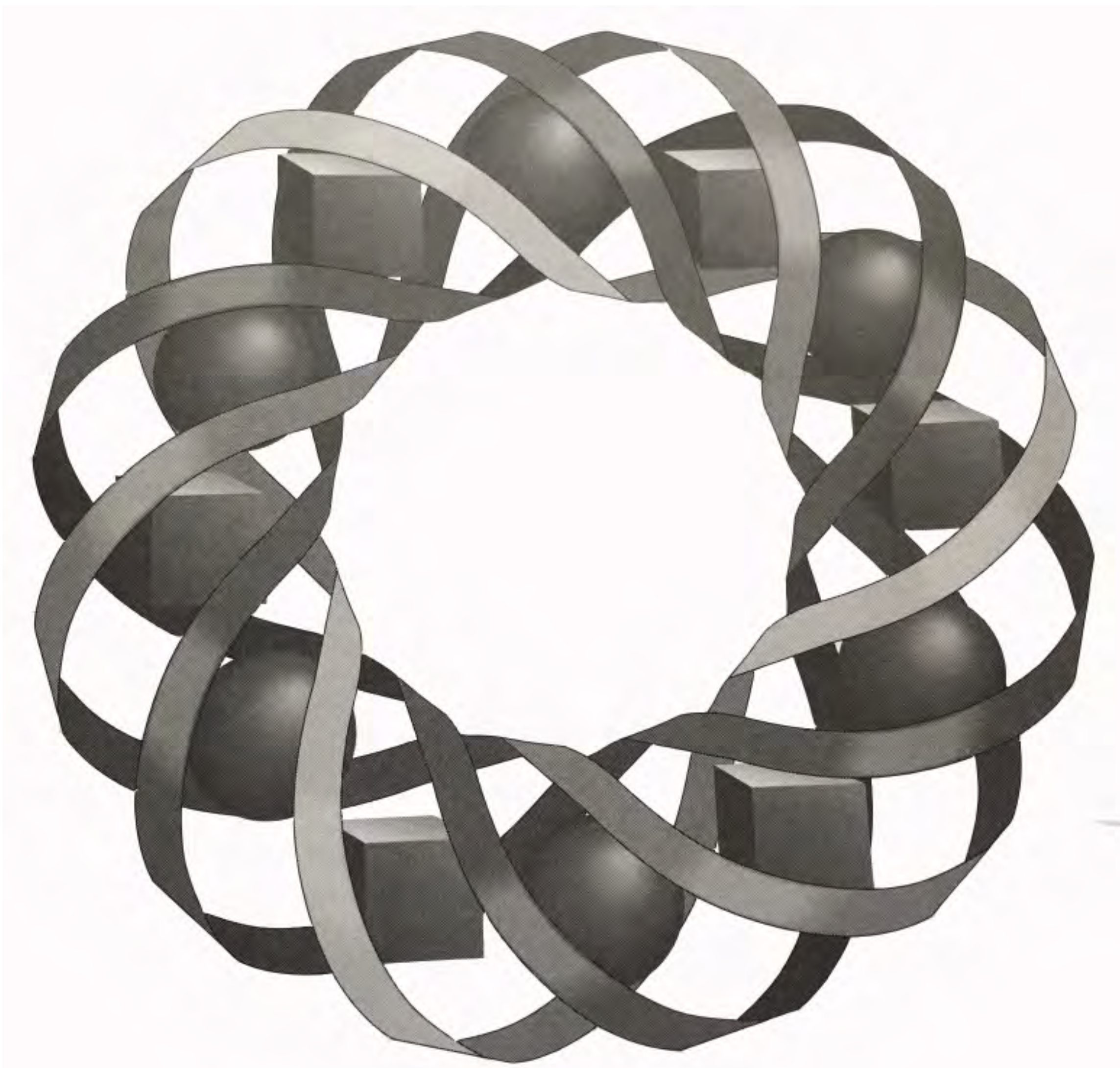
SIMMS Integrated Mathematics:

A Modeling Approach Using Technology



Level 6 Volumes 1-3

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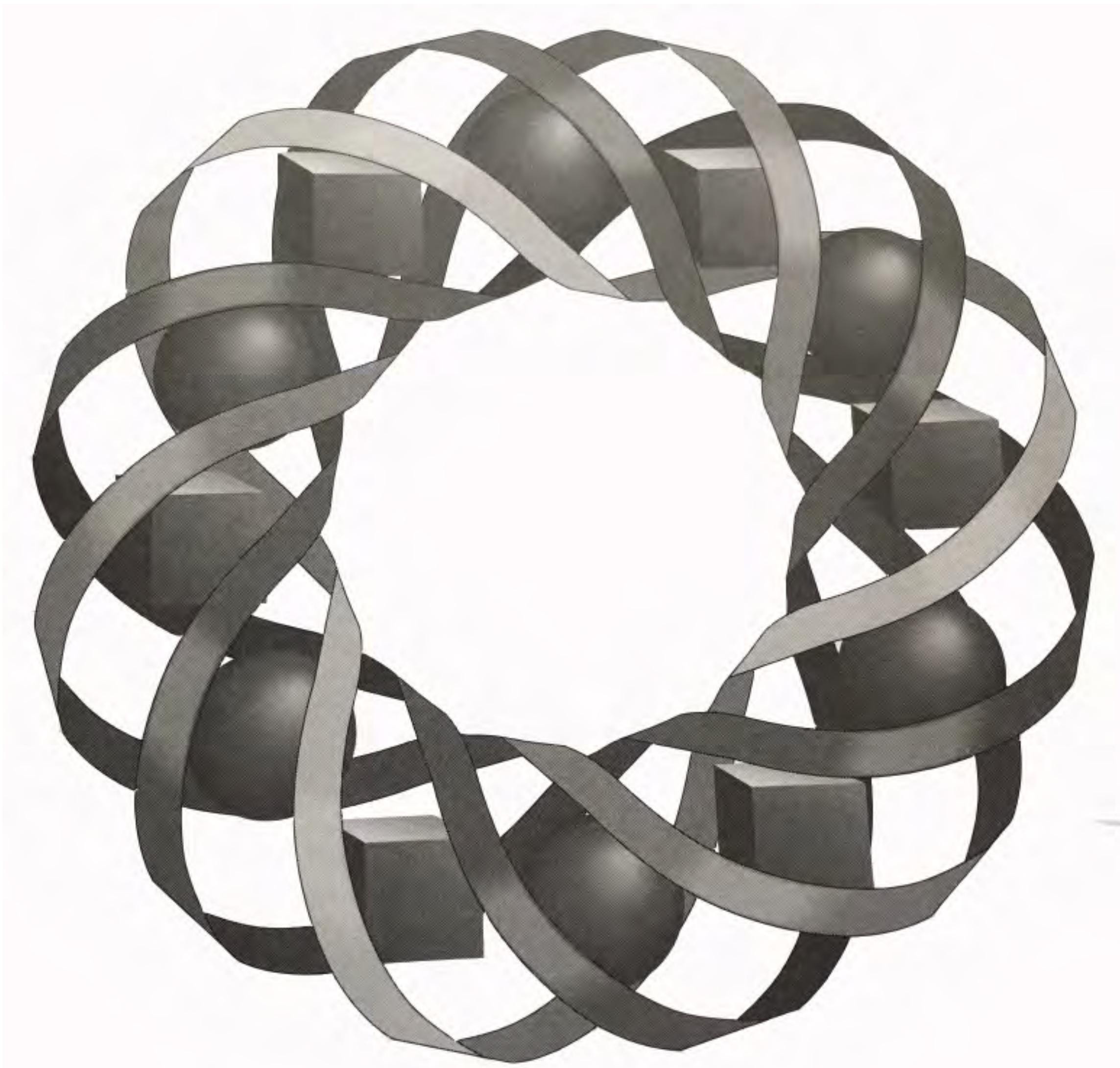
What Shape Is Your World?



Where in the world are you? Where in the world are you going? When answering either question, it may help to use a two-dimensional projection of the earth's three-dimensional surface.

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What Shape is Your World?

Introduction

Cartography, the science of mapmaking, dates back at least to the time of the ancient Greeks. Humans are travelers, and travelers have always needed directions. Historically, most maps have been drawn on a flat surface, despite the approximately spherical shape of the earth. Over short distances, this projection of a three-dimensional surface onto two dimensions is relatively accurate. However, when mapping large regions covering thousands of square kilometers, some distortion is inevitable.

For example, each map of Greenland shown below was created using a different mapping technique. The map in Figure **1a** was made using a **stereographic projection**, a projection whose center is at one of the earth's poles. The map in Figure **1b** was made using a cylindrical projection, a projection with its center at the earth's center.

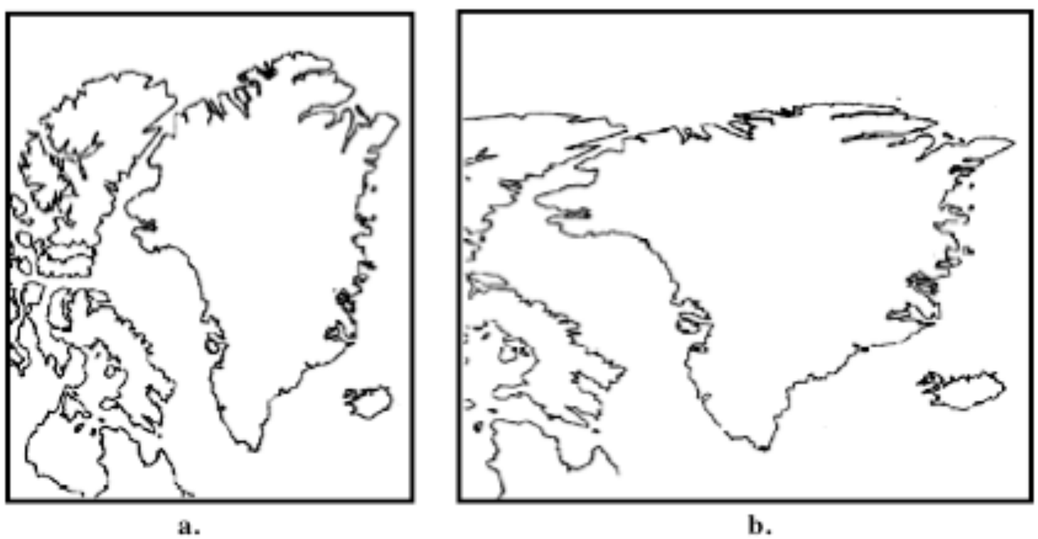


Figure 1: Two maps of Greenland made using different projections

While distortions can make flat paper maps somewhat misleading, such maps have been quite useful in the past, and will certainly remain so in the future. In the following activities, you investigate some of the inconsistencies and distortions that may result from representing a three-dimensional surface on a flat sheet of paper.

Exploration

a. Obtain two sheets of centimeter graph paper from your teacher. Also get the templates of the two maps in Figure **1**. Use each of the templates to make an estimate of Greenland's area, in square kilometers.

b.
1. Compare your two estimates with those of your classmates.
2. Use the class estimates to find a mean value for Greenland's area on each template.

c. Determine which template of Greenland more closely resembles the shape and proportional size of Greenland on a globe.

Discussion

a. What differences did you observe in the two maps of Greenland?

b.
1. The actual area of Greenland is about $2{,}175{,}600 \text{ km}^2$. How does this compare to the two values you obtained in Part **b** of the exploration?
2. What might account for any differences between your estimates and the actual area?

c. How do you think that geographers determined the actual area of Greenland?

d. Despite the distortions in flat maps of Greenland, what useful information may still be obtained from them?

e. Do you think a map of Greenland on a globe is more accurate than the maps in Figure **1**? Explain your response.

Activity 1

One method of giving directions from a starting place to a destination involves specifying a distance east or west, followed by a distance north or south. In other words, the destination is described as a pair of distances along perpendicular lines beginning from a point of origin. This task can be accomplished using a rectangular (Cartesian) coordinate system.

Another method of giving directions involves specifying the distance "as the crow flies" from the starting point to the destination as well as an angle measured from a fixed ray. This task can be accomplished using a **polar coordinate system**. In this activity, you use polar coordinate systems to identify locations on different types of maps.

Mathematics Note

A **polar coordinate system** describes the location of a point P in a plane using an ordered pair consisting of a **radius** r and a **polar angle** θ.

The plane containing a polar coordinate system is the **polar plane**. The polar angle is an angle measured from a fixed ray, called the **polar axis**. The endpoint of the polar axis is the **pole**. The distance from the pole to point P, measured in the polar plane, is r.

To establish a polar coordinate system, a point in the plane is designated as the pole O. Any ray with endpoint O can be designated as the polar axis for the plane. In a standard polar coordinate system, the pole corresponds to the origin of a rectangular coordinate system, while the polar axis corresponds to the positive x-axis, as shown in Figure **2**.

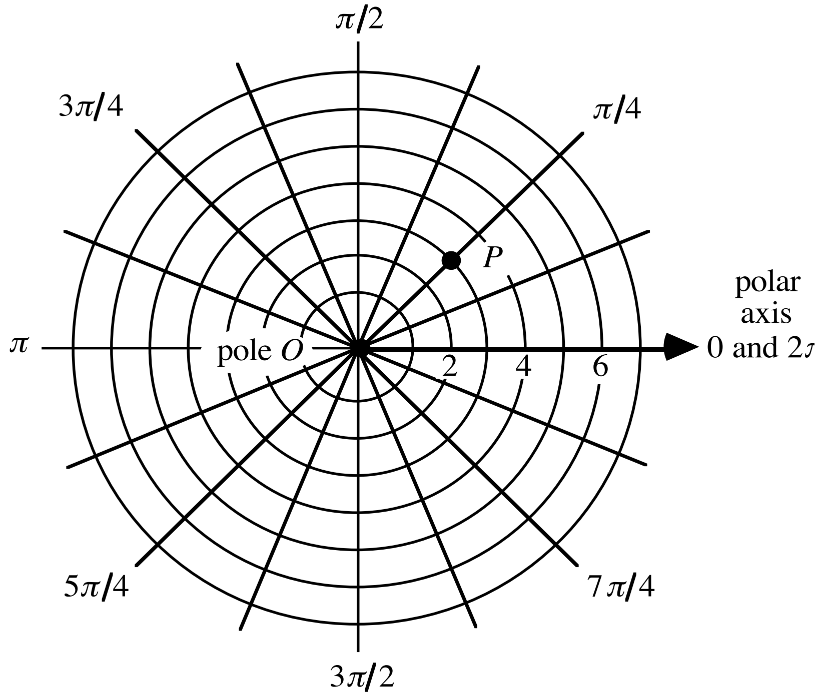


Figure 2: A polar coordinate system

In this system, any point P in the plane may be represented as an ordered pair (r, θ). The variable r represents the distance between O and P, while θ represents the directed measure of the angle formed by the polar axis and $\overrightarrow{OP}$. The measure of the polar angle may be given in either radians or degrees.

In Figure **2**, for example, the coordinates of point P may be given as $(3, 45°)$ using degrees or as $(3, \pi/4) \approx (3, 0.79)$ using radians.

Exploration 1

Figure **3** shows a map of a popular hiking area. In this exploration, you use a polar coordinate system to describe some of the destinations on the map.

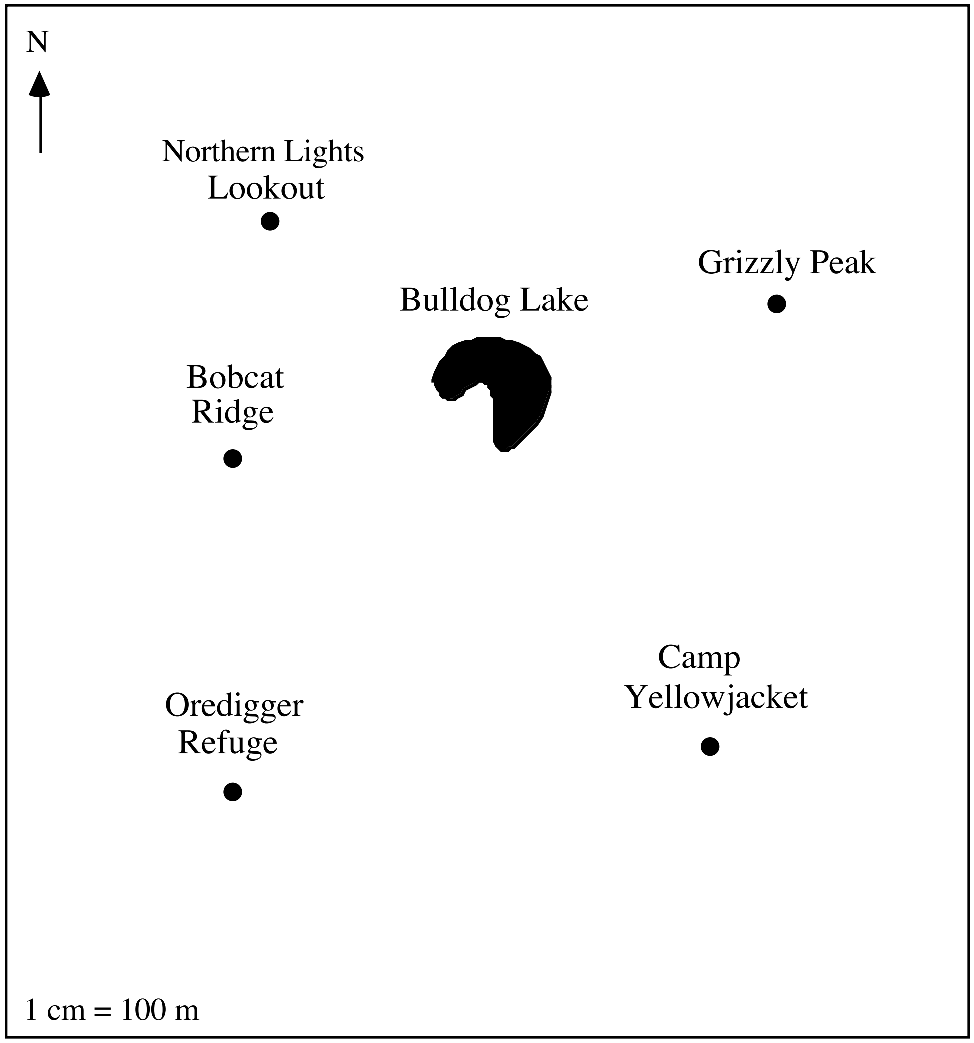


Figure 3: A hiking area

a. Using a copy of Figure **3**, create a polar coordinate system with the pole at Oredigger Refuge and the polar axis extending due east.

b. Use polar coordinates to describe the locations of Grizzly Peak and Camp Yellowjacket.

c. **1.** A polar angle describes an amount of rotation about the pole. This allows the coordinates of a single point to be represented by more than one positive value of θ.

Find two different polar representations for the locations of Grizzly Peak and Camp Yellowjacket, using positive values of θ different from those in Part **b**.

2. Since the polar angle is a directed angle, θ also can have negative values. (A positive value of θ represents an angle measured counterclockwise from the polar axis, a negative value of θ represents an angle measured clockwise from the polar axis.)

Find two different polar representations for the locations of Grizzly Peak and Camp Yellowjacket using negative values of θ.

d. Many hikers use Northern Lights Lookout as a base camp for day trips in the region. To help these hikers plan their trips, create another polar coordinate system with the pole located at Northern Lights Lookout and the polar axis extending due west.

Repeat Parts **b** and **c** using this polar coordinate system.

e. Repeat Part **d** on a polar coordinate system with the pole at Northern Lights Lookout and the polar axis extending due east.

f. Compare the polar coordinates you found for Grizzly Peak and Camp Yellowjacket in Parts **b–e** with those of your classmates.

Discussion 1

a. Describe how to locate the point (r, θ) on a polar coordinate system when θ is positive.

b. How many different ordered pairs of the form (r, θ) can be used to represent a given point on a specific polar coordinate system? Explain your response.

c. **1.** How do the locations of the pole and the polar axis affect the coordinates of a point?

2. Why might it be desirable for all users of a map to agree on the same locations of the pole and polar axis?

Exploration 2

Because of the dramatic changes in elevation between destinations, many hikers purchase a topographic map of the region you examined in Exploration **1**. A topographic map is a two-dimensional representation of a three-dimensional surface. On such maps, **contour lines** are used to depict points of equal elevation. In the map in Figure **4**, for example, each labeled contour line indicates points at the corresponding elevation (in meters) above sea level.

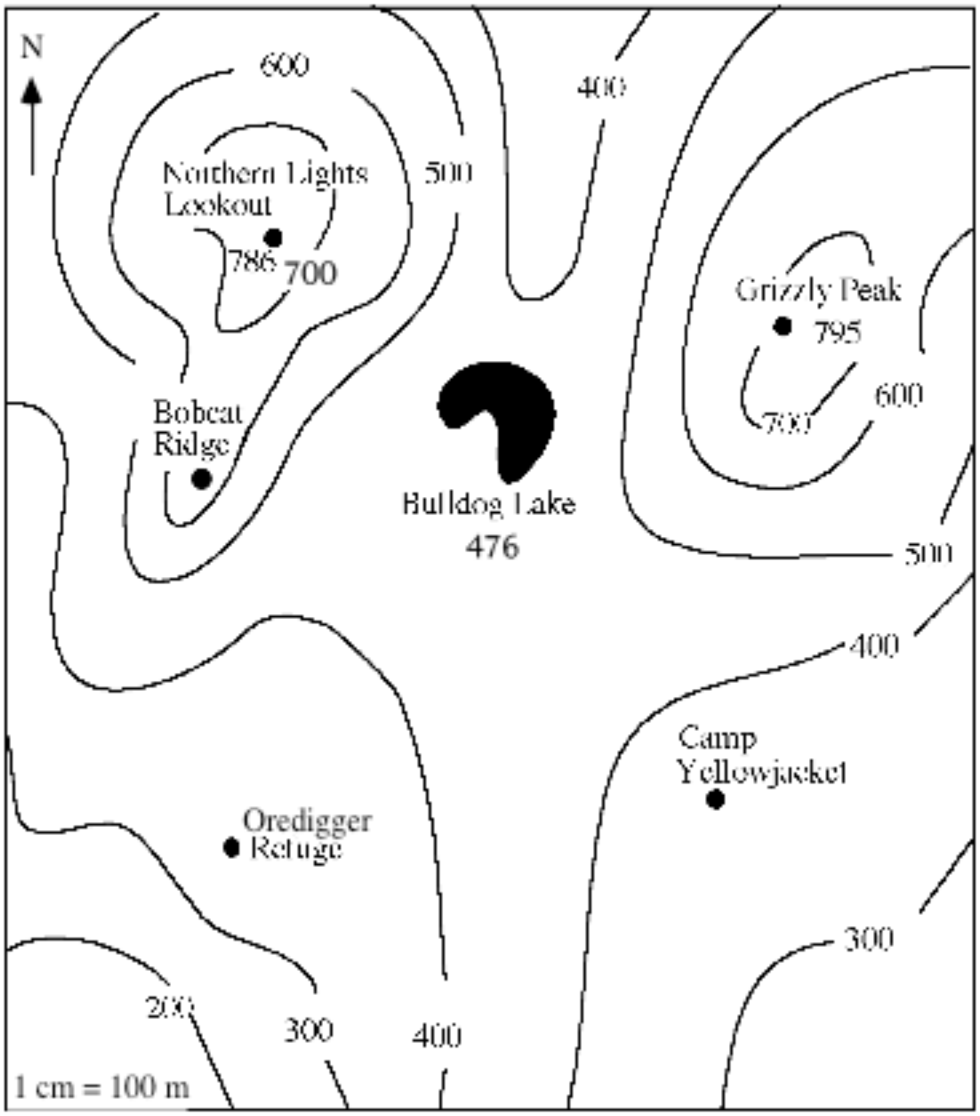


Figure 4: Topographic map of a hiking area

Mathematics Note

Like a rectangular coordinate system, a polar coordinate system can be extended to three dimensions by adding a third dimension z.

In a **cylindrical coordinate system**, a point in space is represented by an ordered triple of the form (r, θ, z). The values of r and θ are measurements in the polar plane. The value of z is the directed distance between the point and the polar plane (the plane containing the polar axis). A positive value for z represents a distance above the polar plane. When a cylindrical coordinate system is used to describe locations on earth, the polar plane is typically located at sea level.

For example, Figure **5** shows the locations of three points on a cylindrical coordinate system. The coordinates of point A are $(5, \pi/3, 1)$, the coordinates of point B are $(4, 5\pi/6, -3)$, and the coordinates of point C are $(2, -\pi/6, 2)$.

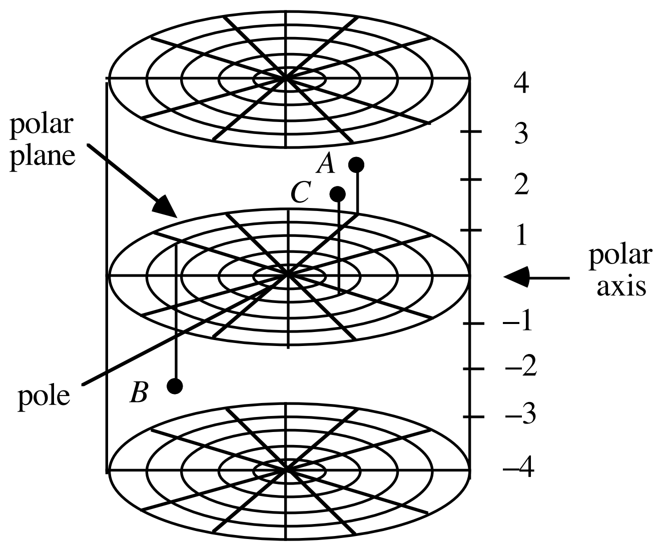


Figure 5: A cylindrical coordinate system

a. On a copy of Figure **4**, create a cylindrical coordinate system with the pole located at the southwest corner of the map, the polar axis extending due east, and the polar plane at sea level.

b. Using your coordinate system from Part **a**, determine cylindrical coordinates for Oredigger Refuge in which:

1. r, θ, and z are all positive
2. r is positive, θ is negative, and z is positive.

Discussion 2

a. How can hikers use the contour lines on a topographic map to help them determine the character of the terrain?

b. Considering the map in Figure **4**, which point do you think would provide the best location for the pole of a cylindrical coordinate system?

c. In a cylindrical coordinate system, describe the geometric figure formed by all points that have the same value for each of the following:

1. r
2. θ
3. z

d. In Part **b** of Exploration **2**, you expressed the location of Oredigger Refuge using both positive and negative values for θ. Are there any points on a cylindrical coordinate system that could not be represented using only positive coordinates? Explain your response.

Assignment

1.1 Re-express the coordinates for each point in Parts **a** and **b** using only positive values.

a. $C\left(7, -\pi/2\right)$

b. $D\left(5, -19\pi/6\right)$

1.2 **a.** Graph and label each of the following points on the same cylindrical coordinate system.

1. $A\left(5, \pi/4, 4\right)$
2. $B\left(5, 7\pi/4, -2\right)$
3. $C\left(3, \pi/4, 3\right)$
4. $D\left(2, \pi/2, -2\right)$
5. $E\left(5, 3\pi/4, -4\right)$
6. $F\left(1, \pi/4, 1\right)$
7. $G\left(4, 3\pi/4, 0\right)$

b. Which of the points in Part **a** lie on the same cylinder centered about the z-axis? Explain your response.

c. Which two points in Part **a** lie in a plane parallel to the polar plane? Explain your response.

d. Which three points in Part **a** determine a plane that contains the pole and is perpendicular to the polar plane? Explain your response.

1.3 Obtain a copy of the map in Figure **4**. Locate the pole of a cylindrical coordinate system at Northern Lights Lookout with the polar axis extending due east. Using this system, find cylindrical coordinates to describe the locations of Grizzly Peak and Camp Yellowjacket. Describe the process you used in each case.

1.4 Imagine that you are camped at Bobcat Ridge on the map in Figure **4**.

a. Using your camp as the pole and a polar axis extending due east, find polar coordinates for each of the other landmarks on the map: Northern Lights Lookout, Grizzly Peak, Camp Yellowjacket, Oredigger Refuge, and the center of Bulldog Lake.

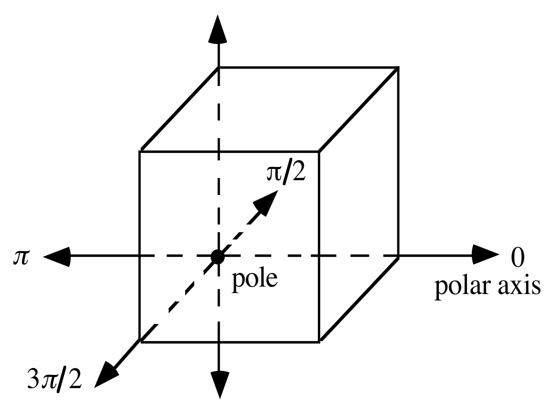
b. Assume Bobcat Ridge is 625 m above sea level. Find cylindrical coordinates for each of the five landmarks named in Part **a**.

c. There are four other points of interest near your camp: Argonaut Alley, Bear Crossing, Saint's Cave, and Devil's Den. Using a copy of Figure **4** and the following polar coordinates, find and label each of these points: Argonaut Alley $(250, \pi/3)$, Bear Crossing $(400, 5\pi/3)$, Saint's Cave $(150, 7\pi/6)$, and Devil's Den $(125, 3\pi/4)$.

d. Write the approximate cylindrical coordinates for each of the four points of interest named in Part **c**.

* * * * *

1.5 The diagram below shows a cube with a volume of s^3. The lower left-hand vertex of the cube's rear face is located at the pole of a cylindrical coordinate system.



a. Determine cylindrical coordinates for each vertex of the cube.

b. Determine cylindrical coordinates for the center of the cube.

1.6 Suppose that the pole in Problem **1.5** were moved to the cube's center.

a. Describe how this affects the r-coordinates of the cube's vertices.

b. Describe how this affects the z-coordinates of the cube's vertices.

* * * * * * * * * *

Activity 2

There are many methods for creating a flat map of a spherical surface—all of which involve mapping points with coordinates in three dimensions to points with coordinates in two dimensions. As you saw in the introduction to this module, flat maps of three-dimensional surfaces may contain distortions in shape, area, and distance. The distortions produced by a particular type of projection depend on several factors. In this activity, you investigate the distortion caused by a stereographic projection of points on a sphere to a flat map.

Mathematics Note

A **stereographic projection** is a projection of the points on a sphere onto a plane perpendicular to a given diameter of the sphere. The plane is the **plane of projection**. The endpoints of the diameter are the **poles** of the sphere.

The image of a point on the sphere is the point of intersection of a ray and the plane perpendicular to the diameter that contains the poles. The ray contains one of the poles, designated as the **center of projection**, and the point being projected. In any stereographic projection, there are points that have no image in the plane.

In Figure **6**, for example, point C' is a stereographic projection of point C, where point N is the center of projection and the plane of projection is perpendicular to the diameter $\overline{NS}$ at point S. In this projection, point N has no image on the plane, while the image of point S is itself.

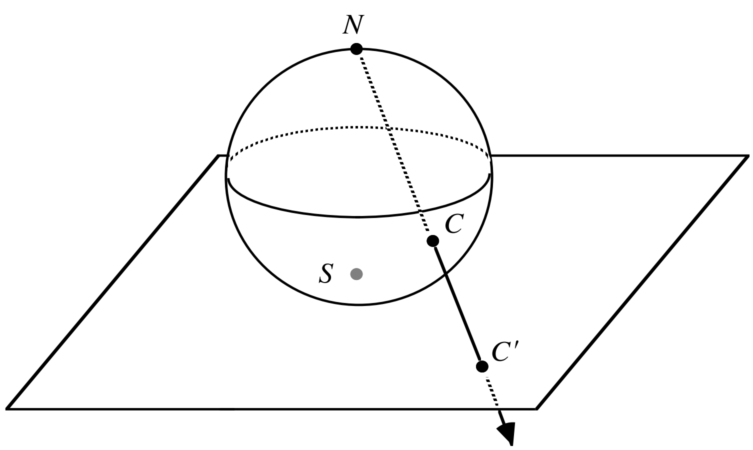


Figure 6: Stereographic projection of point *C*

Exploration 1

Because a sphere is a closed surface and a plane is not, you should expect to observe some differences between a figure on a sphere and its projected image on a plane. In this exploration, you examine how a stereographic projection affects the image of a line on a sphere.

a. Label the two points at opposite ends of a diameter of a sphere *N* and *S*. Let these points represent the north and south poles, respectively. In this exploration, *N* will serve as the center of projection.

b. On a sphere, lines are defined as great circles. The great circles that contain the poles are lines of longitude. Complete the steps below to represent the projection of a line of longitude.

1. Stretch a piece of string from *N* to *S* to represent part of a line of longitude. Mark the string where it touches these two points.
2. Remove the string from the sphere. Mark three other points on the string so that the length determined in Step **1** is divided into four equal parts.
3. Stretch the string from *N* to *S* again. Mark the three points from Step **2** on the sphere. Label these points *A*, *B*, and *C*.

c. As shown in Figure **7**, position the sphere so that a large sheet of paper is tangent to it at *S*. Insert three guides into the sphere and mark the points where the tips of the guides touch the paper. These marks will serve as reference points to maintain the same position of the sphere throughout the exploration.

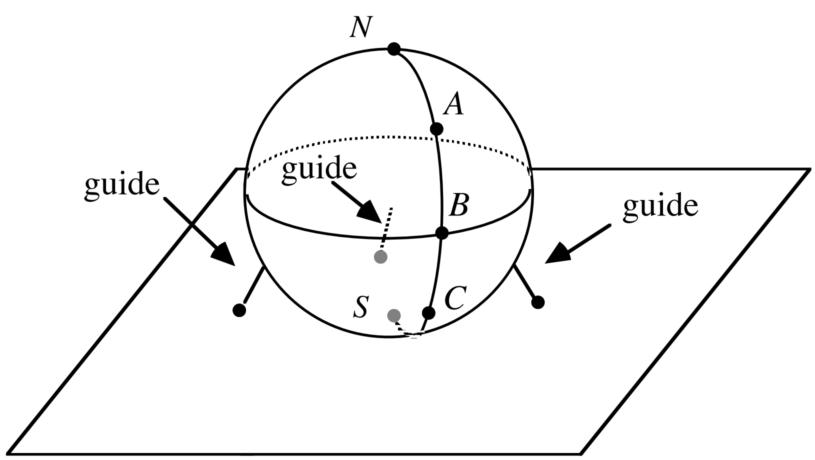


Figure 7: Sphere with points *A*, *B*, and *C* marked

d. To find C', the image of point *C* under a stereographic projection, use a skewer to model a ray. Carefully pass a skewer through the sphere from point *N* through point *C* until the tip of the skewer touches the paper. Mark the point of intersection of the skewer with the paper.

e. Repeat Part **d** for points *A*, *B*, and *S*.

f. Measure $\overline{C'S'}$, $\overline{B'C'}$, and $\overline{A'B'}$.

g. Since the equator is a great circle, it also represents a line on a sphere. Use the steps below to project an image of the equator on the same sheet of paper used above.

1. Stretch a piece of string around the sphere to represent the equator.
2. Remove the string from the sphere. Mark five points on the string so that the distance along the equator is divided into six equal parts.
3. Stretch the string around the equator again. Mark the five points and the point where the endpoints of the string meet. Label these points D, E, F, G, H, and I.
4. Use skewers to determine the image of each point under a stereographic projection where N is the center of projection and the plane is tangent to the sphere at S. **Note:** Save your work for use in the assignment.

Discussion 1

a. In Part **e** of Exploration **1**, what is the relationship between the south pole S and its image S'?

b. Consider a flat map of a globe created using a stereographic projection like the one in Exploration **1**.

1. Describe how the images of the lines of longitude and the equator would appear on the flat map.
2. What appears to be the relationship between the size of the equator and the size of its image?
3. Why must the image of a line of longitude be a line in the plane?

c.

1. Recall that in a one-to-one correspondence, each element in the domain is paired with exactly one element in the range, and each element in the range is paired with exactly one element in the domain.

   Does the mapping of the points of a sphere to a plane as described in Exploration **1** represent a one-to-one correspondence? Defend your answer.
2. A stereographic projection can be considered a function. Explain why this is true.

d. **1.** Are lines preserved under a stereographic projection? In other words, are the projected images of lines on a sphere also lines in the plane? Explain your response. (Remember that a line on a sphere is defined as a great circle.)

2. Is collinearity preserved under a stereographic projection? In other words, if the preimage points lie on the same great circle, do the image points lie on the same line in the plane? Explain your response.

e. **1.** On a sphere, lines of longitude are perpendicular to the equator. Is perpendicularity preserved under a stereographic projection? Justify your response.

2. Two lines of longitude can be perpendicular to each other at the poles. Would their images under a stereographic projection also be perpendicular? Explain your response.

f. Consider three points on a sphere—A, B, and C—where A and B are equidistant from C. Under a stereographic projection, would A' and B' be equidistant from C'? In other words, is distance preserved in a stereographic projection? Justify your response.

g. **1.** Considering your results in Exploration **1**, which regions on a sphere appear to be most distorted in a stereographic projection?

2. What does this imply about a map of Greenland created using this type of projection?

3. Which regions on a sphere appear to be least distorted in a stereographic projection?

4. In general, where would you place the point of tangency of the plane to obtain the least distortion of a preimage?

h. Consider a stereographic projection in which the globe's south pole is the center of projection and the plane is located tangent to the globe at the north pole. How would the amount of distortion in the image of Iceland compare with the amount of distortion in the image of Florida?

i. Describe the map created by a stereographic projection in which the globe's north pole is the center of projection and the plane contains the equator.

j. On a sphere, there are an infinite number of lines (great circles) that do not contain the poles. Describe the images of these lines under a stereographic projection like the one in Exploration **1**.

Exploration 2

In Exploration **1**, you discovered that the distance between points on a sphere generally is not preserved when those points are projected onto a plane. In this exploration, you create a mathematical model of a stereographic projection and use it to investigate these distortions.

a. Figure **8** shows a cross section of a sphere, where $\overline{NS}$ is a diameter of the sphere, and $\overrightarrow{NP}$ intersects the sphere in the plane of the cross section.

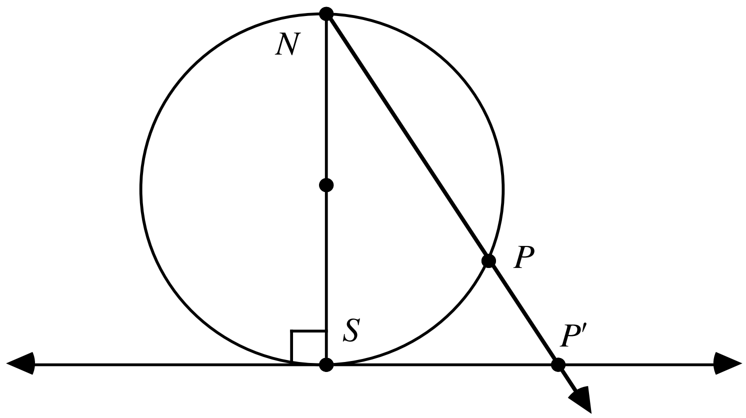


Figure 8: Cross section of a stereographic projection

Reproduce this diagram using a geometry utility. Anchor $\overrightarrow{NP}$ at N, allowing point P to move around the circle. Make sure that the position of point P' changes as P moves around the circle.

b. Describe what happens to P' as you move P around the circle.

c. Construct the three points that correspond with A, B, and C in Part **b** of Exploration **1**. Make sure that $\overset{\frown}{NA} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC} \cong \overset{\frown}{CS}$.

d.
1. Move point P so that it is concurrent with point C.
2. Record the length of $\overline{P'S}$.
3. Repeat Steps **1** and **2** for points A, B, S, and N.

e. Construct $\overline{PZ}$ perpendicular to $\overline{NS}$ as shown in Figure **9**. Make sure $\overline{PZ}$ changes length as point P moves around the circle.

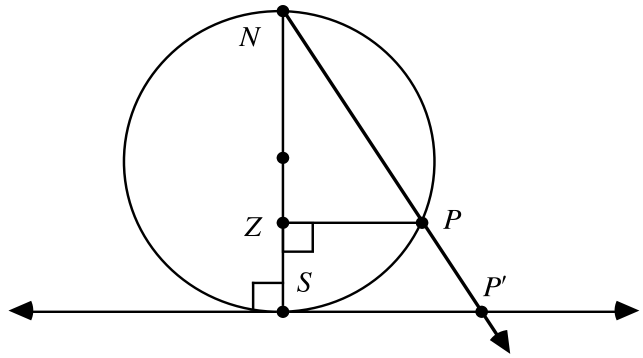


Figure 9: Cross section with $\overline{PZ}$

f. Express the length of $\overline{P'S}$ in terms of the lengths of $\overline{NZ}$, $\overline{NS}$, and $\overline{PZ}$.

Discussion 2

a. What do the results found in Part **d** of Exploration **2** tell you about distances on a map made using a stereographic projection?

b. Describe what happens to the length of $\overline{P'S}$ as the length of $\overline{SZ}$ gets close to the length of $\overline{NS}$.

c. Triangles NPZ and $NP'S$ in Figure **9** are similar triangles. Explain how you know this is true.

d. In Exploration **1**, you observed that the image of the equator under a stereographic projection with the plane tangent to the sphere at the south pole appeared to be twice the diameter of the preimage. Use similar triangles to explain why this is true.

e. The image of a point P on a sphere under a stereographic projection like the one in Exploration **2** can be described using polar coordinates.

Describe how the distance from point S to point P' can be used to help find the polar coordinates of P', where the pole of the graph is at S and the polar axis is opposite of $\overrightarrow{SP'}$.

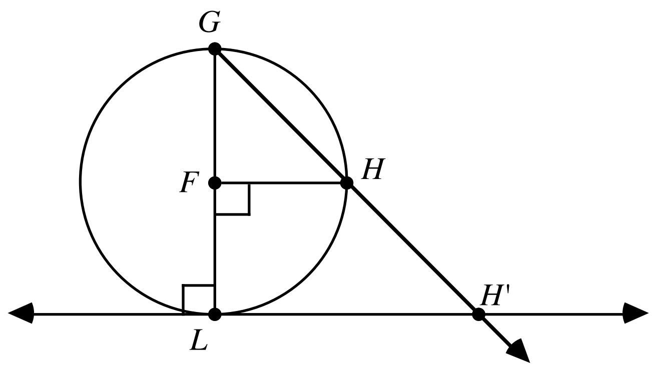
Assignment

2.1 In Exploration **1**, you found that distance and collinearity are not preserved in a stereographic projection. Given this fact, do you think that a figure's perimeter would be preserved? Describe how your response affects the use of flat maps to compare boundaries of countries and continents.

2.2 Virtually every map includes a scale to help users find the distance from one point to another. Describe the dangers in using this scale to determine precise distances.

2.3 Lines of latitude on a globe are not considered lines on a sphere. Using a stereographic projection like the ones in the explorations, what do the images of lines of latitude look like?

2.4 An angle whose vertex is a point on a circle and whose sides contain chords of a circle is an **inscribed angle**. In the diagram below, for example, $\angle LGH$ is an inscribed angle.

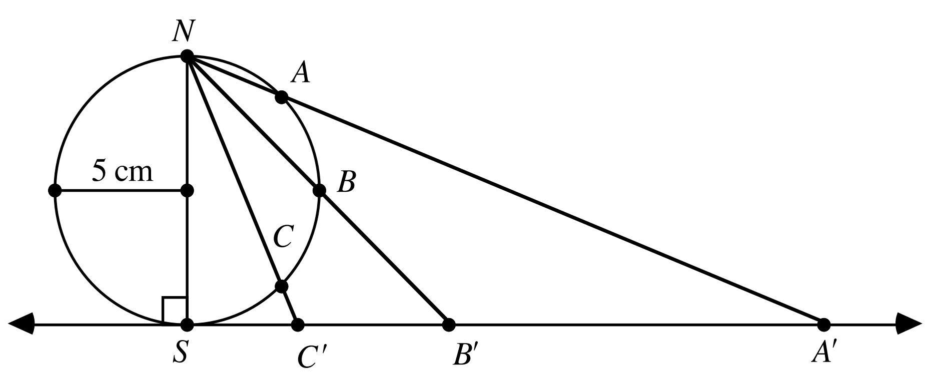


a. Given that circle F in the diagram above has a radius of 3 cm, determine the length of $\overline{LH'}$.

b. Determine the relationship between the measure of $\angle LGH$ and its intercepted arc HL.

c. The relationship you found in Part **b** is true for any inscribed angle and its intercepted arc. Use the diagram above to help prove why this is so.

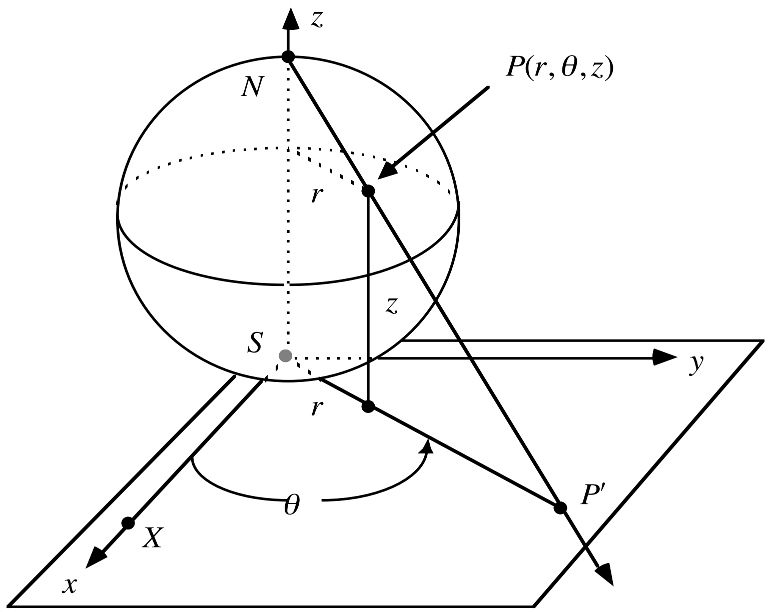
2.5 The following diagram shows a cross section of a sphere like the one in Exploration **1**, where A, B, and C divide the length of $\overset{\frown}{NS}$ into four equal parts. Use this diagram to find the lengths of $\overline{SA'}$, $\overline{SB'}$, and $\overline{SC'}$. Hint: Use a technique similar to that described in Exploration **2**.



2.6 **a.** On your sheet of paper from Exploration **1**, mark a polar axis with its endpoint at point S'.

b. Using this polar axis, find the polar coordinates of points A', B', C', D', E', and F'.

2.7 In the following diagram, S is the pole and $\overrightarrow{SX}$ is the polar axis of a polar coordinate system. Suppose point P represents a city on the globe with coordinates (r, θ, z) and P' is the stereographic projection of P. The diameter d of the sphere equals the length of $\overline{NS}$.



a. Describe how to find the polar coordinates of P' on the plane.

b. Determine the polar coordinates of P' in terms of d, r, z, and θ.

* * * * *

2.8 Consider a stereographic projection in which a globe's north pole is the center of projection and the plane contains the equator. In this case, is the image of a line of longitude also a line on the plane? Explain your response.

2.9 Points A, B, and C lie on a sphere of radius 10 cm. The pole of a cylindrical coordinate system is located at the south pole of the sphere. The approximate cylindrical coordinates of the three points are: A (8.66,1.13,15.00), B (10.00,1.13,10.00), and C (8.66,1.13,5.00).

a. Describe the arrangement and position of A, B, and C on the sphere.

b. Determine the distance along the line of longitude from A to B and from B to C.

c. Find the polar coordinates of the images of A, B, and C under a stereographic projection in which the center of projection is at the north pole and the plane is tangent to the sphere at the south pole. Assume that the same pole and polar axis are used for both the cylindrical and the polar coordinate systems.

d. Describe the arrangement and position of the images A', B', and C' on the plane.

e. Find the distance from A' to B' and from B' to C'. Is distance preserved under this mapping?

2.10 Points D, E, and F lie on a sphere of radius 10 cm. When the pole of a cylindrical coordinate system is located at the south pole of the sphere, their approximate coordinates are: D (9.06,0.69,14.22), E (9.06,2.95,14.22), and F (9.06,5.25,14.22).

a. Describe the arrangement and position of D, E, and F on the sphere.

b. Find the distance along the line of latitude from D to E and from E to F.

c. Find the polar coordinates of the images of D, E, and F under a stereographic projection in which the center of projection is at the north pole and the plane is tangent to the sphere at the south pole. Assume that the same pole and polar axis are used for both the cylindrical and the polar coordinate systems.

d. Describe the arrangement and position of the images D', E', and F' on the plane.

e. Find the distance from D' to E' and from E' to F'. Is distance preserved under this mapping?

* * * * * * * * * *

Activity 3

Over the history of map making, satellite imagery and supercomputers have replaced captain's logs and quill pens as the tools of choice. However, many centuries-old techniques are still useful and relevant to modern cartography.

For example, one common type of flat map is based on the work of the 16th-century Flemish cartographer Gerardus Mercator. A Mercator projection maps the surface of a sphere to a cylinder tangent to the sphere along a great circle (often the equator).

Mathematics Note

A **cylindrical projection** is a projection of the points of the sphere onto a tangent right circular cylinder. The image of a point on the sphere is the intersection of a ray and the cylinder. The ray contains the center of the sphere, designated as the **center of projection**, and the point being projected. In any cylindrical projection of a sphere, there are points that have no images on the cylinder.

For example, Figure **10** illustrates the cylindrical projection of point C on a sphere. Note that $\overrightarrow{OC}$ intersects the cylinder in at most one point.

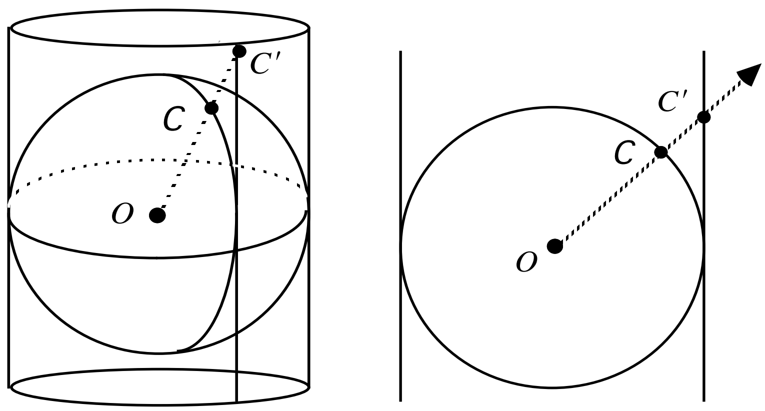


Figure 10: A cylindrical projection and its cross-sectional view

Exploration 1

In this exploration, you investigate cylindrical projections of lines of longitude and of the equator. To simplify locating the sphere's center, you work with a hemisphere, as shown in Figure **11**.

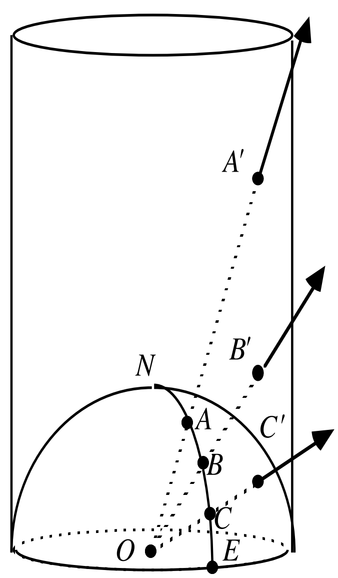


Figure 11: Cylindrical projections of points on a hemisphere

a. Cut a sphere in half along a great circle. **Note:** Save the other half of the sphere for use in the assignment.

b. Label the center of the flat face of the hemisphere O. The outer edge of the hemisphere represents the equator.

c. Label the north pole N. Label a point on the equator E.

d. Label three equally spaced points—A, B, and C—along the line of longitude from point N to point E, as shown in Figure **11**.

e. Wrap a sheet of paper around the hemisphere to make a right circular cylinder that fits over the hemisphere tangent to the equator. Place a mark on the bottom edge of the cylinder at point E and label it E'. Keep points E and E' aligned throughout the exploration.

f. Pass a skewer from O through point C on the hemisphere to model the corresponding ray. Mark and label the intersection of the skewer with the cylinder on the cylinder's outside surface.

g. Repeat Part **f** for points A and B.

h. Repeat the mapping process for at least two other lines of longitude and for the equator.

i. Open the cylinder and lay it flat with the outside surface facing up. This is the map produced by a cylindrical projection.

Discussion 1

a. Describe the map you produced using a cylindrical projection.

b.
1. What would happen if you used the process described in the exploration to project a point close to N onto the cylinder?
2. Where is the image of point N using this projection? Explain your response.

c. Suppose that all the points on a sphere that have an image under a cylindrical projection were mapped onto one cylinder.
1. Describe the surface that would result when the cylinder was "unrolled" to produce a flat map.
2. How would the image of a line of longitude appear on the map?

d. On a sphere, any pair of distinct lines (great circles) intersect in exactly two points. In other words, there are no parallel lines on a sphere.
1. Under a stereographic projection, can the images of great circles form parallel lines on the flat map?
2. Under a cylindrical projection, can the images of great circles form parallel lines on the flat map?

e. Is perpendicularity preserved under a cylindrical projection? Justify your response.

f. Describe a map of the lines of longitude and lines of latitude in the northern hemisphere created under a cylindrical projection.

g. Consider a cylindrical coordinate system in which the polar plane contains the equator of a sphere.

1. What points on the sphere would have negative z-coordinates?
2. Using a cylindrical projection, where would points with negative z-coordinates be projected on the flat map?

Exploration 2

In this exploration, you use a geometry utility to continue your investigation of cylindrical projections. Figure **12** shows a cross section of a sphere and a tangent right circular cylinder. Lines l and m represent parallel lines in the surface of the cylinder and $\overline{NS}$ is a diameter of the sphere.

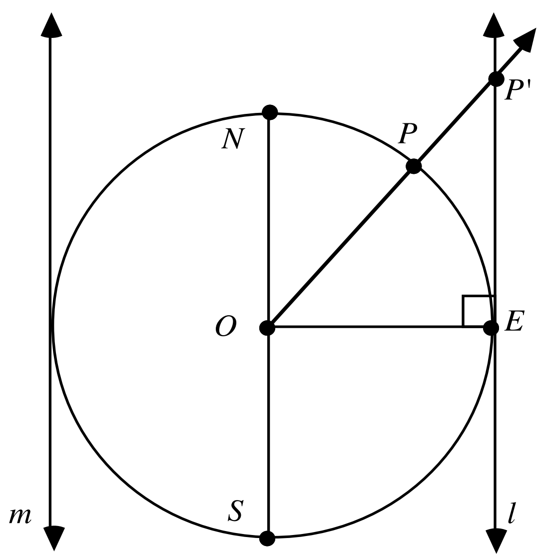


Figure 12: Cross section of cylindrical projection

a. Use a geometry utility to reproduce the diagram in Figure **12**. Anchor $\overrightarrow{OP}$ at O, allowing point P to move around the circle. Make sure that the position of P' changes as P moves around the circle.

b. Describe what happens to P' as you move P from E to N.

c. Construct points A, B, and C so that $\overset{\frown}{NA} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC} \cong \overset{\frown}{CE}$.

d.
1. Move point P so it is concurrent with point C.
2. Record the length of $\overline{EP'}$.
3. Repeat Steps **1** and **2** for points A, B, E, and N.

e. Construct $\overline{PZ}$ as shown in Figure **13** below. Make sure that $\overline{PZ}$ changes length as P moves around the circle.

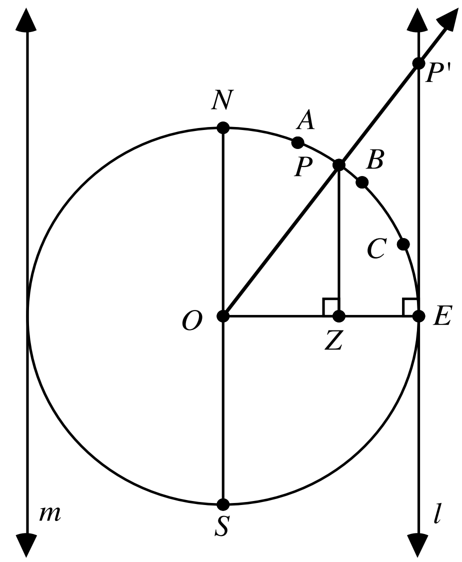


Figure 13: Cross section with $\overline{PZ}$

f. Express the length of $\overline{EP'}$ in terms of the lengths of $\overline{OZ}$, $\overline{PZ}$, and $\overline{OE}$, the radius of the sphere.

Discussion 2

a. Describe what happens to the length of $\overline{EP'}$ as P gets closer and closer to N.

b. In Figure **13**, triangles POZ and $P'OE$ are similar triangles. Explain how you know this is true.

c. A point P on a sphere can be described by cylindrical coordinates of the form (r, θ, z), with the pole located at the sphere's center. Its image on a flat map under a cylindrical projection can be described by rectangular coordinates of the form (x, y).

Suppose that the image of the intersection of the polar axis and the equator has the coordinates (0,0)

1. What is the image of the equator on the flat map?
2. Describe how the value of θ in the cylindrical coordinates of P is related to the x-coordinate of P'.
3. Describe how the distance EP' in Figure **13** can be used to find the y-coordinate of P'.

Assignment

3.1 Are distance, area, or perimeter preserved under a cylindrical projection? Explain your response.

3.2 **a.** Using a cylindrical projection with the cylinder tangent to the equator, would the image of Venezuela be more or less distorted than the image of Greenland? Explain your response.

b. Which would produce a greater distortion of Greenland—a stereographic projection through the south pole or a cylindrical projection? Explain your response.

3.3 Lines of longitude and latitude are important aids for navigation. When navigators use flat maps, what type of projection would you expect them to prefer—stereographic or cylindrical? Justify your response.

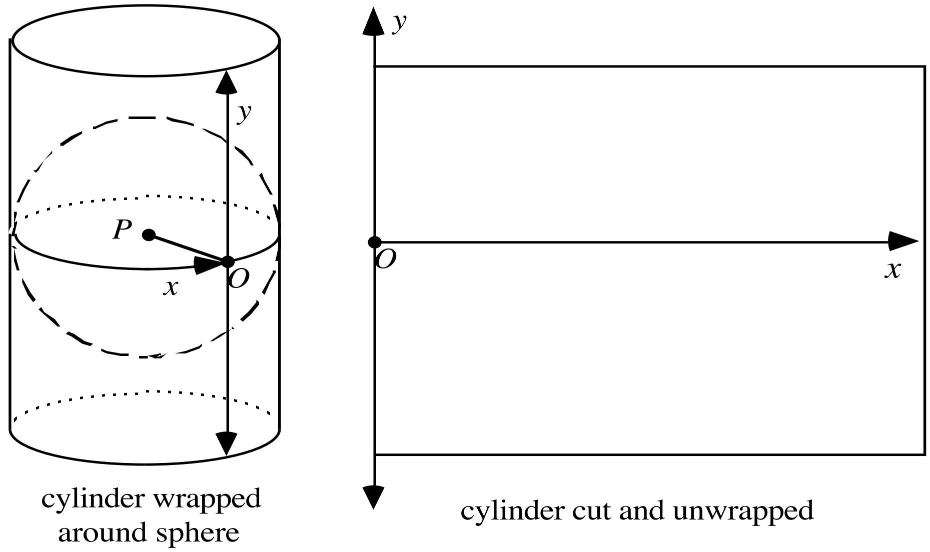
3.4 Consider a figure on a sphere whose image, under a cylindrical projection, is a rectangle. One of the sides of this figure lies along the equator. Describe a possible shape for the preimage.

3.5 Consider a sphere with a paper cylinder wrapped around it, tangent to the equator. Any point on the sphere can be represented by cylindrical coordinates of the form (r, θ, z), with the pole located at the sphere's center.

When points on the sphere are projected onto the cylinder, and the cylinder is cut and unwrapped, a flat map is produced. The position of each point on the map can be described by rectangular coordinates of the form (x, y).

As shown in the following diagram on the left, a line drawn on the outside of the cylinder along the equator can represent the positive x-axis. The origin O can be located at the point where the polar axis ($\overrightarrow{PO}$) of the cylindrical coordinate system intersects the equator.

When the paper cylinder is cut through the origin perpendicular to the x-axis, then unwrapped and laid flat with its outside surface facing up, it resembles the diagram on the right.



a. If the sphere's radius is 10 cm, what is the length of the portion of the x-axis on the unwrapped cylinder?

b. Point A on the sphere has cylindrical coordinates $(10.00, \pi/6, 0.00)$. What is the x-coordinate of A', the image of A under a cylindrical projection?

c. Point B on the sphere has coordinates $(2.59, 5.40, -9.66)$. What is the x-coordinate of B', the image of B under a cylindrical projection?

d. Let P represent any point on a sphere with radius w. If P has cylindrical coordinates (r, θ, z), find the x-coordinate of P', the image of P under a cylindrical projection.

e. Describe how to determine the value of the y-coordinate for the image point in a cylindrical projection.

3.6 **a.** Point A lies on a sphere of radius 20 cm. Using a cylindrical coordinate system with the pole at the sphere's center, the coordinates of A are $(7.32, \pi/2, 10)$.

Using a cylindrical projection as described in Problem **3.5**, what are the rectangular coordinates of A'?

b. Point B lies on a sphere of radius 20 cm. Using a cylindrical coordinate system with the pole at the sphere's center, the coordinates of B are $(1.74, 4.96, -19.92)$.

1. In what region is B located on the sphere?
2. Using a cylindrical projection as described in Problem **3.5**, what are the rectangular coordinates of B'?

* * * * *

3.7 Points A, B, and C lie on a sphere of radius 25 cm. Using a cylindrical coordinate system with the pole at the sphere's center, their coordinates are: A (13.62,5.43,20.97), B (25.00,5.43,0.00), and C (13.62,5.43,–20.97).

a. Describe the arrangement and position of A, B, and C on the sphere.

b. Find the distances along the line of longitude from A to B and from B to C.

c. Using a cylindrical projection as described in Problem **3.5**, find the rectangular coordinates of A', B', and C'.

d. Describe the arrangement and position of A', B', and C' on the flat map.

e. Find the distance from A' to B' and from B' to C'. Is distance preserved under this mapping?

3.8 Points D, E, and F lie on a sphere of radius 25 cm. Using a cylindrical coordinate system with the pole at the sphere's center, their coordinates are: D (19.15,0.25,–16.07), E (19.15,2.01,–16.07), and F (19.15,3.77,–16.07).

a. Describe the arrangement and position of D, E, and F on the sphere.

b. Find the arc lengths along the line of latitude from D to E and from E to F.

c. Using a cylindrical projection as described in Problem **3.5**, find the rectangular coordinates of D', E', and F'.

d. Describe the arrangement and position of D', E', and F' on the flat map.

e. Find the distance from D' to E' and from E' to F'. Is distance preserved under this mapping?

* * * * * * * * * *

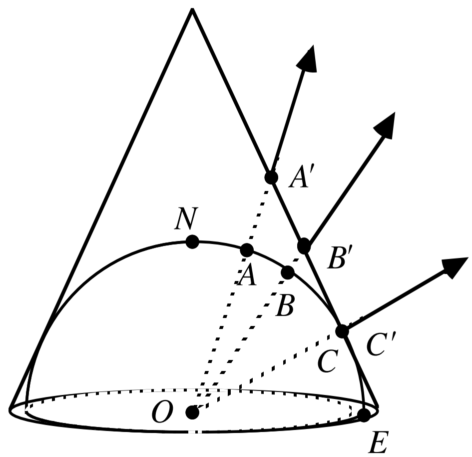
Research Project

Mapmakers use many different types of projections. Write a report that describes three projections not presented in this module. Your report should include the following information:

- how each projection is used to make a map
- how lines of latitude and longitude appear on the resulting map
- the geometric properties that are preserved in each projection
- the types of situations in which each projection is most useful
- the historical background of each projection.

Summary Assessment

Maps made using a conic projection are similar to those made with a cylindrical projection. In a conic projection, however, points are projected onto a tangent right circular cone, instead of a tangent right cylinder, as shown in the diagram below.



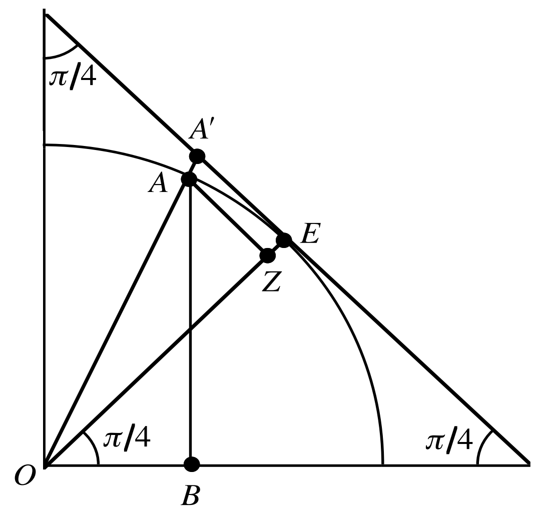
1. **a.** What figure is formed by the intersection of the sphere and the cone?

b. How would the images of lines of latitude and longitude appear in a conical mapping?

c. What geometric properties appear to be preserved under a conic projection? Justify your response.

2. The diagram below shows part of a cross section of a sphere and a tangent cone. The cone's vertex angle is the angle formed by the intersection of the cone and a plane perpendicular to the cone's base and passing through the apex.

Suppose the vertex angle of the cone measures $\pi/2 \approx 1.57$ radians and A is a point on the sphere with cylindrical coordinates $(13.62, 5.43, 20.97)$. The radius of the sphere is 25 cm.



If A' is the image of A under a conical projection, determine the ratio of the length of $\overline{A'E}$ to the length of $\overset{\frown}{AE}$.

3. Does the measure of the vertex angle of the cone affect the amount of distortion in a conic mapping? If so, how? Hint: Use a geometry utility to investigate this situation.

Module Summary

- A **polar coordinate system** describes the location of a point P in a plane using an ordered pair consisting of a **radius** r and a **polar angle** θ.

  The plane containing a polar coordinate system is the **polar plane**. The polar angle is an angle measured from a fixed ray, called the **polar axis**. The endpoint of the polar axis is the **pole**. The distance from the pole to point P, measured in the polar plane, is r.

- In a **cylindrical coordinate system**, a point in space is represented by an ordered triple of the form (r, θ, z). The values of r and θ are measurements in the polar plane. The value of z is the directed distance between the point and the polar plane (the plane containing the polar axis). A positive value for z represents a distance above the polar plane.

- A **stereographic projection** is a projection of the points on a sphere onto a plane perpendicular to a given diameter of the sphere. The plane is the **plane of projection**. The endpoints of the diameter are the **poles** of the sphere.

  The image of a point on the sphere is the point of intersection of a ray and the plane perpendicular to the diameter that contains the poles. The ray contains one of the poles, designated as the **center of projection**, and the point being projected. In any stereographic projection, there are points that have no image in the plane.

- A **cylindrical projection** is a projection of the points of the sphere onto a tangent right circular cylinder. The image of a point on the sphere is the intersection of a ray and the cylinder. The ray contains the center of the sphere, designated as the **center of projection**, and the point being projected. In any cylindrical projection of a sphere, there are points that have no images on the cylinder.

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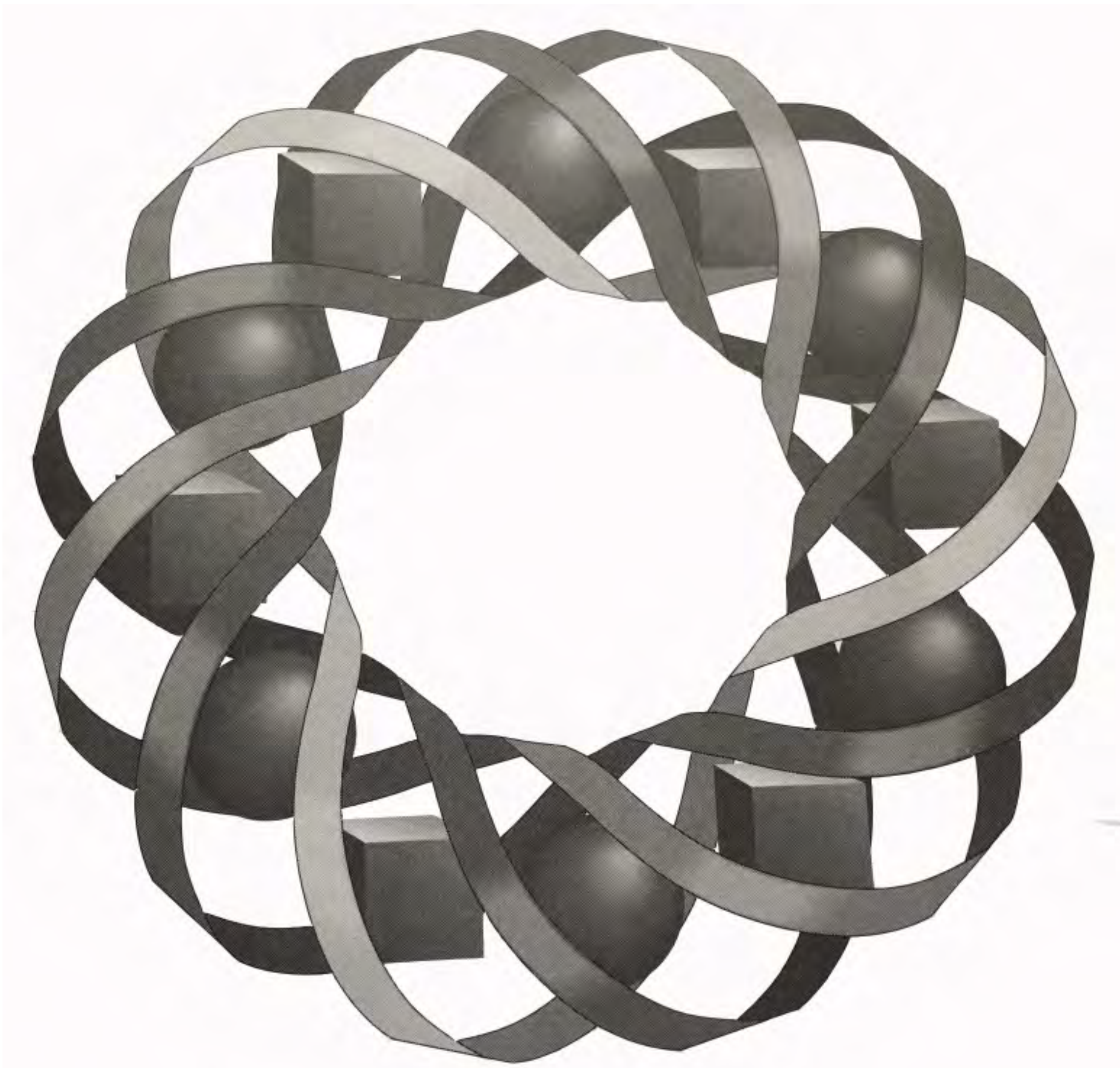
Naturally Interesting



How much money would you have to invest now in order to be a millionaire at age 65? In this module, you use exponentials and logarithms to answer this question.

Todd Fife • Sue Moore • Pete Stabio

Naturally Interesting

Introduction

Banks and investment companies offer a variety of accounts to help customers reach their financial goals. These accounts may offer different rates of interest, based on the initial amount invested. How much money would a person need to invest today in order to be a millionaire at age 65? How long will it take an investment to double in value? These are questions financial advisers must be able to answer for their clients.

In banking, **principal** refers to the amount of money invested or loaned. **Interest** is the amount earned on invested money, or the fee charged for loaned money. The amount of interest received or paid depends on three quantities: principal, interest rate, and time. Interest also varies according to the method used to calculate it. In the following activities, you investigate how savings accounts earn money.

Mathematics Note

One method for determining the amount of interest earned or owed involves **simple interest**. In this case, interest is paid or charged only on the original principal. The formula for calculating simple interest, where I represents interest, P represents principal, r represents the interest rate per time period, and t represents the number of time periods, is shown below:

$$I = Prt$$

To use this formula, t must be expressed in the same units as time in the interest rate, r. For example, if the interest rate is 5% per year, then t must be expressed in years. If \$1000 is invested at an annual interest rate of 5% for 3 yr, the interest earned can be calculated as follows:

$$I = 1000(0.05)(3) = \$150$$

Discussion

a. What types of loans are available in your community? What are the current interest rates and terms available for these loans?

b. Describe the opportunities available for investing or saving money in your community. What are the current interest rates and terms available?

c. Consider an investment account that offers an annual interest rate of 10%. If you invest \$1000 in this account, the interest earned after 1 yr is \$100. If you reinvest the **account balance** (the original principal plus the \$100 interest earned) at the same rate, how much interest will you earn in the second year?

Activity 1

Most savings accounts pay interest not only on the original principal, but also on the interest earned and deposited in any previous time periods. This is an example of **compound interest**. Each time compound interest is calculated, the interest earned is added to the principal. This sum (the account balance) becomes the new principal for the next interest calculation.

Exploration 1

In this exploration, you develop a method for determining the balance of an account that earns compound interest.

a. Imagine that you have invested \$500 at a simple interest rate of 6% per year and plan to make no withdrawals for the next 20 yr.

Use the formula for simple interest to determine the account balance after 20 yr.

b. When interest is compounded annually, the interest earned each year is added to the account at the end of that year. Predict the account balance after 20 yr if interest is compounded annually.

c. To determine the actual balance of the account after 20 yr when interest is compounded annually, you must examine what happens to the account balance at the end of each year.

1. Determine the account balance at the end of the first year by adding the interest earned for 1 yr to the principal.
2. Using the account balance at the end of the first year as the new principal, determine the account balance at the end of the second year.

d. Use a spreadsheet to repeat the process described in Part **c** for each of the next 18 yr. Record your data in a table similar to Table **1** below.

Table 1: \$500 invested at 6% for 20 yr, compounded annually

Years (t)	Principal at Beginning of Year (\$)	Account Balance at End of Year (\$)
1	500	
2		
3		
⋮		
20		

e. Use the spreadsheet to investigate how account balances are affected by changes in the interest rate. Record your observations.

f. Let P_t represent the principal at the end of t years in an investment with an interest rate of 6% per year, compounded annually. Write an expression that describes P_t in terms of the principal for the previous year. (In other words, write a recursive formula for the account balance after t years.)

Mathematics Note

The principal at the end of each time period in an investment or savings account can be thought of as a sequence.

For example, consider an initial principal of \$1000 invested at an interest rate of 8% per year, compounded annually. Assuming that no withdrawals are made and any interest earned is deposited in the account, the following geometric sequence is formed, where P_0 represents the initial principal, P_1 represents the principal after 1 yr, and so on:

$$\begin{aligned} P_0 &= 1000 \\ P_1 &= 1080 \\ P_2 &= 1166.40 \\ &\vdots \\ P_{10} &\approx 2158.92 \end{aligned}$$

In this case, the account balance at the end of 10 yr is approximately \$2158.92.

Such a sequence can be defined recursively by the following formula:

$$P_t = P_{t-1} + r \bullet P_{t-1} = P_{t-1}(1 + r)$$

where P_t is the principal at the end of t years, r is the annual interest rate, and P_{t-1} is the principal for the previous year.

For example, given an initial principal $P_0 = \$345$ and an annual interest rate of 8%, the account balance at the end of 1 yr (P_1) can be determined as shown below (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_1 = P_0 \bullet (1 + 0.08) = (345) \bullet (1 + 0.08) = \$372.60$$

g. To use the recursive formula given in the mathematics note, you must know the account balance in the previous year. An explicit formula, however, would allow you to find P_t without having to determine P_{t-1} .

1. Write P_1 in terms of P_0 (the original principal) and *r* (the annual interest rate).
2. Using substitution and the recursive formula, $P_t = P_{t-1}(1+r)$, determine an explicit formula for P_2 , the principal at the end of 2 yr, in terms of P_0 and *r*.
3. Repeat Step **2** for P_3, the principal at the end of 3 yr.

h. Determine an explicit formula that could be used to find the account balance after *t* years (P_t) for an initial investment of P_0 at an annual interest rate of *r*, compounded annually.

i. Use your explicit formula to calculate the account balance, after 20 yr, of an investment of $500 at an annual interest rate of 6%, compounded annually. Compare this value to the one you determined using the spreadsheet.

Discussion 1

a. What advantages are there to using an explicit formula for account balance rather than a recursive formula?

Mathematics Note

When interest is compounded annually, the yearly account balances that result can be thought of as a sequence defined explicitly by the following formula (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_t = P_0(1+r)^t$$

where P_t is the account balance after *t* years, P_0 is the initial principal, *r* is the annual interest rate, and *t* is the time in years.

For example, given an initial principal of $2000 and an annual interest rate of 4%, compounded annually, the account balance after 12 yr can be determined as follows:

$$\begin{aligned} P_{12} &= 2000(1+0.04)^{12} \\ &\approx 3202.06 \end{aligned}$$

b.
1. How do the terms in the formula $P_t = P_0(1+r)^t$ correspond with the terms of the following general formula for a geometric sequence?
$$g_n = g_1 r^{n-1}$$
2. How does $P_t = P_0(1+r)^t$ compare with the formula you wrote in Part **h** of Exploration **1**?

c. How will doubling the initial investment affect the account balance after 20 yr?

d. Describe two ways to determine the time required for a \$500 investment to double at an annual interest rate of 6%, compounded annually.

e. In previous modules, you modeled population growth with the equation $P_n = P_0(1+r)^n$, where P_n is the population after n time periods, P_0 is the initial population, and r is the growth rate per time period.

1. Compare the equation for population growth with the explicit formula for account balance.
2. What does the expression $(1+r)$ represent in each equation?

Exploration 2

In this exploration, you examine how the number of compoundings per year affects the amount of interest earned in an account.

a. Imagine that the \$500 invested in Exploration **1** is deposited in an account in which interest is compounded semiannually, or twice a year. Since the annual interest rate is 6%, the rate for each half year is $0.06/2$ or 3%.

1. What is the account balance after 2 compounding periods, or 1 yr?
2. Write an expression which describes the balance after 1 yr in terms of the original investment of \$500.
3. Repeat Steps **1** and **2** for the balance after 2 yr (4 compounding periods) and the balance after 3 yr (6 compounding periods).

b. Using an annual interest rate r and an initial principal of P_0, write a formula for P_n, the account balance after n compounding periods, when interest is compounded semiannually for t years.

c. Repeat Parts **a** and **b** for an account in which interest is compounded quarterly, or four times a year.

d. Using an annual interest rate r and an initial principal of P_0, write a general formula for the account balance after n time periods, when interest is compounded c times a year for t years.

e. The number of compounding periods per year affects the account balance at the end of the year. Investigate this effect by using a spreadsheet and your formula from Part **d** to complete Table **2** below.

Table 2: $500 invested at 6%, with different compoundings

Initial Principal: $500		
Annual Interest Rate: 6%		
Type of Compounding	**No. of Compoundings per Year**	**Account Balance (P_n)**
annually	1	$530.00
semiannually		
quarterly		
monthly		
daily		
hourly		
by the minute		
by the second		

f. Predict the account balance in Table **2** after 1 yr if interest is compounded continuously.

g. Change the initial principal and the interest rate in your spreadsheet. For each change in principal or interest rate, observe how the balance is affected as the number of compoundings increases.

Mathematics Note

When compounding interest c times per year for t years, the formula for the account balance after n compounding periods is:

$$P_n = P_0\left(1+\frac{r}{c}\right)^n = P_0\left(1+\frac{r}{c}\right)^{ct}$$

where P_n represents the principal after n compounding periods, P_0 represents the initial principal, and r is the annual interest rate. Note that, in this formula, $n = ct$.

For example, consider an initial investment of $1000 at an annual interest rate of 5%, compounded quarterly. Assuming that no withdrawals are made and any interest earned is deposited in the account, the principal after 3 yr (or $4 \bullet 3 = 12$ compounding periods) can be calculated as follows:

$$P_{12} = 1000\left(1+\frac{0.05}{4}\right)^{12} = \$1160.75$$

Discussion 2

a. To find the account balance after *t* years when interest is compounded annually, you used the formula $P_t = P_0(1+r)^t$. How does this formula differ from the one you wrote in Part **b** of Exploration **2**, when interest is compounded semiannually?

b. Describe what happens to the account balance in Table **2** as the number of compoundings per year increases.

c. How does increasing the number of compoundings per year appear to affect the total amount of interest earned after 1 yr?

d. As the number of compoundings increases, do you think that the sequence of account balances approaches a limit? Explain your response.

Assignment

1.1 Consider an initial investment of \$700 at an annual interest rate of 7.5% for 1 yr. Assuming that no withdrawals are made and any interest earned is deposited in the account, determine the account balance when interest is compounded:

a. annually

b. quarterly

c. monthly

d. daily.

1.2 **a.** On his 18th birthday, a student invests \$112.50 in an account with an annual interest rate of 9%, compounded annually. Assuming that he makes no withdrawals and any interest earned is deposited in the account, determine the account balance on his 65th birthday.

b. Repeat Part **a** for an account in which interest is compounded monthly.

1.3 **a.** Imagine that, on the day you were born, someone deposited \$5000 in an account with an annual interest rate of 4.8%, compounded monthly. Determine how old you would be when the balance of the account is \$20,000.

b. If the interest is compounded monthly, what annual interest rate would be required for the account to have a value of at least \$30,000 on your 18th birthday?

1.4 **a.** Describe reasonable domains for r and c in the formula for account balance using compound interest (shown below).

$$P_n = P_0\left(1 + \frac{r}{c}\right)^{ct}, \quad n = ct$$

b. **1.** Choose values for P_0, c, and r and substitute these values into the equation.

2. To what family of functions does the equation belong?

c. **1.** Graph the equation from Part **b** as a function of t for 3 different values of r. Use the set of real numbers as the domain for t.

2. What effect does the magnitude of r have on the graphs?

1.5 As a financial advisor, you offer investment advice to your clients. One of your clients must decide whether to invest \$1500 at an annual interest rate of 15%, compounded quarterly, or \$1600 at an annual interest rate of 15.5%, compounded annually. Both investments have a 10-year term. Which one would you recommend? Explain your response.

* * * * *

1.6 **a.** In 1991, China's estimated population was 1,151,300,000. The annual growth rate at that time was 1.4%. Assuming that this growth rate remains constant, write an equation that models China's population since 1991.

b. Use your model to estimate China's current population.

c. In 1991, India had a population of 859,200,000 and an annual growth rate of 4%. If this growth rate remains constant, predict the year in which India's population will surpass that of China.

1.7 Most durable goods, such as cars and computers, decrease in value over time. This is known as **depreciation**.

Consider a car which cost \$17,000 new and loses 15% of its value each year.

a. Write both recursive and explicit formulas to represent the depreciation of this car.

b. What is the car's value after 5 yr?

* * * * * * * * * *

Activity 2

With the development of calculators and computers, the determination of compound interest has become quick and easy. This allows banks to compound interest on an account balance up to the instant in which it is withdrawn. This method of calculating interest, known as **compounding continuously**, means that the number of compoundings per year approaches **infinity**, denoted by ∞. Written as $+\infty$ or $-\infty$, this symbol also may be used to depict boundlessness in either a positive or a negative direction.

Exploration

As you saw in Activity **1**, the number of compoundings per year can affect the balance of a savings account. What happens to this balance when the number of compoundings increases without bound? In this exploration, you investigate what happens as c, the number of compoundings, changes for specific values of P_0, r, and t.

a. Consider an investment of \$1.00 at an annual interest rate of 100%, compounded continuously, for 1 yr. Predict the account balance at the end of the year.

b. Create a spreadsheet with columns similar to those in Table **3**. Use the formula for account balance when interest is compounded c times a year to complete the spreadsheet for an investment of \$1.00 at an annual interest rate of 100%.

Table 3: Account balances for different compoundings

No. of Compoundings per Year (c)	Account Balance at End of Year (\$)
1	
10	
100	
1000	
10,000	
100,000	
1,000,000	
10,000,000	
100,000,000	

c. As the number of compoundings per year increases, what happens to the sequence of account balances?

d. Since $P_0 = 1$, $r = 1$, and $t = 1$ in this situation, an explicit formula for the sequence of balances found in Table **3** is:

$$P_1 = 1\left(1+\frac{1}{c}\right)^{c \bullet 1} = \left(1+\frac{1}{c}\right)^{c}$$

In the context of this problem, c is a non-negative integer. However, this formula can be represented more generally as the function below:

$$y = \left(1+\frac{1}{x}\right)^{x}$$

What are the domain and range of this function?

e. Graph the function from Part **d**. As x increases without bound, what limiting value does the graph appear to approach?

Mathematics Note

The limit of the following expression, as n approaches infinity, is an irrational number approximately equal to 2.71828:

$$\left(1+\frac{1}{n}\right)^{n}$$

This irrational number, represented as ***e***, is sometimes called Euler's number in honor of Swiss mathematician Leonhard Euler. The value of e can be represented mathematically as shown below:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

Another way to describe e is as the infinite series below:

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

The value of e can also be derived from continued fractions as follows:

$$e = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\vdots}}}}}$$

f. In the formula for account balance after n compounding periods, $n = c \bullet t$, where c represents the number of compoundings per year and t represents time in years. Considering a period of 1 yr, therefore, $n = c \bullet 1 = c$. Given an initial principal of \$1.00, the formula for account balance can be written as follows:

$$P_n = \left(1 + \frac{r}{n}\right)^n$$

To investigate how a change in the value of r affects the limit of this expression, create and complete a spreadsheet with columns like those in Table **4** below.

Table 4: Balance in dollars for different interest rates

n	$P_n = \left(1 + \frac{1}{n}\right)^n$	$P_n = \left(1 + \frac{2}{n}\right)^n$	$P_n = \left(1 + \frac{3}{n}\right)^n$
1			
10			
100			
1000			
10,000			
100,000			
1,000,000			
10,000,000			
100,000,000			

g.
1. Calculate e^2. Compare it to the values in the spreadsheet in Part **f**.
2. Calculate e^3. Compare it to the values in the spreadsheet in Part **f**.

Discussion

a.
1. Describe the relationship between e^2 and the following expression:

$$\lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n$$

2. Describe the relationship between e^3 and the expression below:

$$\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n$$

3. What conjecture can you make about the value of this expression?

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$$

b. The value of n has no effect on the value of the constant 25 in the product below:

$$25\left(1+\frac{1}{n}\right)^n$$

In fact, the following equation is true:

$$\lim_{n\to\infty} 25\left(1+\frac{1}{n}\right)^n = 25\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] = 25e$$

Use this fact to evaluate each of the expressions below:

1. $\lim_{n\to\infty} 10\left(1+\frac{2}{n}\right)^n$

2. $\lim_{n\to\infty} 20\left(1+\frac{3}{n}\right)^n$

3. $\lim_{n\to\infty} 8\left(1+\frac{r}{n}\right)^n$

4. $\lim_{n\to\infty} P_0\left(1+\frac{r}{n}\right)^n$

Mathematics Note

When compounding interest c times per year for t years, the formula for the account balance after n compounding periods, where $n = ct$, P_0 represents the initial principal, and r is the annual interest rate, is:

$$P_n = P_0\left(1+\frac{r}{c}\right)^{ct}$$

When the number of compoundings per year approaches infinity, then the interest is **compounded continuously**. In this case, the formula for account balance P can be written as follows:

$$P = \lim_{c\to\infty} P_0\left(1+\frac{r}{c}\right)^{ct} = P_0\left[\lim_{c\to\infty}\left(1+\frac{r}{c}\right)^c\right]^t = P_0e^{rt}$$

where P_0 represents the initial principal, r represents the annual interest rate, c represents the number of compoundings per year, and t represents number of years.

For example, consider an initial investment of \$500 at an annual interest rate of 6%, compounded continuously. Assuming that no withdrawals are made and any interest earned is deposited in the account, the account balance after 5 yr can be calculated as follows:

$$P = 500e^{0.06\bullet 5} = \$674.93$$

c. What can you conclude about an investment whose account balance is calculated by the equation below?

$$P = 750e^{0.01t}$$

Assignment

2.1. Imagine that you have deposited \$5000 in a savings account at an annual interest rate of 3%, compounded continuously. Assuming that you make no withdrawals and any interest earned is deposited in the account, how old will you be when the account balance is \$20,000?

2.2 Determine the value of a \$1000 investment at the end of 1 yr if the annual interest rate is 9% and interest is compounded:

a. annually

b. quarterly

c. monthly

d. daily

e. continuously.

2.3 **a.** Based on your responses to Problem **2.2**, find an annual interest rate that, when compounded annually, will produce the same balance after 1 yr as an annual interest rate of 9% compounded:

1. monthly

2. daily

3. continuously.

b. Describe how you determined your responses to Part **a**.

Business Note

To help consumers compare interest rates, banks often report **annual percentage yield (APY)** for savings accounts and **annual percentage rate (APR)** for loans. The APY or APR is the interest rate that, when compounded annually, will produce the same account balance as the advertised interest rate, which is typically compounded more often.

For example, the annual percentage yield of an initial investment of P_0 at an annual interest rate of 9%, compounded quarterly, can be found as follows:

$$
\begin{aligned}
P_0\left(1+\frac{r_{\text{APY}}}{1}\right)^1 &= P_0\left(1+\frac{0.09}{4}\right)^4 \\
\left(1+r_{\text{APY}}\right) &\approx 1.09308 \\
r_{\text{APY}} &\approx 0.09308 = 9.308\%
\end{aligned}
$$

This means that, for any given initial investment, an annual interest rate of 9%, compounded quarterly, produces the same account balance as an annual interest rate of 9.308%, compounded annually.

2.4 **a.** Consider an investment of \$1000 at an annual interest rate of 7.7%. Determine the annual percentage yield (APY) if interest is compounded:

1. quarterly
2. daily
3. hourly.

b. The APY reaches its maximum when interest is compounded continuously. Determine the maximum APY for an investment with an annual interest rate of 7.7%.

c. Write a formula for determining maximum APY.

2.5 The previous business note used an example to determine the annual percentage yield (APY) for a specific annual interest rate (r) and a given number of compoundings per year (c).

a. To determine the general relationship among APY, r, and c, solve the equation below for r_{APY}.

$$P_0\left(1+\frac{r}{c}\right)^c = P_0\left(1+\frac{r_{\text{APY}}}{1}\right)^1$$

b. As the number of compoundings per year increases without bound, what would you expect to find as a formula for r_{APY}?

c. How does the initial principal affect the relationship among APY, r, and c?

2.6 **a.** After 1 yr, will the account balance increase by a significant amount if interest is compounded every hour rather than every day? Use an example to support your response.

b. In general, what effect does increasing the number of compoundings per year have on account balance?

* * * * *

2.7 One general equation used to model the growth or decay in a quantity is $N_t = N_0 e^{nt}$, where N_t represents the final amount, N_0 represents the initial amount, n represents some constant, and t represents time. When $n > 0$, the equation can be used to model growth; when $n < 0$, the equation can be used to model decay.

A population of bacteria has a constant n of 0.538 when t is measured in days. How many days will it take an initial population of 8 bacteria to increase to 320?

2.8 As mentioned in a previous mathematics note, e also can described using the infinite series shown below:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

or by using continued fractions as follows:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\vdots}}}}$$

Use both of these expressions to approximate the value of e to six decimal places.

2.9 In the Level 4 module "Nearly Normal," you learned that a normal probability distribution is symmetric about the mean and tapers to the left and right like a bell. A normal curve is defined by the following equation:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ and σ are the mean and standard deviation, respectively, of a normal distribution.

a. Select a value for μ, then choose several different values for σ and graph the resulting equations.

b. Select a value for σ, then choose several different values for μ and graph the resulting equations.

c. Use your graphs from Parts **a** and **b** to discuss the effects of μ and σ on a normal curve.

* * * * * * * * * *

Research Project

In his studies of infinite sets of numbers, Georg Cantor (1845–1918) developed **transfinite numbers**. A transfinite number is the cardinal number of an infinite set. Cantor also described a method for determining when one infinite set of numbers was larger than another by comparing their cardinal numbers. Write a report on transfinite numbers and their relationship to infinite sets.

Activity 3

In Activity **2**, you wrote equations for determining account balances and annual percentage yields in which a value for time was used as an exponent. In this activity, you use logarithms to determine the amount of time required for an investment to reach a particular amount.

Exploration 1

In the Level 4 module "Log Jam," you investigated **common logarithms**, base-10 logarithms which can be written either as $\log_{10} x$ or $\log x$. In this exploration, you examine logarithms that have bases other than 10.

a. Complete Table **5** below, which relates corresponding exponential and logarithmic equations.

Table 5: Logarithmic and exponential equations

Logarithmic Equation	Related Exponential Equation
$\log_2 8 = 3$	$2^3 = 8$
$\log_4 16 = 2$	
	$6^4 = 1296$
$\log_{1.5} 3.375 = 3$	
	$0.8^2 = 0.64$
$\log_{0.81} 0.9 = 0.5$	

b. Recall from the Level 4 module "Log Jam," that $y = \log_a x$ is equivalent to $a^y = x$. Knowing the relationship between *a* (the base) and *x* for various values of *y* can help you determine an unknown base of a logarithm.

1. Select a value for *x* greater than 1.
2. Using the value for *x* from Step **1**, determine the value of *a* in the equation $a^y = x$ when $y = 1$.
3. Compare the value of *a* to the value of *x*.
4. Repeat Steps **2** and **3** for several values of *y* greater than 1.
5. Repeat Steps **2** and **3** for several values of *y* less than 1.

c. Table **6** shows the logarithms, using two unknown rational bases *a* and *b*, for various values of *x*. Use your results from Part **b** to help determine the approximate value of each base.

Table 6: Two logarithms of *x*

x	$\log_a x$	$\log_b x$
2	0.431	0.333
3		0.528
4	0.861	
5		0.774
6	1.113	0.862
7		0.936
8	1.292	
9		
10		
11	1.490	1.153
12	1.544	1.195

Given the logarithmic equation $\log_{10} 10 = 1$, the related exponential equation is $10^1 = 10$. Write similar equations for $\log_a x$ and $\log_b x$ in Table **6**.

d. Most calculators and computers offer a feature for determining the **natural logarithm** of a number, denoted as $\mathbf{ln}\boldsymbol{x}$.

Determine the natural logarithm of various values of *x*. Use your results to approximate the value of the base of the natural logarithm to five decimal places.

Discussion 1

a. In Part **a** of Exploration **1**, you found that $x < a$ when $\log_a x < 1$ and $x > a$ when $\log_a x > 1$. Explain why this must be true.

b. What value did you determine for the approximate base of the natural logarithm?

Mathematics Note

Logarithms with base e are referred to as **natural logarithms**. The natural log of x is denoted by $\ln x$, where $x > 0$. The equation $\ln x = y$ is true if $e^y = x$.

For example, $\ln 7 \approx 1.9$. The related exponential equation is $e^{1.9} \approx 7$.

c. Describe how you might estimate each of the following:

1. $\log_7 52$
2. $\log_{19.7} 18$
3. $\ln 2$
4. $\ln 20$ (Hint: $e^3 \approx 20$.)

d. Why does the definition of natural logarithms given in the previous mathematics note restrict x to values greater than 0?

Exploration 2

In this exploration, you use natural logarithms to determine the time required for an account to reach a desired balance when interest is compounded continuously.

a. Consider an initial investment of \$500 at an annual interest rate of 6%, compounded continuously. Write a function that describes the account balance after t years.

b. In the module "Log Jam," you used the properties of logarithms to solve equations such as the one below for x.

$$y = 3 \bullet 10^x$$
$$y/3 = 10^x$$
$$\log(y/3) = \log 10^x$$
$$\log(y/3) = x$$

Use natural logarithms and the properties of logarithms to solve the equation in Part **a** for t.

Mathematics Note

The properties that are true for $\log_b x$ also are true for $\ln x$. Therefore, for $b > 0$, $b \neq 1$, $x > 0$, and $y > 0$:

- $\log_b b = 1$ and $\ln e = 1$
- $\log_b b^x = x$ and $\ln e^x = x$
- $\log_b x^y = y \log_b x$ and $\ln x^y = y \ln x$
- $\log_b (xy) = \log_b x + \log_b y$ and $\ln(xy) = \ln x + \ln y$
- $\log_b (x/y) = \log_b x - \log_b y$ and $\ln(x/y) = \ln x - \ln y$
- $b^{\log_b x} = x$ and $e^{\ln x} = x$

c. Use your response to Part **b** to determine the approximate number of years required for the account balance to reach each of the following amounts:

1. $1660.10
2. $3024.80

Discussion 2

a. In the example given in Part **b** of Exploration **2**, the equation $y = 3 \bullet 10^x$ is solved for *x*. Describe how the properties of logarithms were used to solve this equation.

b. Explain how you used the properties of logarithms to solve the equation in Part **a** of Exploration **2** for *t*.

c. In Part **c** of Exploration **2**, you used natural logarithms to determine that an initial investment of $500 at an annual interest rate of 6%, compounded continuously, would require approximately 20 yr to reach a balance of $1660.10.

It also is possible to determine this solution using common logs, as shown below.

$$1660.10 = 500e^{0.06t}$$
$$1660.10/500 = e^{0.06t}$$
$$\log(1660.10/500) = \log e^{0.06t}$$
$$\log(1660.10/500) = 0.06t \log e$$
$$\frac{\log(1660.10/500)}{0.06 \log e} = t$$

1. What advantages are there to using natural logs to solve this equation for *t*?

2. Consider another equation in which the variable to be solved for is an exponent. Do you think it would be possible to solve this equation using logarithms of any base?

Assignment

3.1. **a.** Using natural logarithms, convert each of the following equations from exponential form to logarithmic form.

1. $e^5 = x$
2. $e^0 = 1$
3. $e^{0.06x} = 3$

b. Solve $e^{0.06x} = 3$ for x.

3.2 Convert each of the following equations from logarithmic to exponential form.

a. $\ln x = 2$

b. $\ln e = 1$

c $\ln(y/750) = 0.05x$

3.3 Imagine that a young child invests 100 pennies in an account which compounds interest continuously. Using the equation $P_0e^{rt} = 2P_0$, determine how long it will take the child's initial investment to double at each of the following annual interest rates.

a. 6%

b. 8%

c. 10%

3.4 **a.** To help pay for their newborn child's future education, two parents decide to open a savings account. They make an initial deposit of \$1275 at an annual interest rate of 7%, compounded quarterly.

1. How long will it take for the initial deposit to double?
2. How long will it take for the initial deposit to triple?
3. If the parents make no further deposits or withdrawals, what will the account balance be when the child is ready to enter college?

b. Repeat Part **a** for an initial deposit of \$1275 at an annual interest rate of 7%, compounded continuously.

c. Compare your responses to Part **b** with your responses to Part **a**. Describe any differences you observe.

3.5 Consider an initial investment of P_0 at an annual interest rate of r, compounded continuously.

a. Write an equation that describes the value of the investment after t years.

b. Solve the equation in Part **a** for t.

3.6 LaSasha wants to purchase a new stereo system. The one she has selected costs \$715. At the moment, however, she has only \$500 available to spend. While exploring her options, LaSasha examines an investment account that offers an annual interest rate of 8%, compounded continuously. Use natural logarithms to complete Parts **a–c** below.

a. If LaSasha decides to invest in this account, how long will it take for the account balance to reach \$715?

b. LaSasha would like to buy the stereo system within 6 months. What annual interest rate would she have to earn to make this possible?

c. Is it reasonable to expect an interest rate of this size?

* * * * *

3.7 At last count, the population of Central City was 410,000. City planners expect the population to increase at a rate of 4.25% each year.

a. Write a function to model the number of years t it will take for the city to grow to a given population p.

b. The city prefers to employ one law enforcement officer for every 1000 people. If its growth rate remains constant, determine when the city will need to employ each of the following numbers of officers:

1. 500

2. 1000

3.8 **a.** Describe how different values of k affect the graphs of the following equations:

1. $y = \ln x + k$

2. $y = \ln x + \ln e^k$

b. Repeat Part **a** for the equations below.

1. $y = k \ln x$

2. $y = \ln x^k$

c. **1.** Describe a relationship between the pair of equations in Part **a**.

2. Describe a relationship between the pair of equations in Part **b**.

d. Use laws of exponents to support your responses to Part **c**.

* * * * * * * * * *

Summary Assessment

1. As part of his savings plan, Vonzel invested \$5000 in a one-year certificate of deposit (CD) at an annual interest rate of 7%, compounded daily. When Vonzel told his Aunt Theresa about his investment, she advised him to withdraw the money. Another bank in town, she said, advertises the same interest rate, compounded continuously.

a. Since Vonzel's bank charges a \$150 penalty for early withdrawal, he decided not to move the money. Did Vonzel make the right decision? Explain your response.

b. How long will it take Vonzel's CD to earn \$150 (the cost of the penalty) in interest?

c. The total value of the certificates of deposit at each bank is \$5 million. In this situation, how much more does it cost a bank to pay interest compounded continuously rather than daily?

2. Shortly after buying the \$5000 certificate of deposit, Vonzel purchases a \$1400 stereo system with his credit card. His credit card company charges interest at an annual rate of 13%, compounded daily. Another company has offered him a credit card with the same annual interest rate, compounded monthly. The annual fee for the new card is \$55; the annual fee for his current card fee is \$50. Should Vonzel change credit cards? Explain your response.

3. Vonzel's credit card company offers a no-minimum payment option to some customers with excellent credit ratings. Using this option, customers may carry any balance due until the end of the next month. The interest charged on the balance, however, continues to be compounded daily.

Due to some unforeseen expenses, Vonzel can't afford to pay his \$1400 credit bill. Should he withdraw his \$5000 certificate of deposit, pay the \$150 penalty, then use some of the remaining cash to pay his credit card bill? Explain your response.

Module Summary

- **Principal** is the amount of money invested or loaned.
- **Interest** is the amount earned on invested money, or the fee charged for loaned money.
- The formula for calculating **simple interest**, where I represents interest, P represents principal, r represents the interest rate per time period, and t represents the number of time periods is shown below:

$$I = Prt$$

- The principal at the end of each time period in an investment or savings account can be thought of as a sequence. Assuming that no withdrawals are made and any interest earned is deposited in the account, such a sequence can be defined recursively by the following formula:

$$P_t = P_{t-1} + r \bullet P_{t-1} = P_{t-1}(1 + r)$$

  where P_t is the principal at the end of t years, r is the annual interest rate, and P_{t-1} is the principal for the previous year.

- When interest is compounded annually, the yearly account balances that result can be thought of as a sequence defined explicitly by the following formula (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_t = P_0(1 + r)^t$$

  where P_t is the account balance after t years, P_0 is the initial principal, r is the annual interest rate, and t is the time in years.

- When compounding interest c times per year for t years, the formula for the account balance after n compounding periods is:

$$P_n = P_0\left(1 + \frac{r}{c}\right)^n = P_0\left(1 + \frac{r}{c}\right)^{ct}$$

  where P_n represents the principal after n compounding periods, P_0 represents the initial principal, and r is the annual interest rate. Note that, in this formula, $n = ct$.

- **Infinity**, represented by the symbol ∞, depicts an unlimited quantity or an amount larger than any fixed value. Written as $+\infty$ or $-\infty$, it also may be used to depict quantities that extend without bound in either a positive or a negative direction.

- The value of *e* can be represented mathematically as shown below:

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$$

- When the number of compoundings per year approaches infinity, then the interest is **compounded continuously**. In this case, the formula for account balance *P* can be written as follows:

$$P = \lim_{c \to \infty} P_0\left(1 + \frac{r}{c}\right)^{ct} = P_0\left[\lim_{c \to \infty}\left(1 + \frac{r}{c}\right)^{c}\right]^{t} = P_0 e^{rt}$$

  where P_0 represents the initial principal, *r* represents the annual interest rate, *c* represents the number of compoundings per year, and *t* represents number of years.

- To help consumers compare interest rates, banks often report **annual percentage yield (APY)** for savings accounts and **annual percentage rate (APR)** for loans. The APY or APR is the interest rate that, when compounded annually, will produce the same account balance as the advertised interest rate, which is typically compounded more often.

- Logarithms with base *e* are referred to as **natural logarithms**. The natural log of *x* is denoted by $\ln x$ where $x > 0$. The equation $\ln x = y$ is true if $e^y = x$.

- The properties that are true for $\log_b x$ also are true for $\ln x$. Therefore, for $b > 0$, $b \neq 1$, $x > 0$, and $y > 0$:

  - $\log_b b = 1$ and $\ln e = 1$
  - $\log_b b^x = x$ and $\ln e^x = x$
  - $\log_b x^y = y \log_b x$ and $\ln x^y = y \ln x$
  - $\log_b (xy) = \log_b x + \log_b y$ and $\ln(xy) = \ln x + \ln y$
  - $\log_b (x/y) = \log_b x - \log_b y$ and $\ln(x/y) = \ln x - \ln y$
  - $b^{\log_b x} = x$ and $e^{\ln x} = x$

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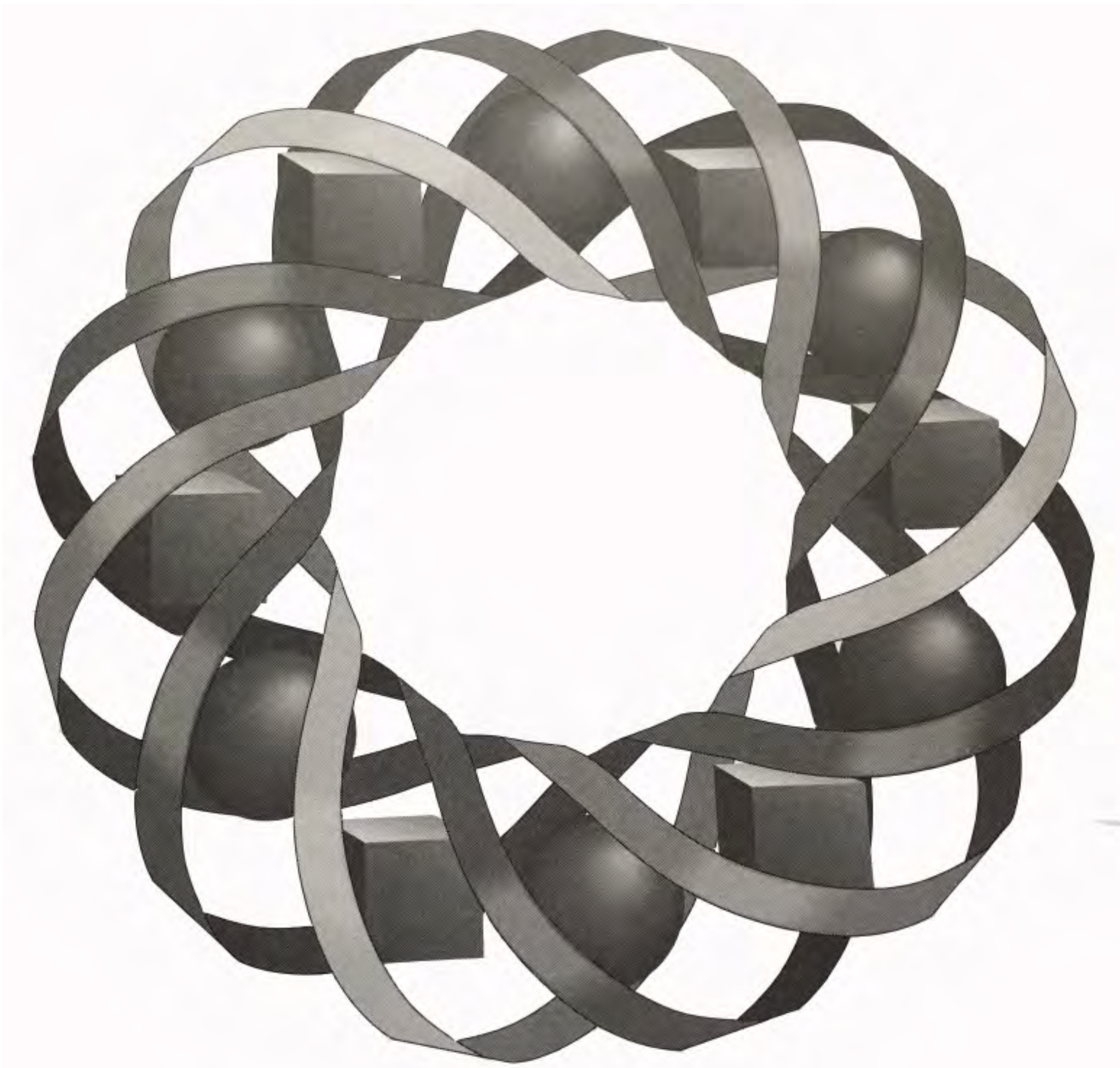
Functioning on a Path



In this module, you use your knowledge of polynomial and rational functions to play a mathematics game.

Gary Bauer • Russ Killingsworth • Margaret Plouvier

Functioning on a Path

Introduction

In a mathematics video game called Gates, there are three different levels of play. In the first level, the object of the game is to determine the characteristics of a **continuous** function that passes through several "gates," represented by a pair of squares on the screen. For example, Figure **1** shows a first-level screen with a second-degree polynomial (or quadratic) function passing through five gates.

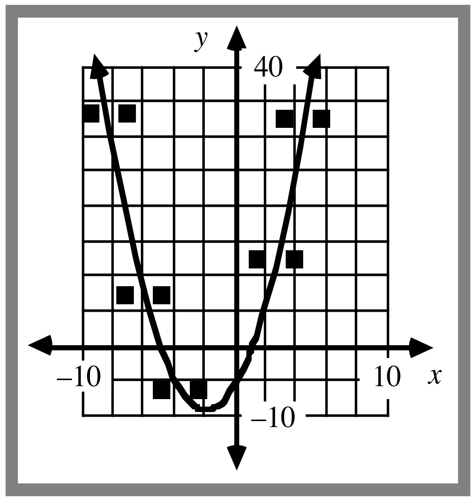


Figure 1: Quadratic function passing through gates

Mathematics Note

A function is **continuous** at a point c in its domain if the following conditions are met:

- the function is defined at c, or $f(c)$ exists
- the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
- the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$

A function is continuous over its domain if it is continuous at each point in its domain.

A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.

For example, a function is discontinuous at $x = c$ if the function is undefined at c, as shown in Figure **2**.

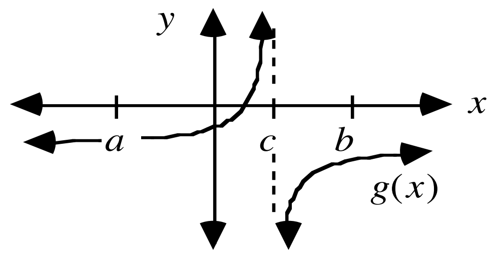


Figure 2: Graph of the discontinuous function $g(x)$

A function is also discontinuous at $x = c$ if the limit of the function does not exist at c, as shown in Figure **3**.

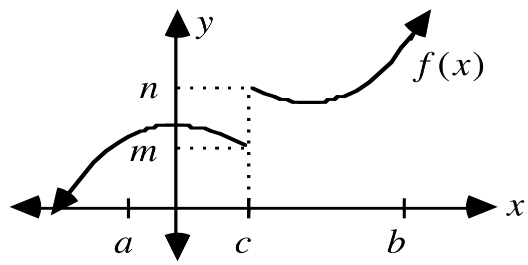


Figure 3: Jump discontinuity in the function $f(x)$

In this case, $f(x)$ approaches m as x approaches c from the left. As x approaches c from the right, $f(x)$ approaches n. Since $m \neq n$, the limit of $f(x)$ as x approaches c does not exist. This kind of discontinuity is referred to as a **jump discontinuity**.

A function is also discontinuous at $x = c$ if the value of the function at c does not equal the limit of the function as x approaches c, as shown in Figure **4**. In this case, the limit of $h(x)$ as x approaches c is m, while $h(c) = n$, and $m \neq n$.

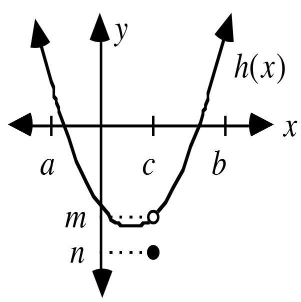


Figure 4: Hole in the graph of the function $h(x)$

Discussion

a. In the first level of Gates, players may use polynomial functions. Recall from the Level 4 module "Drafting and Polynomials" that a **polynomial function** can be written in the following general form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

where a_n is a real number and n is a non-negative integer.

Are the functions below polynomial functions? Explain your responses.

1. $f(x) = x^2 + \sqrt{3} \bullet x - 5$

2. $f(x) = x^2 + \dfrac{4}{x} - 5$

b. The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. The coefficient of that variable is the **leading coefficient**. For a polynomial written in the general form in Part **a**, the degree is n and the leading coefficient is a_n.

Identify the degree and leading coefficient of the polynomial below.

$$x^2 + 5x^4 - 5x^5$$

c. Do you think that all polynomial functions are continuous? Explain your response.

d. The graph in Figure **1** is a second-degree polynomial function. It is only one of many polynomial functions that could be drawn through the desired gates. Describe how you could use polynomial regressions to determine three other continuous functions that pass through these gates.

Activity 1

Figure **1** showed a screen from the first level of a game in which players must identify a polynomial function that passes through gates. One way to determine such an equation involves the relationship among the degree of a polynomial, its roots (or zeros), and the characteristics of its graph.

Exploration 1

A completed screen from a game of Gates is shown in Figure **5**. The degree of the polynomial function that produced this path can be predicted from the characteristics of the graph. In this exploration, you examine polynomial functions of several different degrees and attempt to identify the characteristics of their graphs.

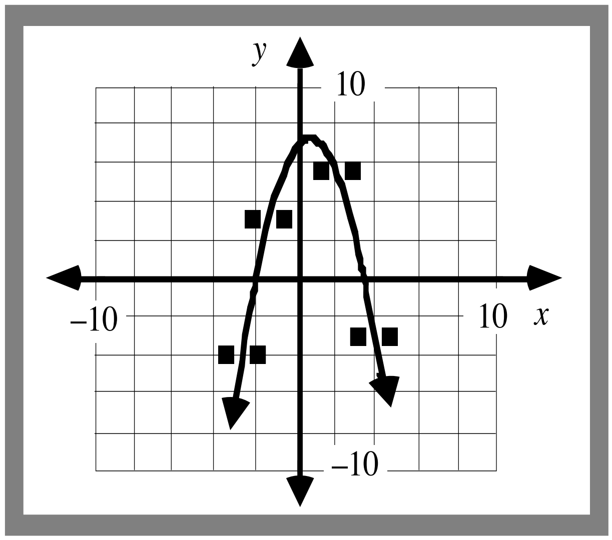


Figure 5: Screen in first level of Gates

a. Make a conjecture about the least possible degree of a polynomial function that could pass through the gates in Figure **5**.

b. Choose the coordinates of any three noncollinear points on a two-dimensional coordinate system.

c. Use a quadratic regression to determine an equation for a polynomial function passing through those three points.

d. A quadratic function can be written in the following general form:

$$q(x) = a_2x^2 + a_1x + a_0$$

1. Use the coordinates of the three points in Part **b** to write a system of equations involving a_2, a_1, and a_0.
2. Use technology to solve the system for a_2, a_1, and a_0.
3. Use the solutions to write a quadratic equation whose graph contains the original three points.

e. On the same coordinate system, plot the points selected in Part **b** and the function found in Part **d**.

Discussion 1

a. **1.** If a quadratic function models the graph in Figure **5**, what do you know about its leading coefficient? Why?

2. If any polynomial function with an even degree models the graph in Figure **5**, what do you know about its leading coefficient?

b. **1.** Why should you expect the functions found in Parts **c** and **d** of Exploration **1** to be equal?

2. Why might the actual equations be slightly different?

c. Considering the class results to Part **d**, do you think that any three noncollinear points can be contained in the graph of a quadratic polynomial?

Exploration 2

a. Sketch the graph of a quadratic function whose leading coefficient is positive.

b. Use your sketch to predict the maximum number of times function of this degree can intersect the x-axis.

Mathematics Note

The **end behavior** of the graph of a polynomial function describes the characteristics of the graph as $|x|$ approaches infinity.

For example, Figure **6** shows the graph of a fourth-degree polynomial function. As $|x|$ approaches infinity, $f(x)$ also increases without bound.

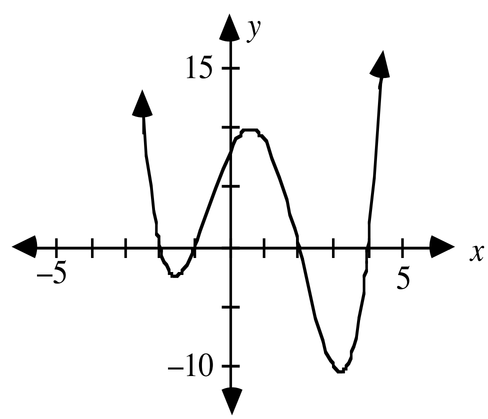


Figure 6: Graph of $f(x) = 0.5x^4 - 1.5x^3 - 4x^2 + 6x + 8$

c. Describe the end behavior of the graph you sketched in Part **a**.

d. Sketch graphs of quadratics that intersect the x-axis in 0, 1, … n times, where n is the number you identified in Part **b**.

e. Using a graphing utility, repeat Parts **a–d** for each of the following:

1. a third-degree polynomial function with four different coefficients and a positive leading coefficient
2. a fourth-degree polynomial function with five different coefficients and a positive leading coefficient.

Discussion 2

a. Recall from the Level 4 module "Can It," that the **absolute maximum** of a function is the greatest value of the range, while the **absolute minimum** is the least value of the range.

1. Of the functions that you graphed in Exploration **2**, which ones had an absolute maximum?
2. Which functions had an absolute minimum?

b. If the leading coefficients of the equations in Exploration **2** had been negative, how would this have affected the absolute maximums or minimums of the functions identified in Part **a**?

c.
1. Describe a function that has both an absolute maximum and an absolute minimum.
2. Describe some functions that have neither an absolute maximum nor an absolute minimum.

d. The x-coordinate of each point where the graph of a polynomial intersects the x-axis is a **zero** or a **root** of the polynomial. What do you think is the maximum number of zeros a polynomial of degree n can have? Explain your response.

e. Consider a function $f(x)$. What is the value of $f(x)$ when x is a root of the function?

f. Given the graph of a polynomial function with three zeros, what do you think is the least possible degree for the polynomial?

g. Recall that a polynomial function also can be written as follows, where each expression of the form $(x - a_i)$ represents a factor of the polynomial:

$$f(x) = (x - a_n)(x - a_{n-1})(x - a_{n-2})\cdots(x - a_0)$$

Consider a polynomial function with zeros of –6, 8, and 2. Write an equation for this function using the least possible degree.

h. How could you determine a polynomial of higher degree than the one in Part **g** that has exactly the same zeros?

Assignment

1.1 Consider a function that has exactly one zero. Identify the polynomial of least degree that could describe such a function. Justify your response with a graph and with a general equation for the polynomial.

1.2

a. Sketch the graph of a function with an absolute maximum of 7 when $x = 2$.

b. Do you think every quadratic function has either an absolute maximum or minimum? Explain your response.

c. What can you tell about a quadratic function if its graph has an absolute maximum? an absolute minimum?

d. Sketch a polynomial of degree greater than 2 that has an absolute maximum.

e. Do you think that any polynomial of degree 3 has an absolute maximum or an absolute minimum? Explain your response.

1.3

a. Figure **6** shows a graph of $f(x) = 0.5x^4 - 1.5x^3 - 4x^2 + 6x + 8$. Do you think this function has an absolute maximum or an absolute minimum? Explain your response.

b. The "peaks" and "valleys" of the graph in Figure **6** might be used to identify "relative maximums" and "relative minimums." How would you define these terms?

1.4 Consider the polynomial $f(x) = 3(x-2)^2(x+4)(x-\pi)$.

a. Describe how to determine the zeros of this function.

b. Identify the degree of this polynomial.

c. Determine whether the polynomial has an absolute minimum or absolute maximum.

d. Describe the end behavior of the polynomial.

1.5

a. Identify the general form of a polynomial function that has the least possible odd degree.

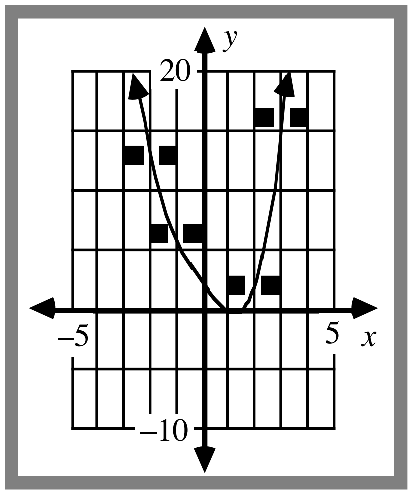
b. Describe the end behavior of this type of polynomial.

c. What do you think is true about the end behavior of any polynomial function with an odd degree? Explain your response.

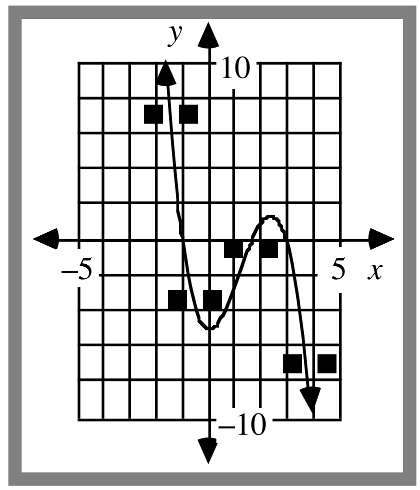
1.6 Given the second-degree polynomial $f(x) = x^2 + 5x + 6$, describe how you could find the exact values of its zeros.

1.7 Parts **a** and **b** below show two screens in the first level of Gates. Explain whether you think the degree of each polynomial shown is odd or even. Describe the characteristics that support your choice.

a.



b.



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1.8 Write an equation of a polynomial function whose graph contains the points with the following coordinates:

a. $(-8,-7)$ and $(5,10)$

b. $(0,0)$, $(-1,1)$, and $(-8,64)$

1.9 The data in the following table was collected during the flight of a model rocket. The values for time represent the number of seconds after the engine burned out. The values for distance represent height in meters. Find an equation that closely models this data and describe how you identified your model.

Time (sec)	**Distance (m)**
2	90.4
3	85.9
4	71.6
5	47.5
6	13.6

* * * * * * * * * *

Activity 2

In the second level of Gates, players use **piecewise functions** to describe graphs. Like any function, a piecewise function has a domain, a range, and a rule relating the two. In a piecewise function, however, different parts of the domain correspond with different rules.

For example, consider the absolute-value function, $f(x) = |x|$. In this function, the rule $f(x) = -x$ applies to the domain interval $(-\infty, 0)$, while the rule $f(x) = x$ applies to the domain interval $[0, \infty)$. This can be written as shown below:

$$f(x) = \begin{cases} -x \text{ if } x \in (-\infty, 0) \\ x \text{ if } x \in [0, \infty) \end{cases}$$

Exploration

In this exploration, you examine the use of piecewise functions to create successful functions for the game of Gates. For example, the screen in Figure **7** shows a piecewise function that passes through five gates.

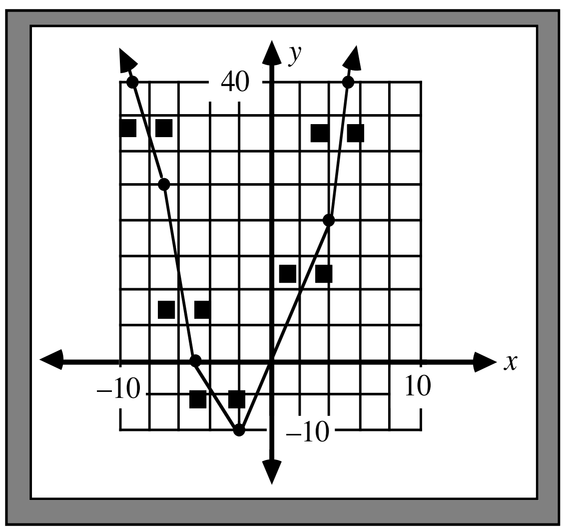


Figure 7: Screen with piecewise function

The graph of this function consists of two rays and three segments. The ray in the upper left-hand portion of the screen has its endpoint at $(-7, 25)$ and contains the point $(-9, 40)$. The ray in the upper right-hand portion of the screen has its endpoint at $(4, 20)$ and contains the point $(5, 40)$.

The three segments have the following pairs of endpoints: $(-7, 25)$ and $(-5, 0)$; $(-5, 0)$ and $(-2, -10)$; and $(-2, -10)$ and $(4, 20)$.

a. The equation of a ray or a segment can be found by determining the equation of the line that contains it and restricting the domain to an appropriate interval.

1. Write an equation for the ray that has its endpoint at $(-7,25)$ and contains the point $(-9,40)$. State the domain of the equation.
2. Write an equation for the segment with endpoints at $(-7,25)$ and $(-5,0)$. State the domain of the equation.
3. Write an equation for the segment with endpoints at $(-5,0)$ and $(-2,-10)$. State the domain of the equation.
4. Write an equation for the segment with endpoints at $(-2,-10)$ and $(4,20)$. State the domain of the equation.
5. Write an equation for the ray that has its endpoint at $(4,20)$ and contains the point $(5,40)$. State the domain of the equation.

b. Use the equations from Part **a** to write a piecewise function whose graph is the one shown in Figure **7**.

c. Use technology to graph the function defined in Part **b**.

Discussion

a. Describe how you used technology to graph the piecewise function in Part **c** of the exploration.

b. Describe how you could show algebraically that each two consecutive parts of a continuous piecewise function contain a common point.

c. Explain why the function in Part **b** of the Exploration is continuous.

d. Consider the following:

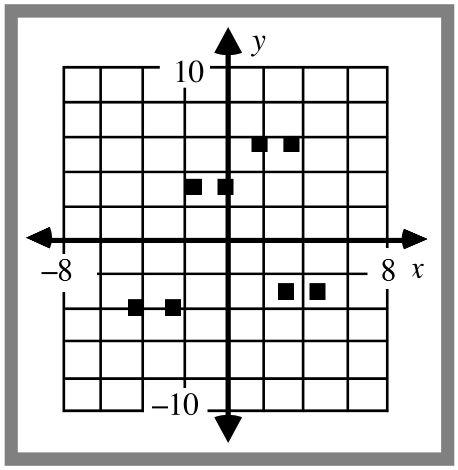
$$f(x) = \left\{ \begin{array}{l} 1/x \text{ if } x \in (-\infty,0) \cup (0,+\infty) \\ 0 \text{ if } x = 0 \end{array} \right\}$$

1. Is f a function?
2. Is f a piecewise function?
3. Is f continuous at 0?

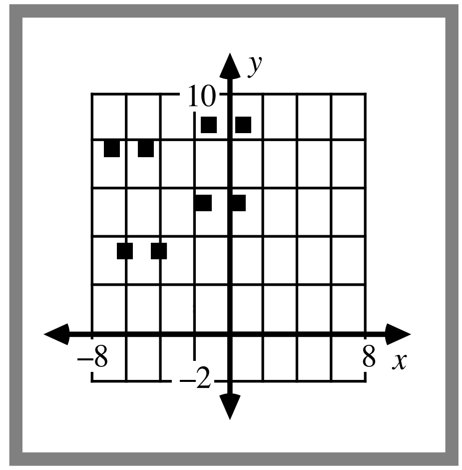
Assignment

2.1 For each screen shown in Parts **a–c** below, determine a continuous piecewise function that passes through all the gates.

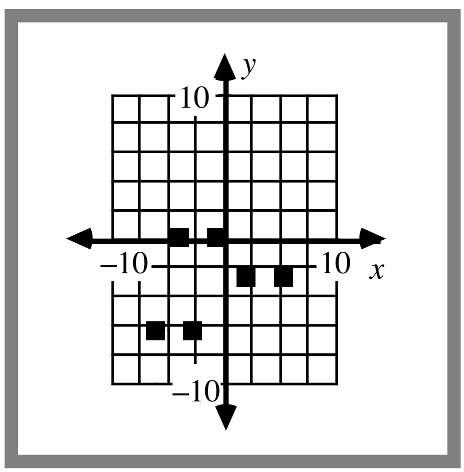
a.



b.

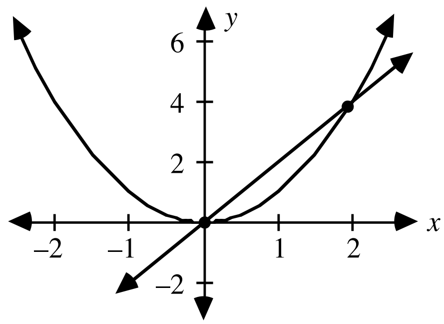


c.



2.2 The greatest integer function, $f(x) = [x]$, pairs every element x in the domain with the greatest integer less than or equal to x. Explain why this function can be considered a piecewise function.

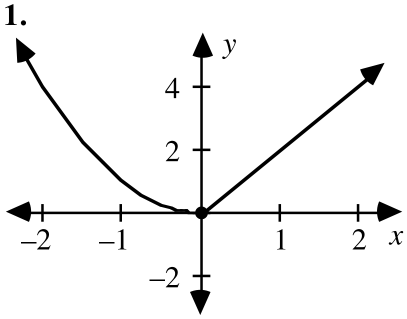
2.3 The graphs of the functions $f(x) = x^2$ and $g(x) = 2x$ are shown below.



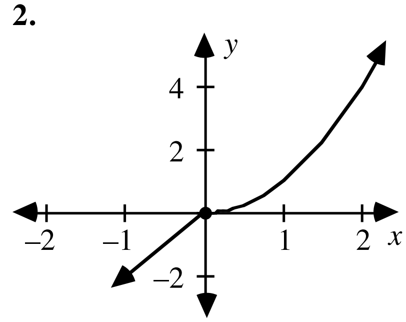
a. Determine the points of intersection of the two functions.

b. Each graph below shows a piecewise function that uses the rules for $f(x)$ and $g(x)$ over parts of its domain. Write a continuous piecewise function to describe each graph.

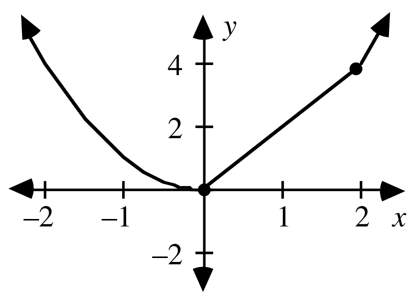
1.



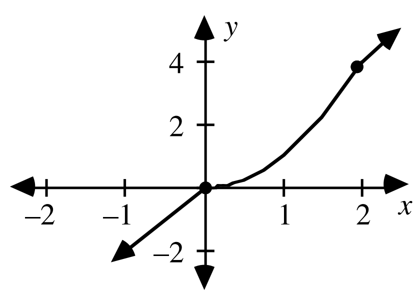
2.



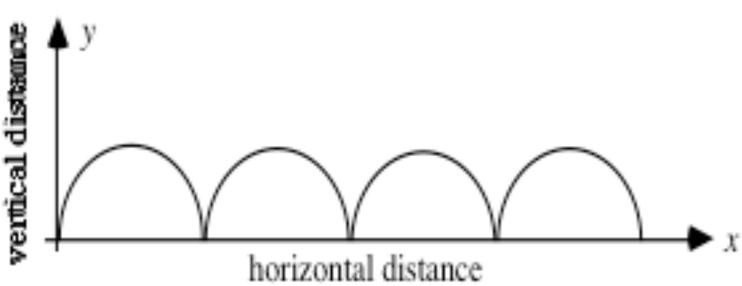
3.



4.



2.4 The graph below models the border to a flowerbed. Each piece of the border is made of a semicircular slab of concrete. Describe this curve using a piecewise function.



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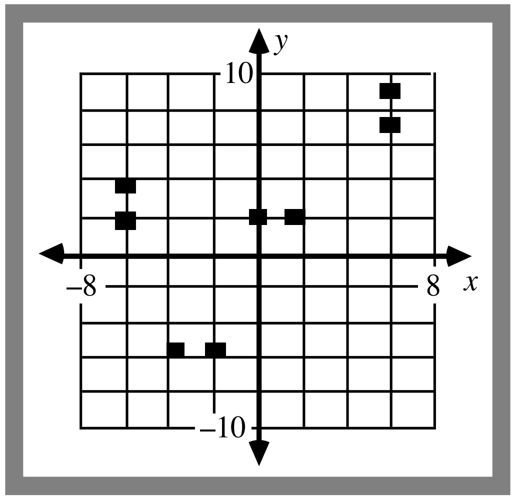
2.5 Use piecewise functions to describe a graph that passes through the gates in the following screen.



2.6. Recall that the height of a freely falling object can be described by the following function:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0$$

where g represents the acceleration due to gravity $\left(9.8 \text{ m/sec}^2\right)$, t is time in sec, v_0 is the object's initial velocity, and h_0 is its initial height.

a. Consider a ball dropped from a height of 1 m. On each successive bounce, it rises to two-thirds the height of the previous bounce. Describe the graph that you think would model the first three bounces of the ball. Justify your response.

b. The ball hits the ground at approximately 0.45 sec, 1.19 sec, 1.79 sec, and 2.29 sec. Determine a piecewise function that models the first three bounces of the ball.

c. Create a graph of your piecewise function.

* * * * * * * * * *

Activity 3

In the previous activities, you investigated some characteristics of polynomial functions and piecewise functions while playing the first two levels of Gates. In this activity, you examine the graphs of yet another type of function. Your discoveries should help you develop a strategy for the next level of Gates.

Exploration 1

Figure **8** shows a screen from the third level of Gates. The line represents the graph of the polynomial function below:

$$f(x) = \frac{1}{3}x + 5$$

Your challenge is to alter the function so that its graph passes through all four gates.

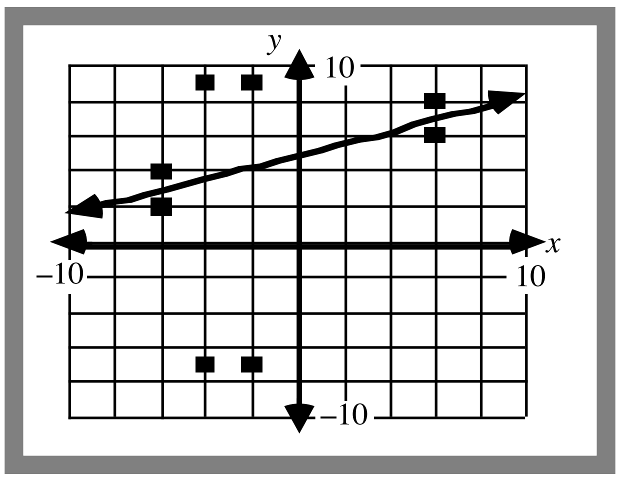


Figure 8: Screen in the third level of Gates

One way to accomplish this task involves rational functions. In this exploration, you examine how the graph of a linear function is affected by the addition of a rational expression.

Recall from the Level 4 module "Big Business" that a **rational function** can be written in the following general form:

$$r(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions and $g(x) \neq 0$.

a. Any rational function also can be expressed in the form below, where $q(x)$ and $h(x)$ are polynomial functions such that $q(x)$ is the quotient and $h(x)$ is the remainder when $f(x)$ is divided by $g(x)$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

1. Choose a polynomial function $q(x)$ of degree 1.

2. Choose a polynomial function $g(x)$ of degree 1.

b. **1.** Create a rational function $r(x)$ by adding the expression $1/g(x)$ to $q(x)$ as follows:

$$r(x) = q(x) + \frac{1}{g(x)}$$

2. Determine the domain of $r(x)$.

3. Recall that a discontinuity occurs in a rational function at any point where the value of the function is undefined. Predict the behavior of the graph of $r(x)$ near its point of discontinuity.

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point *P* on the curve to the line approaches zero as the distance from *P* to the origin increases without bound, where *P* is on a suitable part of the curve.

For example, Figure **9** shows a graph of the function $f(x) = \log x$. In this case, the suitable part of the curve lies below the *x*-axis. As a point *P* moves farther away from the origin on this part of the curve, the distance between *P* and the *y*-axis approaches 0. Therefore, the line $x = 0$ is an asymptote for the curve.

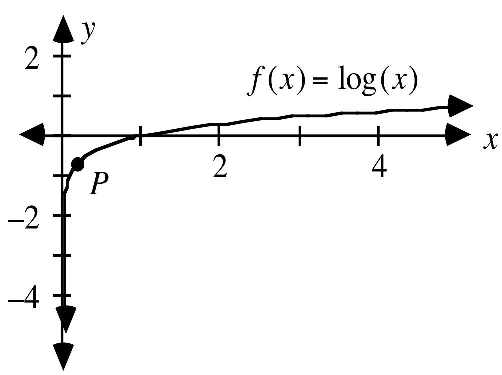


Figure 9: Graph of $f(x) = \log(x)$

c. Graph $r(x)$ and $q(x)$ on the same coordinate system. Use an interval of the domain that includes the point where the graph is discontinuous.

d. Determine what happens to the values of the functions $q(x)$, $r(x)$, and $1/g(x)$ as each of the following occurs:

1. x approaches the point of discontinuity
2. $|x|$ increases without bound.

e. Select a new function $q(x)$ with degree 2 and a new function $g(x)$ with degree 1. Repeat Parts **b–d**.

f. Select a new function $q(x)$ with degree 3 and a new function $g(x)$ with degree 1. Repeat Parts **b–d**.

Discussion 1

a. Describe the graphs of $r(x)$ and $q(x)$ as x approaches the point of discontinuity. Why does this behavior occur?

b. How does the addition of the rational expression $1/g(x)$ affect the graph of a polynomial function $q(x)$?

c. Describe a rational function that has more than one vertical asymptote.

d. Suppose that, in Part **b** of Exploration **1**, you had added the additive inverse of $1/g(x)$ to $q(x)$. How would this have affected the graph of the resulting function $r(x)$?

e. What rational expression could you add to the following polynomial function in order to create a path that passes through all four gates in Figure **8**?

$$f(x) = \frac{1}{3}x + 5$$

f. Consider the rational function

$$r(x) = f(x) + \frac{1}{g(x)}$$

1. Describe the graph of $r(x)$ when $g(x)$ is a linear function.
2. How does the end behavior of $r(x)$ compare to that of $f(x)$?

g. Consider a rational function written in the following general form, where $q(x)$, $h(x)$, and $g(x)$ are all polynomial functions and $g(x) \neq 0$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

Describe how you could rewrite this function in the form below:

$$r(x) = \frac{f(x)}{g(x)}$$

Exploration 2

Figure **10** shows another screen in the third level of Gates. To complete this screen, you must alter the function $f(x) = 2$ so that its graph passes through all three gates without running into the brick wall.

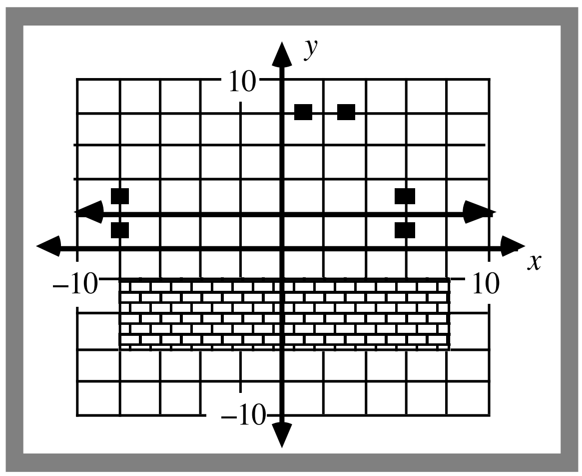


Figure 10: Screen in third level of Gates

As you discovered in Exploration **1**, a rational function of the form below has the same end behavior as $f(x)$, when $g(x)$ is a first-degree polynomial.

$$r(x) = f(x) + \frac{1}{g(x)}$$

In this exploration, you experiment with other degrees for the denominator of the added rational expression.

a. Select a first-degree polynomial $k(x)$ and a constant function $h(x)$. Graph each of the following pairs of functions on a different set of axes. Note any similarities or differences between the two graphs, including their end behaviors.

1. $f(x) = h(x) + \dfrac{1}{k(x)}$ and $g(x) = h(x) + \dfrac{1}{(k(x))^2}$

2. $f(x) = h(x) + \dfrac{1}{(k(x))^3}$ and $g(x) = h(x) + \dfrac{1}{(k(x))^4}$

b.

1. Compare the graphs of the two functions below for a positive value of *n,* where *n* is an integer other than 1.

$$f(x) = h(x) + \frac{1}{(k(x))^2} \text{ and } g(x) = h(x) + \frac{n}{(k(x))^2}$$

2. Repeat Step **1** for several different positive values of *n*.

3. Repeat Step **1** for several different negative values of *n*.

Discussion 2

a. What similarities or differences did you observe in the graphs of the two functions below?

$$f(x) = h(x) + \frac{1}{k(x)} \text{ and } g(x) = h(x) + \frac{1}{(k(x))^2}$$

b.

1. How did the graphs of the following two functions compare when *n* was a positive integer?

$$f(x) = h(x) + \frac{1}{(k(x))^2} \text{ and } g(x) = h(x) + \frac{n}{(k(x))^2}$$

2. How did the graphs compare when *n* was a negative integer?

c. Consider the following two rational functions:

$$f(x) = 3 + \frac{1}{(x+6)^2} \text{ and } g(x) = 3 + \frac{-1}{(x+6)^2}$$

1. As $|x|$ increases without bound, the values of both $f(x)$ and $g(x)$ approach 3. Explain why this occurs.

2. As x approaches –6, however, $f(x)$ approaches $+\infty$ while $g(x)$ approaches $-\infty$. Explain why this occurs.

d. Consider a function of the form below, where c is a constant and n is a positive integer:

$$f(x) = h(x) + \frac{c}{\left(k(x)\right)^n}$$

1. Describe how raising the denominator of the added rational expression from an odd power to an even power affects the graph of the resulting function.
2. Describe how changing the numerator of the added rational expression affects the graph of the resulting function.

e.

1. What rational expression could you add to $f(x) = 2$ to create a graph that passes through all the gates in Figure **10** but misses the brick wall?
2. Are there other rational expressions that would accomplish this task? Explain your response.

f. Consider a rational function written in the form below, where $g(x) \neq 0$:

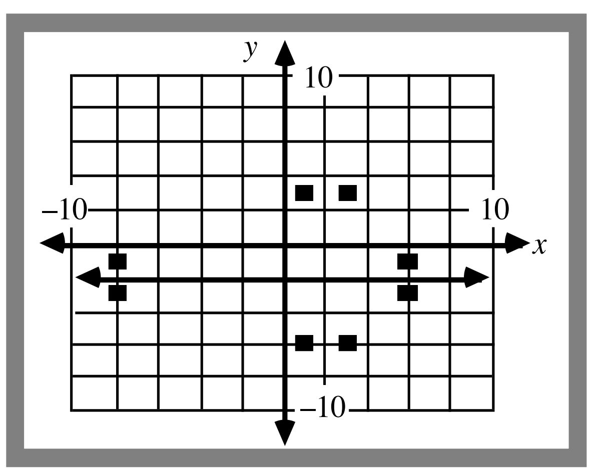
$$r(x) = \frac{f(x)}{g(x)}$$

Describe how you could express this function in the following form, where $q(x)$ is a polynomial function and the degree of $h(x)$ is less than or equal to the degree of $g(x)$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

Assignment

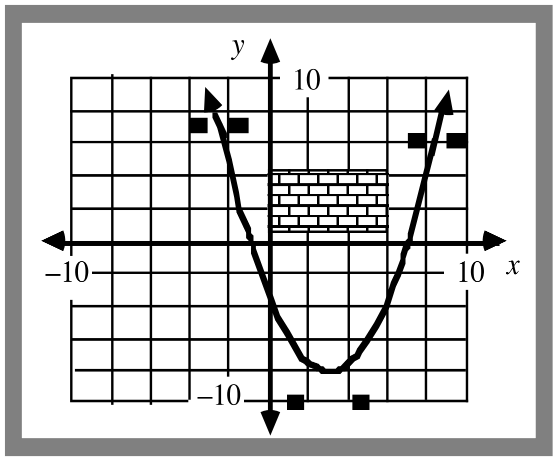
3.1 The figure below shows a screen in the third level of Gates. Add a rational expression to $f(x) = -2$ so that the graph of the resulting function passes through all four gates. Identify the domain and range of your function.



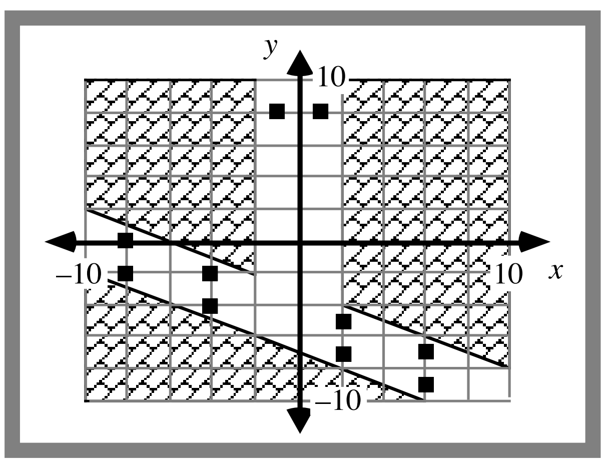
3.2 The following screen in the third level of Gates includes three gates and a brick wall. The curve shown on the screen represents a graph of the function

$$f(x) = \frac{1}{2}x^2 - 3x - \frac{7}{2}$$

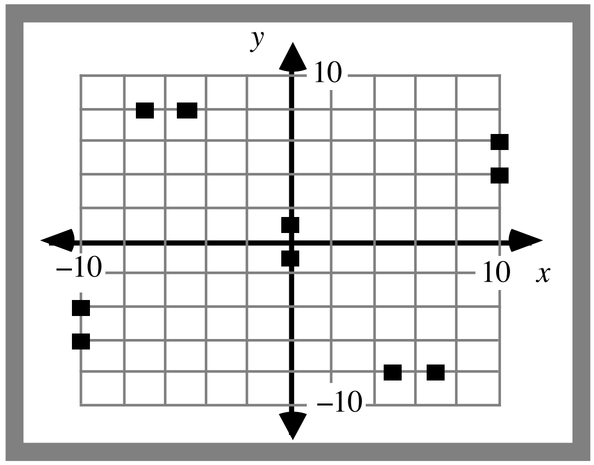
Add a rational expression to $f(x)$ so that the graph of the resulting function passes through all three gates while avoiding the brick wall. Identify the domain and range of your function.



3.3 The figure below shows another screen in the third level of Gates. Determine a rational function that passes through all five gates without touching the brick walls.



3.4 **a.** Create a rational function that passes through all the gates shown on the screen below.



b. Create a continuous piecewise function that passes through all the gates shown on the screen in Part **a**.

c. Compare the domains and ranges of the two functions created in Parts **a** and **b**.

3.5 A screen in the third level of Gates has six gates and a brick wall. The locations of the gates are defined by the following pairs of points: (–7,–1) and (–5,–1); (–5,0) and (–3,0); (–3,1) and (–1,1); (1,–2) and (3,–2); (5,5) and (7,5); (7,6) and (9,6).

The region occupied by the brick wall is defined by the following constraints: $1 \leq x \leq 3$ and $4 \leq y \leq 6$.

a. Use a graphing utility to recreate this screen.

b. Find a rational function whose graph passes through all the gates without hitting the brick wall.

3.6 Create a screen for the third level of Gates that includes five gates and two brick walls. Find a rational function whose graph passes through all the gates without hitting the walls. Identify the domain and range of your function.

3.7 Rational functions are generally written as the quotient of two polynomials. In this activity, however, you wrote rational functions in a form that makes it easier to determine the locations of any vertical asymptotes. Use a symbolic manipulator to convert each of the following functions to the form $r(x) = f(x)/q(x)$. Test your answers by graphing both forms of each equation on the same coordinate system.

a. $h(x) = 5x^2 + x - 10 + \dfrac{3}{x-7}$

b. $g(x) = -11x + 4 + \dfrac{-11}{3x+2}$

c. $q(x) = x^3 + x^2 + \dfrac{3}{x+1}$

* * * * *

3.8 Write each of the following rational functions as a polynomial expression plus a rational expression. Describe where asymptotic behavior might occur in the graph of each function.

a. $h(x) = \dfrac{3x^2 + 29x - 39}{x+11}$

b. $g(x) = \dfrac{-16x^3 + 40x^2 - 8x + 18}{2x-5}$

c. $q(x) = \dfrac{x^5 - 7x^4 - x^3 + 4x^2 + 25x - 31}{x-7}$

3.9 A cattle rancher would like to create a rectangular corral with an area of 150 m^2 using the least possible amount of fencing materials.

a. Let x represent the width of the rectangle. Write a rational function that describes the perimeter of the rectangle in terms of x.

b. Graph the function and determine the location of any asymptotes that occur. If an asymptote occurs, describe what its location represents in this situation.

c. Considering the context, identify a reasonable domain and range for the function.

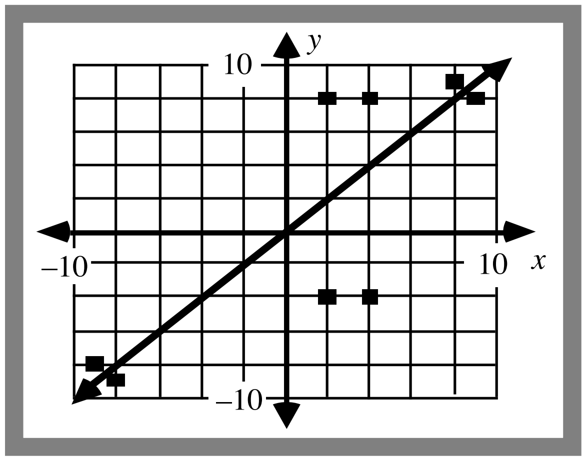
d. What is the minimum value for the perimeter of the rectangular corral? What are the corresponding values for the width and length of the corral?

e. Describe the shape of the rectangle that minimizes the perimeter for an area of 150 m^2. Do you think that this shape will minimize the perimeter for any rectangular region of a given area? Use examples to support your response.

* * * * * * * * * *

Summary Assessment

1. **a.** The figure below shows a screen in the third level of Gates. The line represents the graph of the polynomial function $f(x) = x$. Add a rational expression to $f(x) = x$ so that the graph of the resulting function passes through all four gates.

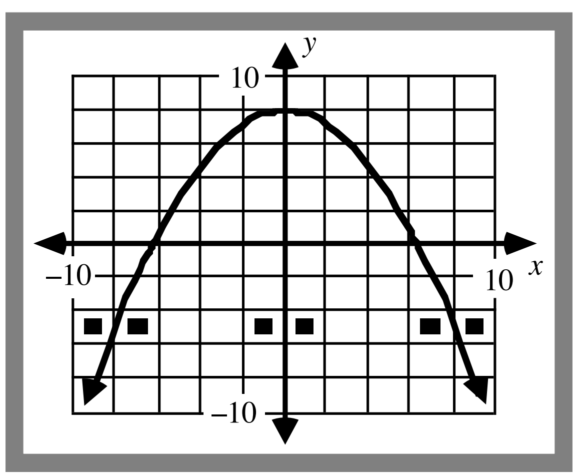


b. Write your function from Part **a** in the general form of a rational function, shown below:

$$r(x) = \frac{f(x)}{g(x)}$$

c. Use piecewise functions to create a graph that passes through the four gates shown in Part **a**.

2. The figure below shows another screen in the third level of Gates.



a. The curve shown represents the graph of the function below:

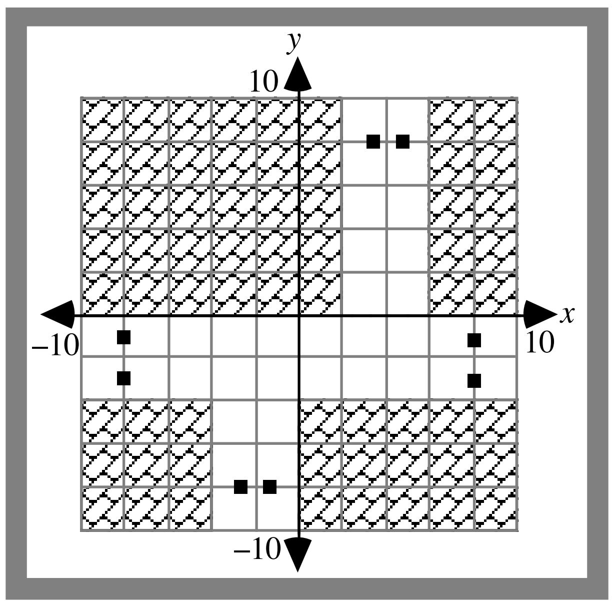
$$f(x) = -\frac{1}{5}x^2 + 8$$

Does this function have an absolute maximum or minimum? Justify your response.

b. Add a rational expression to $f(x)$ so that the graph of the resulting function passes through all the gates.

c. Write the new function in the general form of a rational function and identify its domain and range.

3. The figure below shows a screen in the third level of Gates. Determine a function whose graph passes through all four gates without touching the brick walls.



Module Summary

- A function is **continuous** at a point c in its domain if the following conditions are met:
    1. the function is defined at c, or $f(c)$ exists
    2. the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
    3. the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$.
- A function is continuous over its domain if it is continuous at each point in its domain.
- A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.
- A **polynomial function** is a function of the form below, where a_n is a real number and n is a non-negative integer.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

- The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. The coefficient of that variable in the expression is the **leading coefficient**.
- The **end behavior** of the graph of a polynomial function describes the characteristics of the graph as $|x|$ approaches infinity.
- The **absolute maximum** of a function is the greatest value of the range, while the **absolute minimum** is the least value of the range.
- The x-coordinate of each point where the graph of a polynomial intersects the x-axis is a **zero** or a **root** of the polynomial.
- As $|x|$ approaches infinity, the graph of a polynomial function of even degree has the same behavior at both ends. As $|x|$ approaches infinity, the graph of a polynomial function of odd degree has opposite behavior at each end.
- In a **piecewise function**, different parts of the domain correspond with different rules.
- A **rational function** is a function of the form

$$r(x) = \frac{f(x)}{g(x)}$$

  where $f(x)$ and $g(x)$ are polynomial functions and $g(x) \neq 0$.

- An **asymptote** to a curve is a line such that the distance from a point *P* on the curve to the line approaches zero as the distance from *P* to the origin increases without bound, where *P* is on a suitable part of the curve.
- The graph of a rational function of the form

  $$r(x) = f(x) + \frac{g(x)}{h(x)}$$

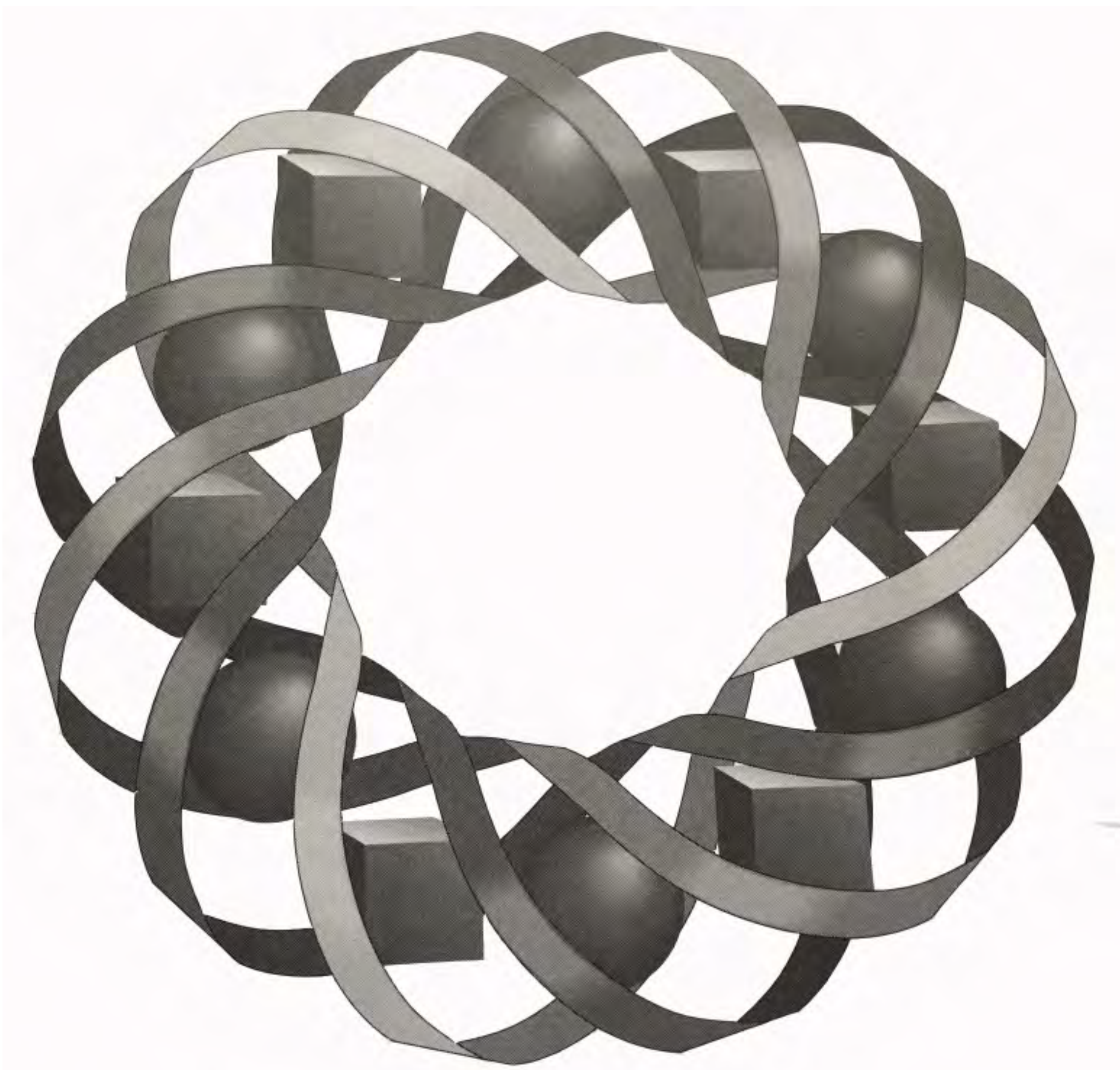
  where $h(x) \neq 0$ and the degree of $g(x)$ is less than the degree of $h(x)$, approaches the graph of $f(x)$ as $|x|$ increases without bound. This function also may be asymptotic to the vertical line $x = k$, where $h(k) = 0$, or it may have a point of discontinuity at $x = k$.

Selected References

Demana, F., B. K. Waits, and S. R. Clemens. *College Algebra & Trigonometry*. New York: Addison-Wesley, 1992.

Faires, J. D., and B. T. Faires. *Calculus and Analytic Geometry*. Boston: Prindle, Weber & Schmidt, 1983.

Changing the Rules Changes the Game



In the past century, women's basketball has changed from a game of six-person teams and half-court play to a game of five-person teams and full-court play. In this module, you investigate how changing the rules of geometry affects the game of mathematics.

Bill Chalgren • Janet Kuchenbrod • Tom Teegarden

Changing the Rules Changes the Game

Introduction

About 300 B.C., the Greek mathematician Euclid recorded a set of basic notions and **axioms** for geometry. Axioms are statements that are assumed to be true. Euclid's axioms described the properties of geometric figures and the relationships among them, including the concepts that a line is straight, that lines could be parallel, and that there is only one parallel line to a given line through a point not on the line (called the Parallel Postulate).

Since Euclid's era, his ideas about geometry have become a part of our everyday lives. But what would happen to a geometry if its basic notions were not those described by Euclid? For example, Euclid tacitly assumed that a line had infinitely many points. This is not the case in finite geometries.

To understand the coordinate system associated with a finite geometry, you must be able to perform arithmetic using a finite set of integers. One place to explore finite arithmetic systems is on the face of a clock. These arithmetic operations on a clock then can be linked to a non-Euclidean finite geometry. In this module, you investigate some properties of a finite geometry and a finite arithmetic.

Exploration

In **clock arithmetic**, an n-hour clock contains the digits 1, 2, 3, ..., n. In such a system, addition is accomplished by moving clockwise around the dial, while subtraction is accomplished by moving counterclockwise around the dial.

a. Draw a 12-hour clock face.

b.
1. Describe a method for representing integers greater than 12 on a 12-hour clock.

2. Describe a method for representing integers less than 1 on a 12-hour clock.

3. Use your method to determine the 12-hour clock values for 15 and –5.

c. To distinguish the symbols for operations in clock arithmetic from those used in real-number arithmetic, they are often drawn with circles around them. The symbol $\oplus$, for instance, indicates clock addition. On a 12-hour clock, 11 hours after 4 o'clock can be symbolized as $4 \oplus 11$, or 3 o'clock. Similarly, 3 hours before 2 o'clock can be written as $2 \ominus 3$, or 11 o'clock.

1. Describe a method to represent addition of integers on a 12-hour clock.

2. Describe a method to represent subtraction of integers on a 12-hour clock.

3. Use your method to determine the sum $8 \oplus 7$ and the difference $5 \ominus 10$.

d. In real-number arithmetic, 0 is the additive identity since, for any real number a, $a + 0 = 0 + a = a$.

An additive identity also exists for 12-hour clock arithmetic. Determine its value.

e. In real-number arithmetic, multiplication can be thought of as "multiple additions." This is also the case in 12-hour arithmetic. For example, the multiplication $7 \otimes 3$ can be considered as shown below:

$$\begin{aligned} 7 \otimes 3 &= (7 \oplus 7) \oplus 7 \\ &= 2 \oplus 7 \\ &= 9 \end{aligned}$$

1. Determine the value of $4 \otimes 2$ in 12-hour arithmetic.

2. Determine the value of $4 \otimes 5$ in 12-hour arithmetic.

Discussion

a. Compare the method you described for representing positive and negative integers on a 12-hour clock with others in your class.

b. What number is the additive identity for 12-hour arithmetic? Justify your response.

c. **1.** Two numbers are said to be additive inverses if their sum is the additive identity. Identify an additive inverse for each number in 12-hour arithmetic.

2. In real-number arithmetic, each number has exactly one additive inverse. Is the corresponding statement true in 12-hour arithmetic? Justify your answer.

d. In real-number arithmetic, 1 is the multiplicative identity since, for any real number a, $a \bullet 1 = 1 \bullet a = a$.

A multiplicative identity also exists for 12-hour arithmetic. What number do you think is this identity? Explain your response.

e. Two numbers are said to be multiplicative inverses if their product is the multiplicative identity. Do you think that each number in 12-hour arithmetic has a multiplicative inverse? If so, identify the multiplicative inverse for each number on a 12-hour clock. If not, describe a number that does not have a multiplicative inverse.

Activity 1

To explore other finite arithmetic systems, mathematicians developed **modular arithmetic**. Modular arithmetic can provide some basic tools for exploring finite geometries. In this activity, you examine modular arithmetic and determine some of its properties.

Mathematics Note

The modular arithmetic system of **modulo *n*** (or **mod *n***) contains the digits $0, 1, 2, 3, ..., n-1$. For example, a modulo 8 (or mod 8) clock contains the numbers 0, 1, 2, 3, 4, 5, 6, and 7.

Like clock arithmetic, modular arithmetic can be thought of as taking place on a circular dial, as shown in Figure **1** below.

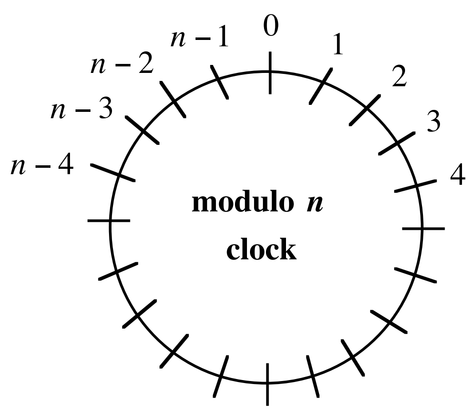


Figure 1: Mod *n* clock

Exploration 1

One way to visualize a modular arithmetic system is to consider a number line of integers "wrapped" around a mod *n* clock. Using this analogy, you can determine which modulo *n* values correspond with each integer on the number line.

For example, the integer 0 on the number line corresponds with 0 on the modulo clock. Moving clockwise around the clock face corresponds to moving along the positive portion of a number line.

a. Sketch a circle on a sheet of paper. Mark and label the circle to form a modulo 5 clock.

b. **1.** Use your mod 5 clock from Part **a** to complete Table **1** for the integers 0 through 12.

Table 1: Integers and their corresponding mod 5 values

Integer on Number Line	Mod 5 Value
0	0
1	
2	
⋮	
12	

2. Moving counterclockwise around the modulo clock face corresponds to moving along the negative portion of a number line. Use this notion to complete Table **1** for the integers –1 through –12.

c. By examining the values in Table **1**, you should observe that the same mod 5 value corresponds with more than one integer.

1. To help visualize this relationship, create a scatterplot of the data in Table **1**. Represent the integers along the *x*-axis and the corresponding mod 5 values along the *y*-axis.

2. Use the scatterplot to identify all the integers from –12 to 12 that correspond with the same value in mod 5.

d. In Part **c**, you should have observed that 12 and –8 both correspond with 2 (mod 5). This fact can also be illustrated using the division algorithm. When using the division algorithm, the remainder must be a non-negative integer less than the divisor.

As shown below, for example, 12 and –8 both have a remainder of 2 when divided by 5.

$$\begin{array}{r} 2 \;\; \text{R2} \\ 5\overline{)12} \\ \underline{-10} \\ 2 \end{array} \qquad \begin{array}{r} -2 \;\; \text{R2} \\ 5\overline{)-8} \\ \underline{-(-10)} \\ 2 \end{array}$$

The division algorithm allows you to determine the mod 5 values that correspond with large integers.

Identify two integers with absolute values greater than 500, one positive and one negative, that correspond with the same mod 5 value.

Mathematics Note

In modulo n, two integers are **congruent** (symbolized by $\equiv$) if they have the same remainder when divided by n.

In mod 5, for example, the integers 12 and 2 are congruent because 12 divided by 5 and 2 divided by 5 both have a remainder of 2. This can be written symbolically as $12 \equiv 2 \pmod 5$.

Discussion 1

a. How is congruence illustrated on the scatterplot you created in Part **c** of Exploration **1**?

b. Describe the process you would follow when using the division algorithm to determine the congruent mod n value of a negative number.

c. Wrapping a number line of integers around the mod 5 clock can be thought of as a function. What are the domain and range of this function?

Exploration 2

In real-number arithmetic, the numbers 1 and 0 play special roles. In this exploration, you create addition and multiplication tables and use them to identify numbers that play similar roles in mod n arithmetic. You then use these numbers to solve some mod n equations.

a. Complete Table **2**, a table of addition facts for mod 5.

Table 2: Addition facts for modulo 5

+	**0**	**1**	**2**	**3**	**4**
0					
1		2			0
2					
3				1	
4					

b. Complete Table **3**, a table of multiplication facts for mod 5.

Table 3: Multiplication facts for modulo 5

×	**0**	**1**	**2**	**3**	**4**
0					
1		1			
2					3
3				4	
4					

c. Create a table of subtraction facts for mod 5. Each entry in the table should represent the row value minus the column value.

d. Determine the additive identity for mod 5.

e. Identify the additive inverse for each element in mod 5.

f. Determine the multiplicative identity for mod 5.

g. The multiplicative inverse of x is also referred to as the reciprocal of x. For the set of real numbers, $1/x$ is the multiplicative inverse of x, for $x \neq 0$, since

$$x \bullet \frac{1}{x} = \frac{1}{x} \bullet x = 1$$

The multiplicative inverse of any element x (other than the additive identity) can be denoted by x^{-1}.

Identify the multiplicative inverse for each element in mod 5.

Mathematics Note

Many of the properties of congruence are comparable to properties of equality.

The **substitution property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$ and $b \equiv c$, then $a \equiv c$.

The **addition property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$, then $a + c \equiv b + c$.

The **multiplication property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$, then $a \bullet c \equiv b \bullet c$.

h. Congruences in mod n can be solved using methods similar to those used to solve algebraic equations involving real numbers. For example, the solution to the congruence $5x - 6 \equiv 4 \pmod 7$ is shown below.

$5x - 6 \equiv 4 \pmod 7$	given
$5x \equiv 4 + 6 \pmod 7$	addition property of congruence
$5x \equiv 3 \pmod 7$	definition of congruence mod 7
$3(5x) \equiv 3(3) \pmod 7$	multiplication property of congruence
$1x \equiv 2 \pmod 7$	definition of congruence mod 7
$x \equiv 2 \pmod 7$	multiplicative identity

The solution can be checked by substituting 2 into $5x - 6 \equiv 4 \pmod 7$. Since $5(2) - 6 \equiv 3 - 6 \equiv 4 \pmod 7$, 2 is a solution to the equation.

Use the process described above to solve the equation $3x + 1 \equiv 2 \pmod 4$. Record the justification for each step in your solution.

Discussion 2

a. Which of the following modular operations are commutative?

1. addition

2. subtraction

3. multiplication

b. **1.** How can you define division in mod 5 arithmetic?

2. Is division commutative in mod 5 arithmetic? Explain your answer.

c. Consider the following congruence equation: $16 \bullet 6 \equiv x \pmod{5}$. One way to determine a solution that is a mod 5 value is to multiply the two factors, then convert the product to mod 5.

Is the solution affected by converting both factors to mod 5 before multiplying?

d. For real numbers, addition and multiplication are both associative. This means that for any real numbers a, b, and c:

$$(a+b)+c = a+(b+c)$$

and

$$(a \bullet b) \bullet c = a \bullet (b \bullet c)$$

Are addition and multiplication associative in modular arithmetic?

e. For real numbers, multiplication is distributive over addition. In other words, for any real numbers a, b, and c:

$$a(b+c) = ab + ac$$

Do you think that multiplication (mod 5) is distributive over addition (mod 5)? Use an example to illustrate your response.

f. **1.** Why is there no multiplicative inverse for 2 (mod 6)?

2. Because there is no multiplicative inverse for 2 (mod 6), the equation $2x - 5 \equiv 4 \pmod{6}$ has no solution. To verify that this is true, substitute each number in mod 6 into the equation.

g. **1.** Why is there no multiplicative inverse for 3 (mod 6)?

2. Although there is no multiplicative inverse for 3 (mod 6), the equation $3x \equiv 3 \pmod{6}$ has three solutions: 1 (mod 6), 3 (mod 6), and 5 (mod 6).

Describe how you might find these solutions.

h. Describe some situations in which you might expect to use a modular arithmetic.

Assignment

1.1 Describe how to determine the number in mod 5 that is congruent to 33.

1.2 Calculate each of the following in mod 5:

a. $3+2$

b. $12 \bullet 8$

c. $13-20$

1.3 Complete the following addition and multiplication tables for mod 3.

+	**0**	**1**	**2**
0			
1			
2			

×	**0**	**1**	**2**
0			
1			
2			

1.4 **a.** Evaluate each of the following expressions.

1. $2+1 \pmod 3$

2. $2 \bullet 2 \pmod 3$

3. $16+9 \pmod 8$

4. $11 \bullet 7 \pmod{10}$

b. Find each of the following inverses.

1. the additive inverse of 3 (mod 5)

2. the additive inverse of 2 (mod 3)

3. the multiplicative inverse of 3 (mod 5)

4. the multiplicative inverse of 0 (mod 5)

1.5 **a.** What is the additive identity in mod 3?

b. Find the additive inverse in mod 3 for each of the following: 0, 1, and 2.

c. Recall that **proof by exhaustion** is the process of examining all possibilities to prove a statement. Use proof by exhaustion to show that 1 is the multiplicative identity for mod 3.

d. Prove or disprove the statement: "Every element in mod 3 has a multiplicative inverse."

1.6 Division in modular arithmetic may be defined as follows: $a \div b \equiv c \pmod n$ if and only if $b \bullet c \equiv a \pmod n$. Use this definition to find each of the following:

a. $1 \div 2 \pmod 5$

b. $3 \div 2 \pmod 5$

c. $4 \div 0 \pmod 5$

1.7 Prove that 2 does not have a multiplicative inverse in mod 4.

1.8 Solve for x in each of the following.

a. $x + 3 \equiv 1 \pmod{7}$

b. $4x \equiv 1 \pmod{7}$

c. $3x - 5 \equiv 4 \pmod{6}$

d. $2x + 1 \equiv 0 \pmod{3}$

e. $x^2 \equiv 1 \pmod{3}$

* * * * *

1.9 Solve each of the following equations:

a. $11x - 6 \equiv 8 \pmod{13}$

b. $3x + 4 \equiv 5 \pmod{11}$

1.10 A monitoring device uses 0.5 m of paper per hour. Each roll of paper is 200 m long. If a new roll is installed at 9:00 A.M., at what time will the device run out of paper?

* * * * * * * * * *

Research Project

One common application of modular arithmetic is the Universal Product Code (UPC) found on nearly every consumer product. Each UPC bar code represents a 12-digit number.

Figure **2** shows a typical UPC bar code and number. The first number on the left (0) identifies the product. The last number on the right (8) is the check digit. To make certain that each code is read correctly, bar-code readers (such as those at supermarket cash registers) use an algorithm to perform an internal check.

0 75520 57070 8

Figure 2: A UPC bar code

For this research project, find an algorithm that performs a check on a bar code. Write an explanation of the algorithm. Collect some samples of UPC bar codes and verify that the algorithm works. Then create one valid and one invalid UPC bar code of your own.

Activity 2

In Activity **1**, you examined some of the basic principles of modular arithmetic. In this activity, you investigate a finite geometry coordinatized with a modular arithmetic system.

Mathematics Note

Finite geometries are unlike traditional Euclidean geometry because they use only finite numbers of points.

For example, one finite geometry is based on a modulo 3 arithmetic system. In this system, each point of a lattice has coordinates (x, y) where x and y are elements of the set $\{0, 1, 2\}$. Figure **3** shows the nine-point lattice used to construct this finite geometry.

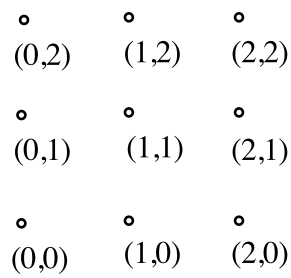


Figure 3: Coordinatized nine-point lattice

In this geometry, a line is defined as the set of all points that satisfies a mod n equation of the form $Ax + By + C = 0 \pmod 3$ where A, B, and C are elements of the set $\{0, 1, 2\}$ with A and B not both 0. For example, one line is identified with the equation $x + y + 0 = 0 \pmod 3$ or $x + y = 0 \pmod 3$. This line contains only the points with coordinates (0,0), (2,1), and (1,2). A graph of this line is shown in Figure **4**.

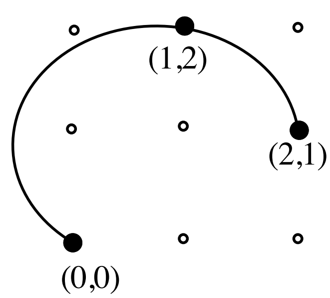


Figure 4: Graph of the line $x + y = 0 \pmod 3$

Although Figure **4** shows the three points on the line connected by an arc, the line contains only those three points. There are no other coordinates that satisfy the equation. **Note:** For the remainder of this module, mod n equations will be written with equals signs rather than congruence signs.

Discussion 1

a. Compare the characteristics of a line in Euclidean geometry with the characteristics of a line in the finite geometry described in the previous mathematics note.

b. How do these characteristics of a line compare with its characteristics in spherical geometry?

c.
1. How might a triangle be defined in this nine-point geometry?
2. Give an example of a triangle that satisfies your definition.
3. How does your definition compare with the definition of a triangle in Euclidean geometry?

Exploration

In this exploration, you continue to investigate the nine-point geometry described in the mathematics note.

a. Determine the number of possible equations of the form $Ax + By + C = 0 \pmod{3}$. List each of these equations.

b. Determine the coordinates of all the points in the nine-point geometry that satisfy each equation identified in Part **a**.

c. Graph each of the equations on a copy of the lattice template (available from your teacher). Connect each set of points in the solution with segments or arcs.

d.
1. It is possible for more than one equation to define the same line. Identify the equations of the unique lines in this nine-point system.
2. Label the points in a nine-point lattice A through I, as shown in Figure **5** below.

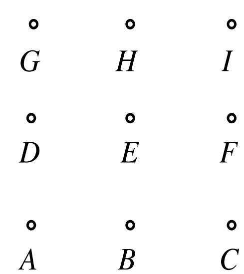


Figure 5: A nine-point lattice

3. Record both the coordinates and the letters that correspond with the points which satisfy each unique line. **Note:** Save this information for use in Problem **2.2**.

e. Determine if each of the following properties of lines in traditional Euclidean geometry is true in this nine-point geometry.

1. Two points determine a unique line.

2. If two distinct lines contain a common point, they contain exactly one common point.

f. Lines in a plane are **parallel** if they have either no points in common or all points in common. Are there parallel lines in this geometry? If so, identify them.

g. According to the Parallel Postulate (mentioned in the introduction), there is exactly one line parallel to a given line through a point not on that line.

1. Select a line in the nine-point geometry and a point not on the line. Determine if the parallel postulate is true for your selections.

2. Repeat Step **1** for each point not on the line until you have checked all appropriate points.

3. Select another line in the finite geometry and repeat Steps **1** and **2**.

4. Repeat Step **3** until all lines have been checked.

Discussion 2

a. What patterns do you observe among the values of A, B, and C in the equations of parallel lines?

Mathematics Note

In the nine-point geometry, a line given by $Ax + By + C = 0 \pmod 3$, where $B \neq 0$, may be expressed in the form $y = mx + b \pmod 3$ where m and b are elements of the set {0, 1, 2} and m represents the **slope** of the line.

For example, consider the line defined by the equation $1x + 2y + 1 = 0 \pmod 3$. This equation may be rewritten using mod 3 arithmetic as follows:

$$\begin{aligned} 1x + 2y + 1 &= 0 \pmod 3 \\ 2y &= -1x + -1 \\ 2(2y) &= 2(-1x + -1) \\ 1y &= -2x + -2 \end{aligned}$$

However, –2 may be rewritten as 1 in mod 3 since $-2 \equiv 1 \pmod 3$. Therefore,

$$\begin{aligned} 1y &= 1x + 1 \\ y &= x + 1 \end{aligned}$$

In this case, the slope of the line is 1.

b. When the equation for a line in the form $Ax + By + C = 0 \pmod 3$ is rewritten in slope-intercept form, it becomes:

$$y = -\frac{A}{B}x - \frac{C}{B} \pmod 3$$

1. Describe the slope of the line when $A = 0$.
2. Describe the slope of the line when $B = 0$.

c. In a real-number coordinate plane, the slope of a line can be found using the coordinates of two points on the line. Is it possible to determine the slope of a line in the nine-point geometry using the coordinates of two points on the line? Justify your response.

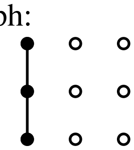
d. Compare the slope of a line in Euclidean geometry to the slope of a line in nine-point geometry.

Assignment

2.1 Write each of the distinct equations found in the exploration in the form $y = mx + b \pmod 3$ or $x = a \pmod 3$.

2.2 Use the equations in Problem **2.1** and the information you recorded in Part **d** of the exploration to complete the chart supplied by your teacher. The following diagram shows one completed cell in the chart. **Note:** Save this chart for use throughout the remainder of this module.

a.
(0,0), (0,1), (0,2)
Equation:
$x = 0$
Graph:



2.3 Consider a line in the nine-point geometry that contains the point (0,1) and has a slope of 1.

a. Write an equation for the line.

b. Use the completed chart from Problem **2.2** to verify your equation from Part **a** and identify the other points on the line.

2.4 How many triangles are there in the nine-point geometry? Explain your response.

2.5 Each row of the chart in Problem **2.2** contains three lines. By considering them in pairs, prove that the lines in each row are parallel. Recall that in coordinate geometry, two lines are parallel if they either have the same slope or both have undefined slopes and are vertical.

2.6 In Euclidean geometry, two lines with non-zero slopes are perpendicular when the product of their slopes is -1. A line with an undefined slope is perpendicular to a line with a slope of 0.

a. Consider two perpendicular lines with non-zero slopes in the nine-point geometry. If a comparable definition of perpendicular lines is true, what must be the product of their slopes?

b. Identify all pairs of perpendicular lines in the nine-point geometry.

c. In Euclidean geometry, the following properties involving perpendicular lines in a plane are true.

1. At a point on a line, there is exactly one line perpendicular to the given line.
2. From a point not on a line, there is exactly one line perpendicular to the given line.
3. Two lines perpendicular to the same line are parallel.

Determine if these properties are true in the nine-point geometry by considering every possible case.

2.7 Consider the lines defined by the following mod 3 equations:

$$y = 2$$
$$y = 2x$$

a. Find the intersection, if any, of these two lines.

b. Repeat Part **a** for the mod 3 equations below:

$$y = x$$
$$y = 2x$$

2.8 Is every pair of intersecting lines in the nine-point geometry perpendicular? Explain your response using a proof.

* * * * *

2.9 Consider a geometry based on a modulo 4 number system in which each point of a lattice has coordinates (x,y) where x and y are elements of the set $\{0, 1, 2, 3\}$. In this geometry, a line is defined as the set of all points on the lattice that satisfies a mod 4 equation of the form $Ax + By + C = 0$.

a. Construct a lattice and graph the equation $y = 2x + 3$ (mod 4).

b. Write the equation (in mod 4) of a line parallel to the line given in Part **a** and containing point (0,0).

c. Find the equation (in mod 4) of a line perpendicular to the line given in Part **a** and containing point (0,0).

2.10 Does every pair of perpendicular lines in the nine-point geometry intersect? Verify your response using proof by exhaustion.

* * * * * * * * * *

Research Project

A triangle with two sides perpendicular is a **right triangle**. By this definition, the points with coordinates (0,0), (0,2), and (2,1) in the nine-point geometry determine a right triangle. This fact can be proven as described below.

The slope of the line through the points with coordinates (0,0) and (2,1) can be calculated as follows:

$$\frac{1-0}{2-0}=\frac{1}{2}$$

Since $1 \div 2 \equiv 2 \pmod 3$, the slope of the line is 2. Similarly, the slope of the line through (0,2) and (2,1) is 1. The product of the two slopes is 2.

As mentioned in Problem **2.6**, two lines with non-zero slopes are perpendicular when the product of their slopes is –1. Since –1 is congruent to 2 (mod 3), the lines are perpendicular. Therefore, the triangle is a right triangle.

A picture of the triangle formed by (0,0), (0,2), and (2,1) is shown in Figure **6**.

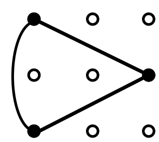


Figure 6: A right triangle in the nine-point geometry

How many other right triangles can be formed in the nine-point geometry?

Activity 3

In Activity **2**, you investigated a finite geometry using algebra and the coordinates of points. In this activity, you explore a finite geometry as an **axiomatic system**. In other words, you use undefined terms, definitions, axioms, and proven theorems to describe the system.

Mathematics Note

An **axiomatic system** is a mathematical system that contains:

- **undefined terms** (terms assumed without definition)
- **definitions** (terms defined using undefined terms and other definitions)
- **axioms** (rules assumed to be true that describe relationships among terms)
- **theorems** (statements proven true using logic)

In Euclidean geometry, for example, both *line* and *point* are undefined terms. The statement, "A line extends indefinitely in two directions" is an axiom, since it is assumed to be true. The statement, "The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs" is a theorem since it can be proven.

Exploration

Gino Fano was one of the first mathematicians to study a finite geometry. In 1892, he built a geometry to satisfy the following five axioms, leaving the terms *point*, *line*, and *on* undefined. (In these axioms, the words *contains* and *has* also are undefined terms.)

1. There exists at least one line.

2. Every line has exactly three points.

3. Not all points are on the same line.

4. For any two points, there exists exactly one line that contains both of them.

5f. Every two different lines have at least one point in common. (The *f* in 5*f* stands for Fano.)

In this exploration, you use Fano's axioms to deduce the properties of his geometry.

a. Considering only Axioms **1** and **2**, determine the minimum number of points in this geometry. Hint: Start listing points by designating each one in order with the letters A, B, C, and so on.

b. Considering only Axioms **1–3**, determine the minimum number of points in this geometry.

c. Now consider all five of Fano's axioms. Determine the number of points and the number of lines in this geometry.

d. Draw a model of Fano's geometry using the number of points and lines from Part **c**.

e. By changing Fano's fifth axiom, John Wesley Young developed another finite geometry (referred to as "Young's geometry" in this module). Young's fifth axiom reads as follows:

5*y*. For each line *l* and each point *B* not on line *l*, point *B* is on one line that does not contain any other points from line *l*. (The *y* in 5*y* stands for Young.)

Young's geometry has nine points. Use a lattice similar to the one shown in Figure **7** to draw a model of Young's geometry.

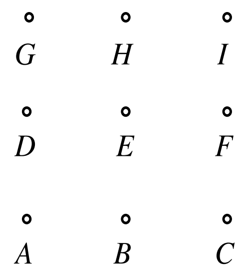


Figure 7: A nine-point lattice

Discussion

a. In Fano's geometry, does Axiom **1** tell you that there are any points on a line? Does it give any hint about how a line might look?

b. What does it mean to say that Fano's five statements are axioms?

c. In either Fano's or Young's geometry, is it possible for two distinct lines to contain the same two points? Explain your response.

d. How does Young's geometry compare to Fano's geometry?

Mathematics Note

The process of **deductive reasoning** begins with a hypothesis, then uses a logical sequence of valid arguments to reach a conclusion.

In mathematical proofs by deductive reasoning, each argument is typically supported by an axiom, definition, or previously proven theorem. A **direct proof** makes direct use of the hypothesis to arrive at the conclusion.

For example, consider the following statement: "In Fano's geometry, each point on a line is a member of at least three lines." To prove this statement using a direct proof, it should first be restated in if-then form: "If a point is on a line, then it is a member of at least three lines."

Assuming that the hypothesis, "If a point is on a line," is true, it can be symbolized as follows: Point B is on one line, l_1.

By Axiom **2**, l_1 must contain two other points: A and C.

By Axiom **3**, there must exist a point D not on l_1.

By Axiom **4**, there must be a line through points B and D: l_2.

From Part **c** of Discussion **1**, both A and B cannot be on l_2. Similarly, both A and C cannot be on l_2. By Axiom **2**, however, l_2 must contain one other point: E.

By Axiom **4**, there exists a line, l_3, that contains A and D.

To satisfy Axiom **2**, l_3 also must contain a third point, F.

According to Axiom **4**, another line, l_4, must contain B and F.

Lines l_1, l_2, and l_4 all contain B. Therefore, point B is contained in at least three lines.

In conclusion, the following statement is true in Fano's geometry. "If a point is on a line, then it is a member of at least three lines."

e. How does a theorem differ from an axiom?

f. The points and lines used to prove the statement in the mathematics note were given arbitrary names. Describe how this helps prove that the theorem is true for all points and lines in Fano's geometry.

Assignment

3.1 Consider the following theorem in Young's geometry: "For any point, there is a line not containing it."

a. Rewrite this theorem as an if-then statement. Identify the hypothesis and the conclusion.

b. When considering this theorem given any point D and a line m, there are two possible cases. If D is not on line m, then there is nothing to prove. If D is on line m, the theorem can be proved by Steps **1–3** below. Give a reason for each step in this proof.

1. Point D is on line m.
2. There exists a point E not on line m.
3. Through E there is a line l not containing any points of line m.
4. In conclusion, given any point, there is a line not containing it.

3.2 For two different lines to be parallel, they must not intersect. In other words, the two lines must have no points in common. Prove that there are no parallel lines in Fano's geometry. Begin your proof with the hypothesis "Lines l_1 and l_2 are different lines in Fano's geometry." Conclude your proof with the statement, "Line l_1 is not parallel to l_2."

3.3 In Young's geometry, prove that every point is contained in at least four lines. Draw a sketch to support your proof.

3.4 In Young's geometry, prove that given any line, there is a different line parallel to it.

3.5 How would your model of Young's geometry change if a line contained four points?

3.6 Consider the points in Young's geometry with coordinates (0,0) and (1,1). Do you think that the Pythagorean theorem could be used to find the distance between these two points? Explain your response. If so, can the distance be expressed in mod 3?

* * * * *

3.7 Use a direct proof to prove the following statement: "If n is an even number, then n^2 is an even number."

3.8 Use a direct proof to prove the following statement: "If n is an integer, then $n^3 - n$ is even." Hint: you will need to prove two cases, one in which n is even and one in which n is odd.

* * * * * * * * * *

Activity 4

In Activity **3**, you examined some properties of Fano's and Young's geometries using proofs by exhaustion and direct proofs. In this activity, you use **indirect proofs** to continue your investigations of these two geometries.

Exploration

Indirect proofs are based on the notion of *reductio ad absurdum*, or "reduction to the absurd." In an indirect proof, the property to be proven true is assumed to be false. From this assumption, statements are argued logically with supporting reasons (axioms, definitions, and proven theorems) until a contradiction to either a known fact or an assumption is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

For example, consider a number n that is an even perfect square. In the following exploration, you prove that the square root of n also is even using an indirect proof.

a. Suppose that m is the square root in question—in other words, that $m^2 = n$. Assume that m is not even. Use an algebraic equation to express m in terms of a, another natural number. Your equation should show that m is indeed odd.

b. Square both sides of the equation from Part **a**. Is the result an even or an odd natural number?

c.
1. Can the statement that n is an even perfect square and the result in Part **b** both be true?
2. Do you believe that the following statement is true: "The square root of an even perfect square also is even"? Explain your response.

Discussion

a. What is the negation of the statement: "The square root of 2 is an irrational number"?

b. In general, the negation of a statement "if p, then q" is "p and not q." How do truth tables verify this relationship?

Mathematics Note

In an **indirect proof**, a statement is proven true by proving that its negation cannot be true.

For example, consider this statement in Young's nine-point geometry: "If a line intersects one of two parallel lines, then it intersects the other." This statement may be rewritten as follows: "If line l_1 is parallel to l_2, and lines l_2 and l_3 each contain point B, then lines l_1 and l_3 intersect."

To prove this statement using an indirect proof, assume that line l_1 is parallel to l_2, that lines l_2 and l_3 each contain point B, and that lines l_3 and l_1 do not intersect. A sketch of this situation is shown in Figure **8**. (The three points in each line are connected for organizational purposes.)

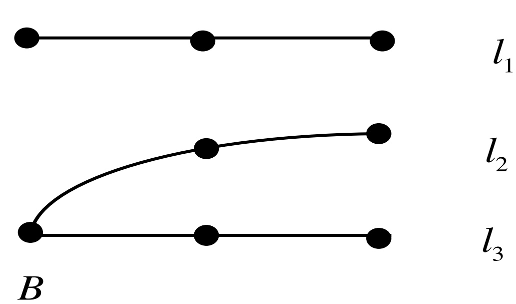


Figure 8: Sketch created using an assumption

If l_3 and l_1 do not intersect, then l_3 and l_1 have no points in common because of the meaning of non-intersecting.

Line l_1 is parallel to l_2, by the hypothesis.

Point *B* is on l_3 and l_2, by the hypothesis.

Now there are two lines, l_3 and l_2, that both contain point *B* and both are parallel to l_1. This contradicts Axiom **5y**, which states that through a point *B* not on a line *l*, there is exactly one line that has no points in common with the given line.

Therefore, the assumption that lines l_3 and l_1 do not intersect must be false because this would mean there could be more than one line that has no point in common with the given line. Consequently, lines l_3 and l_1 intersect.

In conclusion, the following statement is true: "If a line intersects one of two parallel lines, then it intersects the other."

Assignment

4.1 Consider the following theorem in Young's nine-point geometry: "If two lines intersect, then they intersect in exactly one point."

- **a.** Identify the hypothesis and the conclusion in this statement.
- **b.** To prove this statement using an indirect proof, what must you assume to be true?
- **c.** Sketch a picture of the situation that includes your assumption.
- **d.** Which axiom does your assumption contradict?
- **e.** What does this contradiction indicate about your assumption?
- **f.** What can you now conclude about the theorem to be proved? Explain your response.

4.2 Prove indirectly the following statement in Young's geometry: "If two lines are parallel to the same line, then they are parallel to each other." Hint: Use the theorem proven in the mathematics note.

4.3 Prove that every two different lines in Fano's seven-point geometry have exactly one point in common. Begin your proof with the hypothesis that l_1 and l_2 are different lines. Conclude your proof with the statement that lines l_1 and l_2 have exactly one point in common.

* * * * *

4.4 The following paragraph provides an indirect proof of the statement, "The $\sqrt{2}$ is an irrational number." Describe the contradiction which shows that the assumption must be false.

Assume $\sqrt{2}$ is a rational number. This means that $\sqrt{2} = p/q$ where p and q are whole numbers and p/q is in lowest terms. Square both sides of the equation, as shown below:

$$\left(\sqrt{2}\right)^2 = \left(p/q\right)^2$$
$$2q^2 = p^2$$

Since p^2 is even, p must be even. Since p is even, it can be written in the form $p = 2r$, where r is a whole number. By substitution,

$$(2r)^2 = 2q^2$$
$$4r^2 = 2q^2$$
$$2r^2 = q^2$$

Since q^2 is even, q also must be even. Therefore, the square root of 2 must be irrational.

4.5 Use an indirect proof to prove the following: "If a cash register contains $1.45 in nickels and dimes, there must be an odd number of nickels."

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Summary Assessment

1. **a.** Construct addition and multiplication tables for mod 4.

b. Use these tables to describe the existence of additive inverses, an additive identity, multiplicative inverses, and a multiplicative identity in mod 4. Use specific examples in your response.

2. Consider a four-point geometry that has the following three axioms.

- There are exactly four points.
- Through any two points there is exactly one line.
- Given two points there is exactly one line containing them.

In this geometry, the terms *point*, *line*, and *contains* are undefined.

a. Draw a model to represent this geometry.

b. Create a coordinate system in mod 2 for this geometry.

c. Construct addition and multiplication tables for mod 2.

d. Find the equations for all distinct lines in this geometry.

e. If parallel lines are defined as having no points in common, prove that this system has at least three pairs of parallel lines.

f. Use a direct proof to prove that any point in the system is contained in at least three lines.

3. Use an indirect proof to show that if n is an integer and n^2 is odd, then n is odd.

Module Summary

- In **clock arithmetic**, an n-hour clock contains the digits $1, 2, 3, ..., n$. In such a system, addition is accomplished by moving clockwise around the dial, while subtraction is accomplished by moving counterclockwise around the dial.
- To distinguish the symbols for operations in clock arithmetic from those used in real-number arithmetic, they are often drawn with circles around them. The symbol $\oplus$, for instance, indicates addition.
- A modular arithmetic system of **modulo *n*** (or **mod *n***) contains the digits $0, 1, 2, 3, ..., n-1$. In such a system, addition and subtraction are accomplished in a manner similar to clock arithmetic.
- In modulo n, two numbers are **congruent** if they have the same remainder when divided by n. The symbol $\equiv$ denotes congruence.
- Given a set and the operation of addition defined on that set, an **additive identity** is the unique element a of the set such that when a is added to any element x, the result is that element x. In other words, $x + a = a + x = x$.
- Two elements whose sum is the additive identity are **additive inverses**. In other words, b is an additive inverse of x if $x + b = b + x = a$. The additive inverse of x is denoted by $-x$.
- Given a set and the operation of multiplication defined on the set, a **multiplicative identity** is the unique element c of the set such that when any element x is multiplied by c, the result is that element x. In other words, $x \bullet c = c \bullet x = x$.
- Two elements whose product is the multiplicative identity are **multiplicative inverses**. In other words, d is the multiplicative inverse of x if $x \bullet d = d \bullet x = c$. The multiplicative inverse of any element x (other than the additive identity) can be denoted by x^{-1}. The multiplicative inverse of x is also referred to as the reciprocal of x.
- The **substitution property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$ and $b \equiv c$, then $a \equiv c$.
- The **addition property of congruence** states that if a, b, and c are any real numbers with $a = b$, then $a + c \equiv b + c$.
- The **multiplication property of congruence** states that if a, b, and c are any real numbers with $a = b$, then $a \bullet c \equiv b \bullet c$.

- An **axiomatic system** is a mathematical system that contains:
  - **undefined terms** (terms assumed without definition)
  - **definitions** (terms defined using undefined terms and other definitions)
  - **axioms** (rules assumed to be true that describe relationships among terms)
  - **theorems** (statements proven true using logic).
- A **finite geometry** is an axiomatic system which, unlike traditional Euclidean geometry, uses a finite number of points.
- The process of **deductive reasoning** begins with a hypothesis, then uses a logical sequence of valid arguments to reach a conclusion.

  In mathematical proofs by deductive reasoning, each argument is typically supported by an axiom, definition, or previously proven theorem. A **direct proof** makes direct use of the hypothesis to arrive at the conclusion.
- In an **indirect proof**, the property to be proven true is assumed to be false. From this assumption, statements are argued logically with supporting reasons (axioms, definitions, and proven theorems) until a contradiction to either a known fact or an assumption is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

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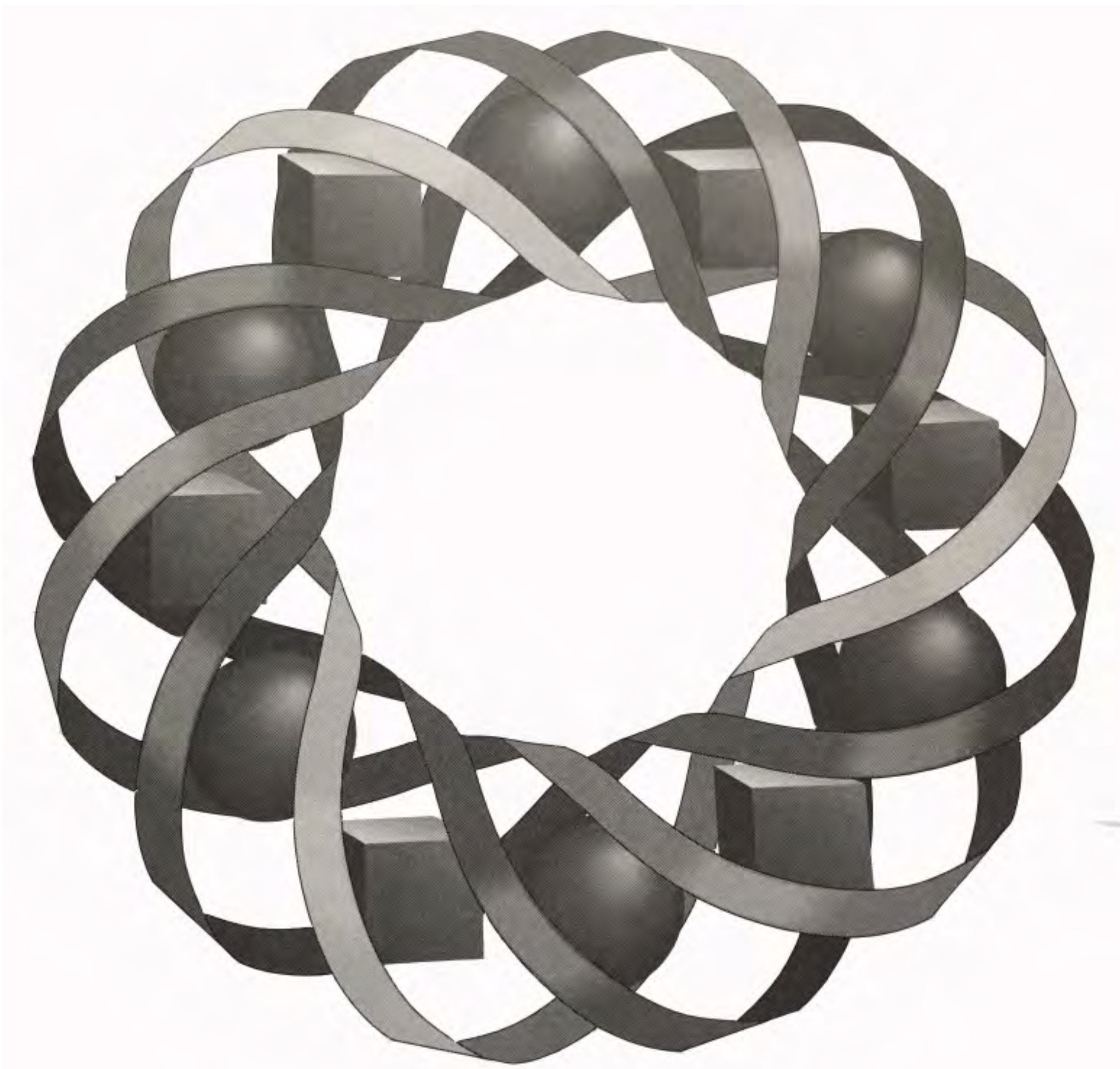
To Null or Not to Null



Imagine that you are a member of a jury in a criminal trial. What kinds of mistakes are possible in your verdict? In this module, you explore how statistics can help you analyze this situation.

Cliff Bara • Pat Mauch • Lisa Wood

To Null or Not to Null

Introduction

Most of the choices you make each day contain some degree of uncertainty. When things go wrong, the consequences of a mistake can range from almost insignificant to very grave. The process of using statistics to help make these choices—and measure the consequences—is known as **hypothesis testing**. Although the use of statistics cannot guarantee that you will never make a mistake, it can allow you to measure the risk of an error.

Discussion

a. Consider each of the following scenarios. For each one, describe the possible consequences of a wrong decision.

1. You are a member of a jury. The defendant in the case has been accused of murder. You must decide if this person is innocent or guilty.
2. You are the president of a tire manufacturing company. A consultant has recommended that you increase the tread life of your tires. You must decide whether to approve the additional spending required for this upgrade.
3. You are editor of a high school newspaper. On the recent Scholastic Aptitude Test (SAT), the senior class scored slightly higher than the national average on the mathematics portion. A reporter has submitted an article claiming that the school's seniors are better at mathematics than students nationally. You must decide whether or not to publish the article.

b. Which of the scenarios described above could be analyzed statistically? If an analysis is possible, explain briefly how it might be done.

c. When deciding how to treat a patient's illness, doctors often rely on test results. Many of these tests are not 100% accurate. Given this fact, why do doctors use such tools?

d. During Olympic and other world-class competitions, athletes must undergo testing for drugs and other banned substances.

1. If an athlete tests positive for a specific substance, does this guarantee that he or she has used the drug?
2. How would you expect a rules committee to react to the news of a positive drug test?

Mathematics Note

Statisticians often make hypotheses or claims about the parameters of a population, then use sampling techniques to test their claims. If a researcher assumes that a population parameter has a specific value, then a hypothesis can be formed about the consequences of that assumption.

In statistical analysis, there are two types of hypotheses. A **null hypothesis** (**H_0**) is a statement about one or more parameters. The **alternative hypothesis** (**H_a**) is the statement that must be true if the null hypothesis is false. The null hypothesis usually involves a claim of no difference or no relationship. In many situations, the null hypothesis and alternative hypothesis are negations of each other, but this is not necessarily the case.

For example, consider a study of tread life for two types of tires: A and B. In this case, the null hypothesis would be that there is no difference in tread life between tire A and tire B, or H_0: $A = B$. An alternative hypothesis could be that the two tires have different tread lives, or H_a: $A \neq B$. If the study revealed that the tread life on tire A appear to be longer than tire B, the alternative hypothesis could be H_a: $A > B$.

e. Suppose that a consumer group wanted to test a manufacturer's claim that its light bulbs have a mean life of 1000 hr. The study team formulates the null hypothesis "H_0: $\mu = 1000$," where μ represents the population mean.

1. What is the negation of this null hypothesis?
2. In this situation, the study team decides to use the alternative hypothesis "H_a: $\mu < 1000$." Why do you think these researchers did not use the negation of the null hypothesis as their alternative hypothesis?

f.

1. Suggest null and alternative hypotheses for a situation in which an athlete undergoes a drug test.
2. If your null hypothesis (H_0) is false, what type of evidence would you expect to observe in the drug test?

Activity 1

Statisticians are seldom 100% confident of their findings. In this activity, you explore how uncertainty affects hypothesis testing.

Mathematics Note

A **hypothesis test** may consist of the following steps.

- State null and alternative hypotheses about a parameter of a population.
- If the null hypothesis is true, predict what this implies about a sample of the population.
- Take a sample of the population and compare the results with your prediction.
- If the results are inconsistent with the prediction, then you can conclude, with some level of certainty, that the null hypothesis is false and, therefore, reject it.
- If the results are consistent with the prediction, you fail to reject the null hypothesis. The failure to reject the null hypothesis does not guarantee that the null hypothesis is true, only suggests that it might be true.

Exploration

In the following exploration, you use a population of coins to investigate how uncertainty affects your interpretation of test results.

a. Place a penny in each of 18 envelopes. Place a nickel in one additional envelope, and a quarter in another envelope. **Note:** For the remainder of this exploration, these will be referred to as the "test envelopes."

b. Randomly select one of the test envelopes. Then obtain an envelope from your teacher containing an unknown coin. In this situation, the null and alternative hypotheses can be stated as follows:

H_0: The unknown coin is a penny .

H_a: The unknown coin is not a penny .

1. If the unknown coin is a penny, what would you expect to occur when this envelope and a test envelope are placed on opposite sides of a balance? What percent of the time would you expect the test envelope to contain a penny?
2. What would you expect to occur if the unknown coin is not a penny? What percent of the time would you expect the envelope to contain the nickel or quarter?

c. Place the test envelope on one side of a balance. Place the envelope containing the unknown coin on the opposite side of a balance. Use your observations to decide whether to reject, or fail to reject, the null hypothesis.

d. Describe the conclusions you can make as a result of your hypothesis test.

Discussion

a. Do you think that the unknown coin is a penny? Explain your response.

b. What is the probability of selecting a test envelope that contains a penny?

c. **1.** If the two envelopes in Part **c** of the exploration balance, can you be sure that the coin is a penny?

2. If the two envelopes do not balance, can you be sure that the coin is not a penny?

d. Given only the results of a single test, is there any way to remove the uncertainty from your conclusions? Explain your response.

Mathematics Note

Whenever a sample is taken from a population about which there is some uncertainty, it is possible that the sample is not representative of the population. Therefore, when performing a hypothesis test by sampling, it is always possible to make an incorrect decision about the null hypothesis H_0. These two possible errors, rejecting a true null hypothesis and failing to reject a false null hypothesis, are shown in Figure **1** below.

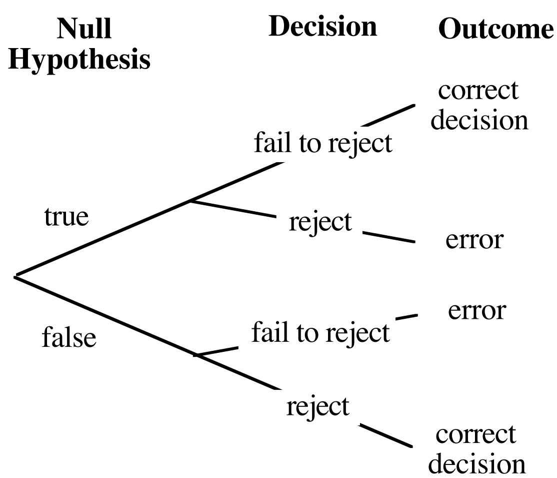


Figure 1: Possible errors for hypothesis test

In the exploration, for example, it is possible to draw a test envelope that contains a nickel rather than a penny. If the null hypothesis states that the unknown coin is a penny, then you would expect the two envelopes to balance. If the test envelope contains a nickel, however, it will not balance with an envelope containing a penny. This would lead you to reject a true null hypothesis.

e. **1.** Why does the rejection of the null hypothesis result in the acceptance of the alternative hypothesis?

2. Does acceptance of the alternative hypothesis guarantee that it is true?

f. **1.** If you fail to reject a null hypothesis, does that prove that it is true? Use the tree diagram in Figure **1** to support your answer.

2. If you reject a null hypothesis, does that prove that it is false? Explain your response.

Assignment

1.1 While studying a developing economy, researchers formulated the following null hypothesis: "The mean annual salary in the population is at least $10,000." What is the alternative hypothesis in this situation?

1.2 A mail-order catalog claims that customer satisfaction is guaranteed. Write the null and alternative hypotheses that you would use in testing this claim.

1.3 Explain why it is not typically possible to guarantee a correct decision when failing to reject a null hypothesis.

1.4 Consider a set of 20 test envelopes: 19 contain a penny and 1 contains a quarter. Using the procedure described in the exploration and an envelope containing an unknown coin, you test the null hypothesis: "The unknown coin is a penny." If the two envelopes balance, how certain can you be of your conclusion?

* * * * *

1.5 In 1994, a U.S. state began collecting a tax on tourism. The state's current governor wants to compare spending by tourists before and after the tax was enacted.

a. Write null and alternative hypotheses for this situation.

b. Describe the types of errors that might occur when testing the null hypothesis in Part **a**.

1.6 A standard deck of 52 playing cards contains 26 red cards and 26 black cards. Suppose that you wanted to determine the proportion of red cards in an unknown deck by sampling. In this situation, the null hypothesis might be: "The proportion of red cards in the deck is 0.5."

a. What is the alternative hypothesis that must be accepted if the null hypothesis is rejected?

b. To test your null hypothesis, you select a random sample of 6 cards from the deck. What outcomes might lead you to suspect that the proportion of red cards in the deck is not 0.5?

c. What is the probability of drawing a random sample of 6 cards, all of which are the same color, from a standard deck?

d. If your sample of 6 cards from this deck were all the same color, would you reject the null hypothesis? Explain your response.

* * * * * * * * * *

Activity 2

The counselor at John F. Kennedy High School has just reported the results of the SAT to 32 seniors. Their scores on the mathematics portion are normally distributed, with a mean of 529. One senior, Dena, scored 610. Although Dena knows she performed better than the mean, she would like to know approximately how many of her classmates she outscored.

In Activity **1**, you formed decisions about hypotheses with various levels of certainty. In this activity, you use a normal curve to determine how an individual observation, such as a test score, compares to a population parameter, such as the mean. You also discover how to compare individual observations from different populations, such as scores from two different tests.

Mathematics Note

The curve that describes the shape of a normal distribution is the **normal curve**. The equation of the curve that models a normally distributed population depends on the mean and standard deviation of the population. The general equation for a normal curve, where μ represents the population mean and σ represents the population standard deviation, is:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

Recall that both π and e are constants with values of approximately 3.14159 and 2.71828, respectively.

For example, Figure **2** shows the normal curve for a population of test scores where $\mu = 80$ and $\sigma = 5$:

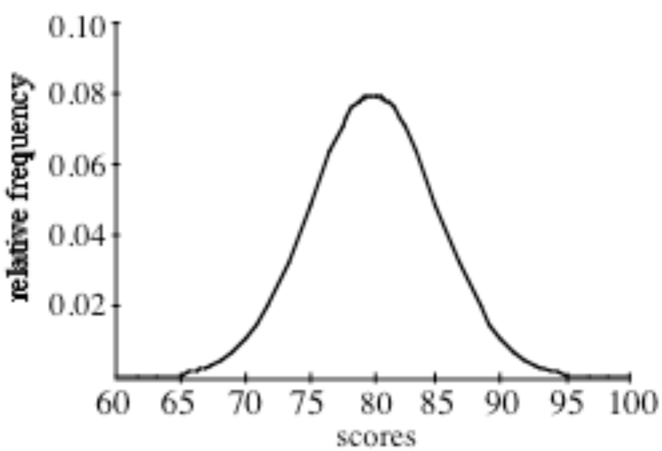


Figure 2: A normal curve

A normal curve describes a continuous probability distribution. As in all such distributions, the area between the horizontal axis and the curve is 1.

Exploration

a. The scores on a recent English examination at JFK High School are normally distributed, with a mean of 80 ($\mu = 80$) and a standard deviation of 5 ($\sigma = 5$). Recall that the **standard deviation** (σ) is a measure of the variability, or spread, within a population. Create a graph that models the distribution of these scores by substituting these values into the general equation for a normal curve.

b.
1. Locate the point on the *x*-axis that corresponds with the mean. Draw a vertical segment from the *x*-axis to the curve at this point.
2. Determine the percentage of the area between the *x*-axis and the curve that lies to the left of the segment in Step **1**.

c.
1. Draw vertical segments from the *x*-axis to the curve at $x = \mu - 1\sigma$ and $x = \mu + 1\sigma$.
2. Find the percentage of the area between the *x*-axis and the curve that lies between these two segments.

d.
1. Draw vertical segments from the *x*-axis to the curve at $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$.
2. Find the percentage of the area between the *x*-axis and the curve that lies between these two segments.

e. Repeat Parts **a–d** using two different values for σ, when $\mu = 80$. Record your observations.

f. Repeat Parts **a–d** using two different values for μ, when $\sigma = 5$. Record your observations.

g. The scores on a recent physics exam at JFK High School are normally distributed with $\mu = 72$ and $\sigma = 3$. Repeat Parts **a–d** for this set of scores.

Discussion

a. When using a continuous curve like the one in Figure **2** to approximate the distribution of test scores, what assumptions are made about the scores?

b. How does the value of σ for a particular normal distribution affect the shape of the corresponding normal curve?

c. How does the value of σ for a particular normal distribution affect the percentage of area above the *x*-axis and below the normal curve in each of the following intervals?

1. $[\mu - 1\sigma, \mu + 1\sigma]$
2. $[\mu - 2\sigma, \mu + 2\sigma]$

d. How does the value of μ for a particular normal distribution affect the shape of the corresponding normal curve?

e. Describe how the percentages you determined in Parts **b–d** of the exploration relate to probabilities.

Mathematics Note

The predictable variability of individual observations is summarized by the **68–95–99.7 rule**, which states that approximately 68% of the area between the x-axis and the normal curve is within 1 standard deviation of the mean, approximately 95% is within 2 standard deviations of the mean, and approximately 99.7% is within 3 standard deviations of the mean. This rule is represented graphically in Figure **3**.

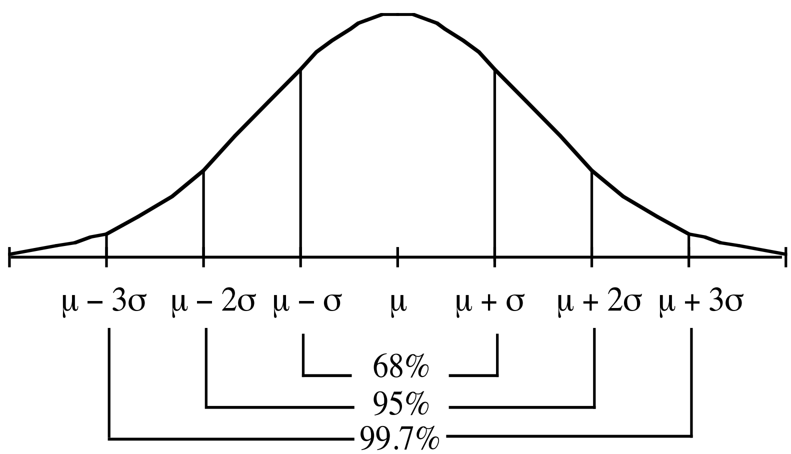


Figure 3: The 68-95-99.7 rule

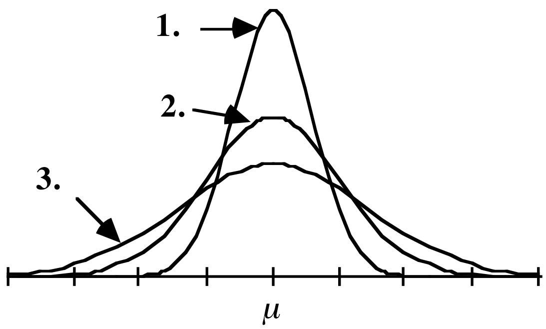
The 68–95–99.7 rule can be used to make predictions about populations that are normally distributed. For example, consider a set of test scores that are normally distributed with a mean of 80 and a standard deviation of 5. About 68% of these scores would fall between $80 - 5 = 75$ and $80 + 5 = 85$. Similarly, about 95% of the scores would fall in the interval [70, 90], while about 99.7% would fall in the interval [65, 95].

f. If a set of scores is normally distributed, describe how the 68-95-99.7 rule can be used to estimate the probability that a random score from the set is:

1. contained in the interval $[\mu, \ \mu + 1\sigma]$
2. contained in the interval $[\mu - 2\sigma, \ \mu]$
3. is not contained in the interval $[\mu - 2\sigma, \ \mu + 2\sigma]$

g. Why does the area between the x-axis and the curve for all continuous probability distributions equal 1?

h. The graph below shows three normal curves with the same mean (μ), but different standard deviations.



Which curve represents the population with the largest standard deviation? Justify your response.

i. Consider a set of scores with a mean of μ and a standard deviation of σ. Describe how to determine the number of standard deviations a score of x is above or below the mean.

Mathematics Note

Any value x from a normally distributed population with mean μ and standard deviation σ can be represented by a ***z*-score**. A *z*-score describes the number of standard deviations that the value is above or below the mean. The formula for determining a *z*-score is shown below:

$$z = \frac{x - \mu}{\sigma}$$

Because the percentage of values lower than x in a normally distributed population can provide useful information, these percentages are commonly available in books and tables. A portion of such a table is shown in Table **1**.

Table 1: Portion of a *z*-score table

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633

In this table, the left-hand column lists *z*-scores from 1.5 to 1.7 by tenths. The hundredths place for each of these values is displayed in the top row. Each four-digit decimal value in the table represents the area between the *x*-axis and a normal curve that lies to the left of the corresponding value of *x*. **Note:** A complete table of *z*-scores and their corresponding areas appears at the end of this module.

For example, consider a set of normally distributed test scores with a mean of 80 and a standard deviation of 3. In this situation, a test score of 85 can be represented by the following z-score: $z = (85 - 80)/3 \approx 1.67$. From the table, a z-score of 1.67 corresponds with a decimal of 0.9525. This means that approximately 95.25% of the test scores in this population are below a test score of 85. In other words, the probability that a test score from this population is less than 85 is approximately 95.25%. Conversely, the probability that a test score from this population is greater than 85 is $1 - 0.9525 = 0.0475$, or approximately 4.75%.

j. Is it possible for z to be negative? If so, when? If not, why not?

k. Dena scored 610 on the mathematics portion of the SAT. The average score for the 32 seniors was 529, with a standard deviation of 125. About how many seniors did Dena outscore?

l. Consider two sets of test scores. In set A, the mean is 70 and the standard deviation is 5. In set B, the mean is 72 and the standard deviation is 3.

1. Based on the number of standard deviations above the mean, which is the better test score: a 76 in set A or a 76 in set B?

2. Based on the percentage of scores below 76, which is the better score?

m. A set of 50 English examination scores is normally distributed with a mean of μ and a standard deviation of σ. Describe how to find the probability that a score selected at random from this set is less than or equal to 64, given each of the following values for μ and σ.

1. $\mu = 71$ and $\sigma = 4$

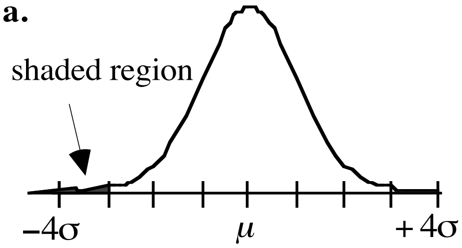
2. $\mu = 75$ and $\sigma = 5$

3. $\mu = 48$ and $\sigma = 5$

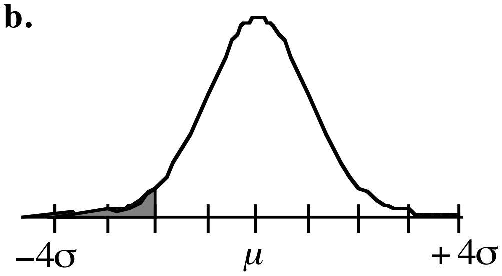
Assignment

2.1 Determine the percentage of the total area under the curve represented by the shaded region(s) in each of the following graphs.

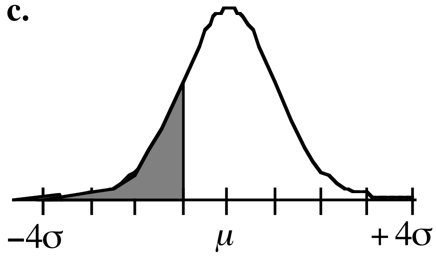
a.



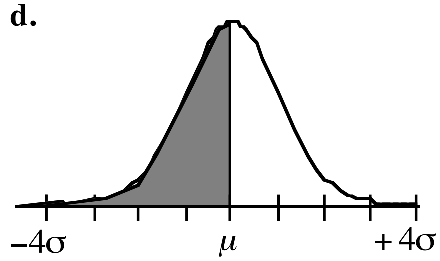
b.



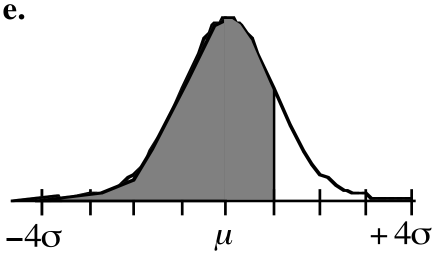
c.



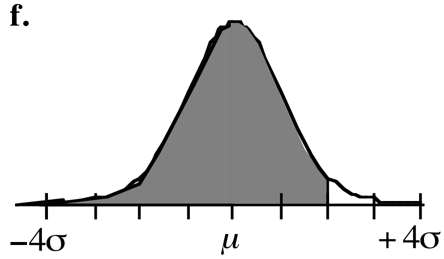
d.



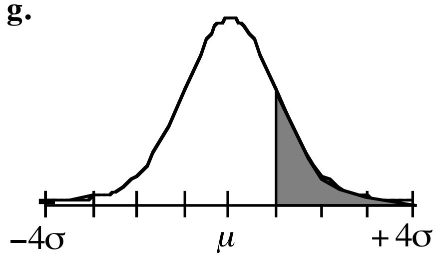
e.



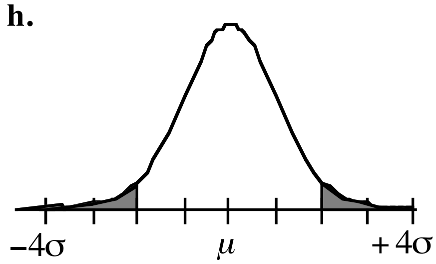
f.



g.



h.



2.2 Determine the area between the x-axis and a normal curve for $x < \mu - 1.25\sigma$.

2.3 Use the tables for Area under Normal Curve found at the end of this module to determine eachof the following:

a. Find the area between the *x*-axis and a normal curve that lies to the left of each of the following values:

1. $\mu - 3\sigma$

2. $\mu - \sigma$

3. $\mu + 2\sigma$

b. Find the area between the *x*-axis and a normal curve that lies to the right of each of the following values:

1. $\mu - 2\sigma$

2. μ

3. $\mu + \sigma$

4. $\mu + 3\sigma$

2.4 Rolf earned an 82 on an English exam that was normally distributed with $\mu = 80$ and $\sigma = 5$. Dena received a 77 on a physics exam that was normally distributed with $\mu = 72$ and $\sigma = 3$.

Rolf claims, "My score is better because 82 is greater than 77." Dena says, "No way! My score is better because the physics exam was tougher." Defend either Rolf's or Dena's position.

2.5 Consider a set of normally distributed test scores with $\mu = 79$ and $\sigma = 5$. What is the probability that a randomly selected score from this set is:

a. less than 70?

b. less than 85?

c. between 74 and 84?

d. between 75 and 90?

e. greater than 83?

* * * * *

2.6 In 1993, 1,044,465 students took the mathematics portion of the SAT. The test scores were normally distributed with $\mu = 478$ and $\sigma = 125$.

a. What percentage of students scored below 500 on the exam? (This is the percentile rank associated with a score of 500.)

b. Approximately how many people scored above 500 on the exam?

c. To be considered for a distinguished scholarship at the local university, a candidate must score in the 90th percentile or better (without rounding) on the mathematics portion of the SAT. What score must a candidate obtain to be considered for this award?

2.7 Is it possible for two sets of examination scores to exist in which a score of 90 on one test is not as good as a score of 47 on the other? Justify your response with examples.

2.8 A set of test scores is normally distributed with $\mu = 72$ and $\sigma = 3$.

a. If one test score is selected at random from this set, what is the probability that this score is:

1. greater than 74?
2. greater than 82?
3. greater than 68?
4. greater than 70?

b. Maeve scored 74 on this test. She claims she performed significantly better than the others who took the exam. Support or refute her claim.

2.9 The mean body temperature of a healthy person is 37˚C. Assume that body temperatures are normally distributed about this mean, with a standard deviation of 0.23˚C. Within what range of values would you expect the temperature of 95% of healthy people fall?

* * * * * * * * * *

Activity 3

In Activity **2**, Dena used a normal curve to compare her individual test score with the mean of 32 scores from her class. However, this set of scores did not include the scores of all the seniors at John F. Kennedy High School.

In an article for the school newspaper, Dena would like to claim that this year's seniors did significantly better on the SAT than last year's seniors. Can she use the mean of 32 scores to estimate the mean for the entire class?

Mathematics Note

The **sampling distribution of sample means** contains the means ($\overline{x}$) of *all* possible samples of size *n* from a population.

The mean of the sampling distribution of sample means, denoted by $\mu_{\overline{x}}$, equals the population mean μ.

The standard deviation of the sampling distribution, denoted by $\sigma_{\overline{x}}$, equals $\sigma/\sqrt{n}$, where σ is the population standard deviation and *n* is the sample size. When σ is unknown, the standard deviation of the sample (*s*) may be used as an estimate of σ.

The **central limit theorem** states that, even if the population from which samples are taken is not normally distributed, the sample means tend to be normally distributed. The approximation to the normal curve becomes more accurate as the sample size n increases. For $n \geq 30$, the distribution of sample means can be modeled reasonably well by a normal curve. This requirement is not necessary if the population from which samples are taken is normally distributed.

For example, suppose you take a sample of 32 boxes of cereal and determine the mean of their masses. Since $n \geq 30$, the central limit theorem applies and the sampling distribution of sample means can be modeled by a normal curve. Using the 68–95–99.7 rule, approximately 68% of the sample means fall within 1 standard deviation of the population mean μ; approximately 95% of the sample means fall within 2 standard deviations of μ; and approximately 99.7% of the sample means fall within 3 standard deviations of μ.

Discussion 1

a. How do $\bar{x}$, μ, and $\mu_{\bar{x}}$ compare?

b. How do σ and $\sigma_{\bar{x}}$ compare?

c.
1. Consider a random sample of 50 test scores taken from a population of scores. Should the 68–95–99.7 rule be used to compare the sample mean to the population mean? Why or why not?

2. Consider a random sample of 20 test scores taken from a population of scores. Should the 68–95–99.7 rule be used to compare this sample mean to the population mean? Why or why not?

d. Consider a sampling distribution with a mean of $\mu_{\bar{x}}$ and a standard deviation of $\sigma_{\bar{x}}$. Describe how to determine the probability that a sample mean $\bar{x}$ selected at random from this distribution is:

1. more than 1.6 standard deviations below the mean

2. more than 2.3 standard deviations above the mean.

Exploration

The scores for 32 of this year's seniors had a mean of 529 and a standard deviation of 125. Assuming that these students represent a random sample of the class, Dena can use their mean and standard deviation to estimate the mean and standard deviation of the scores for the entire class.

The mean score on the mathematics portion of the SAT for last year's seniors was 478. Dena would like to claim that this year's seniors did significantly better than last year's class.

Through hypothesis testing, you can determine if the difference between a sample mean and a population mean is expected or not. If the difference is greater than what might be expected due to the predictable variability among samples, the difference is said to be "significant."

a. Formulate the null and alternative hypotheses for the situation described above.

b. Just as any value x from a normally distributed population with mean μ and standard deviation σ can be represented by a z-score, so can any value $\bar{x}$ from a normally distributed population with mean $\mu_{\bar{x}}$ and standard deviation $\sigma_{\bar{x}}$. In this case, the z-score (denoted by $z_{\bar{x}}$) can be determined as follows:

$$z_{\bar{x}} = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Calculate the z-score for the sample mean of 529. Use the sample standard deviation of 125 as an estimate of the population standard deviation.

Mathematics Note

The **significance level** of a hypothesis test is an arbitrarily assigned probability that distinguishes a significant difference from a chance variation. Traditionally, researchers use significance levels of 0.10, 0.05, or 0.01.

Using a 0.10 significance level, for example, requires that the sample give evidence against the null hypothesis so strong that this evidence would occur no more than 10% of the time, assuming the null hypothesis is true. In other words, the chances of making the error of rejecting a true null hypothesis are less than 0.10, or 10%.

The **critical region** represents the set of all values that would lead a researcher to reject the null hypothesis. For example, suppose that a researcher has obtained a positive z-score for a statistic and selected a 0.10 significance level. From a table, the positive z-score associated with an area of 0.10 is 1.28. This value defines the boundary of the critical region, as shown in Figure **4**.

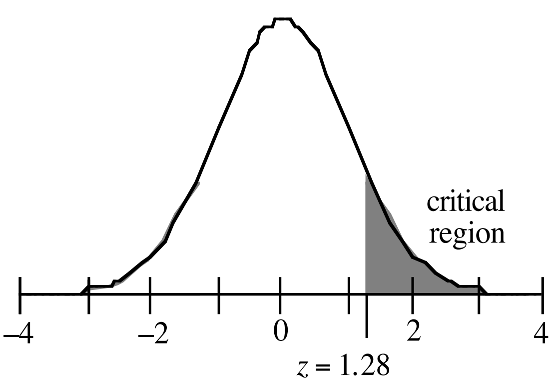


Figure 4: Critical region for 0.10 significance level

If the z-score of the individual observation (or sample mean) falls in the critical region, the researcher should reject the null hypothesis. This indicates that the difference between the statistic and the population parameter is not due to predictable variability at the given level of significance.

For example, suppose that a sample of 50 test scores has a mean of 81.5. Is this sample significantly better than a population with a mean of 80 and a standard deviation of 5? In this situation, the null hypothesis is that there is no difference between the mean of the population from which the sample is taken and 80.

Since the sample size is greater than 30, the central limit theorem applies, and the z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{81.5 - 80}{5/\sqrt{50}} \approx 2.11$$

The critical region for a 0.10 significance level is $z > 1.28$. This indicates that z-scores greater than 1.28 would occur by chance less than 10% of the time. Since the z-score of 2.11 is greater than 1.28, it falls in this critical region. The researcher should reject the null hypothesis at the 0.10 significance level.

If the researcher wanted to test at the 0.01 significance level, the critical region is $z > 2.33$. This indicates that z-scores greater than 2.33 would occur by chance less than 1% of the time. Since 2.11 is less than 2.33, the z-score falls outside the critical region. The researcher should fail to reject the null hypothesis at a 0.01 significance level.

c. Using a 0.05 significance level, determine the critical region for Dena's hypothesis test.

d. Determine whether Dena should reject, or fail to reject, the null hypothesis. Justify your choice.

e. Based on the results of the hypothesis test, what conclusion can you make?

Discussion 2

a. How does the formula for the z-score of an individual observation (x) compare to the formula for the z-score of a sample mean ($\bar{x}$)?

b. How did you determine the null and alternative hypotheses in Part **a** of the exploration?

c. Could Dena use the null hypothesis in the exploration to show that this year's class did the same as last year's class? Justify your response.

d. Describe how you determined the critical region for a hypothesis test at a 0.05 significance level.

e. In some hypothesis tests, the critical region is determined by two values, not one. This occurs when the null hypothesis states that a parameter equals a given value.

For example, to test the null hypothesis, H_0: $\mu = 5$, a researcher must consider possible values for μ that are both above and below 5. If a 0.05 significance level is desired, then the critical region is defined by $z = 2$ and $z = -2$, since approximately 95% of possible sample means fall within 2 standard deviations of μ.

1. Describe a graph of the critical region in this situation.
2. If the z-score for the statistic falls in the critical region, what should the researcher decide?

f. In what kinds of situations would a researcher select a 0.01 significance level rather than a 0.05 or 0.10 significance level?

g. How did you justify your decision to reject, or fail to reject, the null hypothesis?

h. Do you think that Dena should claim that this year's seniors did significantly better than last year's class on the SAT? Explain your response.

Assignment

3.1 In 1993, 2234 students in the state of Montana took the mathematics portion of the SAT. A random sample of 100 of these students had a mean score of 516 with a standard deviation of 125.

The mean score on this test for students around the nation was 478. Can the governor of Montana claim, with 90% certainty, that Montana students who took this test scored significantly better than students nationally? To make this decision, complete the following steps.

a. State the null and alternative hypotheses in this situation.

b. Determine $z_{\bar{x}}$ for the sample statistic.

c. Decide whether to reject, or fail to reject, the null hypothesis. Justify your reasoning.

d. Use the results of the hypothesis test to state a conclusion.

3.2 The mean score on the mathematics portion of the SAT for one class of 32 students was 529, with a standard deviation of 110. The mean score statewide was 516. A parent would like to know, with 95% certainty, if this class did significantly better on the test than the rest of the students in the state.

a. Formulate null and alternative hypotheses for this situation.

b. Find the z-score for the sample mean.

c. Decide whether to reject, or fail to reject, the null hypothesis. Justify your reasoning.

d. Explain what your decision means in this situation.

3.3 The following table shows the scores for 40 seniors at Washington High School on the mathematics portion of the SAT.

530	610	520	440	490	530	500	480
770	530	520	510	460	450	460	500
420	490	530	500	510	560	540	620
500	600	550	470	580	460	520	530
500	500	540	640	610	450	670	540

a. Determine the mean and standard deviation for this data.

b. The national mean for this test was 478. Assuming that these 40 students represent a random sample of the entire class, do you think Washington High's seniors did significantly better than the rest of the nation? Justify your response.

3.4 A set of scores on a physics exam are normally distributed with a mean of 72 and a standard deviation of 3. Four physics tests are found without names on them. The mean score of the four tests is 77. Is it reasonable to believe that these four tests came from this set of scores?

a. Formulate the null and alternative hypotheses for this situation.

b. Do you think that these four tests came from the physics class with a mean of 72? Explain your response.

3.5 An article in the *Daily News* reported that the mean height of women between the ages of 19 and 32 is 168 cm. In a letter to the editor, one reader insisted that this statement was untrue, arguing that the actual mean is less than 168 cm.

a. State the null and alternative hypotheses for the reader's claim.

b. To test the claim, the reader measured the heights of a random sample of 100 women between the ages of 19 and 32. The mean height in this sample was 164.5 cm, with a standard deviation of 16.2 cm. Find the z-score for the sample mean.

c. Using a 0.10 significance level, decide whether to reject, or fail to reject, the null hypothesis. Justify your reasoning.

d. What does your decision mean in this situation?

* * * * *

3.6 A liquid detergent company bottles their product in 1500-mL containers. After analyzing a random sample of 48 containers, a consumer group found a mean of 1488 mL. When the group called the company for an explanation, the customer relations department admitted that the product has a population standard deviation of 47.5 mL. Decide whether or not the detergent company is cheating its customers, with a 0.10 significance level. Justify your reasoning.

3.7 A breakfast cereal company claims that each box of its product contains at least 397 g of cereal. To test the manufacturer's claims, some students select a random sample of 40 boxes. The table below shows the mass of the cereal in each box, rounded to the nearest 1 g.

402	397	404	384	390	395	397	385	392	399
380	390	408	403	389	389	393	381	402	401
383	403	383	392	400	392	395	395	406	396
408	383	381	390	401	385	382	404	409	387

a. State the null and alternative hypotheses in this situation.

b. Determine the critical region for this hypothesis test at the 0.05 significance level.

c. Find the z-score for the sample statistic. **Note:** Use the standard deviation of the sample as an estimate of σ.

d. Decide whether to reject or fail to reject the null hypothesis. Justify your reasoning.

e. Explain what your decision means in this situation.

3.8 An elevator has a recommended capacity of 15 people and a maximum load limit of 1200 kg. The masses of the population that uses this elevator are normally distributed with a mean of 75 kg and a standard deviation of 10 kg.

a. What is the probability that a random sample of 15 people from this population will exceed the maximum load limit?

b. What is the probability that a random sample of 16 people will exceed the load limit?

* * * * * * * * * *

Summary Assessment

In a golden rectangle, the ratio of the measures of the longer side to the shorter side is the number $\left(1+\sqrt{5}\right)/2$, or about 1.618. The proportions of the golden rectangle are believed to be particularly pleasing to the human eye. For example, the outline of the Greek Parthenon resembles a golden rectangle, as does the face of each stone block in the Egyptian pyramids. In more modern times, the shapes of credit cards, driver's licenses, and the screens of many graphing calculators also are rough approximations of a golden rectangle.

The table below shows the length-to-width ratios of 30 beaded rectangles used to decorate Crow Indian leather goods:

1.706	1.706	1.704	1.178	1.176	1.175	1.916	1.502	1.919	1.912
1.504	1.499	1.890	1.701	1.894	1.704	1.887	1.698	1.880	1.942
1.159	2.237	1.751	2.242	2.232	1.754	1.748	2.066	2.075	2.070

From this data, does it appear that the Crow Indians also incorporated the golden ratio in their beaded designs? Use statistical analysis to support your belief. Assume that the data are normally distributed. **Note:** Use the standard deviation of the sample as an estimate of σ.

Module Summary

- In statistical analysis, there are two types of hypotheses. A **null hypothesis** ($\mathbf{H_0}$) is a statement about one or more parameters. The **alternative hypothesis** ($\mathbf{H_a}$) is the statement that must be true if the null hypothesis is false. The null hypothesis usually involves a claim of no difference or no relationship. In many situations, the null hypothesis and alternative hypothesis are negations of each other, but this is not necessarily the case.
- A **hypothesis test** may consist of the following steps.
  1. State null and alternative hypotheses about a parameter of a population.
  2. If the null hypothesis is true, predict what this implies about a sample of the population.
  3. Take a sample of the population and compare the results with your prediction.
  4. If the results are not consistent with the prediction, then you can conclude, with some level of certainty, that the null hypothesis is false and, therefore, reject it.
  5. If the results are consistent with the prediction, you fail to reject the null hypothesis. The failure to reject the null hypothesis does not guarantee that the null hypothesis is true, only that it might be true.
- Whenever a sample is taken of a population about which there is some uncertainty, it is possible that the sample is not representative of the population. Therefore, when performing a hypothesis test by sampling it is always possible to make an incorrect decision about the null hypothesis H_0. The two possible errors are rejecting a true null hypothesis and failing to reject a false null hypothesis.
- The curve that describes the shape of a normal distribution is the **normal curve**. The equation of the normal curve that models a normally distributed population depends on the mean and standard deviation of the population. The general equation for a normal curve, where μ represents the population mean and σ represents the population standard deviation, is:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

- The **standard deviation** σ is a measure of the variability, or spread, within a population. In a normally distributed population, the size of the standard deviation affects the height and width of the normal curve.

- The predictable variability of individual observations is summarized by the **68–95–99.7 rule**, which states that approximately 68% of the area between the x-axis and the normal curve is within 1 standard deviation of the mean, approximately 95% is within 2 standard deviations of the mean, and approximately 99.7% is within 3 standard deviations of the mean.
- Any value x from a normally distributed population with mean μ and standard deviation σ can be represented by a ***z*-score**. A z-score describes the number of standard deviations that the value is above or below the mean. The formula for determining a z-score is:

  $$z = \frac{x - \mu}{\sigma}$$

  The value in a table associated with this z-score represents the area between the x-axis and normal curve to the left of the value x.
- The **sampling distribution of sample means** is the distribution of sample means $\bar{x}$ of *all* possible samples of size n from a population.

  The mean of the sampling distribution of *all* sample means, denoted by $\mu_{\bar{x}}$, equals the population mean μ. The standard deviation of the sampling distribution, denoted by $\sigma_{\bar{x}}$, equals $\sigma/\sqrt{n}$, where σ is the population standard deviation and n is the sample size.
- The **central limit theorem** states that, even if the population from which samples are taken is not normally distributed, the sample means tend to be normally distributed. This approximation becomes more accurate as the sample size n increases. For $n \geq 30$, the distribution of sample means can be modeled reasonably well by a normal curve. This requirement is not necessary if the population from which samples are taken is normally distributed.
- Any value $\bar{x}$ from a normally distributed population with mean μ and standard deviation $\sigma_{\bar{x}}$ can be represented by a ***z*-score** as follows:

  $$z_{\bar{x}} = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$$

- The **significance level** of a hypothesis test is an arbitrarily assigned probability that distinguishes a significant difference from a chance variation. It describes the maximum probability of making the error of rejecting a true null hypothesis.
- The **critical region** represents the set of all values that would lead a researcher to reject the null hypothesis. If the z-score of the individual observation (or sample mean) falls in the critical region, this indicates that the difference between the individual observation (or sample mean) and the population mean is significant, and not due to predictable variability. Therefore, the researcher should reject the null hypothesis.

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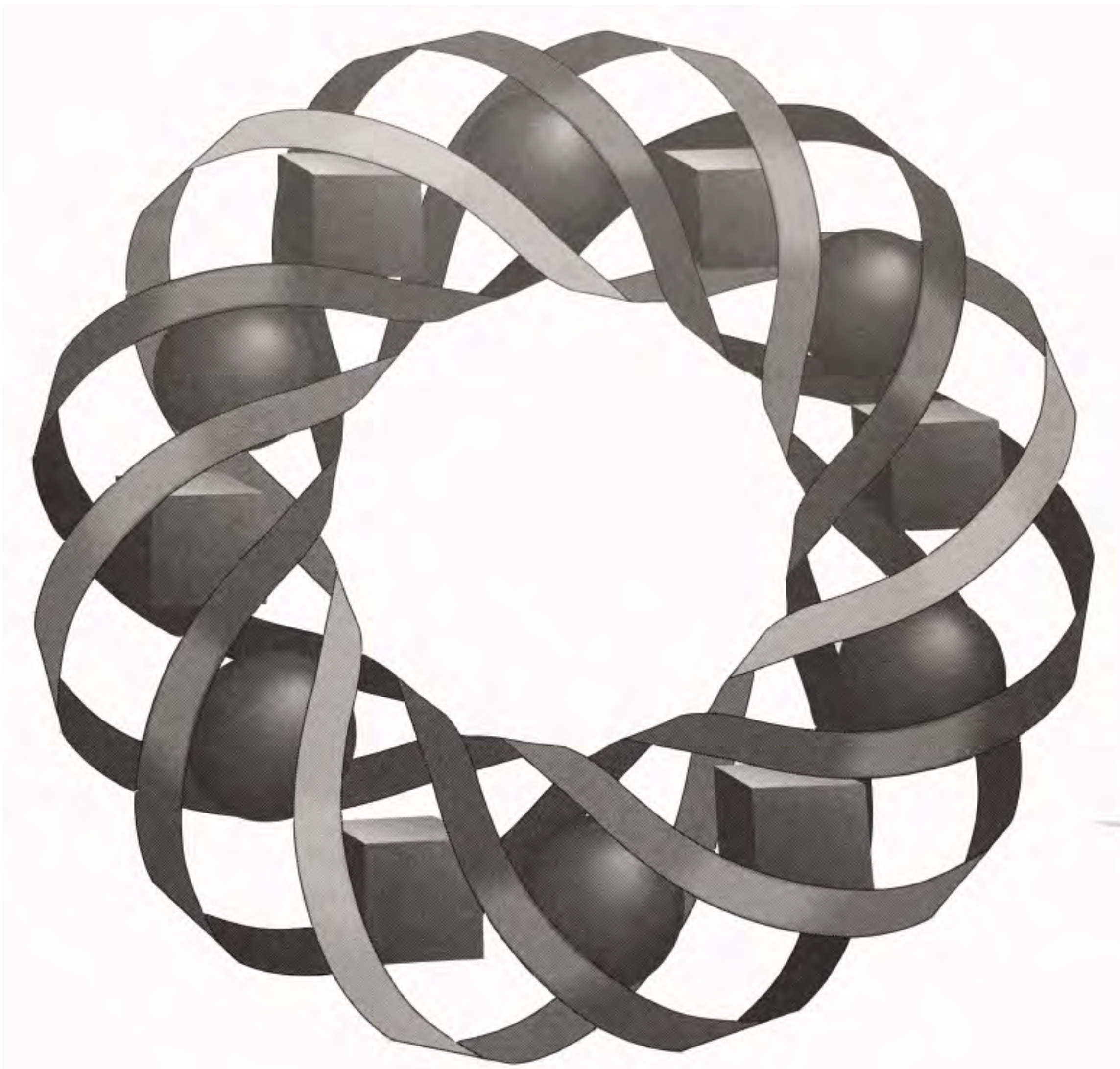
Area under Normal Curve to Left of *Z*-Score										
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Area under Normal Curve to Left of *Z*-Score										
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Ostriches Are Composed



Although ostrich ranching has flourished in South Africa since the mid-1800s, it has only recently become popular in North America. In this module, you use compositions of functions to investigate the business of raising big birds.

Todd Fife • John Gebert • Sue Moore

Ostriches Are Composed

Introduction

Sal and Guinn are partners in a new venture: an ostrich ranch. Although ostrich ranching has flourished in South Africa for over a century, it has only recently become popular in North America.

After months of research, Sal and Guinn buy several pairs of breeding ostriches. Shortly thereafter, the first chicks hatch. The two partners plan to market both unhatched eggs and adult birds. To help attract customers, they decide to come up with a catchy phrase for their new toll-free number—something like 1-800-OSTRICH or 1-800-BIG-BIRD. Unfortunately, both of these numbers are already in use. The telephone company assigns them the number 1-800-825-2445. With a little bit of creative thought, Sal and Guinn plan to use this number to remind their customers of ostriches.

Discussion

a. Using the keypad in Figure **1** for reference, what telephone numbers correspond to OSTRICH and BIG BIRD?

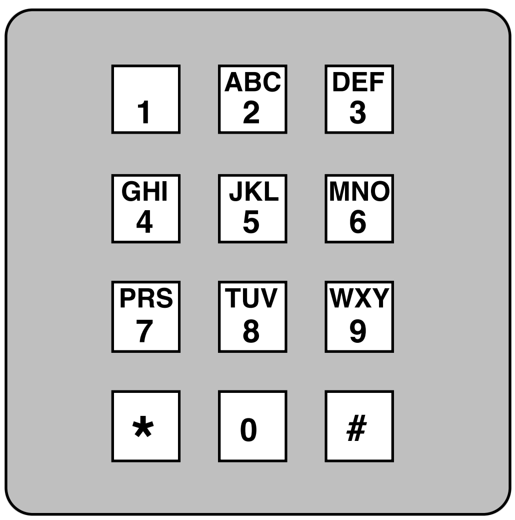


Figure 1: A telephone keypad

b. What catchy phrase might Sal and Guinn use for their assigned number, 1-800-825-2445?

c. Compare the processes you used to respond to Parts **a** and **b**.

d. Use ordered pairs to define the pairing of letters of the alphabet to numbers on the telephone keypad in Figure **1**.

e. Use ordered pairs to define the pairings of numbers to letters on the keypad.

f. Describe the differences between the pairings in Parts **d** and **e**.

Activity 1

Sal and Guinn are excited about the potential for profit in their new business. Ostriches are valued for beautiful feathers, leathery hides, and flavorful meat. Since Sal and Guinn plan to sell both unhatched eggs and adult birds, their income will depend on the number of breeding pairs in their flock. In turn, the number of birds they can keep will depend on the availability of such resources as fenced pasture and fresh water.

In applications such as ostrich ranching, it is often important to know how two quantities—such as profit and flock size—are related.

Exploration

Recall that a **relation** is a set of ordered pairs in which the **domain** is the set of first elements and the **range** is the set of second elements. A **function** is a relation from a domain to a range in which each element of the domain occurs in exactly one ordered pair. Both relations and functions are sometimes specified by a rule relating the domain and range.

Note: Functions often are written without listing a domain. In such cases, the domain is considered to contain all elements for which the function is mathematically meaningful. If the function is composed of numerical ordered pairs, the domain is typically either the set of real numbers or one of its subsets.

Determine a possible domain and range for each of the following. Use a graphing utility to check your results.

a. $f(x) = \sqrt{x-1}$

b. $g(x) = -|x|$

c. $h(x) = \sqrt{4-x^2}$

d. $k(x) = \dfrac{\sin x}{x-5}$

Discussion

a. Explain why the domains of some of the relations in the exploration are restricted.

b. How did your graphing utility indicate limitations in the domain for these relations?

c. Which of the relations in the exploration are functions?

d. Describe how a vertical line can be used to show that the graph of a relation does not represent a function.

e. Define a relation from a subset of the set of digits on a telephone keypad to the letters on a telephone keypad.

1. What is the domain of the relation?
2. What is the range of the relation?
3. Is this relation a function? Explain your response.

f. Consider the relation of the set of letters on a telephone keypad to the set of digits $\{0, 1, 2, \ldots, 9\}$.

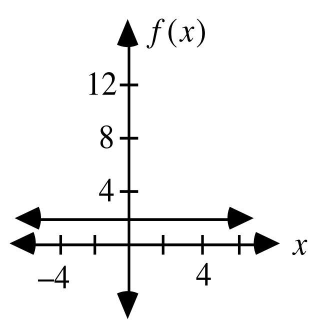
1. What is the domain of the relation?
2. What is the range of the relation?
3. Is this relation a function? Explain your response.

g. Describe a reasonable domain and range for a function Sal and Guinn might use to calculate their profit, if profit is a function of the number of ostriches in their flock. Justify your response.

Assignment

1.1 **a.** Identify the domain, range, and a possible rule for each of the following relations.

1. $h = \{(1,1), (1,-1), (4,2), (4,-2), (9,3), (9,-3), (16,4), (16,-4)\}$
2. $r = \{(1,1), (-2,-8), (3,27), (-4,-64)\}$
3.



b. Which of the relations in Part **a** are functions? Explain your response.

Mathematics Note

One way to represent a function is with a **set diagram**, using an arrow to represent the rule. The set diagram in Figure **2** illustrates the domain and range of the function *f* for $f(x) = x^2$. The domain is the set of real numbers; the range is the set of real numbers greater than or equal to 0.

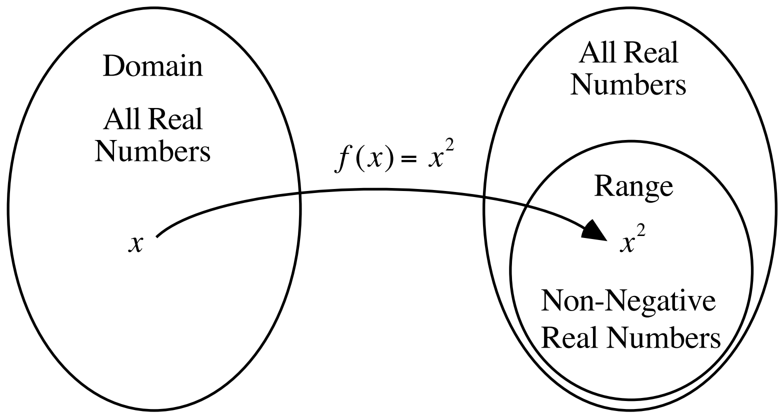


Figure 2: Set diagram illustrating the function $f(x) = x^2$

Another way to represent a function between two sets is a **mapping diagram**. Figure **3** shows a mapping diagram for the function $h(x) = x + 2$. Both the domain and range are the set of real numbers. The arrows indicate pairings of some elements in the domain with the corresponding elements in the range.

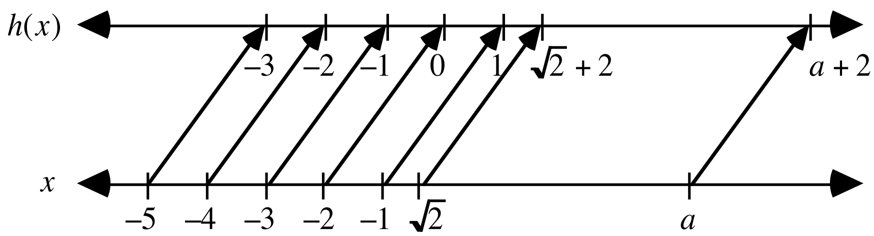


Figure 3: Mapping diagram illustrating the function $h(x) = x + 2$

1.2 Consider the telephone keypad shown in Figure **1**. Select a key that contains both letters and a number.

a. Consider the relationship between the number and the corresponding letters.

1. Draw a set diagram of this relationship.
2. Draw a mapping diagram of this relationship.
3. Write a description of the relationship.

b. Consider the relationship between the letters and the corresponding number. Repeat the steps in Part **a** for this relationship.

1.3 An ostrich can run at a rate of about 50 km/hr.

a. Using function notation, write an equation for the distance an ostrich can run in terms of time measured in minutes.

b. Determine a possible domain and range for the function.

c. Create a graph that could be used to estimate the distance an ostrich can run in a given time.

1.4 The domain of a function f is the set of real numbers. The function is defined by the rule $f(x) = 3x$.

a. What is the range of this function? Justify your response algebraically.

b. Given the restricted domain $[-1,1]$, find the corresponding range for this function.

c. Find the domain for which $f(x) = 3x$ results in the range $[-1,1]$.

1.5 Consider the function defined by the rule $f(x) = 3x - 2$, with domain $[-5,10]$, where x is a real number. Find the range of this function.

1.6 Write a function, giving its domain and range, in which each element of the range is paired with two elements of the domain.

1.7 For each of Parts **a–c** below, determine a rule for a function with the given domain and range. Illustrate each rule graphically.

a. domain: (∞,∞); range: $(-\infty,4]$

b. domain: $(-\infty,0)\cup(0,\infty)$; range: $(-\infty,0)\cup(0,\infty)$

c. domain: $[3,\infty)$; range: $[0,\infty)$

* * * * *

1.8 At standard temperature and pressure, water boils at 100° C or 212° F. Likewise, water freezes at 0° C or 32° F. The relationship between temperatures measured in degrees Fahrenheit and Celsius is linear.

a. Write a function, $f(c)$, which would convert any given temperature in degrees Celsius to its corresponding temperature in degrees Fahrenheit.

b. Find the domain and range for the function.

1.9 The Greek mathematician Diophantus (ca. 250 A.D.) has sometimes been called the "father of algebra." A set of special equations bears his name. The solutions to a Diophantine equation consist of ordered pairs, each of whose elements are integers. For example, the solutions to the linear Diophantine equation $y = x$ is the set of ordered pairs below:

$$\{\ldots,(-3,-3),(-2,-2),(-1,-1)(0,0),(1,1),\ldots\}$$

Find the solutions, if any, to the following Diophantine equations:

a. $x + 5y = 11$

b. $3x + 6y = 71$

1.10 Identify an appropriate domain and range for each function described below.

a. The cost of producing x number of shirts by a clothing company is modeled by the function:

$$c(x) = 23 + 2x$$

b. The distance in meters that an object falls in t seconds can be modeled by the function:

$$d(t) = 4.9t^2$$

c. The quantity of radioactive carbon-14 remaining after t years is modeled by the following function, where $q_\circ$ is the initial quantity:

$$q(t) = q_\circ e^{-0.00012t}$$

* * * * * * * * * *

Activity 2

Sal and Guinn's ranch receives income from two sources: selling unhatched eggs and selling adult breeding pairs. Although the partners could determine a separate profit function for each of these income sources, they would like to simplify the process of projecting potential profits. In this activity, you examine how two functions can be combined into a single function.

Exploration 1

Sal and Guinn sell eggs for \$500 each. The number of eggs that they sell each year depends on the number of adult breeding pairs in their flock. Each breeding pair produces an average of 40 viable eggs per year. Not all of the eggs are sold, however. Four of every 40 eggs are kept for hatching on the ranch. Annually, the ranch spends about \$60,000 in overhead related to egg production.

Mature breeding pairs bring \$10,000 per pair. Like the number of eggs sold, the number of breeding pairs sold is a function of the number of breeding pairs in the flock. Each breeding pair annually produces 4 eggs that are hatched on the ranch and raised to maturity. (Assume that 50% of the hatched chicks are male and 50% are female.) The ranch spends about \$80,000 per year to maintain their adult ostrich flock.

a. **1.** Using the information given above, determine a function $e(x)$ that describes the annual profit on the sale of eggs, where x is the number of breeding pairs in the flock.

2. Determine another function $p(x)$ that describes the annual profit on the sale of breeding pairs, where x is the number of breeding pairs in the flock.

b. **1.** The size of Sal and Guinn's flock varies from 5 to 15 breeding pairs a year. Use appropriate technology to evaluate $e(x)$ for each element of this domain.

2. Evaluate $p(x)$ for each element of the domain described in Step **1**.

c. To find a function for the ranch's total profit, add the two functions in Part **a** to obtain $(e + p)$. Determine a rule for this new function.

d. Assuming the flock varies from 5 to 15 breeding pairs, determine the domain and range for $(e + p)$.

Discussion 1

a. Compare the profit functions *e* and *p* you obtained in Part **a** of Exploration **1** with those of others in the class.

b. Compare the rule you determined for $(e + p)$ with those of others in the class.

c. Describe how the range of the function $(e + p)$ was derived from the ranges of *e* and *p*.

d. Using the function $(e + p)$ as a model, can the total profit for any one year on Sal's and Guinn's ranch ever be exactly \$200? Explain your response.

e. When adding functions, the distributive property of multiplication over addition allows like terms to be combined. For example, consider the sum of the functions $f(x) = 32x$ and $g(x) = 16x$. Since $32x$ and $16x$ have the common factor of x, the sum can be simplified using the distributive property of multiplication over addition as follows:

$$\begin{aligned} f(x) + g(x) &= 32x + 16x \\ &= (32 + 16)x \\ &= 48x \end{aligned}$$

Which of the following sums can be simplified using the distributive property?

1. $16x^2 + 16x^3$

2. $12\sqrt{x} + 3\sqrt{x}$

3. $\cos x + 5\cos x$

4. $2\sin x + \sin(2x)$

Exploration 2

In Exploration **1**, you added two functions together. In this exploration, you examine the results of some other operations on functions.

a. Consider the following functions:

$$f(x) = \sqrt{x}$$
$$g(x) = \sqrt{-x + 5}$$

1. Graph the two functions on the same coordinate system.

2. Determine the domain and range of each function.

3. Find the intersection of the domains of $f(x)$ and $g(x)$.

Mathematics Note

The function $(f + g)$ is defined by $(f + g)(x) = f(x) + g(x)$. Likewise, $(f - g)(x) = f(x) - g(x)$, and $(f \bullet g)(x) = f(x) \bullet g(x)$.

When adding, subtracting, or multiplying two or more functions, the operations are defined only for those values common to the domains of all the functions involved. For example, consider the function $f(x) = 2/x$, with a domain of $(-\infty,0) \cup (0,\infty)$, and the function $g(x) = 5$, with a domain of $(-\infty,\infty)$. The domains have common values of $(-\infty,0) \cup (0,\infty)$. Since addition, subtraction, or multiplication of the two functions is defined only for those common values of x, the domain of $(f + g)(x)$ is $(-\infty,0) \cup (0,\infty)$.

b. Using the functions from Part **a**, graph each of the following on a separate coordinate system and determine its domain and range:

1. $(f+g)(x)$

2. $(f-g)(x)$

3. $(f \bullet g)(x)$

c. Functions may also be divided. When $f(x)$ is divided by $g(x)$, it can be represented as

$$\frac{f(x)}{g(x)}$$

or $(f/g)(x)$, where $g(x) \neq 0$.

1. Graph the function $(f/g)(x)$. Determine the domain and range.

2. Graph the function $(g/f)(x)$. Determine its domain and range.

d. Repeat Parts **a–c** for the functions $f(x)=2$ and $g(x)=\sin x$.

Discussion 2

a. Consider two functions: f and g. The domain of f is $\{1, 2, ..., 10\}$, while the domain of g is $\{-3, -2, ..., 5\}$. The range of each function is a subset of the real numbers. In this case, what is the domain of $(f+g)$? Explain your response.

b. In the function defined by f/g, why is it important that $g(x) \neq 0$?

c. Compare the domain and range of $(f/g)(x)$ and $(g/f)(x)$ when $f(x)=\sqrt{x}$ and $g(x)=\sqrt{-x+5}$.

Assignment

2.1 Consider the functions $f(x)=x^2+3x$ and $h(x)=x-2$.

a. Determine a rule for each of the following:

1. $f+h$

2. $f-h$

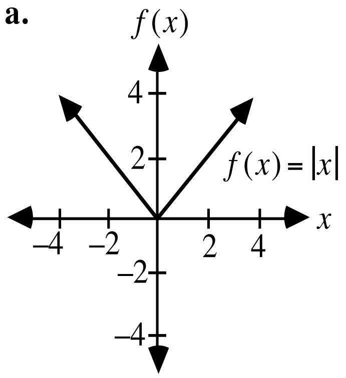
3. $f \bullet h$

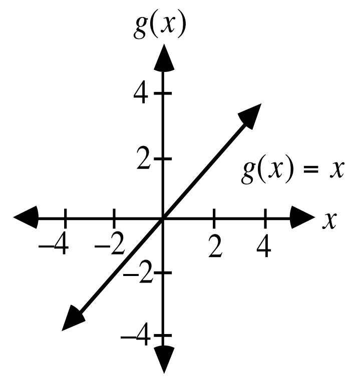
4. f/h

b. Graph each function in Part **a**. Use your graphs to identify the apparent domains and ranges.

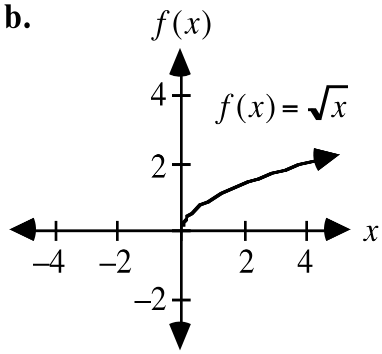
2.2 In each of Parts **a–c** below, determine the domains for which the functions $(f+g)(x)$, $(f-g)(x)$, $(f \bullet g)(x)$, $(f/g)(x)$, and $(g/f)(x)$ are defined.

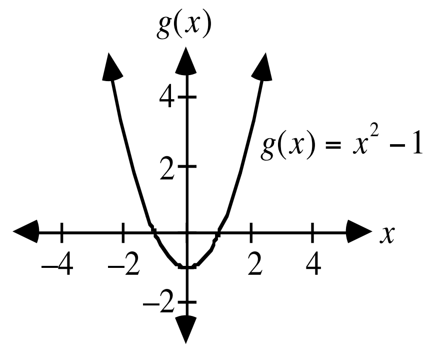
a.



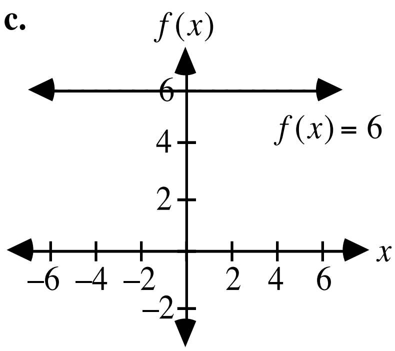


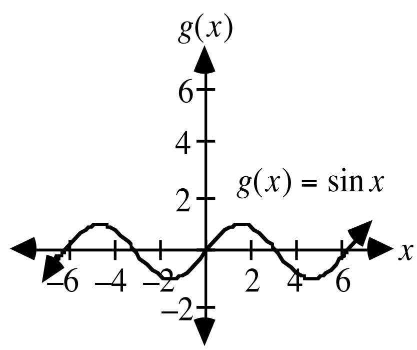
b.





c.





2.3 Use the following functions to complete Parts **a** and **b**.

$$f(x) = x + 5$$
$$g(x) = -2x - 3$$

a. Graph the three functions $(f-g)(x)$, $(f \bullet g)(x)$, and $(f/g)(x)$ on separate coordinate systems.

b. Determine a rule for each function in Part **a** and list its domain and range.

2.4 **a.** If $f(x) = \sin\ x$ and $g(x) = \cos\ x$, graph $(f(x))^2$ and $(g(x))^2$ on the same coordinate system. Determine the domain and range for $(f(x))^2$ and $(g(x))^2$.

b. Assume $(f(x))^2 = (\sin x)^2$ and $(g(x))^2 = (\cos x)^2$. Write an expression for $(f^2 + g^2)(x)$ and graph this new function. Determine the domain and range of this function.

c. Use the graph from Part **b** to write a simplified expression for $(f^2 + g^2)(x)$.

d. An equation involving trigonometric functions that is true for all real numbers in its domain is a **trigonometric identity**. Does the expression you wrote in Part **c** form a trigonometric identity? Explain your response.

e. Recall that the division of $\sin x$ by $\cos x$ results in $\tan x$. Using a graph, determine the domain and range of the tangent function.

f. Is the following expression a trigonometric identity? Explain your response.

$$\tan x = \frac{\sin x}{\cos x}$$

2.5 At another ostrich ranch, the owners sell either newly hatched chicks or breeding pairs. They charge $700 for each chick and have a total of $36,000 in annual overhead expenses related to the hatching operation. The ranch sells breeding pairs for $10,000 per pair and spends a total of $40,000 per year to maintain the adult ostrich flock.

a. **1.** Determine a function $c(x)$ to model the profit on the sale of ostrich chicks, where x represents the number of chicks sold.

2. Determine a function $p(y)$ to model the profit from selling breeding pairs, where y is the number of breeding pairs sold.

b. What do $c(5)$ and $p(5)$ represent?

c. Explain why it is not reasonable to add the two profit functions in this setting to form a new profit function, $(c + p)(x)$.

d. Consider the profit function $r(x,y) = c(x) + p(y)$ where x is the number of chicks sold and y is the number of breeding pairs sold.

1. What does $r(10,8)$ represent?

2 Describe a possible domain for r. What is the corresponding range?

3. Is r a reasonable function to use for determining the ranch's total profit from the sale of chicks and breeding pairs? Explain your response.

2.6 Consider the functions $f(x) = 3x$, $g(x) = 1/(3x)$, and $h(x) = x - 3$ with their appropriate domains and ranges. Given these functions, find the simplest possible rule for each of the following and describe its domain.

a. $(f + g)(x)$

b. $(f - h)(x)$

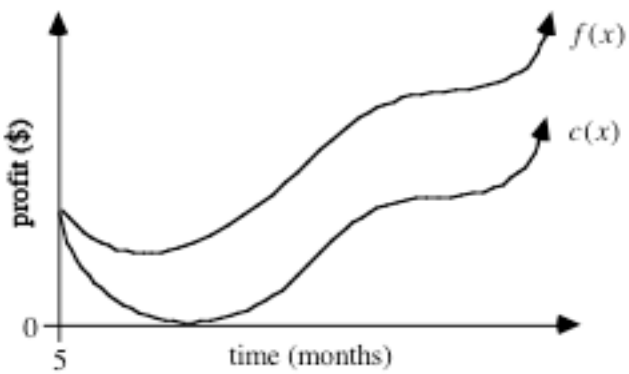
c. $(f \bullet g)(x)$

d. $(h/g)(x)$

e. $(f/h)(x)$

* * * * *

2.7 Since its fifth month in business, Bea's Beauty Salon has earned a profit in every month from at least one of two areas: product sales or customer services. The profit from customer services only can be described by the function $c(x)$, where x represents time in months. The total profit can be described by the function $f(x)$. The following diagram shows graphs of these two functions.



Sketch a graph of the function $p(x)$ that describes the profit from product sales only.

2.8 The function $h(t) = 2 + 15t - 4.9t^2$ models the height (in meters) after t sec of an object thrown straight up in the air at 15 m/sec from a distance of 2 m above the ground. This function can be thought of as the sum of three other functions: $s(t) = 2$, $p(t) = 15t$, and $g(t) = -4.9t^2$.

The function $s(t)$ describes the object's initial height (in meters). The function $p(t)$ describes the effect produced by the object's initial velocity. The function $g(t)$ describes the effect produced by the acceleration due to gravity.

a. Determine a function that models the height (in meters) after t sec of an object thrown straight down from a height of 50 m with an initial velocity of 10 m/sec.

b. How long will it take the object described in Part **a** to hit the ground?

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Activity 3

During their first year of operation, Sal and Guinn found that the size of their flock could be represented as a function of time. They also observed that the number of births at the ranch depended on the number of breeding pairs—more adult ostriches meant that more chicks were hatched. In this respect, the number of births could be thought of as a function of the number of adult ostriches.

Since the number of births is a function of the number of adult ostriches, and the number of adults is a function of time, it is also possible to consider the number of births as a function of time. The diagram in Figure **4** shows how this can be done by forming a **composite function** $f(g(x))$.

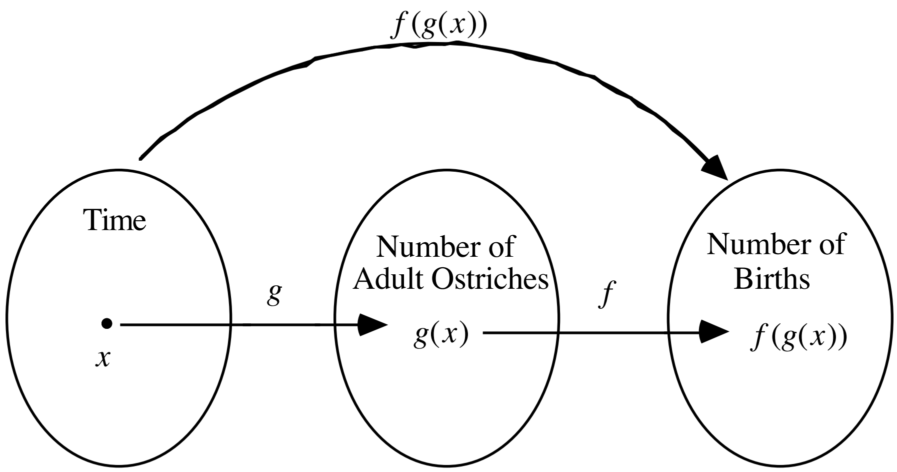

Figure 4: Diagram of a composition of functions

Mathematics Note

Given two functions f and g, the **composite function** $f \circ g$, read as "f composed with g" is defined as

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all values of x in the domain of g such that $g(x)$ is in the domain of f.

For example, if g is the set of ordered pairs $\{(1,3), (2,4), (5,11)\}$, its domain is $\{1,2,5\}$. If f is the set of ordered pairs $\{(3,8), (4,12), (7,13)\}$, then its domain is $\{3,4,7\}$. The domain of $f \circ g$ is $\{1, 2\}$. This is because $\{1,2\}$ is part of the domain of g and $g(1) = 3$ and $g(2) = 4$ are values in the domain of f. Figure **5** illustrates these relationships using a mapping diagram.

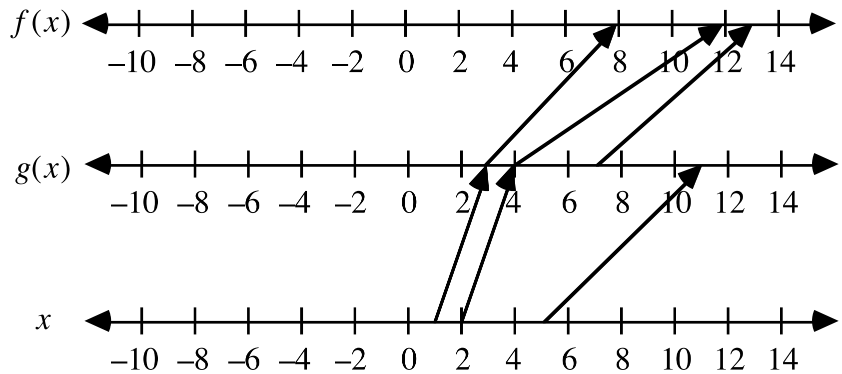

Figure 5: Mapping diagram of $f(g(x))$

Notice in the mapping diagram in Figure **5** that $g(5) = 11$. Since 11 is not in the domain of *f*, however, 5 is not in the domain of $f \circ g$. Figure **6** shows a simplified version of the mapping diagram for $(f \circ g)(x)$.

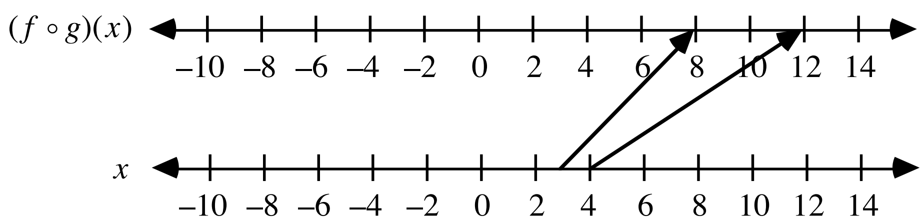

Figure 6: Mapping diagram of $(f \circ g)$

From Figure **6**, the domain of $f \circ g$ is {1,2} and the range is {8,12}.

Exploration

In this exploration, you investigate the composition of functions using $f(x) = 3x$ and $g(x) = x^2$ and the domain $\{-5, -4, -3, ..., 5\}$.

a. **1.** Determine the range for $g(x)$.

2. Find the range for $f(g(x))$ by completing a mapping diagram like the one shown in Figure **7**.

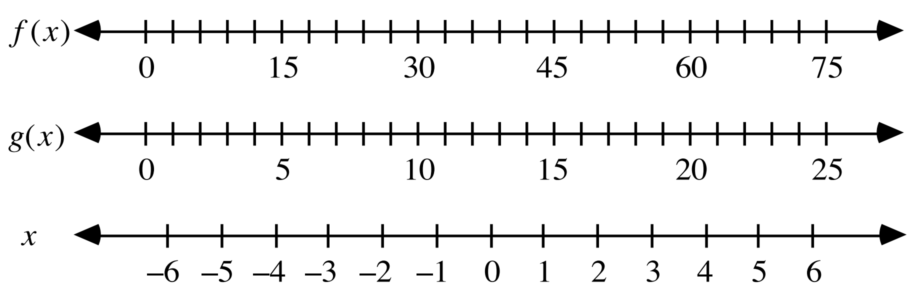

Figure 7: Incomplete mapping diagram for $f(g(x))$

3. Determine a rule for $f(g(x))$. State the domain and range for $f(g(x))$.

b. **1.** Determine the range for $f(x)$.

2. Find the range for $g(f(x))$ by completing a mapping diagram.

3. Determine a rule for $g(f(x))$. State the domain and range for $g(f(x))$.

Discussion

a. Consider the two functions $q(x) = 3x + 2$ and $f(x) = x^2$.

1. Describe what $q(a)$ indicates.

2. Describe what $f(q(a))$ indicates.

b. How do the ranges of $f(g(x))$ and $g(f(x))$ from the exploration compare?

c. Does composition of functions appear to be commutative? Explain your response.

d. If *f* is the set of ordered pairs {(3,5), (4,6), (7,13)} and *g* is the set {(5,10), (6,14), (9,15)}, what ordered pairs would be in $g(f(x))$? Explain your response.

e. Consider the functions $f(x) = \sqrt{x+5}$ and $g(x) = x^3 + 3$.

1. Describe the domain and range for $f(g(x))$.

2. How does the process of finding $g(f(2))$ differ from the process of finding $f(g(2))$?

3. Does the domain of $f \circ g$ include all the values in the domain of $g(x)$? Explain your response.

4. Does the domain of $g \circ f$ include all the values in the domain of $f(x)$? Explain your response.

Assignment

3.1 Suppose an ostrich ranch keeps 4 eggs for hatching each year and sells the rest. The number of eggs for sale can be modeled by the function $h(x) = x - 4$, where *x* represents the total egg production that year. Since about 20% of each year's eggs are lost to breakage, the number of unbroken eggs can be modeled by $g(x) = 0.8x$.

a. If there are 40 eggs in a given year, which of the following correctly describes the number of unbroken eggs that are not kept by the ranch for hatching: $h(g(40))$ or $g(h(40))$?

b. Use your answer from Part **a** to help show that composition of functions is not commutative.

3.2 Given the functions $f(x) = x^2$ and $g(x) = \sin x$, what are the domain and range of the composite function $(f \circ g)(x)$?

3.3 Consider the functions $h(x) = x^2$ and $k(x) = \sqrt{x}$.

a. Find the domain and range of each of the following compositions:

1. $(h \circ k)(x)$

2. $(k \circ h)(x)$

b. Write each of the composite functions below in simplified form:

1. $(h \circ k)(x)$

2. $(k \circ h)(x)$

c. Explain why the compositions in Part **b** result in different equations.

3.4 Given the functions $g(x) = x + 4$ and $f(x) = 4x$, simplify the following:

$$c(x) = \frac{f(g(x)) - f(x)}{4}$$

3.5 Sal and Guinn have agreed to invest 20% of the profit from their ostrich operation in a retirement fund. The remaining amount will be divided evenly between them as salary.

a. Determine a function $s(x)$, where x represents annual profit, that can be used to determine the salary for each partner.

b. In the exploration in Activity **2**, you described Sal and Guinn's annual profit using the function $n(x) = 38{,}000x - 140{,}000$, where x represents the number of breeding pairs in the flock. Use composition of functions to determine the annual salary for each partner if the ranch maintains a flock of 7 breeding pairs.

c. Determine a function that could be used to calculate the partners' annual salaries given the number of breeding pairs.

3.6 Sal and Guinn employ high school students to clean the ostrich barn. The amount that each student earns depends on the number of hours worked. The amount that each saves for college depends on the total earnings.

a. Choose an appropriate wage for a student. Express earnings as a function of the number of hours worked.

b. Select a realistic proportion of earnings for a student to save for college. Express the amount saved as a function of earnings.

c. Use composition of functions to describe the amount saved for college as a function of the number of hours worked.

* * * * *

3.7 Consider the functions $g(x) = \sqrt{16 - x^2}$ and $h(x) = 5x - 1$.

a. Identify the domain and range of each function.

b. Write each of the composite functions below in simplified form:

1. $(g \circ h)(x)$

2. $(h \circ g)(x)$

c. Find the domain and range of each of the following compositions:

1. $(g \circ h)(x)$

2. $(h \circ g)(x)$

3.8 To complete Parts **a** and **b** below, consider the functions, $f(x) = x - 9$, $g(x) = \sqrt{x} + 5$, and $h(x) = x^2$.

a. Find $g(f(h(7)))$, $f(h(g(7)))$, and $g(h(f(7)))$.

b. Determine a rule for $g(f(h(x)))$ and identify the domain.

3.9 Consider a function *m* which connects each child with his or her mother.

a. If the domain is the set of all children, describe a possible range for *m*.

b. Describe a possible range for $m \circ m$.

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Activity 4

Although ostriches can survive for days without water, they prefer to drink and bathe frequently. On Sal and Guinn's ranch, each ostrich consumes about 8 L of water per day. The ranch faces a possible drought this summer. Should Sal and Guinn reduce the size of their flock or should they start looking for more water?

The way in which the partners approach this problem depends on their plans for the business. If they would like to maintain a certain number of birds, they can determine the total amount of water required to sustain a flock of that size. Once that amount is known, they can estimate the volume of water the ranch can provide during a drought, then acquire any additional water needed.

On the other hand, if Sal and Guinn want to use only the available water, they can determine the number of ostriches this supply will maintain and reduce their flock accordingly. The functions involved in analyzing these two different approaches are **inverses** of each other.

Mathematics Note

The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged.

For example, the relation {(0,2), (1,3), (4,–2), (–3,–2)} has an inverse relation {(2,0), (3,1), (–2,4), (–2,–3)}. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.

If a relation is a function, and its inverse is also a function, the inverse is an **inverse function**. The inverse function of $f(x)$ is denoted by $f^{-1}(x)$, often shortened to f^{-1}.

For example, consider the function $f = \{(1,5),(2,4),(-1,0)\}$. Its inverse is the function $f^{-1} = \{(5,1),(4,2),(0,-1)\}$.

Exploration 1

a. Use technology to create a scatterplot of the relation f and its inverse using the domains and ranges given in Table **1**. To simplify comparisons, make the length of one unit on the x-axis the same as the length of one unit on the y-axis.

Table 1: Relation f and the inverse of f

Relation	Domain	Range
$f(x) = 2x - 5$	$\{-2, -1, 0, \ldots, 10\}$	$\{-9, -7, -5, \ldots, 15\}$
Inverse of f	range of f	domain of f

b. Determine the point of intersection of the graph of f and the graph of its inverse.

c. Determine the equation of the line containing the origin and the point identified in Part **b**. Plot this line on the same graph as in Part **a**.

d. Describe any special relationships you observe among the graphs of f, its inverse, and the line graphed in Part **c**.

e. Create another linear function $g(x)$ and repeat Parts **a–d**. You may wish to choose a different domain for $g(x)$.

f. Repeat Part **a** using the quadratic function $h(x) = x^2 + 2$. You may wish to choose a different domain for $h(x)$.

g. Plot the line you found in Part **c** on the graph from Part **f**. Describe any special relationships you observe among the graphs of h, its inverse, and this line.

Discussion 1

a. Describe the graphs of the inverses of *f*, *g*, and *h*.

b. Is each inverse also a function? Explain your responses.

c. What relationships are there among the graphs of each function, its inverse, and the graph of the line $y = x$? Explain your response.

Exploration 2

Each morning when Guinn turns the ostrich chicks into the fenced pasture, she opens the pasture gate, then drives the chicks out of the barn. In the evening, she drives the chicks back into the barn, then closes the gate. In one sense, her chores in the evening represent the "inverse" of her chores in the morning. In the evening, Guinn undoes her actions of the morning and the chicks are returned to the barn.

a. A similar process can be used to determine the equation that represents the inverse of a function, such as $f(x) = 2x + 8$. In this case, the function multiplies a domain value by 2, then adds 8 to the product. The equation that represents the inverse of $f(x)$ undoes these actions, first by subtracting 8, then by dividing the result by 2:

$$y = \frac{x - 8}{2}$$

Since this equation defines a function, it can be denoted as follows:

$$f^{-1}(x) = \frac{x - 8}{2}$$

Use this method to determine an equation that represents the inverse of each of the following relations. Describe any restrictions on the domain of each inverse, and determine whether or not each is a function.

1. $f(x) = 3x - 1$

2. $g(x) = \sqrt{x + 7}$

3. $h(x) = 9x^2$

4. $k(x) = x^3 - 8$

b. Use a symbolic manipulator to check your work in Part **a**. **Note:** Many symbolic manipulators require a function to be denoted as $y = f(x)$. Since the inverse relation of a set of ordered pairs (x,y) is a set of ordered pairs (y, x), *x* and *y* should be interchanged to obtain the rule for the inverse. This new equation should then be solved for *y*, subject to any necessary restrictions.

c. When two functions f and g are inverses of each other, $(f \circ g)(x) = (g \circ f)(x) = x$. Use this definition to verify that each inverse you obtained in Part **a** is correct.

Discussion 2

a. Which of the relations in Part **a** of Exploration **2** have inverses that are not functions? Explain your response.

b. The graphs in Figure **8** below show two relations and their inverses. In each case, the graph of the relation is indicated by a solid curve, while the graph of the inverse is indicated by a dotted curve.

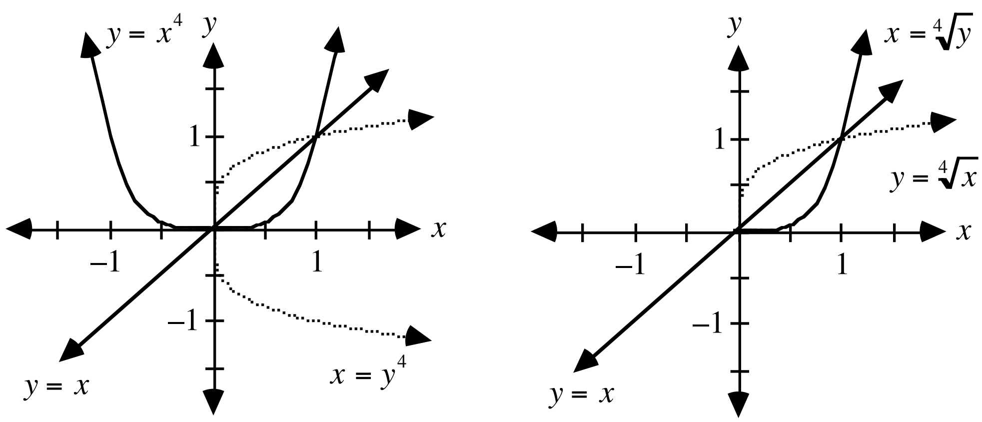


Figure 8: Two relations and their inverses

1. Describe the relationship between the line $y = x$ and the graphs in Figure **8**.
2. Explain why the graphs in Figures **8a** and **b** are identical in the first quadrant but not in the second and fourth quadrants.
3. If a horizontal line intersects the graph of a function in more than one point, then its inverse is not a function. Use this notion to explain whether or not the inverse of each relation in Figure **8** is a function.

Mathematics Note

A **one-to-one function** is a function such that each element in the range corresponds to a unique element of the domain. In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. One-to-one functions are important because they are the only functions whose inverses are also functions.

For example, the one-to-one function $f = \{(-3,-13), (-1,-7), (1,-1), (3,5), (5,11)\}$ has the inverse function $f^{-1} = \{(-13,-3), (-7,-1), (-1,1), (5,3), (11,5)\}$. The function $g = \{(-2,4), (-1,1), (1,1), (2,4)\}$, however, is not a one-to-one function. Its inverse is $\{(4,-2), (1,-1), (1,1), (4,2)\}$, which is not a function.

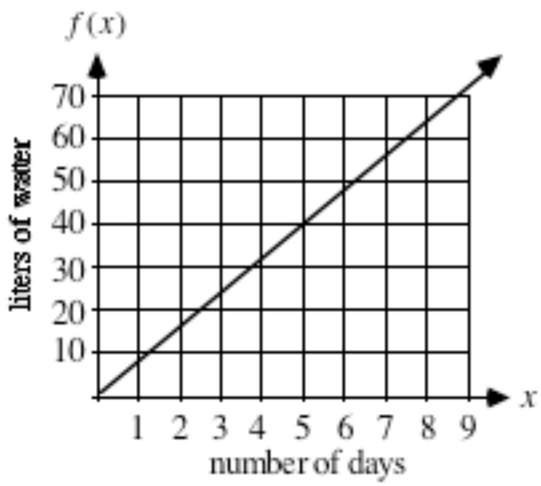
c. Use ordered pairs to illustrate why the function $h(x) = 9x^2$ is not a one-to-one function.

d. Explain why a horizontal line test can be used to determine when a graph is not a one-to-one function.

Assignment

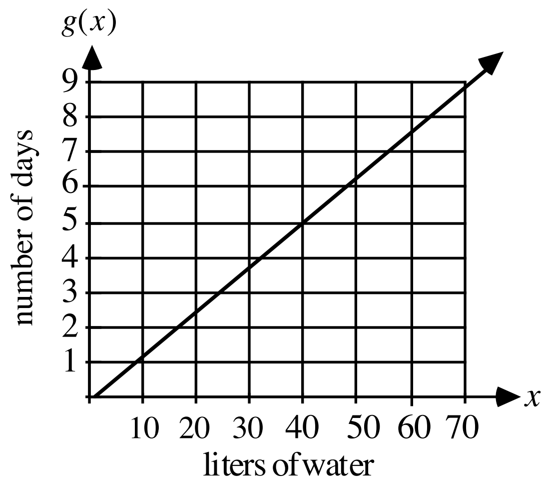
4.1 To help themselves analyze the effects of a potential drought on the ranch, Sal and Guinn decide to create two different sets of graphs.

a. If they want to maintain their current flock of birds, Sal and Guinn must examine the relationship between water consumption and time. Recall that each ostrich uses about 8 L of water per day. The following graph shows the average amount of water needed by one ostrich versus time in days.



1. Use the graph in Part **a** to estimate the amount of water an ostrich will use over 5 days.
2. Estimate the time it will take for one ostrich to use 20 L of water.
3. Write an equation that describes the liters of water used per ostrich as a function of time in days.
4. Suppose that Sal and Guinn want to maintain a flock of 40 birds. If the drought lasts for 60 days, how much water will they need?

b. If Sal and Guinn want to use only the water they currently have available on the ranch, they must determine the number of birds their present resources will maintain during a drought. The following graph shows time in days versus liters of water needed per ostrich.



1. Use the graph above to estimate how many days it will take one ostrich to consume 70 L of water.
2. Estimate the liters of water one ostrich will consume in 2 days.
3. Write an equation that describes time in days as a function of liters of water available per ostrich.
4. The water tank at the ranch holds 16,000 L. If the drought lasts for 60 days, determine the size of the flock that Sal and Guinn can maintain.

c. What relationship exists between the two equations you found in Parts **a** and **b**?

d. Graph the equations from Parts **a** and **b** on the same axes. How does this graph support your response to Part **c**?

4.2 a. Which of the following functions are one-to-one functions?

1. $f(x) = 2x + 3$
2. $f(x) = x^3$
3. $f(x) = (x - 4)/5$
4. $f(x) = x^4$

b. For each one-to-one function identified in Part **a**, write the equation of its inverse using $f^{-1}(x)$ notation.

c. For each one-to-one function identified in Part **a**, list the domain and range of $f(x)$ and $f^{-1}(x)$.

Mathematics Note

The inverse of the exponential function $f(x) = a^x$ where $a > 0$ is $f^{-1}(x) = \log_a x$, where $x > 0$ and read "log of x base a" or "log base a of x."

For example, if $f(x) = 12^x$, then $f^{-1}(x) = \log_{12} x$.

4.3 **a.** Describe the domain and range of $f(x) = 10^x$.

b. Write the inverse of $f(x) = 10^x$ using logarithmic notation.

c. Complete the table below without using technology.

$f(x) = 10^x$		inverse of $f(x)$	
x	10^x	x	$\log x$
–4		0.0001	
–2		0.01	
–1		0.1	
0		1	
1		10	
2		100	
4		10,000	

d. Compare the ordered pairs of $f(x) = 10^x$ to those of its inverse.

e. Graph $f(x)$ and the inverse of $f(x)$ on the same coordinate system.

f. Describe the domain and range of the inverse of $f(x) = 10^x$.

g. Is the inverse of $f(x)$ a function? Explain your response.

* * * * *

4.4 **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable (the parameter). On an *xy*-plane, for example, both x and y can be expressed as functions of a third variable t.

a. Consider the relation $y = 5x^2$. Graph this relation parametrically using the equations $x = t$ and $y = 5t^2$. Graph its inverse using the equations $x = 5t^2$ and $y = t$.

b. How can you verify that the graphs in Part **a** represent inverses?

c. Why does the method outlined in Part **a** result in inverse relations?

4.5 Since only one-to-one functions have inverses that are also functions, it is sometimes desirable to restrict the domain of a function that is not one-to-one in order to create a one-to-one function.

a. Create a scatterplot of the graph of $f(x) = \sin x$ over the domain $[-4\pi, 4\pi]$.

b. On the same coordinate system, create a scatterplot of the inverse of the function defined in Part **a**.

c. Explain why the inverse of $f(x) = \sin x$ over the domain $[-4\pi, 4\pi]$ is not a function.

d. Find a restricted domain of $f(x) = \sin x$ so that its inverse is also a function, and so that all possible values of the range of $f(x) = \sin x$ are values of the domain in the inverse.

e. Graph $f(x) = \sin x$ and its inverse over this restricted domain.

4.6 **a.** Graph the function $f(x) = \tan x$.

b. Identify an interval in which $f(x) = \tan x$ is a one-to-one function.

c. What is the range of the function $f(x) = \tan x$ over the interval in Part **b**?

d. Find the domain and range of the inverse of $f(x)$ over the interval in Part **b**.

e. Graph $f^{-1}(x)$ over the restricted domain in Part **d**.

4.7 What is true about the composition of a function with its inverse function?

* * * * * * * * * *

Summary Assessment

Sal and Guinn have just sold one breeding pair of ostriches to their friend Terry. In approximately two years, this breeding pair will become mature adults and produce approximately 40 eggs per year. Sal and Guinn have recommended that Terry increase the size of his flock by keeping two pairs of ostriches out of each year's eggs and selling the rest. It will take approximately three more years for each new pair of ostriches to mature and produce their own eggs. Function f models the approximate number of ostrich pairs Terry will have in x years:

$$f(x) = 10^{0.1618x}$$

1. Use this function to predict the total number of ostrich pairs Terry will have in seven years.

2. Terry wants to know how many years it will take before he has a certain number of ostrich pairs. Use function f to determine a new function that will give the number of years it will take to produce x ostrich pairs.

3.
- **a.** Graph the functions in Problems **1** and **2** on the same set of axes.
- **b.** Describe the graphs in Part **a**.
- **c.** Explain whether or not the functions in Problems **1** and **2** are one-to-one functions.

4. Terry would like to determine the cost of fencing a pasture for the ostriches. The cost per meter for a chain-link fence is $30 for materials and $10 for labor. According to Sal and Guinn, each pair of ostriches should have 1400 m^2 of pasture.

Assuming that Terry fences a single square pasture, determine functions for each of the following:

- **a.** the length of fence required for x pairs of ostriches
- **b.** the cost of materials for x meters of fence
- **c.** the cost of labor for x meters of fence.

5. Identify the domain and range of each function in Problem **4**.

6. Use the functions in Problems **4b** and **4c** to determine a new function that describes the total cost for x meters of fencing.

7.
- **a.** Compose two or more of the functions in the problems above to determine a function that describes the total cost of fencing sufficient to contain Terry's growing flock of ostriches for x years.
- **b.** Identify the domain and range of the function in Part **a**.

Module Summary

- A **relation** is a set of ordered pairs in which the **domain** is the set of first elements and the **range** is the set of second elements.
- A **function** is a relation from a domain to a range in which each element of the domain occurs in exactly one ordered pair.
- One way to represent a function between two sets is to use a **set diagram** with an arrow to represent the rule. A second way to represent a function between two sets is a **mapping diagram**.
- The function $(f + g)$ is defined by $(f + g)(x) = f(x) + g(x)$. Likewise, $(f - g)(x) = f(x) - g(x)$, $(f \bullet g)(x) = f(x) \bullet g(x)$, and $(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.
- When adding, subtracting or multiplying two or more functions, these operations are defined only for those values common to the domains of all the functions involved.
- Given two functions *f* and *g*, the **composite function** $f \circ g$, read as "*f* composed with *g*" is defined as

  $$(f \circ g)(x) = f\big(g(x)\big)$$

  The domain of $f \circ g$ is the set of all values of *x* in the domain of *g* such that $g(x)$ is in the domain of *f*.
- The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.
- If a relation is a function, and its inverse is also a function, the inverse is an **inverse function**. The inverse function of *f* is denoted by f^{-1}.
- The graph of the inverse is a reflection of the graph of the function in the line $y = x$.
- A **one-to-one function** is a function such that each element in the range corresponds to a unique element of the domain. In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. One-to-one functions are important because they are the only functions whose inverses are also functions.
- The inverse of the exponential function $f(x) = a^x$, where $a > 0$, is $f^{-1}(x) = \log_a x$, where $x > 0$ and read "log of *x* base *a*" or "log base *a* of *x*."

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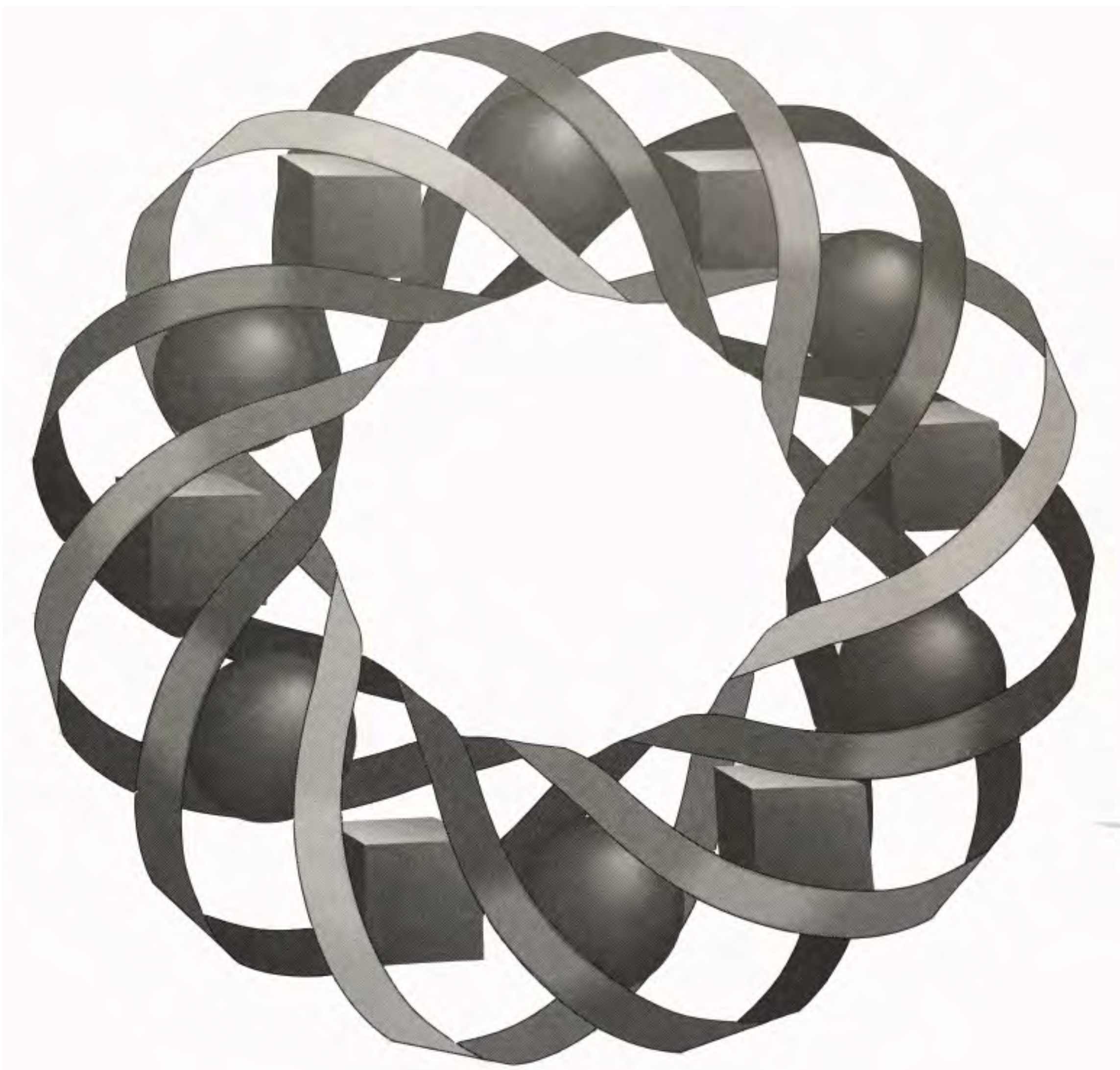
Mathematics in Motion



As an arrow glides toward a target, could you describe its exact location at any time during the flight? In this module, you examine methods for modeling the paths of moving objects.

Ruth Brocklebank • Phil Lieske • Mike Trudnowski

Mathematics in Motion

Introduction

The forest is on fire. Crews on the ground are battling the blaze, but they need more equipment. The dispatcher orders a plane to deliver a crate of supplies.

The crate is designed to be dropped without a parachute. As the plane flies toward the target zone, its crew must decide when to drop the crate. To do this, however, they must be able to predict the path of the falling crate. In this module, you use parametric equations to explore this and other types of motion.

Discussion 1

a. **1.** What factors influence the path of a crate during its fall?

2. Describe the effect of each of these factors on the crate's path.

b. What do you think the path of a falling crate will look like?

c. Should the crew drop the crates when the plane is directly over the target area? Explain your response.

Exploration

In the following exploration, you investigate the motion of two falling objects: one dropped straight down, and one projected horizontally. Both objects begin their fall from the same height.

a. **1.** Fold an index card over a flexible meterstick or ruler. Secure the card to the meterstick with a binder clip.

2. Fold the index card to form a platform on each side of the meterstick, parallel to the ground, as shown in Figure **1**.

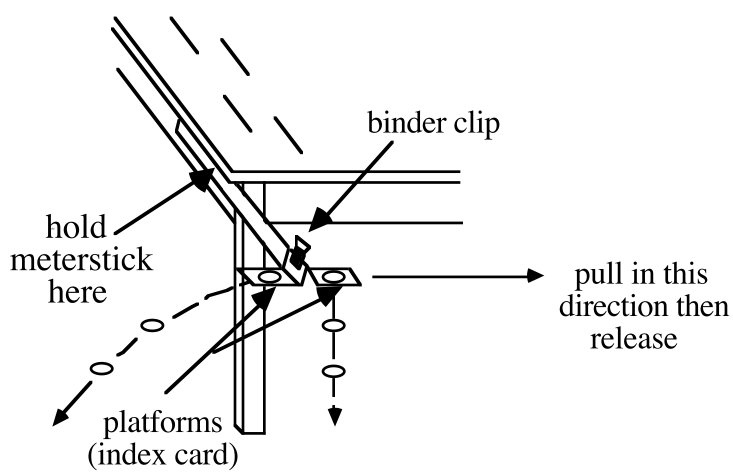


Figure 1: Two falling objects

b.
1. Hold the opposite end of the meterstick against the side of a table.
2. Place a dense object, such as a coin, on each platform. (Using dense objects lessens the effects of air resistance.)
3. Measure and record the height of the objects from the floor.
4. Pull the free end of the meterstick in the direction indicated in Figure **1**, then release it.
5. Observe the path of each object, and note when each hits the floor. Record your observations, including a sketch of each path.

c. Repeat Part **b** two or three times, varying the amount of tension on the meterstick.

d. Repeat Parts **b** and **c** with two dense objects that are not alike.

Discussion 2

a. Compare the paths of the two like objects in Part **b** of the exploration.

b. Did both objects fall from the same height?

c. Compare the time required for the two objects to reach the floor.

d. Does the time required for an object to reach the ground appear to be affected by its path?

e. How did your observations change when using two unlike objects?

f. If a feather and a coin are dropped from a height of 10 m, would you expect them to reach the floor at the same time? Explain your response.

Science Note

One of Galileo Galilei's (1564–1642) more famous accomplishments is his description of the motion of falling objects. While first investigating free fall, he is said to have simultaneously dropped a 10-kg cannonball and a 1-kg stone off the Leaning Tower of Pisa. He discovered that the objects hit the ground at approximately the same time.

About 75 years later, Isaac Newton (1642–1727) developed three laws of motion. Using his own second law of motion and the laws of planetary motion developed by Johannes Kepler (1571–1630), Newton proved that, in the absence of air resistance, any two objects dropped from the same height hit the ground at exactly the same time.

Activity 1

In the introduction, you investigated the paths of freely falling objects. In this activity, you model these paths with parametric equations.

Exploration

Consider two freely falling objects. One is dropped straight down from a height of 10 m. At the same instant, the other is projected horizontally from the same initial height.

a. The graph in Figure **2** shows the position of each object at intervals of 0.2 sec. Use the graph to approximate ordered pairs (x,y) for these positions, where x represents the horizontal distance and y represents the vertical distance.

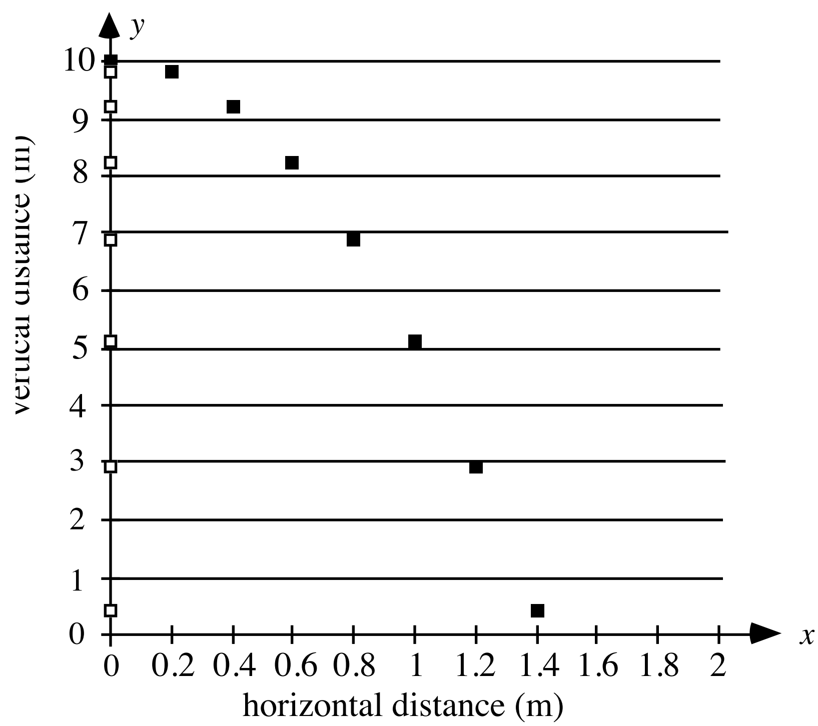


Figure 2: Positions of two freely falling objects

b. Record the values from Part **a**, along with the corresponding times, in a spreadsheet with headings like those in Table **1**.

Table 1: Positions of objects over time

	Object Dropped from Rest (□)		**Object Projected Horizontally** (■)	
Time (sec)	Horizontal Distance (m)	Vertical Distance (m)	Horizontal Distance (m)	Vertical Distance (m)
0.0				
0.2				
$\vdots$				
1.4				

c.

1. Calculate the change in horizontal position between consecutive points for each falling object.
2. The average velocity of an object can be calculated as follows:

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Determine the average horizontal velocity (v_x) between consecutive points for each object.

3. Write a function $x(t)$ that describes each object's horizontal position with respect to time t.

d.

1. Calculate the change in vertical position between consecutive points for each falling object.
2. Determine the average vertical velocity between consecutive points for each object. Record these values in a spreadsheet with headings like those in Table **2** below.

Table 2: Vertical velocity of objects over time

Time Interval (sec)	**Object Dropped from Rest (m/sec)**	**Object Projected Horizontally (m/sec)**
$[0, 0.2)$		
$[0.2, 0.4)$		
$\vdots$		
$[1.2, 1.4)$		

e. **Acceleration** describes an object's change in velocity per unit time. The average acceleration of an object can be calculated as follows:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

Use the spreadsheet to calculate the average vertical acceleration between consecutive points for each object.

f. The acceleration due to gravity near Earth's surface is approximately 9.8 m/sec^2 in a direction toward Earth's center. Compare the average acceleration you determined in Part **e** to this value.

g. When a freely falling object has no initial velocity in the vertical direction, its height after t sec can be described by the following function, where g is the acceleration due to gravity and h_0 is the initial height:

$$y(t) = -\frac{1}{2}gt^2 + h_0$$

1. Write a function $y(t)$ that describes the vertical position of each object in Figure **2** with respect to time t. Recall that the initial height for both objects was 10 m.
2. Check your equations by substituting 0.2, 0.8, and 1.2 for t and comparing the resulting values of $y(t)$ to those in Table **1**.

Mathematics Note

Parametric equations allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. In an xy-plane, for example, both x and y can be expressed as functions of a third variable, t:

$$x = f(t)$$
$$y = g(t)$$

In these parametric equations, the independent variable is the parameter t. The dependent variables are x and y. In other words, each value of t in the domain corresponds with an ordered pair (x,y).

For example, consider an object projected horizontally at a velocity of 15 m/sec off a cliff 20 m high. This object's position after t sec can be described by the following parametric equations, where $x(t)$ represents the horizontal distance traveled and $y(t)$ represents the height above the ground:

$$x(t) = v_x t = 15t$$

$$y(t) = -\frac{1}{2}gt^2 + h_0 = -\frac{1}{2}(9.8)t^2 + 20 = -4.9t^2 + 20$$

At $t = 2$ sec, the ordered pair generated by these equations is (30,0.4). This indicates that 2 sec after leaving the cliff, the object has traveled 30 m horizontally and is 0.4 m off the ground.

h. Write parametric equations to describe the position of each object in Figure **2** with respect to time.

i. Set your graphing utility to graph parametric equations simultaneously.

1. Using appropriate intervals for x, y, and the parameter t, graph both pairs of equations from Part **h**.

2. Experiment with different increments for t. Record your observations.

3. Use the trace feature to observe and record the values of x, y, and t at various locations on each graph.

Discussion

a. Describe the graphs you created in the exploration.

b. Does the speed with which the graphs are drawn appear to be related to the actual speed of the objects? Explain your response.

c. **1.** How could you determine the time required for each object to reach the ground?

2. Describe how you could find the location of each object after half this time has passed.

d. Describe how you could determine the maximum horizontal distance traveled by the object that was projected horizontally.

e. Considering an object whose height above the ground can be described by the function $y(t)$, is it reasonable to consider negative values for $y(t)$? Explain your response.

Assignment

1.1 While practicing at a target range, an archer shoots an arrow parallel to the ground at a velocity of 42 m/sec. At the moment the arrow is released, the strap on the archer's wristwatch breaks and the watch falls toward the ground. The initial height of both the arrow and the watch is 1.6 m.

a. Write a pair of parametric equations, $x(t)$ and $y(t)$, to describe each of the following:

1. the position of the watch after t sec

2. the position of the arrow after t sec.

b. Graph the equations from Part **a**.

c. Determine the height of each object after 0.25 sec.

d. Determine how long it will take for each object to hit the ground.

e. Determine the horizontal distance traveled by the arrow at the time it hits the ground.

1.2 In the introduction to this module, you discussed the airlift of a crate of supplies to some firefighters. Suppose that the plane is traveling at a horizontal velocity of 250 km/hr and the crate is dropped from a height of 100 m.

a. Write a set of parametric equations, $x(t)$ and $y(t)$, to model the path of the crate, where *t* represents time in seconds. Hint: The units for distance should be the same in each equation.

b. Determine how long it will take for the crate to hit the ground.

c. Determine the horizontal distance traveled by the crate during its time in the air.

d. If the plane continues to travel at the same velocity, where will it be located in relation to the crate when the crate hits the ground?

1.3 Two mountain climbers are stranded by a blizzard at an elevation of 1690 m. A search-and-rescue plane locates the climbers but cannot land to pick them up. Flying due east at a velocity of 90 m/sec and an elevation of 1960 m, the crew drops a package of food and supplies.

a. How long (to the nearest 0.1 sec) will it take for the package to reach the ground if it lands at the same elevation as the climbers?

b. How far should the plane be from the target site when the rescue team releases the package?

* * * * *

1.4 Under the watchful eye of your skydiving instructor, you step out of a plane. The plane is traveling at a constant velocity of 65 m/sec and an altitude of 1300 m. You wait 10 sec before pulling the ripcord of your parachute.

a. Ignoring air resistance, describe your path during the 10 sec of free fall.

b. Write a set of parametric equations that models your path during this interval.

c. Determine how far you have fallen vertically before pulling the ripcord.

d. Determine the horizontal distance you have traveled before pulling the ripcord.

e. At the time you pull the ripcord, where is the airplane relative to your position? Explain your response.

1.5 The object of the game "Sure-Aim" is to roll a marble off a table and into a cup. The table is 0.8 m high. The cup is 0.1 m high, with a diameter of 5 cm. The horizontal distance from the table to the cup's rim is 0.75 m.

Determine the approximate velocity at which a marble must leave the table in order to land in the cup. Defend your response.

* * * * * * * * * *

Activity 2

In Activity **1**, you explored the motion of objects falling from rest or projected with a horizontal velocity. In this activity, you investigate the motion of objects projected into the air at an angle.

Discussion 1

a. When a batter hits a ball, what forces are involved?

b. What factors influence the distance traveled by the ball?

Exploration

While watching a videotape of herself in the batting cage, Kami noticed that she hit the ball at many different angles of elevation, from line drives to pop-ups. After speaking with her fast-pitch softball coach, she wondered what angle of elevation would make her hits travel as far as possible.

You may recall from the Level 4 module, "Flying the Big Sky with Vectors," that it is possible to analyze this situation using vectors. In this exploration, you develop a vector model to help answer Kami's question.

Mathematics Note

A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically symbolized by a bold, lowercase letter, such as vector **u**. In handwritten work, the same vector can be symbolized by $\vec{u}$. The magnitude of a vector **u** is denoted by $|\mathbf{u}|$.

The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **u** is denoted by $\mathbf{u}_x$ (read "**u** sub *x*"), while its vertical component is denoted by $\mathbf{u}_y$.

For example, the arrowhead on vector **a** in Figure **3** indicates its direction. The length of vector **a** indicates its magnitude. Its horizontal and vertical components are $\mathbf{a}_x$ and $\mathbf{a}_y$, respectively.

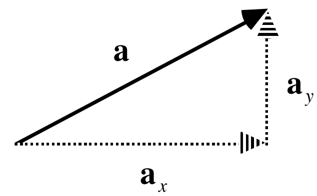


Figure 3: Vector a and its components

a. To analyze the paths of the hit balls, Kami ignores air resistance and assumes that each ball leaves the bat at the same speed of 40 m/sec.

When the initial velocity of a hit ball is represented by a vector **v**, the vector's direction is determined by the angle θ at which the ball leaves the bat. Its magnitude is the velocity at which the ball is hit. Figure **4** shows vector **v** and its components.

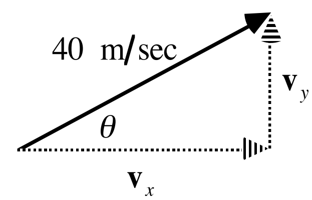


Figure 4: Vector v and its components

1. Write an expression for the horizontal velocity $\mathbf{v}_x$ in terms of the initial velocity of 40 m/sec and the angle θ.
2. Write an expression for the vertical velocity $\mathbf{v}_y$ in terms of the initial velocity and θ.

b. Complete Table **3** for softballs hit at angles of elevation between 0° and 90°, in increments of 5°.

Table 3: Component velocities of a softball

Initial Velocity (m/sec)	**Angle of Elevation (degrees)**	**Horizontal Component $\mathbf{v}_x$**	**Vertical Component $\mathbf{v}_y$**
40	0	40	0
40	5	39.85	3.49
⋮	⋮	⋮	⋮
40	90		

c. In general, the height of a projectile above the ground at any time t can be modeled by the following function:

$$h(t) = -\frac{1}{2}gt^2 + \mathbf{v}_y t + h_0$$

where g is the acceleration due to gravity, v_y is the vertical component of the initial velocity, and h_0 is the initial height.

1. Consider a softball hit with an initial velocity of 40 m/sec at a 30° angle of elevation from an initial height of 1 m. Write a function that models the height of this softball with respect to time.
2. Determine the height of the softball 4 sec after it is hit.

d. The softball's horizontal motion can be analyzed independently of its vertical motion. In general, the horizontal distance traveled at any time t can be modeled by the following function:

$$x(t) = \mathbf{v}_x t$$

where $\mathbf{v}_x$ is the horizontal component of the initial velocity.

1. Write a function that models the horizontal distance traveled by the softball described in Part **c**.
2. Find the horizontal distance traveled by the softball 4 sec after it is hit.

e. Graph the parametric equations from Parts **c** and **d**. Use the graph to determine the horizontal distance traveled by the softball before it hits the ground.

f. Repeat Parts **c–e** using several different values for θ, the angle of elevation. Estimate the measure of the angle that will allow a hit ball to travel the farthest distance.

Discussion 2

a. Describe the paths of the softball in Part **e** of the exploration.

b. Given the initial velocity and angle of elevation for a hit softball, how could you determine each of the following?

1. the maximum horizontal distance traveled by the softball
2. the time required for the softball to reach its maximum height
3. the maximum height reached by the softball.

c. What angle of elevation appears to result in the maximum horizontal distance for a hit ball?

d. Suppose that the wind is blowing when Kami hits the ball.

1. Does wind affect the horizontal or vertical component of a ball's velocity? Explain your response.
2. How would you adjust your parametric equations if the wind was blowing toward Kami?
3. How would you adjust your parametric equations if the wind was blowing away from Kami?

Assignment

2.1 While watching the videotape of herself in the batting cage, Kami noticed that she hit one pitch especially well. Estimating that the angle of elevation measured 20°, she wondered if that hit would have been a home run.

Assume that the softball left the bat with an initial velocity of 40 m/sec at a height of 1 m.

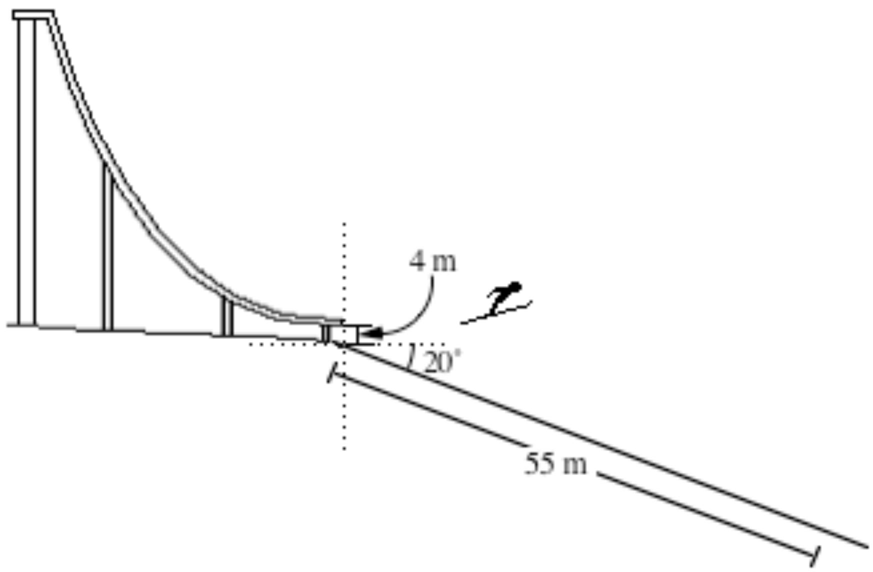
a. At what time would the softball have reached its maximum height?

b. What would have been its maximum height?

c. The outfield fence is 2 m high and 80 m from home plate. Would the ball have cleared the fence? If so, determine the distance by which the ball would have cleared the fence. If not, determine the distance by which the ball would have fallen short.

2.2 Imagine that the wind is blowing directly toward home plate at 8.5 m/sec. If Kami hits the ball as in Problem **2.1**, will the ball clear the fence? Check your response using a graph of the appropriate parametric equations.

2.3 The distance traveled by a ski jumper is measured from the base of the ramp to the landing point. As shown in the diagram below, the end of the ramp is 4 m above the snow. The angle formed by the plane of the landing area and the horizontal is 20˚. Ignoring air resistance, find the horizontal velocity that the skier would need to jump 55 m.



2.4 Imagine that you are an engineer for the Buildaroad Construction Company. In order to widen a highway, the company must blast through a mountain. You have been asked to determine a safe distance from the blast for the construction workers on the site. The charge of dynamite will propel rocks and debris at a maximum initial velocity of 55 m/sec. Write a report explaining your recommendations, including a minimum "safe" distance.

* * * * *

2.5 At the circus, Rowdy the Riot is shot out of a cannon and into a square net which measures 10 m on each side. To land safely, Rowdy must land at least 2 m from the edge of the net. The barrel of the cannon is 2 m off the ground and has a 40˚ angle of elevation. The net is 1 m off the ground. Its nearest edge is 30 m from the cannon.

Ignoring air resistance, determine an interval of initial velocities that will allow Rowdy to land safely in the net. Justify your response by showing an appropriate vector analysis of the situation.

2.6 When Lief hits a golf ball, the distance it travels depends on which golf club he uses. The following table shows the ball's angle of elevation and initial velocity when hit with four different golf clubs.

Golf Club	Angle of Elevation	Initial Velocity
six iron	32°	44.5 m/sec
seven iron	36°	41.5 m/sec
eight iron	40°	38.5 m/sec
nine iron	44°	36.5 m/sec

From his position on the fairway, Lief wants to hit a golf ball so that it lands in the middle of the green. The front of the green is 162 m away, while the back is 181 m away. Use the information in the table to determine which club Lief should select.

* * * * * * * * * *

Activity 3

In the previous activities, you used parametric equations to model parabolic paths. In this activity, you use parametric equations to investigate circular and elliptical paths.

Exploration 1

Recall from the Level 4 module, "Controlling the Sky with Parametrics," that a circle with center at point (h,k) and radius r can be defined by the following pair of parametric equations:

$$x(\theta) = h + r\cos\theta$$
$$y(\theta) = k + r\sin\theta$$

where θ is the measure of the central angle formed by two radii of the circle, one of which is parallel to the x-axis. As shown in Figure **5**, each value of θ corresponds with a specific point on the circle.

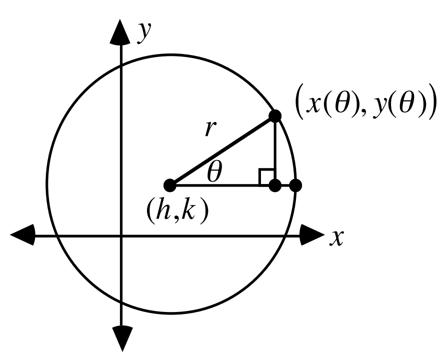


Figure 5: A circle with center at (*h*,*k*) and radius *r*

a. Figure **6** shows a toy airplane attached by a string to a weighted base. As the plane flies, it follows a circular path whose radius is the length of the string.

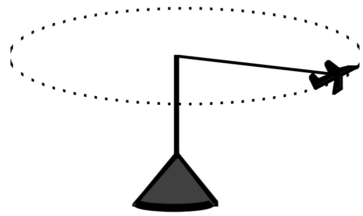


Figure 6: A toy airplane

1. Use parametric equations to model the path of the airplane, given that the length of the string is 2 m long and the end attached to the base is at the origin.
2. Graph your equations from Step **1**. **Note:** Remember to set your graphing utility to measure angles in radians.

b. The plane completes 1 revolution in 1 sec. Determine its average speed in meters per second.

Mathematics Note

The average **angular speed** of a moving point *P*, relative to a fixed point *O*, is the measure of the angle θ through which the line containing *O* and *P* passes per unit time. For example, consider the wheel in Figure **7** below.

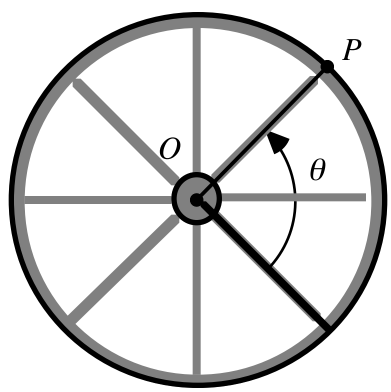


Figure 7: Point *P* on a wheel

Suppose *P* moves $1/4$ the circumference of the wheel in 2 sec. In this case, the line containing *O* and *P* has passed through an angle measure of $2\pi/4$ or $\pi/2$ radians in 2 sec. Therefore, the average angular speed of *P* is:

$$\frac{\pi/2 \text{ radians}}{2 \text{ sec}} = \frac{\pi}{4} \text{ radians/sec}$$

c. Determine the plane's average angular speed in radians per second.

Mathematics Note

The position of an object traveling counterclockwise at a constant angular speed c, on a circle with center at point (h,k) and radius r, can be modeled by the following parametric equations:

$$x(t) = h + r\cos(ct)$$
$$y(t) = k + r\sin(ct)$$

where t represents time.

For example, consider a chair on a Ferris wheel with a radius of 10 m, where the center of the wheel is 12 m off the ground. The Ferris wheel completes 1 revolution every 20 sec. In this case, the angular speed c is $2\pi/20$, or $\pi/10$ radians/sec. If the origin is located on the ground directly below the wheel's center, the chair's position with respect to time can be modeled by the following parametric equations:

$$x(t) = 10\cos\left(\frac{\pi}{10}t\right)$$
$$y(t) = 12 + 10\sin\left(\frac{\pi}{10}t\right)$$

d. Each of the following pairs of parametric equations models the movement of a toy airplane at the end of a 2-m string, where t represents time in seconds. Determine how long it takes each plane to complete 1 revolution.

1. $x(t) = 2\cos(\pi t)$
$y(t) = 2\sin(\pi t)$

2. $x(t) = 2\cos(2\pi t)$
$y(t) = 2\sin(2\pi t)$

3. $x(t) = 2\cos\left(\frac{2\pi}{3}t\right)$
$y(t) = 2\sin\left(\frac{2\pi}{3}t\right)$

e. Determine a pair of parametric equations that models the motion of a toy airplane that completes 1 revolution in each of the following intervals:

1. 2.5 sec

2. 0.8 sec

3. a sec.

f. For each pair of parametric equations in Part **e**, determine the average speed of the plane.

Discussion 1

a. Is the speed at which the graph is plotted related to the actual speed of the object moving around the circle? Explain your response.

b. Using parametric equations of the form given in the previous mathematics note, what is the position of the object when $t = 0$? Justify your response.

c. Figure **8** below shows three different points on a circle: A, B, and C. How would you model the movement of an object whose initial position is at one of these points?

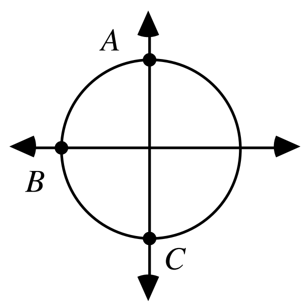


Figure 8: A circle with center at the origin

d. Consider an object moving on a circle with center at (–4,3) and radius 7 units. If the object completes 1 revolution every 5 sec, describe how to use parametric equations to model its position over time.

e. Describe how to determine the speed of an object whose position with respect to time can be modeled by the parametric equations below, where t represents time in hours:

$$x(t) = 9\cos(6t)$$
$$y(t) = 9\sin(6t)$$

f. Figure **9** below shows two concentric circles and a segment OP containing a point Q.

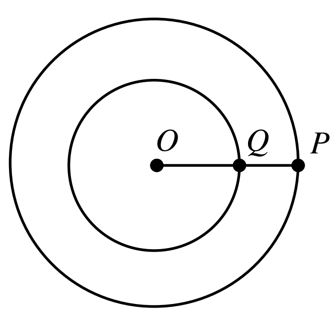


Figure 9: Two concentric circles

1. Compare the speeds of P and Q as the segment rotates about O.
2. Compare the angular speeds of P and Q as the segment rotates about O.

Exploration 2

Parametric equations also can be used to model elliptical paths. In this exploration, you discover how to use parametric equations to define an ellipse.

a. Use a geometry utility to complete the following steps.

1. Construct two circles with center at the origin *O* and different radii. Create a moveable point on the outer circle. Label this point *A*.
2. Draw a ray from *O* through *A*. Locate the point of intersection of the ray and the inner circle. Label this point *B*.
3. From *A*, construct a segment perpendicular to the *x*-axis. Locate the intersection of the perpendicular and the *x*-axis. Label this point *C*.
4. From *B*, construct a segment perpendicular to the *x*-axis and a line perpendicular to the *y*-axis. Label the line perpendicular to the *y*-axis *m*. Locate the intersection of the perpendicular segment and the *x*-axis. Label this point *E*.
5. Locate the point of intersection of $\overline{AC}$ and line *m*. Label this intersection *D*. This point represents one point on your graph of an ellipse. Your construction should now resemble the diagram in Figure **10**.

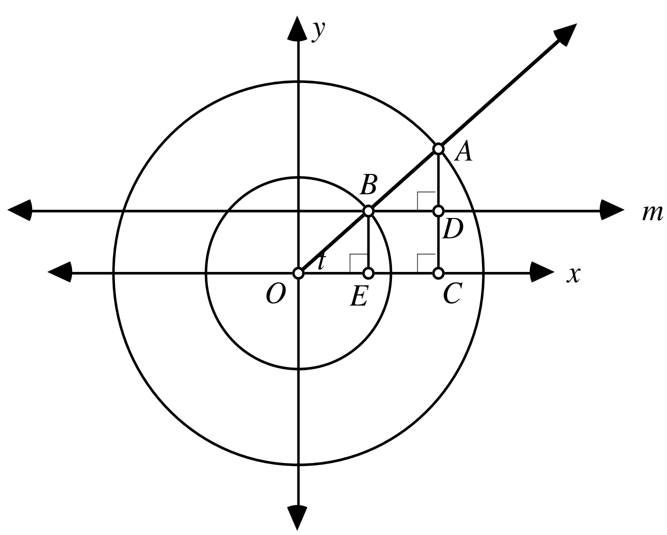


Figure 10: Construction for modeling an ellipse

b. Trace the locus of point *D* as point *A* moves about the outer circle.

c. Using your construction, let *t* represent $m\angle BOE = m\angle AOC$. Let (x,y) represent the coordinates of point *D*.

1. Express *x* in terms of *t* and OA
2. Express *y* in terms of *t* and OB.

d. Suppose that the radius of the larger circle in Figure **9** is 4 units, while the radius of the smaller circle is 2 units. Write parametric equations that model the paths of points A, B, and D as A moves about the larger circle.

Discussion 2

a. Compare your construction with those of your classmates. What differences do you observe?

b.
1. Compare the parametric equations you found in Part **d** of Exploration **2** with those of your classmates.
2. Describe a method you could use to determine these equations.

Mathematics Note

An ellipse with center at the origin can be defined parametrically by the equations $x(t) = a\cos t$ and $y(t) = b\sin t$, where a is the positive x-intercept of the ellipse and b is the positive y-intercept.

For example, consider an ellipse with center at the origin that intersects the x-axis at (12,0) and the y-axis at (0,7). In this case, $a = 12$ and $b = 7$. Therefore, the parametric equations of the ellipse are $x(t) = 12\cos t$ and $y(t) = 7\sin t$.

c. Consider the locus of points traced by D in Part **b** of Exploration **2**. Does a graph of these points appear to be a function? Explain your response.

d. When the locus of points traced by point D is expressed using parametric equations, its graph is a function of t.

1. Describe the domain and range of this function.
2. Explain why it is a function.

e. How are the values of a and b in the equations $x(t) = a\cos t$ and $y(t) = b\sin t$ related to the lengths of the axes of an ellipse?

f. Describe the type of ellipse formed when $a = b$.

g. What advantages are there in using parametric equations to sketch ellipses?

h. The parametric equations $x = a\cos t$ and $y = b\sin t$ define the coordinates of the points of an ellipse. Solving these equations for $\cos t$ and $\sin t$, respectively, results in the following: $\cos t = x/a$ and $\sin t = y/b$.

If these equations are squared and added together, how is the resulting equation related to an ellipse? Hint: Recall from the Level 6 module, "Ostriches are Composed," that $\sin^2 x + \cos^2 x = 1$ is true for all values of x.

Assignment

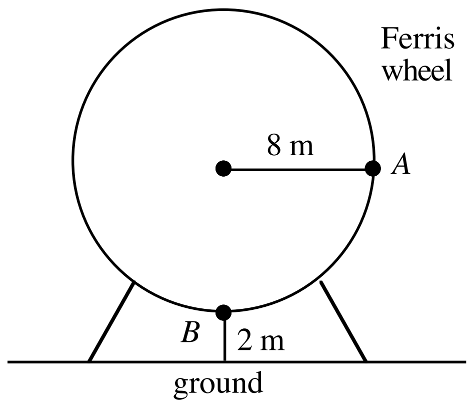
3.1 Assume that the radius of the larger circle in Exploration **2** is 5 units, while the radius of the smaller circle is 3 units.

a. Write parametric equations that describe the locus of points traced by D in terms of the sine and cosine of the angle t. (See Figure **10** for reference.)

b. Graph the equations from Part **a** on a graphing utility. Describe the resulting figure, including the locations of its foci.

c. Graph the parametric equations $x(t) = 3\cos t$ and $y(t) = 5\sin t$. Describe the resulting figure, including the locations of its foci.

3.2 The diagram below shows a Ferris wheel with a radius of 8 m. The bottom of the Ferris wheel is 2 m above the ground.



a. The wheel completes 1 revolution every 20 sec. Determine the speed and angular speed of a chair on this Ferris wheel.

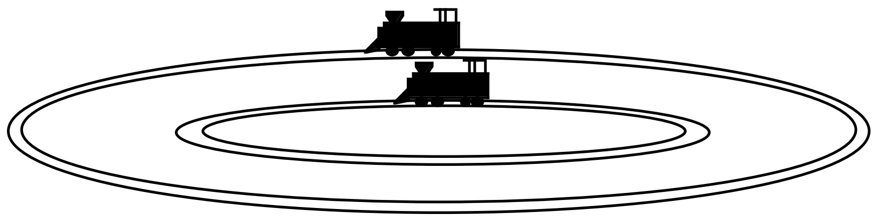
b. **1.** Consider a chair whose initial position is at point A. Use parametric equations to model the position of this chair over time.

2. How high above the ground will this chair be if the wheel stops 10 sec after the chair passes point A? Explain your response.

3. How long will it take for this chair to reach a height of 16 m?

c. Describe how you could model the movement of a chair whose initial position is at point B.

3.3 The following diagram shows two toy trains traveling on concentric sets of circular tracks.



The train on the outer track is 1 m from the center, while the one on the inner track is 0.5 m from the center.

a. Suppose that each train completes 1 lap around its respective track in 15 sec.

1. Determine the angular speed of each train.

2. Determine the speed of each train.

3. Model each train's position over time with parametric equations, given that at $t = 0$, both trains are located on the x-axis of a two-dimensional coordinate system.

b. Suppose that each train travels at a constant speed of 0.25 m/sec.

1. Model each train's position over time with parametric equations.

2. How long will it take the train on the inner track to gain a one-lap lead over the train on the outer track?

3.4 Use your response to Part **h** of Discussion **2** to complete the following.

a. An ellipse can be represented parametrically by the equations $x = 12\cos t$ and $y = 7\sin t$. Write the equation of this ellipse in standard form.

b. Write a set of parametric equations for the ellipse defined by the following equation:

$$\frac{x^2}{36} + \frac{y^2}{4} = 1$$

3.5 **a.** Write a set of parametric equations that define an ellipse with center at (2,3), a major axis with a length of 7 units, and a minor axis with a length of 3 units.

b. At what points does a graph of this ellipse intersect the lines $x = 2$ and $y = 3$?

c. Write a set of parametric equations for an ellipse with center at (h,k), a major axis with length $2a$, and a minor axis with length $2b$.

Mathematics Note

The **area** of an ellipse can be calculated using the formula $A = \pi ab$, where $2a$, and $2b$ are the lengths of the axes.

For example, consider the ellipse defined parametrically by $x(t) = 6\cos t$ and $y(t) = 11\sin t$. In this case, the length of the major axis is 22 units, while the length of the minor axis is 12 units. The area of this ellipse is $6(11)\pi = 66\pi$ units^2.

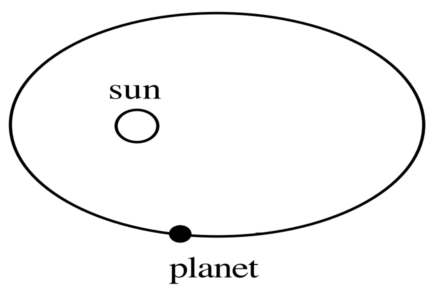
3.6 **a.** Graph an ellipse defined by parametric equations of the form $x(t) = a\cos t$ and $y(t) = b\sin t$, where $a \neq b$.

b. Use the formula $A = \pi ab$ to calculate the area of this ellipse.

c. How is the formula for the area of an ellipse related to the formula for the area of a circle?

* * * * *

3.7 Johannes Kepler's (1571–1630) first law of planetary motion states that the planets move in elliptical orbits in which the sun is located at one focus of the ellipse, as shown in the following diagram.



The shape of Earth's elliptical orbit can be modeled parametrically by the equations $x(t) = \left(1.4958 \bullet 10^8\right)\cos t$ and $y(t) = \left(1.4955 \bullet 10^8\right)\sin t$, where *x* and *y* represent distances in kilometers and *t* represents angle measures.

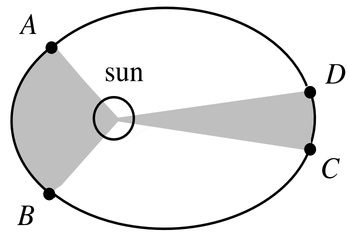
Johannes Kepler approximated the circumference of an ellipse using the equation $\pi(a + b)$.

a. Explain why this formula provides a reasonable approximation for Earth's orbit by comparing it to the formula for the circumference of a circle.

b. Using Kepler's approximation, how far does Earth travel in its yearly orbit?

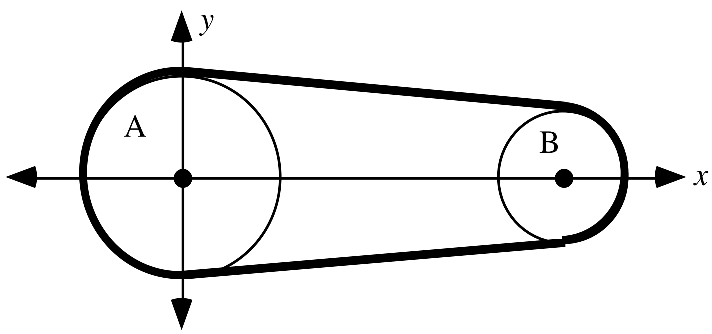
3.8 Kepler's second law of planetary motion states that a ray drawn from the sun to a planet will sweep out equal areas in equal times.

In the diagram below, for example, the time required for a planet to travel from A to B equals the time it takes for the planet to move from C to D. Therefore, according to Kepler's second law, the two shaded areas are equal.



Use Kepler's second law to approximate the area that a ray drawn from the sun to Earth would sweep in 30 days.

3.9 The following diagram shows a belt and two circular pulleys. The radius of pulley A is 10 cm, while the radius of pulley B is 6 cm.



The center of pulley A is located at the origin of a two-dimensional coordinate system. The center of pulley B is 40 cm to the right of the origin on the x-axis.

a. Suppose that pulley A completes 1 revolution every 0.1 sec. Determine the speed of a point on the circumference of pulley A.

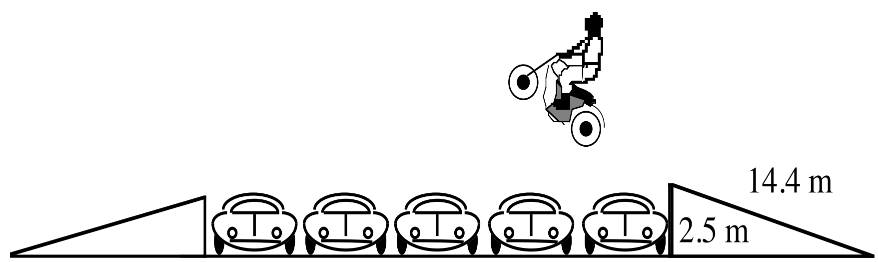
b. Write parametric equations to model the movement of a point on pulley A.

c. When either pulley turns, the belt causes the other pulley to turn also. Given this fact, which quantities would you expect to be equal: the pulleys' speeds, or their angular speeds? Explain your response.

d. Write parametric equations to model the movement of a point on pulley B.

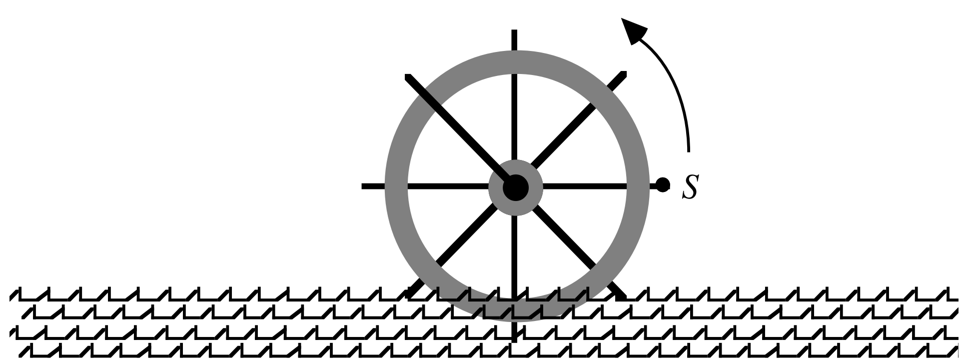
Summary Assessment

1. A motorcycle stunt rider is planning to jump a line of cars arranged side by side, as shown in the diagram below. The approach ramp is 14.4 m long and 2.5 m high, and the motorcycle will have a velocity of 130 km/hr when it leaves the ramp.



The average width of each car is 1.7 m, and the last car in line is 1.5 m high. Determine the maximum number of cars that the stunt rider could clear (ignoring air resistance). Justify your response.

2. The following diagram shows a water wheel with eight paddles.



The center of the wheel is 1.2 m above the water's surface. The distance from the wheel's center to the end of each paddle is 1.8 m. The current flows at a speed of 4.5 km/hr .

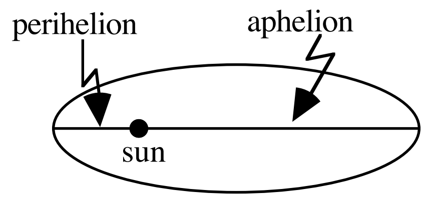
a. Assuming that the speed of point *S* equals the speed of the current, use parametric equations to model the position of *S* with respect to time.

b. Determine how long point *S* is under water during each revolution of the wheel.

c. Given that the eight paddles are evenly spaced, how long are two consecutive paddles under water during a single turn of the wheel? Explain your response.

3. The orbits of planets can be modeled by ellipses with one focus at the sun. Orbits are often described by their *aphelion* (farthest point from the sun), *perihelion* (closest point to the sun), and orbital *eccentricity*.

Eccentricity is a measure of the orbit's elongation, and is equal to the ratio of the distance c between the center and one focus to half the length of the major axis. In other words, $e = c/a$.



In our solar system, Pluto has the most elongated orbit. Its orbital eccentricity is 0.2482. Pluto's aphelion and perihelion are $7.3812 \bullet 10^9$ km and $4.4458 \bullet 10^9$ km, respectively.

Determine parametric equations to model Pluto's orbit. Graph these equations and describe the shape of the orbit.

Module Summary

- **Acceleration** describes an object's change in velocity per unit time. The average acceleration of an object can be calculated as shown below:

  $$\text{average acceleration} = \text{change in velocity} / \text{change in time}$$

- The height h of a falling object after t sec can be described by the function:

  $$h(t) = -\frac{1}{2}gt^2 + h_0$$

  where g is the acceleration due to gravity and h_0 is the object's initial height. The acceleration due to gravity on earth is about 9.8 m/sec^2 in a direction toward Earth's center.

- **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. In an *xy*-plane, for example, both x and y can be expressed as functions of a third variable, t:

  $$x = f(t)$$
  $$y = g(t)$$

  In these parametric equations, the independent variable is the parameter t. The dependent variables are x and y. In other words, each value of t in the domain corresponds with an ordered pair (x, y).

- A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically symbolized by a bold, lowercase letter, such as vector **u**. In handwritten work, the same vector can be symbolized by $\vec{\mathrm{u}}$. The magnitude of a vector **u** is denoted by $|\mathbf{u}|$.
- The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **u** is denoted by $\mathbf{u}_x$ (read "**u** sub *x*"), while its vertical component is denoted by $\mathbf{u}_y$.
- In general, the height of a projectile above the ground at any time t is described by the function:

  $$h(t) = -\frac{1}{2}gt^2 + \mathbf{v}_y t + h_0$$

  where g is the acceleration due to gravity, $\mathbf{v}_y$ is the magnitude of the vertical component of the velocity, and h_0 is the initial height.

- A circle with center at point (h, k) and radius r can be defined by the following pair of parametric equations:

$$x(\theta) = h + r\cos\theta$$
$$y(\theta) = k + r\sin\theta$$

  where θ is the measure of the central angle formed by two radii of the circle, one of which is parallel to the x-axis.

- The average **angular speed** of a moving point P, relative to a fixed point O, is the measure of the angle θ through which the line containing O and P passes per unit time.

- The position of an object traveling counterclockwise at a constant angular speed c, on a circle with center at point (h, k) and radius r, can be modeled by the following parametric equations:

$$x(t) = h + r\cos(ct)$$
$$y(t) = k + r\sin(ct)$$

  where t represents time.

- An ellipse with center at the origin can be defined parametrically by the equations $x(t) = a\cos t$ and $y(t) = b\sin t$, where a is the positive x-intercept of the ellipse and b is the positive y-intercept.

- The **area** of an ellipse can be calculated using the formula $A = \pi ab$, where $2a$, and $2b$ are the lengths of the axes.

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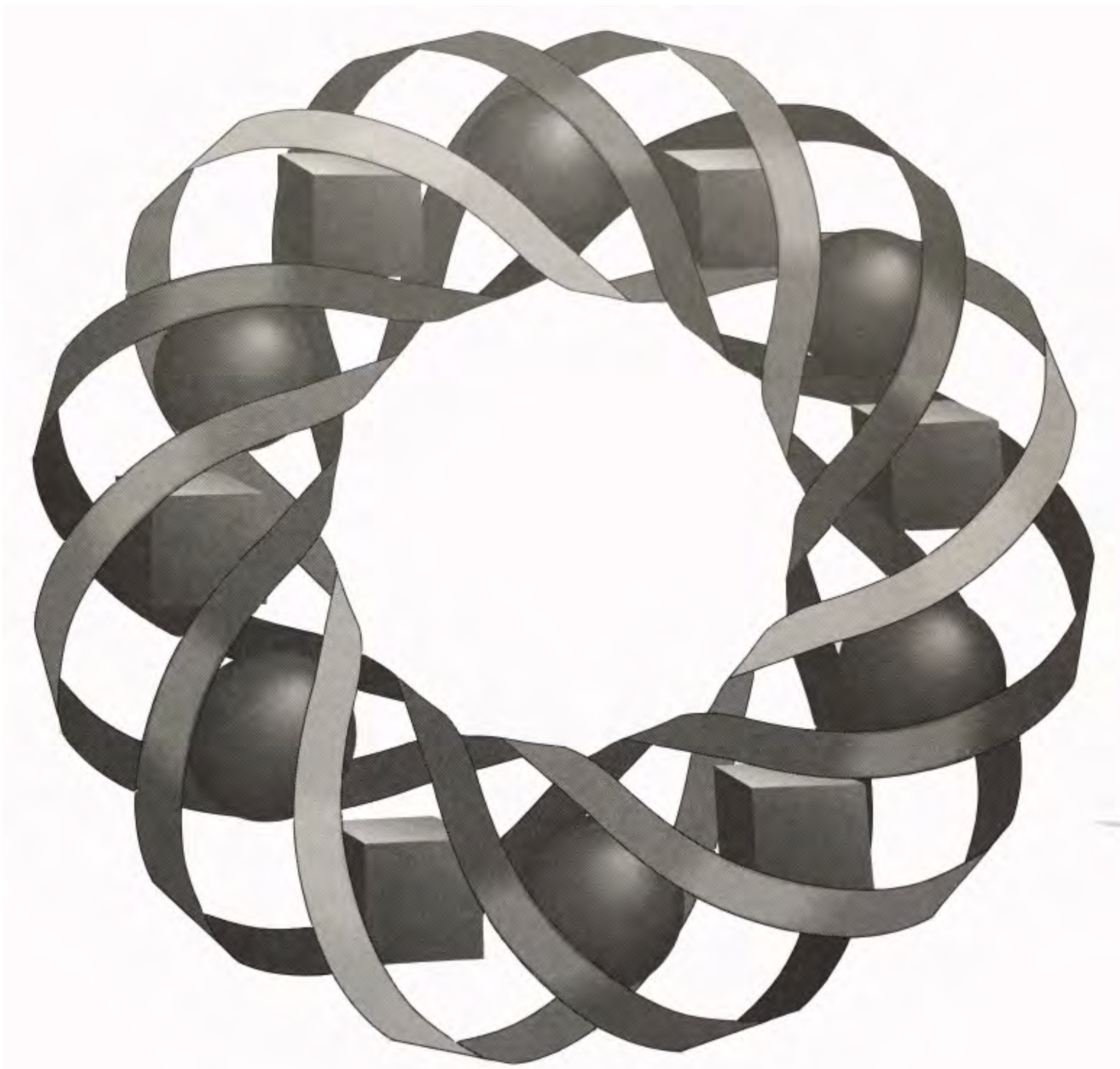
Here We Go Again!



What do teeter-totters, pendulums, and trumpet notes have in common? These real-life phenomena can all be modeled by the same type of functions.

Byron Anderson • Glenn Blake • Anne Merrifield

Here We Go Again!

Introduction

Breathe in and out, in and out, in and out. Your breathing is an event that repeats itself, over and over. A person inhaling and exhaling air, a satellite orbiting Earth, a Ferris wheel in motion, and a city bus following the same route all have at least one thing in common: they involve events that repeat over time.

Discussion

a. Describe several other events that repeat over time.

b. What other characteristics do the events you described in Part **a** have in common?

Activity 1

A repeating event often has a definite beginning and a definite ending. Each swing of a pendulum, for example, has a beginning and an ending. How can this characteristic help you model a pendulum's motion? In the following exploration, you use sand to trace the movement of a pendulum and model its motion.

Exploration

a. Construct a pendulum using a paper cup and string, as shown in Figure **1**. Suspend the pendulum from a solid support arm so that the bottom of the cup is no more than 3 cm above the base.

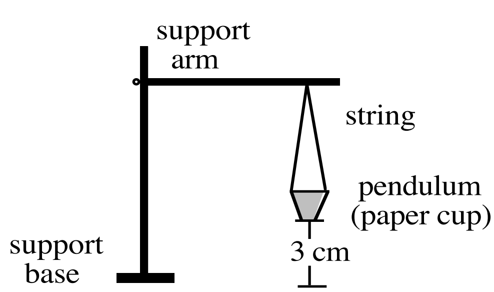


Figure 1: A pendulum

b.

1. As shown in Figure **2**, tape eight pieces of graph paper together along the longer sides to create a single, long sheet.
2. Create a coordinate system with its *x*-axis along the length of the sheet and its *y*-axis along the width of the sheet.
3. Place one end of the sheet underneath the pendulum, so that the pendulum's motion will be parallel to the *y*-axis.

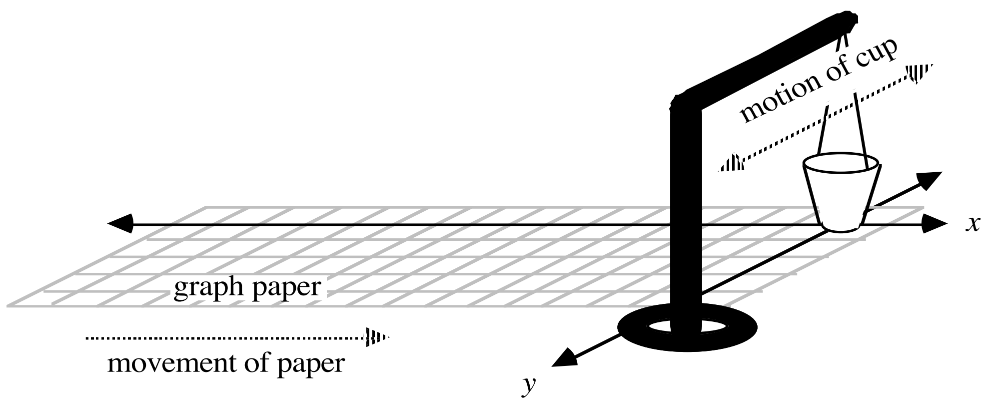


Figure 2: Pendulum and graph paper

c. Make a small hole in the bottom of the paper cup. Cover the hole with your finger, then fill the cup with sand.

d. You are now ready to create a sand graph of the pendulum's motion. While one member of your group uncovers the hole in the cup and gently starts the cup swinging, another should simultaneously begin to move the sheet of graph paper at a constant speed underneath the moving pendulum.

Mathematics Note

A **periodic function** is a function in which values from the range repeat at equal intervals across the domain.

The **period** is the smallest interval of the domain over which the function repeats. A **cycle** is the portion of the function included in one period.

If a periodic function has a maximum M and a minimum m, its **amplitude** is defined as:

$$\left|\frac{M - m}{2}\right|$$

For example, Figure **3** shows a periodic function with a period of 3. This indicates that the function completes 1 cycle every 3 units.

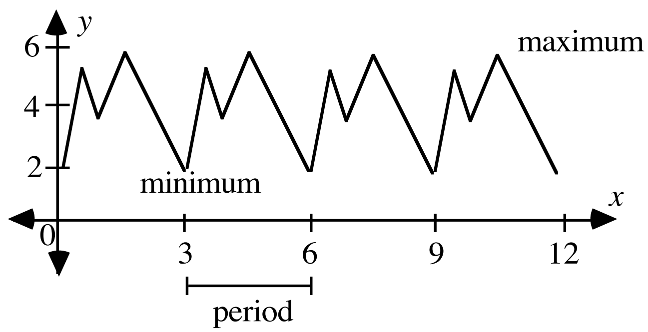


Figure 3: A periodic function

Because the maximum is 6 and the minimum is 2, the amplitude of the function is

$$\left|\frac{6 - 2}{2}\right| = 2$$

e. Determine the period and amplitude of your sand graph.

f. Determine the coordinates of several points on the sand graph.

g. Enter these data points in a graphing utility and create a scatterplot that models the sand graph.

Mathematics Note

There are many types of periodic functions. One example is the sine function, $g(x) = \sin x$. In fact, all functions of the form below, where $a \neq 0$, are periodic functions.

$$f(x) = a\sin[b(x-h)] + k$$

For all periodic functions of this form, the amplitude is $|a|$, the number of cycles in 2π radians is b, and the period is $2\pi/b$. The value of h describes the horizontal translation of the graph from the parent function $g(x) = \sin x$, while the value of k describes the vertical translation from the parent.

For example, the function $f(x) = 4\sin 2(x-3) + 7$ has 2 cycles in 2π radians, a period of π, and an amplitude of 4 units. Its graph is translated 3 units to the right and 7 units up from the graph of $g(x) = \sin x$.

Figure **4** shows the graphs of $f(x)$ and $g(x)$. Note that point P, the y-intercept of $g(x)$, is translated 3 units to the right and 7 units up to the point P' on $f(x)$.

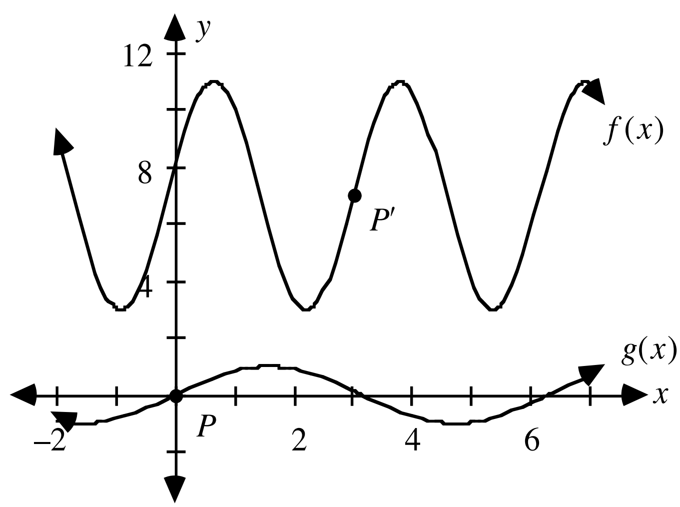


Figure 4: Graphs of $f(x) = 4\sin[2(x-3)] + 7$ and $g(x) = \sin x$

h. Determine a function of the form $f(x) = a\sin[b(x-h)] + k$ that models the data points in the pendulum experiment.

i. Figure **4** shows a graph of the function $f(x) = 4\sin[2(x-3)] + 7$. Find another sine function that has the same graph.

Discussion

a. How does the sand graph compare to the graph of $g(x) = \sin x$?

b.
1. In the pendulum experiment, what is the greatest possible value for the amplitude?
2. What is the smallest possible value for the amplitude?

c. How do you think you could change the period of the pendulum?

d. Did the function you wrote in Part **h** of the exploration include a horizontal or vertical translation of the function $g(x) = \sin x$? If so, describe these translations.

e. The sine function is one example of a periodic function. Name some other periodic functions.

f.
1. Can the function $f(x) = 0$ be considered a periodic function? Explain your response.
2. Is the greatest integer function a periodic function?

g. How does each constant in the function $f(x) = 6\cos[0.5(x+5)] - 2$ affect its graph in comparison to the graph of $g(x) = \cos x$?

h. How could you determine the value of b in $g(x) = \sin(bx)$ by examining its graph? Hint: The period of $f(x) = \sin x$ is 2π.

i. The graph in Figure **5** is a transformation of $f(x) = \sin x$. Describe how to determine a possible value for the horizontal translation in this transformation.

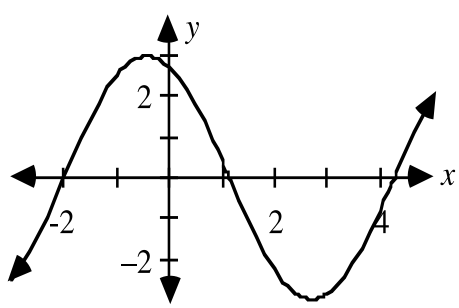
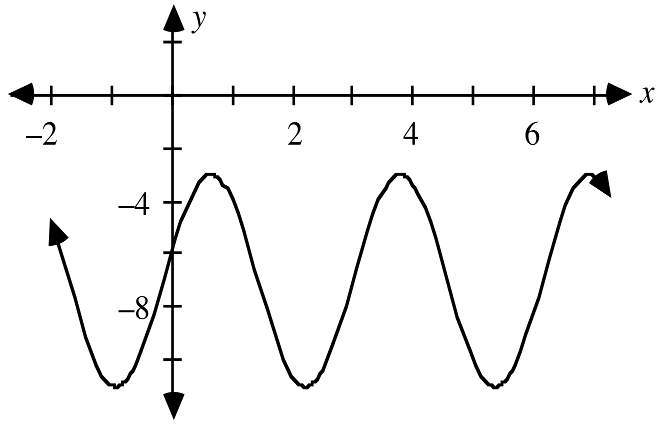


Figure 5: A transformation of $f(x) = \sin x$

j. How many different equations can be written whose graphs are the ones shown in Figure **5**?

Assignment

1.1 The graph shown below is a transformation of the function $f(x) = \sin x$.

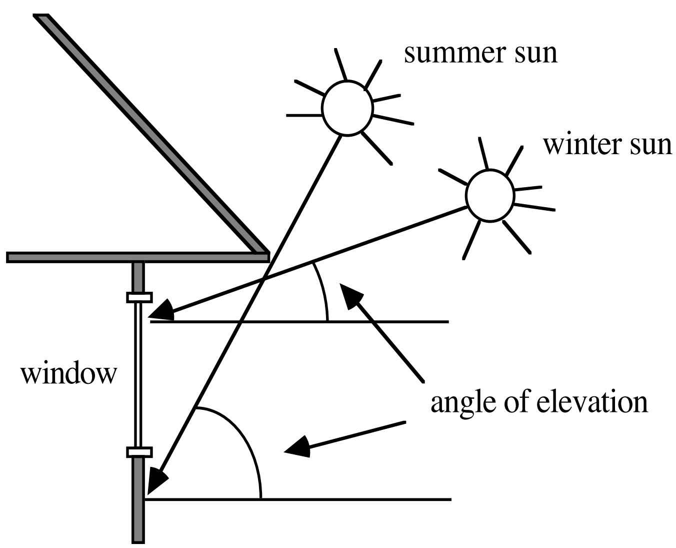


a. Describe how to determine each of the following for this graph:

1. the period
2. the amplitude
3. the vertical shift
4. the horizontal shift

b. Find two functions that model the curve.

1.2 When designing a solar house, the sun's angle of elevation above the horizon at midday is an important concern. As shown below, the sun's angle of elevation is greater in the summer than in the winter.



Imagine that you work for a construction company called Solar Sensation. To improve heating and cooling efficiency, the company designs the eaves on its houses so that windows receive full sun during the winter months, but are shaded during the summer months.

a. The table below shows the sun's angle of elevation at midday for the 15th day of each month in 1994 for Spokane, Washington. Graph this data in a scatterplot.

Date	Angle of Elevation	Date	Angle of Elevation
January 15	21.60°	July 15	63.77°
February 15	29.92°	August 15	56.08°
March 15	40.87°	September 15	44.97°
April 15	52.72°	October 15	33.45°
May 15	61.65°	November 15	23.63°
June 15	65.83°	December 15	19.13°

b. Find a periodic function that models the scatterplot.

c. Use your equation from Part **b** to estimate the sun's angle of elevation (to the nearest degree) on each of the following dates:

1. June 1, 1994

2. October 31, 1995.

d. How could you use your equation to predict the sun's midday angle of elevation for any day of any year?

1.3 **a.** Make a prediction about the relationship between the sun's angle of elevation and mean monthly temperature.

b. The following table shows the mean monthly temperature in degrees Celsius for Spokane, Washington. Graph this data in a bar graph.

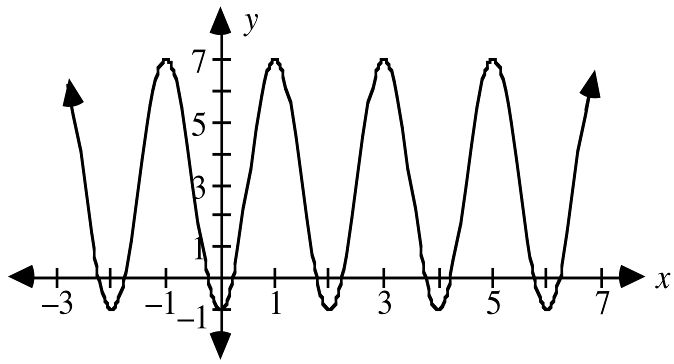
Month	Mean Temperature	Month	Mean Temperature
January	–6.0°	July	20.1°
February	–2.5°	August	19.4°
March	0.7°	September	13.6°
April	6.4°	October	8.6°
May	11.7°	November	1.0°
June	16.4°	December	–4.5°

c. **1.** Locate the midpoint of the segment at the top of each bar in the bar graph and place a dot there. Then connect the dots.

2. Compare the shape formed with the scatterplot of the sun's angle of elevation from Problem **1.2a**.

3. Do the graphs support your prediction from Part **a**? Explain your response.

1.4 The following graph shows a transformation of the parent function $f(x) = \cos x$. Write two possible equations for this function.



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1.5 **a.** A yo-yo rises and falls from a player's hand. Suppose that the distance from the yo-yo to the player's hand can be modeled with a transformation of the sine function.

1. In this situation, what measurement(s) correspond to the amplitude of the function?
2. What measurement corresponds to the period of the function?

b. Suppose that the length of a yo-yo string is 79 cm, and that it takes 1.2 sec to complete 1 cycle. Assume that at 0 sec, the yo-yo is 0 cm from the player's hand.

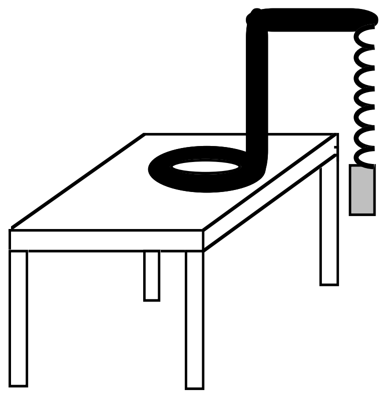
1. Sketch the graph of a function that could model this relationship.
2. Find a transformation of the sine function that models the motion of this yo-yo.

c. Use the periodic function from Part **b** to estimate the distance from the yo-yo to the player's hand after 2 sec.

d. At what times before 3 sec will the yo-yo be 60 cm from the player's hand?

1.6 Consider an object attached to a spring, as shown in the following diagram. The object, after being pulled straight down and released, bounces up and down. Approximately 0.5 sec after the object is released, it reaches its highest point, 50 cm above the plane of the tabletop. It reaches its lowest point, 30 cm below the plane of the tabletop, 1.5 sec after its release.

By ignoring friction (which would eventually stop the motion of the spring), a transformation of the cosine function can be used to model the object's distance above or below the plane of the tabletop versus time.



a. Sketch a graph of a function that models the object's distance above or below the plane of the table top versus time.

b. Write an equation that models the graph in Part **a**.

c. Where is the object located 3.2 sec after its release?

* * * * * * * * * *

Activity 2

In Activity **1**, you used simple periodic functions to model physical phenomena such as a swinging pendulum and the action of a spring. Other physical phenomena can be modeled by periodic functions that are much more complex. For example, the sound waves produced by musical instruments can be modeled by sums of simple trigonometric functions.

Exploration

In this exploration, you investigate the behavior of functions that are formed by adding sine and cosine functions.

a.

1. Determine the period and amplitude of $f(x) = \sin(3x)$ and $g(x) = 2\sin(3x)$.

2. Define $h(x) = f(x) + g(x)$. Compare the period of $h(x)$ to the periods of $f(x)$ and $g(x)$.

3. Compare the amplitude of $h(x)$ to the amplitudes of $f(x)$ and $g(x)$.

b. Let $f(x) = a\sin(bx)$, $g(x) = c\sin(bx)$, and $h(x) = f(x) + g(x)$. Repeat Part **a** for two different sets of values for a, b, and c.

c. Let $f(x) = a\sin(bx)$, $g(x) = c\cos(bx)$, and $h(x) = f(x) + g(x)$. Repeat Part **a** for two different sets of values for a, b, and c.

d. **1.** Determine the period and amplitude of $f(x) = \sin(2x)$ and $g(x) = 2\sin(3x)$.

2. Assuming that $x = 0$ is the beginning of a cycle, list several positive values of x at which each function will start a new cycle. Identify the smallest positive value common to both lists. This is the shortest interval between common starting points for the cycles of these two functions.

3. Define $h(x) = f(x) + g(x)$. How is the period of $h(x)$ related to the periods of $f(x)$ and $g(x)$ and your response to Step **2**?

4. Compare the amplitude of $h(x)$ to the amplitudes of $f(x)$ and $g(x)$.

e. Let $f(x) = a\sin(bx)$, $g(x) = c\sin(dx)$, and $h(x) = f(x) + g(x)$. Repeat Part **d** for two different sets of values for a, b, c, and d.

f. Let $f(x) = a\sin(bx)$, $g(x) = c\cos(dx)$, and $h(x) = f(x) + g(x)$. Repeat Part **d** for two different sets of values for a, b, c, and d.

Discussion

a. Based on what you discovered in the exploration, if $f(x)$, $g(x)$, and $f(x) + g(x)$ are all periodic functions, when will the period of $f(x) + g(x)$ equal the period of $f(x)$?

b. For two periodic functions $f(x)$ and $g(x)$, when does the period of $f(x) + g(x)$ equal the least common multiple of the periods of the two original functions?

c. Describe how to determine the period of $h(x) = f(x) + g(x)$, where $f(x) = \cos(3x)$ and $g(x) = 5\sin(2x)$.

d. Do you think that the function $g(x) = \sin(2x) + \sin(\pi x)$ could have a period of 2π? Explain your response.

e. How does the sum of the amplitudes of $f(x)$ and $g(x)$ compare with the amplitude of $f(x) + g(x)$?

f. Give an example of two non-periodic functions whose sum is a periodic function.

Assignment

2.1 **a.** Find the amplitude and period of the functions $f(x) = 3\cos(4\pi x)$, $g(x) = 5\cos(4\pi x)$, and $f(x) + g(x)$.

b. Find the amplitude and period of the functions $f(x) = 2\cos(3x)$, $g(x) = 4\cos(5x)$, and $f(x) + g(x)$.

c. When cosine curves are added, do you think that the effects on amplitude and period will be similar to those that occur when sine curves are added? Explain your response.

2.2 One electronic test signal is produced by adding the terms of a series based on the sine function. The test signal can be modeled by $h(x)$, where $h(x)$ is the sum of the following five functions:

$$f_1(x) = -\sin(x),\ f_2(x) = -\frac{1}{2}\sin(2x),\ f_3(x) = -\frac{1}{3}\sin(3x),$$

$$f_4(x) = -\frac{1}{4}\sin(4x),\ f_5(x) = -\frac{1}{5}\sin(5x)$$

a. Graph $h(x)$ and describe the shape of the graph.

b. Determine the period and amplitude of $h(x)$.

c. List the next three terms in the series.

d. Graph the sum of the first eight terms of the series.

e. Describe any differences you observe between the graphs in Parts **a** and **d**.

2.3 The force required to stretch or compress a spring is proportional to the distance that spring is stretched or compressed from its natural length. The constant of proportion is referred to as the spring constant.

Ignoring friction, the movement of an object attached to the spring can be modeled with periodic functions. The displacement (in meters) at any given time can be modeled by the following function:

$$d(t) = A\cos\left(t\sqrt{k/m}\right) + B\sin\left(t\sqrt{k/m}\right)$$

where t represents time in seconds, m is the mass of the object in kilograms, and k is the spring constant.

The velocity (in meters per second) at any given time can be modeled by the function below:

$$v(t) = -A\left(\sqrt{k/m}\right)\sin\left(t\sqrt{k/m}\right) + B\left(\sqrt{k/m}\right)\cos\left(t\sqrt{k/m}\right)$$

where t represents time in seconds, m is the mass of the object in kilograms, and k is the spring constant.

a. Consider a mass of 10 kg attached to a spring which has a spring constant of 1.6 N/m. At $t = 0$, the initial displacement of the object is 25 cm, while its initial velocity is 2 m/sec. Determine the values of A and B in this situation.

b. Determine the object's displacement and velocity at $t = 3.6$ sec.

c. The object's displacement over time is described by a sum of two periodic expressions. Represent this sum as a single periodic expression.

d. The object's velocity over time also is described by a sum of two periodic expressions. Represent this sum as a single periodic expression.

e. Substitute $t = 3.6$ into your equations from Parts **c** and **d** and compare the results with your responses to Part **b**.

* * * * *

2.4 Consider the functions $f(x) = a\sin x$ and $g(x) = b\cos x$.

a. Select values for a and b. Create a graph of $f(x) + g(x)$.

b. Determine the domain and range of the function $h(x) = f(x) + g(x)$.

c. It is possible to represent the sum $f(x) + g(x)$ as a single function of the form $h(x) = c\sin(x + d)$. Determine a function of this form whose graph matches the graph of the sum created in Part **a**.

2.5 Some people believe that biological rhythms influence our personal lives. In its simplest form, the biorhythm theory states that, from birth to death, each of us is influenced by three internal cycles: the physical, the emotional, and the intellectual. The 23-day physical cycle affects a broad range of bodily functions, including resistance to disease and strength. The 28-day emotional cycle governs creativity, sensitivity, and mood. The 33-day intellectual cycle regulates memory, alertness, receptivity to knowledge, and other mental processes.

According to this theory, each of the cycles starts at a neutral baseline or zero point on the day of birth. From that zero point, the three cycles enter a rising, positive phase, during which the energies and abilities associated with each are high, and then gradually decline. Each cycle crosses the zero point midway through its period, entering a negative phase in which physical, emotional, and intellectual capabilities are somewhat diminished, and energies are recharged.

Since the three cycles have different periods, the highs, lows and baseline crossings rarely coincide. As a result, the theory predicts that people are usually subject to a mix of biorhythms and that their behavior—from physical endurance to academic performance—is a composite of these varying influences.

a. On the same set of axes, use sine curves to model the three internal cycles for the first 60 days after birth. Represent time on the x-axis and energy level on the y-axis. Let the energy level vary from 2 to –2.

b. According to biorhythm theory, the most vulnerable days occur when each cycle crosses the baseline, switching from positive to negative or vice versa. These are "critical days." Identify the coordinates of the first critical days after birth for each of the three cycles on the graph.

c. Determine your biological rhythms for today.

d. In an average human life span of 70 years, when will the three cycles coincide on the baseline?

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Research Project

Sound waves are created by vibrations. The changes in pressure created by these vibrations can be modeled by a sine function, as shown in Figure **6**.

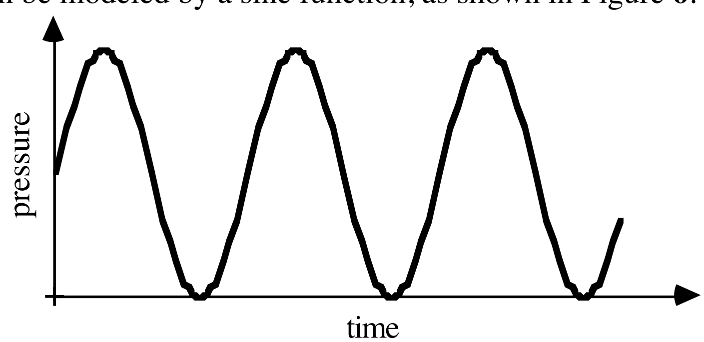


Figure 6: Graph of pressure versus time for a sound wave

The **frequency** of a sound wave describes the number of cycles per unit time (the reciprocal of the period). Frequency is usually measured in hertz (Hz), where 1 Hz equals 1 cycle per second. For example, a sound wave with a frequency of 5 Hz (or $5 \text{ cycles}/\text{sec}$) has a period of $1/5 \text{ sec}$.

The **fundamental frequency** of an object is the lowest frequency at which it can vibrate. **Harmonics** are whole-number multiples of the fundamental frequency. For example, if an object has a fundamental frequency of 40 Hz, the third harmonic is 3 times the fundamental frequency, or 120 Hz.

Find out more about harmonics and write a report explaining how they affect the quality of sounds produced by musical instruments. Include graphs of the first, second, and third harmonics, and explain why the graph of a sound created by a musical instrument might represent a sum of two or more periodic functions.

Activity 3

As you have seen in Activity **2**, periodic functions can be used to model things as diverse as seasonal temperature changes and sound waves. Some periodic functions are also important as parts of larger mathematical systems. In this activity, you investigate the reciprocals of some familiar trigonometric functions.

Exploration 1

Models for periodic phenomena can be written in several different forms by using identities. Recall that a **trigonometric identity** is an equation involving trigonometric functions that is true for all real numbers in its domain.

a.

1. Graph $f(x) = \sin x$ and $g(x) = \cos x$ on the same coordinate system.

2. Determine a transformation of the graph of $g(x)$ for which the image is $f(x)$.

3. Use your response to Step **2** to write an identity involving the sine and cosine functions.

4. Test your answer to Step **3** by graphing the two expressions in the identity.

5. Given $h(x) = 3\sin(2x)$, use your identity to write an equivalent expression for $h(x)$ involving the cosine function.

b.

1. Determine a transformation of the graph of $f(x) = \sin x$ for which the image is $g(x) = \cos x$.

2. Use your response to write an identity involving the cosine and sine functions.

3. Test your answer by graphing the two expressions in the identity.

c.

1. Graph the functions $f(x) = \sin x$ and $g(x) = \sin(x + \pi)$ on the same coordinate system.

2. Determine a relationship between the two graphs and use that relationship to write an identity involving $\sin(x + \pi)$ and $\sin x$.

3. Test your answer by graphing the two expressions.

Discussion 1

a. Use the identities you determined in Exploration **1** to describe an equivalent expression for each of the following:

1. $\sin(\pi/3)$

2. $\cos(\pi/4)$

b. Consider the function $f(x) = \cos(x) + \sin(x + \pi)$.

1. How can this function be written in terms of sine only?

2. How can it be written in terms of cosine only?

Exploration 2

In this exploration, you investigate the reciprocals of some familiar trigonometric functions.

Mathematics Note

The **cosecant** of x, denoted $\csc x$, is the reciprocal of $\sin x$, or $\csc x = 1/\sin x$, where $\sin x \neq 0$. For example, Table **1** shows some sample values for $f(x) = \sin x$ and $g(x) = \csc x$.

Table 1: Sample values for the sine and cosecant functions

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f(x) = \sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$g(x) = \csc x$	undefined	$2/1$	$2/\sqrt{2}$	$2/\sqrt{3}$	1

The **secant** of x, denoted $\sec x$, is the reciprocal of $\cos x$, or $\sec x = 1/\cos x$, where $\cos x \neq 0$. For example, if $\cos x = 1/2$, then

$$\sec x = \frac{1}{1/2} = 2$$

The **cotangent** of x, denoted $\cot x$, is the reciprocal of $\tan x$, or $\cot x = 1/\tan x$, where $\tan x \neq 0$. For example, if $\tan x = \sqrt{3}/2$, then

$$\cot x = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

a. Graph at least three periods of the function $f(x) = \sin x$.

b. Graph $g(x) = \csc x$ on the same coordinate system as in Part **a**.

c. **1.** Determine any points of intersection for the graphs of $f(x)$ and $g(x)$.

2. Verify that the points identified in Part **c** satisfy both functions.

d. Determine the equations of the vertical asymptotes for the graph $g(x)$.

e. Given a function of the form $f(x) = a\sin(bx)$, you know that a affects the amplitude of the graph and b affects the period. In the following steps, you examine the effects of a and b on a function of the form $g(x) = a\csc(bx)$.

1. Select values other than 0 and 1 for both a and b. Graph at least three periods of the function $f(x) = a\sin(bx)$.
2. On the same coordinate system—and using the same values selected above for a and b—graph $g(x) = a\csc(bx)$.
3. Determine any points of intersection for the graphs of $f(x)$ and $g(x)$.
4. Verify that these points satisfy both functions.
5. Determine the equations of the vertical asymptotes for the graph of $g(x)$.

Discussion 2

a. The functions $f(x) = \sin x$ and $g(x) = \csc x$ are reciprocals. What other information about $f(x)$ helps to verify that $g(x)$ is undefined at the asymptotes found in Part **d** of Exploration **2**?

b. How does the domain of $f(x) = \sin x$ compare to the domain of $g(x) = \csc x$?

c. How does the range of $f(x) = \sin x$ compare to the range of $g(x) = \csc x$?

d. What is the period of $g(x) = \csc x$? Defend your response.

e. Will the period of $g(x) = a\csc(bx)$ always be the same as the period of $f(x) = a\sin(bx)$? Defend your response.

Assignment

3.1. Consider a graph of a function of the form $g(x) = a\csc(bx)$, where $a \neq 0$. Each branch that lies between consecutive asymptotes intersects the graph of $f(x) = a\sin(bx)$ in exactly one point.

Describe the significance of these points of intersection.

3.2. Find an equivalent expression in terms of the sine function for each of the following.

a. $\sin(x - \pi)$

b. $\sin(\pi - x)$

c. $\sin\left(x - (3\pi/2)\right)$

d. $\sin\left(x + (3\pi/2)\right)$

e. $\sin(-x)$

3.3. For each expression given in Problem **3.2**, find an equivalent expression in terms of the cosine function.

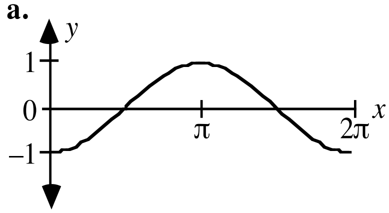
3.4 Use a graph of each of the following equations to rewrite it as an identity.

a. $y = \cos^2 x + \sin^2 x$

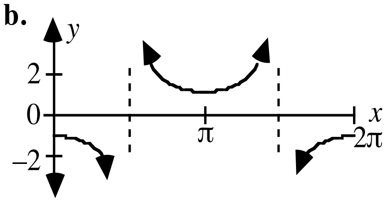
b. $y = \dfrac{1 - \cos^2 x}{\sin x}$; $\sin x \neq 0$

3.5 Which of the following graphs best illustrates the graph of $g(x) = \sec x$ over the domain $[0, 2\pi]$? Defend your choice.

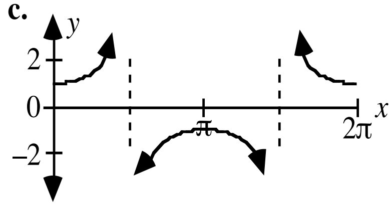
a.



b.



c.



3.6 **a.** Graph the following two functions on the same coordinate system.

$$f(x) = 5\cos\left(\frac{\pi}{2}x\right) \text{ and } g(x) = 5\sec\left(\frac{\pi}{2}x\right)$$

b. Determine the domain and range of each function.

c. Is the secant function periodic? Defend your response.

d. Graph $g(x)$ and $1/f(x)$ on the same coordinate system. Describe any relationships you observe between the two graphs.

e. In general, what is the relationship between $1/f(x)$, where $f(x) = a\cos(bx)$, and $g(x) = a\sec(bx)$?

3.7 **a.** The ratio for the tangent, $\tan\theta = \sin\theta/\cos\theta$, is true for individual values of θ. If this relationship is true for all values of θ, then it is an identity. Is the tangent ratio an identity? Justify your response.

b. Demonstrate algebraically that $\tan x = 1/\cot x$.

c. Consider the functions $f(x) = \tan x$ and $g(x) = \cot x$. Determine the domain and range of each function.

d. Is the cotangent function periodic? Explain your response.

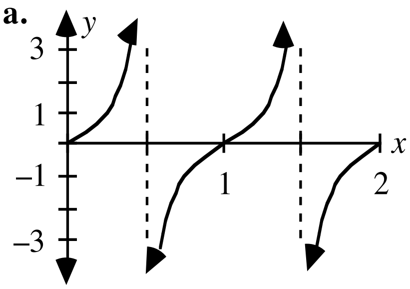
* * * * *

3.8 **a.** Determine the relationship between $f(x) = \cos x$ and $g(x) = \cos(x + \pi)$. Express this relationship as an identity.

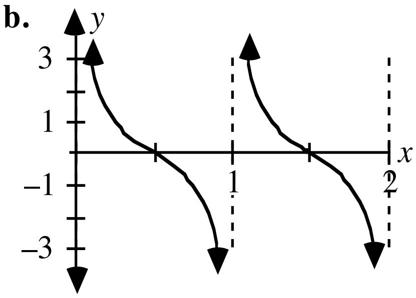
b. Determine the relationship between $f(x) = \cos(x - \pi)$ and $g(x) = \cos(x + \pi)$. Express this relationship as an identity.

3.9 Which of the following graphs best illustrates the graph of $f(x) = \cot(\pi x)$ over the domain $[0, 2]$? Defend your choice.

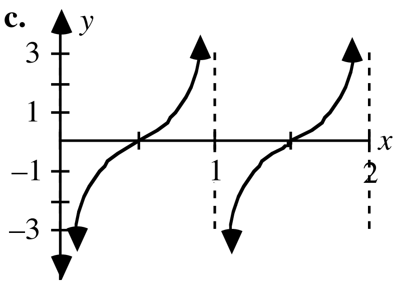
a.



b.



c.



3.10 Use reciprocal properties and identities to simplify each of the following expressions:

a. $\sin x \bullet \sec x \bullet \cot x$

b. $\csc x \bullet \tan x \bullet \cos x$

* * * * * * * * * *

Summary Assessment

1. Imagine that you are a wildlife biologist for the state of North Dakota. As part of an ecosystem restoration project, you have been assigned to reintroduce a colony of prairie dogs to a 50-acre wildlife preserve.

The preserve already has a resident population of voles, which feed on the same plants as prairie dogs. It is also home to a population of owls, which prey on both voles and prairie dogs. The most recent data on these populations is shown in the table below.

Owl	**Population**	**Date**
Maximum	37	July 1, 1992
Minimum	11	July 3, 1993
Vole	**Population**	**Date**
Maximum	1752	January 1, 1993
Minimum	976	July 1, 1994

In order to give the prairie dogs a good chance at establishing themselves, you would like to introduce them when both competition from voles and predation from owls are at a minimum.

The fluctuations in the owl and vole populations can be modeled using transformations of the sine function. Use the information above to determine the best times and the worst times to introduce the prairie dogs between January 1992 and January 1999. Write a report on your findings. Use graphs and functions to support your reasoning.

2. In the northern hemisphere, the earliest sunset occurs near December 21 each year, while the latest sunset occurs near June 21. (The actual dates for the earliest and latest sunsets may vary by a few days from December 21 and June 21, respectively.)

Assume that a graph of the time of sunset versus the day of the year can be modeled with a transformation of the sine or cosine curves.

a. Sketch a curve that models a graph of the time of sunset in your town versus the day of the year.

b. Write two different equations that model the graph from Part **a**.

c. Use one of your models to predict the time the sun will set today and compare your prediction to the actual time of sunset.

d. Do you think that your model could accurately estimate the time of sunset on any given day? Justify your response.

Module Summary

- A **periodic function** is a function in which values repeat at constant intervals.
- The **period** is the smallest interval of the domain over which the function repeats.
- A **cycle** is the portion of the function included in one period.
- If a periodic function has a maximum *M* and a minimum *m*, its **amplitude** is defined as

$$\left|\frac{M-m}{2}\right|$$

- For all periodic functions of the form $f(x) = a\sin[b(x-h)] + k$, where $a \neq 0$, the amplitude is $|a|$, the number of cycles in 2π radians is *b*, and the period is $2\pi/b$.

  The value of *h* describes the horizontal translation of the graph from the parent function $g(x) = \sin x$, while the value of *k* describes the vertical translation from the parent.

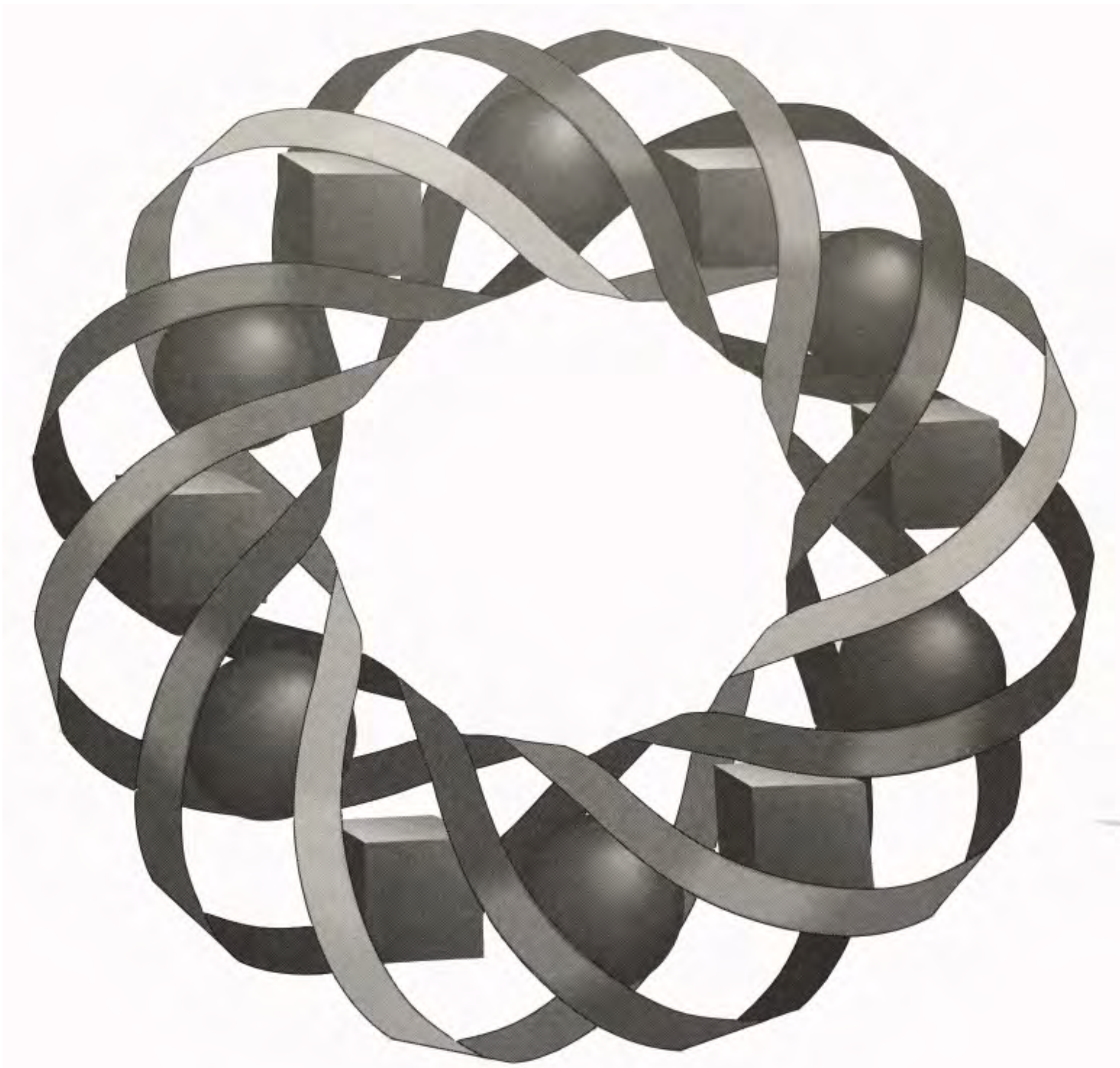
- A **trigonometric identity** is an equation involving trigonometric functions that is true for all real numbers in its domain.
- The **cosecant** of *x*, denoted $\csc x$, is the reciprocal of $\sin x$, or $\csc x = 1/\sin x$, where $\sin x \neq 0$.
- The **secant** of *x*, denoted $\sec x$, is the reciprocal of $\cos x$, or $\sec x = 1/\cos x$, where $\cos x \neq 0$.
- The **cotangent** of *x*, denoted $\cot x$, is the reciprocal of $\tan x$, or $\cot x = 1/\tan x$, where $\tan x \neq 0$.

Selected References

Giancoli, D. *Physics*. Englewood Cliffs, NJ: Prentice Hall, 1991.

Hewitt, P. *Conceptual Physics*. Menlo Park, CA: Addison-Wesley, 1987.

The Sequence Makes the Difference



Some number patterns are easy to recognize, while others are more difficult. In this module, you'll explore one strategy for recognizing number patterns described by polynomials.

Darlene Pugh • Tom Teegarden

The Sequence Makes the Difference

Introduction

In this module, you continue your explorations with number patterns. Some of these patterns are easily recognizable, while others are not. If you can recognize a number pattern, then you may be able to identify functions that can generate it.

For example, consider the function $f(n) = 2n$. Over a domain of the natural numbers, this function generates the following sequence:

$$2, 4, 6, 8, \ldots$$

The function $f(n) = 2n$ is a first-degree polynomial (or linear) function. Polynomial functions of other degrees also can be used to generate sequences.

Discussion

a. Recall that the general form of a polynomial function is:

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \cdots + a_1 n^1 + a_0$$

where k is a natural number. What is the degree of this polynomial?

b. Using a function $f(n)$ and the domain of natural numbers to generate a sequence, what value of n corresponds with the first term t_1 of the sequence? What value of n corresponds with t_k?

c. Over a domain of the natural numbers, a zero-degree polynomial of the form $f(n) = a_0$ generates a **constant sequence**. Give an example of a constant sequence.

d. What is another name for a sequence generated by a linear function, such as $2, 4, 6, 8, \ldots$? Explain your response.

e. Sequences also can be generated by second-, third-, and fourth-degree polynomials. Give an example of a sequence generated by each of the following:

1. a quadratic function

2. a cubic function

3. a quartic function.

f. Compare the use of polynomial functions to the use of explicit formulas in defining the terms of sequences.

Activity 1

A scatterplot of a sequence can often give you a clue about the types of functions that may have generated its terms. However, scatterplots alone may not provide enough information. In this activity, you examine another useful tool: the **finite-difference process**.

Exploration 1

As you have seen in earlier modules, an arithmetic sequence always has a common difference. For example, the common difference for the following arithmetic sequence is 6:

$$14, 20, 26, 32, 38, 44, \ldots$$

In this exploration, you identify differences between successive terms of non-arithmetic sequences.

a. Table **1** lists four different degrees of polynomial functions and shows their general forms. Select values for the coefficient(s) and constant in each of these polynomials.

Table 1: Four types of polynomials

Type	General Form
linear	$f(n) = a_1 n + a_0$
quadratic	$f(n) = a_2 n^2 + a_1 n + a_0$
cubic	$f(n) = a_3 n^3 + a_2 n^2 + a_1 n + a_0$
quartic	$f(n) = a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0$

b. Finding the differences between consecutive terms in a sequence can help you identify how the sequence may have been formed. Listed in order, these differences themselves form a sequence: a **sequence of differences**.

For example, consider the finite sequence 4, 7, 24, 60, 120, 209. Its first sequence of differences is (7 – 4), (24 – 7), (60 – 24), (120 – 60), (209 – 120), or 3, 17, 36, 60, 89.

The terms of its second sequence of differences are the differences between consecutive terms in this new sequence: (17 – 3), (36 – 17), (60 – 36), (89 – 60), or 14, 19, 24, 29.

Figure **1** shows the results of continuing this process for a third sequence of differences.

Sequence	4	7	24	60	120	209
1st Sequence of Differences	3	17	36	60	89	
2nd Sequence of Differences	14	19	24	29		
3rd Sequence of Differences	5	5	5			

Figure 1: Successive sequences of differences

1. List at least the first six terms of the sequence generated by the cubic function you wrote in Part **a**.
2. Determine the first sequence of differences for this sequence.
3. Determine the second sequence of differences.
4. If possible, continue the process of finding successive sequences of differences until you obtain a constant sequence.
5. If you obtained a constant sequence in Step **4**, identify the sequence of differences which corresponds with the constant sequence.

c. Repeat Part **b** for the other polynomials you created in Part **a**.

d.
1. Recall that a geometric sequence can be defined using a recursive formula of the form $t_n = rt_{n-1}$, where $n > 1$ and r is the common ratio. Select values for t_1 and r, then repeat Part **b** for at least six terms of the resulting sequence.
2. Recall that a Fibonacci-type sequence can be defined recursively as $t_n = t_{n-1} + t_{n-2}$ where $n > 2$. Select values for t_1 and t_2, then repeat Part **b** for at least six terms of the resulting sequence.

Discussion 1

a. In Exploration **1**, what type(s) of sequences eventually resulted in a constant sequence of differences?

b. What appears to be the relationship between the degree of a polynomial function that generates a sequence and the corresponding sequences of differences?

Mathematics Note

A sequence generates a constant sequence of differences if and only if that sequence can be generated by a polynomial function.

If the first constant sequence of differences is the nth sequence of differences, there exists a unique polynomial function of degree n that generates the original sequence.

For example, consider the sequence 4, 7, 24, 60, 120, 209 and its successive sequences of differences (shown in Figure **1**). The second sequence of differences is 14, 19, 24, 29. The third sequences of differences is 5, 5, 5. Since the first constant sequence of differences occurs at the third sequence of differences, there is a unique third-degree polynomial that generates the original sequence. In this case, that polynomial function is:

$$f(n) = \frac{5}{6}n^3 + 2n^2 - \frac{53}{6}n + 10$$

where the domain is 1, 2, 3, 4, 5, 6.

c. Do you think that it would be possible to generate an infinite geometric sequence, where $r \neq 1$, using a polynomial function?

d. Given any finite sequence, can you determine with certainty the degree of the polynomial function that generates it? Explain your response.

e. Why would you expect it to be possible to generate every finite sequence using polynomial functions?

Exploration 2

Each sequence of differences that results from a sequence generated by a polynomial function also can be generated by a polynomial function. In this exploration, you explore the relationship among the degree(s) of polynomial functions that generate successive sequence(s) of differences.

a. Determine the first 10 terms of the sequence generated by the quartic function $f(n) = n^4 - 2n^2 + n$.

b. **1.** The terms of this sequence's first sequence of differences can be generated by the function $f_1(n) = f(n+1) - f(n)$. Use this function to calculate the first nine terms of the first sequence of differences.

2. The function $f_1(n)$ can be expanded, as shown below:

$$f_1(n) = f(n+1) - f(n)$$
$$= \left((n+1)^4 - 2(n+1)^2 + (n+1)\right) - \left(n^4 - 2n^2 + n\right)$$

Use the distributive and associative properties to simplify this expression. Record the degree of the function in a table with headings like those in Table **2**.

Table 2: Degrees of functions that generate sequences of differences

Function	Degree
$f_1(n)$	
$f_2(n)$	
$f_3(n)$	
$f_4(n)$	

c. 1. Use the function $f_2(n) = f_1(n+1) - f_1(n)$ to calculate the first eight terms of the second sequence of differences.

2. Using the procedure described in Part **b**, determine the degree of $f_2(n)$ and record it in Table **2**.

d. 1. Use the function $f_3(n) = f_2(n+1) - f_2(n)$ to calculate the first seven terms of the third sequence of differences.

2. Determine the degree of $f_3(n)$ and record it in Table **2**.

e. 1. Use the function $f_4(n) = f_3(n+1) - f_3(n)$ to calculate the first six terms of the fourth sequence of differences. (This should be a constant sequence.)

2. Determine the degree of $f_4(n)$ and record it in Table **2**.

f. Describe a relationship between the degree of a polynomial that generates a sequence and the degrees of polynomials that generate successive sequences of differences.

Discussion 2

a. Given a finite sequence generated by a polynomial function of degree 7, what is the least degree of a polynomial function that generates each of the following:

1. the first sequence of differences?
2. the third sequence of differences?
3. the fifth sequence of differences?
4. the sixth sequence of differences?

b. What is the relationship between the degree of a polynomial function that generates a finite sequence and the degrees of the polynomial functions that generate its successive sequences of differences?

c. Describe how you could use your knowledge of the finite-difference process to determine the terms of a sequence given the following information.

- The first term of the sequence is 17.
- The first term of the first sequence of differences is 9.
- The first term of the second sequence of differences is 6.
- The terms of the third sequence of differences are 5, 5, 5.

Assignment

1.1 **a.** Complete a copy of the following table for the finite sequence generated by $t_n = 3n^3 + 4n - 3$, for $n = 1, 2, 3, \ldots, 10$.

n	**Sequence**	**Sequences of Differences**		
1		**First**		
2			**Second**	
3				**Third**
4				
⋮				
10				

b. Identify the least degree of polynomial function that could generate the sequence in each column of the table.

1.2 Create a polynomial function that generates a finite sequence in which the fourth sequence of differences is the first constant sequence of differences.

1.3 **a.** Identify the missing term in the following finite sequence and describe how you determined your answer.

$$2, 9, ?, 65, 126, 217, 344, 513$$

b. Identify the least degree of polynomial function that could generate the sequence in Part **a**.

1.4 Explain why a sequence generated by a fifth-degree polynomial eventually results in a constant sequence of differences.

1.5 Each of the following sequences was generated by a cubic function. Identify the missing term in each one.

a. 4, 15, 38, 79, 144, 239, ?

b. –144, –286, –420, –540, –640, –714, –756, –760, ?

* * * * *

1.6 Determine the least degree of polynomial function that could have generated each sequence below.

a. 3, 11, 31, 69, 131, 223, 351, 521, 739

b. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

1.7 Use the infinite sequences defined by the following recursive rules to complete Parts **a–c**.

$$\begin{cases} p_1 = 1 \\ p_n = p_{n-1} + n, \ n > 1 \end{cases} \qquad \begin{cases} q_1 = 1 \\ q_n = q_{n-1} + n^2, \ n > 1 \end{cases} \qquad \begin{cases} r_1 = 1 \\ r_n = r_{n-1} + n^3, \ n > 1 \end{cases}$$

a. Determine the first eight terms of each sequence.

b. Determine whether or not each sequence could be generated by a polynomial function.

c. If the sequence could be generated by a polynomial function, determine the least degree of this polynomial.

1.8 **a.** Identify the missing term in the following sequence, given that it was generated by a polynomial:

5, –22, –147, ?, –1219, –2550, –4747, –8122, –13035, –19894

b. Determine the least degree of polynomial function that could have generated the sequence in Part **a**.

* * * * * * * * * *

Activity 2

Retailers and builders often stack objects in pyramids. For example, a grocer might display a pyramid of fruit in the produce department, while a builder might store a pyramid of concrete blocks on a job site. In this activity, you examine how sequences generated by polynomial functions can be used to model these situations.

Exploration

a. Create a stack of blocks shaped like a pyramid with a square base, using 25 blocks on the bottom level and 1 block on the top level. When completed, your stack should resemble the diagram in Figure **2**.

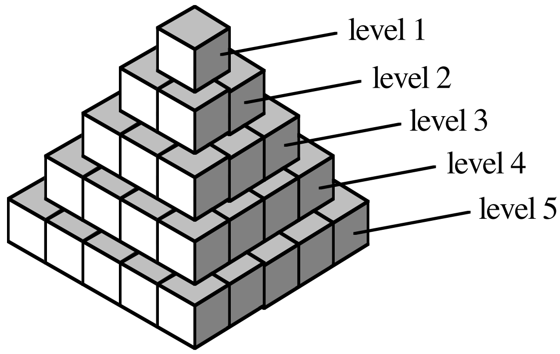


Figure 2: A stack of blocks

b. **1.** Write a sequence to model the total number of blocks in the pyramid after each level, starting with the top. Let the term number of the sequence equal the level of the pyramid, and let the value of the term equal the total number of blocks in the pyramid after each level.

2. Determine a recursive formula for the sequence.

3. Find the next three terms of the sequence.

c. One method for finding an explicit formula for a polynomial sequence involves using finite differences. Use the finite-difference process to determine the least degree of the polynomial that might generate the sequence found in Part **b**.

d. Once the least degree of the polynomial has been identified using finite differences, the polynomial function itself can be found by solving a system of equations.

For example, Table **3** shows the results of using the finite-difference process for the sequence 49, 72, 99, 130, 165.

Table 3: Differences for a sequence created by a quadratic

n	Sequence	Sequences of Differences	
1	49	**First**	
2	72	23	**Second**
3	99	27	4
4	130	31	4
5	165	35	4

Since the second sequence of differences is the first constant sequence, the least degree of the polynomial that generates this sequence is 2.

From Table **3**, when $n = 1$, $f(n) = 49$; when $n = 2$, $f(n) = 72$; and when $n = 3$, $f(n) = 99$.

These values, along with the general equation for a quadratic polynomial, $f(n) = a_2n^2 + a_1n + a_0$, can be used to create the following system of equations:

$$\begin{cases} 49 = a_2(1)^2 + a_1(1) + a_0 \\ 72 = a_2(2)^2 + a_1(2) + a_0 \\ 99 = a_2(3)^2 + a_1(3) + a_0 \end{cases}$$

Solving this system for a_2, a_1, and a_0, results in the polynomial function that generates the sequence: $f(n) = 2n^2 + n - 6$.

Use this process to determine a polynomial that generates the sequence in Part **b**.

e. Another method for identifying a polynomial that generates a sequence involves curve-fitting techniques.

1. Create a scatterplot of term value versus term number for the sequence in Part **b**.
2. Considering your response to Part **c**, use a polynomial regression to find an explicit formula for the sequence.

Discussion

a. Compare the two polynomial functions you identified in Parts **d** and **e** of the exploration.

b.
1. How can a recursive formula be used to find the 100th term of a sequence?
2. How can an explicit formula be used to find the 100th term of a sequence?

c. Given only the first few terms, what assumption must be made in order to predict the 100th term of a sequence?

d. Consider the sequence $1, 2, 4, 8, 16\ldots$.

1. Predict the sixth term of this sequence and describe how you made your prediction.
2. The sequence given above could be generated by the following polynomial function:

   $$f(n) = \frac{41}{120}n^5 - \frac{61}{12}n^4 + \frac{691}{24}n^3 - \frac{911}{12}n^2 + \frac{1393}{15}n - 40$$

   In this case, what would be the sixth term of the sequence?

e. What do your responses to Part **d** above imply about the reliability of making predictions about subsequent terms in a given sequence?

f. Suppose that you know the recursive formula for an infinite sequence. The pattern revealed by the first several terms indicates that it can be generated by a polynomial. Would you expect there to be more than one polynomial that could generate this sequence?

Assignment

2.1 Suggest a polynomial function for generating each sequence below and use it to predict the next three terms.

a. **1.** 1, 6, 18, 40, 75, 126, 196, ...

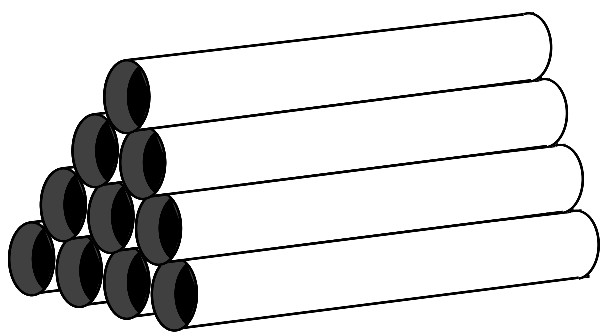
2. 6.5, 17, 36.5, 65, 102.5, …

3. 8.9, –43.6, –215.1, –631.6, –1469.5, –2955.6, …

4. –1, 2, 3, 2, –1, –6, –13, –22, …

b. Are the functions you suggested in Part **a** the only polynomials that could have generated these sequences? Explain your response.

2.2 A pipe manufacturer stores its products in piles like the one shown in the diagram below.



a. The total number of pipes after any row, beginning with the top row, defines a sequence. For example, after 1 row, there is 1 pipe; after 2 rows, there are 3 pipes; and so on.

Do you think that this sequence can be generated by a polynomial function? Explain your response.

b. The warehouse manager would like to develop a method for determining the total number of pipes in a pile by counting the number of pipes in the bottom row.

Find a formula that describes the total number of pipes in a pile given the number of pipes in the bottom row.

c. The warehouse has a pile with 40 pipes in the bottom row. How many pipes are in this pile?

2.3 Consider the sequence 1, 2, 3, ...

a. Predict the fourth term in the sequence.

b. Determine a polynomial function that could be used to generate the sequence containing your four terms.

c. Suggest a value for the fourth term other than the one you predicted in Part **a**.

d. Use the first four terms of the sequence from Part **c** and the general form of a cubic function, $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, to determine a polynomial that could be used to generate these four terms.

e. Compare the fifth terms of the sequences generated by the polynomials in Parts **b** and **d**.

2.4 Consider an arrangement of the natural numbers in a triangular pattern, as shown below.

1 — row 1

2 3 — row 2

4 5 6 — row 3

… *etc.* …

a. Generate the next two rows of the triangular pattern.

b. Which row will contain the number 1000?

c. What is the sum of the numbers in the 100th row?

2.5 Consider the sequence –5, 7.5, –11.25, 16.875, … .

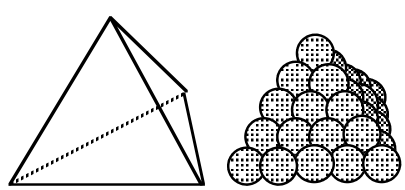
a. Suggest a recursive formula for this sequence.

b. Predict the next three terms.

c. Suggest an explicit formula for this sequence.

d. Predict the 50th term.

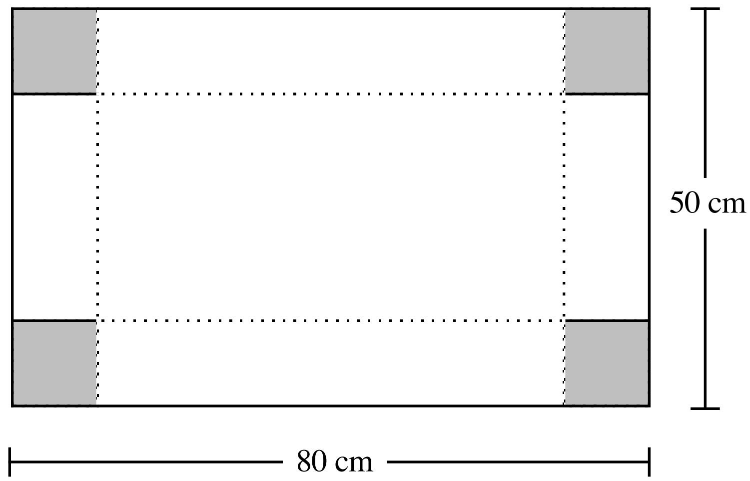
2.6 Grocery stores often display fruit in stacks shaped like tetrahedrons. The diagram below shows such a display, with 15 oranges on the bottom level and 1 orange on the top level.



a. Determine the number of oranges that could be displayed in a tetrahedral stack with 10 levels.

b. A produce department has 7 cases of oranges. Each case contains 48 oranges. If the manager decides to display these oranges in a tetrahedral stack, how many levels will it have?

2.7 A box manufacturer creates clothing boxes from a template like the one shown below.



Cuts are made along the solid lines and folds are made along the dotted lines. The square tabs (shaded areas) are folded and glued inside to form the box. The machinery that makes the cuts will cut only square tabs with dimensions measured in whole centimeters.

a. Create a sequence in which the term number represents the length of the side of the square tab and the terms describe the surface area of the resulting box.

b. Create a sequence in which the term number represents the length of the side of the square tab and the terms describe the volume of the resulting box.

c. Determine the dimensions of the square tabs that result in a box with the largest possible surface area.

d. Determine the dimensions of the square tabs that result in a box with the largest possible volume.

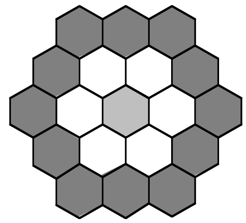
* * * * *

2.8 According to a popular holiday song, "The Twelve Days of Christmas," a "true love" gives an increasing number of gifts on 12 consecutive days. On the first day, the true love gives a partridge in a pear tree. On the second day, the true love gives two turtle doves and a partridge in a pear tree. On the third day, the true love gives three French hens, two turtle doves, and a partridge in a pear tree. This pattern continues.

a. On the 12th day, how many gifts does the true love give? Justify your response.

b. During the entire 12 days, how many gifts does the loved one receive? Explain your response.

2.9 Quilts are sometimes created by starting with a hexagonal piece of cloth, then adding rings of hexagons, as shown below.



a. Write a sequence for the number of hexagons in each of the first 10 rings of this quilt pattern.

b. How many hexagons would there be in the 21st ring of such a quilt?

2.10 A person starts a savings plan by saving \$10 in the first month, \$12 in the second month, \$14 in the third month, and so on.

a. How much should be saved in the 12th month of the fifth year?

b. What is the total amount of money saved after 5 years?

2.11 **a.** Create a sequence by following the steps below.

1. Construct a circle. Construct one point on the circle. Record the number of regions in the interior of the circle.
2. Construct a circle. Construct a diameter of the circle. Record the number of regions in the interior of the circle.
3. Construct a circle. Inscribe an equilateral triangle in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.
4. Construct a circle. Inscribe a square in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.
5. Construct a circle. Inscribe a regular pentagon in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.

b. Determine a pattern in the numbers of interior regions formed.

c. Predict the numbers of regions formed by each of the following:

1. inscribing a regular hexagon in a circle and connecting each vertex to every other vertex with a line segment
2. inscribing a regular heptagon in a circle and connecting each vertex to every other vertex with a line segment.

d. Verify your predictions from Part **c**.

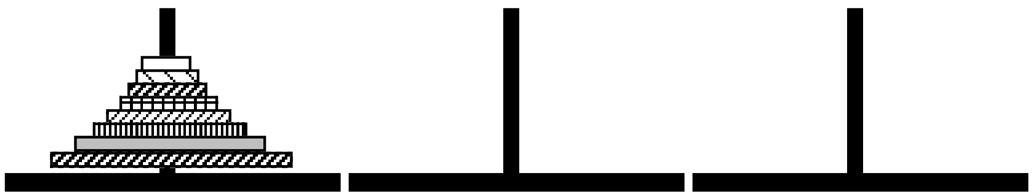
* * * * * * * * * *

Research Project

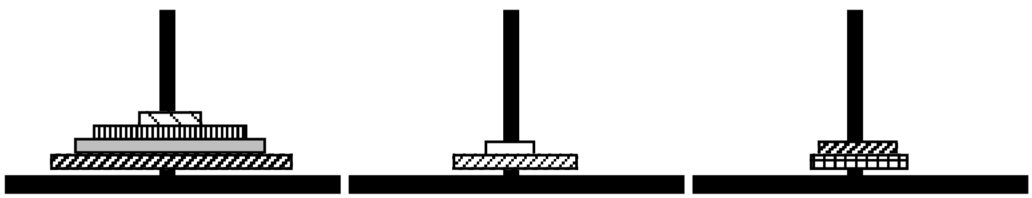
The Tower of Hanoi is a classic game involving sequences. The goal of the game is to move all the rings from one peg to another peg, retaining the same order, in a minimum number of moves. The game is subject to the following restrictions:

- Only one ring may be moved at a time.
- Once a ring is removed from one peg, it must be placed on one of the other two pegs.
- A ring may not be placed on top of a smaller ring.

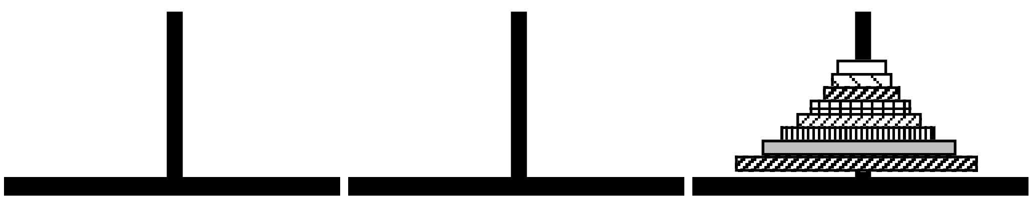
Figure **3** shows the starting positions for 8 rings, an intermediate step, and the final positions of the rings.



Starting Positions



Intermediate Step



Ending Positions

Figure 3: Playing the Tower of Hanoi with eight rings

a. Determine the minimum number of moves needed to move a stack of n rings from one peg to another while retaining the same order.

b. Determine the relationship between the number of rings and the minimum number of moves necessary to complete the task. Express this relationship as:

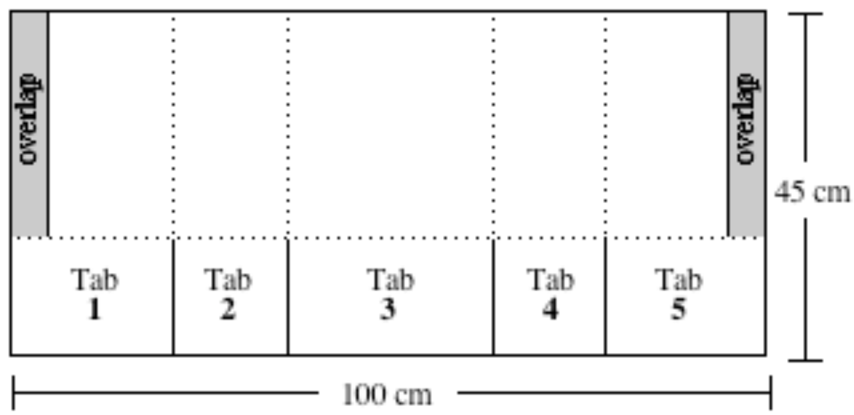
1. a recursive formula

2. an explicit formula.

Summary Assessment

Imagine that you are an engineer for a company that manufactures paper bags. Your department has been asked to design a bag with a capacity of at least 12,500 cm^3. For advertising purposes, the bag also must have the largest possible outside surface area.

The bag must be cut and folded from a rectangular sheet of paper measuring 100 cm by 45 cm, as shown in the following diagram.



The solid lines on the template indicate cuts; the dotted lines indicate folds. Tabs **2** and **4** are squares of equal measure. Tabs **1** and **5** are $3/4$ the length of Tab **3**. (This creates an overlap for gluing and strength.) The dimensions of the base of the bag are the length of Tab **2** by the length of Tab **3**. Due to the nature of the machinery that makes the cuts, the dimensions of the tabs must be in whole centimeters.

Once you have designed an acceptable bag, you must make a presentation to the company's board of directors. Your presentation should include the dimensions of the bag, a description of the method you used to determine these dimensions, and a paragraph that explains why you believe your bag is the best one possible. As part of your presentation, include a scale model made from a sheet of notebook paper.

Module Summary

- A **sequence of differences** can be generated from a finite sequence $t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ \ldots,\ t_n$ by taking the differences of consecutive terms. The first sequence of differences is $t_2 - t_1,\ t_3 - t_2,\ t_4 - t_3,\ t_5 - t_4,\ \ldots,\ t_n - t_{n-1}$.
- The process of finding successive sequences of differences is called the **finite-difference process**, which continues until the first constant sequence of differences is found.
- A sequence generates a constant sequence of differences if and only if that sequence can be generated by a polynomial function.
- If the first constant sequence of differences is the *n*th sequence of differences, there exists a unique polynomial function of degree *n* that generates the original sequence.

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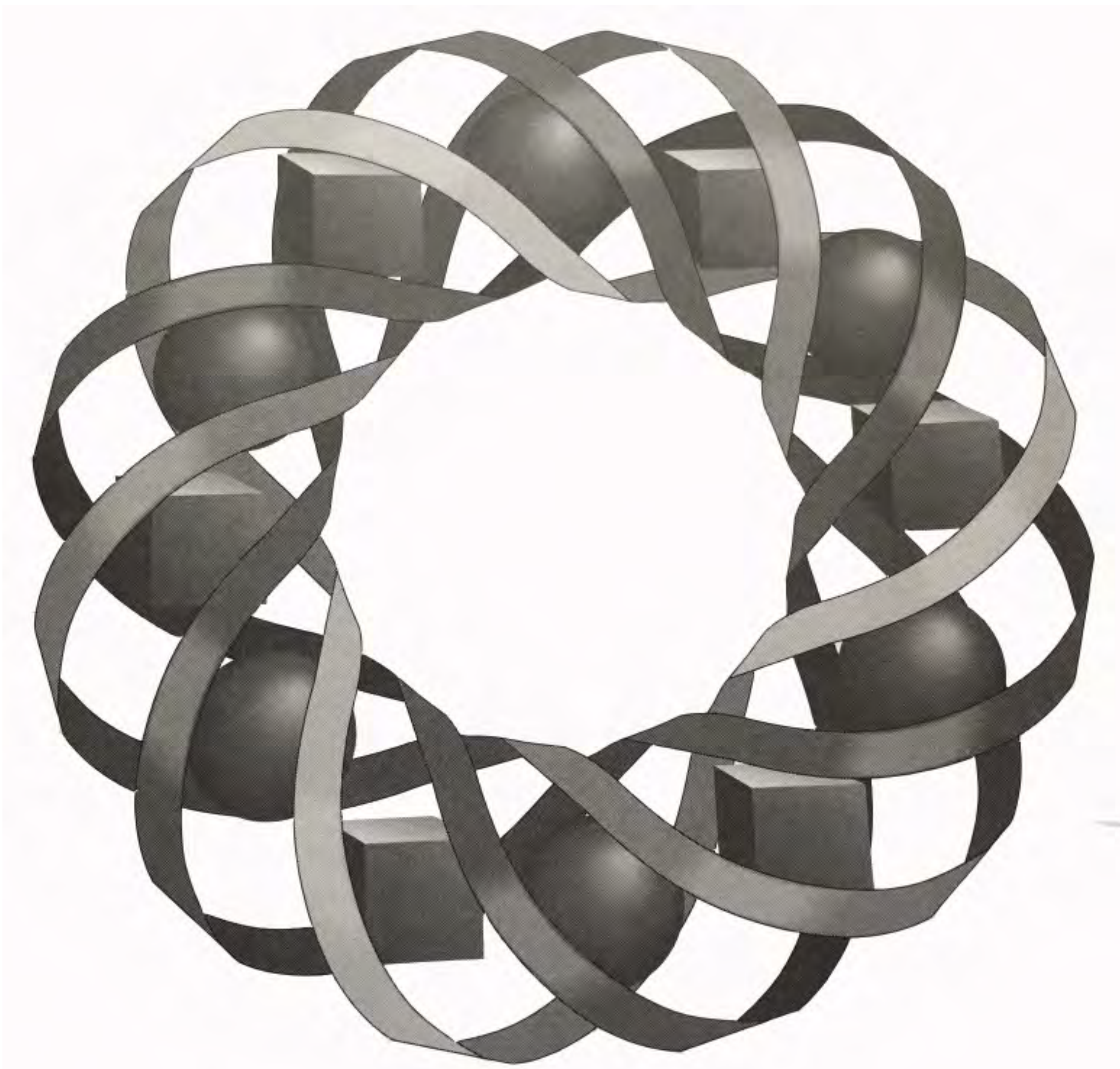
Brilliant Induction



In this module, you use the principle of mathematical induction to establish the validity of mathematical statements.

Staci Auck • Monty Brekke • Tim Skinner

Brilliant Induction

Introduction

When dominos are stood on end, each one slightly behind another, tipping over the first domino will cause the second one to fall. As the second domino falls, it will cause the next one to fall, and so on. Figure **1** shows this chain of events.

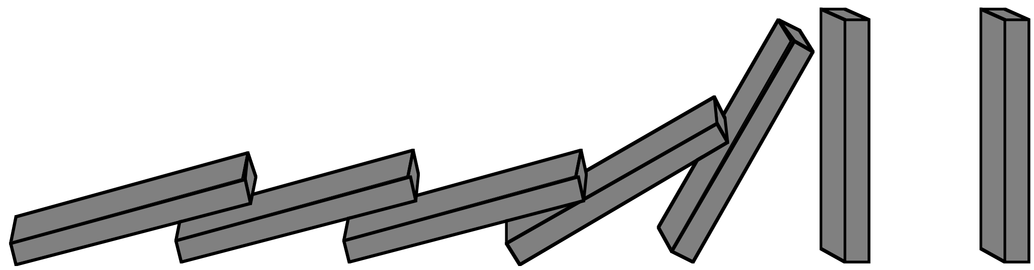


Figure 1: Falling dominos

A process in which each falling domino causes the next one to fall resembles in some ways a method of proof known as **mathematical induction**. This technique has been in use at least since the 16th century, and may have been recognized much earlier, perhaps by the Pythagoreans. In this module, you explore the conditions under which such a method of proof might work, as well as investigate situations where it does not apply.

Discussion

a. When 100 dominos are stood on end, what conditions are necessary for all the dominos to fall when the first one is knocked over?

b. The process that causes the 50th domino to fall is similar to the process that causes the 5th domino to fall. Describe the similarities.

c. How could you prove, without actually knocking the first domino over, that all the dominos will fall if the first one falls?

d. Many situations involve the successful completion of a chain of similar events. In a 400-m relay, for example, the first person must run 100 m, then successfully pass the baton to the second person. The second person also must run 100 m and successfully pass the baton to the next person, and so on, until the race ends. Describe the different ways in which a 400-m relay team might not finish a race.

e. To climb to the top of a ladder, you must start on the first rung, then advance to the second. Once on the second rung, you can advance to the third, and so on. Describe how this process is similar to the one which causes dominoes to fall.

Activity 1

In this activity, you consider the conditions necessary to prove a statement using mathematical induction.

Exploration 1

Figure **2** shows a point on a line. Disregarding the point itself, it can be thought of as separating the line into two regions R_1 and R_2 : one on either side of the point.

Figure 2: A point separating a line into two regions

In the following steps, you consider the number of regions into which *n* distinct points separate a line.

a. Draw a picture showing the number of regions formed when a second distinct point is placed on the line. Label each region as in Figure **2**.

b. Suppose that a third distinct point is placed on the line. Does the number of regions formed depend on the location of the point?

c. Repeat the process described in Parts **a** and **b** for three more points. Record the results in a table with headings like those in Table **1** below.

Table 1: Number of regions into which *n* distinct points separate a line

Number of Distinct Points (*n*)	Number of Regions Added with Each Additional Point	Total Number of Regions
1		2
2		
3		
4		
5		
6		

Discussion 1

a. Judging from your results in Exploration **1**, what happens to the number of regions formed when an additional point is placed on the line?

b. Does it matter where each additional point is placed on the line, as long as each one is distinct from any previous points?

c. Do you think that your response to Part **b** is true regardless of the number of points already placed on the line?

d. If you knew that your conjecture in Part **a** was true for all points up to and including some *k*th point, how would you argue that the conjecture was true when a $(k + 1)$st point is placed on the line?

Exploration 2

Figure **3** shows three rectangles constructed with toothpicks. Each rectangle has dimensions $1 \times n$, where n is 1, 2, or 3 toothpicks. The total number of toothpicks required to build each rectangle with dimensions $1 \times n$ can be described by the following formula: $a_n = 2n + 2$ for $n = 1,\ 2,\ 3$.

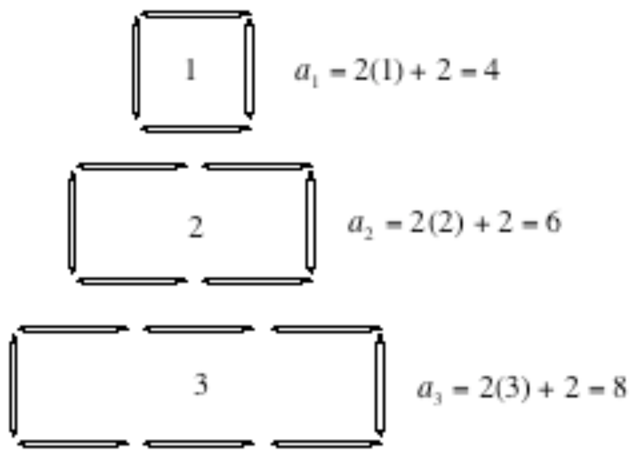


Figure 3: Rectangles constructed with toothpicks

Suppose that the pattern shown in Figure **3** is continued for all natural numbers. Does the formula still work for $n = \{1,\ 2,\ 3,\ \ldots\}$?

a. To argue that the formula $a_n = 2n + 2$ is correct for all natural numbers n, you must start by examining the first rectangle of dimensions $1 \times n$. When $n = 1$, the rectangle requires four toothpicks. Therefore, $a_1 = 4$.

1. What happens to the first rectangle in order to create the second rectangle?
2. How is a_2 related to a_1?

b. Explain how the process you described in Part **a** can be used to create a third rectangle given the second rectangle, and determine a_3 given a_2.

c. Suppose that the process you described in Parts **a** and **b** continues to work for all natural numbers up to k. Do you think that it can then be used to determine a_{k+1} given a_k? Explain your response.

d. The number of toothpicks required to build each rectangle in Figure **3** describes a sequence: $a_1,\ a_2,\ a_3$, where $a_1 = 4$, $a_2 = 6$, and $a_3 = 8$. The corresponding series S_3 represents the total number of toothpicks required to build three rectangles.

Determine $S_1,\ S_2,\ S_3,\ S_4$, and S_5.

e. Using the techniques you learned in the Level 6 module, "The Sequence Makes a Difference," verify that one possible formula for S_n is $S_n = n(n+3)$.

f. Note that $S_1 = a_1$ and $a_n = 2n + 2$. Use this fact to verify that the formula suggested in Part **e** is true for S_1.

g. Since $S_2 = S_1 + a_2$ and $a_n = 2n + 2$, the algebraic process below demonstrates that the formula $S_n = n(n+3)$ is true for $n = 2$, given that it is true for $n = 1$.

$$\begin{aligned} S_2 &= S_1 + a_2 \\ &= 1(1+3) + 2(2) + 2 \\ &= 4 + 6 \\ &= 10 \\ &= 2(2+3) \end{aligned}$$

1. Use the same process to verify that the formula $S_n = n(n+3)$ is true for $n = 3$, given that it is true for $n = 2$.
2. Verify that the formula is true for S_4, given that it is true for S_3.

h.

1. Assuming that the formula is true for S_{100}, the total number of toothpicks required to build the first 100 rectangles, verify that it also is true for S_{101}.
2. Assuming that the formula is true for S_{752}, verify that it also is true for S_{753}.

Discussion 2

a. Is it possible to prove that the formulas in Exploration **2** are true by checking them for every possible value of n?

b. Describe how you could verify that the formula $S_n = n(n+3)$ is true for S_{k+1} given that it is true for S_k. Hint: The process is the same as the one used in verifying that the formula is true for S_2 given that it true for S_1.

Assignment

1.1 At a very young age, the mathematician Karl Gauss (1777–1855) devised a method for adding consecutive natural numbers $1+2+3+\cdots+n$. This method has many applications.

a. By examining a pattern or using the methods developed in "The Sequence Makes a Difference," suggest a formula for the sum of the first n natural numbers.

b. **1.** Assume that the formula is true for $n = 100$. Use this assumption to verify that the formula also is true for $n = 101$.

2. Assuming that the formula is true for $n = 101$, show that it also is true for $n = 102$.

3. Assuming that the formula is true for $n = 102$, show that it also is true for $n = 103$.

c. Describe how your work in Part **b** is similar to the following statement: "If the 100th domino falls, so will the 101st, the 102nd, and the 103rd."

1.2 **a.** A student has suggested the following formula for the sum of the first n natural numbers:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}+1$$

Show that if this conjecture is true for some natural number k, then it also is true for the next natural number $k + 1$.

b. Show that the conjecture is in fact false when $n = 1$.

c. Your responses to Parts **a** and **b** are comparable to showing that if the first domino in a row of dominoes is knocked down, then all the others will fall, when in fact, the first one cannot be knocked down.

Use a truth table to illustrate how a false hypothesis leads to a true conditional statement.

d. Is there any way to prove that the conjecture in Part **a** is true? Explain your response.

1.3 Consider the following inequality: $2^{(n+1)} < 3^n$, where n is a natural number. This relationship is not true for $n = 1$ but is true for $n = 2$.

a. Graph the sequences $t_n = 2^{(n+1)}$ and $t_n = 3^n$ on the same coordinate system for $n = \{2, 3, 4, 5\}$.

b. Does the inequality appear to be true for $n \geq 2$?

c. Assuming that the inequality is true for $n = 577$, explain how the following steps verify that it also is true for $n = 578$.

$$\begin{aligned} 2^{(578+1)} &= 2^{(577+1)} \bullet 2^1 \\ &< 3^{577} \bullet 2^1 \\ &< 3^{577} \bullet 3^1 = 3^{578} \end{aligned}$$

d. Make a conjecture about the set of natural numbers for which the inequality is true.

1.4 **a.** Determine an explicit formula for finding the sum of the first n positive even integers: $2 + 4 + 6 + \cdots + 2n$.

b. Assume that the formula is true for $n = 50$. Write the equation that is implied by this assumption.

c. Show that if the assumption from Part **b** is true, then the formula also is true for $n = 51$.

1.5 Consider a meeting room containing the members of a civic group. Each person shakes hands with every other person in the room. When 2 people are in the room, 1 handshake occurs. When 3 people are in the room, 2 handshakes occur.

Use the process described in Problem **1.1** to suggest a formula for the number of handshakes that occur when n people are in the room.

1.6 Consider the following conjecture: "The quantity $3^n + 1$ is divisible by 2 for all natural numbers n."

a. Assuming that the conjecture is true for $n = 10,003$, show that it also is true for 10,004 using the steps described below.

1. The expression $3^{10,004} + 1$ can be rewritten as follows:

$$\begin{aligned} 3^{10,004} + 1 &= 3^{10,003+1} + 1 \\ &= 3^{10,003} \bullet 3^1 + 1 \end{aligned}$$

Using the fact that $3^1 = 2 + 1$, rewrite the above expression.

2. Argue that the result is divisible by 2, given that the conjecture is true for $n = 10,003$.

b. Does your work in Part **a** alone guarantee that the conjecture is always true? Explain your response.

1.7 Each of the following mathematical statements is false. To prove that each is false, identify a counterexample for each one.

a. $1 \bullet 2 \bullet 3 \bullet \cdots \bullet n = n^n - 2^{n-1}$ for all natural numbers n

b. $5^n \geq n^5$ for all natural numbers n

c. 8 is a factor of $12^n - 8^n$ for all natural numbers n

d. $2n+1$ is prime for all natural numbers n

* * * * *

1.8 A sequence can be defined by the recursive formula below:

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \ n > 1 \end{cases}$$

a. Determine the first five terms of the sequence.

b. Write an explicit formula for the sequence.

c. Assume that the explicit formula is true for $n = 35$. Use this assumption to show that the formula also is true for $n = 36$.

1.9 Consider the inequality $(n-3)^2 \leq 3n$, where n is a natural number.

a. Assume that this inequality is true for $n = 7$. Use this assumption to show that it also is true for $n = 8$.

b. Graph the sequences $t_n = (n-3)^2$ and $t_n = 3n$ on the same coordinate system.

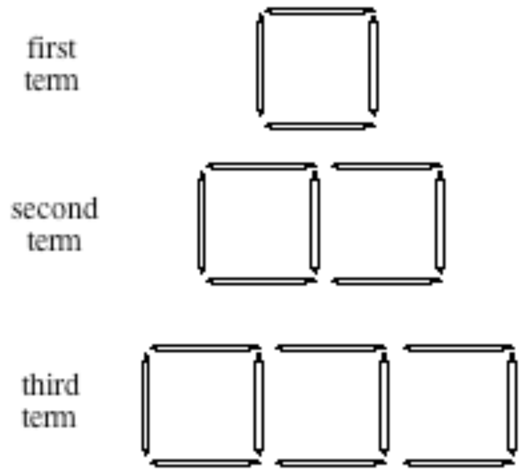
c. Make a conjecture about the set of natural numbers for which the inequality is true.

1.10 Consider the conjecture: "The quantity $2^n - 2$ is divisible by n whenever n is a positive odd integer."

a. Find a value of n that supports this conjecture.

b. Does the evidence you provided in Part **a** constitute a proof? Explain your response.

1.11 The following diagram shows a sequence of three figures constructed with toothpicks.



a. Describe the process required to create successive terms in this sequence.

b. Develop a formula for the number of toothpicks needed to construct a figure with n congruent squares.

c. Assuming that the formula is true for $n = 50$, show that it also is true for $n = 51$.

* * * * * * * * * *

Activity 2

As you observed in Activity **1**, many conjectures can be verified for a finite number of cases. However, this does not necessarily prove that a conjecture is true for all cases. In this activity, you use what you have learned to investigate a proof by mathematical induction.

Discussion 1

a. Consider an endless row of dominos standing on end, each one slightly behind another. Describe how this arrangement ensures that if the first domino in the row is tipped over, then:

1. the millionth domino will fall
2. the rest of the dominoes will continue to fall as well.

b. Describe the results in Part **a** if the fifth domino in the row is tipped over instead of the first.

c. Consider a non-empty subset T of the natural numbers with the following property: for any natural number that is in T, the next consecutive natural number also is in T.

Explain how the "domino effect" described in Part **a** guarantees that T contains all natural numbers greater than the least natural number in T.

Mathematics Note

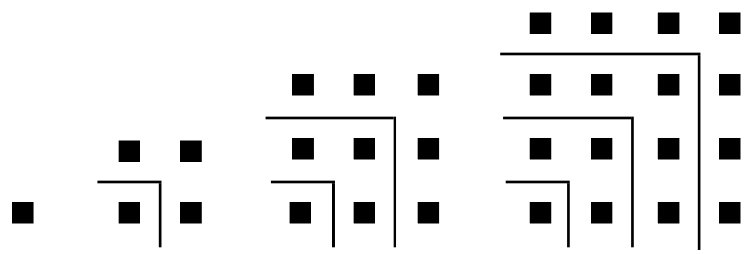
The principle of **mathematical induction** can be described as follows:

Suppose that for any natural number n, $P(n)$ is a mathematical statement involving n. If,

- $P(1)$ is true, and
- whenever k is a natural number such that $P(k)$ is true, $P(k+1)$ is also true

then $P(n)$ is true for all natural numbers n**.**

For example, consider the following conjecture: "The square of each natural number n is the sum of the first n odd numbers." Figure **4** shows a geometric representation of this conjecture for $n = \{1,\ 2,\ 3,\ 4\}$.



1   1 + 3   1 + 3 + 5   1 + 3 + 5 + 7

Figure 4: Geometric depiction of square numbers

The numbers of dots in the terms of this sequence are 1, 4, 9, and 16, respectively, or $1^2,\ 2^2,\ 3^2$, and 4^2. Since the sum of the first n odd numbers can be represented as the series $S_n = 1 + 3 + 5 + \cdots + (2n-1)$, the conjecture can be expressed as follows: $1 + 3 + 5 + \cdots + (2n-1) = n^2$**.**

This conjecture can be proven true for all natural numbers n using mathematical induction, as described below.

- Show $P(1)$ is true:

$$S_1 = 2(1) - 1 = 1 = 1^2$$

- Showing that $P(1)$ implies that $P(2)$ is true may suggest a method for proving that $P(k)$ implies that $P(k+1)$ is true. In this case, $P(1)$ can be used to prove that $P(2)$ is true as follows:

$$\begin{aligned} S_2 &= S_1 + (2 \bullet 2 - 1) \\ &= 1 + (2 \bullet 2 - 1) \\ &= 2 \bullet 2 + 1 - 1 \\ &= 2 \bullet 2 \\ &= 2^2 \end{aligned}$$

- Let k be a natural number such that whenever $P(k)$ is true,

$$1 + 3 + 5 + \cdots + (2k - 1) = S_k = k^2$$

Use this assumption to prove that $P(k+1)$ is true.

$$\begin{aligned} S_{k+1} &= S_k + \big(2(k+1) - 1\big) \\ &= k^2 + \big(2(k+1) - 1\big) \\ &= k^2 - 1 + 2k + 2 \\ &= (k-1)(k+1) + 2(k+1) \\ &= (k - 1 + 2)(k+1) \\ &= (k+1)^2 \end{aligned}$$

Since it has been shown that $P(1)$ is true, and that if $P(k)$ is true, then $P(k+1)$ also is true, $P(n)$ is true for all natural numbers n.

d. **1.** Consider a non-empty subset T of the integers with the same property described in Part **c** of this discussion: for any integer that is in T, the next consecutive integer also is in T.

Could the principle of mathematical induction be used to show that a set contains all integers greater than the least integer in the set? Explain your response.

2. Could it be used to show that a set contains all real numbers greater than the least in the set? Explain your response.

e. Consider the following conjecture: "The inequality $2^{n+1} < 3^n$ is true for all natural numbers n." Could the principle of mathematical induction be used to prove this conjecture? Justify your response.

f. Describe how you could prove the following conjecture using a process similar to mathematical induction: "The inequality $2^{n+1} < 3^n$ is true for all natural numbers greater than 1."

Exploration

In Exploration **2** of Activity **1**, you examined the series $S_n = 4 + 6 + 8 + \cdots + (2n + 2)$ and suggested a possible formula for it. In the following exploration, you use mathematical induction to prove that $S_n = n(n + 3)$ for all natural numbers n.

a. In Activity **1**, you showed that the following conjecture is true for $n = 1$ and $n = 2$, as well as for some other natural numbers.

$$S_n = 4 + 6 + 8 + \cdots + (2n + 2) = n(n + 3)$$

Assume that this conjecture is true for any natural number k. Write the equation that is implied by this assumption.

b. Use the equation you wrote in Part **a** to show that if k is a natural number and $P(k)$ is true, then $P(k + 1)$ also is true.

Hint: Begin by adding the next term of the sequence, $\big(2(k + 1) + 2\big)$, to both sides of the equation. Then manipulate the right-hand side of the equation until it is equal to $(k + 1)\big((k + 1) + 3\big)$.

Discussion 2

a. How does manipulating the right-hand side of the equation in Part **b** of the exploration until it is equivalent to $(k + 1)\big((k + 1) + 3\big)$ verify that $P(k + 1)$ is true?

b. In Activity **1**, you verified that $P(1)$ is true. You also showed that if $P(1)$ is true, then $P(2)$ is true. Do these verifications, along with the steps in the exploration, constitute a proof that the conjecture is true for all natural numbers?

c. Consider the false conjecture: "The inequality $(n + 1)! > 2^{n+3}$ is true for all natural numbers n." How could this conjecture be disproved?

d. Describe the steps needed for a proof by mathematical induction.

e. How do the requirements for proof by mathematical induction guarantee that a conjecture is true for all natural numbers?

Assignment

2.1 Consider the following conjecture: $2+4+6+\cdots+2n=n(n+1)$, for all natural numbers n. Complete the following steps to prove, by mathematical induction, that this conjecture is true.

a. Show that $P(1)$ is true.

b. Show that your response to Part **a** implies that $P(2)$ also is true.

c. Suppose the conjecture is true for a natural number k. Write the equation that is implied if $P(k)$ is true.

d. Write the equation which is implied if $P(k+1)$ is true.

e. Prove that if $P(k)$ is true, then $P(k+1)$ also is true. Hint: Manipulate the right-hand side of the equation from Part **d**.

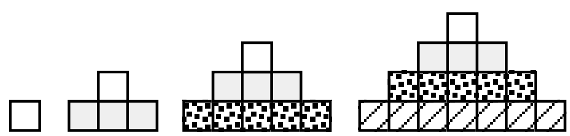
2.2 In Problem **1.6**, you examined the conjecture, "The quantity 3^n+1 is divisible by 2 for all natural numbers n." This can be restated as follows: "For all natural numbers n, $3^n+1=2p$, where p is some integer."

a. Show that $P(1)$ is true.

b. Show that your response to Part **a** implies that $P(2)$ also is true.

c. Continue using the principle of mathematical induction to prove that the conjecture is true for all natural numbers.

2.3 The diagram below shows the first four terms of a sequence generated by combining unit squares into triangular patterns.



a. Make a conjecture about an explicit formula for S_n, the number of unit squares in the nth term of the sequence.

b. Use mathematical induction to prove that your conjecture is true for all natural numbers n.

2.4 Consider the conjecture below for all natural numbers n:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Explain what is wrong with the following proof of this conjecture.

- As shown below, $P(1)$ is true:

$$S_1=1=\frac{1(1+1)}{2}$$

- Given that $P(1)$ is true, $P(2)$ also is true:

$$\begin{aligned}S_2&=S_1+2\\&=\frac{1(1+1)}{2}+2\\&=3\\&=\frac{2(2+1)}{2}\end{aligned}$$

- Assuming that $P(k+1)$ is true, it can be shown that $P(k)$ is true:

$$\begin{aligned}1+2+3+\cdots+n+(n+1)&=\frac{(n+1)\big((n+1)+1\big)}{2}\\1+2+3+\cdots+n+(n+1)&=\frac{(n+1)(n+2)}{2}\\1+2+3+\cdots+n+(n+1)&=\frac{n(n+1)+2(n+1)}{2}\\1+2+3+\cdots+n+(n+1)&=\frac{n(n+1)}{2}+(n+1)\\1+2+3+\cdots+n&=\frac{n(n+1)}{2}\end{aligned}$$

- Therefore, by the principle of mathematical induction, the conjecture is true for all natural numbers.

2.5. Use mathematical induction to prove that the following conjecture is true for all natural numbers n.

$$\begin{bmatrix}a&0\\0&b\end{bmatrix}^n=\begin{bmatrix}a^n&0\\0&b^n\end{bmatrix}$$

2.6 Use mathematical induction to prove that the following conjecture is true for all natural numbers n: “3 is a factor of n^3+5n+6.”

2.7 To prove a conjecture using mathematical induction, you must first prove that the statement $P(1)$ is true. However, some conjectures may be true only for a subset of the natural numbers (for example, $n \geq 2$).

In such cases, it may be possible to prove the conjecture for a particular subset of natural numbers using a form of induction in which the first statement is not $P(1)$. After showing that the conjecture is true for some initial natural number, the conjecture is proven true for the next natural number. From there, you can generalize and prove that if $P(k)$ is true, then $P(k+1)$ also is true.

a. Consider the following conjecture: $n! > 2^n$. This conjecture is not true for $P(1)$, since $1! \ngtr 2!$.

1. Find the first value of n for which the conjecture is true by graphing the sequences $t_n = n!$ and $t_n = 2^n$ on the same coordinate system for $n \geq 1$.
2. Show that the conjecture is true for the value of n you identified in Step **1**. This is the first step of the induction process.
3. Does the conjecture appear to be true for all values of n greater than the number you identified in Step **1**?

b. The second step of the induction process is to show that $P(5)$ is true, given that $P(4)$, or $4! > 2^4$, is true. This can be done as follows:

$$(4+1)! = 5! = 5 \bullet 4! > 2 \bullet 4! > 2^1 \bullet 2^4 = 2^5$$

So, $5! > 2^5$ is true.

Use the same method to show that if $P(k)$ is true, then $P(k+1)$ also is true. This is the final step of the induction process.

* * * * *

2.8 Use mathematical induction to prove that the following conjecture is true for all natural numbers n:

$$7 + 11 + 15 + \cdots + (4n+3) = 2n^2 + 5n$$

2.9 Consider the conjecture: $2^{(n+1)} < 3^n$. Complete the following steps to prove that the conjecture is true for all natural numbers greater than 1.

a. Show that the inequality is true for $n = 2$.

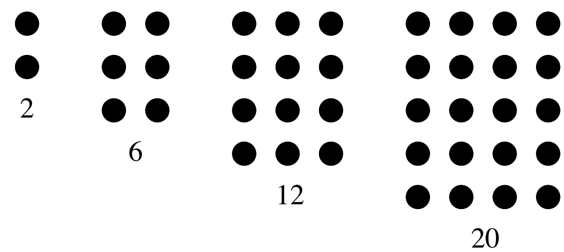
b. Use the fact that the inequality is true for $n = 2$ to show that it is also true for $n = 3$.

c. Write the inequality implied by the assumption that $P(k)$ is true.

d. Write the inequality that is implied if $P(k+1)$ is true.

e. Use the inequality from Part **c** to show that if $P(k)$ is true, then $P(k+1)$ also is true.

2.10 The diagram below shows a geometric model of a sequence. In each rectangular array of dots, the length is always 1 greater than the width.



a. Determine an explicit formula for a_n, the number of dots in each array.

b. Use mathematical induction to prove that your formula is true for all natural numbers n.

2.11 **a.** A diagonal of a polygon connects two non-adjacent vertices. As the number of sides of a polygon increases, the number of diagonals also increases. To explore the patterns created by this situation, complete the following table.

Term No. (n)	**No. of Sides in Polygon**	**No. of Additional Diagonals**	**Total No. of Diagonals (a_n)**
1	3		0
2	4		
3	5		
4	6		
5	7		

b. Determine a recursive formula for a_n, the total number of diagonals.

c. Either prove or disprove the conjecture that an explicit formula for the total number of diagonals is as follows:

$$a_n = \frac{(n+2)(n-1)}{2}$$

2.12 Consider the following argument and determine what is wrong with the proof.

Prove that $4n+3$ is divisible by 4 for all natural numbers n.

Assume the conjecture is true for some natural number k. This means $4k+3=4p$ for some integer p. So,

$$4(k+1)+3 = 4k+4+3 = 4k+3+4 = 4p+4 = 4(p+1)$$

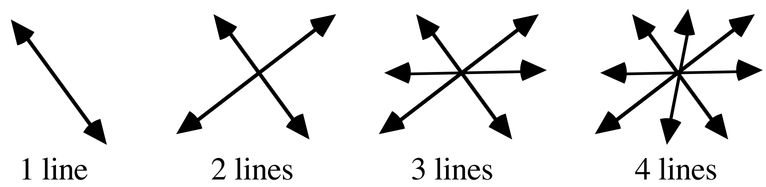
Since $4(p+1)$ is divisible by 4, $4(k+1)+3$ is also divisible by 4. Therefore, $4n+3$ is divisible by 4 for all natural numbers n.

* * * * * * * * * *

Summary Assessment

1. **Concurrent lines** are two or more lines that intersect at a common point. Two angles are **supplementary** if the sum of their measures is 180°.

a. Given n concurrent lines, how many pairs of supplementary angles are formed, if none of the angles are right angles? To identify a pattern, examine the diagram below and complete the following table.



No. of Lines (n)	Additional Pairs of Supplementary Angles	Total No. of Pairs of Supplementary Angles (a_n)
1		0
2	4	4
3		
4		

b. Describe the recursive pattern in the number of pairs of supplementary angles formed.

c. Use the pattern described in Part **b** to find the number of pairs of supplementary angles for five concurrent lines.

d. Write a recursive formula for a_n.

2. Prove the conjecture that the explicit formula for the number of pairs of supplementary angles for n concurrent lines is $a_n = 2n(n-1)$.

Module Summary

- The principle of **mathematical induction** can be described as follows:

  Suppose that for any natural number n, $P(n)$ is a mathematical statement involving n. If,

  - $P(1)$ is true, and
  - whenever k is a natural number such that $P(k)$ is true, $P(k+1)$ is also true

  then $P(n)$ is true for all natural numbers n**.**

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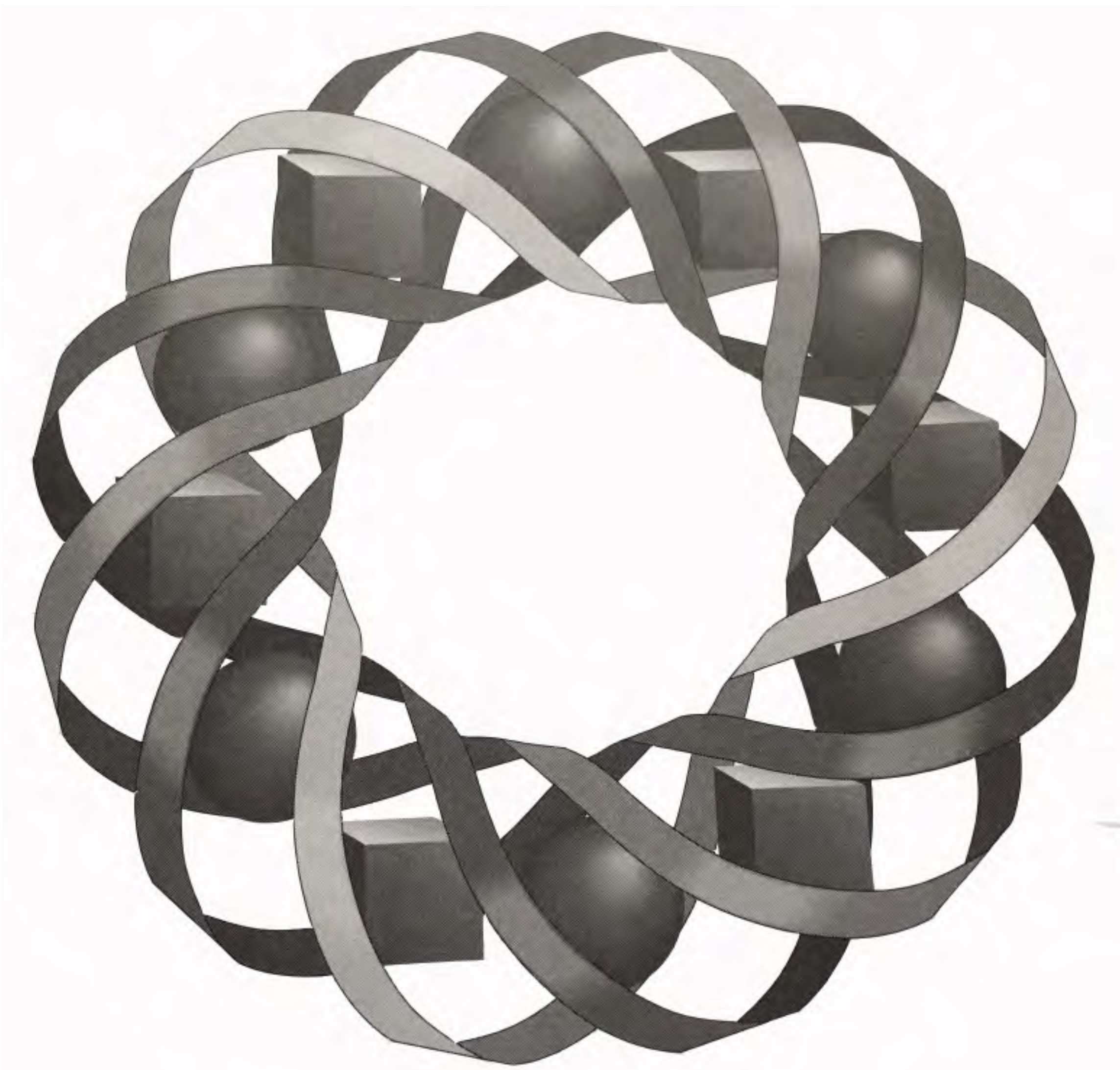
Cards and Binos and Reels, Oh My!



The turn of a card or the spin of a reel can add suspense to a game. In this module, you use your knowledge of randomness, probability, and combinations to explore how these games challenge players.

Sherry Horyna • Satinee Lightbourne • Teri Willard

Cards and Binos and Reels, Oh My!

Introduction

High Tech Games designs and markets video gaming machines. As shown in Figure **1**, the video screen for Cards of Chance displays 16 cards face down. These cards include four from each suit, placed in a random order.

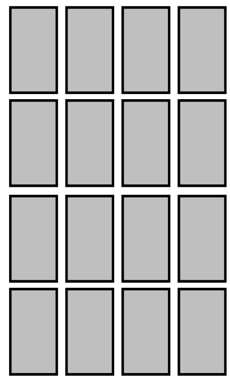


Figure 1: Arrangement of cards for Cards of Chance

Cards of Chance offers the possibility of winning at three different levels involving two, three, and four cards, respectively.

To play Cards of Chance, a player inserts tokens in the machine. When the opening screen appears, the player selects one card and turns it face up. The player then selects any one of the remaining cards and turns it face up. (This constitutes play at the two-card level.)

A player wins credits at each level if all cards turned face up are the same color. If the combination of cards does not meet this condition, the game is over. More credits may be earned at the third level than the second, and more at the fourth level than at the third.

Successful players at the two-card level may collect the credits they have earned or draw a third card, beginning play at the three-card level. Successful players at the three-card level may collect their credits or draw a fourth card. The game ends at the four-card level. Winning players may then collect their earned credits. If a player loses at any level, then all credits earned to that point are lost. Figure **2** shows some possible card combinations for three sample games:

	two-card level	three-card level	four-card level
Game 1 win at all levels	♦ ♥	♦ ♥ ♦	♦ ♥ ♦ ♥
Game 2 win at all levels	♦ ♦	♦ ♦ ♦	♦ ♦ ♦ ♦
Game 3 win at two-card level, loss at three-card level	♠ ♠	♠ ♠ ♦	play stopped with loss at three-card level

Figure 2: Card combinations in three sample games

Exploration

In this exploration, you investigate the probabilities of winning Cards of Chance at the two-card level.

a. Select 16 cards, 4 from each suit, from a deck of ordinary playing cards. To simulate the game, shuffle the 16 cards and draw 2 randomly.

b. Play 10 games of Cards of Chance, stopping each game at the two-card level and shuffling the cards between games. Record your wins and losses.

c. Determine the experimental probability of winning a game of Cards of Chance at the two-card level.

d. Compile the class results. Use the class data to determine the experimental probability of winning at the two-card level. **Note:** Save your data for use in Activity **1**.

e. List all the possible outcomes for Cards of Chance at the two-card level.

f. Determine the theoretical probability of each possible outcome.

g. Recall that for two events A and B, the theoretical probability of either A or B occurring can be found as follows:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

Given this fact, determine the theoretical probability of winning Cards of Chance at the two-card level.

Discussion

a. Describe how you determined the theoretical probabilities in Part **f** of the exploration.

b. What is the theoretical probability of winning at the two-card level? Explain your response.

Mathematics Note

According to the **law of large numbers**, as the number of trials increases, the experimental probability of an event tends to approach its theoretical probability.

For example, when tossing a fair coin, the theoretical probability that the coin lands heads up is 0.5. As the number of tosses increases, the experimental probability that the coin lands heads up will tend to get closer to 0.5.

c. How should increasing the number of times you play the game affect the experimental probability of winning?

Activity 1

Video gaming machines are designed to be durable, not portable. They are usually too big and too heavy for easy transportation. Imagine that you are a sales representative for High Tech Games. To make a sales pitch to potential buyers, you need a quick, creative simulation of the game that can be presented easily.

In this activity, you develop your own simulation of Cards of Chance and determine theoretical probabilities for the three-card level.

Exploration

a. Develop a simulation of Cards of Chance that could be used in a presentation to prospective buyers. The model must demonstrate the rules of the game, provide representative sample results, and be easy to transport.

b. Use your simulation to play 10 games at the three-card level. Use the results to calculate the experimental probability of winning at the three-card level.

c.
1. Compile the class results. Use this data to determine the experimental probability for winning at the three-card level.
2. Compare this probability with the experimental probability for winning at the two-card level obtained in the introduction.

Mathematics Note

Conditional probability is the probability of an event occurring, given that an initial event, or **condition**, has already occurred. The probability of event B occurring, given that event A has already occurred, is denoted $P(\mathrm{B}|\mathrm{A})$.

In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(\mathrm{A \text{ and } B}) = P(\mathrm{A}) \bullet P(\mathrm{B}|\mathrm{A})$$

For example, consider drawing two playing cards from a standard deck, one at a time, without replacement, and observing their colors. The cards may be either red (R) or black (B). The tree diagram in Figure **3** shows the probabilities in this situation.

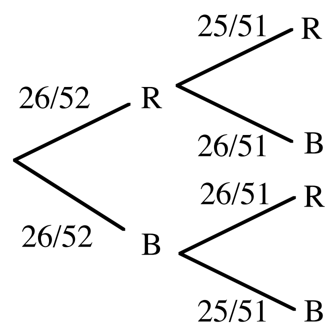


Figure 3: Tree diagram for drawing two cards without replacement

In this case, the conditional probability of obtaining a red card on the second draw, given that the first card is black, is $P(\mathrm{R}|\mathrm{B}) = 26/51$. The conditional probability of obtaining a red card on the second draw, given that the first card is also red, is $P(\mathrm{R}|\mathrm{R}) = 25/51$.

The probability of obtaining two red cards is:

$$P(\mathrm{RR}) = P(\mathrm{R}) \bullet P(\mathrm{R}|\mathrm{R}) = \frac{26}{52} \bullet \frac{25}{51} = \frac{25}{102}$$

d. Create a tree diagram showing each possible outcome and its probability for the three-card level of Cards of Chance. **Note:** Save your work for use in the assignment.

Mathematics Note

An experiment is **random** if individual outcomes are chance events.

A **random variable** *X* is a variable that takes on each of its possible values with a specific probability. Given possible values for *X* of x_1, x_2, …, x_k, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.

For example, rolling an ordinary six-sided die is a random experiment because the outcome of the roll is uncertain. If the outcomes of the experiment are assigned to the random variable *X*, then the possible values for *X* are 1, 2, 3, 4, 5, or 6. The probability of each outcome is $1/6$.

A **probability distribution** for a random variable *X* assigns probabilities $p_1, p_2, \ldots, p_k$ to the values x_1, x_2, …, x_k for *X*.

Table **1** shows the probability distribution of the random variable *X* when rolling a six-sided die.

Table 1: Probability distribution of random variable *X*

Value of *X* (x_i)	1	2	3	4	5	6
Probability (p_i)	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

e. Make a table that shows the probability distribution for all possible outcomes of Cards of Chance at the three-card level.

f. Calculate the theoretical probability of winning at the three-card level.

Discussion

a. Is the theoretical probability of winning Cards of Chance at the two-card level the same as winning at the three-card level? Explain your response.

b. What is the sum of all the probabilities in a probability distribution? What does this sum represent?

c. In Cards of Chance, the first draw and the second draw are not independent events. How does this affect the probabilities of drawing two cards of the same color?

d. How could you change the rules for Cards of Chance so that the game involves independent events?

e. What is the probability of each of the following events in the three-card level of Cards of Chance?

1. drawing a black card, given that the first two cards are black
2. drawing a black card, given that the first two cards are red
3. drawing a black card, given that the first two cards are different colors

Assignment

1.1 **a.** What is the theoretical probability of drawing three red cards at the three-card level of Cards of Chance?

b. Describe how you could use your tree diagram from the exploration to determine the probability in Part **a**.

1.2 **a.** Extend the tree diagram from the exploration to the four-card level of Cards of Chance.

b. Make a table that shows the probability distribution for all possible outcomes at the four-card level.

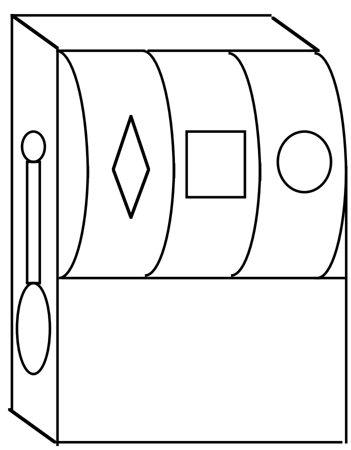
1.3 **a.** Determine the probability of drawing a fourth card of the same color, given that three cards of the same color were drawn at the three-card level.

b. Determine the probability of losing at the four-card level, given that three cards of the same color were drawn at the three-card level.

c. Determine the probability of winning at the four-card level, with no previous conditions given. Justify your response.

d. Determine the probability of losing at the four-card level, with no previous conditions given. Justify your response.

1.4 The Reel Game is another product of High Tech Games. As shown below, each machine has three spinning reels. Each reel has three equally likely symbols—a diamond, a square, and a circle. A player wins by matching the symbols on all three reels when the reels stop.



a. Draw a tree diagram that shows all the possible outcomes of the Reel Game. Label each branch with the appropriate probability.

b. Find the probability of getting a diamond on the second reel.

c. Find the probability of getting a diamond on the third reel.

d. Are getting a diamond on the second reel and a diamond on the third reel independent events? Explain your response.

e. Determine the probability of winning the Reel Game described with three diamonds, given that two diamonds have already appeared on the first two reels.

f. How does $P(3 \text{ diamonds} | 2 \text{ diamonds})$ compare to the probability of getting a diamond on the third reel?

1.5 **a.** Consider a game that involves drawing two cards from a standard deck, one at a time with replacement. Is obtaining an ace on the first draw independent of obtaining an ace on the second draw?

b. For two independent events A and B, $P(\text{B}|\text{A}) = P(\text{B})$. How does this fact support your response to Part **a**?

c. If each card is not replaced after it is drawn, does the game still involve independent events? Explain your response.

1.6 Given rules similar to those of the Reel Game in Problem **1.4**, determine the probability of winning each of the following:

a. a game with two spinning reels, each having two symbols, a diamond and a triangle

b. a game with three spinning reels, each having three symbols, a square, a diamond, and a circle.

1.7 Consider a game that involves rolling two standard dice. If the sum of the faces is greater than 10, the player earns 10 points. If the sum equals 7, the player earns 5 points. Any other roll of the dice is worth 0 points. Show the probability distribution for the random variable S, where S represents the number of points won.

* * * * *

1.8 A history test contains 5 multiple-choice questions. Each question has 5 possible responses, only 1 of which is correct. To pass this test, students must answer at least 4 of the questions correctly.

If a student selects responses at random, what is the probability of each of the following events?

a. All 5 questions are answered correctly.

b. None of the questions are answered correctly.

c. Exactly 4 of the questions are answered correctly.

d. The student passes the test.

e. The student does not pass the test.

1.9 Consider a game in which 26 cards, each labeled with a different letter of the alphabet, are placed in a container and mixed thoroughly. If cards are drawn without replacement, what is the probability of each of the following?

a. When three cards are drawn, all three cards are vowels.

b. When five cards are drawn, all five cards are consonants.

c. Given that two vowels have been drawn, the next card is a vowel.

d. Given that three vowels and two consonants have been drawn, the next card is a vowel.

e. When seven cards are drawn, they include the letters of the word *fortune*.

1.10 Louis has four $1 bills, three $5 bills, and two $10 bills in his pocket. If he randomly draws two bills from his pocket, one at a time without replacement, what is the probability that the total is $15?

* * * * * * * * * *

Activity 2

To expand its share of the market, High Tech Games is developing another kind of gaming machine, the binostat. As shown in Figure **4**, a binostat game involves dropping a ball through a triangular grid. At each level of the grid, the ball deflects to the right or left of a peg with equal probability, until it enters a numbered slot. For example, the ball in Figure **4** passed through the grid and landed in slot 2.

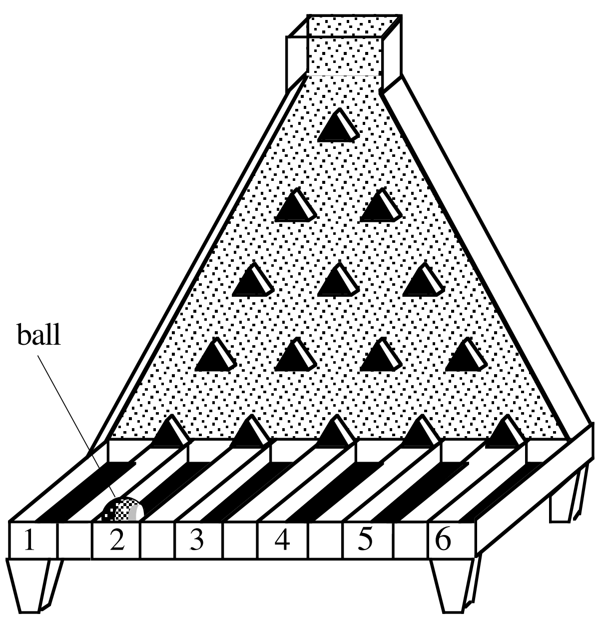


Figure 4: A binostat game

To play a binostat game, a player inserts tokens in the machine and selects a slot number. The player then drops the ball into the top of the machine and watches as it falls through the grid. If the ball lands in the selected slot, the player wins the game and earns credits. If the ball lands in any other slot, the game is over and the player loses the tokens or credits played.

Exploration

Figure **5** shows a simple binostat with only one level. In this binostat, a ball falling through the grid deflects either to the right or to the left. If it deflects to the left, it ends up in slot 1. If it deflects to the right, it ends up in slot 2.

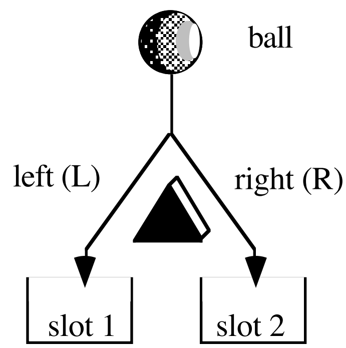


Figure 5: A one-level binostat

a. In the one-level binostat game shown in Figure **5**, a ball falling through the grid has only two possible paths: right (R) or left (L).

For a ball to reach slot 1 in the two-level binostat game shown in Figure **6**, the ball must deflect left and then left again (LL). To reach slot 2, a ball can deflect left and then right (LR), or it can deflect right and then left (RL). To reach slot 3, the ball must deflect right and then right again (RR).

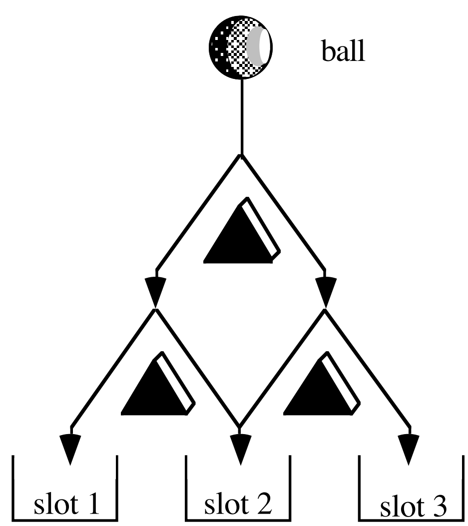


Figure 6: A two-level binostat

In other words, there are a total of four ways—LL, LR, RL, and RR—to reach the three slots. Two of these paths lead to slot 2.

List all the paths in a binostat game with each of the following:

1. three levels
2. four levels.

b. At each junction in the binostat, the probability of going left is equal to the probability of going right. Determine the probability of the ball landing in each slot in a one-level binostat. Express each probability as a fraction in which the denominator equals the number of possible paths.

c. Repeat Part **b** for two-level, three-level, and four-level binostat games.

d. Obtain a template of the binostat paths shown in Figure **7**.

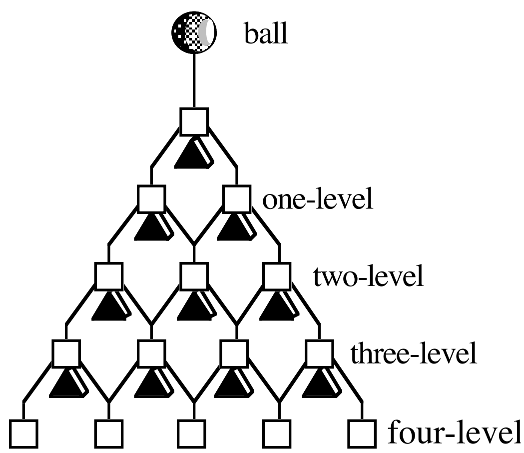


Figure 7: Number of paths for binostat games

1. Fill in each square with the number of paths that can be taken to get to that particular slot.
2. Describe any patterns you observe in this figure.

e. In a one-level binostat, there are a total of 2 possible paths. In a two-level binostat, there are a total of 4 possible paths.

1. Use your results from Part **d** to express the total number of paths as a power of 2 for one-level, two-level, three-level, and four-level binostats.
2. Describe any relationship you observe between the total number of paths and the corresponding level of the binostat.

f. Use the patterns you observed in Parts **d** and **e** to extend the template through the 10th level.

Mathematics Note

The diagram you developed in the exploration is part of a pattern of numbers known as **Pascal's triangle**. Figure **8** shows the first five rows of Pascal's triangle.

				1					Row 0
			1		1				Row 1
		1		2		1			Row 2
	1		3		3		1		Row 3
1		4		6		4		1	Row 4

Figure 8: A portion of Pascal's triangle

Discussion

a. Describe how you could generate any row of Pascal's triangle, given the previous row.

b. Explain how you could use your diagram to find the number of paths a ball can take to land in a particular slot of a binostat game.

c. Explain how you could use your diagram to find the total number of paths in a binostat game of any level.

d. In Part **e** of the exploration, the total number of paths in a given row was expressed as a power of 2. Explain why 2 was chosen as the base.

e. Explain how you could use your diagram to find the probability of a ball landing in slot x of an n-level binostat.

f. Why is a 1-level binostat like tossing a fair coin?

Assignment

2.1 Consider a three-level binostat game.

a. How many slots does this game have?

b. How many different paths are possible for the ball?

c. **1.** How many paths can the ball take to land in slot 2?

2. What is the probability of the ball landing in slot 2?

2.2 **a.** What is the sum of the theoretical probabilities of a ball landing in the slots of:

1. a one-level binostat?

2. a two-level binostat?

3. a three-level binostat?

b. Explain your responses to Part **a**.

2.3 **a.** How could you use exponential notation to express the theoretical probability of a ball landing in slot 1 in a one-, two-, or three-level binostat game?

b. What is the theoretical probability of a ball landing in slot 1 of a 10-level binostat?

2.4 **a.** Complete the probability distribution table below for each slot in a three-level binostat game.

Slot	1	2	3	4
Probability				

b. Can you use Pascal's triangle to create a probability distribution for each slot in an n-level binostat game? If so, how?

2.5 Create a probability distribution table for a six-level binostat.

* * * * *

2.6 An industrial plant has 8 robots for use on its assembly line. Six of the robots must be functioning for the plant to be operational.

a. If "I" represents an inoperable robot and "O" represents an operable robot, then one possible situation can be written as [OIOOIOOI]. Find the total number of situations that can occur.

b. Which row of Pascal's triangle corresponds with your response to Part **a**?

c. Use Pascal's triangle to find the number of outcomes in which exactly 6 out of 8 robots are operational.

d. Use Pascal's triangle to find the number of outcomes in which at least 6 out of 8 robots are operational.

2.7 Consider an experiment in which 10 supermarket customers are selected at random and asked to taste 2 different brands of tomato soup. Each person must state a preference for one brand or the other. The company sponsoring the taste test would like to claim that 80% of consumers prefer their brand over their competitor's brand.

a. Find the number of possible outcomes for the experiment. Which row of Pascal's triangle corresponds with this total?

b. Use Pascal's triangle to find the number of ways that exactly 8 of the 10 customers could choose a specific brand.

c. Use Pascal's triangle to find the number of ways that at least 8 of 10 customers could choose a specific brand.

* * * * * * * * * *

Activity 3

Although Pascal's triangle can be used to calculate the probabilities of winning and the numbers of possible paths in binostat games, High Tech Games wants a mathematical model to determine the probabilities for its machines.

Mathematics Note

A **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure. (The prefix *bi-* means "two.")
- The probability of a success remains constant from trial to trial.
- The total number of successes is observed.

For example, consider an experiment that consists of tossing a fair coin 3 times and observing the number of heads that occur. In this case, there is a fixed number of trials, 3. For each trial, there are only two possible outcomes: either heads or tails. The probability that heads occurs remains constant for each toss, and the result of one toss does not influence the result of any other. Therefore, this represents a binomial experiment.

Discussion 1

a. Explain whether or not each of the following represents a binomial experiment.

1. A fair coin is flipped six times. The total number of heads is recorded.
2. Two cards are drawn from a standard deck of playing cards, one at a time with replacement. The total number of diamonds is recorded.
3. Two cards are drawn from a standard deck of playing cards, one at a time without replacement. The total number of diamonds is recorded.

b. Given $P(\text{success})$ for any trial in a binomial experiment, describe how to find $P(\text{failure})$.

Exploration

Consider a ball falling through the four-level binostat in Figure **9**. Before reaching a slot, the ball makes a total of four deflections. At each peg, the ball has an equal probability of deflecting to the right or to the left.

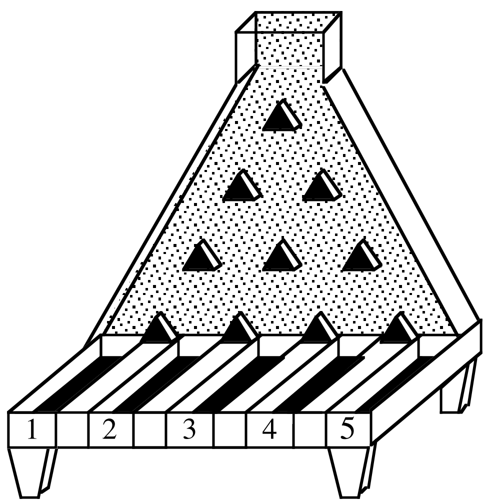


Figure 9: Four-level binostat game

Using Pascal's triangle, you can determine that there is one possible path to slot 1, four possible paths to slot 2, six possible paths to slot 3, four possible paths to slot 4, and one possible path to slot 5.

a. To land in slot 2, the ball can make only one deflection to the right out of a total of four deflections. The four possibilities are RLLL, LRLL, LLRL, and LLLR.

Recall that an arrangement of r items out of n items, where order is not important, is a **combination**. Use the notation $C(n,r)$, where n is the total number of deflections and r is the number of deflections to the right, to describe the number of paths the ball can take to land in slot 2.

b. List the possible paths the ball can take to land in each of the other slots of a four-level binostat. What patterns exist between the number of deflections to the left or right for each path to a given slot?

c. Use combination notation to describe the number of paths possible for each of the following slots in a four-level binostat.

1. slot 1
2. slot 3
3. slot 4
4. slot 5

d. As noted previously, there are four possible paths to slot 2. Each path has a total of four deflections, with only one deflection to the right: RLLL, LRLL, LLRL, and LLLR.

Since the probability of a deflection to the left or to the right is $1/2$, the probability that the ball will take path RLLL can be found as follows:

$$\begin{aligned} P(\text{RLLL}) &= P(\text{R}) \bullet P(\text{L}) \bullet P(\text{L}) \bullet P(\text{L}) \\ &= [P(\text{R})]^1 \bullet [P(\text{L})]^3 \\ &= \left(\frac{1}{2}\right)^1 \bullet \left(\frac{1}{2}\right)^3 = \frac{1}{16} \end{aligned}$$

Similarly, the probability for each of the other three paths to slot 2 is also $[P(\text{R})]^1 \bullet [P(\text{L})]^3$, or $1/16$. Because there are 4 possible paths, each with a probability of $[P(\text{R})]^1 \bullet [P(\text{L})]^3$, the probability of the ball landing in slot 2 can be expressed as follows:

$$\begin{aligned} P(\text{slot } 2) &= 4 \bullet [P(\text{R})]^1 \bullet [P(\text{L})]^3 \\ &= 4 \bullet \left(\frac{1}{2}\right)^1 \bullet \left(\frac{1}{2}\right)^3 \\ &= \frac{4}{16} = \frac{1}{4} \end{aligned}$$

Using the process described for slot 2, determine the probabilities for slots 1, 3, 4, and 5.

e. Use combinations and exponents to write a formula for the probability of a ball ending in slots 1, 2, 3, 4, and 5.

Discussion 2

a. How can you use Pascal's triangle to determine the number of possible paths that include *r* deflections to the right in an *n*-level binostat?

b. How can you use the notation $C(n,r)$ to represent any element of Pascal's triangle?

Mathematics Note

A **binomial distribution** is the probability distribution associated with repeated trials of a binomial experiment.

The probability of obtaining r successes in n trials can be determined using the following formula, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

For example, consider a binomial experiment that involves rolling a fair die. In this experiment, rolling a six is designated a success while any other roll is designated a failure. In 7 trials, the probability of getting 3 sixes is:

$$P(3 \text{ successes}) = C(7,3) \bullet \left(\frac{1}{6}\right)^3 \bullet \left(\frac{5}{6}\right)^4 = 35 \bullet \frac{625}{279{,}936} \approx 8\%$$

c. Using the process described in Part **e** of the exploration, how can you use combinations to express the probability of a ball landing in a slot that requires each of the following?

1. r deflections to the right in an n-level binostat
2. r deflections to the left in an n-level binostat

d. Can you use the formula for a binomial distribution to calculate the probability of getting at least 3 twos with 5 rolls of a fair die? If so, how?

Assignment

3.1 Which of the following are binomial experiments? Justify your responses.

- **a.** A die is tossed three times. After each toss, the number that appears on the top face is recorded.
- **b.** Two playing cards are drawn from a deck of 16 cards, one at a time, without replacement. After each card is drawn, its color is recorded.
- **c.** A coin is flipped until it lands heads up.

3.2 Use the formula for a binomial distribution to calculate the probabilities of a ball landing in each of the slots of a five-level binostat game. Check your answers using Pascal’s triangle.

3.3 Consider a reel game with five reels. Each reel has the same five symbols—a diamond, a club, a heart, a spade, and an automobile. Each of the five symbols is equally likely to appear on each reel.

- **a.** If you wanted to get an automobile on each reel, how would you define a “success” in this game?
- **b.** What is the probability of a “success” on any one reel?
- **c.** What is the probability of a “failure” on any one reel?

3.4 Consider the reel game described in Problem **3.3**.

a. What is the probability of getting a diamond on every reel?

b. What is the probability of getting a heart on exactly three of the reels?

c. What is the probability of getting a heart on at least three reels?

d. The middle reel on one machine always sticks, displaying a diamond. Calculate the probability of getting three diamonds on this machine.

3.5 Binomial expressions such as $(x+y)^3$ can be expanded using the distributive property. For example, the expanded forms of several binomial expressions are shown below:

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

a. Examine the expanded form of each of the binomial expressions above. Describe the sum of the exponents on each term in the expanded form with respect to the exponent of the original expression.

b. For each expression, describe the relationship between the coefficients of each term and Pascal's triangle.

c. Rewrite the expanded form of $(x+y)^4$ using the notation for combinations, $C(n,r)$.

d. Use your results in Parts **a–c** to suggest an expanded form for the general binomial $(x+y)^n$.

* * * * *

3.6 Experimental data shows that when a thumbtack is tossed in the air, it will land point up 75% of the time, and point down 25% of the time.

a. Does tossing a thumbtack in the air and observing the outcome represent a binomial experiment?

b. What is the probability that when a tack is tossed 10 times, it lands point down exactly 6 times?

3.7 A married couple plans to have four children. Assuming that the probability that each child is a girl is 0.5, what is the probability that their four children will include:

a. 3 boys and 1 girl?

b. 4 girls?

c. 2 boys and 2 girls?

3.8 Consider a test which contains 10 multiple-choice questions. Each question offers 4 possible choices for the answer, but only one of them is correct. To pass this test, students must obtain at least 6 correct answers. If a student guesses the answer to each question at random, what is the probability of passing the test?

3.9 A certain airplane has two engines. The probability that any one engine will fail during a transcontinental flight is 0.001. Assuming that the event of one engine failing is independent of the other engine failing, determine the probability of each of the following.

a. a transcontinental flight will be completed without engine failure

b. both engines will fail

c. at least one engine will fail.

3.10 On January 28, 1986, the space shuttle *Challenger* exploded shortly after launch. The cause of this tragedy was traced to the failure of 1 of the 6 sealed joints on the booster rockets. Assuming that each joint has a 0.977 success rate, and that the failure of any one joint is independent of the failure of any of the others, calculate the probability that at least 1 of the 6 joints fails.

3.11 According to the **binomial theorem**, the expansion of $(x+y)^n$, where n is a whole number, is the sum of $(n+1)$ terms, as follows:

$$(x+y)^n = C(n,n)x^n + C(n,n-1)x^{n-1}y + C(n,n-2)x^{n-2}y^2 + \cdots + C(n,0)y^n$$

Use the binomial theorem to expand each of the binomials below.

a. $(x-y)^6$

b. $(x^2+3y)^3$

c. $(2xy-5)^4$

* * * * * * * * * *

Activity 4

The video gaming machine industry is highly competitive. Besides designing games that keep players interested and entertained, High Tech Games must also ensure that its products generate a profit.

Whenever someone uses a gaming machine, there are costs involved. For the player, the cost is the price required to play. For the machine's owners, the costs include paying off the credits earned by successful players.

Mathematics Note

The **expected value** or **mean** of a random variable *X*, denoted $E(X)$, is the sum of the products of each possible value of *X* and its corresponding probability.

$$E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k$$

In mathematics, a sum is often denoted using the Greek letter *sigma*, Σ. Using this notation, the expected value of *X* can be written as follows:

$$E(X) = \sum_{i=1}^{k} x_i p_i$$

This indicates that the values of $x_i p_i$ are added as *i* increases from 1 to *k*.

For example, consider a game that consists of rolling a fair die. A roll of one is worth 1 point, a roll of two is worth 2 points, and so on. The expected value for this game can be calculated as shown below:

$$E(X) = \sum_{i=1}^{6} x_i p_i = 1 \bullet \frac{1}{6} + 2 \bullet \frac{1}{6} + 3 \bullet \frac{1}{6} + 4 \bullet \frac{1}{6} + 5 \bullet \frac{1}{6} + 6 \bullet \frac{1}{6} = \frac{21}{6} = 3.5$$

In this case, an exact value of 3.5 cannot be obtained on any single roll of the die. However, according to the law of large numbers, if the die is rolled many times, the mean of all the rolls is likely to be close to 3.5.

Exploration

In the following exploration, you investigate a five-level binostat that costs 100 tokens to play.

a. To players, the expected value for a game is its expected payoff. This can be determined by multiplying the payoff for each outcome by its theoretical probability, then finding the sum of these products.

The payoffs (in tokens) for each slot in the five-level binostat are shown in Table **2** below. From the exploration in Activity **2**, the probabilities of the ball landing in slot 1, 2, 3, 4, 5, or 6 are 1/32, 5/32, 5/16, 5/16, 5/32, or 1/32, respectively. Determine the expected payoff for this game.

Table 2: Payoffs for five-level binostat game

Slot	**1**	**2**	**3**	**4**	**5**	**6**
Payoff	200	20	10	10	20	200

b. In a **fair game**, the expected payoff equals the cost of playing the game. The game described in Part **a** is not a fair game. Use a spreadsheet to determine a set of payoffs that make this five-level binostat game a fair one.

c. In High Tech's home market, state law requires a 70% minimum return to players. Determine a set of payoffs that satisfies the state requirements, while providing a profit for the owners of the gaming machines.

Discussion

a. For the binostat game represented in Table **2**, the payoffs for landing in slots 1 or 6 are much higher than those of the other slots. What is the relationship between the probabilities for each of the six outcomes and their corresponding payoffs?

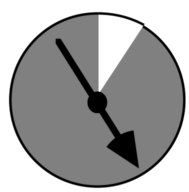
b. Is there more than one way to assign payoffs to make the game fair? Explain your response.

c. What changes did you make in your payoff scheme to turn your fair game into a legal and profitable one?

d. Describe a payoff scheme that might attract players, yet still make the game profitable for owners.

Assignment

4.1 The spinner below is used in a carnival game. If the arrow lands in the unshaded sector, the player receives 50 points. If it lands in the shaded sector, the player receives 5 points. The central angle of the unshaded sector measures 30°. Determine the expected value (in points) for this game.



4.2 In most games, outcomes with lower probabilities have higher payoffs than outcomes with higher probabilities. This can make the game attractive to players, yet still profitable.

a. Suppose that the payoff for the first slot of a six-level binostat is 457 tokens, as shown in the table below. Determine the payoffs for each remaining slot using an inverse relationship between the probability and the payoff. For example, the probability of landing in slot 2 is 6 times the probability of landing in slot 1. To determine the payoff for slot 2, multiply the payoff for slot 1 by $1/6$, or $457 \bullet 1/6 \approx 76$.

Slot	1	2	3	4	5	6	7
Probability	1/64	6/64	15/64	20/64	15/64	6/64	1/64
Payoff	457	76					

b. What should be the cost to play to make this a fair game?

4.3 Consider a six-level binostat that costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs that would make this game both legal and profitable. As in Problem **4.2**, the payoff for each slot should be inversely related to its probability.

4.4 Suppose that Cards of Chance costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs for the entire game—from the two-card level to the four-card level—that would make it both legal and profitable.

4.5 Consider a three-level binostat game that costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs that would make this game both legal and profitable. The payoff for each slot should be inversely related to its probability.

4.6 As part of a market analysis for High Tech Games, you have been asked to compare Cards of Chance, reel games, and binostat games in terms of complexity, player interest, and potential profitability. Write a summary of your comparisons.

* * * * *

4.7 A three-reel game costs 50 tokens to play. Each reel has three distinct symbols, and players must match at least two symbols to earn tokens. If state law requires a minimum of 70% return for players, determine a set of payoffs that would make this game both legal and profitable. The payoff for each slot should be inversely related to its probability.

4.8 An insurance company offers an accident/illness policy with the following benefits.

- If a policyholder becomes seriously ill during the year, the company will pay $1000.
- If a policyholder has an accident, the company will pay $2000.
- If a policyholder has an accident and becomes seriously ill, the company will pay $7500.

The annual premium for this policy is $200.

a. According to the company's statistics, the probability of becoming seriously ill in any one year is 0.06, while the probability of having an accident is 0.04. Assuming that these are independent events, determine the probability of each of the following:

1. a policyholder does not become ill or have an accident
2. a policyholder becomes ill, but does not have an accident
3. a policyholder does not become ill, but does have an accident
4. a policyholder becomes ill and has an accident.

b. What is the company's expected annual profit per policyholder?

* * * * * * * * * *

Summary Assessment

High Tech Games is developing a new video gaming machine called Flip-o-Mania. To begin the game, players insert tokens in the machine. The game then electronically "flips" a coin. The player continues flipping coins electronically. For play to continue, each successive coin flipped must match the first coin. When a coin appears that does not match the others, the game is over. The player wins credits based on the number of coins flipped successfully.

Although High Tech has not yet determined how many coins players should have the option of flipping, the company plans to market a test version of Flip-o-Mania soon. As director of marketing, you must make a presentation at next week's board meeting about the new machine.

1. Design a simulation for Flip-o-Mania that you could use to demonstrate how the game works.

2. **a**. An initial study suggests that players should be offered the chance to flip at least 4 coins, but no more than 10. If each flip is considered a level in the game, determine at what level Flip-o-Mania should conclude.

b. Determine the probabilities of winning at each level.

3. If state law requires a 70% minimum return to players, determine an appropriate set of payoffs for Flip-o-Mania, given that the cost to play is 10 tokens.

4. Write a report that defends your recommendations and findings in Problems **1–3**. Your report should include a discussion of probability distribution tables, binomial experiments, binomial probabilities, and expected value.

Module Summary

- According to the **law of large numbers**, as the number of trials increases, the experimental probability of an event tends to approach its theoretical probability.
- **Conditional probability** is the probability of an event occurring, given that an initial event, or **condition**, has already occurred. The probability of event B occurring, given that event A has already occurred, is denoted $P(\mathrm{B}|\mathrm{A})$.
- In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(\mathrm{A\ and\ B}) = P(\mathrm{A}) \bullet P(\mathrm{B}|\mathrm{A})$$

- An experiment is **random** if individual outcomes are chance events.
- A **random variable** X is a variable that takes on each of its possible values with a specific probability. Given possible values for X of x_1, x_2, …, x_k, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.
- A **probability distribution** for a random variable X assigns probabilities $p_1, p_2, \ldots, p_k$ to the values x_1, x_2, …, x_k for X.
- The triangular pattern of numbers shown below is called **Pascal's triangle**:

$$\begin{array}{ccccccccccl}
 & & & & & 1 & & & & & \text{Row 0} \\
 & & & & 1 & & 1 & & & & \text{Row 1} \\
 & & & 1 & & 2 & & 1 & & & \text{Row 2} \\
 & & 1 & & 3 & & 3 & & 1 & & \text{Row 3} \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \text{Row 4} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

- A **binomial experiment** has the following characteristics:
  1. It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
  2. The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
  3. Each trial has only two possible outcomes: a success or a failure. (The prefix *bi-* means "two.")
  4. The probability of a success remains constant from trial to trial.
  5. The total number of successes is observed.
- A **binomial distribution** is the probability distribution associated with repeated trials of a binomial experiment. The probability of obtaining *r* successes in *n* trials can be determined using the following formula, where *p* is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

- The **expected value** or **mean** of a random variable *X*, denoted $E(X)$, is the sum of the products of each possible value of *X* and its corresponding probability.

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_kp_k$$

  In mathematics, a sum is often denoted using the Greek letter *sigma*, Σ. Using this notation, the expected value of *X* can be written as follows:

$$E(X) = \sum_{i=1}^{k} x_ip_i$$

  This indicates that the values of x_ip_i are added as *i* increases from 1 to *k*.
- In a **fair game**, the expected payoff equals the cost of playing the game.

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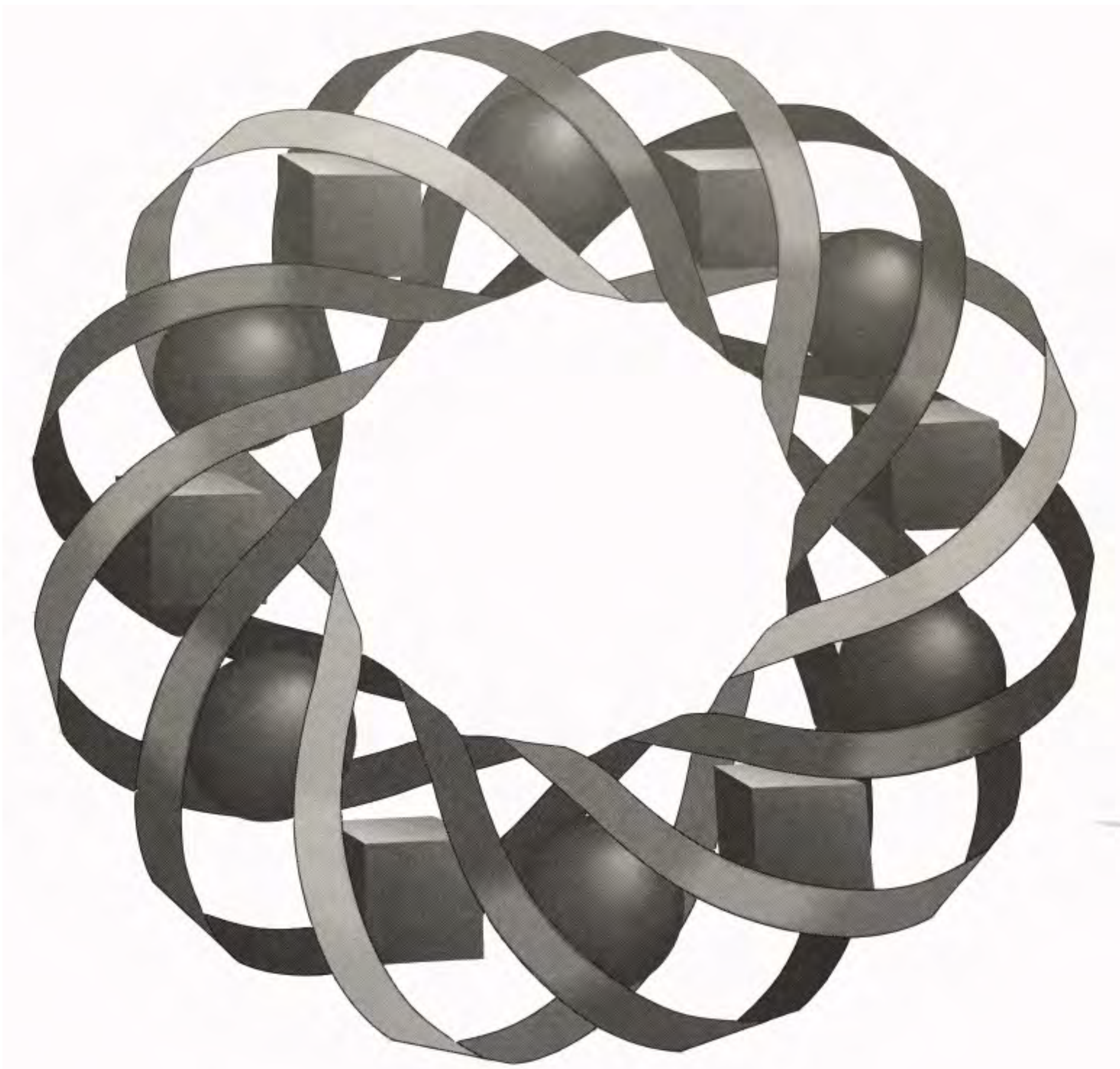
An Imaginary Journey Through the Real World



Can you find the square root of a negative number? In this module, you discover that you can!

Lee Brown • John Freal • Anne Watkins

An Imaginary Journey Through the Real World

Introduction

You are familiar with many different sets of numbers: the natural numbers, the whole numbers, the integers, the rational numbers, and the real numbers. Each set was developed as a social—or mathematical—need arose.

Written symbols for the natural numbers 1, 2, 3, … are at least as old as the pyramids. Around 200 A.D., the number 0 was introduced in India to represent an empty column in a counting board that resembled an abacus. The set of numbers consisting of 0 and the natural numbers make up the set of whole numbers.

The need for negative numbers emerged in China in the 6th and 7th centuries, though they were not used in Europe until the 15th century. Negative numbers were useful for representing quantities above or below a given level. The natural numbers, their opposites (negatives), and zero make up the set of integers.

The ancient Greeks introduced the positive rational numbers to represent fractional parts of a quantity. The term *rational* was coined to describe numbers that are ratios of two natural numbers, where the denominator is not 0. During this time, the Greeks believed that rational numbers could be used to describe exactly all measurements in the physical world.

This hypothesis about rational numbers was incorrect. When Greek mathematicians tried to find a rational number to describe the length of the diagonal of a square like the one shown in Figure **1**, they realized that no such rational number existed.

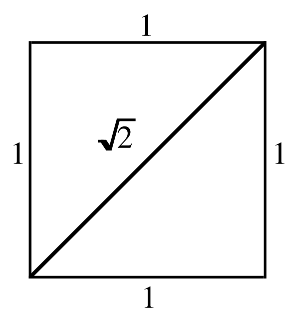


Figure 1: A square and one of its diagonals

As a result, the Greeks extended their number system to include irrational numbers. Eventually, the sets of rational and irrational numbers were combined to form the set of real numbers.

Activity 1

Is the set of real numbers sufficient to describe everything in the physical—or mathematical—world? In this module, you will investigate situations in which another set of numbers is useful.

Discussion

a. **1.** What are the solutions to the equation $x^2 - 2 = 0$?

2. How are these solutions related to the factors of the polynomial $x^2 - 2$?

b. In general, the **difference of squares** $x^2 - a^2$ has two factors: $(x - a)$ and $(x + a)$. In other words, $x^2 - a^2$ can be factored as $(x - a)(x + a)$, or $x^2 - a^2 = (x - a)(x + a)$.

Given this fact, what are the solutions to a polynomial equation of the form $x^2 - a^2 = 0$?

c. Are there any real-number solutions to the equation $x^2 + 1 = 0$? Explain your response.

Mathematics Note

The notation for the **imaginary unit *i***, where $i = \sqrt{-1}$ and $i^2 = -1$, was first introduced by Swiss mathematician Leonhard Euler (1707–1783). The adoption of i by Gauss in his classic *Disquisitiones arithmeticae* in 1801 secured its use in mathematical notation. This notation was generalized to define the square root of any negative number as: $\sqrt{-a} = \sqrt{-1} \bullet \sqrt{a} = i\sqrt{a}$ for any number $a > 0$.

For example, $\sqrt{-3} = \sqrt{-1} \bullet \sqrt{3} = i\sqrt{3}$ and $\sqrt{-9} = \sqrt{9} \bullet \sqrt{-1} = 3i$.

A **complex number** is any number in the form $a + bi$, where both a and b are real numbers. For example, $4.3 + i\sqrt{5}$ and $\pi - 2i$ are complex numbers. So are $7i$ and 11, since they may be represented as $0 + 7i$ and $11 + 0i$, respectively.

A **pure imaginary number** is a complex number $a + bi$ for which $a = 0$ and $b \neq 0$. For example, $5i$, $8i$, and $i\sqrt{5}$ are pure imaginary numbers.

A real number is a complex number $a + bi$ for which $b = 0$. For example, $5 + 0i = 5$ and $-3 + 0i = -3$ are real numbers.

The Venn diagram in Figure **2** shows the relationships among the sets of complex numbers, pure imaginary numbers, and real numbers.

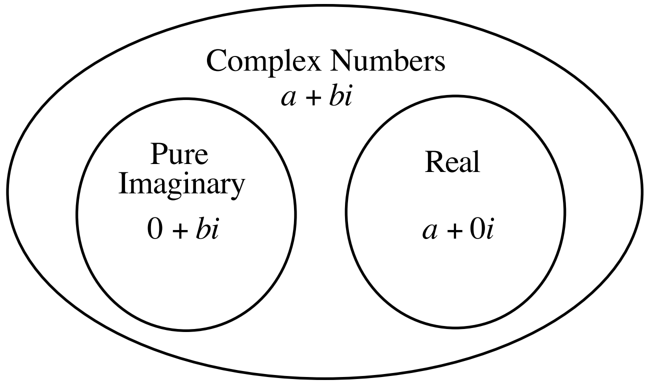


Figure 2: Venn diagram of complex numbers

In the set of complex numbers, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

d. **1.** Considering the information given in the mathematics note, along with the factors of a difference of squares, determine the factors of $x^2 + 4$.

2. What are the solutions to the equation $x^2 + 4 = 0$?

e. Describe how you could factor any expression of the form $x^2 + a^2$ and identify its zeros.

f. **1.** The Pythagorean theorem states that in a right triangle with legs of lengths a and b and hypotenuse of length c, $a^2 + b^2 = c^2$. Another way to describe this relationship is to say that the sum of the areas of two squares is the area of a third square.

Given the lengths of the sides of two of these squares, can you always find the length of the side of the third square? Explain your response.

2. In the set of real numbers, can you always find a value for b given the values of a and c in the equation $a^2 + b^2 = c^2$? Explain your response.

Assignment

1.1 Using your knowledge of the distributive property of multiplication over addition and subtraction, find the sum and the difference of each pair of complex numbers below. (Find the difference by subtracting the second complex number from the first.) Write each result in the form $a + bi$.

a. i and $9i$

b. 4 and $7 + 3i$

c. $21 - 6i$ and $15i$

d. $-13 + 4i$ and $3 - i$

e. $12 + 5i$ and $12 - 5i$

f. $a + bi$ and $c + di$

1.2 Complex numbers also can be multiplied. Use technology to multiply each of the following pairs of complex numbers:

a. i and 3

b. $2i$ and i

c. 7 and $6 - 11i$

d. $5 + i$ and $6 - 3i$

e. $a + bi$ and $c + di$

1.3 Use your results in Problem **1.2e** to show a pencil-and-paper method for multiplying complex numbers.

1.4 **Complex conjugates** are pairs of complex numbers in the form $a + bi$ and $a - bi$.

a. Create a pair of complex conjugates.

b. Find the sum and product of the numbers in Part **a**.

c. Suggest a method for finding the sum and product of complex conjugates.

1.5 In the set of real numbers, the multiplicative identity is 1. In other words, when a is a real number, $a \bullet 1 = 1 \bullet a = a$. Demonstrate that $1 + 0i$ is the multiplicative identity in the set of complex numbers.

* * * * *

1.6 Show that the solutions to the equation $x^2 + 27 = 0$ are $x = 3i\sqrt{3}$ and $x = -3i\sqrt{3}$.

1.7 **a.** Describe the solutions to $x^2 + 12 = 0$ where the domain is the set of real numbers.

b. Describe the solutions to $x^2 + 12 = 0$ where the domain is the set of complex numbers.

* * * * * * * * * *

Activity 2

Some sets of numbers have the **closure property** under certain operations. For example, consider the set of even natural numbers. If you add any two even natural numbers, you obtain an even natural number. Therefore, this set of numbers is closed under addition.

Does the set of complex numbers have the closure property under addition? Do you think that this set is closed under multiplication? Do you think that each complex number has a multiplicative inverse? In this activity, you answer these questions and more.

Exploration

If complex numbers behave like real numbers, then the reciprocal of $3 + 4i$ can be written as $1/(3 + 4i)$. By definition, the reciprocal of $3 + 4i$ is $a + bi$ if and only if $(a + bi)(3 + 4i) = 1 + 0i$. In the following exploration, you discover how this reciprocal also can be represented in the form $a + bi$.

a. **1.** Expand the left side of the equation below by multiplying the complex numbers.

$$(a + bi)(3 + 4i) = 1 + 0i$$

2. Write the product on the left in the form $m + ni$.

b. In order for the complex number $m + ni$ found in Part **a** to equal $1 + 0i$, the real part (m) must equal 1 and the imaginary part (n) must equal 0.

1. Write each of these relationships as an equation.

2. Solve these two equations to find the values of a and b in the complex number $a + bi$ that is the reciprocal of $3 + 4i$.

c. Verify that the complex number found in Part **b** is the reciprocal of $3 + 4i$ by determining that its product with $3 + 4i$ is 1.

d. The conjugate plays an important role in writing the reciprocal of a complex number in the form $m + ni$.

Use technology to evaluate $1/(3 + 4i)$. Write the result so that m and n are reduced fractions.

e. **1.** Evaluate the following expression:

$$\left(\frac{1}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right)$$

2. Compare the result to the complex number determined in Part **b** and your response to Part **d**.

3. Suggest a method for finding the reciprocal of a complex number $a + bi$ using the conjugate.

f. In the set of real numbers, division by a non-zero number can be interpreted as the product of the dividend and the reciprocal of the non-zero divisor. In other words,

$$a \div b = a \bullet \frac{1}{b}$$

where $b \neq 0$. Division among the complex numbers can be interpreted in the same way. Use this information to perform the following division:

$$\frac{7-5i}{3+4i}$$

Discussion

a. Does the set of natural numbers have the closure property under subtraction? Explain your response.

b. In the real-number system, the commutative property of addition is stated as $a + b = b + a$. How could you show that the commutative property of addition is preserved in the set of complex numbers?

c. In the real-number system, the associative property of addition is stated as $a + (b + c) = (a + b) + c$. How could you show that the associative property of addition is preserved in the set of complex numbers?

d. In the real-number system, the commutative property of multiplication is stated as $ab = ba$. How could you show that the commutative property of multiplication is preserved in the set of complex numbers?

e. In the real-number system, the associative property of multiplication is stated as $a(bc) = (ab)c$. How could you show that the associative property of multiplication is preserved in the set of complex numbers?

Assignment

2.1 Write each of the following expressions in the form $a + bi$.

a. $\sqrt{-49} + \sqrt{-1} + \sqrt{-9}$

b. $\sqrt{-25} - 5\sqrt{-9}$

c. $(3 - 4i) + (-8 + 6i)$

d. $(8 - 7i) - (2 + 6i)$

2.2 **a.** Determine the values of $i^1, i^2, \ldots, i^{10}$.

b. Describe any patterns you observe in your response to Part **a**.

c. Evaluate i^{90}.

d. Write a rule for evaluating i^n for any positive integer n.

2.3 Write each of the expressions below in the form $a + bi$.

a. $(-9i)(22i)$

b. $(4 - i)(7 + 2i)$

c. $i^3(5 + 7i)(3 - 4i)$

2.4 Using the method developed in the exploration, simplify each expression below to the form $a + bi$.

a. $\dfrac{3}{5 - 6i}$

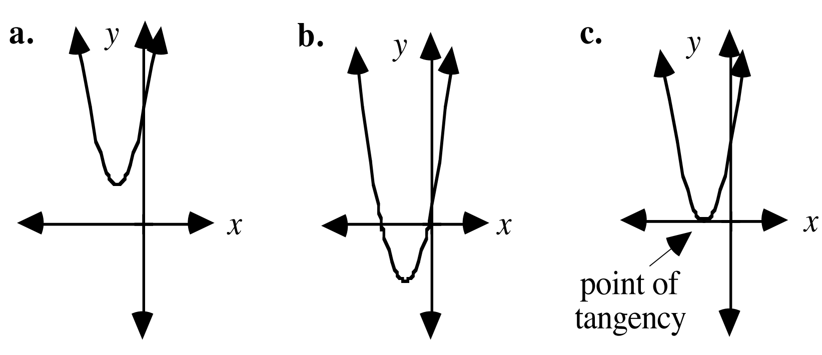
b. $\dfrac{-8 - i}{-3 - 9i}$

2.5 Determine the roots of each equation below in the set of complex numbers and write the equation in factored form.

a. $y = x^2 - 28$

b. $y = x^2 + 28$

2.6 Describe the roots of each quadratic equation graphed below.



2.7 Write an equation with real coefficients for which each expression below is a solution.

a. $5i$

b. $2i\sqrt{7}$

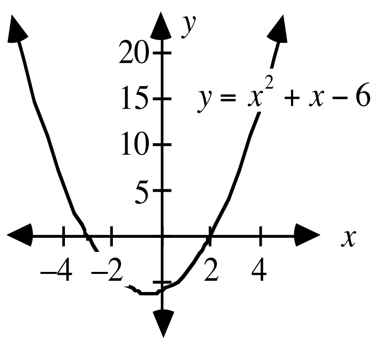
c. $6 - 7i$

* * * * *

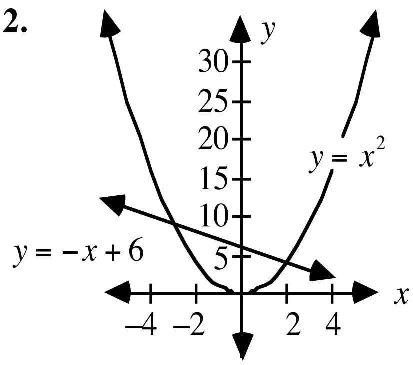
2.8 **a.** Explain why finding the zeros for $y = x^2 + x - 6$ is equivalent to solving the equation $-x + 6 = x^2$.

b. Describe how the zeros of $y = x^2 + x - 6$ are represented in each of the following graphs:

1.

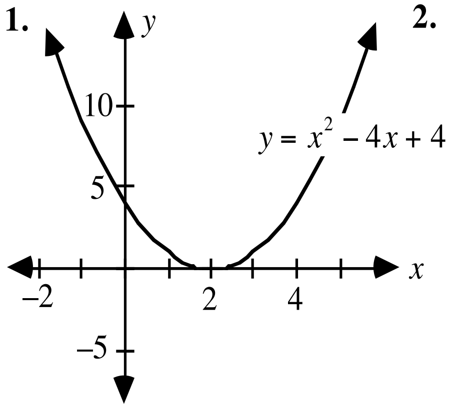


2.

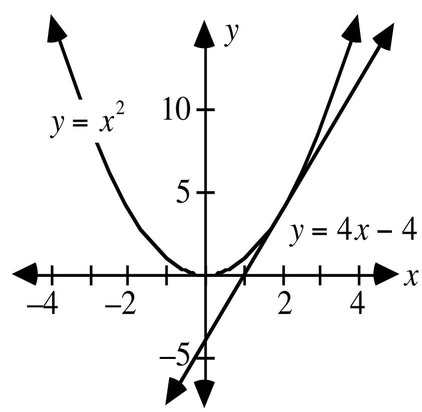


c. Describe how the zeros of $y = x^2 - 4x + 4$ are represented in each of the graphs below:

1.



2.



d. Create a pair of graphs like those shown in Parts **b** and **c** to illustrate the zeros of $y = x^2 + x + 4$.

e. Given an equation of the form $y = ax^2 + bx + c$, there are three possible cases for the roots of the equation:

1. two real roots

2. one real root

3. two non-real roots.

For each of these cases, sketch the graph of $y = ax^2 + bx + c$ on one set of axes. Sketch the corresponding graph of $y = ax^2$ and $y = -bx - c$ on a second set of axes.

* * * * * * * * * *

Activity 3

Many quadratic equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$, have no real-number solutions. However, when using the set of complex numbers, solving an equation such as $x^2 + 0x + 1 = 0$ results in two solutions: $x = \sqrt{-1} = i$ and $x = -\sqrt{-1} = -i$.

Exploration 1

In this exploration, you investigate the solutions to second-degree polynomial equations of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

a. **1.** Find integer values for a, b, and c so that $y = ax^2 + bx + c$ has two real-number roots, r_1 and r_2. **Note:** Recall that if $ax^2 + bx + c = 0$ has solutions r_1 and r_2, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

2. To check your results, substitute the values you used for a, b, and c into the general equation $y = ax^2 + bx + c$ and graph it.

b. Repeat Part **a** so that $ax^2 + bx + c = 0$ has one real root r. **Note:** If r is the only solution, then $ax^2 + bx + c = a(x - r)(x - r)$. This solution is a double root.

c. Repeat Part **a** so that $ax^2 + bx + c = 0$ has no real roots.

d. Use technology to solve for x in the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.

e. Substitute the values of a, b, and c chosen for each case in Parts **a–c** into the solutions found in Part **d**. Confirm that the results agree with the values for the roots in Parts **a–c**.

f. Determine the value of $b^2 - 4ac$ in each equation from Parts **a–c**.

Discussion 1

a. Describe how you could use a graph to demonstrate that a quadratic function has each of the following numbers of roots:

1. two real roots

2. one double root

3. no real roots.

b. How are the two complex-number solutions to the equation in Part **c** of Exploration **1** related?

Mathematics Note

Second-degree polynomial equations of the form $ax^2 + bx + c = 0$ with $a \neq 0$, always have two solutions when solved over the complex numbers:

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

These two solutions make up the **quadratic formula**.

When a, b, and c are real numbers and $b^2 - 4ac < 0$, the solutions are complex and occur in conjugate pairs. For example, for $x^2 + 2x + 5 = 0$, $b^2 - 4ac = -16$. Since $-16 < 0$, $x^2 + 2x + 5 = 0$ has two complex-number solutions:

$$x = \frac{-2}{2} + \frac{\sqrt{2^2 - 4 \bullet 5}}{2} = -1 + 2i \text{ and } x = \frac{-2}{2} - \frac{\sqrt{2^2 - 4 \bullet 5}}{2} = -1 - 2i$$

c. The expression $b^2 - 4ac$, which appears under the radical sign in the solution of the general quadratic equation $ax^2 + bx + c = 0$, is known as the **discriminant**.

As described in the mathematics note above, the discriminant can help you determine whether the roots of a quadratic equation are real or complex. Explain why this is true.

d. A polynomial is **reducible** over the real numbers if it can be expressed as the product of two or more polynomials of degree 1 and with real coefficients. Is every second-degree polynomial reducible over the real numbers? Explain your response.

Exploration 2

In this exploration, you investigate the solutions to polynomial equations of degrees 3 and 4 with real-number coefficients.

a. If the third-degree polynomial equation $ax^3 + bx^2 + cx + d = 0$ has roots r_1, r_2, and r_3, the equation can be expressed in the form:

$$a(x - r_1)(x - r_2)(x - r_3) = 0$$

Using this fact, find a combination of three distinct, real-number roots r_1, r_2, and r_3, such that $a(x - r_1)(x - r_2)(x - r_3)$ results in a third-degree polynomial with real coefficients.

Check your response by graphing the resulting equation.

b. Repeat Part **a** where r_1 is a real root and r_2 and r_3 are complex roots that are not real.

c. Consider fourth-degree polynomial equations of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ where a, b, c, d, and e are real numbers and $a \neq 0$.

Find a combination of four distinct, real-number roots r_1, r_2, r_3, and r_4, such that $a(x - r_1)(x - r_2)(x - r_3)(x - r_4)$ results in a fourth-degree polynomial with real coefficients.

Check your response by graphing the resulting equation.

d. Repeat Part **c** where r_1 and r_2 are distinct, real roots and r_3 and r_4 are complex roots that are not real.

e. Repeat Part **c** where r_1, r_2, r_3, and r_4 are complex roots that are not real.

Discussion 2

a.

1. How may graphs be used to check the number of real solutions to a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are real numbers and $a \neq 0$?
2. How many real-number solutions are possible for an equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are real numbers and $a \neq 0$?
3. How many complex-number solutions are possible?

b.

1. How many real-number solutions are possible for an equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ where a, b, c, d and e are real numbers and $a \neq 0$?
2. How many complex-number solutions are possible?

c. Describe the relationship among the complex solutions of the form $a + bi$, where $b \neq 0$, of a polynomial equation when that equation has real-number coefficients.

Mathematics Note

The **fundamental theorem of algebra** states that every polynomial equation of degree $n \geq 1$ with complex coefficients has at least one root which is a complex number.

One consequence of the fundamental theorem of algebra is that *n*th-degree polynomial equations have exactly *n* roots in the set of complex numbers. This total may include some multiple roots. For example, the roots of the fifth-degree polynomial $x^5 - 4x^4 - 15x^3 + 50x^2 + 64x - 96 = 0$ are –3, –2, 1, and 4. One of these (4) is a double root. Therefore, the polynomial has a total of five roots in the set of complex numbers.

d. Describe the number of real solutions possible for polynomial equations of the form $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, where the coefficients are real numbers and $a \neq 0$.

e. In general, how many complex roots of the form $a + bi$, where $b \neq 0$, can an *n*th-degree polynomial equation with real coefficients have? How are these complex roots related?

f. For what type of polynomial equations must there always be at least one real root? Explain your response.

Assignment

3.1 Determine the solutions to each of the following equations over the complex numbers. Use these solutions to express each equation as the product of first-degree polynomials.

a. $9x^2 + 12x + 4 = 0$

b. $9x^2 + 35x - 4 = 0$

c. $x^2 + 4x + 9 = 0$

d. $3x^3 - 12x^2 + 12x - 48 = 0$

e. $2x^4 - 6x^3 + 12x^2 + 4x - 120 = 0$

3.2 When considering solutions to polynomial equations with real coefficients, the fact that the product of a complex number and its conjugate is a real number has special significance.

a. Find a polynomial equation in the form $ax^2 + bx + c = 0$ with real coefficients that has solutions $r_1 = 2 + i$ and $r_2 = 2 - i$.

b. Find the solutions to $x^2 - 6x + 13 = 0$. Describe the relationship between the two solutions.

c. Determine four complex-number solutions for an equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ that result in coefficients that are real numbers. Give the values of these coefficients.

3.3 If $2 + 3i$, 2, and -5 are solutions to the polynomial equation $x^4 - x^3 + cx^2 + 79x - 130 = 0$, determine the value of c. Describe how you made this determination.

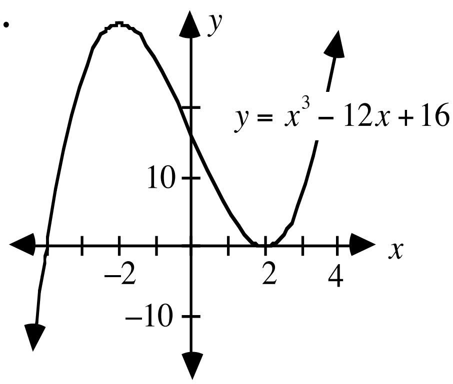
3.4 Write a paragraph describing the different numbers of real solutions that are possible for sixth-degree polynomial equations of the form $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$, where the coefficients are real numbers and $a \neq 0$.

* * * * *

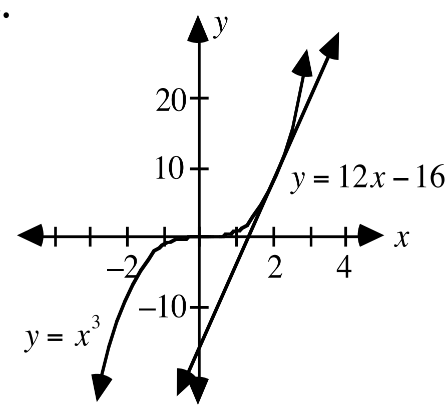
3.5 **a.** Explain why finding the zeros for $y = x^3 - 12x + 16$ is equivalent to solving the equation $x^3 = 12x - 16$.

b. Describe how the zeros of $y = x^3 - 12x + 16$ can be interpreted using each of the following graphs:

1.



2.



c. Changing the constant in the equations graphed in Part **b** will change the zeros of the equation. Suppose that the constant is changed from 16 to 10, resulting in the equation $y = x^3 - 12x + 10$.

1. Predict the number of times that the graph of $y = x^3$ intersects the graph of $y = 12x - 10$.

2. Predict the number of real zeros for $y = x^3 - 12x + 10$.

3. Confirm your predictions by finding the zeros for $y = x^3 - 12x + 10$.

d. In Part **b**, the graph of $y = x^3 - 12x + 16$ is tangent to the x-axis at one point, while the graph of $y = 12x - 16$ is tangent to $y = x^3$ at one point.

Use the graphs to predict how the zeros of $y = x^3 - 12x + 16$ would change if the constant 16 is increased.

e. Given an equation of the form $y = ax^3 + bx^2 + cx + d$, there are three possible cases for the roots of the equation. For each of these cases, sketch the graph of $y = ax^3 + bx^2 + cx + d$ on one set of axes. Sketch the corresponding graph of $y = ax^3 + bx^2$ and $y = -cx - d$ on a second set of axes. Then describe the types of roots represented by the graphs.

* * * * * * * * * *

Activity 4

The use of the word *imaginary* reflects some of the original uneasiness that mathematicians had with numbers involving $\sqrt{-a}$, where a is a positive real number. However, the phrase "imaginary numbers" seems inappropriate in today's world, where such numbers are routinely used in analyzing electrical circuits, in cartography, and in quantum mechanics.

Mathematics Note

Swiss clerk Jean Robert Argand (1768–1822) and Danish mathematician Caspar Wessel (1745–1818) were the first two people to graph complex numbers on a plane. They represented a complex number $a + bi$ as an ordered pair (a, b) where a is the real part and b is the imaginary part.

Each complex number can be graphed as a point in the complex plane. Any point on the horizontal axis is a real number and any point on the vertical axis is a pure imaginary number.

For example, Figure **3** shows the graphs of the ordered pairs (2,3), (0,3), (3,0) and (4,–3), which represent the complex numbers $2 + 3i$, $0 + 3i$, $3 + 0i$ and $4 - 3i$, respectively.

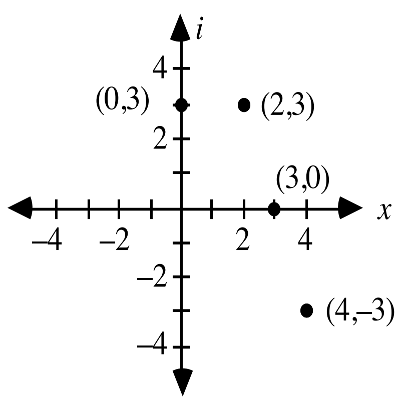


Figure 3: The complex plane

Exploration

Argand's geometric interpretation of complex numbers provides many advantages when exploring the complex-number system.

a. Graph a complex number of the form $a + 0i$ on the complex plane.

b.
1. Multiply the number by i and graph the result as a point.
2. Multiply the result from Step **1** by i and graph the resulting point. Continue this process of multiplying by i to obtain two more points.

c. Make a conjecture about the effects of multiplying by i with respect to the movement of a point on the complex plane.

d.
1. Predict the result of multiplying $a + 0i$ by $-i$.
2. Test your prediction by repeating Parts **a** and **b** using the factor $-i$.

e.
1. Multiply $a + bi$ by i.
2. Does your conjecture from Part **c** appear to apply to all complex numbers? If not, revise it so that it does.

Discussion

a. Describe the transformation that occurs when a complex number is multiplied by each of the following:
1. i
2. $-i$

b. How do the transformations in Part **a** affect the ordered pair that represents $a + 0i$?

c. Describe what occurs when $0 + 0i$ is multiplied by $-i$.

Assignment

4.1 **a.** Multiply the complex number $3 + 2i$ by the each of the following numbers. Write the products as ordered pairs.
1. i
2. i^2
3. i^3
4. i^4

b. Plot the products from Part **a** in the complex plane. What is the geometric relationship among these points?

4.2 Describe how complex conjugates are related in terms of their graphs in the complex plane.

4.3 **a.** Let $u = 4 + 9i$ and $v = 5 + 4i$. Find $u + v$ and $u - v$.

b. Graph the four complex numbers from Part **a**, u, v, $u + v$, and $u - v$ as ordered pairs on the complex plane.

c. Define the addition and subtraction of complex numbers $a + bi$ and $c + di$ using ordered pairs.

4.4 **a.** Consider the polynomial equation $x^3 - x^2 + x - 1 = 0$. Determine which of the following are solutions to this equation: $1, -1, i$, or $-i$.

b. Rewrite $x^3 - x^2 + x - 1 = 0$ as a product of factors in the form $(x - k)$ where k is a root of the equation.

c. Multiply the factors to verify your response to Part **b**.

* * * * *

4.5 Julia sets are sets of complex numbers that often make interesting patterns when graphed on the complex plane. Julia sets are generated by the recursive formula: $a_n + b_n i = (a_{n-1} + b_{n-1}i)^2 + a_1 + b_1 i$, where n is a natural number.

a. The second term of the Julia set where $a_1 + b_1 i = 2 + 3i$ is:

$$\begin{aligned} a_2 + b_2 i &= (2 + 3i)^2 + 2 + 3i \\ &= 4 + 6i + 6i + 9i^2 + 2 + 3i \\ &= -3 + 15i \end{aligned}$$

Find the third term of this Julia set.

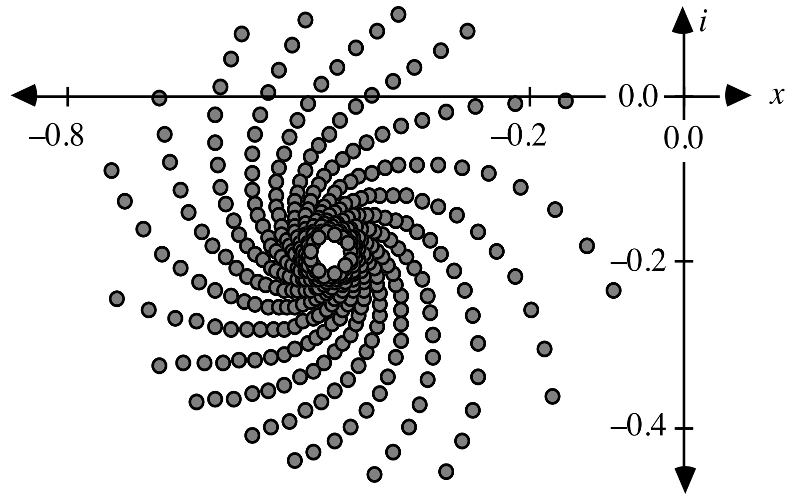
b. In order to create a scatterplot of a Julia set using technology, each term is written as an ordered pair. The first two terms of the Julia set from Part **a** can be written as (2,3) and (–3,15). Write the third term of this Julia set as an ordered pair.

c. **1.** Expand the recursive formula for Julia sets, writing the result in the form indicated below:

$$\begin{aligned} a_n + b_n i &= (a_{n-1} + b_{n-1}i)^2 + a_1 + b_1 i \\ &= \text{real part} + \text{imaginary part} \end{aligned}$$

2. Write the nth term as an ordered pair in the form (a_n, b_n).

d. For certain values of a_1 and b_1, the scatterplots of Julia sets make interesting patterns. For example, the following graph shows a scatterplot of the first 400 terms of the Julia set where $a_1 = -0.63$ and $b_1 = -0.37$:



The table below shows the first six terms (rounded to the nearest 0.0001) of the Julia set with $a_1 = -0.63$ and $b_1 = -0.37$.

Term Number (n)	**Real Part (a_n)**	**Imaginary Part (b_n)**
1	–0.63	–0.37
2	–0.3700	0.0962
3	–0.5024	–0.4412
4	–0.5723	0.0733
5	–0.3079	–0.4539
6	–0.7412	–0.0906
⋮	⋮	⋮

1. Use a spreadsheet to extend the table to 200 terms.
2. Create a scatterplot of the ordered pairs (a_n, b_n) on the complex plane.

e. Small changes in the first term of a Julia set result in a dramatically different set.

1. Create a scatterplot of the first 200 terms of the Julia set where $a_1 = -0.63$ and $b_1 = -0.38$.
2. Create new scatterplots by making small modifications in a_1 and b_1.

f. The table below shows the first terms in some other Julia sets. Use these first terms to explore patterns in the resulting scatterplots.

a_1	b_1
0.2477	0.56
–0.61	–0.405
0.29	0.45
–1.195	0.45
–1.2	0.15

4.6 Consider the set of matrices of the form below, where a and b are real numbers.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

a. Show that this set of matrices has the closure property under addition. In other words, show that if any two matrices in this set are added together, the sum is a matrix from the set.

b. Show that the set of matrices is closed under multiplication.

c. Find an additive identity and a multiplicative identity for this set of matrices.

d. Find the multiplicative inverse of the following matrix, if it exists:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

e. The arithmetic of the set of matrices defined in Parts **a–d** behaves almost exactly the same as arithmetic with real numbers. However, this arithmetic has one property that arithmetic with real numbers does not. Square the following matrix and describe how the result compares to the multiplicative identity for the set of matrices.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

f. Considering your results in Part **e**, the arithmetic of this set of matrices behaves like the arithmetic of the set of complex numbers. Now suppose that every complex number of the form $a + bi$ can be identified with the matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

What is the matrix representation of the complex number $0 + i$?

g. Recall that multiplication by a matrix of the form below produces a rotation of θ about the origin.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What matrix produces a 90° rotation about the origin?

h. Compare the matrices found in Parts **f** and **g**.

* * * * * * * * * *

Activity 5

In Activity **4**, you used ordered pairs of the form (a, b) to represent complex numbers. However, when performing multiplication of complex numbers, it can be more convenient to use their **trigonometric form**. Using the trigonometric form also can simplify finding powers of complex numbers.

Mathematics Note

Figure **4** shows the complex number $a + bi$ represented as the ordered pair (a, b).

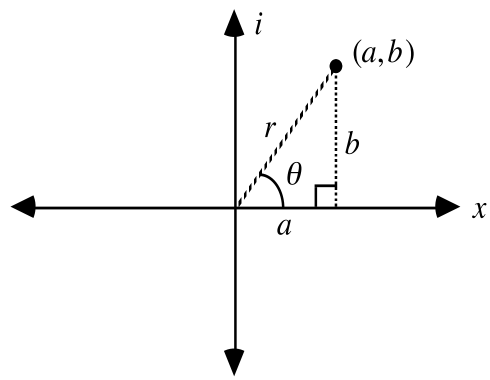


Figure 4: Graph of *a + bi*

A complex number $a + bi$ can be written in **trigonometric form** as follows:

$$a + bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

The value of r is the **absolute value** or **modulus** of the complex number and is determined by $r = \sqrt{a^2 + b^2}$. Note that r is always a non-negative number.

The angle θ is an **argument** of the complex number and is measured from the positive portion of the real axis. Angles generated by counterclockwise rotations are assigned positive measures; those generated by clockwise rotations are assigned negative values. The ray passing through the point (a, b) representing the number $a + bi$ is the **terminal ray** of the argument.

For example, consider the complex number $\sqrt{3} + i$ represented by the point $\left(\sqrt{3}, 1\right)$. Using right-triangle trigonometry, an argument is $\theta = \tan^{-1}\left(1/\sqrt{3}\right) = \pi/6$. The absolute value is $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$. Therefore, the trigonometric form of $\sqrt{3} + i$ is:

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Because θ can be any angle measured from the positive portion of the real axis whose terminal ray contains $(\sqrt{3},1)$, the trigonometric form of $\sqrt{3}+i$ is not unique. In this case, θ can be any angle of the form $\theta = \pi/6 + 2n\pi$, where n is an integer. Two additional representations of $\sqrt{3}+i$ in trigonometric form are $2(\cos(13\pi/6) + i\sin(13\pi/6))$, where $\theta = \pi/6 + 2\pi$, and $2(\cos(-11\pi/6) + i\sin(-11\pi/6))$, where $\theta = \pi/6 - 2\pi$.

Exploration 1

a. The following complex numbers are written in the form $a + bi$. Evaluate $\tan^{-1}(b/a)$ for each number. Use the resulting angle measure to determine an argument of the given number.

1. $1+\sqrt{3}i$

2. $-1+\sqrt{3}i$

3. $-1-\sqrt{3}i$

4. $1-\sqrt{3}i$

b. Determine the measure of two additional positive arguments and two additional negative arguments for each of the complex numbers in Part **a**.

Discussion 1

a. Given the complex number $a + bi$, in which quadrants can the point (a,b) lie if $\theta = \tan^{-1}(b/a)$ is an argument of $a + bi$?

b. If $\theta = \tan^{-1}(b/a)$ is not an argument of $a + bi$, how can an argument be determined using $\tan^{-1}(b/a)$?

c. Describe the methods you used to determine the additional positive and negative arguments in Part **b** of Exploration **1**.

Mathematics Note

If the graph of the complex number $a + bi$ is in the first or fourth quadrants, $\theta = \tan^{-1}(b/a)$ is an argument of the number and every argument is represented by the expression $\theta = \tan^{-1}(b/a) + 2n\pi$, where n is any integer.

For complex numbers represented by points in the second and third quadrants, arguments have the form $\theta = (\tan^{-1}(b/a) + \pi) + 2n\pi$, where n is an integer.

For example, the graph of the complex number $-2 + 2i$ is in the second quadrant. In this case, its arguments can be found as follows, where n is an integer:

$$\theta = (\tan^{-1}(2/-2) + \pi) + 2n\pi = 3\pi/4 + 2n\pi$$

Exploration 2

In Activity **4**, you learned that points on the complex plane are rotated when multiplied by i or $-i$. In this activity, you multiply complex numbers to discover other patterns.

a. Multiply each of the following pairs of complex numbers. Plot each pair of complex numbers and their product as points on a complex plane.

1. $v = 2 + i$ and $s = 2 + 3i$
2. $t = 0 + 2i$ and $u = -1 + i$
3. $m = -1 + 2i$ and $w = -2 - i$
4. $g = 3 + 2i$ and $h = 2 + 4i$

b. For each complex number in Part **a**, determine its absolute value, as well as an argument (to the nearest 0.01 radians) between -2π and 2π. Leave the absolute value in radical form, even if the square root is an integer. Record these values in a table with headings like those in Table **1** below.

Table 1: Complex numbers and their products

Number	$a + bi$	Absolute Value	Argument
v	$2 + i$		
s	$2 + 3i$		
$v \bullet s$			
t	$0 + 2i$		
u	$-1 + i$		
$t \bullet u$			
m			
w			
$m \bullet w$			
g			
h			
$g \bullet h$			

c. Select at least three conjugate pairs of complex numbers. Repeat Parts **a** and **b** using these pairs.

d.
1. Use a symbolic manipulator to verify the following rule for multiplying complex numbers in trigonometric form:

$$\left[a(\cos x + i\sin x)\right] \bullet \left[b(\cos y + i\sin y)\right] = ab\left[\cos(x + y) + i\sin(x + y)\right]$$

2. Using the terms *absolute value* and *argument*, describe the rule for multiplying complex numbers in trigonometric form.
3. Determine if this rule is illustrated in Table **1**.

e.

1. Write $1-i$ and $-2+i$ in trigonometric form, rounding both r and θ to the nearest 0.01.
2. Use the rule from Part **d** to multiply the trigonometric forms of $1-i$ and $-2+i$.
3. Write the product in the form $a+bi$.
4. Use the distributive property to multiply $1-i$ and $-2+i$.
5. Compare the products in Steps **2** and **4**.

Discussion 2

a. Describe the relationship between conjugates when they are expressed in trigonometric form.

b. What is the argument of the product when a complex number and its conjugate are multiplied?

c. Is multiplication of complex numbers in trigonometric form commutative? Justify your response.

d. Compare the process of multiplying complex numbers in the form $a+bi$ with the process of multiplying the same numbers in trigonometric form.

Assignment

5.1 The complex numbers in Parts **a–c** are given in trigonometric form. Multiply each number by its respective conjugate and write the products in trigonometric form.

a. $3(\cos(\pi/12)+i\sin(\pi/12))$

b. $11(\cos(8.83)+i\sin(8.83))$

c. $\sqrt{7}(\cos(-\pi/15)+i\sin(-\pi/15))$

5.2 Use the rule developed in Exploration **2** to multiply the following pairs of complex numbers. Write each product in trigonometric form.

a. $3(\cos(\pi/4)+i\sin(\pi/4))$ and $2(\cos(8.85)+i\sin(8.85))$

b. $r_1(\cos\theta_1+i\sin\theta_1)$ and $r_2(\cos\theta_2+i\sin\theta_2)$

5.3 The ordered pairs (–1,2) and (3,–2) represent two complex numbers on the complex plane.

a. Find their product using two different methods.

b. Compare the results and explain any differences you observe.

5.4 Multiplication by the complex number $2(\cos(\pi/6) + i\sin(\pi/6))$ can be thought of as a dilation by a scale factor of 2 and a rotation of $\pi/6$ with center at (0,0). What complex number produces the same dilation but the opposite rotation? Describe the relationship between these two numbers.

5.5 **a.** Evaluate $(3-4i)^2$ by converting the expression to trigonometric form before multiplying.

b. Find the trigonometric form of $(3-4i)^3$ by multiplying the trigonometric form of $(3-4i)^2$ by the trigonometric form of $(3-4i)$.

c. Write $(3-4i)^5$ in trigonometric form.

d. Express $(3-4i)^n$, where n is an integer, in trigonometric form.

5.6 Given that the trigonometric form of $a+bi$ is $r(\cos\theta + i\sin\theta)$, write the trigonometric forms of $(a+bi)^2$, $(a+bi)^3$, $(a+bi)^4$, and $(a+bi)^n$.

* * * * *

5.7 In previous modules, you have used several different methods to draw regular polygons. In this assignment, you investigate another way to construct regular polygons.

a. Consider the complex number $3+4i$. What is the modulus r of this number?

b. Plot the complex number $3+4i$ on a grid.

c. What is the radius of the circle with center at the origin that contains the point $3+4i$?

d. What is the measure θ of a central angle of a regular pentagon?

e. To construct a regular pentagon with the point representing $3+4i$ as one of the vertices, one could rotate the point (and its successive images) by θ, with center at the origin.

To do this, multiply $3+4i$ by a complex number in the form $\cos\theta + i\sin\theta$. Plot the coordinates of the product. Continue this process to find the five vertices of the regular pentagon.

5.8 Use the process described in Problem **5.7** to construct a regular hexagon whose sides measure 3 units.

5.9 When designing circuits for use with alternating current, electrical engineers use the complex-number form of Ohm's law:

$$I = \frac{V}{Z}$$

where I is the effective current (a measure of the number of electrons moving in the wires), V is the effective voltage (a measure of the force moving the electrons), and Z is the impedance (a measure of the resistance to the flow of electrons caused by magnetic fields). In this relationship, I, V, and Z are complex numbers.

In Parts **a–c**, write each response in the same form as the original numbers.

a. Determine the effective voltage in a circuit if the effective current is $4(\cos(\pi/18) + i\sin(\pi/18))$ and the impedance is $29(\cos(\pi/9) + i\sin(\pi/9))$.

b. Find the effective current in a circuit if the effective voltage is $120(\cos 0 + i\sin 0)$ and the impedance is $44(\cos(11\pi/36) + i\sin(11\pi/36))$.

c. Find the impedance of a circuit when the effective voltage is $77 + 77i$ and the effective current is $2.9 - 0.35i$.

* * * * * * * * * *

Activity 6

Using the trigonometric form of a complex number, you can explore some other interesting properties of complex numbers.

Mathematics Note

De Moivre's theorem states that the non-zero powers of any complex number $a + bi$ can be found in the following manner:

$$(a + bi)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Abraham De Moivre (1667–1754) developed this theorem for positive integer values of n. Later work found it to be true for all real-number values of n. For example,

$$(2 + 0i)^3 = [2(\cos 0^\circ + i\sin 0^\circ)]^3 = 2^3(\cos(3 \bullet 0) + i\sin(3 \bullet 0))$$

Exploration

In previous activities, you examined the square roots of negative numbers. In this activity, you investigate some additional roots of complex numbers.

a. **1.** Graph $2(\cos 0 + i\sin 0)$ on a complex coordinate plane.

2. On the same plane, graph the image of $2(\cos 0 + i\sin 0)$ under a counterclockwise rotation of $\pi/2$ with center at (0,0).

3. Repeat Step **2** with each new image until points begin to repeat. Using this process, determine the coordinates of each unique point generated.

b. **1.** Describe the geometric relationships among the points in Part **a**.

2. Determine the trigonometric form of the complex number represented by each point.

c. Use De Moivre's theorem to raise each complex number in Part **b** to the fourth power. Convert each result to a number in the form $a + bi$.

d. **1.** If the graph of the complex number $2(\cos 0 + i\sin 0)$ represents one vertex of a regular pentagon centered at the origin, determine the angle of counterclockwise rotation about the origin required to locate the next consecutive vertex of the pentagon.

2. Determine the complex numbers that correspond to the five vertices obtained by repeating this rotation on each image. Write these numbers in trigonometric form. **Note:** Save this data for use in the assignment.

e. Use De Moivre's theorem to raise each complex number from Part **c** to the fifth power. Convert each result to a number in the form $a + bi$.

f. Select any complex number $a + bi$ where $a \neq 0$ and $b \neq 0$. Determine the cube roots of this number and write them in trigonometric form.

g. Graph the cube roots found in Part **f** on the complex plane. Describe any geometric relationship among these points.

Mathematics Note

From the fundamental theorem of algebra, the equation $x^n - z = 0$ has n roots in the set of complex numbers.

For example, $x^3 - 8 = 0$ has roots 2, $-1 + i\sqrt{3}$, and $-1 - i\sqrt{3}$. The solutions to this equation are the cube roots of 8. Thus, there are 3 cube roots of 8 in the set of complex numbers. In a similar manner, there are 4 fourth roots of 8 and 5 fifth roots of 8. In general, there are exactly n distinct nth roots of any complex number.

Discussion

a. Describe the significance of your results in Part **c** of the exploration in terms of the fourth roots of a number.

b. Describe the significance of your results in Part **e** of the exploration in terms of the fifth root of a number.

c. Using the examples from the exploration, describe the relationship between the modulus of a complex number in trigonometric form and the modulus of its roots in trigonometric form.

d. Using the examples from the exploration, describe the relationship between the argument of a complex number and the argument of its roots.

e.
1. When $\theta = 0$, what is $r(\cos\theta + \mathrm{i}\sin\theta)$?
2. When $\theta = \pi/2$, what is $r(\cos\theta + \mathrm{i}\sin\theta)$?

f. What effect did the initial value of θ have on the polygons formed in the exploration?

Mathematics Note

If a complex number $z = a + bi$ is written as $z = r(\cos\theta + i\sin\theta)$, then the nth roots of z can be found using the following formula:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right)\right)$$

where $k = 0,\ 1,\ 2,\ \ldots,\ n-1$. There will be exactly n of these roots.

For example, $z = 8 + 0i$ can be written as $z = 8(\cos 0 + i\sin 0)$. The three cube roots of z are:

$$r_1 = 2\left(\cos\left(\frac{0}{3} + \frac{0 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{0 \bullet 2\pi}{3}\right)\right) = 2\left(\operatorname{cis}\frac{0\pi}{3}\right) = 2 + 0i$$

$$r_2 = 2\left(\cos\left(\frac{0}{3} + \frac{1 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{1 \bullet 2\pi}{3}\right)\right) = 2\left(\operatorname{cis}\frac{2\pi}{3}\right) \approx -1 + 1.73i$$

$$r_3 = 2\left(\cos\left(\frac{0}{3} + \frac{2 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{2 \bullet 2\pi}{3}\right)\right) = 2\left(\operatorname{cis}\frac{4\pi}{3}\right) \approx -1 - 1.73i$$

Assignment

6.1 **a.** Evaluate $\left(2\sqrt{3}+2i\right)^3$ using the distributive property.

b. Convert $2\sqrt{3}+2i$ to trigonometric form and cube it using De Moivre's theorem. Round the argument to the nearest 0.001 radians.

6.2 The 4 fourth roots of 81 are evenly spaced on a circle in the complex plane centered at the origin.

a. What is the radius of the circle?

b. By how many radians are consecutive roots separated?

c. Graph all 4 fourth roots. Include the circle containing the vertices in your graph.

d. Write each fourth root in the form $a+bi$.

6.3 **a.** Pick a complex number of the form $a+bi$ where $a \neq 0$ and $b \neq 0$.

b. Find the three cube roots of this complex number.

c. Write each cube root in trigonometric form.

d. Sketch a graph of the cube roots on a complex coordinate plane.

6.4 Solve the equation $z^8 = -2+3i$ for z and write the roots as ordered pairs (a,b).

6.5 One of the cube roots of 64 is 4, which can be written as $4\cos 0 + 4i\sin 0$.

a. The point (4,0) in the complex plane represents the complex number $4\cos 0 + 4i\sin 0$. Determine the counterclockwise rotation of the point (4,0) about the origin required to locate the next consecutive cube root of 64.

b. Graph the three cube roots of 64 on the complex plane.

* * * * *

6.6 **a.** Find the three cube roots of 1 in the form $r(\cos\theta + \text{i}\sin\theta)$ and write each cube root in the form $a+bi$.

b. Verify that the roots obtained are in fact cube roots of 1.

6.7 **a.** If the graph of $3\left(\cos(3\pi/4) + i\sin(3\pi/4)\right)$ defines one vertex of a regular octagon centered at the origin of the complex plane, what numbers correspond to the other seven vertices?

b. What do the complex numbers corresponding to each vertex of this octagon represent?

* * * * * * * * * *

Research Project

Besides introducing the imaginary unit i, Swiss mathematician Leonhard Euler made many other contributions to mathematics. Read more about Euler's life and work. Complete the following tasks in your report.

a. Describe Euler's formula.

b. Explain how Euler's formula can be used to represent complex numbers in exponential form.

c. Show how substituting π into Euler's formula results in a natural logarithm for -1.

d. Demonstrate that natural logarithms of negative numbers exist in the set of complex numbers.

Summary Assessment

1. Write a paragraph describing the number of real roots possible for the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$, where the coefficients are real numbers and $a_n \neq 0$.

2.
a. Use a symbolic manipulator to find the roots of the polynomial equation $x^4 - 2x^3 + x^2 + 4x - 6 = 0$.

b. Use the results from Part **a** to express the polynomial $x^4 - 2x^3 + x^2 + 4x - 6$ as a product of one or more polynomials such that the constant term in each factor is:

1. a complex number
2. a real number
3. a rational number.

3. According to the binomial theorem:

$$(a+b)^n = C(n,n) \bullet a^n b^0 + C(n,n-1) \bullet a^{n-1} b^1 + C(n,n-2) \bullet a^{n-2} b^2 + \cdots + C(n,1) \bullet a^1 b^{n-1} + C(n,0) \bullet a^0 b^n$$

where $C(n,r)$ is the combination of *n* things, taken *r* at a time.

a. Expand $(1+i)^8$ using the binomial theorem.

b. Simplify the expression from Part **a**.

c. Write $(1+i)$ in trigonometric form.

d. Use the trigonometric form to evaluate $(1+i)^8$. Write the result in standard form.

e. Compare the binomial theorem and De Moivre's theorem as methods for raising complex numbers to a power.

4. The coordinates of the vertices of ΔABC in a complex coordinate plane are $A(2,1)$, $B(4,1)$, and $C(3,2)$. The image of ΔABC has vertices with coordinates $A'(-5,5)$, $B'(-7,9)$, and $C'(-8,6)$.

The transformation from ΔABC to $\Delta A'B'C'$ is produced by multiplying by the complex number z, then adding z. For example, A is transformed to A' by $(2,1) \bullet z + z = (-5,5)$.

a. Plot ΔABC and its image in the complex plane. Describe the geometric relationship between these two triangles.

b. What is the ratio of $A'B'/AB$? What does this ratio reveal about the number z that produced the transformation?

c. Find the trigonometric forms of the $B - A$ and $B' - A'$. What do these reveal about the number z that produced the transformation?

d. Find z. Pick a point on ΔABC and show that it is transformed appropriately.

Module Summary

- The **imaginary unit** is ***i*** where $i = \sqrt{-1}$ and $i^2 = -1$.
- A **complex number** is defined as any number in the form $a + bi$, where both a and b are real numbers.
- A **pure imaginary number** is a complex number $a + bi$ for which $a = 0$ and $b \neq 0$.
- A real number is a complex number $a + bi$ for which $b = 0$.
- In the set of complex numbers, $a + bi = c + di$ if and only if $a = c$ and $b = d$.
- **Complex conjugates** are pairs of complex numbers of the form $a + bi$ and $a - bi$. The sum of complex conjugates is a real number. The product of complex conjugates also is a real number.
- The **reciprocal** of a complex number $a + bi$ is $1/(a + bi)$. To express this reciprocal in complex form $m + ni$, it can be multiplied by

  $$\frac{a - bi}{a - bi}$$

  where $a - bi$ is the conjugate of $a + bi$.
- A complex number $a + bi$ can be represented by the ordered pair (a, b) where a is the real part and b is the imaginary part. Using the horizontal axis as the real axis and the vertical axis as the imaginary axis, this ordered pair can be graphed as a point on the complex plane.
- Second-degree polynomial equations of the form $ax^2 + bx + c = 0$ with $a \neq 0$, always have two solutions when solved over the complex numbers:

  $$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

  These two solutions make up the **quadratic formula**. When a, b, and c are real numbers and $b^2 - 4ac < 0$, the solutions are complex and occur in conjugate pairs.
- The **fundamental theorem of algebra** states that every polynomial equation of degree $n \geq 1$ with complex coefficients has at least one root, which is a complex number (real or imaginary).
- One consequence of the fundamental theorem of algebra is that *n*th-degree polynomial equations have exactly *n* roots in the set of complex numbers.

- A complex number $a + bi$ can be written in **trigonometric form** as:

$$r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$$

The value of r is the **absolute value** or **modulus** of the complex number and is determined by $r = \sqrt{a^2 + b^2}$. Note that r is always a non-negative number. The angle θ is an **argument** of the complex number and is measured from the positive portion of the real axis to the point (a,b) in the complex plane.

If the graph of the complex number $a + bi$ is in the first or fourth quadrants, $\theta = \tan^{-1}(b/a)$ is an argument of the number and every argument is represented by the expression $\theta = \tan^{-1}(b/a) + 2n\pi$ where n is any integer.

For complex numbers $a + bi$ represented by points in the second and third quadrants, arguments have the form $\theta = \left(\tan^{-1}(b/a) + \pi\right) + 2n\pi$ where n is any integer.

- **De Moivre's theorem** states that the powers of any complex number $a + bi$ can be found in the following manner:

$$(a + bi)^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

- Another consequence of the fundamental theorem of algebra is that the equation $x^n - z = 0$ has n roots in the set of complex numbers.
- If a complex number $z = a + bi$ is written as $z = r(\cos\theta + i\sin\theta)$, then the nth roots of z can be found using the following formula:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right)\right)$$

where $k = 0,\ 1,\ 2,\ \ldots,\ n - 1$. There will be exactly n of these roots.

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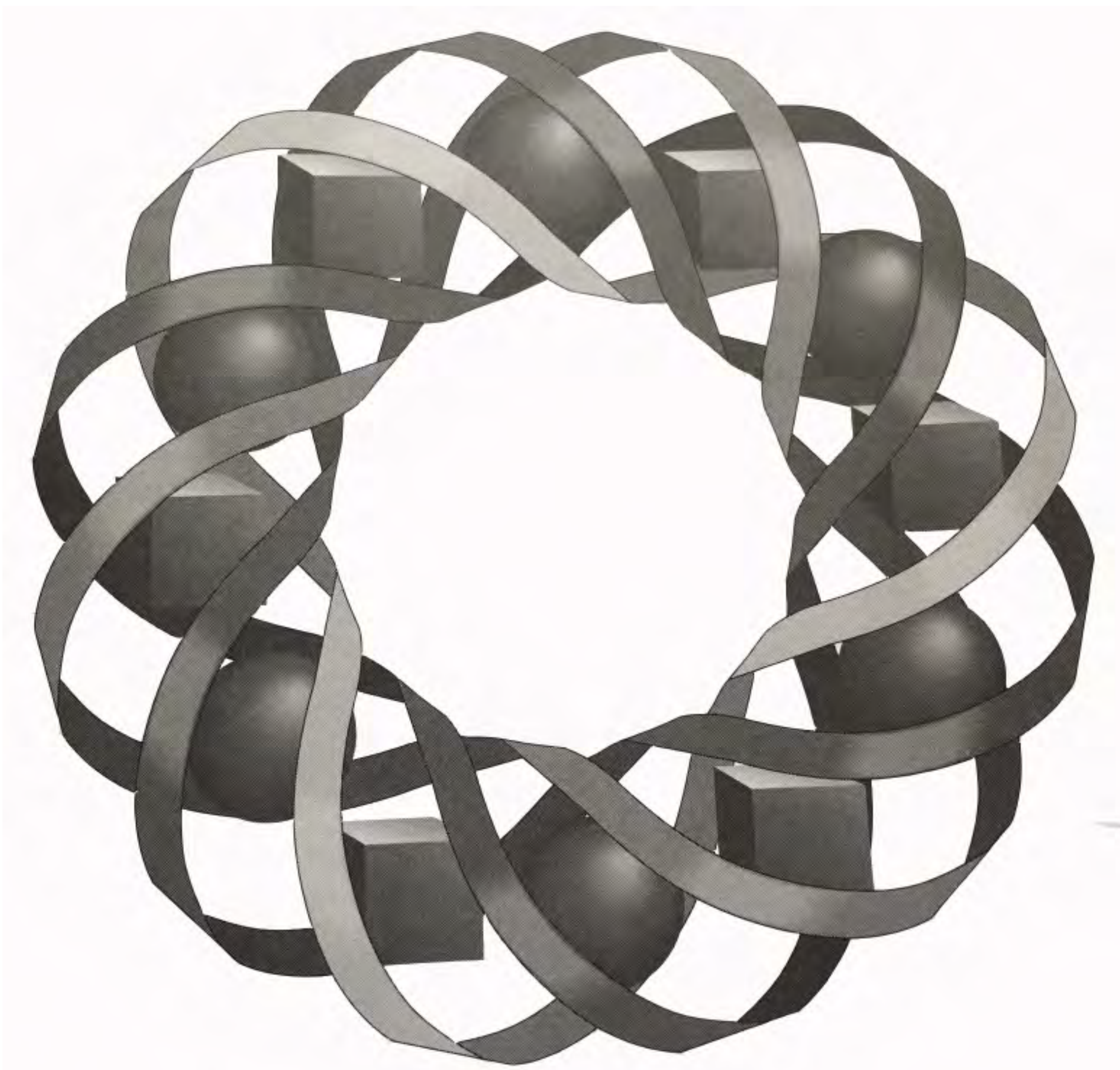
What Did You Expect, Big Chi?



How did the U.S. Surgeon General determine that cigarette smoking is hazardous to your health? In this module, you investigate the statistical tests that medical researchers—and others—use to assess the information they collect.

Bill Chalgren • Robbie Korin • Pete Stabio

What Did You Expect, Big Chi?

Introduction

The Office of the U.S. Surgeon General requires cigarette packages to carry warning labels. Before concluding that cigarette smoking is a health hazard, researchers collected and analyzed thousands of pieces of information. For example, Table **1** lists the cause of death, along with the smoking habits, for 1000 randomly selected males, ages 45–64.

Table 1: Causes of death for 1000 males, ages 45–64

	Cancer	Heart Disease	Other
Nonsmokers	56	153	141
Smokers	136	308	206

Source: U.S. Department of Health, Education, and Welfare.

Using similar information, researchers attempted to determine whether smoking increased an individual's chances of dying from cancer or heart disease.

Discussion

a. Researchers compared the observed number of deaths for smokers with the number of deaths that would be expected if smoking did not increase the risk of contracting cancer or heart disease.

At the time the data in Table **1** was gathered, about 40% of the adult male population of the United States were smokers. Based on this statistic, how many smokers would you expect to find in a random sample of 1000 adult males?

b. After researchers identified differences between what they observed and what they expected, they had to decide if these differences were the result of increased risk, or were due to the chance variations in outcomes that occur in the sampling process.

1. Of the 1000 deceased adult males represented in Table **1**, how many had been smokers?
2. How does this value compare with your response to Part **a**?
3. Do you think the discrepancy between the expected number of smokers and the observed number of smokers is due to chance variation, or due to the possibility that smoking increases a male's chances of dying between the ages of 45 and 64?

c. Describe some real-world events that have an extremely small probability of occurring and yet do occur.

Activity 1

Before conducting an experiment, researchers typically state a hypothesis about the possible outcomes. Once the data has been collected, the actual results may differ from their expectations. The researchers then must decide if the differences between the observed frequencies and the expected frequencies are due to chance, or to an incorrect hypothesis.

Mathematics Note

The **expected frequency** of an outcome in an experiment is the number of times the outcome should theoretically occur. The **observed frequency** is the actual number of times (a non-negative integer) that the outcome occurs. When describing an experiment in this module, the following notation will be used:

- n represents the number of trials in an experiment
- k represents the number of different outcomes possible in each trial
- O_i represents the observed frequency of the ith outcome, where $i \in \{1, 2, 3, \ldots, k\}$
- E_i represents the expected frequency of the ith outcome
- p_i represents the theoretical probability of the ith outcome

For example, consider an experiment that involves rolling a fair die 30 times. Since there are 30 trials, each with 6 possible outcomes, $n = 30$ and $k = 6$. Since each outcome is equally likely, $p_i = 1/6$ for $i = 1, 2, \ldots, 6$. The expected frequency of each outcome is $(1/6) \bullet 30 = 5$, so $E_1 = 5, E_2 = 5, \ldots, E_6 = 5$. Table **2** shows the expected frequency of each outcome, along with some sample results for this experiment.

Table 2: Expected and observed frequencies for 30 rolls of a die

Outcome	1	2	3	4	5	6
Expected Frequency	5	5	5	5	5	5
Observed Frequency	7	3	5	8	2	5

Discussion 1

a. In an experiment that involves 200 flips of a fair coin, what is the expected frequency of heads? Explain your response.

b. The frequency of an outcome can be thought of as a random variable. Do you think that the observed frequency of heads in the experiment will always be the same as the expected frequency? Explain your response.

c. Is it possible to get 185 heads when tossing a coin 200 times?

d. When the observed number of heads differs from the expected number, at what point might you suspect that the coin is not fair?

e. Using the data in Table **2**, how would you analyze the differences between the observed frequencies and the expected frequencies to determine if these variations are due to chance?

Exploration 1

In most areas of research, repeating an experiment again and again would require too much time and cost too much money. To reduce the need for repeating experiments, statisticians have developed a method for determining whether observed differences are significant, or simply due to chance.

Mathematics Note

The **chi-square statistic**, denoted χ^2, is a measure of the difference between what actually occurred in an experiment and what was expected to occur. In an experiment with k possible outcomes, the chi-square statistic is the sum of the ratios of the squared differences of the observed and the expected frequencies $(O_i - E_i)^2$ to the expected frequency, where $i \in \{1, 2, 3, \ldots, k\}$. This can be denoted as:

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

For example, the chi-square statistic for the data in Table **2** can be found as follows:

$$\begin{aligned}
\chi^2 &= \sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i} \\
&= \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \cdots + \frac{(O_6 - E_6)^2}{E_6} \\
&= \frac{(7-5)^2}{5} + \frac{(3-5)^2}{5} + \frac{(5-5)^2}{5} + \frac{(8-5)^2}{5} + \frac{(2-5)^2}{5} + \frac{(5-5)^2}{5} = 5.2
\end{aligned}$$

Once the chi-square value is calculated, it can be used to determine whether the differences between observed and expected frequencies are significant or the result of chance variation.

In the following exploration, you use histograms to investigate chi-square probability distributions. These distributions can be used in much the same way as the normal distribution described in the Level 6 module, "To Null or Not to Null."

a. Create a table to record the outcomes for an experiment that involves tossing a six-sided die. Your table should have rows and columns similar to those in Table **3** below.

Table 3: Calculation of chi-square

Outcome	**1**	**2**	**3**	**4**	**5**	**6**	**Sum of Row**
Expected Frequency (E_i)							
Observed Frequency (O_i)							
$O_i - E_i$							
$(O_i - E_i)^2$							
$(O_i - E_i)^2/E_i$							$\chi^2 =$

b. Determine the expected frequency for each outcome if the die is tossed 30 times. Record these values in your table.

c. Perform the experiment by tossing a die 30 times. Record the observed frequency of each outcome in your table and calculate the chi-square statistic for the data.

d. Use technology to simulate the experiment in Part **c** 99 more times and calculate the corresponding chi-square values.

e. Create a histogram of the frequencies of the chi-square values from Part **d**. Sketch a smooth curve that closely fits the histogram and describe its shape and characteristics. **Note:** Save your work for use in Exploration **2**.

f. Consider an experiment that involves tossing five coins 160 times and counting the number of heads that appear each time. Table **4** shows the chi-square values for 99 such experiments. Create a histogram of the frequencies of these chi-square values. Sketch a smooth curve that closely fits the histogram and describe its shape and characteristics.

Table 4: 99 chi-square values for tossing 5 coins 160 times

3.16	2.36	5.08	3.98	10.70	1.26	6.12	0.86	4.36
10.20	6.62	10.38	10.72	7.58	7.42	1.66	3.58	8.44
1.72	10.04	10.94	3.72	3.18	4.06	8.44	5.24	2.88
4.92	3.82	3.02	6.36	0.72	6.52	1.82	5.82	8.78
16.70	5.42	0.56	8.86	2.48	1.42	3.52	2.40	15.90
0.94	2.22	4.76	3.10	2.94	19.26	0.80	3.12	3.96
11.66	5.68	3.88	3.66	4.14	4.54	3.92	2.86	2.52
5.64	5.04	7.00	2.86	3.64	6.78	4.06	3.82	1.92
4.06	2.52	1.82	3.9	2.12	3.04	4.14	1.24	0.80
7.58	6.14	5.04	5.64	6.08	2.04	3.34	6.78	6.86
8.64	1.04	7.18	17.36	3.64	3.50	2.16	6.86	3.32

Discussion 2

a. Will the sum of the entries in the $O_i - E_i$ row always equal 0? Explain your response.

b.
1. How many different outcomes are possible when tossing an ordinary die?

2. What is the probability of each outcome?

c.
1. How many different outcomes are possible when tossing five fair coins and counting the number of heads that appear?

2. What is the probability of each outcome?

d. Compare the shapes of the two curves you sketched in Exploration **1**.

e. Do you think that the difference in the probabilities of the outcomes when tossing a die and tossing five coins has any effect on the shape of the two curves? Justify your response.

Exploration 2

a. Consider an experiment that involves 20 tosses of a four-sided die with faces labeled 1, 2, 3, and 4. Determine the number of possible outcomes on each toss, their probabilities, and the expected frequency of each.

b. Use technology to simulate the experiment described in Part **a** 99 times and calculate the corresponding chi-square values.

c. Create a histogram of the frequencies of the chi-square values from Part **b**. Sketch a smooth curve that closely fits the histogram and describe its shape and characteristics.

d. Repeat Parts **a–c** for an experiment that involves tossing a 12-sided die 60 times.

e. Compare the smooth curves you sketched in Parts **c** and **d** with those you sketched in Exploration **1**.

Discussion 3

a. Compare the number of outcomes for each of the four experiments performed in Explorations **1** and **2**.

b. What characteristic of the experiment do you think affects the shape of the curves sketched in the explorations? Explain your response.

Mathematics Note

As shown in Figure **1**, a probability distribution of chi-square values, unlike a normal distribution, is not symmetric.

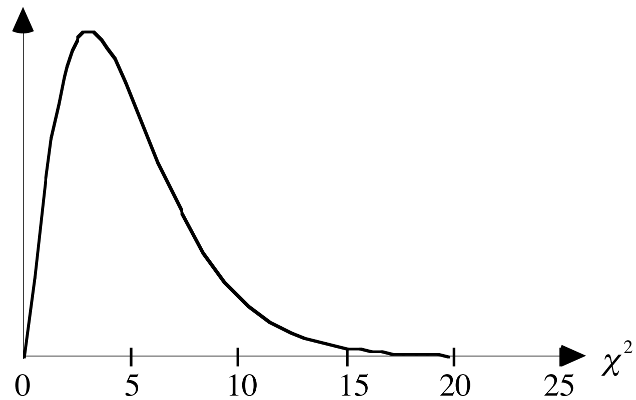


Figure 1: A chi-square distribution

As with a normal curve, however, the area of the region under the curve is 1. The shape of a chi-square distribution is determined by the **degrees of freedom** of the experiment. The degrees of freedom is based on the number of outcomes in an experiment.

For an experiment in which the events are independent and the theoretical probability for each outcome remains the same every time the experiment is performed, the degrees of freedom is equal to the number of possible outcomes minus 1, or $k - 1$.

For example, consider an experiment in which a single trial involves tossing three coins and counting the number of heads that appear. The four possible outcomes are 3 heads, 2 heads, 1 head, or 0 heads. Each trial is independent of other trials and the probability of obtaining each outcome remains the same. Since there are four possible outcomes, the degrees of freedom for the chi-square statistic in this experiment is $4 - 1 = 3$.

Note: In other types of experiments, the degrees of freedom must be determined using a different method and may not equal $k - 1$. You will investigate some examples in Activity **3**.

c. Explain how you know that the events in each experiment performed in the explorations are independent.

d. What are the degrees of freedom for each experiment in Explorations **1** and **2**?

e. As the degrees of freedom increase, what shape do you think the chi-square distribution will approach? Explain your response.

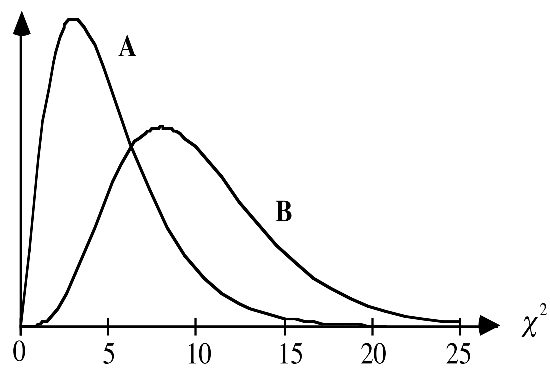
Assignment

1.1 **a.** Determine the degrees of freedom in an experiment that involves tossing one coin.

b. How many degrees of freedom are there in an experiment that involves drawing one card from an ordinary deck of playing cards and recording its suit? Explain your response.

c. How would the number of degrees of freedom in the playing-card experiment change if the face value of the card were recorded? Explain your response.

1.2 **a.** Based on your experience in the explorations, which of the chi-square probability distributions shown below involves more degrees of freedom? Justify your selection.



b. On each curve, estimate the chi-square value that divides the area under the curve in half. How do these values compare?

c. Consider the degrees of freedom and the chi-square values that divide the area under a curve in half. Based on your response to Part **a**, make a generalization about the relationship between the two.

d. Explain whether you think your generalization from Part **c** holds true for a value that divides the curve into any proportion.

1.3 **a.** Use the population in your mathematics class to complete the following table. Assume that males and females are equally likely to enroll in the class.
Note: Save your work for use in Problem **2.2**.

	Males	**Females**
Expected (E)		
Observed (O)		
$O - E$		
$(O - E)^2$		
$(O - E)^2/E$		

b. Determine the value of χ^2.

c. Determine the degrees of freedom for this experiment.

1.4 **a.** Develop a simulation for an experiment that involves flipping a fair coin 50 times.

b. Use the data from one simulation to complete the following table and calculate the corresponding chi-square value.

	Heads	**Tails**
Expected (*E*)		
Observed (*O*)		
$O - E$		
$(O - E)^2$		
$(O - E)^2/E$		

c. Repeat the simulation 40 times to obtain 40 chi-square values.

d. Create a stem-and-leaf plot of the 40 values for χ^2. On your plot, locate the χ^2 that best estimates the value that separates the upper 10% of the data from the lower 90%.

e. Describe how you might modify the simulation to improve your estimate of the chi-square value that divides the upper 10% of the data from the lower 90%.

* * * * *

1.5 A chamber of commerce estimates that the numbers of small, medium, and large businesses in the city are approximately equal. To verify this estimate, the group conducts a survey. The results of a survey of 200 randomly selected businesses are shown in the following table.

Size of Business	**Small**	**Medium**	**Large**
Observed Frequency	42	67	91
Expected Frequency			

a. Determine the degrees of freedom for this situation.

b. Calculate the value of χ^2 for this data.

1.6 The student council estimates that 30% of the high school population hold part-time jobs. To support their estimate, the council selects a random sample of 32 students. The results of the survey are shown in the following table.

Part-time Job	**Yes**	**No**
Observed Frequency	14	18
Expected Frequency		

a. Determine the degrees of freedom for this situation.

b. Calculate the value of χ^2 for this data.

* * * * * * * * * *

Activity 2

At first glance, the information shown in Table **5** seems to indicate that smoking is related to cancer and heart disease. By itself, however, this intuitive observation does not provide sufficient reason to accept the connection.

Table 5: Causes of death for 1000 males, ages 45–64

	Cancer	Heart Disease	Other
Nonsmokers	56	153	141
Smokers	136	308	206

When a claim is tested using statistical methods, it is stated in the form of a hypothesis. Recall that the **null hypothesis** (H_0) is a statement about one or more parameters. The **alternative hypothesis** (H_a) is a hypothesis that is true if the null hypothesis is false.

In the Level 6 module "To Null or Not to Null," you learned to test hypotheses that compare a sample mean with a population mean. That test allowed you to compare a single observed value with the expected one. The chi-square statistic provides a tool for testing how well a set of observed data matches the expected frequencies for two or more categories.

Mathematics Note

The results of hypothesis testing are not expressed as certainties or absolutes. When testing a hypothesis, it is customary to limit the maximum probability of rejecting a true null hypothesis. This probability is the **level of significance** or **significance level**.

On a chi-square distribution with the appropriate degrees of freedom, the significance level is indicated by the area under the curve to the right of a given chi-square value. The significance level identifies the set of values that would lead to the rejection of the null hypothesis. The corresponding region under the curve is the **critical region**.

For example, Figure **2** shows a chi-square distribution with 5 degrees of freedom. The area under the curve to the right of 9.24 is approximately 10% of the total area. At a 0.10 level of significance, therefore, the shaded portion under the curve represents the critical region. For any chi-square value greater than 9.24, the null hypothesis should be rejected at the 0.10 significance level. This also means that the probability of incorrectly rejecting the null hypothesis is 10%. In other words, approximately 10% of true null hypotheses would be rejected using this significance level.

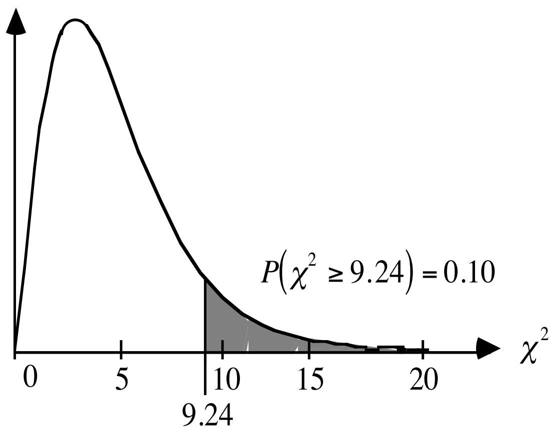


Figure 2: Chi-square distribution with 5 degrees of freedom

Chi-square values for various significance levels and degrees of freedom are often given in table form. Table **6** shows a portion of such a table. The value in the fifth row of the first column indicates that, in an experiment with 5 degrees of freedom, there is a 0.10 probability that a chi-square value greater than or equal to 9.24 will occur. This corresponds to the graph in Figure **2**. **Note:** A table with more values appears at the end of this module.

Table 6: Portion of a chi-square distribution table

	Significance Level				
Degrees of Freedom	**0.10**	**0.050**	**0.025**	**0.010**	**0.005**
1	2.71	3.84	5.02	6.63	7.88
2	4.61	5.99	7.38	9.21	10.60
3	6.25	7.81	9.35	11.34	12.84
4	7.78	9.49	11.14	13.28	14.86
5	9.24	11.07	12.83	15.09	16.75

Discussion 1

a.

1. Describe how to use a chi-square distribution table to determine if a chi-square value should result in a rejection of the null hypothesis at the 0.10 significance level with 5 degrees of freedom.
2. Explain how to use a graph to make the decision described above.

b. One traditional method of making decisions about statistical data is to formulate a hypothesis, then choose a significance level on which to base the decision to reject or fail to reject that hypothesis. At which significance level, 0.10 or 0.05, are you more likely to reject a true null hypothesis? Explain your response.

c.
1. If you select a significance level of 0.05, approximately what percentage of the time would you fail to reject a valid null hypothesis? Explain your response.
2. If you select a significance level of 0.01, approximately what percentage of the time would you fail to reject a valid null hypothesis? Explain your response.

d. A null hypothesis that is rejected at one significance level might not be rejected at another significance level. Why does this occur?

e. In Figure **2**, why would a chi-square value greater than 12.83 indicate that the results of the experiment may not be due to chance?

f. Explain why a given significance level can also be described as the probability of incorrectly rejecting a true null hypothesis.

g. Table **7** contains some information on fatal automobile accidents in Montana in 1993, according to the day of the week on which they occurred.

Table 7: Fatal accidents, by day of the week

Day	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Sun.
Fatal Accidents	15	16	15	23	34	38	25

Source: *1993 Montana Highway Patrol Annual Report*.

1. At first glance, the information in Table **7** seems to indicate that more fatal accidents occur on weekend days than on weekdays. What factors might contribute to higher numbers of fatal accidents on weekends?
2. State the hypothesis that you would test if you wished to determine whether this difference in number of fatal accidents is due to chance.
3. If you can reject your null hypothesis, would that imply anything about the potential causes for the observed data?

Exploration

In the following exploration, you use the chi-square statistic to decide whether or not an altered die is still fair. After collecting data, you test the null hypothesis H_0: The probability of each face occurring equals $1/6$.

a. Obtain a fair die, an altered die, and a cup. To ensure unbiased outcomes, shake both dice in the cup, then release the dice so that they roll several times before coming to rest. Roll the dice 60 times and record the observed frequencies for each die separately.

b. To determine if the altered die is still fair, the frequency of its outcomes must be compared with the expected frequencies from a fair die.

1. Calculate the expected frequencies for each outcome for a fair die.
2. Then calculate the chi-square statistic for your rolls of the fair die.

c. **1.** Collect the class data for χ^2 from Part **b**.

2. Create a stem-and-leaf plot of the class data.

3. On your plot, locate the χ^2 value that best estimates the value that divides the upper 10% of the data from the lower 90%. (This approximates the 0.10 significance level.)

d. **1.** Calculate the chi-square statistic for your altered die, using expected frequencies for a fair die.

2. Plot the chi-square value for the altered die on the stem-and-leaf plot from Part **c**.

e. Based on the position of the chi-square value for the altered die, decide if you believe the null hypothesis should be rejected. Write a statement defending your decision.

Discussion 2

a. If the altered die is still fair, should the chi-square value found in the experiment be large or small? Explain your response.

b. Compare the chi-square value you obtained for the altered die with those of others in your class.

c. If the observed frequencies equaled the expected frequencies for all six faces of the die, what would be the value of χ^2?

Assignment

2.1 Using the chi-square distribution table at the end of this module, write a statement that describes the approximate probability of obtaining a chi-square value at least as great as the value you obtained for the altered die in Part **d** of the exploration.

2.2 **a.** In Problem **1.3**, you recorded the numbers of males and females in your mathematics class and determined the value of a chi-square statistic. Can this statistic be used to test the null hypothesis below? Explain your response.

H_0: Males and females are equally likely to enroll in this math class.

b. Using the chi-square distribution table, write a statement that describes the approximate probability of obtaining a chi-square value that is greater than the value in Problem **1.3**.

2.3 **a.** Describe the relationship between the chi-square values and the degrees of freedom in an experiment. Is this relationship true for all significance levels?

b. Explain why this relationship occurs.

2.4 The table below shows one year's data for the number of fatal accidents by day of the week.

Day	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**	**Sun**
Observed (O)	15	16	15	23	34	38	25
Expected (E)							

Complete Parts **a–d** below to test H_0: The probability of a fatal accident occurring is the same for each day of the week.

a. Calculate the expected number of fatal accidents for each day and record these values in the table.

b. Calculate χ^2 for this data.

c. Test H_0 at the 0.05 significance level. Summarize the results of your test, indicating whether you rejected or failed to reject H_0, and explaining whether or not the differences between the observed and expected frequencies are due to chance at this significance level.

2.5 A greeting card distributor recommends that stores carry the following percentages of cards: 25% love/friendship, 30% birthday, 20% wedding/anniversary, 10% sympathy/get well, and 15% other/special occasion.

The new manager of a gift shop wants to see if actual sales closely follow the distributor's recommended percentages. The data collected for cards sold during one week are shown in the following table.

Type	**Love**	**Birthday**	**Wedding**	**Sympathy**	**Other**
Sales	54	71	42	19	43

a. State H_0 and H_a for this situation.

b. Find the value of χ^2 that could be used to test the null hypothesis.

c. Based on the value of χ^2 from Part **b**, would you reject or fail to reject H_0 at the 0.05 significance level? Make a sketch of the approximate chi-square distribution to justify your response. Include the significance level, the area that indicates rejection of H_0, and the location of the chi-square value from Part **b**.

2.6 According to genetic theory, when red snapdragons and white snapdragons are crossed, the first generation of hybrids will have pink flowers. When these hybrids are crossed, the second generation will produce flowers in the following ratio of colors: 1 red to 2 pink to 1 white (1:2:1).

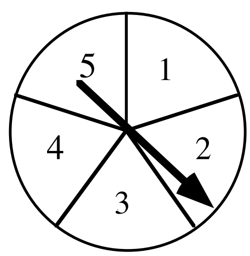
a. In a laboratory experiment involving snapdragons, the second generation consisted of 19 red, 59 pink, and 22 white flowers. Are these results consistent with genetic theory at the 0.05 significance level?

b. Can you be absolutely certain that your response to Part **a** is true? Explain your response.

* * * * *

2.7 A candy company claims that 30% of its popular candy mix is red, 30% is green, 20% is yellow, and 20% is brown. A random sample of 200 pieces of the mix contains 50 red, 54 green, 46 yellow, and 50 brown. Should the company adjust its current claims? Use a 0.05 significance level in your test.

2.8 A board game uses a spinner in order to determine the number of spaces a player may advance on each turn. The spinner is divided into five congruent sectors, as shown below.



One of the players thinks that the spinner is not fair. She tallies the results of 100 spins. The results of her experiment are shown in the following table.

Outcome	**1**	**2**	**3**	**4**	**5**
Observed Frequency	19	20	25	26	10

Determine an appropriate null hypothesis for this situation. Use a chi-square test with a 0.10 significance level to determine whether to reject or fail to reject your null hypothesis.

2.9 Toss a thumbtack into the air 100 times and record your observations. Is the probability that the tack lands point up the same as the probability that it lands point down? Use a chi-square statistic to test an appropriate hypothesis.

* * * * * * * * * *

Research Project

Formulate a null hypothesis regarding some topic of interest to you. Collect data and use the chi-square statistic to test your hypothesis. Write a report that includes the following:

- the null hypothesis
- a description of the method used to gather data
- the collected data
- the chi-square statistic generated
- your reasons for rejecting or failing to reject the null hypothesis
- a summary statement.

Activity 3

A clothing manufacturer plans to launch a new line of unisex apparel. To help maximize its appeal, the designers want to know if there is any relationship between gender and color preference. Do males tend to like blue or red, while females prefer purple or green?

In the previous activity, you used chi-square values to determine how well expected outcomes matched observed outcomes. In this activity, you use chi-square values to determine if two variables are dependent or independent.

Statistical dependence does not imply a cause-and-effect relationship between two variables. For example, consider an experiment that involves drawing 2 cards from a set of 4 red cards and 4 black cards. Although the probability of getting a red card on the second draw depends on the outcome of the first draw, the color of the first card does not *cause* the second draw to produce either a red or a black card.

Exploration

When the chi-square statistic is used to test whether or not two variables are dependent, the null hypothesis is stated assuming that the two variables are independent. If the null hypothesis is rejected, you treat the two variables as if they are dependent. In the following exploration, you will test the null hypothesis H_0: Color preference is independent of gender.

a. Do you think that color preference is independent of gender? For instance, given the colors blue, green, purple, and red, do you think that the distribution of color preferences in males will be about the same as in females? Record your prediction.

b. In order to test H_0, you must first design a survey and collect a sufficient amount of data. Record the gender and color preference for each person surveyed in a two-way table with headings like those in Table **8** below.

In order for the chi-square probability distribution to provide a reasonable model for an experiment, the expected frequency E_i for each possible outcome must be at least 5. If any cell in your expected frequency table contains a value less than 5, you must increase the sample size in the experiment. In other words, you must survey enough people to obtain expected frequencies of at least 5 for each outcome.

Table 8: Observed frequencies of color preferences

	Blue	**Green**	**Purple**	**Red**	**Total**
Female					
Male					
Total					

c. To record expected frequencies, create another two-way table with headings like those in Table **8**.

Since the totals in the right-hand column of the new table must be the same as those in Table **8**, enter these totals in the table now.

d. To find the expected frequency for females who prefer blue, complete the following steps.

1. Use the totals of the observed frequencies to determine the probability that a person in the sample is female.
2. Use the totals of the observed frequencies to determine the probability that a person in the sample prefers blue.
3. Your responses to Steps **1** and **2** are probabilities for events that are assumed to be independent. Use them to determine the probability that a person in the sample is female and prefers blue.
4. Use the sample size along with your response to Step **3** to determine the expected number of females who prefer blue. Record this number in the table from Part **c**.

e. Determine the expected frequencies of the other possible outcomes in the experiment and record them in the table.

f. In order to decide if color preference and gender are independent, you must next determine the degrees of freedom in the experiment. Table **9** shows some sample data for a survey of 70 people.

Table 9: Gender versus color preference

	Blue	**Green**	**Purple**	**Red**	**Total**
Female					34
Male					36
Total	22	15	16	17	70

Given these totals for the observed frequencies, determine the least number of values needed to allow you to find every other value in Table **9**. This value is the number of degrees of freedom for the experiment.

Mathematics Note

In an experiment to test the independence of two variables, the results can be displayed in a two-way table. In this case, the degrees of freedom can be calculated as follows, where r is the number of rows and c is the number of columns (not including the totals):

$$(r-1)\bullet(c-1)$$

For example, consider a survey in which a random sample of 100 students are asked to name their favorite academic subject. To test the independence of subject preference and gender, the researchers display the collected data in a two-way table, such as Table **10** below.

Table 10: Favorite subject of 100 students

	English	**Math**	**History**	**Science**	**Total**
Female	9	16	14	10	49
Male	17	10	11	13	51
Total	26	26	25	23	100

Considering only the cells that contain observed frequencies, this table has 2 rows and 4 columns. The degrees of freedom for this experiment can be calculated as follows:

$$(r-1)(c-1)=(2-1)(4-1)=1\bullet 3=3$$

g. Use the information given in the mathematics note to determine the degrees of freedom for your experiment involving gender and color preference.

h. Use the eight cells in your table of observed frequencies and the corresponding eight cells in the table of expected frequencies to calculate χ^2.

Discussion

a. When determining expected frequencies in Parts **d** and **e** of the exploration, why are gender and color preference assumed to be independent events?

b. Why would obtaining the same distribution of color preferences for males and females suggest that color preference is independent of gender?

c. In Parts **f** and **g** of the exploration, you determined the degrees of freedom for the experiment using two different methods. How did your answers compare?

d. If color preference and gender are found to be dependent, does this mean that being female causes a person to prefer a certain color? Explain your response.

Assignment

3.1 **a.** Test the hypothesis in the exploration at the 0.05 significance level and the appropriate degrees of freedom.

b. Summarize the results of your test, indicating whether you rejected or failed to reject H_0. Explain whether or not the differences between the observed and expected frequencies are due to chance at this significance level.

c. How would your decision regarding the null hypothesis change for a significance level of 0.025? of 0.005?

3.2 In the introduction to this module, you discussed the statistics shown in the table below, which shows cause of death for 1000 randomly selected males, ages 45–64, along with their smoking habits.

	Cancer	**Heart Disease**	**Other**	**Total**
Nonsmoker	56	153	141	350
Smoker	136	308	206	650
Total	192	461	347	1000

a. Use this data to test the null hypothesis, H_0: Cause of death is independent of smoking habits, at the 0.05 significance level.

b. Write a report on the results of your test, including your decision to reject or fail to reject the null hypothesis, the chi-square value obtained, the degrees of freedom in the experiment, the significance level used to test the null hypothesis, and the corresponding value of χ^2 in the chi-square distribution table.

3.3 A magazine subscription service wants to determine if readers' preferences for two national magazines are independent of geographical location. After selecting a sample from the national population, they classified readers by both magazine preference and geographical region. The table below shows the results of their survey.

	Magazine A	Magazine B	No Preference
Northeast	16	23	5
South	33	18	6
Midwest	15	20	7
West	20	32	5

a. Use this data to test the null hypothesis, H_0: Preference for magazine A or magazine B is independent of geographical region, at the 0.05 significance level.

b. Write a report on the results of your test, including your decision to reject or fail to reject the null hypothesis, the chi-square value obtained, the degrees of freedom in the experiment, the significance level used to test the null hypothesis, and the corresponding value of χ^2 from the chi-square distribution table.

3.4 The table below shows the results of another magazine preference survey, as described in Problem **3.3**.

	Magazine A	Magazine B	No Preference
Northeast	160	230	50
South	330	180	60
Midwest	150	200	70
West	200	320	50

a. In what ways does this data differ from the data in Problem **3.3**?

b. Does the probability that a person will prefer magazine A according to the table in Problem **3.3** differ from the probability according to the data in this table? Explain your response.

c. Do the probabilities of any preferences differ between the two tables? Explain your response.

d. Use the data in this table to test the null hypothesis, H_0: Preference for magazine A or magazine B is independent of geographical region, at the 0.05 significance level.

e. Use your results from Problems **3.3a** and **3.4d** to describe how the value of χ^2 was affected by the increase in sample size.

f. Do large sample sizes always imply large chi-square values? Explain your response.

* * * * *

3.5 Imagine that you are a quality control specialist at a manufacturing plant. As part of the quality control process, you select a sample from a week's production, then collect data on product performance by day manufactured. This information is shown in the table below.

	Mon.	**Tues.**	**Wed.**	**Thurs.**	**Fri.**
Acceptable	182	210	190	186	175
Not Acceptable	18	9	15	20	23

In your next report to the plant manager, you must analyze how product quality is related to the day manufactured. Write an appropriate statistical hypothesis, then use the data from the table to arrive at a decision about this hypothesis.

Your report should include an explanation of your decision and a discussion of the difference between statistical dependence (or independence) and a cause-and-effect relationship.

3.6 In the U.S. Armed Forces, the ready reserve includes those who are intended to assist active forces in a war. The table below shows some data collected in 1994 from a sample of 1000 ready-reserve personnel.

	Enlisted	**Officer**	**Total**
Female	101	21	122
Male	742	136	878
Total	843	157	1000

Consider the null hypothesis H_0: The gender of ready reserve personnel is independent of status. Use the data in the table to make a decision about this hypothesis.

* * * * * * * * * *

Summary Assessment

1. In the following problem, you will use an appropriate tool to generate 100 random digits (from 0 to 9, inclusive), then test to see if the digits have been generated in a truly random manner.

a. If the random number generator is truly random, what would you expect the probability of obtaining each digit to be?

b. Write a null hypothesis for this test.

c. Generate 100 random digits.

d. Record the frequency of each digit.

e. Calculate χ^2 and interpret its value at the 0.10 significance level.

f. Write a summary of your test.

2. In the exploration in Activity **3**, you used the chi-square statistic to test the hypothesis that color preference is independent of gender.

Choose another variable that might be related to gender. Design and conduct a survey to determine if this variable is independent of gender.

Write a report of your findings. Include a description of any factors that might create bias in your sample, an explanation of your decision to reject or not to reject the null hypothesis, the significance level used, the degrees of freedom in the experiment, the chi-square value obtained, and a discussion of the implications of your decision.

Module Summary

- The **expected frequency** of an outcome in an experiment is the number of times the outcome should theoretically occur. The **observed frequency** is the actual number of times the outcome occurs.
- The **chi-square statistic**, denoted χ^2, is a measure of the difference between what actually occurred in an experiment and what was expected to occur.

  In an experiment with *k* possible outcomes, the chi-square statistic is the sum of the ratios of the squared differences of the observed and the expected frequencies $(O_i - E_i)^2$ to the expected frequency, where $i \in \{1,\ 2,\ 3,\ \ldots,\ k\}$. This can be denoted as:

  $$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)}{E_i}$$
- A probability distribution of chi-square values, unlike a normal distribution, is not symmetric. As with a normal curve, however, the area of the region under the curve is 1.
- The shape of a chi-square distribution is determined by the number of **degrees of freedom** of the experiment.

  For an experiment in which the events are independent and the theoretical probability for each outcome remains the same every time the experiment is performed, the degrees of freedom equal the number of possible outcomes minus 1, or $k - 1$.
- In an experiment to test independence of two variables, the results of the experiment can be displayed in a two-way table. In this case, the degrees of freedom can be calculated as follows, where *r* is the number of rows and *c* is the number of columns:

  $$(r - 1) \bullet (c - 1)$$
- On a chi-square distribution with the appropriate degrees of freedom, the **significance level** is the area under the curve to the right of a given chi-square value. This represents the probability that a chi-square value greater than the given value can occur.

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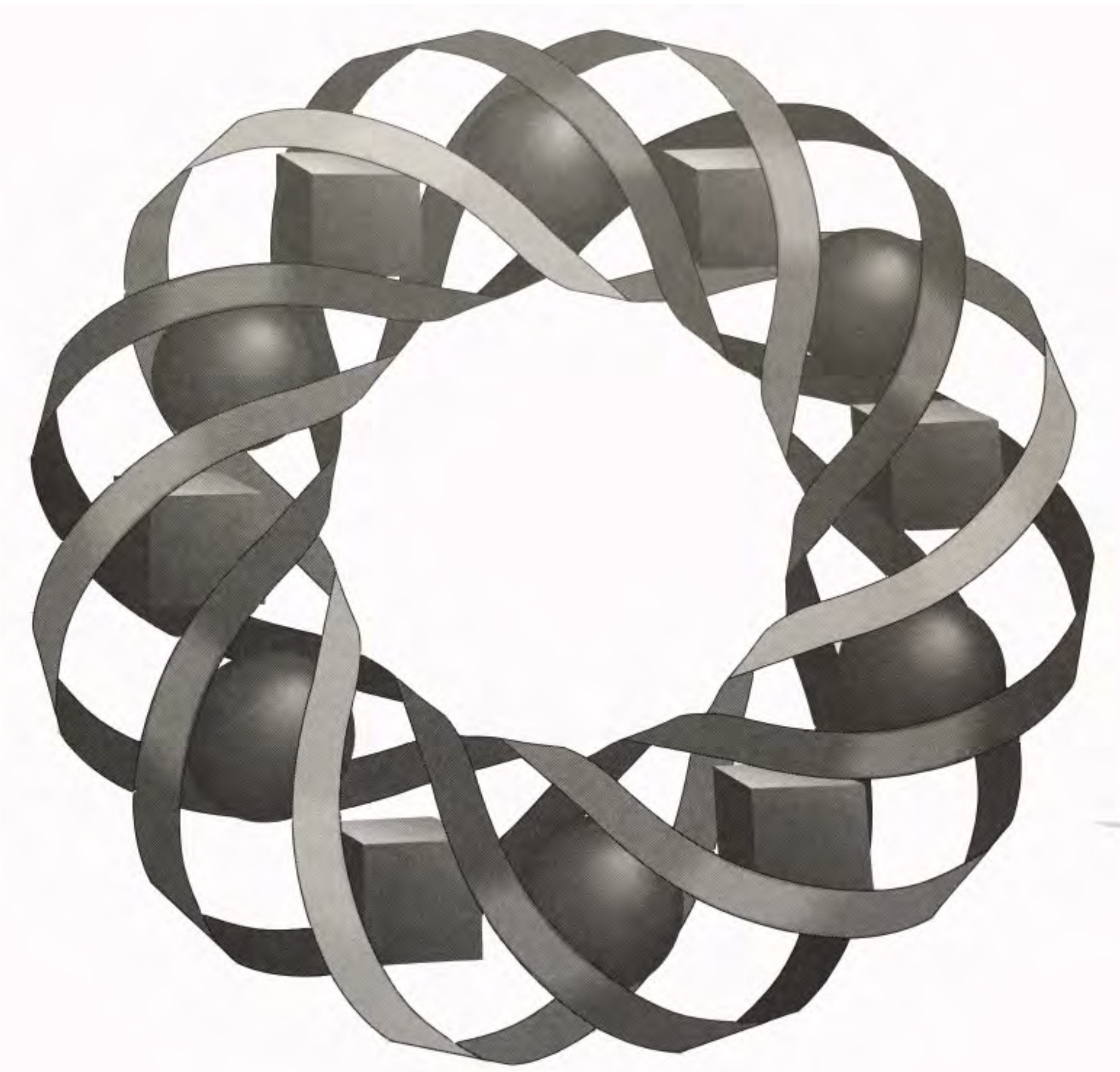
Chi-Square Distribution Table

	Significance Level				
Degrees of Freedom	**0.10**	**0.050**	**0.025**	**0.010**	**0.005**
1	2.71	3.84	5.02	6.63	7.88
2	4.61	5.99	7.38	9.21	10.60
3	6.25	7.81	9.35	11.34	12.84
4	7.78	9.49	11.14	13.28	14.86
5	9.24	11.07	12.83	15.09	16.75
6	10.65	12.59	14.45	16.81	18.55
7	12.02	14.07	16.01	18.48	20.28
8	13.36	15.51	17.53	20.09	21.96
9	14.68	16.92	19.02	21.67	23.59
10	15.99	18.31	20.48	23.21	25.19
11	17.28	19.68	21.92	24.72	26.76
12	18.55	21.03	23.34	26.22	28.30
13	19.81	22.36	24.74	27.69	29.82
14	21.06	23.68	26.12	29.14	31.32
15	22.31	25.00	27.49	30.58	32.80
⋮	⋮	⋮	⋮	⋮	⋮
20	28.41	31.41	34.17	37.57	40.00
30	40.26	43.77	46.98	50.89	53.67
40	51.80	55.76	59.34	63.69	66.77
50	63.17	67.50	71.42	76.15	79.49

Slow Down! You're Deriving over the Limit



How does the velocity of a falling object change over time? And exactly how fast is it traveling at any instant during its fall? In this module, you discover how to answer these questions.

Byron Anderson • Ruth Brocklebank • Mike Trudnowski

Slow Down! You're Deriving over the Limit

Introduction

In the Level 6 module "Mathematics in Motion," you used parametric equations to model the position, with respect to time, of freely falling objects. Recall that freely falling objects are acted on only by the force of gravity. Ignoring air resistance, for example, a ball dropped from some initial height is a freely falling object, as is a ball thrown with some initial velocity. In this module, you continue your exploration of this type of motion.

Discussion

a. Consider a ball dropped from an initial height of 100 m. What would a graph of the ball's height versus time look like? Explain your response.

b. **1.** Recall that velocity describes an object's change in position with respect to time. After 1 sec, the ball described in Part **a** is 95.1 m from the ground. Describe the ball's average velocity during this interval.

2. In this case, what does the sign of the velocity indicate?

c. Will the ball travel the same distance during each second of its fall? Explain your response.

Activity 1

In this activity, you collect data on the motion of a freely falling object and determine an equation that models its height with respect to time.

Exploration

a. Drop an object from an initial height of 2 m. As it falls, use a range finder and science interface device to collect data on the object's height with respect to time.

b. Create a scatterplot of the object's height versus time.

c. Determine an appropriate equation to model the data. **Note:** Save this equation, the scatterplot from Part **b**, and your data for use in Activity **2**.

d. Graph your function from Part **c** on the scatterplot from Part **b**.

e. Use your equation to predict the time required for the object to reach a height of 0 m.

f. The acceleration due to gravity is 9.8 m/sec^2 in a direction toward Earth's center. Estimate the ball's velocity at the time it hit the ground.

Discussion

a. Describe your scatterplot of the data for the falling object.

b. What is represented by the slope of a line containing any two points on the scatterplot?

c.
1. Describe the equation you obtained in the exploration.
2. How well does your equation appear to model the data?

d. Recall that the height of a freely falling object after t sec can be described by the following equation:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0$$

where g is the acceleration due to gravity, v_0 is the object's initial velocity in the vertical direction, and h_0 is the object's initial height.

1. Given this general equation, describe a function that would model the height with respect to time of a ball dropped from the same initial height as in the exploration.
2. Compare this equation to the one you obtained in the exploration.

e. Figure **1** shows a graph of height versus time for a ball thrown straight into the air.

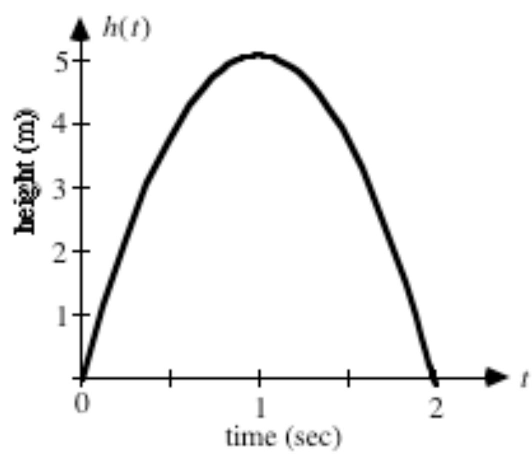


Figure 1: Graph of distance versus time

1. During what time interval is the ball's velocity positive?
2. During what time interval is the ball's velocity negative?
3. What is the ball's velocity when it reaches its highest point?

Assignment

1.1 Consider an object propelled upward at a velocity of 49 m/sec from the top of a 98-m tower.

a. Determine a function $h(t)$ that models the object's height with respect to time and graph it.

b. Identify the interval for which the object's velocity is positive.

c. Identify the interval for which the object's velocity is negative.

d. **1.** What is the greatest height reached by the object?

2. How long will it take the object to reach this height?

3. What is the velocity of the object when it reaches this height?

e. How many seconds will it take the ball to return to its initial height of 98 m?

1.2 A ball dropped from the top of a tower strikes the ground after 3 sec.

a. How tall is the tower?

b. Determine an equation that describes the ball's height with respect to time.

1.3 **a.** If an object is propelled straight up from the ground with an initial velocity of 34.3 m/sec, how long will it remain in the air?

b. What will the object's velocity be just before it strikes the ground?

c. If a similar object remained in the air for 6 sec, what was its initial velocity?

1.4 Consider a ball thrown straight up from the ground with a velocity of 49 m/sec.

a. Write an equation that models the ball's height with respect to time.

b. Determine how long the ball will remain in the air.

c. Determine the height of the ball at the end of each second of its flight.

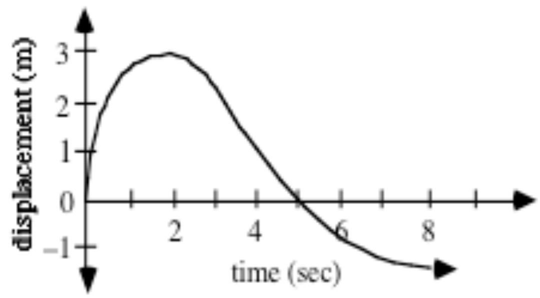
d. Determine the ball's average velocity during each 2-sec interval of its flight. Record these values in a table like the one shown below.

Interval (sec)	**Average Velocity**
$[0, 2)$	
$[2, 4)$	
$\vdots$	

e. What do the values in the table from Part **d** indicate about the ball's flight?

* * * * *

1.5 Consider a wind-up toy moving along a linear track. The graph below shows its displacement over time. In this case, a positive displacement indicates movement to the right of the starting position. Assume that the toy's initial velocity is 0 m/sec.



a. During what time interval is the toy's velocity positive? During what interval is its velocity negative? Explain your response.

b. What is the toy's average velocity during the first 2 sec?

c. When did the toy change direction? Describe how this change is indicated on the graph.

d. When did the toy return to its starting position? Describe how this is indicated on the graph.

e. During which 1-sec interval does the magnitude of the toy's velocity appear to be the greatest? How is this indicated on the graph?

f. Describe the toy's motion from the time it started moving until the time it stopped.

1.6 The Empire State Building in New York City is approximately 381 m high. Consider the motion of an object dropped from the top of this building.

a. Write an equation that models the height of the object over time.

b. Determine the time required for the object to reach the ground.

c. Estimate the object's velocity just before it reaches the ground.

d. Because of the danger to pedestrians, it is illegal to drop objects from tall buildings. Write a statement explaining how such laws protect the public.

1.7 The height of a ball in meters after t seconds can be modeled by the following equation: $h(t) = 24.5 + 19.6t - 4.9t^2$.

a. From what height was the ball thrown?

b. What initial velocity was given to the ball?

c. How long was the ball in the air?

d. What was the maximum height reached by the ball?

e. What was the ball's velocity just before it struck the ground?

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Activity 2

In Activity **1**, you investigated the change in position with respect to time for a falling object, as well as its average velocity over particular intervals of time. In this activity, you examine one method for approximating an object's velocity at any given instant. This is known as **instantaneous velocity**.

Exploration

a. Examine your scatterplot of the falling-ball data from Activity **1**. Select a subset of the data that appears to accurately describe the motion of the ball for approximately 0.5 sec. Identify a data point *A* so that there are an equal number of data points before and after *A*.

b. One way to estimate the ball's instantaneous velocity at point *A* is to examine the average velocities for intervals that include *A*.

1. Using your chosen subset of data, identify the data point *P* that is farthest to the left of *A*.
2. Draw the line that contains *A* and *P*. Your graph should now resemble the one shown in Figure **2** below.

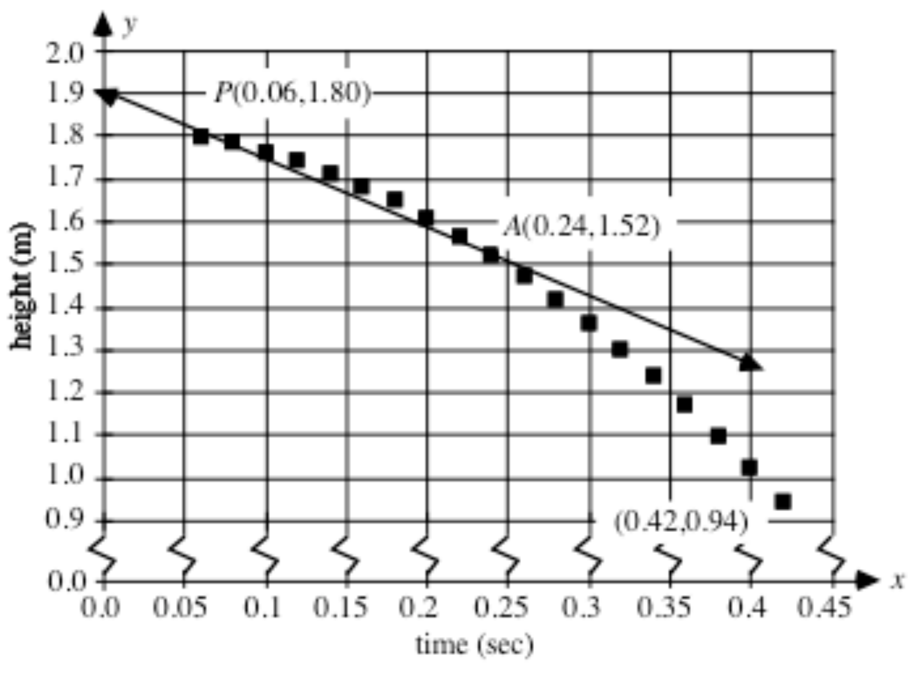


Figure 2: Graph of sample falling-ball data

3. Calculate the difference *d* in the *x*-coordinates of this pair of points. For *A* and *P* in Figure **2**, for example, this value is 0.24 – 0.06 = 0.18 sec.
4. Determine the slope of the line from Step **2** and describe what this value represents in terms of the falling ball.

c. Draw a line through A and a data point Q that is d sec to the right on the scatterplot. In Figure **2**, for example, this is the point (0.42,0.94). Determine the slope of this line and describe what its value represents in terms of the falling ball.

Record your findings from Parts **b** and **c** in a table with headings like those in Table **1**. In this case, $(x - d)$ is the x-coordinate of a point P to the left of A, while $(x + d)$ is the x-coordinate of a point Q to the right of A. (The cells in the first row of Table **1** show the appropriate sample values from Figure **2**.)

Table 1: Approximating velocity at point $A(x, y)$

Value of d	$(x - d)$	y-coord. of P	Slope of $\overleftrightarrow{PA}$	$(x + d)$	y-coord. of Q	Slope of $\overleftrightarrow{AQ}$
0.18	0.06	1.80	–1.56	0.42	0.94	–3.22
0.16						
0.14						
⋮						
0.04						
0.02						

d. What is the relationship among the slope of $\overleftrightarrow{PA}$, the slope of $\overleftrightarrow{AQ}$ and the instantaneous velocity of the ball at A? Explain your response.

e. Repeat Parts **b** and **c**, using a pair of points P and Q that are closer to A. Record your results in Table **1**.

f. Continue the process described in Parts **b** and **c** for each successive pair of points, until you have used the pair closest to A.

g. Use your results in Parts **b–f** to estimate the ball's instantaneous velocity at point A.

Discussion

a. The slope of the line that passes through the first and last data points in Figure **2** is:

$$m = \frac{1.80 - 0.94}{0.06 - 0.42} \approx -2.39 \text{ m/sec}$$

Describe what this value represents in terms of the falling ball.

b. Describe how you approximated the instantaneous velocity of the falling ball at point A.

c. In Activity **1**, you used a regression equation to model the data for the falling ball. Figure **3** shows the graph of a function, $f(x) = -4.65x^2 - 0.14x + 1.82$, that models the scatterplot in Figure **2**.

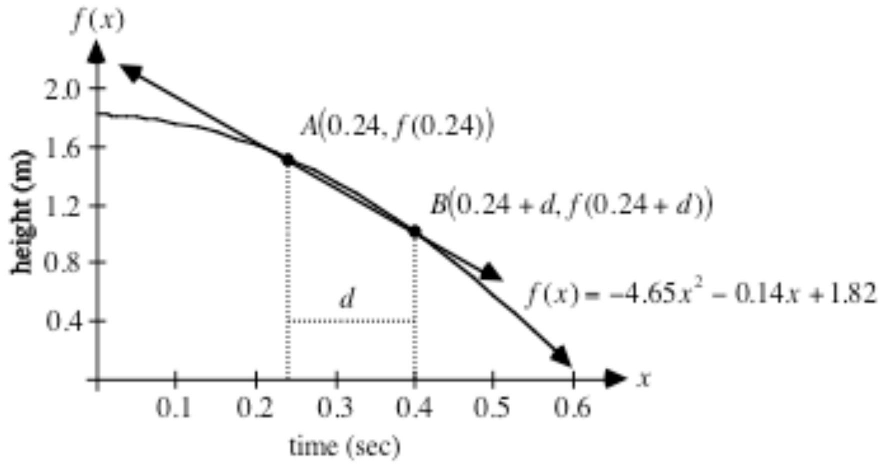


Figure 3: Regression equation for sample data

1. A **secant** is a line that intersects a curve but is not tangent to it. Describe how a secant such as $\overleftrightarrow{AB}$ in Figure **3** can be used to approximate the velocity of the falling ball at 0.24 sec.
2. In Figure **3**, the coordinates of *B* are given in terms of *d* and the coordinates of *A*. How does the value of *d* affect the estimate of instantaneous velocity?
3. As *d* approaches 0, the slope of $\overleftrightarrow{AB}$ approaches the slope of the line tangent to the curve at point *A*. What does the slope of the line tangent to the curve at *A* represent?

d. Table **2** lists the slopes of some secant lines through *A* (0.24,1.52) on the curve in Figure **3**. The values in the columns labeled "Slope" were calculated using the following formulas, where $x = 0.24$:

$$\frac{f(x-d)-1.52}{(x-d)-0.24} \text{ and } \frac{f(x+d)-1.52}{(x+d)-0.24}$$

Describe how you could use this table to approximate the ball's instantaneous velocity at 0.24 sec.

Table 2: Slopes of secant lines through *A*

d	**(*x* – *d*)**	***f*(*x* – *d*)**	**Slope**	**(*x* + *d*)**	***f*(*x* + *d*)**	**Slope**
0.18	0.06	1.798	–1.54	0.42	0.942	–3.21
0.16	0.08	1.781	–1.63	0.40	1.023	–3.11
0.14	0.10	1.762	–1.73	0.38	1.097	–3.02
0.12	0.12	1.739	–1.83	0.36	1.168	–2.93
0.10	0.14	1.711	–1.91	0.34	1.236	–2.84
0.08	0.16	1.682	–2.03	0.32	1.300	–2.75
0.06	0.18	1.646	–2.10	0.30	1.360	–2.67
0.04	0.20	1.608	–2.20	0.28	1.417	–2.58
0.02	0.22	1.566	–2.30	0.26	1.471	–2.45

e. What information would allow you to obtain a better approximation of the velocity of the ball at 0.24 sec? Explain your response.

Assignment

2.1 A group of students obtained the following data during a ball-drop experiment.

Time (sec)	Height (m)
0.22	1.564
0.23	1.542
0.24	1.519
0.25	1.494
0.26	1.469
0.27	1.443
0.28	1.416

a. Determine the quadratic regression equation that models the data above and create a graph of this equation.

b. Choose an instant in time x. Use the quadratic regression equation to predict the ball's height y at that time. Plot this point (x,y) on the graph from Part **a**.

c. Use a spreadsheet with headings like those in Table **2** to approximate the ball's instantaneous velocity at this point.

d. Describe what the instantaneous velocity represents in terms of the line tangent to the curve at this point.

2.2 **a.** Select three more instants in time on the graph of the regression equation from Problem **2.1**. Approximate the ball's instantaneous velocities at these times and record these values, along with the one from Problem **2.1c**, in a table like the one below.

Time (sec)	Velocity (m/sec)

b. Use your results in Part **a** to create a scatterplot of velocity versus time.

c. Describe any trends you observe in the scatterplot and determine an appropriate model for the data.

d. What does your model in Part **c** indicate about the ball's change in velocity with respect to time?

2.3 Consider a projectile shot upward with an initial velocity of 98 m/sec from a height of 196 m above the ground.

a. Determine and graph a function $h(t)$ that models the height of the projectile with respect to time.

b. Find the projectile's average velocity during each of the following intervals:

1. from $t = 0$ sec to $t = 2$ sec

2. from $t = 18$ sec to $t = 20$ sec

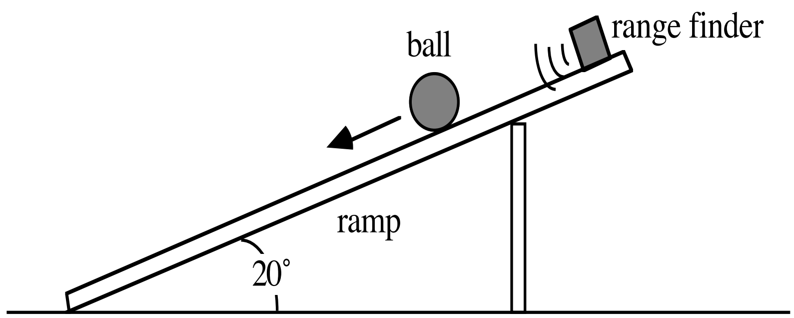
3. from $t = 2$ sec to $t = 18$ sec

c. The total distance traveled by the projectile from $t = 2$ sec to $t = 18$ sec is approximately 628 m. Half of this distance is in the direction away from Earth's surface, while the other half is directed towards Earth's surface. Explain why the average velocity for this interval cannot be calculated as shown below:

$$\frac{628 \text{ m}}{16 \text{ sec}} = 39.25 \text{ m/sec}$$

d. Approximate the projectile's instantaneous velocity at $t = 16$ sec.

2.4 The diagram below shows an experiment designed to measure the acceleration of a ball rolling down a ramp.



The following table shows some of the data collected in this experiment.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	0.443	1.0	2.607
0.1	0.571	1.1	2.931
0.2	0.718	1.2	3.276
0.3	0.885	1.3	3.640
0.4	1.072	1.4	4.024
0.5	1.279	1.5	4.427
0.6	1.505	1.6	4.850
0.7	1.751	1.7	5.293
0.8	2.016	1.8	5.755
0.9	2.302	1.9	6.238

a. Predict the shape of a scatterplot of this data.

b. Create a scatterplot of the data and compare its shape with your prediction.

c. Determine an appropriate regression model for the data and graph it on the scatterplot from Part **b**.

d. Use your model to determine the distance the ball has traveled after 1 sec, 2 sec, 3 sec, and 4 sec. Approximate the ball's instantaneous velocity at each of these points.

e. Use your results in Part **d** to create a scatterplot of velocity versus time.

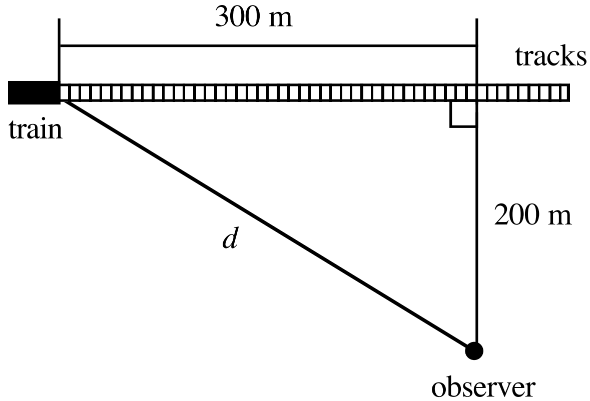
f. Determine an appropriate model for the scatterplot in Part **e**.

g. What does your model in Part **f** indicate about the ball's change in velocity with respect to time?

* * * * *

2.5 As a train sounding its whistle approaches and passes an observer, the pitch of the sound changes. (This is known as the Doppler effect.)

a. The following diagram shows the locations of the observer and the train at the moment when the train was first heard ($t = 0$ sec). What is the distance d between the observer and the train at this time?



b. The train is traveling at a constant velocity of 30 m/sec. When will it be closest to the observer?

c. Write an equation that describes the distance from the train to the observer with respect to time.

d. Do you believe that the rate at which the distance between the observer and the train changes is constant? Explain your response.

e. Graph the equation from Part **c** and explain whether or not the graph confirms your response to Part **d**.

f. Approximate the instantaneous rate of change in the distance from the observer to the train at $t = 5$ sec and at $t = 10$ sec.

* * * * * * * * * *

Activity 3

In the previous activity, you used the average velocity over smaller and smaller intervals to approximate instantaneous velocity. In this activity, you investigate a method of precisely determining a rate of change at any given instant.

Exploration

Figure **4** shows a portion of a graph of a semicircle in the first quadrant, a point on the semicircle $(x, f(x))$, and two secant lines *u* and *t*. As described in Activity **2**, the slope of the line tangent to the curve at $(x, f(x))$ can be approximated using the slope of *u* or *t*. Better approximations of the slope are found for values of *h* near 0.

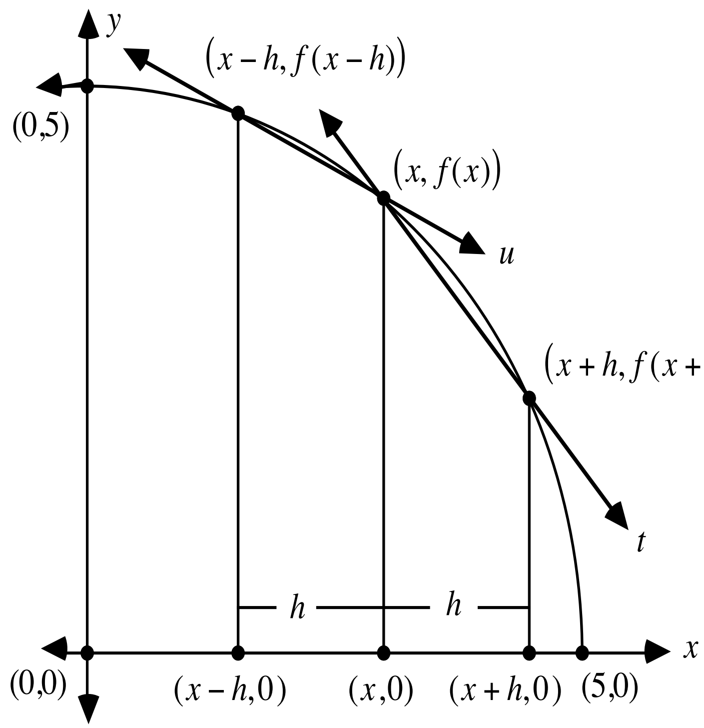


Figure 4: Approximating the slope of a tangent line

a.
1. Construct a circle with center at the origin of a two-dimensional coordinate system and a radius of 5 units.
2. As shown in Figure **4**, construct a point $(x, f(x))$ on the circle. Record its coordinates.
3. Construct a line perpendicular to the *x*-axis through $(x, f(x))$. Label the point of intersection with the axis $(x,0)$.

b. **1.** Construct a moveable point $(x+h,0)$ on the x-axis between $(x,0)$ and (5,0).

2. Construct the point $(x-h,0)$ by reflecting the point $(x+h,0)$ in the line created in Part **a**. (This guarantees that the distance from $(x+h,0)$ and $(x-h,0)$ to $(x,0)$ will be the same, h units.)

c. **1.** The point $(x+h, f(x+h))$ is the intersection of the circle with the line perpendicular to the x-axis passing through $(x+h,0)$. Construct the point $(x+h, f(x+h))$.

2. The point $(x-h, f(x-h))$ is the intersection of the circle with the line perpendicular to the x-axis passing through $(x-h,0)$. Construct the point $(x-h, f(x-h))$.

d. **1.** Construct two secant lines: one passing through $(x, f(x))$ and $(x+h, f(x+h))$, the other through $(x, f(x))$ and $(x-h, f(x-h))$.

2. Measure and record the slopes of the two secants.

e. To decrease the size of h, move the point $(x+h,0)$ toward the point $(x,0)$. As you move $(x+h,0)$, the point $(x-h,0)$ should move the same distance towards $(x,0)$.

As the points move closer together, the secant lines approach the tangent line through the point $(x, f(x))$. Observe the measures of the slopes of the secant lines as h approaches 0.

Use your construction to approximate the slope of the line tangent to the circle at $(x, f(x))$.

f. Use the construction to approximate the slopes of the lines tangent to three other points on the circle.

Discussion

a. If $f(x)$ represents distance and x represents time, how is the slope of a secant line related to average velocity?

Mathematics Note

Figure **5** shows a secant line passing through two points on the graph of a function $f(x)$. Note that *h* is depicted as a positive real number.

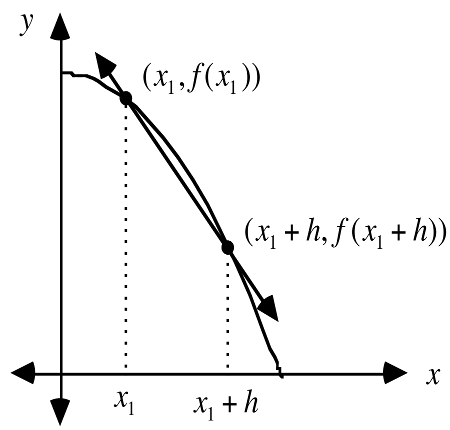


Figure 5: A secant to a curve

The slope of the secant line can be expressed as follows:

$$\frac{f(x_1 + h) - f(x_1)}{h}$$

By assigning a value to *h* that is close to 0, the point with coordinates $(x_1 + h, f(x_1 + h))$ can be moved very close to the point $(x_1, f(x_1))$. When this occurs, the slope of the secant line is a good approximation of the slope of a tangent line at $(x_1, f(x_1))$. A better approximation can be obtained by assigning *h* a value even closer to 0. This process can be repeated indefinitely and provides the basis for the following definition.

The slope of the tangent line at $(x, f(x))$ is defined as:

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists. If this limit exists, it is the **derivative** of the function $f(x)$ at x and represents the **slope of the curve** at the point $(x, f(x))$. The derivative of *f* at x is denoted by $f'(x)$. The slope of the curve at a point is equal to the slope of the tangent at that point.

For example, consider an object dropped from a height of 10 m. The height of the object with respect to time is described by the function $f(t) = -4.9t^2 + 10$, where *t* represents time in seconds. In this case, the slope of the curve at the point $(t, f(t))$ represents the instantaneous velocity at *t*.

To find the function that describes instantaneous velocity with respect to time, you can determine the derivative $f'(t)$ as follows:

$$
\begin{aligned}
f'(t) &= \lim_{h\to 0}\left(\frac{f(t+h)-f(t)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-4.9(t+h)^2+10-\left(-4.9t^2+10\right)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-4.9t^2-9.8ht-4.9h^2+10+4.9t^2-10}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-9.8ht-4.9h^2}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{h(-9.8t-4.9h)}{h}\right)\\
&= \lim_{h\to 0}(-9.8t-4.9h)\\
&= -9.8t
\end{aligned}
$$

b. On Earth, the height of a freely falling object after t sec can be described by the equation $h(t)=-4.9t^2+v_0t+h_0$, where v_0 is the object's initial velocity in the vertical direction, and h_0 is the object's initial height.

In this situation, what does the derivative $h'(t)=-9.8t+v_0$ represent?

c. Consider the linear function $f(x)=-2x+3$. Explain why $f'(x)=-2$.

d. Figure **6** below shows a graph of height versus time for an object propelled upward with an initial velocity of 9.8 m/sec from a height of 196 m. Explain why $h'(t)=0$ when the object reaches its highest point.

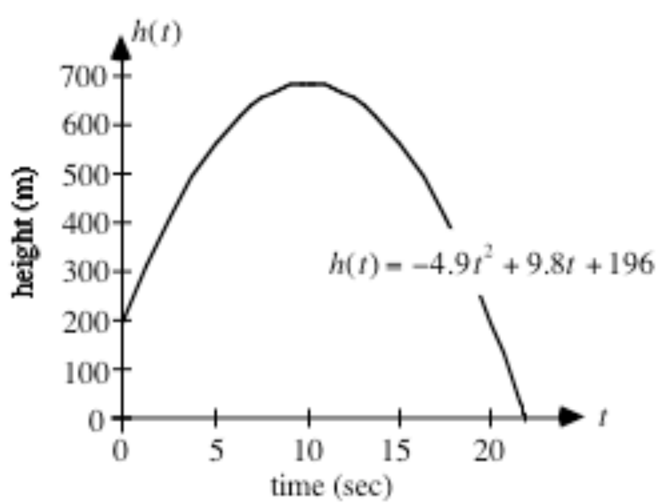


Figure 6: Graph of $h(t)=-4.9t^2+9.8t+196$

Assignment

3.1 **a.** Use the definition given in the previous mathematics note to find the derivative of the linear equation $f(x) = 3x - 5$.

b. How is the derivative of $f(x)$ related to the slope of the line?

c. Compare the degree of $f(x)$ with the degree of its derivative $f'(x)$.

3.2 The regression equation $h(t) = -4.65t^2 - 0.14t + 1.82$ was used to model data collected in a ball-drop experiment.

a. Find its derivative $h'(t)$.

b. Graph $h(t)$ and $h'(t)$ on the same coordinate system.

c. Describe what the coordinates of corresponding points on each graph represent in terms of the falling ball.

d. Compare the degree of $h(t)$ to the degree of its derivative $h'(t)$.

3.3 Consider a quadratic function whose derivative is $f'(x) = 2x - 7$.

a. For what value of x does the slope of the graph of $f(x)$ equal 0?

b. Over what interval are the values of $f(x)$ increasing?

c. Over what interval are the values of $f(x)$ decreasing?

3.4 **a.** Use the definition given in the previous mathematics note to find the derivative of $f(x) = x^2$.

b. Use a symbolic manipulator to verify your response to Part **a**.

c. Find the slope of the graph of $f(x)$ at $x = -3$ and at $x = 15$.

d. For what value of x is the slope of the graph 0? What is the significance of the point that corresponds to this value of x?

3.5 Consider the function $f(x) = 2x^3 - 3x^2 + 4x + 2$.

a. To determine the derivative of this function, expand the expression below, then evaluate the limit of this expression as h approaches 0:

$$\frac{f(x+h) - f(x)}{h}$$

b. Use technology to find the derivative of $f(x)$ and compare it to the limit found in Part **a**.

c. Compare the degree of $f(x)$ to its derivative $f'(x)$.

3.6 **a.** The graph of a semicircle with center at the origin and radius 5 is defined by the function $f(x) = \sqrt{25 - x^2}$. Use technology to find the derivative of this function.

b. Evaluate the derivative at $x = 1.5$, $x = 2.5$, and $x = 3.5$.

c. Recall that a radius of a circle is perpendicular to the tangent at the point of tangency. On a coordinate plane, two lines are perpendicular if the product of their slopes is –1.

Use these facts to demonstrate that the derivative is the slope of the tangent at the three points described in Part **b**.

3.7 **a.** Write a function $f(r)$ that describes the volume of a cylinder in terms of the radius r when its height is twice the radius.

b. Graph the function found in Part **a**.

c. What does $f'(r)$ represent?

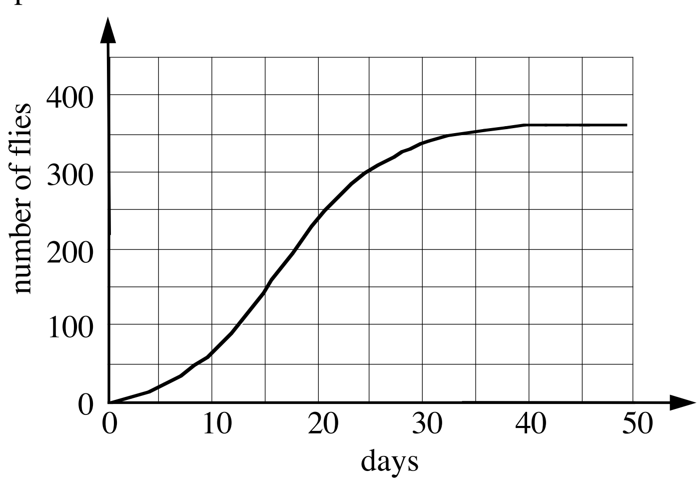
* * * * *

3.8 Consider the general linear equation $f(x) = mx + b$.

a. Use the definition of a derivative to find $f'(x)$.

b. What is the mathematical meaning of $f'(x)$ for a linear equation?

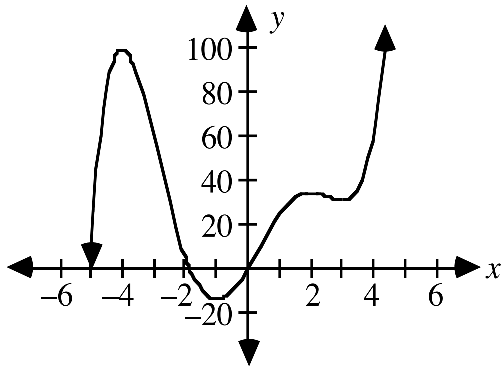
3.9 The following graph models the number of fruit flies in a laboratory population over time.



a. Determine the average rate of change in the fruit fly population from day 15 to day 35.

b. Determine the instantaneous rate of change in the population on day 25.

3.10 Consider the graph of the function $f(x) = 0.2x^5 - 5x^3 + 5x^2 + 24x$:



a. Describe the intervals on the graph where the slopes of the tangents to the curve are:

1. positive

2. negative.

b. Use the intervals you described in Part **a** to predict the points for which the derivative of the function is 0.

c. Use technology to determine the derivative of the function, then test your predictions from Part **b**.

d. Describe the significance of the points where the slope of the tangent to the curve is 0 in terms of the graph of the function.

* * * * * * * * * *

Summary Assessment

Understanding rates of change is important in business and economics. To most companies, for example, the bottom line means profit. The relationship between profit (P), sales or revenue (R), and costs (C) can be described by the equation $P = R - C$. For manufacturers, these quantities can be described by functions in terms of the number of units (x) produced and sold.

The derivative of a profit function determines the instantaneous rate of change in the profit for a given number of units. This is known as *marginal profit*. Similarly, the derivative of a cost function determines marginal cost, the instantaneous rate of change in the cost for a given number of units. In the same manner, the derivative of the revenue function determines the marginal revenue.

1. Consider a company that makes only one product. The cost of producing x of these items can be described by the function $C(x) = x^2 + \$80,000$ for $x > 0$.

- **a.** The company sells each item it produces for $750. Write an expression for R in terms of x.
- **b.** Given that $P = R - C$, write an expression for P in terms of x.

2.

- **a.** Graph $P(x)$, $R(x)$, and $C(x)$ on the same coordinate system.
- **b.** Explain what the graphs show about the relationship between profit, revenue, and cost for the company.

3. Determine the equations for the marginal profit, the marginal revenue, and the marginal cost.

4. Since the derivative of a cost function for a particular x is the instantaneous rate of change in the cost, the marginal cost approximates the additional cost of producing one more item $(x + 1)$. If the company has already produced 200 items, what is the additional cost of producing the 201st item?

5. Determine the number of items for which the company will receive the maximum profit. Compare the marginal revenue and marginal cost for this number of items.

6. Imagine that you are a business consultant. How many items would you advise the company to produce? Explain your response.

Module Summary

- A **secant** is a line that intersects a curve but is not tangent to it.
- The **derivative** of a function at the point $(x, f(x))$, denoted by $f'(x)$, is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This value is the slope of the tangent line to the function at $(x, f(x))$ and represents the instantaneous rate of change in the function with respect to x.

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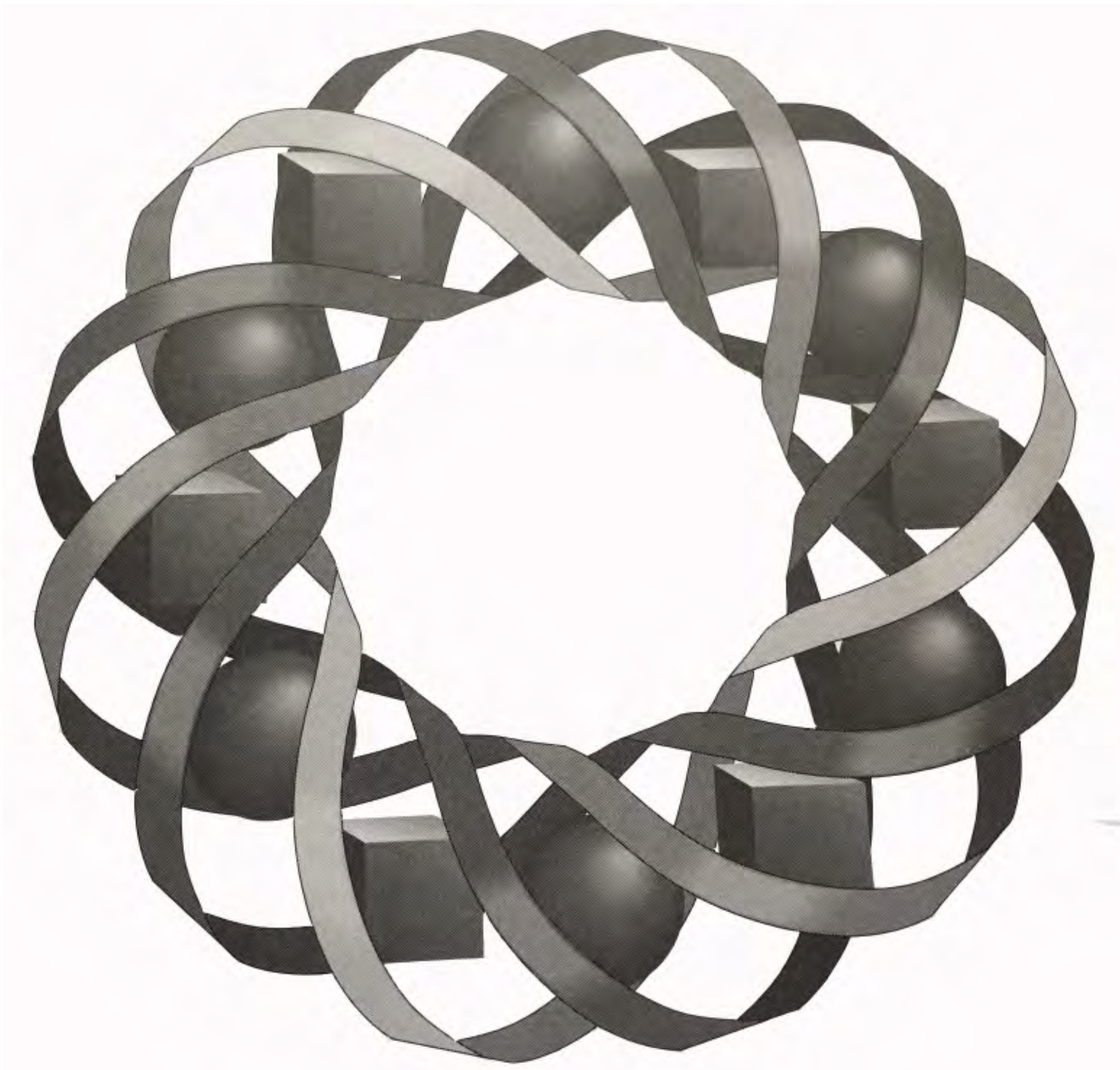
Total Chaos



The world is full of unpredictable and ever-changing systems, including the weather, the stock market, and animal populations. In this module, you investigate chaos theory, a relatively new branch of mathematics that studies the behavior of dynamical systems.

Byron Anderson • Sandy Johnson • Lisa Wood

Total Chaos

Introduction

Imagine a video camera standing on a tripod with its lens facing a television. The camera records a picture of the television screen. The signal for that picture is fed back to the television, where it appears on the screen as an image.

The camera then records an image of this image, and transmits it back to the television, where it appears on screen, and the cycle begins again.

If this process—known as video feedback—continues indefinitely, what would you expect to see on the television screen?

Mathematics Note

Recursion is a repetitive process that uses the output from one sequence of operations or instructions as the input for the next iteration of the same operations.

In general, *iteration* means repetition. Recursion is an iterative process. However, an iterative process is not always recursive.

Figure **1** shows a diagram of a recursive process.

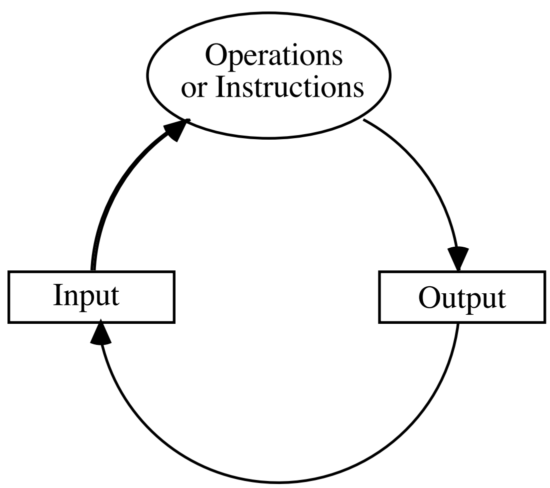


Figure 1: The recursive process

Each complete cycle of the process is a **stage**. For example, Figure **2** shows the result of the iterative process that occurs when an image is reflected between two parallel mirrors. Each individual image is a stage.

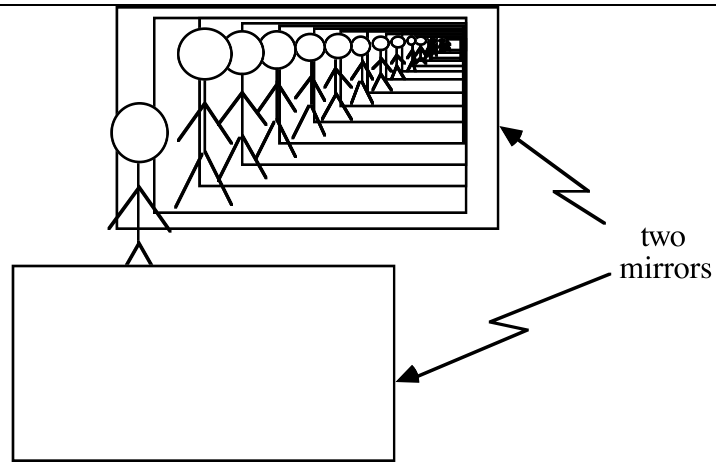


Figure 2: Two mirrors showing the iterative process

Discussion

a. Describe the input and output for video feedback.

b. Describe the process of video feedback.

c. How are the reflections in a set of parallel mirrors similar to the images formed by video feedback?

Activity 1

The word *fractal* was coined by mathematician Benoit Mandelbrot in 1975. He used the term to designate a set that may require three parameters to describe it: length, fractional (or fractal) dimension, and chance. Figure **3** shows an image of a Mandelbrot set, the fractal that bears his name.

Figure 3: A Mandelbrot set

Because of their complexity, fractals can be used to model the geometry of many natural objects, such as mountain ranges, shorelines, and the root systems of plants. For example, Figure **4** shows the image of a fern leaf defined by a fractal.

Like many other fractals, this one exhibits the property of **self-similarity**. In other words, a small portion of the set is similar to the whole set of which it is a part.

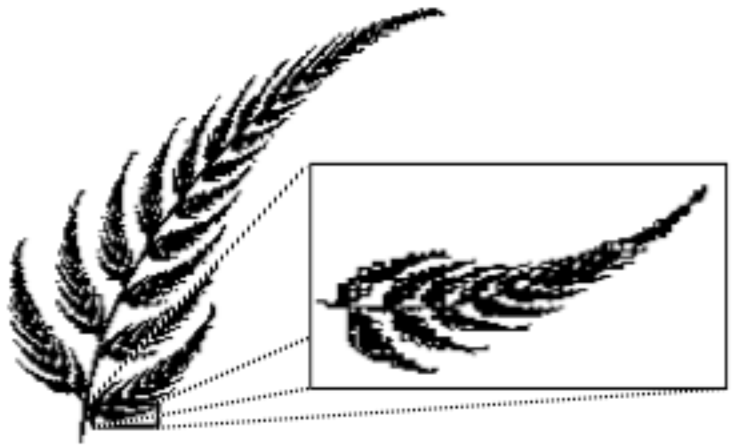


Figure 4: Fractal model of a fern leaf

Exploration

Though Mandelbrot was the first to publish a comprehensive theory of fractals, other mathematicians had been aware of their interesting geometry for many years. In this activity, you explore the initial stages of two classic fractals: the Koch curve, introduced by Helge von Koch in 1904, and the hat curve.

a. The shape of the Koch curve at its initial stage, or **stage 0**, is a line segment.

1. Draw a line segment.
2. Divide the segment into thirds.
3. Draw an equilateral triangle whose base is the middle third of the segment.
4. Remove the base of the triangle to obtain the shape shown in Figure **5**. This is stage 1 of a Koch curve.

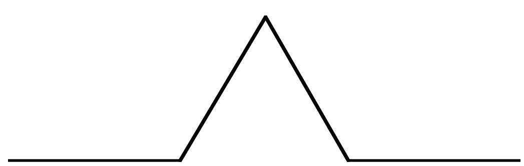


Figure 5: Stage 1 of a Koch curve

5. To produce stage 2 of a Koch curve, repeat the process described in Steps **2–4** on each line segment from stage 1.
6. Continue using the process described in Steps **2–4** on segments from previous stages until you reach stage 4.

b. Figure **6** shows the first three stages of another fractal, the hat curve. As in the Koch curve, stage 0 is a line segment. The shape displayed at stage 1 illustrates the iterative process that is performed on each line segment at each successive stage.

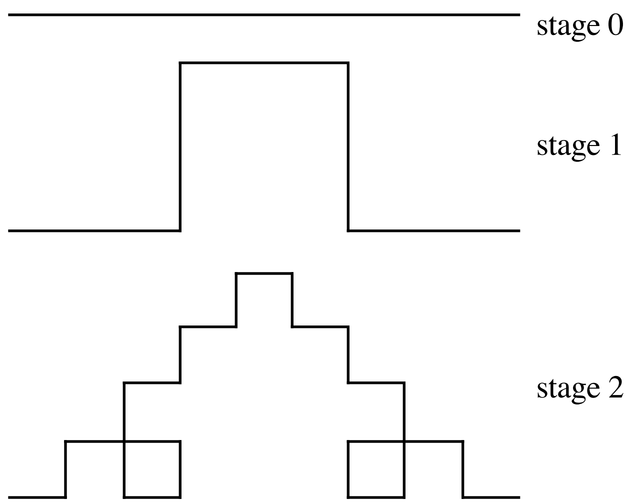


Figure 6: The first three stages of the hat curve

1. Use the pattern of the first three stages of the hat curve to describe the process used to draw stage 3.
2. To help confirm that the process you described is correct, draw stage 3 of the hat curve.

Discussion

a. As noted earlier, one feature common to many fractals is self-similarity under magnification. If your drawings of the Koch curve and the hat curve could be continued indefinitely, do you think the results would be self-similar? Explain your response.

b. As the number of stages increases without bound, describe the limit (if it exists) of the total length of each of the following:

1. a Koch curve
2. a hat curve.

c. How many times do you repeat the appropriate process to reach stage n of a fractal?

Assignment

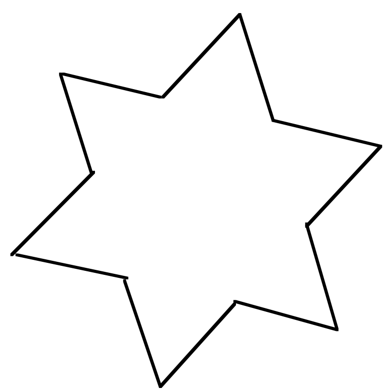
1.1 In the previous exploration, you constructed the first few stages of a Koch curve and a hat curve. Complete Parts **a** and **b** below for each of these fractals.

a. Assuming that stage 0 is 1 unit long, determine the lengths of stages 1–3 and record them in a table like the one below.

Stage	Length
0	1
1	
2	
3	

b. If possible, determine the length of the fractal after *n* stages.

1.2 Stage 0 of the Koch snowflake is an equilateral triangle. Stage 1 of the Koch snowflake is an equilateral triangle whose sides are stage 1 Koch curves, as shown below:

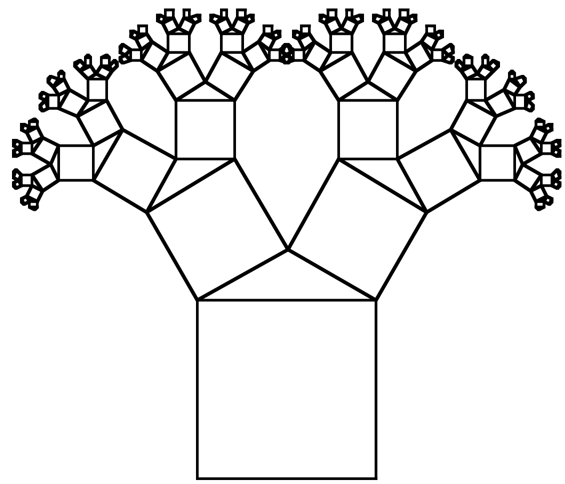


a. Make a sketch of stage 3 of the Koch snowflake.

b. As the number of stages increases without bound, what happens to the perimeter of the Koch snowflake? Explain your response.

c. As the number of stages increases without bound, what happens to the area of the Koch snowflake? Explain your response.

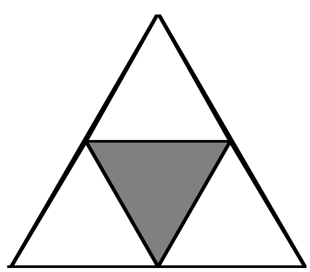
1.3 Describe the initial stage and iterative process used to generate the figure below.



1.4 Polish mathematician Waclaw Sierpinski (1882–1969) first described the fractal now known as Sierpinski's triangle. In the following assignment, you construct the initial stages of Sierpinski's triangle and discover one of its unusual properties.

a. **1.** Construct an equilateral triangle. This is stage 0 of the fractal.

2. Locate and mark the midpoints of each side of the triangle. Construct the line segments that connect the midpoints, creating four equilateral triangles. Shade the "middle" triangle to indicate that it should be removed from the picture. Your drawing should now resemble the figure below. This is stage 1 of the fractal.



3. To obtain the next stage in the fractal, repeat the process described in Step **2** on each remaining unshaded triangle.

4. Continue this iterative process through at least stage 3.

b. Write a sequence that describes the number of unshaded triangles at each stage of Sierpinski's triangle. Describe the limit (if any) to this sequence as the number of stages increases without bound.

c. Use a sequence to describe the perimeter of the fractal at each stage. What is the limit of this sequence as the number of stages increases without bound?

d. The area of the fractal at each stage is equal to the sum of the areas of the unshaded triangles. What happens to the area of the fractal as the number of stages increases without bound?

* * * * *

1.5 **a.** Design a recursive process that could be used to create a fractal, beginning with a line segment.

b. Draw stage 3 of your fractal.

c. Determine the length of the fractal after n stages.

* * * * * * * * * *

Research Project

Figure **7** shows a portion of Pascal's triangle. Recall that each row of Pascal's triangle begins and ends with the term 1. All other terms are found by adding the term above and to the left to the term above and to the right. In row 4, for example, $4 = 1 + 3$ and $6 = 3 + 3$.

					1						Row 0
				1		1					Row 1
			1		2		1				Row 2
		1		3		3		1			Row 3
	1		4		6		4		1		Row 4
1		5		10		10		5		1	Row 5

Figure 7: A portion of Pascal's triangle

Many computer-generated fractal images are obtained by systematically coloring a particular set. Using four copies of a template supplied by your teacher, explore the patterns you can obtain by coloring Pascal's triangle according to the instructions given in each of Parts **a–d**.

a. Shade cells that correspond with even numbers in Pascal's triangle one color. Shade cells that correspond with odd numbers in Pascal's triangle another color.

b. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 3 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

c. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 5 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

d. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 9 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

Describe your findings in a report, including a comparison of your results with the model of Sierpinski's triangle created in Problem **1.4**.

Activity 2

In Activity **1**, you used iterations of geometric processes to produce fractals. In this activity, you investigate the effects of iteration on functions.

Exploration 1

Iterating a function involves using the output of one calculation of the function as the input for the next calculation of the function. Iteration of a simple function can be used to generate a sequence such as x_0, $f(x_0)$, $f\big(f(x_0)\big)$, …, where x_0 is the **initial value**. This type of iterative sequence is an **orbit** of x_0.

a. Using $f(x) = \sqrt{x}$ and an initial value of 1, calculate the first few terms of the orbit.

> ### Mathematics Note
> A **fixed point** *p* of a function $g(x)$ is a value such that $g(p) = p$.
>
> For example, $p = 3$ is a fixed point of $g(x) = (2/3)x + 1$ because $g(3) = 3$. Notice also that $g\big(g(3)\big) = 3$, $g\big(g(g(3))\big) = 3$, and so on.

b. **1.** Explain why 1 is a fixed point of $f(x) = \sqrt{x}$.

2. Identify any other fixed points of $f(x)$.

c. Using three different initial values greater than 1, list the first 15 terms of their respective orbits.

d. Repeat Part **c** for three other initial values between 0 and 1.

e. Compare the six orbits you found in Parts **c** and **d**.

Discussion 1

a. Given that *p* is a fixed point for a function, describe the orbit which uses *p* as the initial value.

b. What are the fixed points for $f(x) = -x^3$?

c. **1.** When an orbit approaches a fixed point, that fixed point is considered an **attractor**. Why is 1 an attractor for $f(x) = \sqrt{x}$?

2. Does 0 also appear to be an attractor for this function? Explain your response.

Exploration 2

In this exploration, you observe the behavior of orbits for several different functions.

a. Using $f(x) = \cos x$ and $x_0 = 2$, where x is an angle measure in radians, find the first 20 terms of the orbit.

b. Create a connected scatterplot of the term value versus the term number.

c. Experiment with different initial values for $f(x) = \cos x$. What changes, if any, do you observe in the term values of the orbits?

d. Does the orbit of $f(x) = \cos x$ appear to approach a limit? If so, use that value as the initial value. Record your results.

e. Identify all the fixed points of $f(x) = \cos x$. Demonstrate graphically that your list is complete.

f. Repeat Parts **a–d** using $f(x) = \sin x$.

g. Repeat Parts **a–d** using $f(x) = 0.5x + 5$.

h. Repeat Parts **a–d** using $f(x) = 2x + 3$.

Discussion 2

a. Which orbits in Exploration **2** have attractor points?

b. Which of the functions in Exploration **2** do not appear to approach a limit under iteration? Do these functions have fixed points?

c. Why must a function that has a fixed point intersect the line $y = x$?

d. Do you think that there are any functions which do not have a fixed point under iteration? Explain your response.

Assignment

2.1 **a.** Using the function below and an initial value of your choice, generate an orbit until an attractor can be identified.

$$f(x) = -\frac{1}{3}x + 12$$

b. Confirm your answer to Part **a** by finding the fixed point for $f(x)$ algebraically.

2.2 Consider the iteration of the function below with $x_0 = 2$:

$$f(x) = \frac{1}{3}x + b$$

a. Investigate the behavior of the orbits for four different values of b. You may wish to organize your data in a table like the one below.

Value of b	**Orbit**	**Behavior of Orbit**

b. What characteristic do the orbits in Part **a** have in common?

c. How is the limit of an orbit related to the value of b?

d. Confirm your response to Part **c** by finding an expression for the fixed point in terms of b.

2.3 Find an expression for the fixed point of the general linear function $f(x) = ax + b$, where a does not equal 0 or 1.

2.4 **a.** Find the fixed points with real values for the quadratic function $f(x) = ax^2 + b$, where $a \neq 0$.

b. Is it possible to determine a real fixed point for every value of a and b? Explain your response.

c. What restrictions are necessary on a and b in order to find real fixed points? Justify your response. As part of your justification, use graphs to show why these restrictions are necessary.

2.5 Find the fixed point of $f(x) = 19x + 6$.

2.6 Identify the fixed points of $f(x) = |x|$.

2.7 Determine the two complex fixed points of $f(x) = x^2 + 1 + i$.

* * * * *

2.8 Describe how the balance in a savings account with an initial investment of $100 and an annual interest rate of 12% could be determined using an iterative process.

2.9 Determine a recursive formula for the following sequence:

$$4,\ \frac{1}{4},\ 4,\ \frac{1}{4},\ 4,\ \frac{1}{4},\ \ldots$$

2.10 Determine the fixed points for $f(x) = x!$

* * * * * * * * * *

Activity 3

In many cases, the graph of a function can help you identify its fixed points and characterize the behavior of its orbits. In this exploration, you investigate the different paths that occur for initial values near a fixed point.

Exploration

In this exploration, you examine four lines that have the same y-intercept but different slopes, then develop a conjecture about the behavior of orbits around fixed points. To create a geometric picture of the orbit of a linear equation, you graph the linear equation along with the equation $y = x$.

a. To produce a geometric picture of the orbit generated by $f(x) = 2x + 3$, complete the following steps on a sheet of graph paper.

1. Graph $f(x)$ and the line $g(x) = x$.

2. Identify the coordinates of the intersection of the two graphs, then select an initial value (x_0) greater than the x-coordinate of the intersection.

3. Draw a vertical line segment from x_0 on the x-axis to $f(x)$.

4. Draw a horizontal line segment from the intersection of the segment from the previous step and $f(x)$ to $g(x) = x$.

5. Draw a vertical line segment from the intersection of the segment from Step **4** and $y = x$ to $f(x)$.

6. Repeat the process described in Steps **4** and **5**.

7. Describe the path of the orbit, as represented by the segments, in relation to the fixed point.

b. Select an initial value less than the x-coordinate of the intersection of $f(x)$ and $y = x$. Repeat Steps **3–7** of Part **a**.

c. Use appropriate technology to repeat Parts **a** and **b** for each of the following functions:

1. $f(x) = 0.5x + 3$

2. $f(x) = -3x + 3$

3. $f(x) = -0.4x + 3$.

Discussion

a. In Part **a** of the exploration, what process is illustrated by drawing a vertical segment to $f(x)$, followed by a horizontal one to the line $y = x$?

b. Describe any similarities you observed in the paths of the orbits for $f(x) = 2x + 3$ and $f(x) = 0.5x + 3$.

c. Describe any similarities you observed in the paths of the orbits for $f(x) = -3x + 3$ and $f(x) = -0.4x + 3$.

d.
1. Which of the functions in the exploration generated orbits that appear to approach a fixed point?
2. What differences did you observe in the paths of the orbits?
3. Compare the coefficients of the functions whose orbits approach fixed points.

e.
1. Which of the functions in the exploration generated orbits that moved away from a fixed point?
2. What differences did you observe in the paths of the orbits?
3. Compare the coefficients of the functions whose orbits do not approach fixed points.

Mathematics Note

A fixed point that is approached by an orbit in the plane is an **attractor**. A fixed point from which an orbit in the plane moves away is a **repeller**.

Web plots model the paths created by a function's orbit. If a fixed point is an attractor, then the path may approach it in either a **staircase** or a **spiral**. If a fixed point is a repeller, then the path may move away from it in either a staircase or a spiral.

Figure **8** shows an example of a path that approaches an attractor in a spiral.

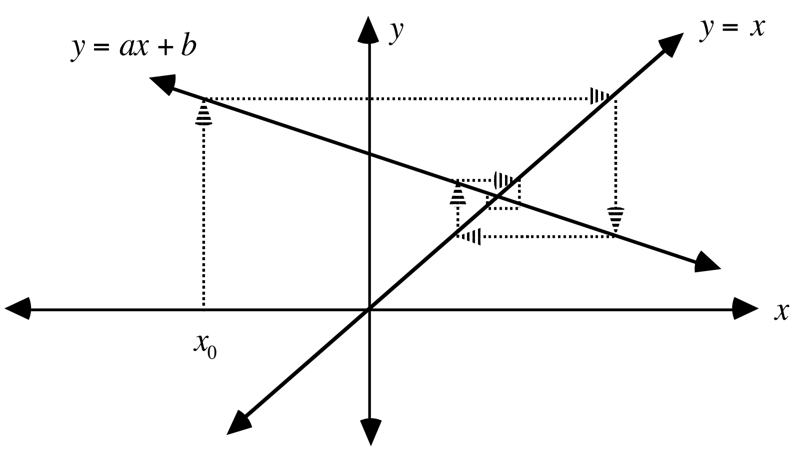


Figure 8: A web plot that spirals in

Figure **9** shows an example of a path that moves away from a repeller in a staircase.

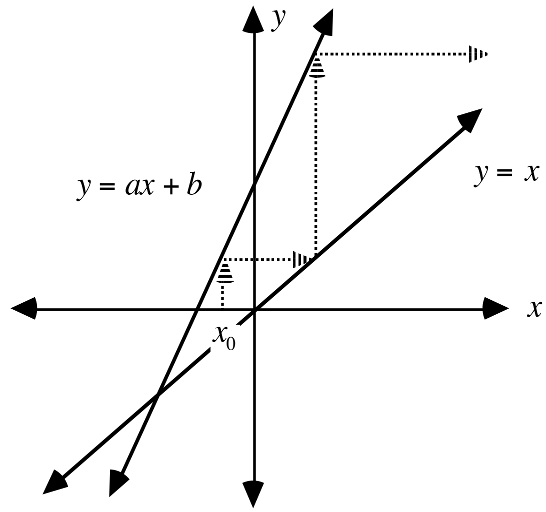


Figure 9: A web plot that staircases out

f. Characterize each fixed point for the orbits in the exploration as an attractor or repeller. Justify your responses.

g. In Exploration **2** of Activity **2**, you observed the behavior of orbits for four different functions and identified those that appeared to approach a limit. How do your observations in that exploration relate to the presence of repellers and attractors?

h. What is the fixed point of the function $f(x) = ax + 3$, where $a \neq 0$?

i. Predict the orbits of $f(x) = ax + 3$ for each of the following:

1. $a < -1$

2. $-1 < a < 1$

3. $a > 1$

Assignment

3.1. Consider the function

$$f(x) = -\frac{1}{3}x + 12$$

a. Find the fixed point of the function.

b. Is this fixed point an attractor or a repeller? Justify your response.

3.2 **a.** Use a spreadsheet or graphing utility to investigate the behavior of the orbits of $f(x) = -1x + b$ for at least three different values of b. For each value of b, experiment with at least two different initial values. Record your results in a table like the one below.

Value of b	**Initial Value x_0**	**Orbit**

b. Write a summary of your findings, including a description of the fixed point and how it was determined, and a description of the orbits for various values of b.

Be sure to mention any common characteristics of the orbits, as well as any differences they exhibit when compared to orbits of a function of the form $f(x) = ax + b$, where $a \neq -1$.

Mathematics Note

An orbit is **periodic** if it eventually cycles between two or more values. If the orbit cycles between n different values, the orbit has a cycle of period n.

For example, given $f(x) = -x + 5$ and the initial value $x_0 = -2$, the orbit is $-2, 7, -2, 7, \ldots$. Since the orbit cycles between two different values, –2 and 7, the orbit has a cycle of period 2. In fact, every orbit for this function—with the exception of those with the fixed point as the initial value—has a cycle of period 2.

3.3 In a paragraph, characterize the fixed point of the general linear function $f(x) = ax + b$, where $a \neq 0$, as an attractor, a repeller, or neither for various values of a.

3.4 Use the function $f(x) = a/x$, where $a \neq 0$, $x \neq 0$, and $x_0 = 1$, to complete Parts **a–d**.

a. Use a spreadsheet to find the orbits for several values of a in the interval $-5 < a < 5$.

b. Find the fixed point(s) algebraically.

c. For what values of a are the terms of the orbit identical?

d. For what values of a are the orbits periodic? Describe the period for each of these orbits.

* * * * *

3.5 Consider the function $f(x) = 16/x^3$, where $x \neq 0$.

a. Find the fixed points algebraically. Hint: There are two real solutions and two imaginary solutions.

b. Verify that all four solutions are fixed points.

* * * * * * * * * *

Activity 4

Dynamical systems are systems that change over time. Such systems often involve iteration. The study of the behavior of dynamical systems under repeated iteration is known as **chaos theory**.

Making predictions about complicated systems can be a difficult, if not impossible, task. In this activity, you examine the orbits of several nonlinear functions which become unpredictable and chaotic.

Exploration

In this exploration, you use the parent function $f(x) = ax(1-x)$, where $a \neq 0$, to investigate the iteration of selected nonlinear functions.

a. To generate orbits for the function $f(x) = ax(1-x)$, where $a \neq 0$, create a spreadsheet with headings like those in Table **1**. Design your spreadsheet so that changing either the value of *a* or the initial value results in updated values for orbit and output.

Table 1: Sequence of terms for iteration on $f(x) = ax(1-x)$

a	2	
x_0	0.1	
Term No.	**Orbit**	**Output**
1		0.1800
2	0.1800	0.2952
⋮	⋮	⋮

b. **1.** Determine the orbit of the function when $a = 2$ using $x_0 = 0.1$. Graph the function and create a web plot.

2. Complete Table **2** below. Record the fixed points, the number and values of the attractors, and the behavior of the orbits (spiral in, staircase out, and so on).

Table 2: Orbits for the function $f(x) = ax(1 - x)$

a	Fixed Point(s)	Number of Attractors	Value of Attractor(s)	Behavior of Orbit
2.0	0, 0.5000	1	0.5000	staircase in
2.2				
2.4				
2.6				

c. Use your table to make predictions about the number of attractors, the values of the attractors, and the behavior of the orbits as a increases.

d. To test your predictions, extend Table **2** to include three additional values of a: 2.8, 3, and 3.2.

e. Summarize the results of your tests from Part **d**.

f. Re-evaluate your predictions about the number of attractors, the values of the attractors, and the behavior of the orbits as a increases.

g. To test your revised predictions, extend Table **2** to include three additional values for a: 3.4, 3.6, and 3.8.

h. Summarize the results of your tests from Part **g**.

i. What do you expect to happen when the value of a is 4? Test your conjecture.

j. The orbit generated by the function $f(x) = 4x(1 - x)$ becomes stable when the initial value is 0, 0.5, 0.75, or 1. Graph the orbits that are generated for values of x_0 slightly greater than or less than each of these values. Describe any changes that occur in the graphs.

Mathematics Note

Dynamical systems may be considered **chaotic** if the following criteria are met.

- There is sensitive dependence on initial conditions. In other words, small changes can create vast differences in long-term behavior.
- The system appears to exhibit random behavior.

For example, the orbit of $0 < x_0 < 1$ for the function $f(x) = 4x(1 - x)$ displays chaotic behavior when $x_0 \neq 0.75$ or $x_0 \neq 0.5$ (the fixed points). When $x_0 = 0.75$, the orbit is stable: 0.75, 0.75, 0.75, 0.75, 0.75, 0.75, 0.75, … . With only a slight change, however, to $x_0 = 0.76$, the orbit is 0.76, 0.730, 0.789, 0.666, 0.890, 0.391, 0.952, 0.182, 0.596, … . This reveals both the system's sensitivity to initial conditions, and the appearance of random behavior.

Discussion

a. Were your predictions from Part **e** of the exploration supported or refuted by your tests? How did this affect your next predictions?

b. How did changing the value of a change the number of attractors?

c. How did the paths of the orbits change as the value of a increased?

d. In Part **j** of the exploration, how did small changes in the initial value affect the orbit generated by $f(x) = 4x(1-x)$?

e. The motion of clouds in the sky is an example of chaotic behavior in a dynamical system. Describe other dynamical systems that display chaotic behavior.

Assignment

4.1 Consider the function $f(x) = x^2 + c$ and the initial value $x_0 = 0$. Analyze the orbits that occur given the following values of c: $0.1, -0.5, -0.8, -1.3, -1.37, -1.4, -1.76, -1.77, -1.95$.

Summarize the results of your analysis. If any of the orbits are periodic, describe the period and the attractors.

4.2 Using the function $f(x) = ax(1-x)$, where $a \neq 0$, and an initial value of 0.1, find values of a in the interval $2.5 \leq a \leq 4.0$ (to the nearest hundredth) that result in each of the long-term behaviors in the table below.

Long-term Behavior	***a***
fixed point	
cycle of period 2	
cycle of period 3	
cycle of period 4	
cycle of period 5	
chaos	

* * * * *

4.3 Consider the function $f(x) = x^2 + (-1 + 0i)$. Determine the behavior of the orbit when the initial value is $0 + 0i$.

4.4 Consider the function $f(x) = x^2 + (-1 - 1i)$. Determine the behavior of the orbit when the initial value is $0 + 0i$.

* * * * * * * * * *

Activity 5

The population of a given species in a particular habitat can be thought of as a dynamical system. In the mid-nineteenth century, Belgian mathematician Pierre François Verhulst developed a formula that could be used to model population dynamics. In Verhulst's formula, the growth rate of a population is proportional to the difference between its present size and the **carrying capacity** of the environment, where carrying capacity is the number of individuals that a given habitat can support.

In this activity, you use one version of Verhulst's formula to model population dynamics as an iterative process.

Science Note

In a very simple model of population growth, the population for the following year can be determined by the formula,

$$P_{t+1} = P_t + (P_t \bullet r)$$

where P_t is the population after t years and r is the annual percentage change in the population.

According to this model, populations that have positive growth rates could continue to increase indefinitely, which is unrealistic. A more complex model of population dynamics takes the carrying capacity of the habitat into account. In Verhulst's formula, for example, the growth rate depends on the percentage of the carrying capacity C unaccounted for by the current population:

$$\frac{C - P_t}{C}$$

This modified equation for population size—sometimes referred to as the **logistic model**—can be expressed as follows:

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C}\right)$$

Exploration

To explore the application of the logistic model to population dynamics, imagine a population of 50 kangaroos in an Australian wildlife preserve. Considering the available land, food, and water, wildlife biologists have determined that the carrying capacity of the preserve is 150 kangaroos. In the past year, the annual growth rate r for the population was 1.9.

a. Using the logistic model and appropriate technology, predict the population of kangaroos in the preserve for each of the next 20 years.

b. Use the results of the iterative process from Part **a** to create a graph of population size versus time.

c. Investigate how the predicted behavior of the population is affected by changing the initial number of kangaroos, while maintaining the same growth rate of 1.9. Include some values greater than the carrying capacity in your experiment. Describe any changes you observe in the graphs of population size versus time.

d. Repeat Parts **a** and **b** for an initial population of 50 kangaroos and a growth rate of 2.1. Record any changes you observe in the graph of population size versus time.

e. Repeat Part **c** using a growth rate of 2.1. Describe any changes you observe in the graphs of population size versus time.

Discussion

a. What patterns did you observe in your graph from Part **b** of the exploration?

b. What factors might contribute to alternating high and low values in an actual animal population?

c. In Part **c** of the exploration, how did the predicted long-term behavior of the kangaroo population change for different values of the initial population?

d. How did the results you described in Part **c** differ from the predicted long-term behavior of the kangaroo population in Parts **d** and **e** of the exploration?

Assignment

5.1 Show that one of the fixed points for any population described by the logistic model is the carrying capacity.

5.2 The wildlife preserve described in the exploration also has a population of 125 wombats. The annual growth rate for this population is 1.25. Biologists believe that the preserve's carrying capacity is 500 wombats.

a. Using the logistic model, predict the population of wombats for each of the next 10 years and create a graph of population size versus time.

b. Would you describe the predicted behavior of the population as fixed, periodic, or chaotic? Explain your response.

5.3 **a.** Using each of the following annual growth rates, create a graph of population versus time over 30 years for an initial population of 125 wombats and a carrying capacity of 500 wombats.

For each growth rate, characterize the predicted behavior of the population as fixed, periodic, or chaotic.

1. $r = 2.25$
2. $r = 2.5$
3. $r = 2.55$
4. $r = 2.57$
5. $r = 2.75$

b. If the logistic model is accurate, what can you conclude about population dynamics from your findings in Part **a**?

5.4 Suppose that wildlife managers want to transplant some additional wombats to the preserve. Assume that the carrying capacity is still 500 and the annual growth rate is 1.25.

Using the logistic model, what is the largest possible initial population which does not result in extinction after 1 year?

* * * * *

5.5 The following equation provides a possible model of the price of gasoline, where p_n is the price on day n, $D(p_n)$ is a function that models the demand for gas, and $S(p_n)$ is a function that models the supply of gas:

$$p_{n+1} = p_n + k(D(p_n) - S(p_n))$$

In this case, the constant k is some positive value.

a. When the supply and demand are equal, the price of the gasoline is the **equilibrium price**. Assume that the equations below represent the supply and demand functions, respectively:

$$S(p) = 900p - 500$$
$$D(p) = 1000 - 100p$$

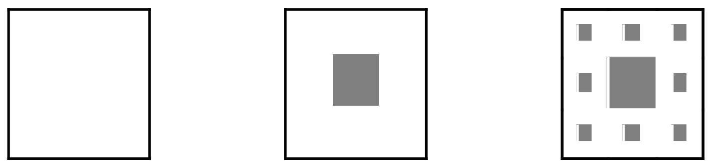
Find the equilibrium price. Hint: This is the price when $S(p) = D(p)$.

b. Over time, the current market price of gasoline approaches the equilibrium price. Use the equation $p_{n+1} = p_n + k\big(D(p_n) - S(p_n)\big)$, with $k = 0.0001$ and $p_1 = \$1.80$, to determine when the market price will equal the equilibrium price.

* * * * * * * * * *

Summary Assessment

1. **a.** Describe the iterative process used to create the following stages of a fractal known as Sierpinski's carpet.



stage 0 stage 1 stage 2

b. Recreate Sierpinski's carpet, showing at least stage 3.

c. Find the limit of the area of the shaded region of the fractal.

2. Design your own model of a fractal. Include a description of the iterative process you used to create your model, as well as a drawing of at least its first three stages.

3. Imagine that 50 koala bears are currently living in a habitat with a carrying capacity of 400 koala bears. Using the logistic model below—where P_t is the population after t years, r is the annual percentage change in the population, and C is the carrying capacity—create several graphs of predicted population size versus time.

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

Find at least one value for r that illustrates each of the following types of long-term behavior: fixed, periodic, and chaotic.

4. The iterative process has been a useful mathematical tool for many centuries. The iteration of simple arithmetic operations, for example, can often provide a good approximation of a more complicated operation. This type of iteration was especially useful prior to the development of the electronic calculator.

The function below is a combination of three simple arithmetic operations: multiplication, addition, and division.

$$f(x) = 0.5\left(\frac{a}{x} + x\right)$$

Iteration of this function can be used to approximate a more complicated function performed on an integer a. To investigate the operation this function approximates, complete Parts **a–g**.

a. Using each of the integers from 1 to 10 for a and an initial value of 0.1 for x, create the orbit generated by $f(x)$ until you can determine its limit, if one exists. If a limit exists, how does its value compare to the value of a?

b. When used in an iterative process, what operation does $f(x)$ appear to approximate?

c. Create the orbits generated by $f(x)$ using three different negative integers for a. Describe your results and explain their significance.

d. Predict what the function $g(x)$ below will approximate when used in an iterative process.

$$g(x) = 0.5\left(\frac{a}{x^2} + x\right)$$

e. Test your prediction from Part **d** using various integers for a and describe the results.

f. Repeat Parts **d** and **e** for the following function:

$$h(x) = 0.5\left(\frac{a}{x^3} + x\right)$$

g. Describe the possible uses of functions of the form below:

$$h(x) = 0.5\left(\frac{a}{x^n} + x\right)$$

Module Summary

- **Recursion** is a repetitive process that uses the output from one sequence of operations or instructions as the input for the next iteration of the same operations. Each complete cycle of the process is a **stage**.
- If a figure contains smaller replicas of the whole, then the figure is **self-similar**. Many fractals are self-similar.
- An **orbit** of a function *f* is an iterative sequence:

  $$x_0\, ,\ f(x_0)\, ,\ f\big(f(x_0)\big), \ldots$$

  where x_0 is the **initial value**.
- A **fixed point** *p* of a function $f(x)$ is a value such that $f(p) = p$.
- A fixed point that is approached by an orbit in the plane is an **attractor**. A fixed point from which an orbit in the plane moves away is a **repeller**.
- **Web plots** model the paths created by a function's orbit.
- If a fixed point is an attractor, then a path may approach it in either a **staircase** or a **spiral**. If a fixed point is a repeller, then a path may move away from it in either a staircase or a spiral.
- An orbit is **periodic** if it eventually cycles between two or more values. If the orbit cycles between *n* different values, the orbit has a cycle of period *n*.
- **Dynamical systems** are systems that change over time. Such systems often involve iteration. The study of the behavior of dynamical systems under repeated iteration is known as **chaos theory**.
- A dynamical system may be considered **chaotic** if the following criteria are met:
  1. There is sensitive dependence on initial conditions. In other words, small changes can create vast differences in long-term behavior.
  2. The system appears to exhibit random behavior.
- The **logistic model** for population growth can be expressed as follows:

  $$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C}\right)$$

  where P_t is the population at time *t*, *r* is the growth rate of the population, and *C* is the carrying capacity of the environment.

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