers will vary. Sample response: $\hat{p} = 12/40 = 0.3$.

e. Using the sample proportion given in Part **d**,

$$\frac{12}{40} \approx \frac{50}{x}$$
$$x \approx 167$$

Therefore, the estimated population is approximately 167.

f. Using the sample proportion given above, the resulting interval is $[0.23, 0.37]$.

g. Sample response: With 68% confidence, you can say that $0.23 \le p \le 0.37$. Therefore,

$$0.23 \le \frac{\text{tagged trout}}{\text{total population}} \le 0.37$$
$$0.23 \le \frac{50}{\text{total}} \le 0.37$$
$$\frac{1}{0.37} \le \frac{\text{total}}{50} \le \frac{1}{0.23}$$
$$135 \le \text{total} \le 217$$

h. Students verify their estimates by counting the number of beans in the population.

Discussion (page 319)

a. Sample response: Some animals may lose their tags or die. Others may be easier or harder to capture, making the sampling process non-random.

b. Sample response: It is likely that the population of trout is either very large or the capturing technique is not random. No estimate of the population size can be made. At this point, you might consider tagging more fish or increasing the number of fish recaptured.

c. Answers will vary, depending on student data.

d. If 100 estimates were made using this method, 68% of the corresponding confidence intervals would be expected to contain the true total.

Assignment (page 319)

4.1 Students should solve for x as shown below:

$$\frac{14}{168} = \frac{350}{x}$$
$$x = 4200$$

4.2 a. Using the proportion shown below, there were approximately 150 wolves in the region.

$$\frac{15}{45} = \frac{50}{x}$$
$$x = 150$$

b. Using the new data, students should estimate the wolf population at approximately 149 wolves.

$$\frac{14}{40} = \frac{52}{x}$$
$$x \approx 149$$

c. Using the results from Parts **a** and **b**, most students would conclude that there has been no significant change in the wolf population. Some may determine the 95% confidence interval for both years for further comparison. The interval for the year before drilling started is $[106, 263]$. The 95% confidence interval for the year after drilling stopped $[104, 260]$.

4.3 Since an estimate for the proportion of tagged fish p is $1/3$, students may conclude that an estimate for the number of fish is 21. However, the 95% confidence interval for p is $[0, 0.718]$. When the sample proportion approaches 0, the upper bound for the confidence interval for population becomes very large. Because of the small sample size, no realistic estimate of the pond population can be given.

<div style="text-align:center">431</div>

*4.4   **a.** Sample response:

$$\frac{N}{1000} = \frac{1000}{120}$$
$$N \approx 8300$$

b. Sample response:

$$\frac{N}{750} = \frac{800}{105}$$
$$N \approx 5700$$

c. Although there appears to be a major decline in the trout population, students should check the confidence intervals before stating any conclusions. The 95% confidence interval for the first sample is [7142, 10,000]. For the second sample, it is [5194, 7547]. Since these intervals overlap slightly, it is reasonable to suppose that the number of fish has not declined.

  Note: Since the number of tagged fish is small in comparison to the total population, the confidence intervals are large. Confidence intervals for population size are sensitive to proportions close to 0. Scientists usually prefer to tag at least 5% of a population before attempting a capture-recapture estimate.

d. Sample response: You could obtain a better estimate by increasing the number of tagged fish.

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4.5   **a.** Since $\hat{p} = 0.024$ and $s = 0.007$, the 68% confidence interval for p is $[0.017, 0.031]$. The corresponding 68% confidence interval for the population size x is $3226 \le x \le 5882$.

b. The 95% confidence interval for p is $[0.010, 0.038]$. The corresponding 95% confidence interval for the population size x is $2632 \le x \le 10,000$.

c. Sample response: It is assumed that both samples are random samples of the population, that the tagged bats mix thoroughly with the others, and that previous capture does not make the bats any easier or harder to recapture.

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Answers to Summary Assessment

Note: You may wish to use the beans from Activity **4** to create populations for the summary assessment. To reduce the number of populations needed, you also may wish to ask half the class to work on Problem **1** while the remaining students work on Problem **2**, then switch.

1. Students should design sampling techniques that produce random samples of sufficient size. Reports should include sample size, the resulting proportion and standard deviation, and at least one confidence interval.

2. Student reports should include the number tagged, the sample size used in the recapture portion, the proportion of those recaptured that were tagged, the standard deviation for this proportion, a confidence interval for the proportion, and a confidence interval for the population size.

1. **a.** To estimate the proportion of rainbow trout in a lake, researchers capture 100 random samples of 10 fish each. The following relative frequency table shows some of the data collected. Use the information given to complete the table.

Sample Proportion of Rainbow Trout	Frequency	Relative Frequency
0/10	0	
1/10	0	
		0.01
	6	
	9	
		0.20
		0.21
	22	
	19	
9/10		
10/10		0.00
Total	100	

b. Determine the mean of the 100 sample proportions.

c. Given the data in this table, what is your best estimate of the actual proportion of rainbow trout in the fish population?

d. Based on the information in this table, what is the probability that a random sample of 10 fish from the lake will contain fewer than 40% rainbow trout?

2. Assume that 50% of a population of bats are male.

a. Construct the sampling distribution of the sample proportions of males for samples of size 6 taken from this population.

b. What is the probability that a sample of 6 bats will contain more than 4 males?

c. What are the mean and standard deviation of this sampling distribution?

d. Draw a histogram of the sampling distribution.

3.   The chemical DDT was a common ingredient in pesticides in the 1950s and 1960s. Because DDT has toxic effects on many birds, fish, and mammals, it has been banned from use in the United States.

   Imagine that you are a researcher testing DDT levels in the bat population of Carlsbad Caverns, New Mexico. In a random sample of 25 bats, you find unsafe levels of DDT in 6 bats. Because the bats must be killed to test their DDT levels, the sampling is done without replacement.

   a. Estimate the mean and standard deviation of the sampling distribution for the proportion of bats with unsafe levels of DDT.

   b. Write a 95% confidence statement for the proportion of bats in the population with unsafe levels of DDT.

   c. Two years ago, the estimated proportion of the bat population with unsafe levels of DDT was 0.10. Do you think that the proportion has changed?

   d. How would increasing the sample size affect your 95% confidence statement?

   e. Why is a larger sample size not always a reasonable option in wildlife studies?

4.   As part of its stocking program, a state fish hatchery releases 5000 tagged trout in a stream. A week later, a random sample of 500 trout from the stream contains 100 tagged fish.

   a. Estimate the number of trout in the stream and describe how you determined your estimate.

   b. Determine a 95% confidence interval for the number of trout in the stream.

Answers to Module Assessment

1.   **a.**   A completed table is shown below.

Sample Proportion of Rainbow Trout	Frequency	Relative Frequency
0.0	0	0.00
0.1	0	0.00
0.2	1	0.01
0.3	6	0.06
0.4	9	0.09
0.5	20	0.20
0.6	21	0.21
0.7	22	0.22
0.8	19	0.19
0.9	2	0.02
1.0	0	0.00
Total	100	1.00

   b.   The mean of the 100 sample proportions is 0.606.

   c.   Sample response: Because the mean of all sample proportions equals the population proportion, the best estimate is 0.606.

   d.   The sum of the probabilities for sample proportions of 0.3, 0.2, 0.1, and 0 is 0.07 or 7%.

2.   **a.**   The sampling distribution is shown in the following table.

Sample Proportion	Probability
0	0.016
1/6	0.094
2/6	0.234
3/6	0.313
4/6	0.234
5/6	0.094
1	0.016

   b.   The probability is $0.094 + 0.016 = 0.11$ or 11%.

   c.   The mean is 0.50; the standard deviation is 0.204.

d. Sample histogram:
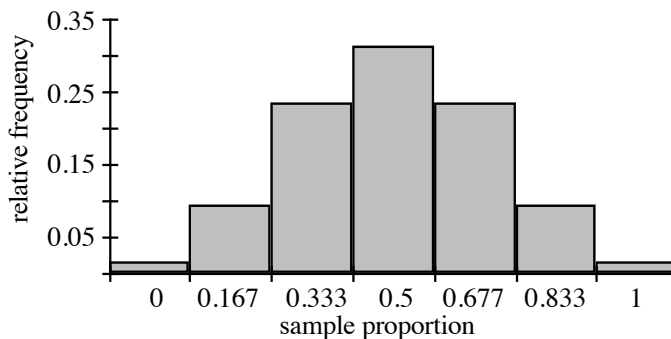
3. **a.** $\hat{p} = 6/25 = 0.24$

$$s = \sqrt{\frac{(0.24)(1 - 0.24)}{25}} \approx 0.085$$

b. Sample response: You can be 95% confident that the proportion of bats with unsafe levels of DDT is between 0.07 and 0.41.

c. Sample response: You cannot say with any certainty that the population proportion has changed because 0.10 is in the interval $[0.07, 0.41]$.

d. Increasing the sample size would reduce the width of the 95% confidence interval and narrow the range in which you would expect to find the true proportion of bats with unsafe levels of DDT.

e. Since the bats must be killed to be tested, and since the tests themselves are often expensive, scientists must decide how important the level of confidence is to the research.

4. **a.** Students should solve for the population size x as follows:

$$\frac{5000}{x} = \frac{100}{500}$$

$$x = 25,000$$

b. Since $\hat{p} = 0.200$ and $s = \sqrt{(0.2)(0.8)/500} = 0.018$, the 95% confidence interval for p is $0.164 \le p \le 0.236$.

Using this interval, the corresponding confidence interval for the population size x can be found as follows:

$$0.164 \le \frac{5000}{x} \le 0.236$$

$$\frac{1}{0.236} \le \frac{x}{5000} \le \frac{1}{0.164}$$

$$21,186 \le x \le 30,487$$

Therefore, you can be 95% confident that the total population is in the interval $[21{,}187, 30{,}487]$.

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Flashbacks

Activity 1

1.1 Consider the following set of data: $\{1, 2, 2, 4, 5, 6, 7, 3, 2, 4, 5, 6\}$.

  a. Find the mean of the data.

  b. Find the standard deviation of the data.

  c. Create a frequency table for the data.

Activity 2

2.1 **a.** Consider an experiment that involves flipping a fair coin three times and recording whether each flip shows heads or tails. List the sample space for this experiment.

  b. What is the probability of obtaining two heads and one tail?

Activity 3

3.1 **a.** Use the following formula to determine the standard deviation of all possible sample proportions for a sample of size 200 and a population proportion of 0.3.

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

  b. What percentage of the sampling proportions would you expect to fall within 1 standard deviation of the population proportion?

Activity 4

4.1 Solve each of the following for x:

  a. $\dfrac{35}{99} = \dfrac{14}{x}$

  b. $0.45 \le \dfrac{14}{x} \le 0.67$

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Answers to Flashbacks

Activity 1

1.1   **a.** The mean of the data is approximately 3.9.

        b. The standard deviation of the data is approximately 1.9.

        c. Sample table:

Number	Frequency
1	1
2	3
3	1
4	2
5	2
6	2
7	1

Activity 2

2.1   **a.** TTT, TTH, THT, HTT, THH, HTH, HHT, HHH

        b. The probability of obtaining 2 heads and 1 tail is $3/8 = 0.375$

Activity 3

3.1   **a.** $\sigma = \sqrt{\dfrac{0.3 \bullet 0.7}{200}} \approx 0.03$

        b. approximately 68%

Activity 4

4.1   **a.** $x = \dfrac{99 \bullet 14}{35} = 39.6$

        b. Sample response:

$$0.45 \leq \frac{14}{x} \leq 0.67$$

$$\frac{1}{0.67} \leq \frac{x}{14} \leq \frac{1}{0.45}$$

$$20.9 \leq x \leq 31.1$$

Catch a Wave

Catchin' a wave and listening to tunes—what a glorious summer day. In this module, you explore the mathematics of waves and musical notes using sine and cosine functions.

Ed Sisolak • Danny Jones

Teacher Edition

Catch a Wave

Overview

Students model periodic events using the sine and cosine functions. Radian measure and the real numbers are used to investigate sine and cosine curves.

Objectives

In this module, students will:

*   pair the real numbers with the points on a unit circle

*   use radian measures for angles

*   graph sine, cosine, and tangent curves

*   identify the amplitude and period of a circular function

*   use the sine and cosine functions to model circular motion and sound waves.

Prerequisites

For this module, students should know:

*   the definition of a circle

*   the trigonometric ratios sine, cosine, and tangent.

Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	3	2	2	2	11

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
large can	X				
scissors	X				
ruler	X				
strips of paper	X				
protractor	X				
bottles of water				X	

Technology

Software	Activity				
	1	**2**	**3**	**4**	**Summary Assessment**
graphing utility	X	X	X	X	X
geometry utility		X			
scientific interface device				X	X
microphone				X	
motion detector (optional)					X

Catch a Wave

Introduction   (page 327)
Students discuss real-world events that occur in cycles.

Discussion   (page 327)

a.   Sample response: The graph would have alternating peaks and valleys between 0 and 40 on the y-axis. The heights of the peaks and the depths of the valleys gradually decrease as time passes. Finally, the graph becomes a horizontal line at $y = 22$. The graph would be similar to the one below.
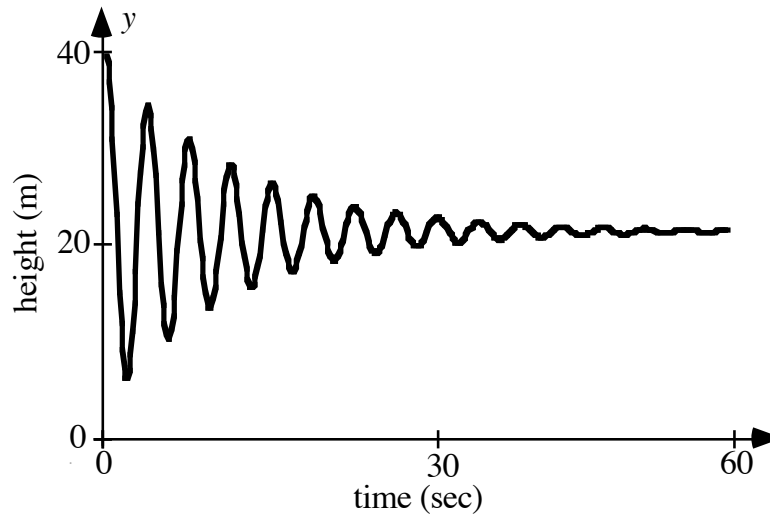
b.   Sample response. The graph would consist of alternating peaks and valleys between the highest and lowest points of the Ferris wheel seats. The peaks and valleys stay the same throughout the graph. The graph would be similar to the one below.
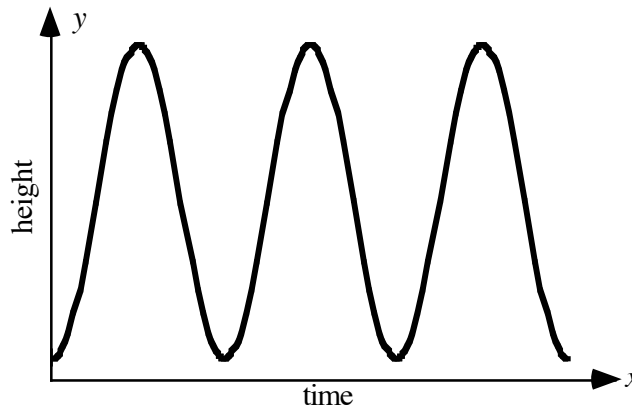
c.   Sample response: In the graph of the bungee jump, the y-values for the peaks and valleys get closer and closer to the height at which the jumper will eventually stop. In the graph for the Ferris wheel, the y-values for the peaks and valleys remain the same.

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Activity 1

Students investigate the relationship between angles measure in radians and arc lengths on unit circles by wrapping a number line around a unit circle.

Materials List

- large can (one per group)
- scissors (one pair per group)
- ruler (one per group)
- strips of paper (one per group)
- protractor (one per group)

Technology

- graphing utility

Exploration   (page 328)

Students are introduced to radian measure with a can-wrapping activity.

a.   Using the radius of a can to define 1 unit, students create a unit circle and a number line.

b.   Students mark the locations of various real numbers on the unit circle, as shown below.
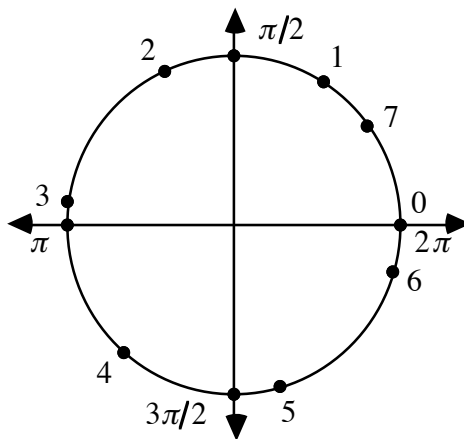
c. Students approximate the number of radii that fit various portions of the unit circle's circumference.

    1. Approximately 6.3 radii fit around the circumference of a circle.

    2. Approximately 4.7 radii fit around three-fourths the circumference of a circle.

    3. Approximately 3.1 radii fit around one-half the circumference of a circle.

    2. Approximately 1.6 radii fit around one-fourth the circumference of a circle.

d. Students approximate the ordered pair that corresponds to each labeled point on the unit circle. They should realize that, as a result of the wrapping process, a real number is paired with an ordered pair of real numbers. Sample table:

Number	x-coordinate	y-coordinate
0	1.0	0.0
1	0.54	0.84
$\pi/2$	0.0	1.0
2	–0.42	0.91
3	–0.99	0.14
π	–1.0	0.0
4	–0.65	–0.76
$3\pi/2$	0.0	–1.0
5	0.28	–0.96
6	0.96	–0.28
2π	1.0	0.0

e. Students construct a central angle with a measure of 1 radian and determine its corresponding measure in degrees (approximately 57°).

Teacher Note

You may wish to point out later in the module that although constant functions can be considered periodic, they have no period.

Discussion
(page 329)

a.   **1.** Students should discover that just over 6 radii will fit around the circumference of the circle.

    2. The circumference of a unit circle is 2π, which is approximately 6.28.

b. **1.** 2π

     2. π

     3. $\pi/2$

c. **1.** Students compare the coordinates of the labeled points on the unit circle. (See sample table given in Part **d** of the exploration.)

     2. Sample response: The negative numbers could be represented by wrapping the number line clockwise around the unit circle.

     3. Sample responses: $5\pi/2$, $9\pi/2$, $-3\pi/2$, $-7\pi/2$.

     4. An infinite number of positive real numbers and an infinite number of negative real numbers can be paired with any point on the unit circle.

d. **1.** 2 units

     2. Sample response: The arc length is equal to the product of the radius of the circle and the radian measure of the central angle.

Assignment (page 331)

1.1 **a.** The circumference of a circle is $2\pi r$, where r is the radius.

     b. The distance around a 12-m wheel is approximately 75.4 m.

1.2 **a.** 4π radians

     b. 6π radians

     c. 16π radians

1.3 48 m

1.4 **a.** Answers may vary. Sample responses are given below.

     1. π

     2. $3\pi/2$

     3. $\pi/2$

     b. **1.** $(-1,0)$

        2. $(0,1)$

        3. $(0,-1)$

        4. $(0,1)$

        5. $(-1,0)$

1.5 **a.** Answers will vary. Sample response: $3 + 2\pi \approx 9.28$.

     b. Sample response: $-2\pi + 3 \approx -3.28$

<div align="center">* * * * *</div>

1.6 **a.** To be spaced evenly, the lights should be $2\pi r/16 \approx 0.4r$ units apart.

   b. The angle formed measures approximately 0.4 radians.

1.7 The proportion of the total area of the circle that is in the sector is $(5\pi/4)/2\pi = 5/8$. The area of the sector is $(5/8)(\pi \cdot 10^2) \approx 196.3 \text{ m}^2$.

$$* * * * * * * * * *$$

Activity 2

In this activity, students discover the relationship between the coordinates of points on a unit circle and the sine and cosine functions.

Materials List

- none

Technology

- geometry utility

Teacher Note

Some geometry utilities may not measure angles greater than π radians.

Exploration

Students use a geometry utility to investigate relationships involving the coordinates of points on a unit circle, real numbers, and the sine and cosine functions.

a. **1.** The sine of the angle is equal to the length of the side opposite the angle divided by the length of the hypotenuse: $\sin t = y/r$.

   2. $y = r \sin t$

   3. The cosine of the angle is equal to the length of the side adjacent the angle divided by the length of the hypotenuse: $\cos t = x/r$.

   4. $x = r \cos t$

b. Students should use centimeters and radians as units of measurement when creating the construction. If a unit circle with a radius of 1.0 cm cannot be obtained, slight adjustments in the precision of the measurements may be needed for students to recognize the outcomes.

c–g. Students should observe that if A is a point on the unit circle, its coordinates are $(\cos \angle DCA, \sin \angle DCA)$.

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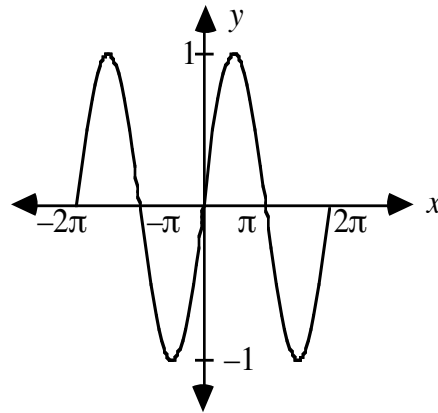
h.   **1.** Sample graph:

2. Sample response: The graph repeats in a pattern where the minimum values are –1 and the maximum values are 1. The pattern repeats twice in the interval graphed.

3. The values of x are approximately $-5.8, -3.7, 0.5,$ and 2.6.

4. $-1 \le \sin x \le 1$

5. The period is 2π.

Discussion

a.   Sample response: Using right-triangle trigonometry, the x-coordinate of A is $\cos\angle BCA$, while the y-coordinate is $\sin\angle BCA$. This relationship exists because the length of the hypotenuse of right triangle ABC is 1 unit. Since $m\angle DCA = m\angle BCA$, the x-coordinate of A is also $\cos\angle DCA$, while the y-coordinate is $\sin\angle DCA$.

b.   Sample response: The x-coordinate of a point A is $\cos\angle DCA$, while the y-coordinate of the point is $\sin\angle DCA$, where C is the origin and D is $(1,0)$.

c.   Sample response: Yes. The ordered pairs $(\cos t, \sin t)$ are close to the approximated ordered pairs from Activity **1**.

d.   Sample response: The sine of a real number is the y-coordinate of a point on the unit circle with center at the origin. The y-coordinates on a unit circle range from –1 to 1, so the range of the sine function is $-1 \le f(x) \le 1$.

e.   Sample response: When you start at $(1,0)$ and move around the circumference of the unit circle in a counterclockwise direction, the y-coordinates increase to 1, then decrease to –1, and then increase to 0. This pattern is illustrated in the graph of the sine function.

f.   There are four solutions over the interval $-2\pi \le x \le 2\pi$. There are an infinite number of solutions over the real numbers.

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g. There are no solutions since 2 is not in the range of values for the sine function.

h. **1.** The number must be paired with a point in the first or fourth quadrants.

 2. The number must be paired with a point in the first or second quadrants.

i. **1.** Sample response: Since $\cos x \neq 0$, $x \neq (\pi/2) + 2k\pi$, where k is an integer.

 2. The number must be paired with a point in the first or third quadrants.

Assignment (page 336)

2.1 **a.** The point paired with 0 is (1,0). Therefore, $\cos 0 = 1$, $\sin 0 = 0$, and $\tan 0 = 0/1 = 0$.

 b. The point paired with $\pi/2$ is (0,1). Therefore, $\cos \pi/2 = 0$, $\sin \pi/2 = 1$, and $\tan \pi/2 = 1/0$, which is undefined.

 c. The point paired with π is (–1,0). Therefore, $\cos \pi = -1$, $\sin \pi = 0$, and $\tan \pi = 0/-1 = 0$.

 d. The point paired with $3\pi/2$ is (0,–1). Therefore, $\cos 3\pi/2 = 0$, $\sin 3\pi/2 = -1$, and $\tan 3\pi/2 = -1/0$, which is undefined.

2.2 Students can use the symmetry of the circle or the sine and cosine functions to complete Parts **a–d**.

 a. The ordered pair (0.5,0.87) corresponds to $\pi/3$, (–0.5,0.87) corresponds to $2\pi/3$, (–0.5,–0.87) corresponds to $4\pi/3$, and (0.5,–0.87) corresponds to $5\pi/3$.

 b. All of the x-coordinates have the same absolute value. All of the y-coordinates have the same absolute value.

 c. The numbers in each of the following pairs correspond to the same point on the unit circle: $\pi/3$ and $7\pi/3$, $2\pi/3$ and $8\pi/3$, $4\pi/3$ and $10\pi/3$, $5\pi/3$ and $11\pi/3$.

 d. Answers may vary. The numbers in each of the following sample pairs correspond to the same point on the unit circle: $\pi/3$ and $-5\pi/3$, $2\pi/3$ and $-4\pi/3$, $4\pi/3$ and $-2\pi/3$, $5\pi/3$ and $-\pi/3$.

2.3 **a.** Sample graph:
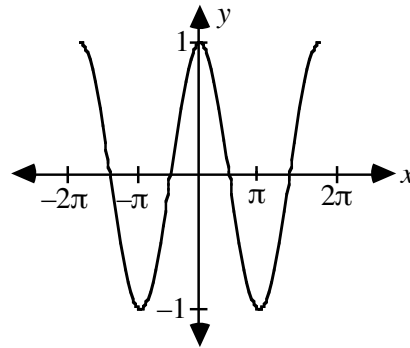
b. Sample response: The graph repeats a pattern where the minimum values are –1 and the maximum values are 1. The pattern repeats twice in the interval graphed.

c. The values of x are approximately $-5.24, -1.05, 1.05$, and 5.24.

d. The period is 2π.

2.4 **a.** Sample graph:
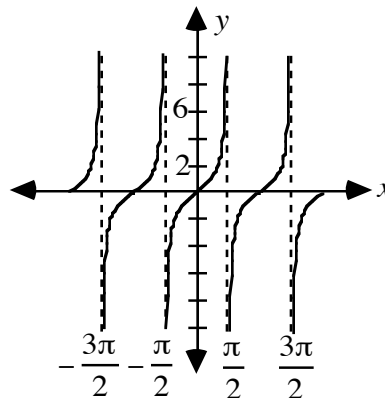
b. Sample response: The graph is a series of congruent curves that are not connected. The curves are π units wide. Breaks in the graph occur where the tangents are not defined because of division by 0.

c. The values are approximately $-4.91, -1.77, 1.37$ and 4.51.

d. The tangent function is periodic, with a period of π.

2.5 **a.** The coordinates of A are $\left(\cos(\pi/3), \sin(\pi/3)\right) \approx (0.5, 0.87)$.

b. Using proportions:

$$\frac{0.5}{1} = \frac{x}{2} \quad \text{and} \quad \frac{0.87}{1} \approx \frac{y}{2}$$
$$x = 1 \qquad\qquad y \approx 1.74$$

c. The coordinates are approximately $(0.5r, 0.87r)$.

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2.6   **a.** $(2\pi/150) \approx 0.04$ radians

   b. $(2.5/2) \cdot 60$ sec $= 75$ sec

   c. The arc length is:

$$\frac{2\pi \cdot 12}{150} \approx 0.50 \text{ m}$$

   d. The coordinates are:

$$\left(12\cos\left(\frac{3\pi}{2} + \frac{2\pi}{150}\right), 12\sin\left(\frac{3\pi}{2} + \frac{2\pi}{150}\right)\right) \approx (0.50, -11.99)$$

* * * * *

2.7   **a.** The coordinates are $(0,-1)$, $(-1,0)$, and $(0,1)$.

   b.   **1.** The coordinates are $(\cos(2\pi/3), \sin(2\pi/3)) \approx (-0.5, 0.87)$.

   2. The hours are $1, 5$, and 7. They have coordinates $(0.5, 0.87)$, $(0.5, -0.87)$, and $(-0.5, -0.87)$, respectively.

   c. The hour hand would be pointing at 9, which has coordinates $(-1,0)$.

2.8   **a.** Sample graph:
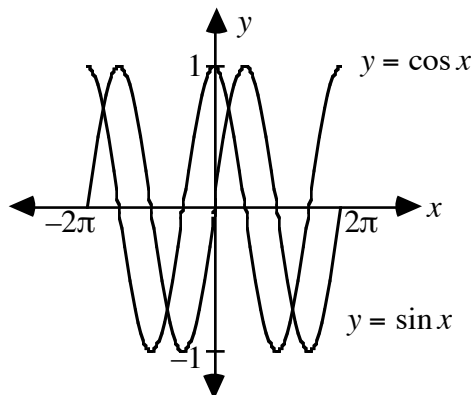
   b. The two functions are equal for the following values of x:

$-7\pi/4 \approx -5.50$, $-3\pi/4 \approx -2.36$, $\pi/4 \approx 0.79$, and $5\pi/4 \approx 3.93$.

2.9   **a.** Sample graph:

   b. The two functions are equal for all values of x.

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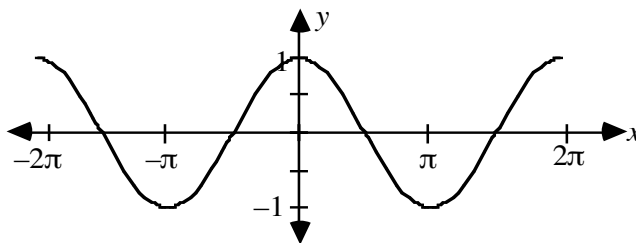
2.10   **a.**   Sample graph:
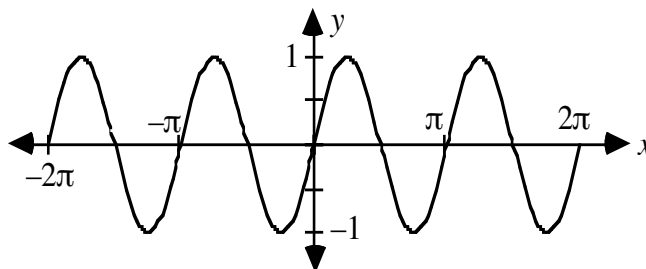
   b.   The two functions are equal for all values of x.

2.11   Sample response: As shown below, the graph is a horizontal line 1 unit above the y-axis. So, $\sin^2 x + \cos^2 x = 1$ for all values of x in the interval.

* * * * * * * * * *
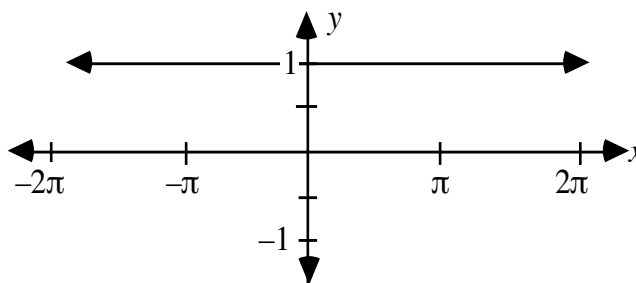
(page 339)

Activity 3

In this activity, students investigate the effects of r and b on the graph of $y = r\sin(bx)$. They are also introduced to the terms *amplitude*, *frequency*, and *cycle*.

Materials List

• none

Technology

• graphing utility

Exploration   (page 339)

Students discover how r and b affect the period, amplitude, and frequency of the graph of $y = r\sin(bx)$.

a. Sample graph:
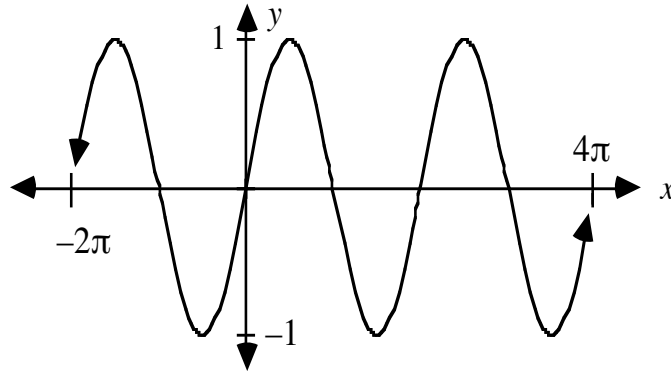
b. For $y = r\sin x$, the maximum is $|r|$ and the minimum is $-|r|$.

c. The frequency is $1/(2\pi) \approx 0.16$ cycles/sec or 0.16 Hz.

d. Sample table:

Function	Period	Frequency
$y = \sin x$	2π	$1/(2\pi)$
$y = \sin(2x)$	π	$1/\pi$
$y = \sin(4x)$	$\pi/2$	$2/\pi$
$y = \sin(\pi x)$	2	$1/2$
$y = \sin(x/3)$	6π	$1/(6\pi)$
$y = \sin(bx)$	$2\pi/b$	$b/(2\pi)$
$y = \sin(2\pi x)$	1	1
$y = \cos x$	2π	$1/(2\pi)$
$y = \tan x$	π	$1/\pi$

Discussion

(page 341)

a. Sample response: The graph of $y = 8\sin x$ has an amplitude 8 times that of $y = \sin x$. This means that the maximum value is 8 and the minimum value is –8. The periods of the two graphs (and therefore their frequencies) are the same.

b. Sample response: The value of b is the number of cycles within an interval of width 2π. In other words, the period is $2\pi/b$.

c. 1. Sample response: $y = \cos(50x)$.

2. Sample response: $y = \sin(24x)$.

d. 1. Sample response: $y = 4\sin(2\pi x)$.

2. Sample response: $y = 0.5\cos(10 \bullet 2\pi x)$.

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Assignment

3.1. **a.** Sample response: $y = 8\sin x$.

b. Sample response: $y = 0.4\cos x$.

3.2 **a.** Sample response: $y = \cos(15 \bullet 2\pi x)$.

b. Sample response: $y = \cos(25 \bullet 2\pi x)$.

c. Sample response: $y = \cos(\pi x)$.

*3.3 **a.** $y = \cos(2\pi x/15)$

b. $y = \cos(2\pi x/25)$

3.4 Students may use sine or cosine functions to describe the sound waves.

a. Sample responses:

  1. $y = \sin(20 \bullet 2\pi x)$

  2. $y = \sin(20{,}000 \bullet 2\pi x)$

b. Sample response: The graphs would have the same general shape, but with frequencies greater than 20 Hz and less than 20,000 Hz.

3.5 One possible function is $y = 7\sin(2\pi x/6)$. A sample graph is given below:
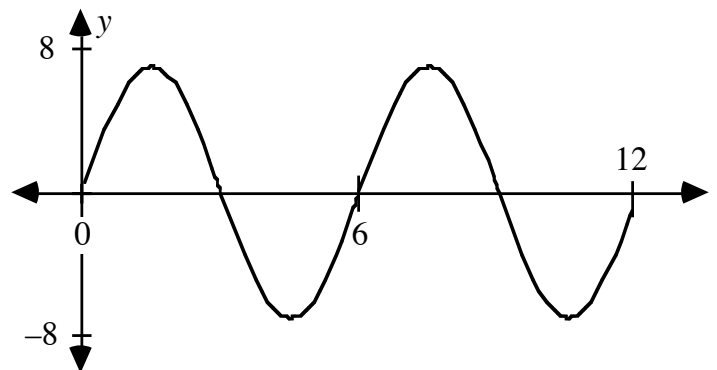
3.6 **a.** 20 cm

b. 5 sec

c. 0.2 Hz

d. Sample response: $y = 20\sin(0.2 \bullet 2\pi x)$.

* * * * *

3.7   **a.** Sample response: $y = 10\cos(2\pi x/0.8)$.

b. See sample graph given in Part **d**.

c. **1.** After 0.1 sec, the weight is approximately 7.07 cm above its starting position.

2. After 0.5 sec, the weight is approximately 7.07 cm below its starting position.

3. After 0.95 sec, the weight is approximately 9.24 cm above its starting position.

4. After 1.75 sec, the weight is approximately 9.24 cm above its starting position.
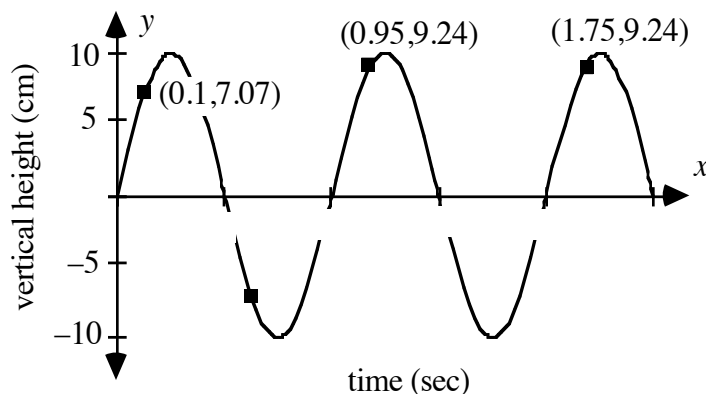
d. Sample graph:

e. Sample response: The weight will be 10 cm below its starting point at 0.6 sec and every 0.8 sec after that.

3.8   Sample response: The value of k in the equation $y = \cos(x + k)$ appears to determine the amount of horizontal translation from the graph of $y = \cos x$. If k is negative, the graph is shifted k units to the right. If k is positive, then the graph is shifted k units to the left.

* * * * * * * * * *

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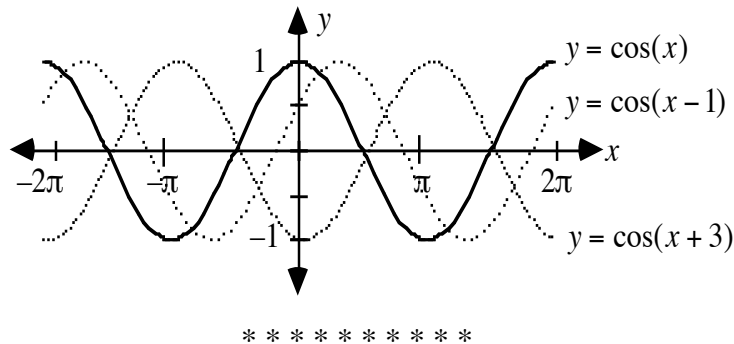
Activity 4

In this activity, students use the tools developed in the previous three activities to model sound with a sine function.

Materials List

- bottles of water (although many types of bottles will work, plastic soda bottles are recommended)

Technology

- graphing utility
- scientific interface device
- microphone

Teacher Note

Students will need some means of recording sound waves. If using Texas Instruments' CBL and calculators, students should use a program described in the CBL manual to collect the sound data. Your band instructor or science teacher may have a tuner that can identify notes.

Students should place the microphone alongside the lip of the bottle as they blow to produce the note, but not directly in the path of the stream of air. Students may not be able to sustain a single note and may produce a graph similar to the one below. In such cases, they should select an interval that contains a reasonable representation of a single note's sound wave. From this single cycle, students should be able to determine a sine or cosine function to model the note.
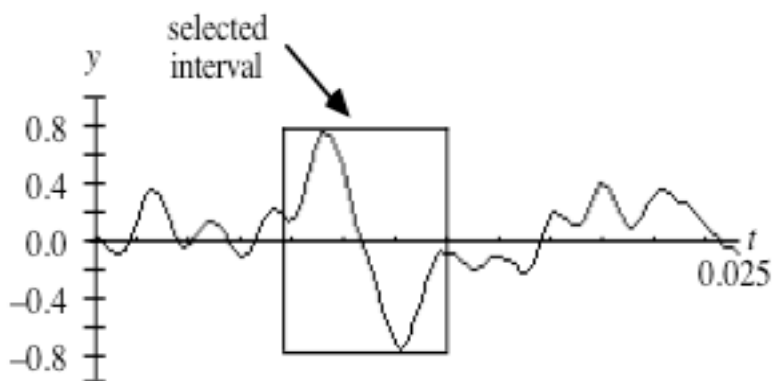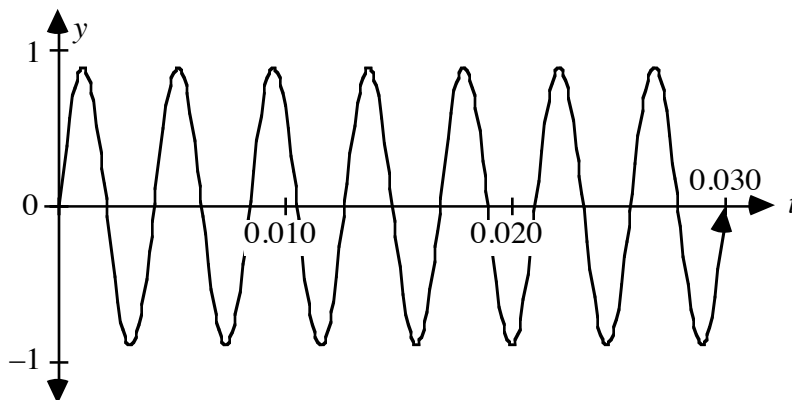
With practice, students may be able to generate graphs similar to the one below. However, it may still be necessary to select a single cycle to determine a function to model the note.
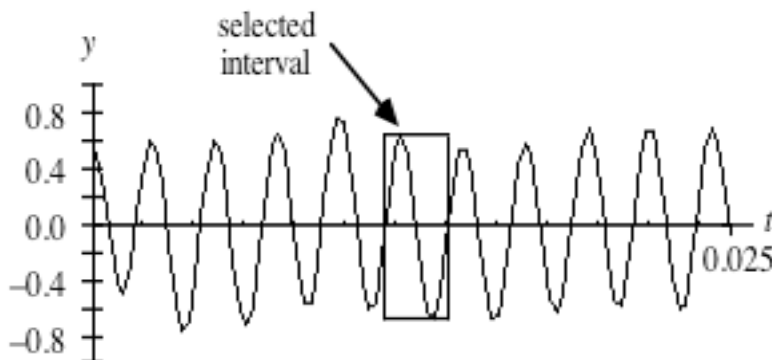
Exploration

a–c. Answers will vary. Sample graph:

d. For the sample graph shown above, the period is approximately 0.0042, the amplitude is 0.89, and the frequency is 238 Hz.

e. One equation for the sample graph shown above is $y = 0.89\sin(238 \bullet 2\pi x)$.

f. Students should obtain similar results with different frequencies (and possibly different amplitudes).

Note: For a given frequency, the intensity of sound is proportional to the amplitude. In general, sound waves of higher intensity are louder. However, the human ear is not equally sensitive to all frequencies.

Discussion

a. Sample response: To find the period, use the trace feature of the graphing utility to approximate the length of 1 cycle of the graph. To find the frequency, divide 1 by the period. To find the amplitude, use the trace feature to approximate the distance between the maximum

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and minimum *y*-values of the curve, then divide that distance by 2.

b.   Answers will vary. For the sample graph given in Part **c** of the exploration, the frequency of 238 Hz is closest to C (262 Hz) in Table **4**. **Note:** Since the table is not complete, values outside the range of the table may occur. The frequency of 238 Hz is actually closer to a B note (247 Hz).

c.   Answers will vary. All functions should be in the form $y = r\sin(bx)$, where *r* is the amplitude and $b/(2\pi)$ is the frequency. **Note:** If a horizontal shift is required, the equation should take the form $y = r\sin(b(x - k))$, where *k* is the required translation.

d.   Sample response: The frequencies are different because the notes are different. Each note varies because of the different water levels in the bottles. The amplitudes also are different.

e.   Sample response: A higher note is produced by adding more water to the bottle.

Assignment   (page 345)

*4.1   a. $1/0.00340 \approx 294$ Hz

        b.   Sample response: $y = 0.005\sin(294 \bullet 2\pi x)$.

        c.   Using Table **4**, the note with a frequency of 294 is D.

4.2   The frequency of middle C is one-half the frequency of C an octave higher.

4.3   Sample response: Any equation of the form $y = r\sin(370 \bullet 2\pi x)$ could model F#.

*4.4   a.   Sample response: The amplitude of the graph is 0.006 and the period is 0.0015 sec. This graph has a frequency of 660 Hz. It represents the note E.

        b.   Sample response: The amplitude of the graph is 0.004 and the period is 0.003 sec. This graph has a frequency of 360 Hz. It also represents the note E, but an octave lower than the graph in Part **a**.

<p align="center">* * * * *</p>

4.5   **a.** The amplitude is 5 amp. The period is $1/60$ sec. The frequency is 60 Hz.

  b. Sample response: $y = 5\cos(60 \bullet 2\pi x)$.

  c. **1.** approximately –4.05 amp

  2. 5 amp

  d. **1.** The current is 0 amperes at $x/240$ sec, where x is an odd positive integer.

  2. The current is –5 amperes at $x/120$ sec, where x is an odd positive integer.

$$* \ * \ * \ * \ * \ * \ * \ * \ * \ *$$

Research Project   (page 347)

The graph of a chord is the sum of the graphs of the individual notes. In the chord CEG, for example, the frequency of E is approximately $5/4$ that of C, while the frequency of G is approximately $3/2$ that of C. This means that the period for E is $4/5$ that of C and the period for G is $2/3$ that of C. As a result, the period of the graph of the chord, the sum of the graphs of the individual notes, is 4 times that of the graph of C.

Teacher Note

You may wish to allow students to use a scientific interface device to collect their own data in Problem **2**.

Answers to Summary Assessment (page 348)

1. **a.** Sample table:

Time (sec)	Height (m)	Time (sec)	Height (m)
0.00	0.00	33.75	5.66
3.75	5.66	37.50	8.00
7.50	8.00	41.25	5.66
11.25	5.66	45.00	0.00
15.00	0.00	48.75	–5.66
18.75	–5.66	52.50	–8.00
22.50	–8.00	56.25	–5.66
26.25	–5.66	60.00	0.00
30.00	0.00		

b. Sample graph:
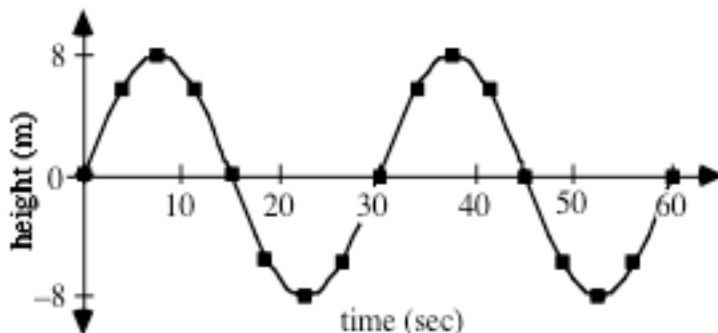
c. $y = 8\sin\left(\dfrac{1}{30}\,2\pi x\right) = 8\sin\left(\dfrac{\pi x}{15}\right)$

d. The amplitude is 8 m, the period is 30 sec, and the frequency is $1/30$ revolutions per second or 2 revolutions per minute.

2. **a.** Sample response: The same amount of string must be rolled and unrolled each time, and the delay when the yo-yo is either in your hand or at the end of the string must be consistent. The speed at which you toss the yo-yo also must be the same each time, or must vary in a regular pattern.

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b. Sample response:

$$y = 40\cos\left(\frac{1}{1.2} \cdot 2\pi x\right) = 40\cos\left(\frac{\pi x}{0.6}\right) \approx 40\cos(5.24x)$$

Sample graph:
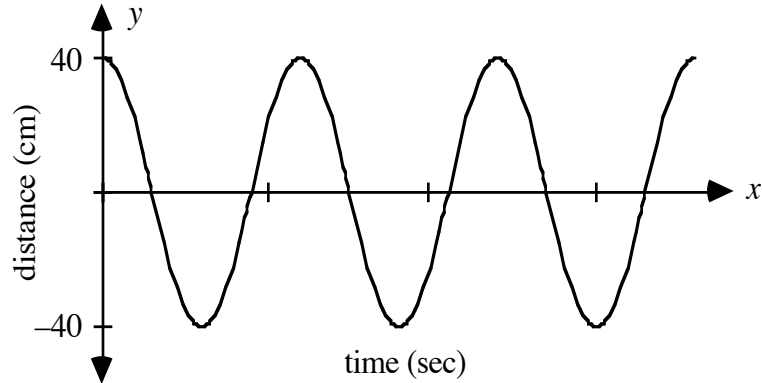
c. The amplitude is 40 cm, the period is 1.2 sec, and the frequency is 1/1.2 ≈ 0.83 cycles/sec.

d. 1. At 1.5 sec, the yo-yo is at the same height as the table top.

    2. At 1.9 sec, the yo-yo is approximately 35 cm below the table top.

    3. At 3.4 sec, the yo-yo is approximately 20 cm above the table top.

1. Ocean tides reoccur over time. On the west coast of the United States, there are generally two high tides and two low tides every day.

    a. Create a connected scatterplot for the data in the following table.

Date	Time	High Tide (m)	Low Tide (m)
February 1	1:20 A.M.	2.2	
February 1	6:55 A.M.		0.5
February 1	1:02 P.M.	2.4	
February 1	7:25 P.M.		–0.1
February 2	1:59 A.M.	2.3	
February 2	7:42 A.M.		0.5
February 2	1:48 P.M.	2.2	
February 2	8:01 P.M.		0.0

    b. Describe a function you could use to model this data.

2. If an arc of a circle has a measure of 4 radians, what is its length?

3. Identify the frequency, period, and amplitude of each of the following functions:

    a. $y = 5\sin(6x)$

    b. $y = 0.25\sin(6\pi x)$

    c. $y = 16\cos(600\pi x)$

4. Determine the frequency, period, and amplitude of each of the following graphs. Then write an equation that could be used to model each curve.
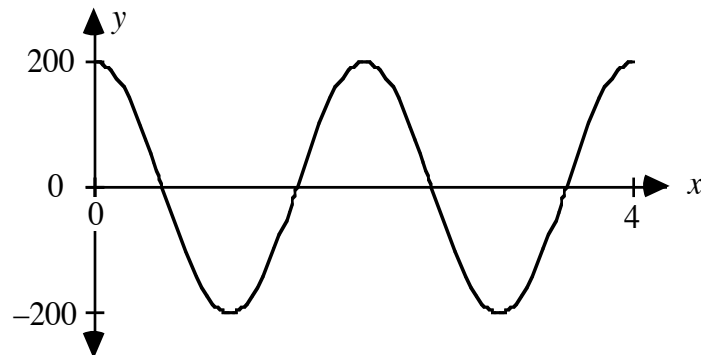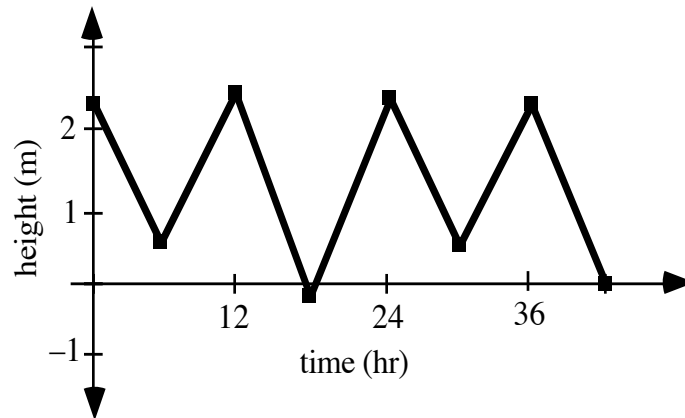
a.
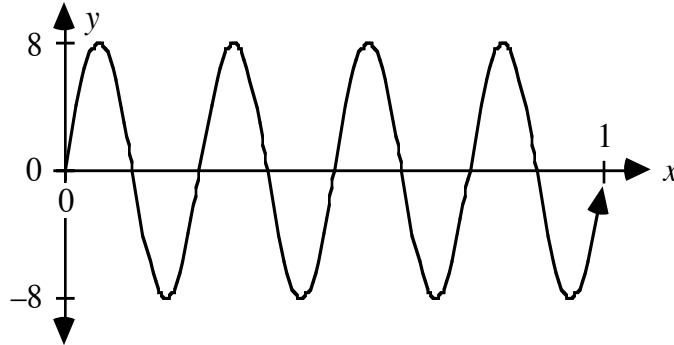
b.

c.
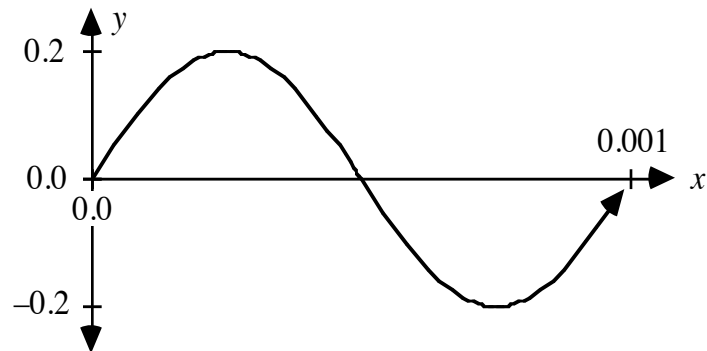
Answers to Module Assessment

1.   **a.** Sample graph:

      b. Answers may vary. Sample response: The water level could be roughly approximated by a cosine function with an amplitude of approximately 1.25 m and a period of approximately 12 hr.

2.   The length of the intercepted arc is $4r$, where r is the radius of the circle.

3.   **a.** The function $y = 5\sin(6x)$ has a frequency of $3/\pi$, a period of $\pi/3$, and an amplitude of 5.

      b. The function $y = 0.25\sin(6\pi x)$ has a frequency of 3, a period of $1/3$, and an amplitude of 0.25.

      c. The function $y = 16\cos(600\pi x)$ has frequency of 300, a period of $1/300$, and an amplitude of 16.

4.   **a.** The graph has a period of 0.25, a frequency of 4, and an amplitude of 8. One possible equation is $y = 8\sin(4 \bullet 2\pi x)$.

      b. The graph has a period of 0.001, a frequency of 1000, and an amplitude of 0.2. One possible equation is $y = 0.2\sin(1000 \bullet 2\pi x)$.

      c. The graph has a period of 2, a frequency of 0.5, and an amplitude of 200. One possible equation is $y = 200\cos(0.5 \bullet 2\pi x)$.
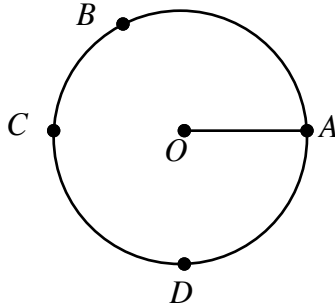
Selected References

Gross, E. E., Jr., and A. P. G. Peterson. *Handbook of Noise Measurement*. Concord, MA: General Radio Company, 1972.

Texas Instruments. *CBL™ System Experiment Workbook*. Dallas, TX: Texas Instruments Inc., 1994.

Flashbacks

Activity 1

1.1   The diagram below shows a circle with center at point O and a radius OA.

    a.   Estimate the measures of central angles AOB, AOC and AOD.

    b.   Determine the circumference of the circle if $OA = 5$ cm .

1.2   Describe the location of the image of point A in Flashback **1.1** under a counterclockwise rotation of $480°$ about O.

Activity 2
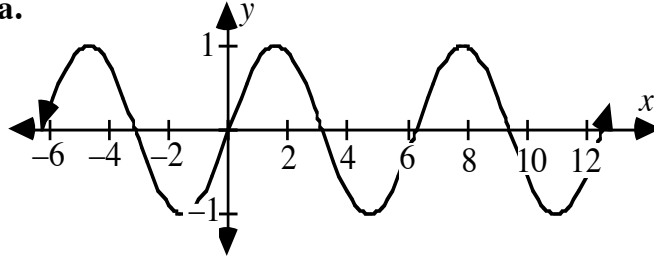
2.1   Consider the following right triangle ABC.
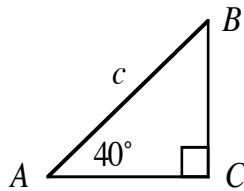
    Write an expression to represent each of the following lengths:

    a. AC

    b. BC

2.2   Describe the image of point $A(3,2)$ under each of the following transformations:

    a.   a reflection in the x-axis
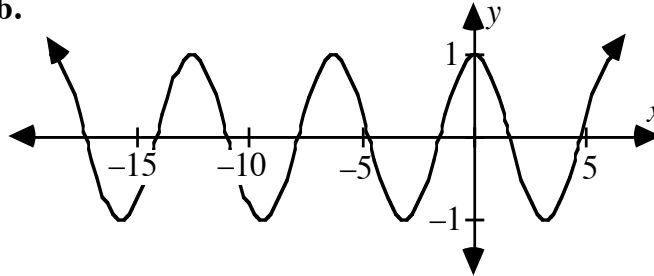
    b.   a reflection in the y-axis.

Activity 3

3.1 Identify each of the graphs below as $y = \sin x$ or $y = \cos x$ and describe its domain and range.
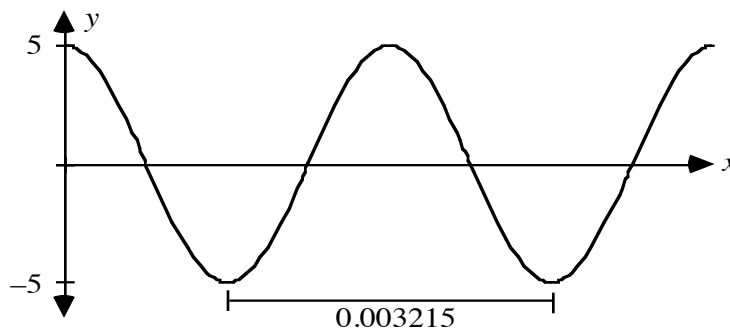
a.

b.

3.2 If a Ferris wheel completes one revolution every 3 min, through how many radians does the wheel rotate in:

  a. 45 sec?

  b. 4 min?

Activity 4

4.1 Determine the amplitude and period of the graph shown below.

0.003215

4.2 Determine the amplitude and period of each of the following functions:

  a. $y = \sin(4x)$

  b. $y = 3\cos(4 \bullet 2\pi x)$

  c. $y = 0.5\sin(100\pi x)$

Answers to Flashbacks

Activity 1

1.1   **a.** $m\angle AOB \approx 120°$; $m\angle AOC \approx 180°$; $m\angle AOD \approx 90°$

        b. $2\pi(5) \approx 31.4$ cm

1.2   The image A' is located at B.

Activity 2

2.1   **a.** $AC = c \bullet \cos(40°) \approx 0.77c$

        b. $BC = c \bullet \sin(40°) \approx 0.64c$

2.2   **a.** $(3,-2)$

        b. $(-3,2)$

Activity 3

3.1   **a.** This is the sine function, $y = \sin x$. Its domain is $-\infty \le x \le \infty$; its range is $-1 \le y \le 1$.

        b. This is the cosine function, $y = \cos x$. Its domain is $-\infty \le x \le \infty$, its range is $-1 \le y \le 1$.

3.2   **a.** $2\pi \bullet \dfrac{45}{3 \bullet 60} = \dfrac{\pi}{2}$ radians

        b. $2\pi \bullet \dfrac{4}{3} = \dfrac{8}{3}\pi$ radians

Activity 2

4.1   The amplitude is 5 and the period is 0.003215.

4.2   **a.** The amplitude is 1 and the period is $\pi/2$.

        b. The amplitude is 3 and the period is 0.25.

        c. The amplitude is 0.5 and the period is 0.02.