SIMMS Integrated Mathematics:

A Modeling Approach Using Technology

Level 6 Volumes 1-3

Teacher Guide
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About *Integrated Mathematics: A Modeling Approach Using Technology*

The Need for Change

In recent years, many voices have called for the reform of mathematics education in the United States. Teachers, scholars, and administrators alike have pointed out the symptoms of a flawed system. From the ninth grade onwards, for example, about half of the students in this country's mathematical pipeline are lost each year (National Research Council, 1990, p. 36). Attempts to identify the root causes of this decline have targeted not only the methods used to instruct and assess our students, but the nature of the mathematics they learn and the manner in which they are expected to learn. In its *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics addressed the problem in these terms:

> When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics. (p. 64)

Some Methods for Change

Among the major objectives of the *Integrated Mathematics* curriculum are:

- offering a 9–12 mathematics curriculum using an integrated inter-disciplinary approach for *all* students.
- incorporating the use of technology as a learning tool in all facets and at all levels of mathematics.
- offering a *Standards*-based curriculum for teaching, learning, and assessing mathematics.

The *Integrated Mathematics* Curriculum

An integrated mathematics program "consists of topics chosen from a wide variety of mathematical fields. . . [It] emphasizes the relationships among topics within mathematics as well as between mathematics and other disciplines" (Beal, et al., 1992; Lott, 1991). In order to create innovative, integrated, and accessible materials, *Integrated Mathematics: A Modeling Approach Using Technology* was written, revised, and reviewed by secondary teachers of mathematics and science. It is a complete, *Standards*-based mathematics program designed to replace all currently offered secondary mathematics courses, with the possible exception of advanced placement classes, and builds on middle-school reform curricula.

The *Integrated Mathematics* curriculum is grouped into six levels. All students should take at least the first two levels. In the third and fourth years, *Integrated Mathematics* offers a choice of courses to students and their parents, depending on interests and goals. A flow chart of the curriculum appears in Figure 1.

Each year-long level contains 14–16 modules. Some must be presented in sequence, while others may be studied in any order. Modules are further divided into several activities, typically including an exploration, a discussion, a set of homework assignments, and a research project.

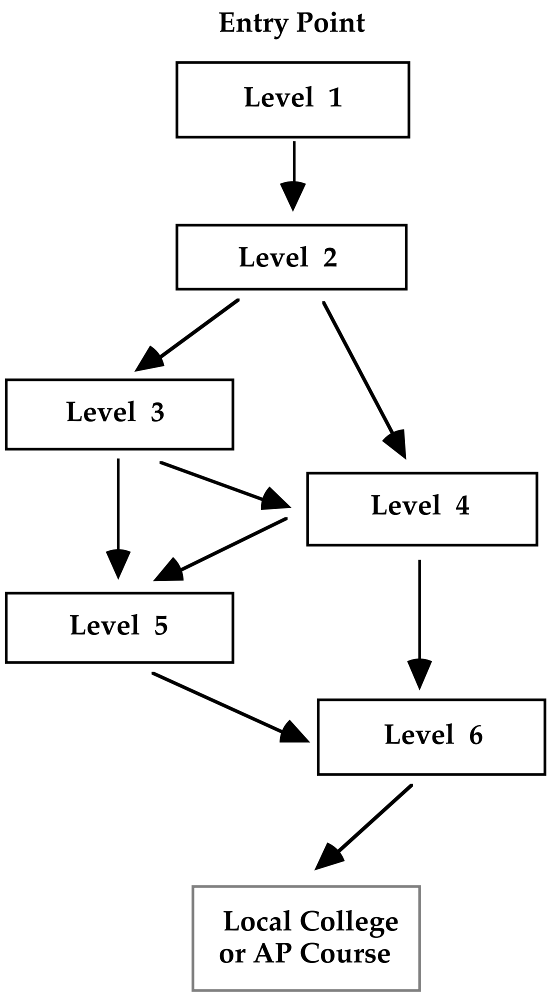

Figure 1: ***Integrated Mathematics*** **course sequence**

Assessment materials—including alternative assessments that emphasize writing and logical argument—are an integral part of the curriculum. Suggested assessment items for use with a standard rubric are identified in all teacher editions.

Level 1: a first-year course for ninth graders (or possibly eighth graders)

Level 1 concentrates on the knowledge and understanding that students need to become mathematically literate citizens, while providing the necessary foundation for those who wish to pursue careers involving mathematics and science. Each module in Level 1, as in all levels of the curriculum, presents the relevant mathematics in an applied context. These contexts include the properties of reflected light, population growth, and the manufacture of cardboard containers. Mathematical content includes data collection, presentation, and interpretation; linear, exponential, and step functions; and three-dimensional geometry, including surface area and volume.

Level 2: a second-year course for either ninth or tenth graders

Level 2 continues to build on the mathematics that students need to become mathematically literate citizens. While retaining an emphasis on the presentation and interpretation of data, Level 2 introduces trigonometric ratios and matrices, while also encouraging the development of algebraic skills. Contexts include pyramid construction, small business inventory, genetics, and the allotment of seats in the U.S. House of Representatives.

Levels 3 and 4: options for students in the third year

Both levels build on the mathematics content in Level 2 and provide opportunities for students to expand their mathematical understanding. Most students planning careers in math and science will choose Level 4. While Level 3 also may be suitable for some of these students, it offers a slightly different mixture of context and content.

Contexts in Level 4 include launching a new business, historic rainfall patterns, the pH scale, topology, and scheduling. The mathematical content includes rational, logarithmic, and circular functions, proof, and combinatorics.

In Level 3, contexts include nutrition, surveying, and quality control. Mathematical topics include linear programming, curve-fitting, polynomial functions, and sampling.

Levels 5 and 6: options for students in the fourth year

Level 6 materials continue the presentation of mathematics through applied contexts while embracing a broader mathematical perspective. For example, Level 6 modules explore operations on functions, instantaneous rates of change, complex numbers, and parametric equations.

Level 5 focuses more specifically on applications from business and the social sciences, including hypothesis testing, Markov chains, and game theory.

More About Level 6

In the modules "What Shape Is Your World," "Changing the Rules Changes the Game," and "Brilliant Induction," students examine non-Euclidean geometric systems and formal and informal proof. "Cards, Binos, and Reels, Oh My!" is an exploration of binomial probability.

In "Functioning on a Path," "Ostriches Are Composed," "Deriving Over the Limit," and "Here We Go Again," students examine derivatives, operations, compositions, and inverses of rational and periodic functions. "To Null or Not to Null" and "What Did You Expect Big Chi" introduce students to statistical hypothesis testing.

"Naturally Interesting" explores natural logarithms. "Total Chaos" uses iteration to introduce the study of dynamical systems.

The Teacher Edition

To facilitate use of the curriculum, the teacher edition contains these features:

Overview /Objectives/Prerequisites

Each module begins with a brief overview of its contents. This overview is followed by a list of teaching objectives and a list of prerequisite skills and knowledge.

Time Line/Materials & Technology Required

A time line provides a rough estimate of the classroom periods required to complete each module. The materials required for the entire module are listed by activity. The technology required to complete the module appears in a similar list.

Assignments/Assessment Items/Flashbacks

Assignment problems appear at the end of each activity. These problems are separated into two sections by a series of asterisks. The problems in the first section cover all the essential elements in the activity. The second section provides optional problems for extra practice or additional homework.

Specific assignment problems recommended for assessment are preceded by a single asterisk in the teacher edition. Each module also contains a Summary Assessment in the student edition and a Module Assessment in the teacher edition, for use at the teacher's discretion. In general, Summary Assessments offer more open-ended questions, while Module Assessments take a more traditional approach. To review prerequisite skills, each module includes brief problem sets called "Flashbacks." Like the Module Assessment, they are designed for use at the teacher's discretion.

Technology in the Classroom

The *Integrated Mathematics* curriculum takes full advantage of the appropriate use of technology. In fact, the goals of the curriculum are impossible to achieve without it. Students must have ready access to the functionality of a graphing utility, a spreadsheet, a geometry utility, a statistics program, a symbolic manipulator, and a word processor. In addition, students should have access to a science interface device that allows for electronic data collection from classroom experiments, as well as a telephone modem.

In the student edition, references to technology provide as much flexibility as possible to the teacher. In the teacher edition, sample responses refer to specific pieces of technology, where applicable.

Professional Development

A program of professional development is recommended for all teachers planning to use the curriculum. The *Integrated Mathematics* curriculum encourages the use of cooperative learning, considers mathematical topics in a different order than in a traditional curriculum, and teaches some mathematical topics not previously encountered at the high-school level.

In addition to incorporating a wide range of context areas, *Integrated Mathematics* invites the use of a variety of instructional formats involving heterogeneous classes. Teachers should learn to use alternative assessments, to integrate writing and communication into the mathematics curriculum, and to help students incorporate technology in their own investigations of mathematical ideas.

Approximately 30 classroom teachers and 5 university professors are available to present inservice workshops for interested school districts. Please contact Kendall Hunt Publishing Company for more information.

Student Performance

During the development of *Integrated Mathematics*, researchers conducted an annual assessment of student performances in pilot schools. Each year, two basic measures—the PSAT and a selection of open-ended tasks—were administered to two groups: students in classes using *Integrated Mathematics* and students in classes using other materials. Students using *Integrated Mathematics* materials typically had access to technology for all class work. During administration of the PSAT, however, no technology was made available to either group. Student scores on the mathematics portion of this test indicated no significant difference in performance.

During the open-ended, end-of-year test, technology was made available to both groups. Analysis of student solutions to these tasks showed that students using *Integrated Mathematics* were more likely to provide justification for their solutions and made more and better use of graphs, charts, and diagrams. They also demonstrated a greater variety of problem-solving strategies and were more willing to attempt difficult problems.

References

Beal, J., D. Dolan, J. Lott, and J. Smith. *Integrated Mathematics: Definitions, Issues, and Implications; Report and Executive Summary*. ERIC Clearinghouse for Science, Mathematics, and Environmental Education. The Ohio State University, Columbus, OH: ED 347071, January 1990, 115 pp.

Lott, J., and A. Reeves. "The Integrated Mathematics Project," *Mathematics Teacher* 84 (April 1991): 334–35.

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What Shape Is Your World?

Where in the world are you? Where in the world are you going? When answering either question, it may help to use a two-dimensional projection of the earth's three-dimensional surface.

John Carter • Anne Merrifield • Karen Umbaugh

Teacher Edition

What Shape Is Your World?

Overview

In this module, students investigate polar and cylindrical coordinates in the context of mapmaking. They then extend these mathematical concepts to create planar maps of spherical surfaces.

Teacher Note

The representation of a three-dimensional spherical surface on a two-dimensional plane has been the central problem of cartography since people first discovered that the earth was not flat. Of the many sorts of maps that have been developed over the centuries, none results in images that are similar to their preimages. All of them contain some sort of distortion. Since each type of projection has its strengths and weaknesses, each serves a different purpose.

When developing maps, cartographers have at least one of four "cardinal virtues" in mind: 1) the map should represent distances correctly; 2) the map should represent shapes correctly; 3) the map should represent areas correctly; or 4) the map should represent lines of longitude (great circles) as straight lines. As it turns out, no projection map can fulfill all four virtues. Hence, they settle for maps that are *conformal*.

A map is conformal if shape, distance, and area are all preserved for a small region under the projection. A projection map can only be strictly conformal in an infinitesimal area. Where exactly a map is practically conformal is not clearly defined. As a result, map users must consider how a map was made and how distortion affects different regions on the map.

Students investigate two types of projections in this module: stereographic and cylindrical. Polar stereographic projections are conformal near the pole. Cylindrical projections, such as the Mercator projection, are conformal near the equator.

Objectives

In this module, students will:

- graph points with polar coordinates (r, θ)
- determine points with cylindrical coordinates (r, θ, z)
- determine the image of a point using a stereographic projection
- determine the image of a point using a cylindrical projection
- compare the distortion resulting from various projections
- identify which projection(s) preserve parallel lines, perpendicular lines, length, area, and shape.

Prerequisites

For this module, students should know:

- how to measure angles using radians
- the properties of similar triangles
- how to use the proportionality of similar triangles to find unknown values
- how to use trigonometric ratios to determine parts of triangles
- how to graph coordinates in three dimensions
- how to determine the length of an arc of a circle.

Time Line

Activity	**Intro.**	**1**	**2**	**3**	**Summary Assessment**	**Total**
Days	1	3	4	2	1	11

Materials Required

Materials	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
templates A and B	X				
world globe	X				
centimeter graph paper	X				
radian protractor		X			
transparency (optional)		X			
metric ruler		X	X	X	
templates C and D		X			
compass (optional)		X			
polar graph paper (optional)		X			
cylindrical graph paper (optional)		X			
foam sphere			X	X	
skewers			X	X	
toothpicks			X		
string			X	X	
serrated knife				X	

Teacher Note

Blackline masters for templates A, B, C, and D, a radian protractor, as well as polar and cylindrical graph paper appear at the end of the teacher edition FOR THIS MODULE. The spheres used in Activities **2** and **3** can be as small as 5 cm in diameter. Although larger spheres also will work, they are more expensive and require larger sheets of paper for the projections.

Technology

Software	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
geometry utility			X	X	X

What Shape Is Your World?

Introduction (page 3)

In the following activities, students investigate planar, topographical, and spherical maps. They use rectangular and polar coordinate systems to locate points on a flat map and a cylindrical coordinate system to locate points on a sphere.

Some mappings preserve area and distance well for some parts of the map, while not for others. The introduction uses two maps of Greenland to illustrate some inconsistencies created by the mapmaking process.

Materials List

- templates A and B (one set per person)
- world globe (one per group)
- centimeter graph paper (two sheets per group)

Exploration (page 4)

a. Students may use any method they wish to estimate areas. Some may lay a sheet of centimeter graph paper over each template and count squares. Others may estimate average height and width or use a geometry utility to draw a polygon that approximates Greenland's coastline.

Using the given scales, the area of Greenland on template A is approximately $3{,}562{,}500 \text{ km}^2$; while the area of Greenland on template B is approximately $1{,}093{,}750 \text{ km}^2$.

b. Students compare their estimates with those of their classmates then find a mean value for the area of Greenland on each template.

c. The shape of Greenland in template A more closely resembles its shape on the globe, especially for the northern half of the island. The shape of Greenland in template B appears much wider than its shape on the globe.

Discussion (page 4)

a. Answers will vary. Students should note that the shapes and the areas of the two maps of Greenland are not the same.

b. **1.** There is considerable distortion in both maps. The area of Greenland in template A is about $1.4 \bullet 10^6 \text{ km}^2$ more than the actual area, while the area of Greenland in template B is about $1.1 \bullet 10^6 \text{ km}^2$ less than the actual area

2. Answers will vary. The distortions created by different mapmaking processes are explored in Activities **2** and **3**.

c. Answers will vary. Sample response: The area of Greenland was determined by surveyors who took actual measurements.

d. Sample response: Both maps are useful for visualizing Greenland's borders, as well as its position relative to other countries.

e. Sample response: The map on the globe is probably more accurate since a globe more closely approximates the actual shape of the earth than a flat surface.

(page 4)

Activity 1

Students represent points on a flat map using polar coordinates and points on a topographic map using cylindrical coordinates.

Materials List

- template C (one per person)
- template D (one per person)
- metric ruler (one per group)
- radian protractor (one per group)
- transparency (for tracing radian protractor template; optional)
- compass (optional)
- polar graph paper (optional)
- cylindrical graph paper (optional)

Teacher Note

Unless otherwise noted, angle measurements in this module are given in radians. References to a *polar axis* correspond with the polar axis of a polar or cylindrical coordinate system, not with the axis passing through the earth's poles.

Exploration 1 (page 6)

In this exploration, students create polar coordinate systems and determine the coordinates of points on a map.

a–b. Sample response: With the pole located at Oredigger Refuge and the polar axis extending due east, one possible set of coordinates for Grizzly Peak is $(970, 5\pi/24)$; one possible set of coordinates for Camp Yellowjacket is $(650, \pi/24)$.

c.
1. Sample response: Two possible sets of approximate polar coordinates for Grizzly Peak are $(970, 53\pi/24)$ and $(970, 101\pi/24)$; two possible sets of approximate coordinates for Camp Yellowjacket are $(650, 49\pi/24)$ and $(650, 97\pi/24)$.
2. Sample response: Two possible sets of approximate polar coordinates for Grizzly Peak are $(970, -43\pi/24)$ and $(970, -91\pi/24)$; two possible sets of approximate coordinates for Camp Yellowjacket are $(650, -47\pi/24)$ and $(650, -95\pi/24)$.

d. Sample response: With the pole located at Northern Lights Lookout and the polar axis extending due west, the approximate location of Grizzly Peak can be described by the coordinates $(690, 23\pi/24)$, $(690, 71\pi/24)$, $(690, 119\pi/24)$, $(690, -25\pi/24)$, or $(690, -73\pi/24)$.

The approximate location of Camp Yellowjacket can be described by the coordinates $(920, 17\pi/24)$, $(920, 65\pi/24)$, $(920, 113\pi/24)$, $(920, -31\pi/24)$, and $(920, -79\pi/24)$.

e. Sample response: With the pole located at Northern Lights Lookout and the polar axis extending due east, the approximate location of Grizzly Peak can be described by the coordinates , $(690, -\pi/24)$, $(690, -97\pi/24)$, $(690, 47\pi/24)$, $(690, 95\pi/24)$, or $(690, 143\pi/24)$. The approximate location of Camp Yellowstone can be described by the coordinates $(920, -\pi/24)$, $(920, -55\pi/24)$, $(920, -103\pi/24)$, $(920, 41\pi/24)$, $(920, 89\pi/24)$, or $(920, 137\pi/24)$,

f. Students compare the polar coordinates they found for Grizzly Peak and Camp Yellowjacket in Parts **b–e** with those of their classmates.

Discussion 1 (page 7)

a. Answers will vary. Sample response: To locate the point (r, θ) on a polar coordinate system, draw a circle of radius r with its center at the pole. This circle represents all points r units from the pole. Then, locate the point on the circle that is θ radians counterclockwise from the polar axis.

b. Any point on a polar coordinate system can be represented by an infinite number of ordered pairs of the form (r, θ). Since $2n\pi$ represents a complete rotation about the axis, θ can be represented as an infinite number of radian measurements $\theta \pm 2n\pi$, where $n = 1, 2, 3, \ldots$.

c. **1.** Since the location of the pole is the starting point for any distances measured, its location affects the r-coordinate. The polar axis is determined using the pole as its endpoint and is the ray from which any angles are measured. Therefore, the location of the polar axis affects the θ-coordinate.

2. Sample response: Unless all agree on the same locations of the pole and polar axis, each person using the map could have different polar coordinates for the same point on the map.

Exploration 2

(page 8)

In this exploration, students create a cylindrical coordinate system to describe the locations of points on topographic maps.

a–b. Using a cylindrical coordinate system with the pole in the southwest corner of the map, the polar axis extending due east, and the polar plane at sea level, some possible coordinates for Oredigger Refuge are:

1. $(450, \pi/4, 325)$

2. $(450, -7\pi/4, 325)$.

Discussion 2

(page 10)

a. Sample response: Contour lines indicate points with the same elevation. The larger numbers represent higher elevations. Contour lines that are relatively close together indicate a more rapid change in elevation than contour lines that are relatively far apart.

b. Some students may argue that the pole should be placed at the point where most hikers begin their trips, so that the direction and distance to other points can be easily interpreted.

Others may argue that placing the pole at the southwest corner with the axis extending due east and the polar half-plane at sea level makes it easier to express all coordinates as positive values.

c. **1.** In a cylindrical coordinate system, all points with the same value for r form a cylinder centered around the z-axis.

2. All points with the same value for θ form a half-plane perpendicular to the polar plane and containing the pole.

3. All points with the same value for z form a plane parallel to the polar plane.

d. Sample response: Yes. Points below the polar plane must have negative z-coordinates. However, it is possible to identify any location using only positive values for r and θ.

Teacher Note

For the remainder of the module, students will be expected to use only positive values for θ when reporting the coordinates of a point.

You may wish to distribute copies of the templates for polar graph paper and cylindrical graph paper for use in the assignment.

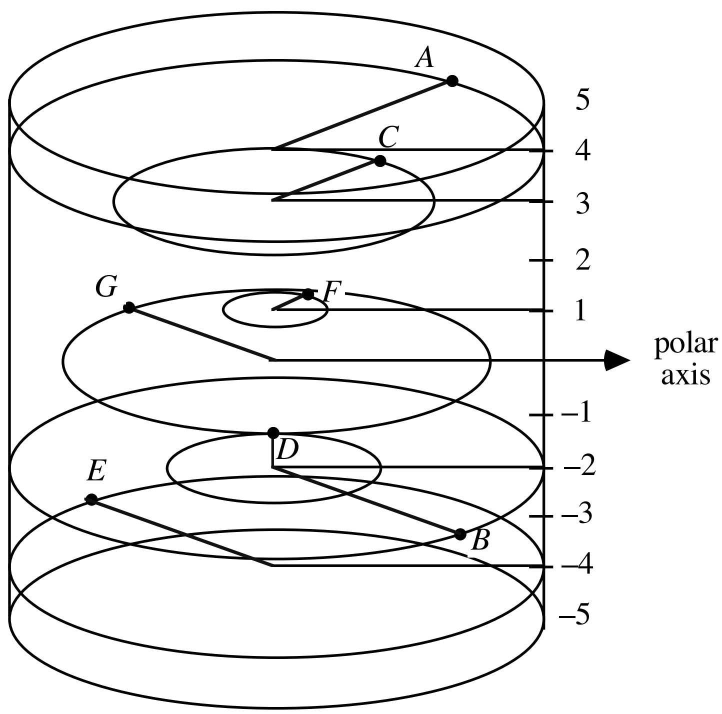
Assignment

(page 10)

1.1 **a.** $C(7, 3\pi/2)$

b. $D(5, 5\pi/6)$

***1.2** **a.** Sample graph:



b. Points A, B, and E lie on the same cylinder because they all have the same value for r, 5 units.

c. Points B and D lie in a plane parallel to the polar plane because they both have the same value for z, –2 units.

d. Points A, C, and F determine a plane that contains the pole and is perpendicular to the polar plane because they all have the same value for θ, $\pi/4$.

1.3 Using Northern Lights Lookout as the pole, one possible set of coordinates for Grizzly Peak is $(690, 47\pi/24, 9)$. The z-value is found by the difference in the elevations $795 - 786 = 9$ m. The values for r and θ are based on the sample response given in Part **b** of Exploration **1**.

A similar method may be used to determine the coordinates for Camp Yellowjacket. Using an estimated elevation of 375 m, one possible set of coordinates is $(920, 41\pi/24, -411)$.

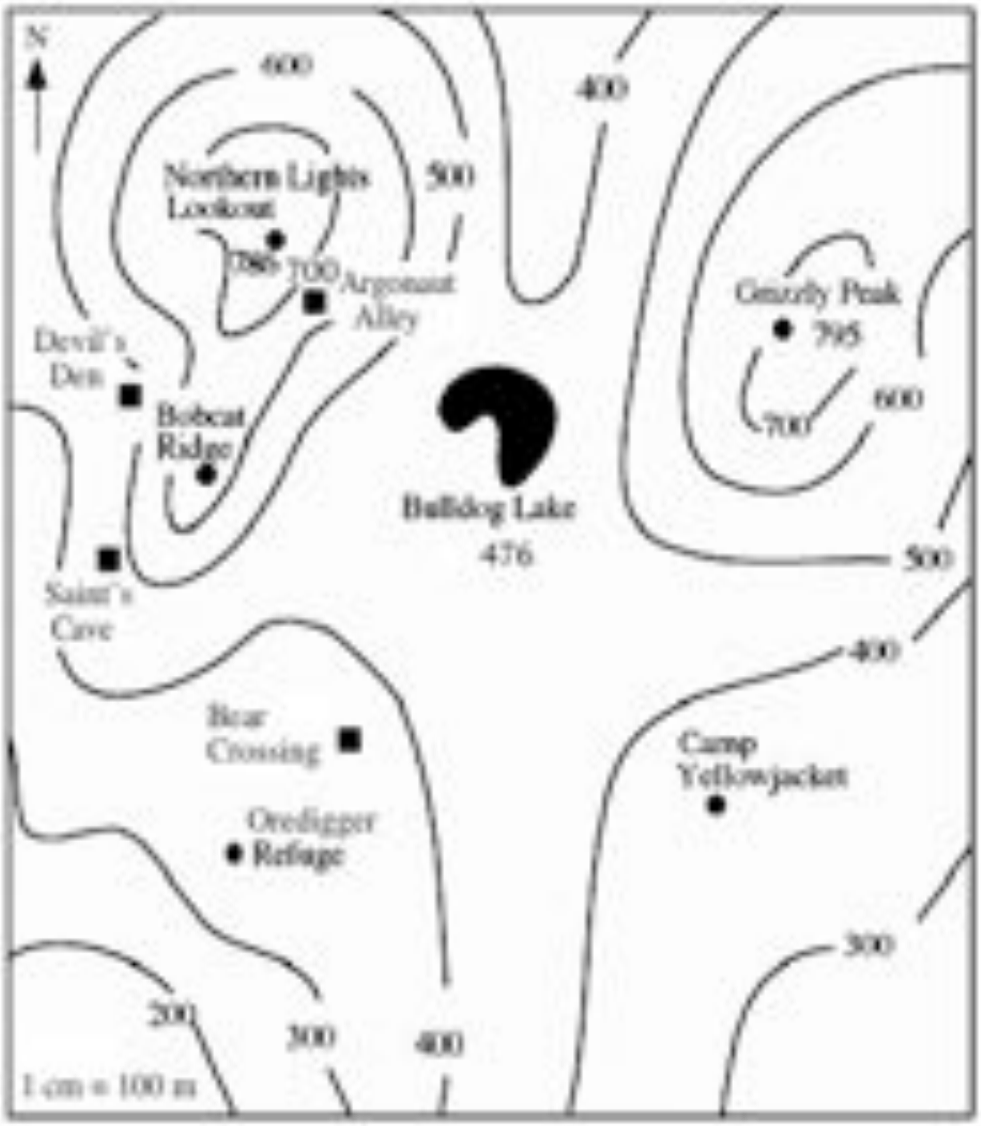
***1.4** **a.** Answers may vary. One possible set of polar coordinates for each location is shown in the table below.

Location	Polar Coordinates
Northern Lights Lookout	$(320, 11\pi/24)$
Grizzly Peak	$(790, \pi/12)$
Camp Yellowjacket	$(780, 11\pi/6)$
Oredigger Refuge	$(460, 3\pi/2)$
Bulldog Lake	$(400, \pi/12)$

b. One possible set of cylindrical coordinates for each location is shown in the following table.

Location	Cylindrical Coordinates
Northern Lights Lookout	$(320, 11\pi/24, 161)$
Grizzly Peak	$(790, \pi/12, 170)$
Camp Yellowjacket	$(780, 11\pi/6, -250)$
Oredigger Refuge	$(460, 3\pi/2, -300)$
Bulldog Lake	$(400, \pi/12, -149)$

c. Sample map:

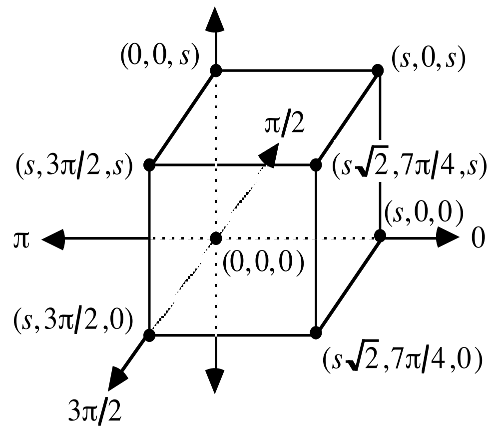


d. One possible set of cylindrical coordinates for each location is shown in the following table.

Location	Cylindrical Coordinates
Argonaut Alley	$(250, \pi/3, 25)$
Bear Crossing	$(400, 5\pi/3, -250)$
Saint's Cave	$(150, 7\pi/6, -150)$
Devil's Den	$(125, 3\pi/4, -125)$

* * * * *

***1.5** **a.** One possible set of cylindrical coordinates for each vertex of the cube is shown in the diagram below.



b. The coordinates for the cube's center are $\left(s\sqrt{2}, 7\pi/4, s/2\right)$.

1.6 **a.** Since the pole is in the center of the cube, the distance to each of the vertices is the same. This means that their radii or r-coordinates must all be equal. All of the r-coordinates would be $s\sqrt{2}/2$.

b. The vertical distance from the cube's center to the plane of the vertices is the same in either direction. Therefore, the absolute values of the z-coordinates of the vertices would be the same. The absolute value of all of the z-coordinates would be $s/2$.

* * * * * * * * * *

(page 12)

Activity 2

Students investigate stereographic projections by projecting points from a foam sphere to a sheet of paper. They then use a geometry utility to model stereographic projections. **Note:** Stereographic projections were invented by Hipparchus in the second century B.C. They are conformal for regions near the point of tangency. In addition, all lines of longitude on a polar aspect are straight lines and all directions from the center of a polar aspect are true.

Materials List

- foam spheres (about 5 cm in diameter; one per group)
- skewers (one per group)
- lengths of string (approximately 15 cm per group)
- toothpicks (three per group)
- metric ruler (one per group)
- radian protractor (one per group)
- transparency (for tracing radian protractor template; optional)

Teacher Note

Foam spheres are relatively inexpensive and available at most craft stores. They are easily pierced with bamboo barbecue skewers.

Technology

- geometry utility

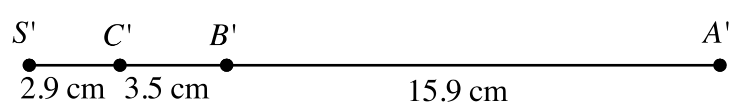
Exploration 1 (page 13)

Students map points from a sphere to a plane and make some observations about the distortions produced by this type of projection.

a–b. Point B should lie on the equator. Point A should lie midway between B and N; point C should lie midway between B and S.

c. To obtain accurate projections, students should keep their spheres aligned with their reference marks and make sure that point S remains tangent to the paper.

d–f. The diagram below shows the results of a projection using a foam sphere with a diameter of 5 cm.



g. The projection of the equator is a circle with center at S. Using a 5-cm sphere, the radius of the circle is 10 cm.

Discussion 1

(page 14)

a. Sample response: The image of S under this projection is itself. In other words, S and S' are the same point.

b.

1. Sample response: The images of lines of longitude would be concurrent lines whose point of intersection is the south pole. The image of the equator would be a circle with center at the south pole.
2. Sample response: The radius of the image of the equator appears to be twice that of the preimage.
3. As shown in the exploration, the closer a point P is to N, the farther P' is from S'. Point N has no image because any attempt to map N on the plane results in a ray parallel to the plane (and thus there is no intersection). This guarantees the infinite length of the image of a line of longitude. All of the points on a line of longitude lie in a plane perpendicular to the plane of the map, as does the ray creating the image points. The intersection of the two planes is a line contained in the plane of the map. It is this line of intersection that contains the image points of a line of longitude.

c.

1. Sample response: No. There is no image for the center of projection under a stereographic projection. Therefore, a one-to-one correspondence between the points in the preimage and the points in the image does not exist with this mapping.
2. Sample response: If the domain of the function is defined as all the points on the sphere except the center of projection, each point on the sphere has exactly one image point in the plane. In this case, the points of the plane are the range of the function.

d.

1. Sample response: No. While the projection of the lines of longitude are lines, the projection of the equator is a circle.
2. Sample response: No. As noted above, the images of some lines on a sphere are not lines in the plane of projection.

e.

1. Sample response: No. In the plane, lines that represent the map of the lines of longitude intersect a circle, the map of the equator. Since only lines are perpendicular in a plane, perpendicularity is not preserved.
2. Sample response: Yes. The angle formed by two lines of longitude at the pole opposite the center of projection is equal to the angle formed by their images. Since the planes forming the angle on the sphere are both perpendicular to the plane of projection, the images of two perpendicular lines of longitude also are perpendicular.

f. Sample response: No. In Exploration **1**, points A and C are equidistant from point B on the sphere. On the flat map, the distance from A' to B' is greater than the distance from C' to B'.

g. **1.** Sample response: The portion of the preimage close to the center of projection, *N*, appears to be most distorted.

2. Sample response: If the north pole is used as the center of projection, the map of Greenland would be very distorted because Greenland is near the north pole. If the south pole were used as the center of projection, with the plane of projection tangent to the globe at the north pole, then a stereographic map of Greenland would be more accurate.

3. Sample response: Regions close to the point of tangency of the sphere and plane of projection are the least distorted.

4. Sample response: The point of tangency should be placed in the center of the region to be mapped.

h. Sample response: In the stereographic projection described, the distortion of Iceland would be much less than that of Florida. Iceland is much farther north and closer to the point of tangency than Florida.

i. Sample response: The image of the equator would be a circle with the same diameter as the globe. Images of lines of longitude would look like two rays extending outward from the circle in opposite directions. None of the points in the southern hemisphere would be on the map.

j. Sample response: The image of a great circle that does not contain the poles would be an elongated closed curve. The smaller the angle formed by the plane containing the great circle and the plane containing the equator, the closer the image would be to a circle.

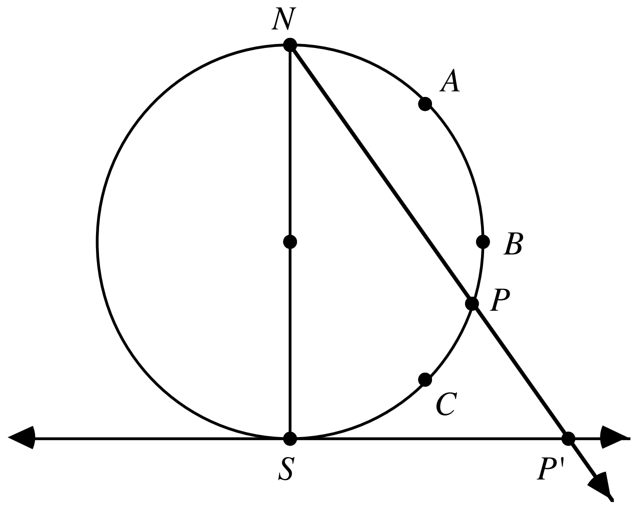
Exploration 2

(page 16)

In this exploration, students use a geometry utility to model a stereographic projection. They then use properties of similar triangles to make some generalizations about stereographic projections.

a–b. Sample response: As *P* moves around the circle, P' moves along the tangent at *S*. As *P* approaches *N*, P' goes to infinity. As *P* approaches *S*, P' also approaches *S*.

c. Points A, B, and C should appear as shown in the diagram below.



d. The following sample data corresponds with a circle of radius of 5 cm.

Segment	Length (cm)
$\overline{SS'}$	0
$\overline{SC'}$	4.15
$\overline{SB'}$	10.03
$\overline{SA'}$	24.21
$\overline{SN'}$	∞

e–f. Using ratios of similar triangles,

$$\frac{P'S}{PZ} = \frac{NS}{NZ}$$

Hence,

$$P'S = \frac{NS}{NZ} \bullet PZ$$

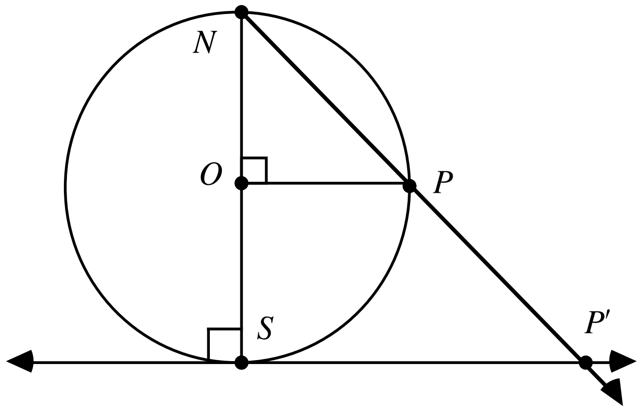
Discussion 2 (page 17)

a. Sample response: For two points that are a fixed distance from each other on a sphere, the distance between the image points increases as the preimage points move closer to the center of projection. This means that distortion of distances on the map becomes greater as the distance from the center of projection decreases.

b. Sample response: The length of $\overline{P'S}$ increases without bound.

c. Both triangles share $\angle SNP$ and both contain a right angle. This means that their corresponding third angles must be equal in measure. By the Angle-Angle-Angle Property, $\Delta NPZ \sim \Delta NP'S$.

d. When the center of projection is at one of the poles and the plane of projection tangent to the opposite pole, the radius of the image of the equator will always be twice that of the preimage. For example, consider the cross section of a sphere and tangent plane shown below.



If P is a point on the equator, two similar isosceles right triangles are formed: NOP and NSP'. One contains radii of the sphere as legs, while the other contains diameters as legs. Therefore, the ratio of corresponding lengths is 1:2.

e. Sample response: If S is the pole, the distance $P'S$ corresponds to the value of r in polar coordinates of the form (r, θ). The value of θ can be found by measuring the angle formed by the polar axis and $\overrightarrow{SP'}$. In the case where the polar axis is opposite $\overrightarrow{SP'}$, θ is $\pi/2$ radians.

Assignment

(page 17)

2.1 Sample response: No. Perimeter cannot be preserved if distance is not. When interpreting a map, it is important to realize that the images are only reasonable approximations of the actual size and shape of the countries represented. The amount of distortion depends on the distance from the country to the center of projection. The closer to the center of projection, the more distorted the map.

2.2 Sample response: Since distance is not preserved, the scale will not accurately represent distances on the entire map. Some distances found using the scale may only be rough approximations.

2.3 Sample response: The projections of lines of latitude are concentric circles. They are centered at the image of the south pole, the sphere's point of tangency with the plane of projection.

2.4 **a.** Using similar triangles, $\frac{GF}{FH} = \frac{GL}{LH'}$, or $\frac{3}{3} = \frac{6}{LH'}$. Therefore $LH' = 6 \text{ cm}$.

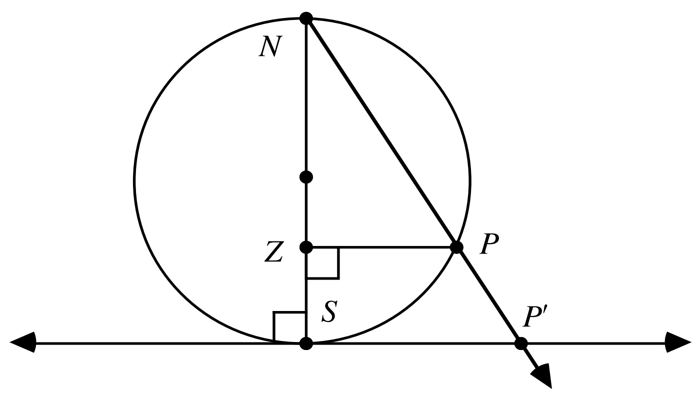
b. The measure of $\angle LGH$ is 45°; the measure of its intercepted arc is 90°. Therefore, the measure of $\angle LGH$ is half the measure of its intercepted arc.

c. In ΔGHF, $\angle LGH \cong \angle FHG$ because they are base angles of an isosceles triangle. It follows that:

$$m\angle LGH + m\angle FHG + m\angle GFH = 180^\circ$$
$$2 \bullet m\angle LGH = 180^\circ - m\angle GFH$$
$$2 \bullet m\angle LGH = m\angle LFH$$
$$m\angle LGH = \frac{m\angle LFH}{2}$$

The measure of $\angle LFH$ is equal to the measure of $\overset{\frown}{HL}$ because the measure of a central angle and its intercepted arc are equal. Therefore, the measure of an inscribed angle is half the measure of its intercepted arc.

2.5 Students should refer to Figure **9**, the cross section that contains $\overline{PZ}$. To determine the length of each segment, the length of $\overline{PZ}$ must be known when *P* is concurrent with the points *A*, *B*, and *C*.



For point *B*, $PZ = 5 \text{ cm}$ (the radius of the circle). For points *A* and *C*, however, the distance must be determined using trigonometry, a geometry utility, or a scale drawing. For a circle with radius 5 cm, $PZ \approx 3.55 \text{ cm}$ when *P* is concurrent with points *A* and *C*.

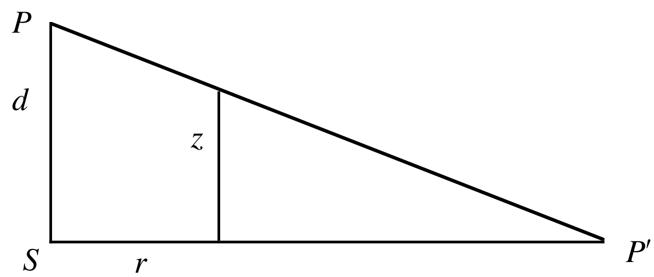
Students must also determine the length of $\overline{NZ}$. For point *B*, $NZ = 5 \text{ cm}$ (the radius of the circle). For point *C*, $NZ \approx 8.52 \text{ cm}$ using the Pythagorean theorem. For point *A*, $NZ = 10 - 8.52 = 1.48 \text{ cm}$.

So $SA' \approx 10(3.55)/1.48 \approx 23.99 \text{ cm}$, $SB' = 10(5)/5 = 10 \text{ cm}$ and $SC' \approx 10(3.55)/8.52 \approx 4.17 \text{ cm}$.

2.6 Answers will vary. In the following sample response, the polar axis is $\overrightarrow{SA}$, r is given in centimeters, and θ is given in radians: A' $(22.3, 0)$, B' $(6.4, 0)$, C' $(2.9, 0)$, D' $(12.7, 5\pi/24)$, E' $(12.7, -5\pi/24)$, F' $(2.5, 5\pi/24)$.

***2.7** **a.** The value of r for P' is the distance SP'. The value of θ is the same as that for point P in the cylindrical coordinate system.

b. The polar coordinates of P' are $(dr/(d-z), \theta)$. The distance SP' can be determined using the similar triangles in the diagram below.



$$\frac{z}{d} = \frac{SP' - r}{SP'}$$

$$z \bullet SP' = d \bullet (SP' - r)$$

$$z \bullet SP' - d \bullet SP' = -d \bullet r$$

$$SP'(z - d) = -d \bullet r$$

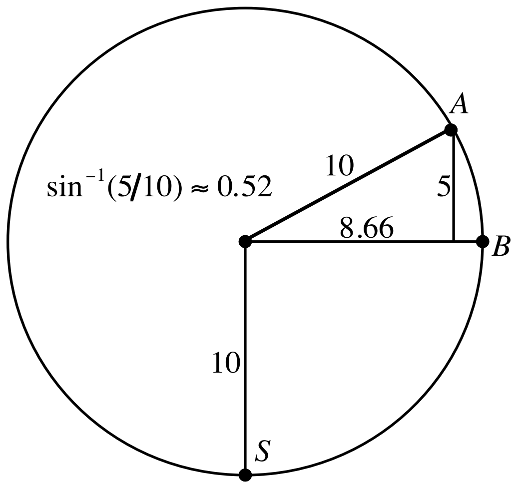
$$SP' = \frac{dr}{d - z}$$

* * * * *

2.8 Sample response: No. The image of a line of longitude would be two rays that point in opposite directions. Their endpoints would be on a circle that is the image of the equator and at opposite ends of a diameter of the circle. This is because no points in the southern hemisphere have an image under this projection.

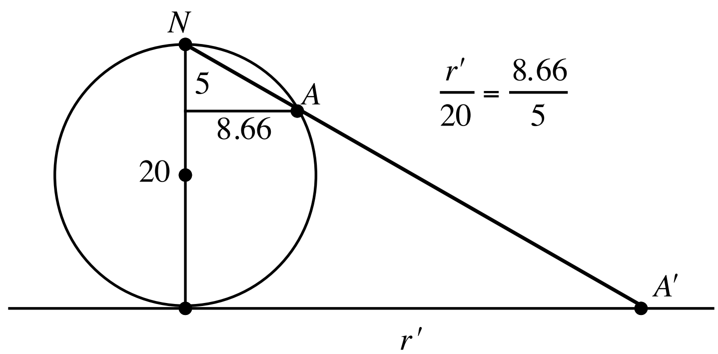
***2.9** **a**. The points are equally spaced along a line of longitude with B at the equator. Point A is in the northern hemisphere and point C is in the southern hemisphere.

b. Both arcs have a central angle of approximately 0.52 radians, as shown in the cross section of the sphere below.



Both distances are $0.52/2\pi \approx 0.08$ of the circumference of the circle. Their measures are $0.08(10)(2\pi) \approx 5.03$ cm .

c. One possible set of approximate polar coordinates for A' is (34.64,1.13), for B' is (20.00,1.13), and for C' is (11.55,1.13). The figure below shows the determination of *r* for A' . The radii for B' and C' can be found in a similar manner.



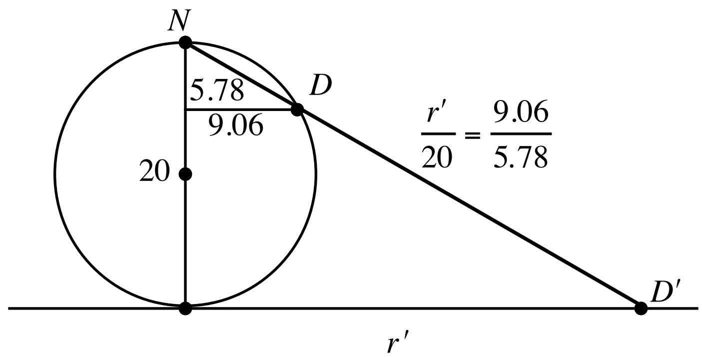
d. Sample response: The points are collinear radiating outward from S' . Successive distances between points are larger.

e. The distance $A'B'$ is about 14.64 cm, while $B'C'$ is about 8.45 cm. Distance is not preserved in this mapping.

***2.10** **a.** The points lie on a line of latitude in the northern hemisphere.

b. Each arc has a central angle of approximately 2.26 radians and a radius of 9.06 cm. Both distances can be calculated as follows: $(2.26/2\pi)(9.06)(2\pi) \approx 20.49$ cm. **Note:** This is not the true distance between the points on the sphere. That distance must be calculated along a great circle.

c. One possible set of polar coordinates for D' is approximately: (31.34,0.69), for E' is (31.34,2.95), and for F' is (31.34,5.25). The polar radii for all three points can be found as shown in the following diagram.



d. The points are mapped onto a circle with center at S' and a radius of 31.34 cm.

e. The distances are both approximately 71.11 cm. Although distance is not preserved, E' is still midway between D' and F'.

* * * * * * * * * *

(page 20)

Activity 3

Students explore cylindrical projections by projecting points from a foam hemisphere to a paper cylinder. They then continue their investigations using a geometry utility. **Note:** Cylindrical projections were invented by Gerardus Mercator in 1569. They are conformal for regions near the line of tangency (typically the equator). Lines of longitude are projected as equally spaced, straight, parallel lines; lines of latitude are perpendicular to lines of longitude.

Materials List

- foam hemispheres (about 5 cm in diameter; two per group)
- skewers (one per group)
- string (approximately 15 cm per group)
- ruler (one per group)
- serrated knife (optional; one per group)

Teacher Note

To create hemispheres, students can cut foam spheres in half with a serrated knife. One hemisphere is used in Exploration **1**; the other is used in the assignment. Foam hemispheres also may be purchased at many craft stores.

Technology

- geometry utility

Exploration 1 (page 21)

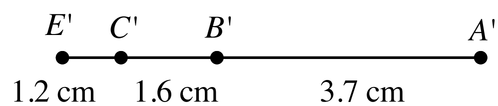
Students map points from a hemisphere to a cylinder and observe the distortion produced by cylindrical projections.

a. Students must cut the sphere along a great circle to obtain a true hemisphere.

b–d. The labeled hemisphere should resemble the one shown in Figure **11** of the student edition.

e. Students should use their reference marks E and E' to keep the hemisphere in a fixed position relative to the cylinder.

f–g. The diagram below shows the results of a projection using a foam sphere with a diameter of 5 cm.



h–i. Students repeat the mapping process for two more lines of longitude and for the equator, then unwrap the cylinder to observe the resulting map.

Discussion 1 (page 22)

a. Sample response: The image of the equator is a segment whose length is equal to the circumference of the sphere. The images of lines of longitude are parallel rays extending perpendicular from the image of the equator.

b.
1. Sample response: The distance from that point's image to the image of a point on the equator would be extremely large.
2. Sample response: Point N has no image on the cylinder, since the projection ray would be parallel to the cylinder's surface.

c.
1. Sample response: The borders of the surface would be defined by two parallel edges separated by a distance equal to the circumference of the sphere. The map would lie on the surface between the parallel edges and extend without bound in both directions.
2. Sample response: The image of a line of longitude would be a line perpendicular to the image of the equator.

d. **1.** Sample response: No. In a stereographic projection with the center of projection at one pole and the plane of projection tangent at the other pole, the images of lines of longitude are concurrent lines intersecting at the image of the pole opposite the center of projection. The image of the equator is a circle. The images of other lines (great circles) are elongated closed curves.

2. Sample response: Yes. The images of lines of longitude under a cylindrical projection are parallel lines. In fact, the projection of each great circle that contains the north and south poles results in two parallel lines separated by a distance equal to half the circumference of the sphere.

e. Sample response: No. It is possible for two lines of longitude to be perpendicular on the sphere. Their images under a cylindrical projection, however, would be parallel lines. Perpendicularity is only preserved between the equator and lines of longitude.

f. Sample response: The images of lines of longitude are parallel rays perpendicular to the image of the equator. The images of lines of latitude are segments parallel to the image of the equator. Their length is equal to the length of the equator. The distance between these parallel segments increases as the line of latitude's distance from the equator on the sphere increases.

g. **1.** Points in the southern hemisphere have negative values for z'.

2. Sample response: If the *x*-axis were located along the image of the equator, the images of points in the southern hemisphere would be below the *x*-axis.

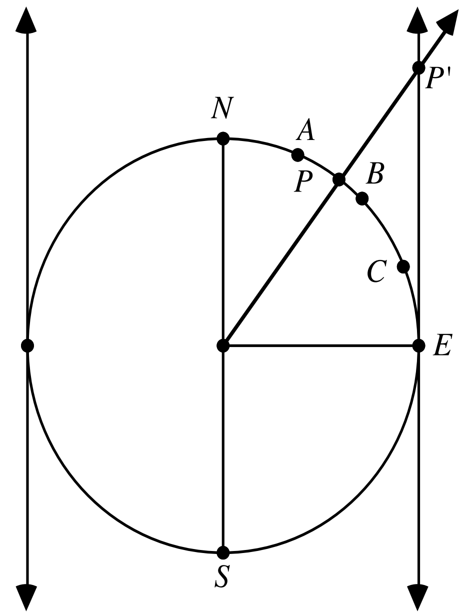
Exploration 2

(page 23)

In this exploration, students use a geometry utility to model a cylindrical projection. They then use properties of similar triangles to make generalizations about cylindrical projections.

a–b. As *P* moves around the circle toward *E*, P' moves along line *l* and approaches *E* as well. As *P* moves toward *N*, the distance from P' to *E* approaches infinity.

c. Points A, B, and C should appear as shown in the diagram below.



d. The following sample data corresponds with a circle of radius of 5 cm.

Segment	Length (cm)
$\overline{EE'}$	0
$\overline{EC'}$	2.07
$\overline{EB'}$	5.00
$\overline{EA'}$	12.08
$\overline{EN'}$	∞

e–f. Using ratios of similar triangles,

$$\frac{EP'}{ZP} = \frac{EO}{ZO}$$

Hence,

$$EP' = \frac{EO}{ZO} \bullet ZP$$

Discussion 2

(page 24)

a. Sample response: The length of $\overline{EP'}$ increases without bound.

b. Sample response: Both triangles share $\angle POE$ and both contain a right angle. This means that their corresponding third angles must be equal in measure. By the Angle–Angle–Angle Property, $\Delta POZ \sim \Delta P'OE$.

c. **1.** Sample response: If the intersection of the polar axis and the equator is the origin, then the image of the equator is a portion of the positive *x*-axis.

2. In this case, the *x*-coordinate of P' can be found by determining the arc length along the equator generated by the θ-coordinate of *P*. For point *P*, where *w* represents the radius of the sphere and $0 \le \theta \le 2\pi$:

$$x = \frac{\theta}{2\pi} \bullet 2\pi w = \theta w$$

3. In this case, the length of $\overline{EP'}$ is the *y*-coordinate of the point.

Assignment (page 25)

3.1 Sample response: No. Although distance is preserved along the equator, it is not preserved anywhere else. For example, on a sphere, lines of longitude have a finite length, the circumference of the sphere. Under a cylindrical projection, their images are lines which have infinite lengths. Since distance is not preserved, neither are perimeter and area.

3.2 **a.** Sample response: Venezuela is very close to the equator, while Greenland is much farther north. If the line of tangency is the equator, a cylindrical projection of Venezuela would have considerably less distortion than one of Greenland.

b. Sample response: The cylindrical projection would have greater distortion than the stereographic projection because Greenland is close to the stereographic map's point of tangency, but distant from the cylindrical map's line of tangency.

3.3 Sample response: Perpendicularity between lines of latitude and lines of longitude is preserved under a cylindrical projection. This gives navigators a convenient grid for determining their locations and plotting their courses.

3.4 Sample response: The preimage consists of two arcs that lie on lines of longitude, one arc that lies along the equator, and an arc of a line of latitude. The angles formed by the two sides intersecting the equator measure 90˚. Since the fourth side is not on a great circle, there is no way to measure the remaining angles.

***3.5** **a.** The length of the portion of the *x*-axis on the unwrapped cylinder is $2(\pi)(10) \approx 62.83$ cm.

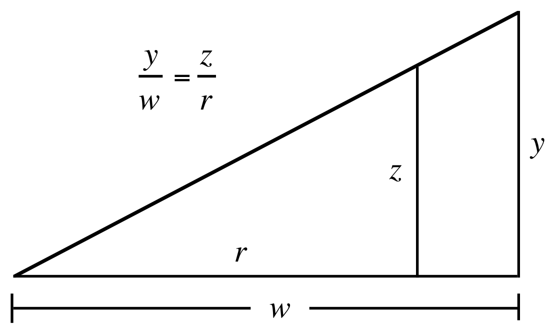
b. The *x*-coordinate of A' can be found by determining the arc length along the equator generated by the θ-coordinate of *A*. For point *A*, where *w* represents the radius of the sphere:

$$x = \frac{0.53}{2\pi} \bullet 2\pi w = 0.53(10) \approx 5.34$$

c. The *x*-coordinate of B' is $(5.40)(10) \approx 54.04$.

d. The *x*-coordinate of P', when $0 \le \theta \le 2\pi$, is $x = \theta w$.

e. The *y*-coordinate can be determined using similar triangles, as shown in the following diagram.



***3.6** **a.** The *x*-coordinate is $(\pi/2)(20) \approx 31.42$. The *y*-coordinate can be found by solving the following proportion:

$$\frac{y}{20} = \frac{10}{7.32}$$

Thus, the approximate rectangular coordinates of A' are (31.42,27.32).

b. **1.** Point *B* is in the southern hemisphere near the south pole.

2. The *x*-coordinate of B' is $(4.96)(20) \approx 99.27$. Using similar triangles, the *y*-coordinate can be determined by solving the following proportion:

$$\frac{y}{20} = \frac{-19.92}{1.74}$$

The rectangular coordinates of B' are approximately (99.27,–228.97).

* * * * *

3.7 **a.** The points are equally spaced along a line of longitude. Point *B* is on the equator.

b. The arcs each have an approximate measure of approximately 1.10 radians. The arc lengths are both approximately 25.13 cm.

c. The rectangular coordinates for A' are approximately (135.87,38.49), for B' are (135.87,0.00), and for C' are (135.87,–38.49).

d. The projected points lie along a vertical line.

e. The distances are each 38.49 cm. Although distance is not preserved, B' is midway between A' and C'.

3.8 **a.** The points lie on a line of latitude in the southern hemisphere.

b. Each arc has a measure of approximately 1.76 radians and a radius of 19.15 cm. The arc lengths are both approximately 33.69 cm. **Note:** This is not the true distance between the points on the sphere. That distance must be calculated along a great circle.

c. The rectangular coordinates for D' are approximately (6.28,–20.98), for E' are (50.27,–20.98), and for F' are (93.46,–20.98).

d. The points lie on a horizontal line 20.98 units below the *x*-axis.

e. The distances are each approximately 43.59 cm. Distance is not preserved in this mapping.

* * * * * * * * * *

Research Project (page 27)

The following paragraphs describe four types of projections used in mapmaking: orthographic azimuthal, orthographic cylindrical, gnomic, and oblique Mercator.

Orthographic azimuthal maps are made by projecting lines through points on the surface of the sphere perpendicular to the plane of the map. This results in a perspective projection from an infinite distance. The lines of latitude and longitude appear as either ellipses, circles, or straight lines. The map is not conformal and is one of the least useful for purposes of measurement. However, since the map looks more like a globe than any other azimuthal projection, it is commonly used in atlases, as well as on posters and book covers. The ancient Egyptians, Greeks, Arabians, and Indians used this type of map for astronomical purposes. An orthographic azimuthal map of the world was first employed by Albrecht Dürer, a 16th-century artist and cartographer.

Orthographic cylindrical maps are made by performing two projections onto one cylinder. First, lines of longitude are point-projected onto the cylinder from the center of the earth. Next, the lines of latitude are perspective-projected using lines parallel to the equator (in other words, orthographically). The result is a map with evenly spaced, straight, parallel lines of longitude and unevenly spaced, straight, parallel lines of latitude that are perpendicular to each other. Although these maps have a high degree of visual distortion, they do preserve area and distance along horizontal and vertical lines. First proposed by Johann Heinrich Lambert in 1772, it is sometimes used as a textbook example of the most easily constructed equal-area map.

Gnomic maps are azimuthal maps made by projecting points on the surface of the sphere onto a plane. The center of projection is the center of the earth. The resulting map has one important and useful feature: all lines of longitude are straight lines, no matter where the plane is tangent to the sphere. The lines of latitude are conic sections, depending on the location of the point of tangency. Gnomic maps are not conformal and have the worst distortion of all the ancient types of projection maps. Only one geometric property can be salvaged from a gnomic map: a route plotted between two points on the same line of longitude is accurate in direction (but not in distance). Gnomic projections were used by the ancient Greeks, such as Thales of Miletus, to make star maps. They are seldom used today.

Oblique Mercator maps are made using a standard cylindrical projection onto a cylinder tangent to the earth at a great circle other than the equator or a line of longitude. This results in a map that is conformal along two lines of longitude (180° apart) that are straight lines, while all other lines of longitude and latitude are complex curves. Distortion increases as the distance from the straight lines of longitude increases. The oblique Mercator projection was first used to obtain conformal maps of specific regions, including the Alaska panhandle, Madagascar, and Borneo. Today, this projection is used to make maps from satellite images, such as the Landsat maps, in which the line of tangency is the orbital path of the satellite.

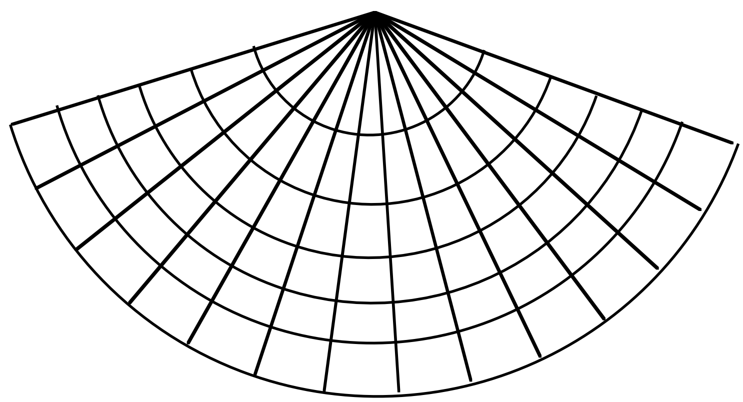
Teacher Note

Students may wish to complete Problems **2** and **3** using a geometry utility.

Answers to Summary Assessment (page 28)

1. **a.** The intersection of the sphere and a tangent cone forms a circle.

b. In a conical mapping, lines of latitude appear as concentric arcs. The distance between any pair of lines of latitude increases as the distance from the image of the points of tangency increases. As shown in the diagram below, lines of longitude appear as concurrent straight lines.

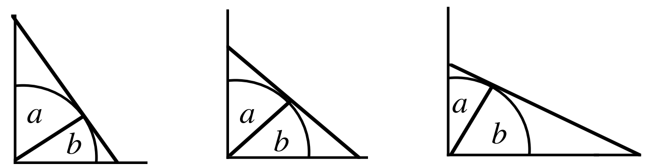


c. Sample response: The only geometric property that appears to be preserved is the intersection of the lines of longitude with the equator and the concurrency of the lines of longitude. There appears to be no other geometric properties preserved. These include perpendicularity, collinearity, distance, perimeter, and area.

2. Using the given coordinates for A, $OB = 13.62$ cm and $AB = 20.97$ cm. Since OE is 25 cm (the radius of the sphere), $m\angle AOB = \cos^{-1}(13.62/25) \approx 1.01$ and $m\angle AOE \approx 1.01 - 0.79 = 0.22$.

Therefore, $A'E = 25\tan(0.22) \approx 5.59$ and the length of arc AE is $0.22(25) \approx 5.50$. The ratio of these two values is approximately 1.02.

3. Sample response: Yes, the measure of the vertex angle affects the amount of distortion. In the cross section, the closer the points are to the point of tangency, the less the distortion. As the measure of the angle increases, the point of tangency moves farther north on the sphere, as shown in the following diagrams.



When the point of tangency is farther north, the images of points in region *a* are less distorted, while images of points in region *b* are more distorted. As the measure of the vertex angle decreases, the point of tangency moves south, resulting in more distortion in images of region *a* and less in images of region *b*.

Module Assessment

1. Point *A* lies on a sphere with radius 100 cm. When the pole for a cylindrical coordinate system is located at the sphere's south pole, the coordinates of *A* are approximately $(79.86, 3.39, 39.82)$. Find the polar coordinates of A' using a stereographic projection, assuming that the same pole and polar axis are used for both coordinate systems.

2.

a. Find the cylindrical coordinates of point *A* in Problem **1** when the pole is located at the sphere's center.

b. Find the rectangular coordinates of the image of *A* using a cylindrical projection in which the cylinder is tangent to the sphere at the equator. Let the projected image of the equator represent a portion of the positive *x*-axis, with the origin at the intersection of the equator and the polar axis.

3. Imagine that you are a cartographer in the year 2050. Some astronomers in your office have just completed a globe of the planet Alpha. The globe has a radius of 30 cm. Your job is to produce a flat map of one region of the globe. Using a cylindrical coordinate system with the pole located at the planet's south pole, the region has four corner points with the following coordinates: $A\ (30.00, 0, 30.00)$, $B\ (30.00, \pi/4, 30.00)$, $C\ (28.19, \pi/4, 40.26)$, and $D\ (28.19, 0, 40.26)$.

a. Describe the shape of region *ABCD* on the globe.

b. Find the polar coordinates of A', B', C', and D' using a stereographic projection. Assume that the same pole and polar axis are used for both coordinate systems.

c. Describe the shape of region $A'B'C'D'$ on the flat map, as well as the distortions that occurred as a result of the projection.

d. Convert the coordinates of points *A*, *B*, *C*, and *D* to cylindrical coordinates with the pole at the center of the globe.

e. Find the rectangular coordinates of A'', B'', C'', and D'' using a cylindrical projection in which the cylinder is tangent to the globe at the equator. Let the projected image of the equator represent a portion of the positive *x*-axis, with the origin at the intersection of the equator and the polar axis.

f. Describe the shape of region $A''B''C''D''$ on the new flat map and the distortions that occurred as a result of the projection.

g. Approximate the area of the two projections in Parts **c** and **f**.

h. Which of the two projections do you feel results in a more accurate mapping of the region of the globe? Explain your response.

Answers to Module Assessment

1. One possible set of polar coordinates for A' is $(99.71, 3.39)$.

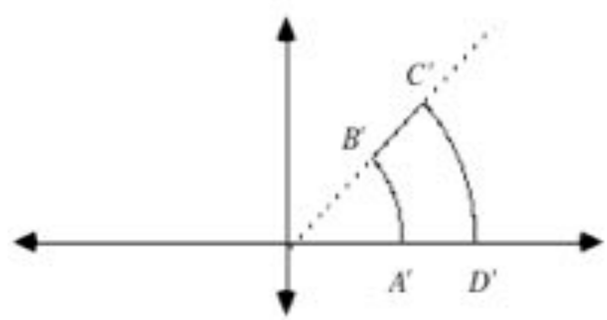
2.
a. With the pole at the center of the sphere, one possible set of cylindrical coordinates for *A* are $(79.86, 3.39, -60.18)$.

b. The rectangular coordinates of the image are approximately $(339.34, -75.36)$.

3.
a. Region *ABCD* is a four-sided figure with its long side on the equator. **Note:** Students should recognize that the sides of figures on a globe are arcs, not segments.

b. The polar coordinates for A' are $(60, 0)$, for B' are $(60, \pi/4)$, for C' are $(85.68, \pi/4)$, and for D' are $(85.68, 0)$.

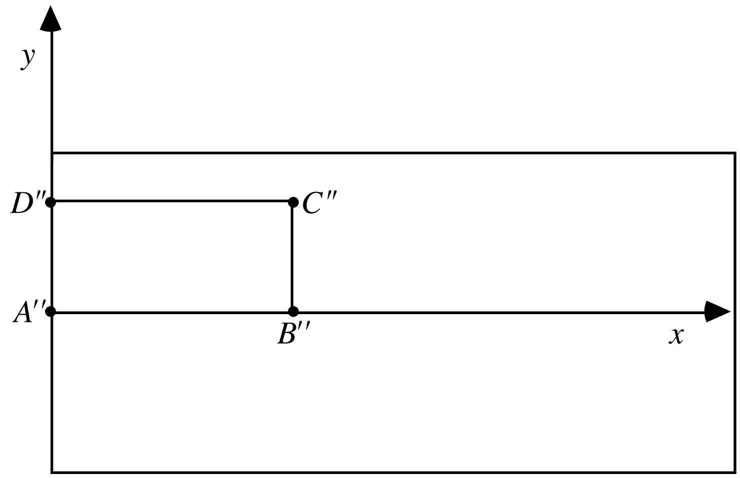
c. As shown in the figure below, $A'B'C'D'$ is a four-sided figure with two sides that are arcs. Points A' and B' lie on a circle of radius 60 cm. Points C' and D' lie on a circle of radius 85.65 cm.



d. The cylindrical coordinates for *A* are $(30.00, 0, 0)$, for *B* are $(30.00, \pi/4, 0)$, for *C* are $(28.19, \pi/4, 10.26)$, and for *D* are $(28.19, 0, 10.26)$.

e. The rectangular coordinates for A'' are (0,0), for B'' are (23.56,0), for C'' are (23.56,10.92), and for D'' are (0,10.92).

f. Region *ABCD* is a four-sided figure with its long side on the equator. Its cylindrical projection, $A''B''C''D''$, is a rectangle.



g. Student methods may vary. The area of the region found using the stereographic projection can be found by subtraction of the areas of two sectors, as shown below:

$\left((\pi/4/2\pi)(\pi)(85.68^2)\right) - \left((\pi/4/2\pi)(\pi)(60^2)\right) \approx 1469.11 \text{ cm}^2$

The area of the rectangular region found using the cylindrical projection is $23.56 \bullet 10.92 \approx 257.28 \text{ cm}^2$.

h. An argument can be made for either method. Considering only shape, a stereographic projection may be the better of the two. Another argument for this type of projection is that only one hemisphere is grossly distorted. Considering area, students may defend the cylindrical projection.

Selected References

Alpha, T. R., and D. Strebe. "Map Projections." U.S. Geological Survey Open-File Report 91-553A. Washington, DC: U.S. Government Printing Office, 1991.

Birch, T. W. *Maps Topographical and Statistical*. Oxford: Oxford University Press, 1966.

Fisher, I., and O. M. Miller. *World Maps and Globes*. New York: J. J. Little & Ives Co., 1944.

Greenhood, D. *Mapping*. Chicago: The University of Chicago Press, 1964.

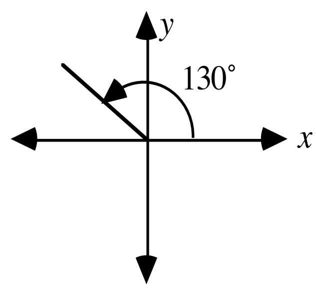
Kellaway, G. P. *Map Projections*. London: Methuen and Co., 1946.

Snyder, J. P. "Map Projections—A Working Manual." U.S. Geological Survey Professional Paper 1395. Washington, DC: U.S. Government Printing Office, 1987.

Flashbacks

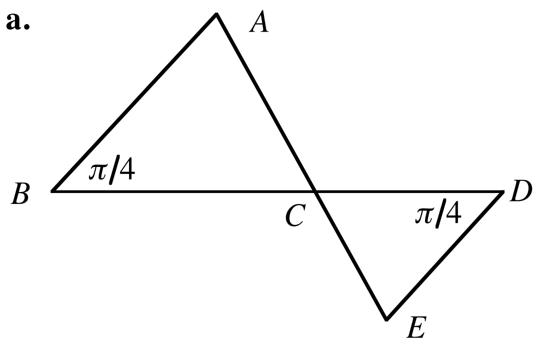
Activity 1

1.1 Identify four different angle measures, in radians, that result in an angle in the same position as the one shown below. Of the four angle measures, two must be positive and two must be negative.

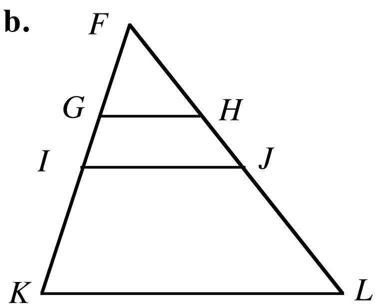


1.2 Identify the similar triangles in Parts **a–c** below.

a.

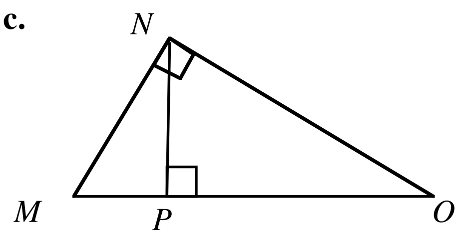


b.



$\overline{GH} \parallel \overline{IJ} \parallel \overline{KL}$

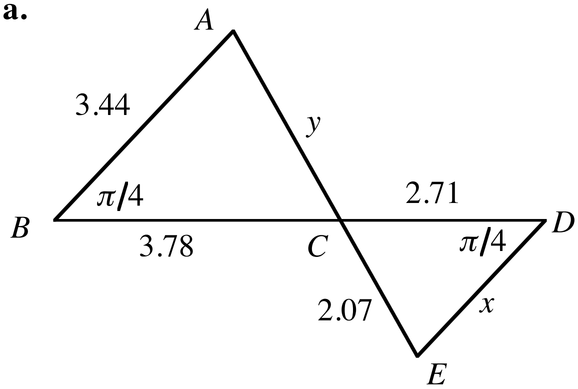
c.



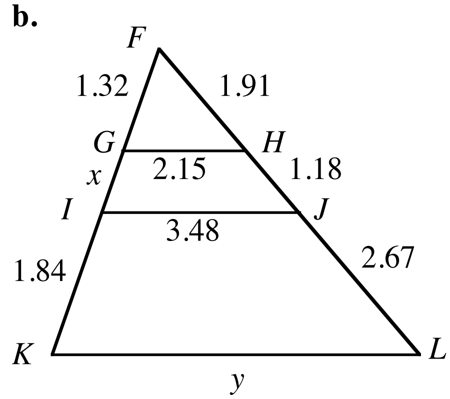
Activity 2

2.1 Determine the values of x and y in each of the following figures.

a.

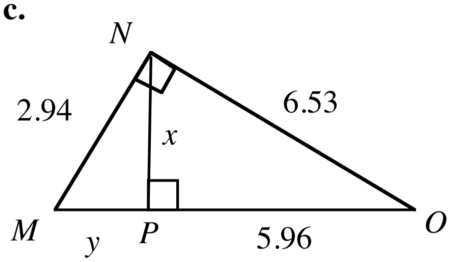


b.

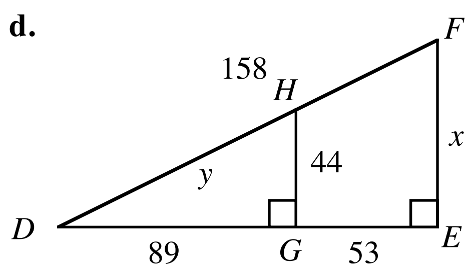


$\overline{GH} \parallel \overline{IJ} \parallel \overline{KL}$

c.



d.



Activity 3

3.1 Determine the length of a great circle on a sphere with each of the following radii:

a. 15 cm

b. 2560 km

3.2 Consider a sphere with a radius of 20 cm. Find the arc lengths of paths along a great circle for each of the following angle measures:

a. π

b. $\pi/5$

c. 30°

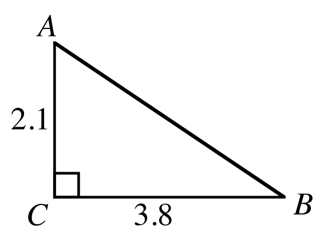
3.3 Assuming that the polar axis is the positive *x*-axis, convert each pair of polar coordinates below to Cartesian coordinates.

a. $(7, \pi/2)$

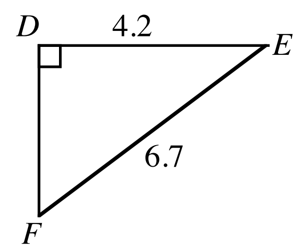
b. $(12, \pi/3)$

3.4 Use trigonometric ratios to determine the unknown angle measures in each triangle below.

a.



b.



Answers to Flashbacks

Activity 1

1.1 Answers will vary. Sample responses:

$$130\pi/180 = 13\pi/18 \approx 2.27$$
$$490\pi/180 = 49\pi/18 \approx 8.55$$
$$-230\pi/180 = -23\pi/18 \approx -4.01$$
$$-590\pi/180 = -59\pi/18 \approx -10.30$$

1.2
- **a.** $\Delta ABC \sim \Delta EDC$
- **b.** $\Delta FGH \sim \Delta FIJ$, $\Delta FIJ \sim \Delta FKL$, and $\Delta FGH \sim \Delta FKL$
- **c.** $\Delta MNO \sim \Delta NPO$, $\Delta MNO \sim \Delta MPN$, and $\Delta MPN \sim \Delta NPO$

Activity 2

2.1
- **a.** $x \approx 2.47$ and $y \approx 2.89$
- **b.** $x \approx 0.82$ and $y \approx 6.48$
- **c.** $x \approx 2.68$ and $y \approx 1.21$
- **d.** $x \approx 69$ and $y \approx 99$

Activity 3

3.1
- **a.** $2\pi(15) = 30\pi \approx 94$ cm
- **b.** $2\pi(2560) = 5120\pi \approx 16{,}085$ km

3.2
- **a.** $\left(\frac{\pi}{2\pi}\right)(2\pi)(20) \approx 20\pi \approx 62.83$ cm
- **b.** $\left(\frac{\pi/5}{2\pi}\right)(2\pi)(20) \approx \frac{20\pi}{5} \approx 12.57$ cm
- **c.** $\left(\frac{\pi/6}{2\pi}\right)(2\pi)(20) \approx \frac{20\pi}{6} \approx 10.47$ cm

3.3
- **a.** $(0,7)$
- **b.** $\left(6, 6\sqrt{3}\right)$

3.4
- **a.** $m\angle B = \tan^{-1}(2.1/3.8) \approx 0.50 \approx 28.9^\circ$; $m\angle A \approx 1.07 \approx 61.1^\circ$
- **b.** $m\angle F = \sin^{-1}(4.2/6.7) \approx 0.68 \approx 38.8^\circ$; $m\angle E \approx 0.89 \approx 51.2^\circ$

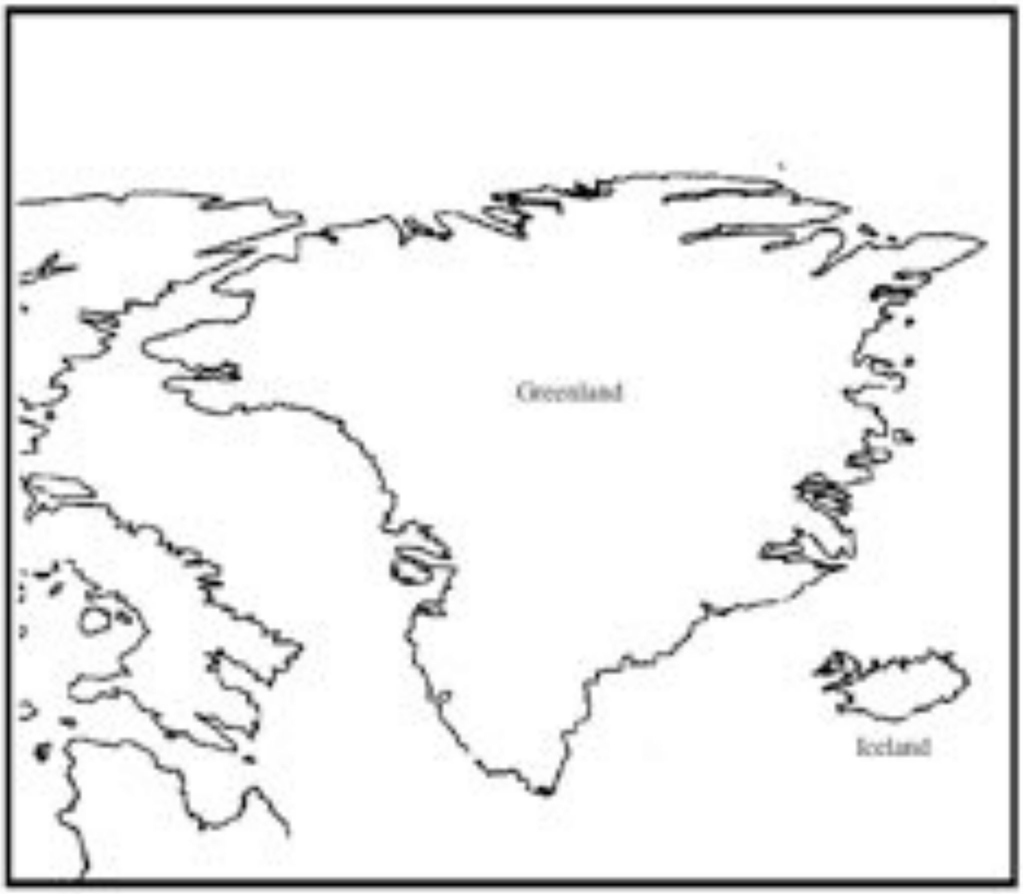
Template A

of Figure **1a** in Student Edition

1 cm = 250 km

Template B

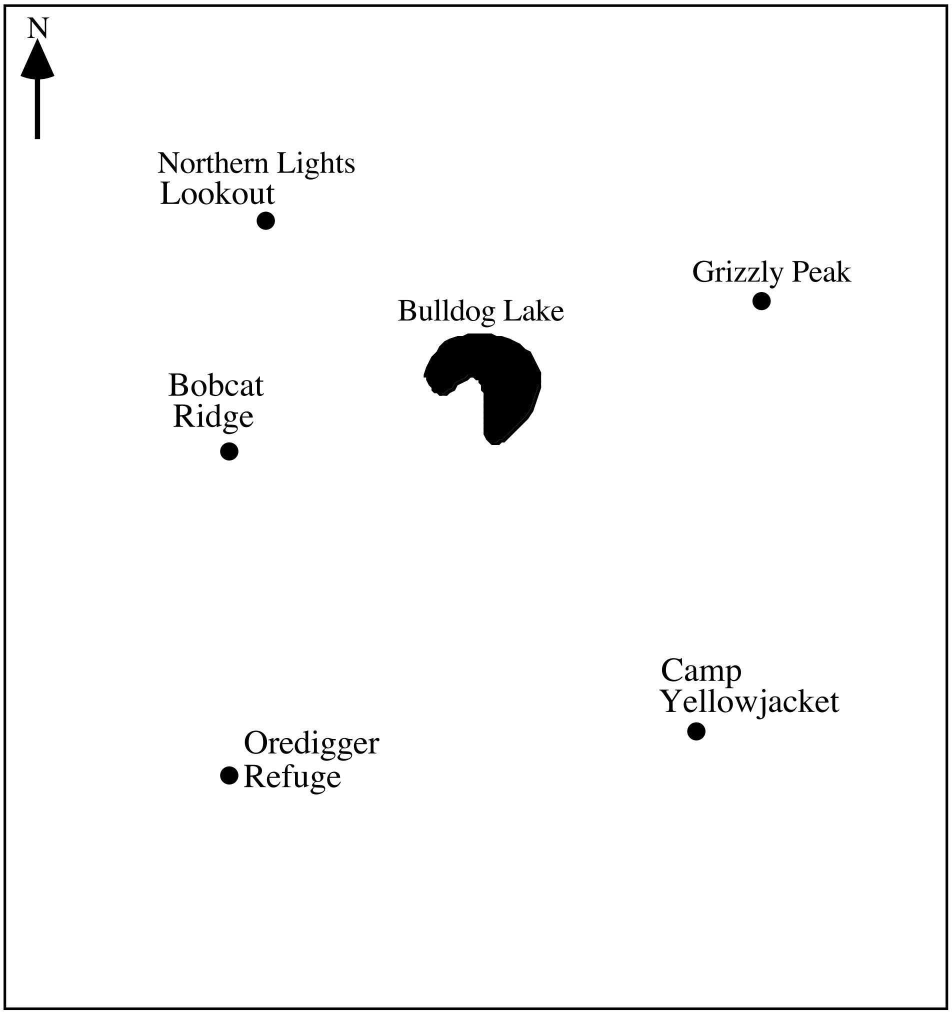
of Figure **1b** in Student Edition



1 cm = 125 km

Template C

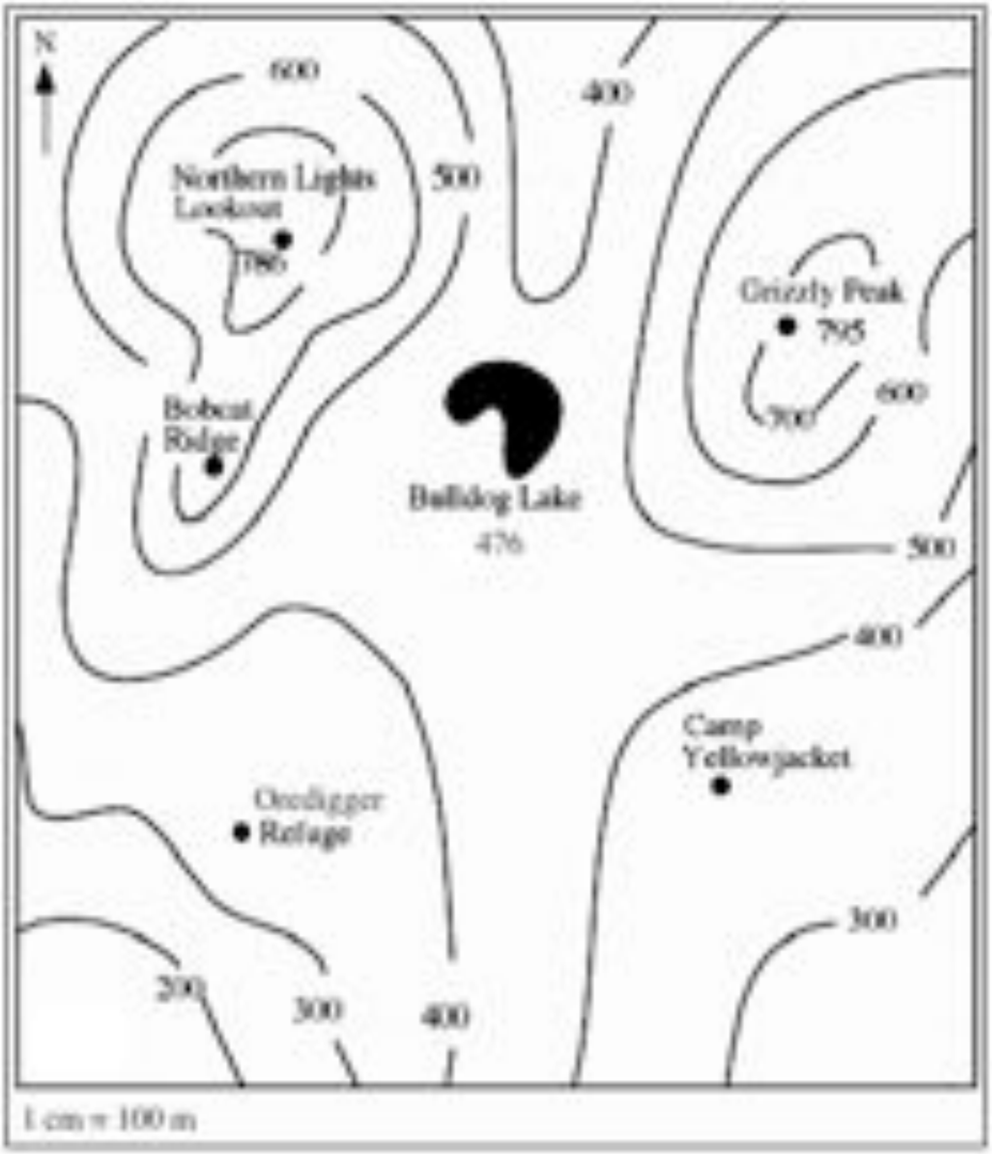
of Figure **3** in Student Edition



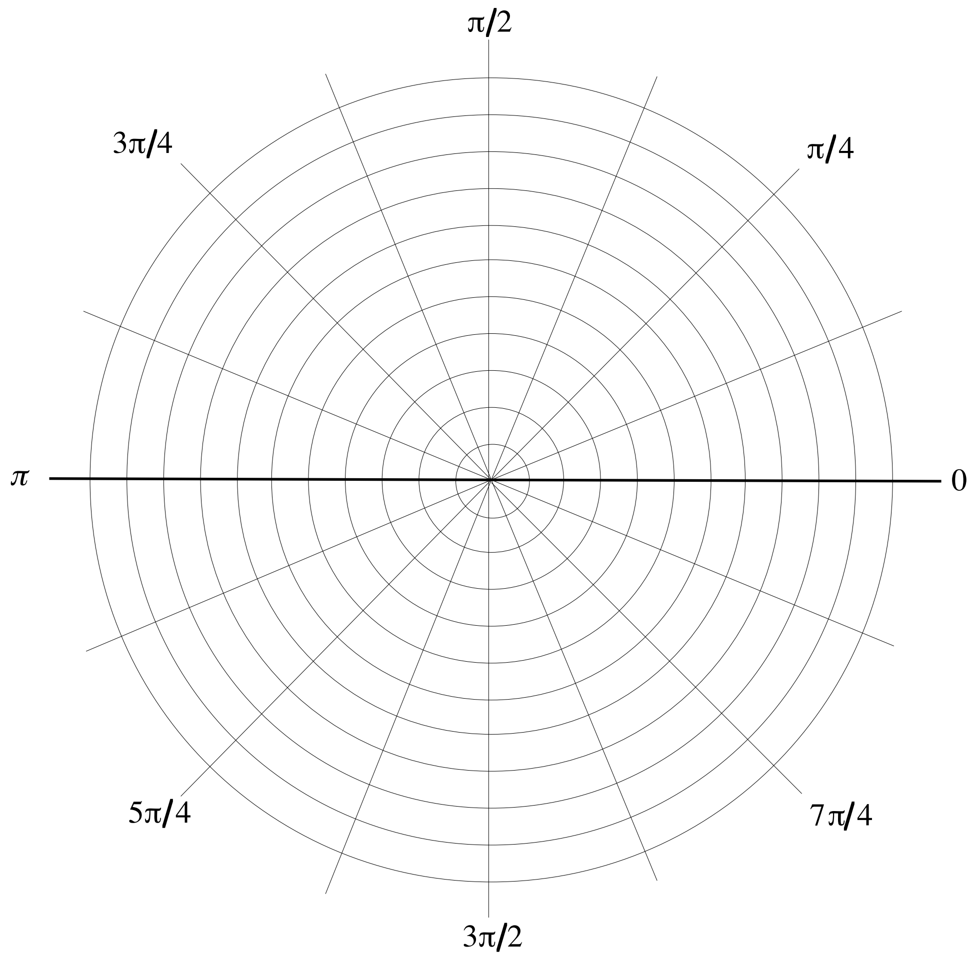
1 cm = 100 m

Template D

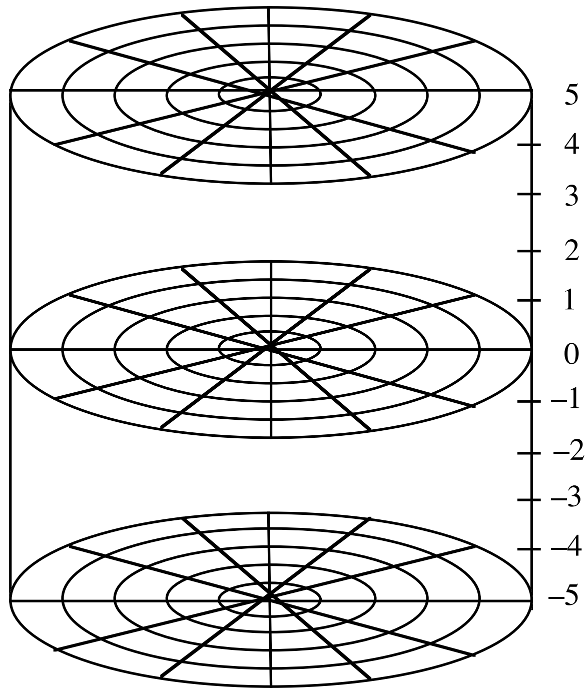
of Figure **4** in Student Edition



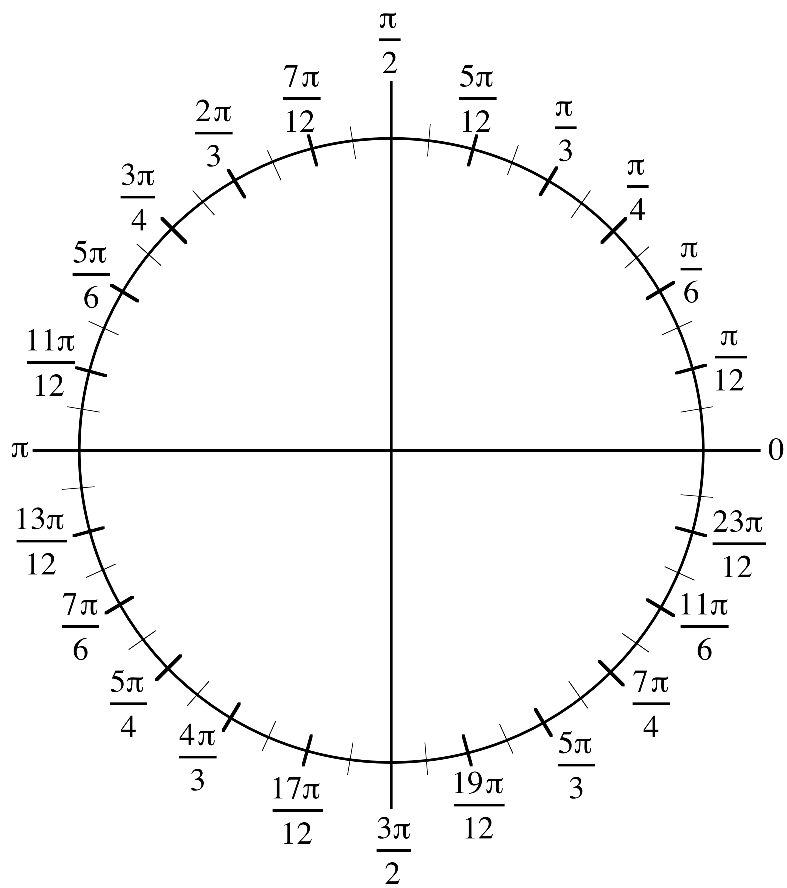
Polar Graph Paper



Cylindrical Graph Paper



Radian Protractor



Naturally Interesting

How much money would you have to invest now in order to be a millionaire at age 65? In this module, you use exponentials and logarithms to answer this question.

Todd Fife • Sue Moore • Pete Stabio

Teacher Edition

Naturally Interesting

Overview

Students explore exponential and logarithmic models through the contexts of savings and investments. The real number e is introduced in the formula for compound interest. The natural logarithm is defined as the logarithm with base e.

Objectives

In this module, students will:

- define e as the limit of a sequence
- solve problems involving compound interest
- define natural logarithms.

Prerequisites

For this module, students should know:

- how to solve problems involving simple interest
- how to estimate base-10 logarithms
- the properties of exponents and logarithms
- the definition of a normal probability distribution.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	2	2	1	8

Materials Required

- none

Technology

Software	Activity			
	1	2	3	Summary Assessment
spreadsheet	X	X	X	X
graphing utility		X	X	
symbolic manipulator			X	X

Naturally Interesting

Teacher Note

You may wish to ask students to research available opportunities for investing or borrowing money before beginning this module. A conversation with a banker or investment analyst may help clarify similarities and differences among banks, credit unions, and other financial institutions. The business section of a daily newspaper is a good source of advertisements that mention current interest rates.

Introduction

(page 35)

Students discuss saving, borrowing, and investment. The mathematics note introduces a formula for simple interest.

Discussion

(page 35)

a. Car loans, home improvement loans, personal loans, and college loans are all commonly available. Current interest rates and terms will vary and can be checked by calling a bank or consulting the daily newspaper. In the case of car loans, terms may vary from 24 months to 72 months and interest rates often depend on whether the car is new or used.

b. Sample response: Money may be invested in certificates of deposit where the longer the money is invested, the higher the interest rate and the greater the return. Savings accounts, interest-bearing checking accounts, stocks, and bonds are other opportunities for investing money.

c. Because the interest is now calculated using a principal of $1100, the interest earned is $(\$1100)(0.10) = \$110$.

(page 36)

Activity 1

Students develop a formula for calculating account balance when interest is compounded and examine how interest rate and number of compoundings affect account balance.

Materials List

- none

Technology

- spreadsheet

Exploration 1

(page 36)

Students examine an initial investment of \$500 at an annual interest rate of 6%, compounded annually, over 20 yr. They identify the yearly account balances as a recursive sequence and generalize a formula for the sequence.

a. Using the formula for simple interest, $I = 500 \bullet (0.06) \bullet 20 = \$600$. The balance of the account would be $\$500 + \$600 = \$1100$.

b. Answers may vary. Sample response: Since interest is earned on the principal plus the previous year's interest, the balance after 20 yr should be more than \$1100.

c. **1.** $500 + 0.06 \bullet 500 = 530$

2. $530 + 0.06 \bullet 530 = 561.80$

d. Sample table:

Years (t)	Principal at Beginning of Year (\$)	Account Balance at End of Year (\$)
1	500.00	530.00
2	530.00	561.80
3	561.80	595.51
⋮	⋮	⋮
18	1346.39	1427.17
19	1427.17	1512.80
20	1512.80	1603.57

e. Sample response: A rate greater than 6% makes the account balance increase faster. At 6%, the money doubles in about 12 yr. It takes less time to double as the rate increases. A rate less than 6% makes the account balance increase more slowly.

f. If P_t represents the principal at the end of t years, the principal at the beginning of the year is P_{t-1}. The recursive formula can be written as follows:

$$P_t = P_{t-1} + 0.06P_{t-1} = P_{t-1}(1 + 0.06) = P_{t-1}(1.06)$$

g. **1.** $P_1 = P_0(1 + r)^1$

2. $P_2 = P_1(1 + r) = P_0(1 + r)(1 + r) = P_0(1 + r)^2$

3. $P_3 = P_2(1 + r) = P_0(1 + r)^2(1 + r) = P_0(1 + r)^3$

h. Given an initial principal P_0 and the time *t* in years, $P_t = P_0(1 + r)^t$.

i. Using the explicit formula $P_t = P_0(1 + r)^t$, the account balance at the end of 20 years is $P_{20} = 500(1 + 0.06)^{20} \approx \$1603.57$. This is the same account balance as calculated using the spreadsheet in Table **1**.

Discussion 1

(page 38)

a. Sample response: An explicit formula allows you to find any term in a sequence without knowing the previous term. All that is needed is the initial principal, the annual interest rate, and the number of years.

b.
1. Sample response: The first term of the sequence is the initial principal. The common ratio is $1 + r$, where *r* is the annual interest rate. The term number is the number of years.
2. Although the notation may be slightly different, student equations should be mathematically equivalent to $P_t = P_0(1 + r)^t$.

c. Sample response: Doubling the initial principal will double the final balance.

d. Sample response: By trial and error, manipulate *t* in the following expression until its value becomes approximately \$1000.

$$500(1 + 0.06)^t$$

Or, solve the following equation for *t* using logarithms:

$$1000 = 500(1 + 0.06)^t$$

In this case, *t* is approximately 12 yr. **Note:** Students will encounter properties of exponents and logarithms again in Activity **3**.

e.
1. Sample response: The equations are identical except for the choice of variables. They both model situations that behave exponentially, with a constant rate of increase or decrease for a given number of time periods.
2. Sample response: In both equations, the 1 represents 100% of the initial value, while *r* represents the rate of increase or decrease.

Exploration 2

(page 39)

Students investigate how changing *c*, the number of compoundings per year, affects the amount of interest earned. As *c* becomes very large, the total number of compoundings in *t* years (or $c \bullet t$) becomes very large and the account balance approaches a limit for given values of P_0, *r*, and *t*.

a. **1.** The interest earned for the first compounding period is \$15, resulting in a balance of \$515.

$$P_1 = 500 + 500\left(\frac{0.06}{2}\right) = 500\left(1 + \frac{0.06}{2}\right) = \$515$$

The interest for the second compounding period is \$15.45, resulting in a year-end balance of \$530.45.

$$P_2 = 515 + 515\left(\frac{0.06}{2}\right) = 515\left(1 + \frac{0.06}{2}\right) = \$530.45$$

2. The account balance after 1 yr is:

$$P_2 = 500\left(1 + \frac{0.06}{2}\right)\left(1 + \frac{0.06}{2}\right) = 500\left(1 + \frac{0.06}{2}\right)^2$$

3. The account balance after 2 yr (or 4 compounding periods) is:

$$P_4 = 500\left(1 + \frac{0.06}{2}\right)^4 \approx \$562.75$$

The account balance after 3 yr (or 6 compounding periods) is:

$$P_6 = 500\left(1 + \frac{0.06}{2}\right)^6 \approx \$597.03$$

b. Sample formula, where $n = 2t$:

$$P_n = P_0\left(1 + \frac{r}{2}\right)^{2t}$$

c. The interest for the first compounding period is \$7.50, resulting in a balance of \$507.50.

$$P_1 = 500 + 500\left(\frac{0.06}{4}\right) = 500\left(1 + \frac{0.06}{4}\right) = \$507.50$$

The interest for the second compounding period is approximately \$7.61, for the third approximately \$7.73, and for the fourth approximately \$7.84, resulting in a year-end balance of \$530.68.

The expression for the balance after 1 yr (4 compounding periods) is:

$$P_4 = 500\left(1 + \frac{0.06}{4}\right)\left(1 + \frac{0.06}{4}\right)\left(1 + \frac{0.06}{4}\right)\left(1 + \frac{0.06}{4}\right) = 500\left(1 + \frac{0.06}{4}\right)^4$$

The account balance after 2 yr (8 compounding periods) is:

$$P_8 = 500\left(1 + \frac{0.06}{4}\right)^8 \approx 563.25$$

The account balance after 3 yr (12 compounding periods) is:

$$P_{12} = 500\left(1+\frac{0.06}{4}\right)^{12} \approx 597.81$$

Sample formula, where $n = 4t$:

$$P_n = P_0\left(1+\frac{r}{4}\right)^{4t}$$

d. Sample formula, where $n = ct$:

$$P_n = P_0\left(1+\frac{r}{c}\right)^{ct}$$

e. Sample spreadsheet:

Initial Principal: $500		
Annual Interest Rate: 6%		
Type of Compounding	**No. of Compoundings per Year (*c*)**	**Account Balance (P_n)**
annually	1	530.00
semiannually	2	530.45
quarterly	4	530.68
monthly	12	530.84
daily	365	530.92
hourly	8760	530.92
by the minute	525,600	530.92
by the second	31,536,000	530.92

f. Answers may vary. Judging from Table **2**, some students may predict a balance of $530.92, while others may predict an increase.

g. Students should observe that, for any given initial principal and annual interest rate, the balance of the account approaches a limiting value as the number of compoundings increases. Sample spreadsheet:

P_0	1000	100	5000	5000
r	4%	4%	8%	2%
c	**Balance**	**Balance**	**Balance**	**Balance**
1	1040.00	104.00	5400.00	5100.00
2	1040.40	104.04	5408.00	5100.50
4	1040.60	104.06	5412.16	5100.75
12	1040.74	104.07	5415.00	5100.92
365	1040.81	104.08	5416.39	5101.00
8760	1040.81	104.08	5416.43	5101.01
525,600	1040.81	104.08	5416.44	5101.01
31,536,000	1040.81	104.08	5416.44	5101.01

Discussion 2 (page 41)

a. Sample response: In the formula for account balance when interest is compounded semiannually, the annual interest rate is divided by 2 and the number of years is multiplied by 2 to obtain the number of compounding periods.

b. Sample response: As the number of compoundings increases, the account balance appears to approach a limiting value.

c. Sample response: As the number of compoundings per year increases, the interest that can be earned also increases, although it eventually approaches a limiting value.

d. Sample response: Yes. The interest earned changes less and less as the number of compoundings increases. **Note:** Some students may respond that there is no limit. This idea is examined further in Activity **2**.

Assignment (page 41)

1.1 **a.** \$752.50

b. \$754.00

c. \$754.34

d. \$754.51

***1.2** **a.** $\$112.50(1+0.09)^{47} = \$6482.45$

b. $\$112.50\left(1+\frac{0.09}{12}\right)^{12\bullet 47} = \$7636.68$

***1.3** **a.** The account balance will be \$20,000 when $t \approx 28.9$ yr.

Note: One method of solution some students may use is graphing the equations $P_n = 20{,}000$ and

$$P_n = 5000\left(1+\frac{0.048}{12}\right)^{12t} = 5000(1.004)^{12t}$$

where $n = 12t$, then determine the intersection of the two graphs.

b. After 18 yr, the balance will be nearly \$30,000 when $r \approx 0.10$, compounded monthly. **Note:** To solve this problem, some students may graph the equations $P_n = 30{,}000$ and

$$P_n = 5000\left(1+\frac{r}{12}\right)^{12\bullet 18} = 5000\left(1+\frac{r}{12}\right)^{216}$$

where $n = 216$, then determine the intersection of the two graphs.

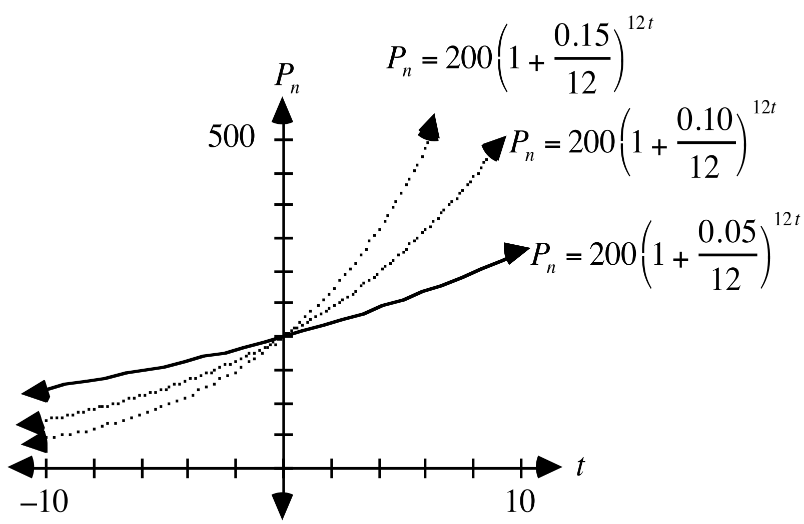
1.4 **a.** Answers will vary. Typically, r is a percentage less than 100% and will be contained in the interval $0 < r \leq 1$. Since most accounts compound at least annually, c must be greater than or equal to 1 and thus $n = ct$ must be greater than or equal to 1. Note that because c is a positive integer, if time t in years also is limited to positive integers, then ct or n is a positive integer.

b. **1.** Answers will vary. Sample response: If $P_0 = 200$, $c = 12$, and $r = 5\%$ then the equation, where $n = 12t$, is

$$P_n = 200\left(1 + \frac{0.05}{12}\right)^{12t}$$

2. Sample response: If t is limited to the positive integers, this is an example of an exponential function with a restricted domain.

c. **1.** Sample graph:



2. Sample response: The graph becomes steeper as r increases.

1.5 Sample response: After 10 yr, the investment of \$1500 at 15%, compounded quarterly, will have a value of \$6540.57. The return on each dollar invested is $\$1.00(6540.57/1500) \approx \$4.36$. The investment of \$1600 at 15.5%, compounded annually, will have a value of \$6759.89. The return on each dollar is $\$1.00(6759.89/1600) \approx \$4.22$. The client should be encouraged to invest in the account that is compounded quarterly.

Note: Some students may compare annual percentage yields, which are described in the assignment in Activity **2**.

* * * * *

1.6 **a.** Using the general equation, $N_t = N_0(1+r)^t$, where N_t is the population t years after 1991, N_0 is the initial population in 1991, and r is the growth rate, the equation is $N_t = 1{,}151{,}300{,}000(1.014)^t$.

b. Sample response: In the year 1997, the estimated population is $N_6 = 1{,}151{,}300{,}000(1.014)^6 = 1{,}251{,}457{,}872$.

c. Using these rates, India will surpass China in total population during the year 2003 ($t = 11.56$ yr). Students may determine this solution by solving for t using the properties of logarithms or by graphing $N_t = 1{,}151{,}300{,}000(1.014)^t$ and $N_t = 859{,}200{,}000(1.04)^t$, then finding their intersection.

1.7 **a.** A recursive formula for depreciation is $V_t = V_{t-1}(1-r)$, where V_t is the value after t years and r is the rate of depreciation. An explicit formula for depreciation is $V_t = V_0(1-r)^t$.

b. Since the car's value is depreciating at a rate of 15%, $r = 0.15$. The value after 5 yr is $\$17{,}000(0.85)^5 \approx \$7{,}543$.

* * * * * * * * * *

(page 43)

Activity 2

This activity introduces students to the mathematical constant e. They also examine the formula used to calculate account balance when interest is compounded continuously: $P = P_0e^{rt}$.

Material List

- none

Technology

- spreadsheet
- graphing utility

Exploration

(page 43)

This exploration numerically illustrates the following limit:

$$\lim_{n \to \infty}\left(1 + \frac{r}{n}\right)^n = e^r$$

This helps students make the connection between the following two equations, where $n = ct$:

$$P_n = P_0\left(1 + \frac{r}{c}\right)^n \text{ and } P = P_0 e^{rt}$$

Note: To indicate that a finite sequence of terms no longer exists, the variable P is used to denote the account balance after t years when compounding continuously.

a. Sample response: With an initial investment of $1.00 at 100% compounded annually, the account balance after 1 yr will be $2.00. The account balance when interest is compounded continuously will be more than $2.00.

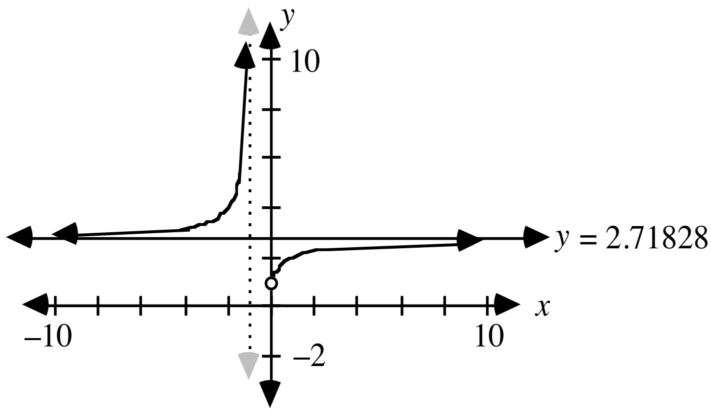
b. **Note:** On some spreadsheets, the account balance may not appear to approach e as n increases beyond $1 \bullet 10^{12}$. This error is due to rounding. Sample spreadsheet:

No. of Compoundings per Year (*c*)	**Account Balance at End of Year ($)**
1	2.00
10	2.59374246
100	2.704813829
1000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281694
100,000,000	2.718281798

c. Sample response: As the number of compoundings increases, the sequence of account balances approaches a limit of 2.71828, or approximately $2.72.

d. Answers may vary. The domain is $(-\infty, -1) \cup (0, +\infty)$ and the range is $(1, e) \cup (e, +\infty)$.

e. Sample response: As x increases without bound, the graph appears to approach the number 2.71828.



Note: As x decreases without bound, the graph also appears to approach 2.71828.

f. Sample spreadsheet:

n	$P = \left(1 + \frac{1}{n}\right)^n$	$P = \left(1 + \frac{2}{n}\right)^n$	$P = \left(1 + \frac{3}{n}\right)^n$
1	2	3	4
10	2.59374246	6.191736422	13.78584918
100	2.704813829	7.244646118	19.21863198
1000	2.716923932	7.37431239	19.99553462
10,000	2.718145927	7.387578632	20.07650227
100,000	2.718268237	7.388908321	20.08463311
1,000,000	2.718280469	7.389041321	20.08544654
10,000,000	2.718281694	7.389054613	20.08552788
100,000,000	2.718281798	7.389056025	20.0855361

g. **1.** As n increases without bound, the following expression approaches e^2, or approximately 7.389056.

$$\left(1 + \frac{2}{n}\right)^n$$

2. As n increases without bound, the following expression approaches e^3, or approximately 20.085537.

$$\left(1 + \frac{3}{n}\right)^n$$

Discussion

(page 45)

a. **1.** The values appear to be the same:

$$e^2 = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n \approx 7.389$$

2. The values appear to be the same:

$$e^3 = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n \approx 20.086$$

3. Sample response:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

Note: You may wish to discuss the following argument with your students:

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n/2}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n/2}\right)^{\frac{n}{2}\bullet 2}$$

$$= \lim_{n/2\to\infty}\left(1+\frac{1}{n/2}\right)^{\frac{n}{2}\bullet 2}$$

$$= \left[\lim_{n/2\to\infty}\left(1+\frac{1}{n/2}\right)^{n/2}\right]^2$$

But,

$$\lim_{n/2\to\infty}\left(1+\frac{1}{n/2}\right)^{n/2} = e$$

So,

$$\left[\lim_{n/2\to\infty}\left(1+\frac{1}{n/2}\right)^{n/2}\right]^2 = e^2$$

b. **1.** $10e^2$

2. $20e^3$

3. $8e^r$

4. P_0e^r

c. Sample response: This investment represents an initial deposit of \$750 at an annual interest rate of 1%, compounded continuously.

Assignment

(page 47)

2.1 This account balance will reach \$20,000 in approximately 46.2 yr. Student responses will vary, depending on their ages. For example, an 18-year-old would be 64 when the balance reaches \$20,000.

Note: One method of solution some students may use is graphing the equations $y = 5000e^{0.03t}$ and $y = 20,000$, then find their point of intersection.

2.2
- **a.** \$1090.00
- **b.** \$1093.08
- **c.** \$1093.81
- **d.** \$1094.16
- **e.** \$1094.17

***2.3**
- **a.**
  - **1.** 9.381%
  - **2.** 9.416%
  - **3.** 9.417%
- **b.** The corresponding interest rate, compounded annually, can be found by setting each value found in Problem **2.2** equal to the following expression, then solving for *r*:

$$1000\left(1+\frac{r}{1}\right)^{1} \text{ or } 1000+1000r$$

This can be done algebraically, graphically, or by using a symbolic manipulator.

2.4
- **a.**
  - **1.** Solving the following equation for r_{APY} results in an APY of about 0.0793 or 7.93%.

$$1000\left(1+\frac{0.077}{4}\right)^{4} = 1000(1+r_{\text{APY}})$$

  - **2.** Solving the equation below for r_{APY} results in an APY of about 0.08 or 8%.

$$1000\left(1+\frac{0.077}{365}\right)^{365} = 1000(1+r_{\text{APY}})$$

  - **3.** Solving the following equation for r_{APY} results in an APY of about 0.08 or 8%.

$$1000\left(1+\frac{0.077}{8760}\right)^{8760} = 1000(1+r_{\text{APY}})$$

b. Solving the equation $1000e^{0.077(1)} = 1000(1 + r_{\text{APY}})$ for r_{APY} results in an APY of about 0.08 or 8%.

c. Solving the equation $P_0 e^{rt} = P_0(1 + r_{\text{APY}})$ for r_{APY} results in the equation $r_{\text{APY}} = e^{rt} - 1$, which gives the maximum APY.

2.5 **a.** Sample response:

$$r_{\text{APY}} = \left(1 + \frac{r}{c}\right)^c - 1$$

b. $r_{\text{APY}} = e^r - 1$

c. Sample response: The initial principal has no effect on this relationship.

***2.6** **a.** Sample response: No. As the number of compoundings gets large, the resulting balance approaches a limit. Therefore, the account balance will not increase by a significant amount if the interest is compounded every hour rather than every day. For example, $100,000 invested at an annual interest rate of 9%, compounded daily, for 1 yr results in a balance of $109,417.38. When compounded hourly, the balance after 1 yr is $109,417.43. This is only a 5-cent difference (or about 0.00005%).

b. Sample response: As the number of compoundings increases, the interest earned, and therefore the account balance, increases. As the number of compoundings approaches infinity, however, the account balance approaches a limit.

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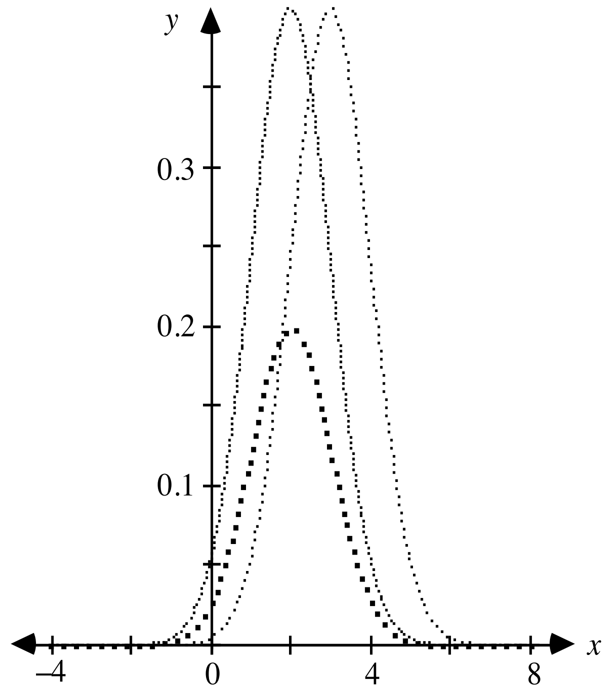
2.7 The population will be 320 when $t \approx 6.86$ days. Some students may graph the equations $y = 8e^{0.538t}$ and $y = 320$, then determine their point of intersection. Others may use a symbolic manipulator to solve the equation $320 = 8e^{0.538t}$ for *t*.

2.8 Students use two different expressions to approximate the value of *e* as 2.71828.

2.9 **a–b.** Sample graphs:



c. Sample response: The values of σ and μ determine the location and "spread" of the normal curve. The larger the mean, the farther the graph is to the right. The smaller the standard deviation, the narrower the graph.

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Research Project (page 50)

Georg Cantor used transfinite numbers to describe the cardinal numbers of infinite sets. For example, he showed that the cardinal number, c, of the set of real numbers is larger than the cardinal number, $\aleph_0$ (read as "aleph null"), of the set of natural numbers.

Cantor also developed an arithmetic for transfinite numbers, which differs significantly from real-number arithmetic. For example, $\aleph_0 + 1 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, and $\aleph_0 \bullet \aleph_0 = \aleph_0$.

(page 50)

Activity 3

Students are introduced to the natural logarithm and review the properties of logarithms.

Exploration 1

(page 50)

Students review logarithms by relating exponential and logarithmic equations. They also examine logarithms with bases other than 10, including the natural logarithm.

a. A completed table is shown below.

Logarithmic Equation	Related Exponential Equation
$\log_2 8 = 3$	$2^3 = 8$
$\log_4 16 = 2$	$4^2 = 16$
$\log_6 1296 = 4$	$6^4 = 1296$
$\log_{1.5} 3.375 = 3$	$1.5^3 = 3.375$
$\log_{0.8} 0.64 = 2$	$0.8^2 = 0.64$
$\log_{0.81} 0.9 = 0.5$	$0.81^{0.5} = 0.9$

b. **1–3.** When $y = 1$, then $a = x$.

4. When $y > 1$, then $a < x$.

5. When $y < 1$, then $a > x$.

c. $a \approx 5,\ b \approx 8$

d. $\log_5 5 = 1$ and $5' = 5$

$\log_8 8 = 1$ and $8' = 8$

e. Sample response: Since $\ln(2) \approx 0.7$, $b > 2$. Since $\ln(3) \approx 1.1$, $b < 3$. Using trial and error, $\ln(2.71828) \approx 0.9999993$. Since this value is closest to 1 for any value of b to five decimal places, $b \approx 2.71828$.

Discussion 1

(page 52)

a. Sample response: When $\log_a x < 1$, this indicates an exponent that would reduce the value of a. Therefore, $a > x$. When $\log_a x > 1$, it indicates an exponent that would increase the value of a. Therefore, $a < x$.

b. Sample response: 2.71828.

c. **1.** Since $\log_7 49 = 2$, $\log_7 52$ is a little more than 2.

2. Since $\log_{19.7} 19.7 = 1$, $\log_{19.7} 18$ is a little less than 1.

3. Since $\ln 2.7 \approx 1$, $\ln 2$ is a little less than 1.

4. Since $\ln e^3 = 3$, $\ln 20$ is approximately 3.

d. Sample response: For any power y, e^y is always positive. Since $\ln x = y$ is true if $e^y = x$, x must always be positive.

Exploration 2 (page 52)

a. The following equation describes the balance (P) after t years:

$$P = 500e^{0.06t}$$

b. Using the equation given in Part **a**,

$$t = \frac{\ln(P/500)}{0.06}$$

c. **1.** 20 yr

2. 30 yr

Discussion 2 (page 53)

a. Sample response: The second step shows the common log of both sides of the equation. The common log uses base 10. According to one of the properties of exponents, $\log_b b^x = x$. Therefore, $\log 10^x = x$.

b. Sample response: Dividing both sides of the equation $P = 500e^{0.06t}$ by 500 results in $P/500 = e^{0.06t}$. Taking the natural logarithm of both sides gives the following:

$$\ln(P/500) = \ln e^{0.06t}$$

Since $\ln e^x = x$, $\ln e^{0.06t} = 0.06t$.

c. **1.** Sample response: Since the equation includes e, the base of natural logs, the equation simplifies more easily using natural logs.

2. Sample response: Yes, you could use any base.

Assignment (page 54)

3.1 **a.** **1.** $\ln x = 5$

2. $\ln 1 = 0$

3. $\ln 3 = 0.06x$

b. $x = \ln 3/0.06 \approx 18.3$

3.2 **a.** $e^2 = x$

b. $e^1 = e$

c. $e^{0.05x} = y/750$

3.3 The equation $e^{rt} = 2$, written in logarithmic form, is equivalent to $\ln 2 = rt$. Solving for *t* results in the equation $t = \ln 2/r$.

a. $(\ln 2)/0.06 \approx 11.6$ yr

b. $(\ln 2)/0.08 \approx 8.7$ yr

c. $(\ln 2)/0.1 \approx 6.9$ yr

3.4 **a.** **1.** Sample response:

$$2550 = 1275\left(1 + \frac{0.07}{4}\right)^{4t}$$
$$t \approx 10.0 \text{ yr}$$

2. Sample response:

$$3825 = 1275\left(1 + \frac{0.07}{4}\right)^{4t}$$
$$t \approx 15.8 \text{ yr}$$

3. Answers may vary. The following equation determines the value of the account at the child's 18th birthday:

$$y = 1275\left(1 + \frac{0.07}{4}\right)^{4 \cdot 18} \approx \$4446.19$$

b. **1.** Sample response:

$$2550 = 1275e^{0.07t}$$
$$t = 9.9 \text{ yr}$$

2. Sample response:

$$3825 = 1275e^{0.07t}$$
$$t = 15.7 \text{ yr}$$

3. The following equation determines the value of the account at the child's 18th birthday: $y = 1275e^{0.07 \cdot 18} \approx \$4494.91$.

c. When compounded continuously, the investment doubles or triples slightly faster than when compounded quarterly. After 18 yr, the account balance is slightly higher (approximately 1%).

***3.5** **a.** $P = P_0 e^{rt}$

b. Using the properties of logarithms,

$$\ln P = \ln P_0 e^{rt}$$
$$\ln P = \ln P_0 + rt$$
$$\ln P - \ln P_0 = rt$$
$$\frac{\ln P - \ln P_0}{r} = t$$

***3.6** **a.** To determine when the account balance will reach \$715, students may solve the following equation for *t*:

$$715 = 500e^{0.08t}$$
$$t \approx 4.47 \text{ yr}$$

b. The interest rate can be determined as follows:

$$715 = 500e^{0.5r}$$
$$r \approx 0.72 \text{ or } 72\%$$

c. Sample response: An interest rate of 72% is extremely unlikely, even in very risky investments.

* * * * *

3.7 **a.** Sample response:

$$p = 410,000(1 + 0.0425)^t$$

b. **1.** A population of 500,000 people will require 500 officers. The solution below results in about 4.8 yr.

$$500,000 = 410,000(1 + 0.0425)^t$$
$$\log 500,000 = \log 410,000 + t \log 1.0425$$
$$5.6990 \approx 5.6128 + t(0.0181)$$
$$t \approx \frac{5.6990 - 5.6128}{0.0181}$$
$$t \approx 4.8$$

2. A population of 1 million people will require 1000 officers. Following a solution similar to that in Part **b1,** 1000 officers will be needed in about 21.4 yr.

3.8 **a.** Sample response: Adding *k* to the equation $y = \ln x$ shifts the graph of the parent horizontally by *k* units. Since $\ln e^k = k$, adding it to the equation $y = \ln x$ has the same effect.

b. Sample response: As *k* increases in either equation, the graph appears to climb more quickly.

c. **1.** The equations in Part **a** are equivalent.

2. The equations in Part **b** are equivalent.

d. Sample response: Since $\ln e = 1$, $\ln e^k = k$. So, $\ln x + \ln e^k$ is equivalent to $\ln x + k$. Likewise, since $\ln x^k = k \ln x$, the equations in Part **b** are equivalent.

* * * * * * * * * *

Answers to Summary Assessment (page 56)

1. **a.** Sample response: Yes, Vonzel would be better off leaving his CD alone. In 1 yr, compounding daily, the CD will be worth

$$\$5000\left(1+\frac{0.07}{365}\right)^{365} \approx \$5362.50$$

Compounding continuously, it would be worth $\$5000e^{0.07} \approx \$5362.54$ or only about \$0.04 more. This is not enough to justify the \$150 penalty.

b. Sample response: The CD will earn \$150 in interest in about 0.42 yr (about 154 days):

$$5000\left(1+\frac{0.07}{365}\right)^{365t} = \$5150$$

$$365t\ln\left(1+\frac{0.07}{365}\right) = \ln(1.03)$$

$$t \approx 0.42$$

c. Including principal and interest, the bank that compounds daily will owe the following total to its customers:

$$\$5,000,000\left(1+\frac{0.07}{365}\right)^{365} \approx \$5,362,504.92$$

The bank that compounds continuously will owe $\$5,000,000e^{0.07} \approx \$5,362,540.91$, or about \$36 more.

2. Sample response: The APR of Vonzel's credit card is about 13.9%, as shown below:

$$\left(1+\frac{0.13}{365}\right)^{365} \approx 1.139$$

The APR for the new card is about 13.8%:

$$\left(1+\frac{0.13}{12}\right)^{12} \approx 1.138$$

This is only about a 0.1% difference. Vonzel's credit balance would have to average approximately \$5000 for an entire year to make up the difference in annual fees of \$5. He should not switch cards.

3. Sample response: Vonzel should keep his money in the CD and avoid the $150 penalty. As the following spreadsheet shows, the total interest earned for the CD grows faster than the total interest owed on the credit card because the principal is larger. Every day the CD stays in the bank the better, even if Vonzel makes no payments on the credit card, as shown in the column headed "net gain or loss."

	CD		Credit Card Bill		Net Gain or Loss
Day	**Total Value**	**Gain if Withdrawn**	**Total Value**	**Interest Paid**	
1	$5001	–$149	$1400	$1	–$150
2	$5002	–$148	$1401	$1	–$149
3	$5003	–$147	$1402	$2	–$149
4	$5004	–$146	$1402	$2	–$148
⋮	⋮	⋮	⋮	⋮	⋮
324	$5320	$170	$1571	$171	–$1
325	$5322	$172	$1572	$172	$0
326	$5323	$173	$1572	$172	$1
⋮	⋮	⋮	⋮	⋮	⋮
364	$5361	$211	$1594	$194	$17
365	$5363	$363	$1594	$194	$169

Module Assessment

1. At age 24, Chris deposits \$100 in an account at an annual interest rate of 7%, compounded monthly.

a. If she makes no further deposits or withdrawals, how much will the account be worth when Chris is 65?

b. How long would it take the account balance to reach \$2400 if the interest is compounded continuously?

2. **a.** How long will it take an investment at an annual interest rate of 9%, compounded monthly, to double?

b. To estimate the time required to reach an investment goal, bankers and financial advisors often use the following formula:

$$\text{years} = \frac{\log\left(\dfrac{\text{future value}}{\text{present value}}\right)}{n \bullet \log\left(1 + \dfrac{r}{n}\right)}$$

where r represents the annual interest rate and n represents the number of compoundings per year. Use this formula to calculate the number of years required for an investment at an annual interest rate of 9%, compounded annually, to double.

3. As a new college freshman, Sarah receives two offers for credit cards. Both have inviting terms, such as no annual fee and a low interest rate. While reading the fine print, however, Sarah notices some differences between the two offers.

Card A features an introductory interest rate of 5.6%. After the first year, the interest rate increases to 12.9%. There is no annual fee for the first year. After the first year, there is an annual fee of \$65. The credit limit is \$2000.

Card B has an introductory interest rate of 4.3%. After the first year, the rate increases to 14.5%. There is no annual fee for the first year. After the first year, there is an annual fee of \$60. The credit limit is \$2000.

If both cards compound interest daily, which credit card do you think provides the better deal if she spends the maximum and makes no payments for 30 days? Justify your response.

4. Kelly currently has $6000 in a savings account. After 5 more years, he hopes to use the account balance to make a $15,000 down payment on a home. If his bank compounds interest continuously, is he likely to have enough money to reach his goal? Explain your response.

5. Like Kelly, Kaylee is saving money to buy a home. She currently has $12,000 in a savings account at an annual interest rate of 5.6%, compounded continuously. Write a formula that describes the time in years that she must leave her money in savings in order to make a 10% down payment on a house that costs *H* dollars.

Answers to Module Assessment

1. **a.** $\$100\left(1+\frac{0.07}{12}\right)^{12\bullet 41} \approx \$1749.06$

b. Solving the equation $\$2400 = \$100e^{0.07t}$ for t gives approximately 45.4 yr.

2. **a.** Solving the equation $2P_0 = P_0e^{0.09t}$ for t gives approximately 7.7 yr.

b. Solving the equation below for t gives approximately 8.04 yr.

$$t = \frac{\log\left(\frac{2P_0}{P_0}\right)}{1 \bullet \log\left(1+\frac{0.09}{1}\right)}$$

3. Sample response: In the first year, Card B would be the better card, since its introductory rate is lower. If Sarah charged \$2000 in purchases on the credit card and did not make any payments during the first 30 days, the bill at the end of 30 days for card A would be:

$$2000\left(1+\frac{0.056}{365}\right)^{30} \approx \$2009.23$$

The bill for card B would be:

$$2000\left(1+\frac{0.043}{365}\right)^{30} \approx \$2007.08$$

After the first year, Card A might be a better way to go. If Sarah had \$2000 on her credit card at the beginning of the second year and did not make any payments during 30 days, the monthly bill for card A would be:

$$2000\left(1+\frac{0.129}{365}\right)^{30} + \frac{65}{12} \approx \$2026.73$$

The bill for card B would be:

$$2000\left(1+\frac{0.145}{365}\right)^{30} + \frac{45}{12} \approx \$2027.72$$

4. Sample response: It is not very likely that Kelly will have enough money to make the down payment in 5 yr unless he puts more money in savings. Using the formula from Problem **2**, he would need a savings account with an annual interest rate of about 18.3%.

$$r = \frac{\ln\left(\frac{P_t}{P_0}\right)}{t} = \frac{\ln\left(\frac{15000}{6000}\right)}{5} \approx 18.3\%$$

5. The following formula gives the time t in years that Kaylee would need to leave her money in savings to afford a 10% down payment for a home that costs H dollars.

$$t = \frac{\ln\left(\frac{H \bullet 0.10}{12000}\right)}{0.56} \approx \ln\left(H^{1.78571}\right) - 167.726$$

Selected References

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Salisbury, D. F. *Money Matters: Personal Financial Decision Making with the Pocket Calculator*. Englewood Cliffs, NJ: Prentice-Hall, 1982.

Vilenkin, N. *Stories about Sets*. New York: Academic Press, 1969.

Flashbacks

Activity 1

1.1 An electronics store purchases a stereo from the manufacturer for \$750. To determine the retail price, the store adds 20% to its cost. What is the retail price of this stereo?

1.2 Use the distributive property to write the following expression as a product: $12 + 16r$.

1.3 Consider the geometric sequence 2, 6, 18, 54, … .

a. Write a recursive formula for this sequence.

b. Write an explicit formula for this sequence.

Activity 2

2.1 What value do the terms of the following sequence appear to approach?

$$\frac{3}{7}, \frac{3}{14}, \frac{3}{21}, \ldots$$

2.2 Solve the following equation for r:

$$800 = 200\left(1 + \frac{r}{3}\right)^2$$

2.3 Consider an investment of \$20,000 at an annual interest rate of 5.5%, compounded monthly. If no further deposits or withdrawals are made, determine the account balance after 20 yr.

Activity 3

3.1 Estimate the value of x in each of the following expressions.

a. $x = \log 105$

b. $x = \log 8$

3.2 Write an equivalent exponential equation for each of the following logarithmic equations.

a. $\log 1000 = 3$

b. $\log 0.01 = -2$

3.3 Solve each of the expressions below.

a. $x^3 = 20$

b. $y^2 = 30$

3.4 Simplify each of the following expressions.

a. $x^3(x^2)^4$

b. x^{10}/x^2

3.5 Simplify each expression below and estimate its value to the nearest 0.01.

a. e^5/e^2

b. πe

Answers to Flashbacks

Activity 1

1.1 $\$750 + \$750 \bullet 0.2 = \$900$

1.2 Sample response: $12 + 16r = 4(3 + 4r)$.

1.3 **a.** $t_n = 3t_{n-1}$

b. $t_n = 2 \bullet 3^{n-1}$

Activity 2

2.1 The terms appear to approach 0. An explicit formula for this sequence is:

$$t_n = \frac{3}{7n}$$

2.2 $r = 3$ or $r = -9$

2.3 $P_{20} = \$20,000 \bullet \left(1 + \frac{0.055}{12}\right)^{12 \bullet 20} \approx \$59,932.51$

Activity 3

3.1 **a.** Sample response: The value of x is a little more than 2.

b. Sample response: The value of x is a little less than 1.

3.2 **a.** $10^3 = 1000$

b. $10^{-2} = 0.01$

3.3 **a.** $x \approx 2.7$

b. $y \approx 5.5$ or -5.5

3.4 **a.** x^{11}

b. x^8

3.4 **a.** $e^3 \approx 20.09$

b. $\pi e \approx 8.54$

Functioning on a Path

In this module, you use your knowledge of polynomial and rational functions to play a mathematics game.

Gary Bauer • Russ Killingsworth • Margaret Plouvier

Teacher Edition
Functioning on a Path

Overview

Students examine rational functions in the context of a hypothetical video game.

Objectives

In this module, students will:

- determine equations for polynomial functions
- identify absolute maxima and minima in polynomial functions
- investigate piecewise functions
- write rational functions as sums of polynomial and rational expressions
- identify asymptotes in the graphs of rational functions
- explore the relationship between the end behaviors of the rational function $r(x)$ and the polynomial function $f(x)$, where

$$r(x) = f(x) + \frac{1}{p(x)}$$

Prerequisites

For this module, students should know:

- how to identify the domain and range of a function
- how to determine the equation of a line
- the definitions of a polynomial and its degree
- how to write a polynomial function in factored form
- how to solve systems of equations
- the definition of continuity at a point
- how to use technology to determine polynomial regressions
- how to simplify functions using a symbolic manipulator
- the definition of a rational function
- the definition of an asymptote.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	3	2	11

Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
graph paper (optional)	X	X	X	X

Technology

Software	Activity			
	1	2	3	Summary Assessment
graphing utility	X	X	X	X
symbolic manipulator		X	X	
spreadsheet		X		

Functioning on a Path

Introduction

(page 63)

The game of Gates provides a context for the study of polynomial, rational, and piecewise functions.

Teacher Note

In the first level of Gates, students may use polynomial functions to create graphs that pass through all the gates on a screen. In the second level of the game, they use continuous piecewise functions. In the third level, students use rational functions to pass through all gates while avoiding barriers.

Discussion

(page 65)

a.

1. This is a second-degree polynomial function. All of its coefficients are real numbers and all of its exponents are non-negative integers.
2. Sample response: This is not a polynomial function. As shown below, the exponent of the second term is not a non-negative integer:

$$\frac{4}{x} = 4x^{-1}$$

b. This is a fifth-degree polynomial. The leading coefficient is –5. **Note:** You may wish to emphasize that the leading coefficient in a polynomial is not always the first coefficient as the expression is read from left to right.

c. Sample response: Each polynomial function is continuous over the domain of real numbers. Each value in the domain of any polynomial function has a single corresponding value in the range.

Note: You may wish to discuss some intuitive notions of the characteristics of continuous functions. For example, one rule of thumb involves testing graphs to see if they can be drawn "without lifting a pencil." This may help students to observe that the limit of the function at each value in the domain is the function value.

d. Sample response: You could determine other polynomial functions that pass through the gates by identifying the coordinates of appropriate points, then using technology to fit those points with different polynomial regression equations. For example, the graphs of the following equations all pass through the gates in Figure **1**: $f(x) = 1.00x^2 + 3.15x - 7.27$; $f(x) = -0.017x^3 + 0.91x - 5.91$; and $f(x) = 0.004x^4 + 0.005x^3 + 0.758x^2 + 3.12x - 4.576$

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Activity 1

Students investigate the relationship among the degree of a polynomial, its roots, and the characteristics of its graph.

Materials List

- graph paper (optional)

Technology

- graphing utility

Teacher Note

Students may find their graphs easier to interpret if a table of values is generated for each one. On a TI-92 graphing calculator, for example, the "table" feature can be used to provide an appropriate aid. Other graphing utilities have similar features.

Exploration 1

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In this exploration, students discover that a quadratic equation can be used to create a graph passing through any three noncollinear points.

a. Answers may vary. From previous experience, most students should predict degree 2.

b. Sample response: (0,2), (0.5,0), and (2,0).

c. The equation $p(x) = 2x^2 - 5x + 2$ fits the sample points given in Part **b**.

d.
1. The system of equations for the sample points given in Part **b** is shown below:

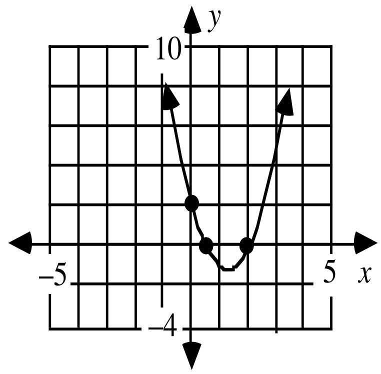
$$\begin{cases} a_2(0)^2 + a_1(0) + a_0 = 2 \\ a_2(0.5)^2 + a_1(0.5) + a_0 = 0 \\ a_2(2)^2 + a_1(2) + a_0 = 0 \end{cases}$$

2. The solution follows:

$$a_0 = 2$$
$$a_1 = -5$$
$$a_2 = 2$$

3. Using the sample points given in Part **b**, the equation is $p(x) = 2x^2 - 5x + 2$.

e. Students should observe that the graph of the function from Part **d** contains all three points selected in Part **b**. Sample graph:



Discussion 1 (page 67)

a.
1. Sample response: The leading coefficient must be negative for the parabola to open downward.
2. Sample response: Because any real number raised to an even power is positive, the graph of an even-degree polynomial will open upward if its leading coefficient is positive and downward if its leading coefficient is negative. Therefore, if the polynomial that models the graph has an even degree, its leading coefficient must be negative.

b.
1. Sample response: Both functions are second-degree polynomials that model the same three points.
2. Sample response: A regression typically reports each coefficient to many decimal places. If the solutions to the system of equations are not reported with the same number of decimal places, then the coefficients may be slightly different.

c. Sample response: Since any three points can be used to determine a set of three equations whose unknowns are the coefficients of a second-degree polynomial function, it should be possible to find a quadratic function that exactly fits any three noncollinear points.

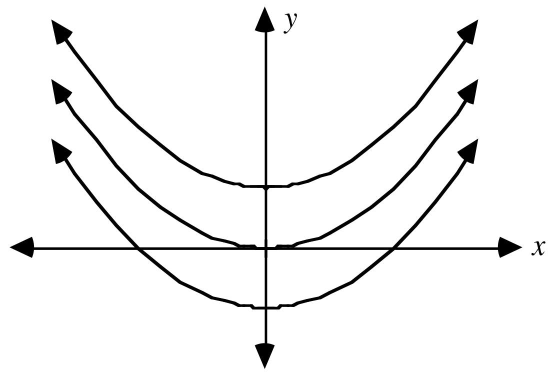
Exploration 2 (page 67)

a. See sample graph in Part **d** below.

b. Students should determine that the maximum number of times a quadratic function can intersect the *x*-axis is 2.

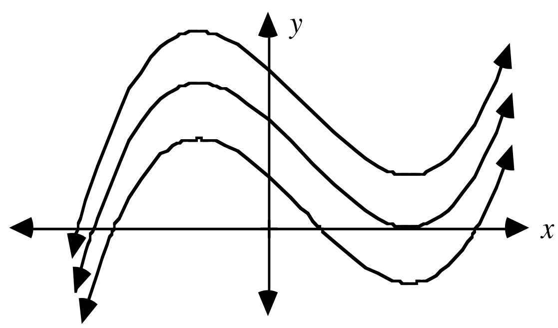
c. Sample response: As the absolute value of *x* increases, the function values increase.

d. Sample graph:



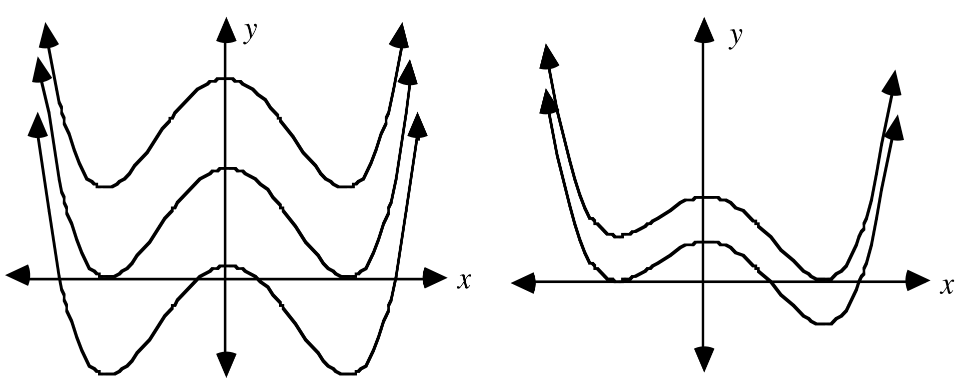
e. **1.** A cubic can intersect the x-axis at most 3 times. For a positive leading coefficient, as x approaches $+\infty$, the function approaches $+\infty$. As x approaches $-\infty$, the function approaches $-\infty$.

The sample graphs below show that the graph of a cubic function can cross the x-axis 1, 2, or 3 times. It is impossible for a cubic polynomial to cross the x-axis 0 times.



2. A quartic can intersect the x-axis at most 4 times. For a positive leading coefficient, as the absolute value of x increases, the function values increase.

The sample graphs below show that the graph of a quartic function can cross the x-axis 0, 1, 2, 3, or 4 times.



Discussion 2

(page 68)

a. **1.** None of the functions in the exploration have absolute maximums.

2. The second- and fourth-degree functions in the exploration have absolute minimums. The third-degree functions do not.

b. Sample response: The functions that had absolute minimums would instead have had absolute maximums.

c. **1.** Sample response: Both the sine and cosine functions contain both an absolute maximum and absolute minimum.

2. Any function whose range is all real numbers contains neither an absolute maximum nor an absolute minimum.

d. Sample response: In Exploration **2**, the maximum zeros for a second degree polynomial function is 2, for a third degree 3 zeros, and for a fourth degree 4 zeros. The maximum number of zeros for a polynomial of degree *n* would be *n*.

e. Sample response: When x is a root of a function, $f(x) = 0$.

f. The least possible degree is 3.

g. Sample response:

$$(x+6)(x-8)(x-2) = x^3 - 4x^2 - 44x + 96$$

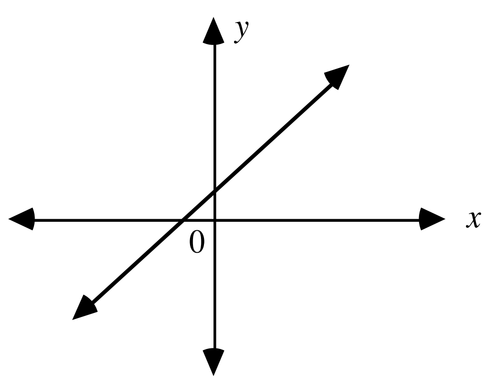
h. Sample response: One way of doing this is to multiply the existing polynomial by a common factor, for example $(x+6)$. The resulting expression would have the same roots but a degree of 4, as shown below:

$$(x+6)^2(x-8)(x-2) = x^4 + 2x^3 - 68x^2 - 168x + 576$$

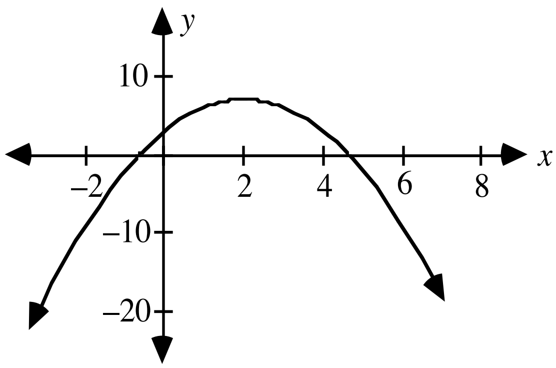
Assignment

(page 69)

1.1 Sample response: The polynomial of least degree that has exactly one zero is a linear polynomial (degree 1). A general equation is $f(x) = a_1x + a_0$.



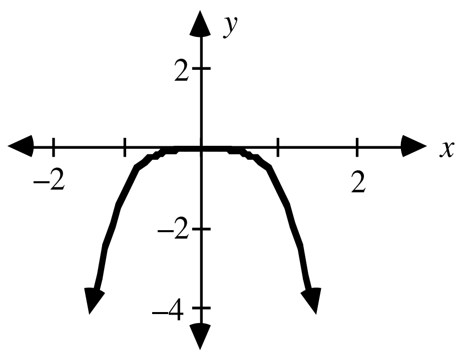
1.2 **a.** Sample response:



b. Sample response: Yes. Every quadratic function has a parabolic shape. The absolute maximum or absolute minimum occurs at the vertex.

c. Sample response: If the quadratic has an absolute minimum, the leading coefficient is positive. If it has an absolute maximum, the leading coefficient is negative.

d. Sample graph:



e. Sample response: Any polynomial of degree 3 has neither an absolute maximum nor an absolute minimum. As $|x|$ increases, one end increases without bound positively, while the other end increases without bound negatively.

1.3 **a.** Sample response: The polynomial in Figure **6** has an absolute minimum of –10 when $x = 3$. Since it is a fourth-degree polynomial with a positive leading coefficient, it has no absolute maximum.

b. Sample response: A relative maximum is a function value that is greater than all "nearby" function values. Similarly, a relative minimum is a function value that is less than all "nearby" function values.

1.4 **a.** Sample response: The zeros are 2,–4, and π because those values of x yield function values of 0.

b. The degree is 4.

c. Sample response: The function has an absolute minimum because it is an even-degree polynomial whose leading coefficient is positive.

d. Both ends of the graph increase without bound as $|x|$ approaches infinity.

1.5 **a.** The polynomial function of the least odd degree is a linear function. One general form is $f(x) = a_1x + a_0$.

b. Sample response: One end increases to positive infinity, while the other end tends to negative infinity.

c. Sample response: One end increases to positive infinity, while the other end tends to negative infinity. This occurs because a positive value raised to an odd power is positive, while a negative value raised to an odd power is negative.

1.6 Sample response: One method is to factor the function as follows,

$$f(x) = (x+3)(x+2)$$

In this form, it is easy to identify the x-values for which $f(x) = 0$. The zeros are –3 and –2. **Note:** Some students may suggest using the quadratic formula to find the exact values of the zeros.

1.7 **a.** Sample response: This function may have an even degree since the end behaviors appear to be the same and there appears to be an absolute minimum.

b. Sample response: This function may have an odd degree since the end behaviors appear to be opposite.

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1.8 **a.** Sample response:

$$f(x) = \frac{17}{13}(x-5) + 10$$

b. $f(x) = x^2$

1.9 Sample response: Since the height of a freely falling object over time can be modeled by a quadratic function, I used a second-degree polynomial regression to determine the following function.

$$f(x) = -4.9x^2 + 20x + 70$$

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Activity 2

In this activity, students experiment with piecewise functions.

Materials List

- graph paper (optional)

Technology

- graphing utility
- spreadsheet (optional)
- symbolic manipulator (optional)

Exploration 1

(page 71)

Students investigate a piecewise function composed of polynomials.

a. **1.** The equation of the ray is shown below. Its domain is $(-\infty,-7]$.

$$f_1(x) = -\frac{15}{2}(x+7)+25 = -\frac{15}{2}x - \frac{55}{2}$$

2. The equation is shown below. Its domain is [–7, –5].

$$f_2(x) = -\frac{25}{2}(x+5) = -\frac{25}{2}x - \frac{125}{2}$$

3. The equation is shown below. Its domain is [–5, –2].

$$f_3(x) = -\frac{10}{3}(x+5) = -\frac{10}{3}x - \frac{50}{3}$$

4. The equation is shown below. Its domain is [–2, 4].

$$f_4(x) = \frac{30}{6}(x+2) - 10 = 5x$$

5. The equation is shown below. Its domain is $[4,+\infty)$.

$$f_5(x) = \frac{20}{1}(x-4) + 20 = 20x - 60$$

b. Sample response:

$$f(x)=\begin{cases}-7.5x-27.5; & x\in(-\infty,-7]\\ -12.5x-62.5; & x\in[-7,-5]\\ -\dfrac{10}{3}x-\dfrac{50}{3}; & x\in[-5,-2]\\ 5x; & x\in[-2,4]\\ 20x-60; & x\in[4,+\infty)\end{cases}$$

c. Students may wish to refer to the appropriate manual to determine how their technology addresses the entry of piecewise functions.

Discussion

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a. Answers will vary, depending on the technology used. On the TI–92, for example, the function may be defined then graphed as follows:

```
y1(x)=Func:If x≤-7 Then:Return -7.5*x-27.5: ElseIf x≥-7
and x≤5 Then: Return -12.5*x-62.5: ElseIf x≥-5 and x≤-2
Then: Return (-10/3)*x-(50/3): ElseIf x≥-2 and x≤4 Then:
Return 5*x: ElseIf x≥4 Then: Return 20*x-60: EndIf:
EndFunc
```

Students also may graph each individual part using the "Func" command or the "When" command.

b. Sample response: If the two consecutive pieces contain a common point, the coordinates of the common point must satisfy the rules that define each of the two parts.

c. Sample response: The function appears to meet all the conditions described in the definition of a continuous function. It is defined at each value in the domain, the limit appears to exist at each value in the domain, and in each case the function value is the same as the limit.

d.

1. Sample response: Yes, $f(x)$ is a function. It consists of a domain, a range, and a rule such that no value of the domain is paired with more than one image.
2. Sample response: Yes, $f(x)$ is a piecewise function since different parts of the domain correspond with different rules.
3. Sample response: No, $f(x)$ is discontinuous at 0. When x is negative and approaches 0, $f(x)$ approaches $-\infty$. When x is positive and approaches 0, $f(x)$ approaches $+\infty$. Therefore, the limit of the function at 0 does not exist.

Assignment

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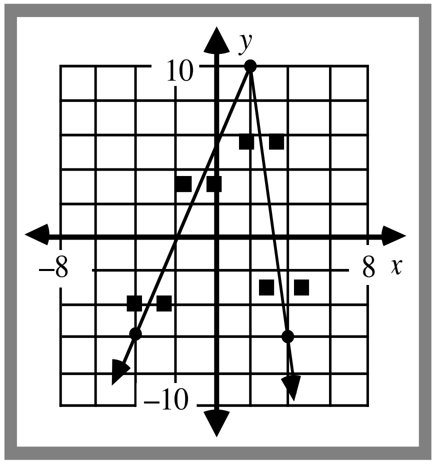
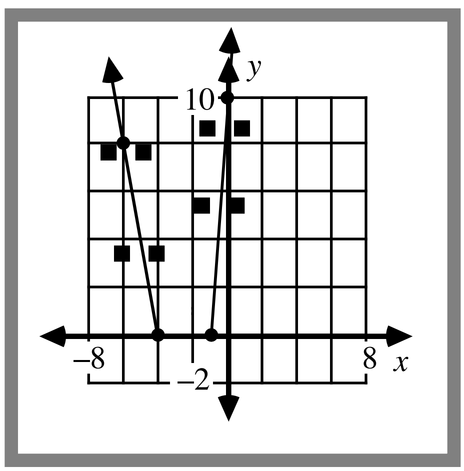
2.1 Answers will vary. Sample responses are given below.

a. One continuous piecewise function that passes through these gates is shown below:



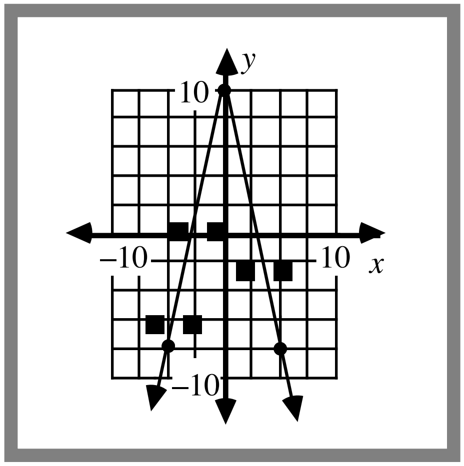
$$f(x)=\begin{cases}\frac{8}{3}\text{x}+\frac{14}{3}, x\in(-\infty,2]\\ -2x+14,\ x\in[2,\infty)\end{cases}$$

b. One continuous piecewise function that passes through these gates is shown below:



$$f(x)=\begin{cases}-4x-16,\ x\in(-\infty,-4]\\ 0,\ x\in[-4,-1]\\ 10x+10,\ x\in[-1,\infty)\end{cases}$$

c. One continuous piecewise function that passes through these gates is shown below:



$$f(x)=\begin{cases}4.5x+10, & x\in(-\infty,0]\\ -4.5x+10, & x\in[0,\infty)\end{cases}$$

2.2 Sample response: The greatest integer function is defined in such a way that each "piece" of the function is a segment with one closed endpoint. This fits the description of a piecewise function. Since there is a "jump" in the graph at each integer, however, it is not a continuous function.

2.3 **a.** The coordinates of the points of intersection are $(0,0)$ and $(2,4)$.

b. **1.** $f(x)=\begin{cases}x^2, & x\in(-\infty,0]\\ 2x, & x\in[0,\infty)\end{cases}$

2. $f(x)=\begin{cases}2x, & x\in(-\infty,0]\\ x^2, & x\in[0,\infty)\end{cases}$

3. $f(x)=\begin{cases}x^2, & x\in(-\infty,0]\\ 2x, & x\in[0,2]\\ x^2, & x\in[2,\infty)\end{cases}$

4. $f(x)=\begin{cases}2x, & x\in(-\infty,0]\\ x^2, & x\in[0,2]\\ 2x, & x\in[2,\infty)\end{cases}$

***2.4.** Responses may vary: The equation of a circle with center at the origin and radius r is of the form $x^2 + y^2 = r^2$. The equation of the corresponding semicircle is of the form $y = \sqrt{r^2 - x^2}$.

The centers of the semicircles on the graph are located at $(r,0)$, $(3r,0)$, $(5r,0)$, and $(7r,0)$. These can be used to define the function as follows:

$$f(x) = \begin{cases} \sqrt{r^2 - (x-r)^2}, & x \in [0, 2r] \\ \sqrt{r^2 - (x-3r)^2}, & x \in [2r, 4r] \\ \sqrt{r^2 - (x-5r)^2}, & x \in [4r, 6r] \\ \sqrt{r^2 - (x-7r)^2}, & x \in [6r, 8r] \end{cases}$$

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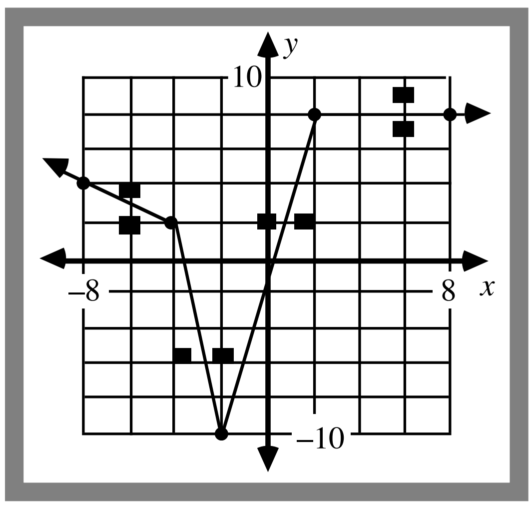
2.5 Answers will vary. One possible response is shown below:



$$f(x) = \begin{cases} -0.5x, & x \in (-\infty, -4] \\ -6x - 22, & x \in [-4, -2] \\ 4.5x - 1, & x \in [-2, 2] \\ 8, & x \in [2, \infty) \end{cases}$$

***2.6** **a.** Sample response: The graph would consist of four pieces. All of them would be pieces of parabolas, since freely falling objects are modeled by quadratic equations. Because the leading coefficient is negative, the parabolas would open downward.

The first part, representing the ball's initial drop, would look like half of a parabola. This part would end when the ball hit the ground the first time. The remaining pieces would look like parabolas that start and stop when the ball comes in contact with the ground. The height of each successive parabola would be two-thirds that of the preceding part, since the ball only recovers two-thirds of its height on each bounce.

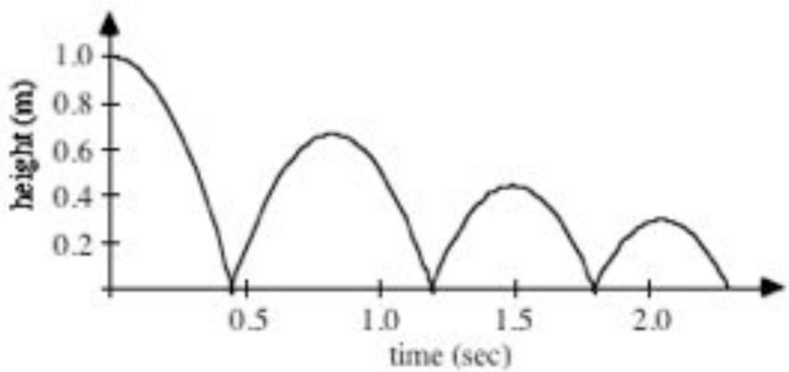
b. Sample response: Since the endpoints of each bounce are given and the shape is a parabola, the equation for each bounce can be determined by finding a third point.

Each parabola is symmetric with respect to a vertical line through its vertex, so the *x*-coordinate of the vertex can be found by determining the midpoint of the segment whose endpoints are the beginning and ending of each bounce. The *y*-coordinate of each vertex is two-thirds of the *y*-coordinate of the previous vertex.

Once three points have been determined for each part, a system of three equations can be used to determine the coefficients of the quadratic function that models the parabola. The piecewise function that describes the ball's height is shown below:

$$h(t) = \begin{cases} -4.9t^2 + 1, & t \in [0, 0.45] \\ -4.9t^2 + 8t - 2.6, & t \in [0.45, 1.19] \\ -4.9t^2 + 14.7t - 10.5, & t \in [1.19, 1.79] \\ -4.7t^2 + 19.3t - 19.4, & t \in [1.79, 2.29] \end{cases}$$

c. Sample graph:



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Activity 3

Students explore the asymptotic behavior of rational functions.

Materials List

- graph paper (optional)

Technology

- graphing utility
- symbolic manipulator

Teacher Note

Different graphing utilities may handle vertical asymptotes in different ways. In many cases, the selected interval of the domain determines how well the asymptote is displayed. Some technology, however, may not display vertical asymptotes at all, even when the specified interval of the domain includes its location.

Exploration 1

(page 76)

Students investigate the effects of adding a rational expression of the form $1/g(x)$, where $g(x)$ is a linear function, to various polynomials. They discover that a vertical asymptote exists at $x = k$, where $g(k) = 0$.

They also should observe that the graph of the rational function created is similar to that of the polynomial function to which $1/g(x)$ has been added at points other than those near the points of discontinuity.

a. Answers will vary. Sample responses are shown below.

1. $q(x) = x + 1$

2. $g(x) = x - 3$

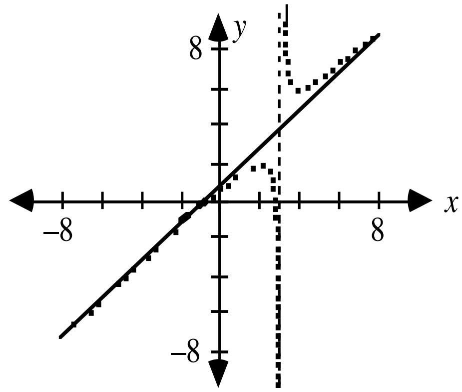
b. **1.** Using the sample functions given in Part **a**:

$$r(x) = x + 1 + \frac{1}{x - 3}$$

2. In general, the domain of $r(x)$ excludes the value of x for which $g(x) = 0$. In this case, the domain of $r(x)$ is all real numbers except 3.

3. Students may examine some points whose x-coordinates are in the vicinity of the excluded value. The absolute values of the y-coordinates will increase without bound as the selected x-values approach the excluded value.

c. Sample graph:



d. **1.** For the sample functions given above, the point of discontinuity occurs at $x = 3$. As x approaches 3, $q(x)$ approaches 4. Both $r(x)$ and $1/g(x)$ approach $+\infty$ as x approaches 3 from the right, and $-\infty$ as x approaches 3 from the left.

Note: Mathematicians generally state such findings using limit notation, as shown in the three examples below. You may wish to discuss this notation with your students. If so, it is important to note that a limit does not exist when $x = 3$ for examples **ii** and **iii**.

i. $\lim_{x \to 3} q(x) = 4$

ii. $\lim_{x \to 3^+} r(x) = \infty$ and $\lim_{x \to 3^-} r(x) = -\infty$

iii. $\lim_{x \to 3^+} \frac{1}{g(x)} = \infty$ and $\lim_{x \to 3^-} \frac{1}{g(x)} = -\infty$

2. For the sample functions given above, as $|x|$ approaches infinity, $q(x)$ also increases without bound.

In general, as $|x|$ approaches infinity, $1/g(x)$ approaches 0 and $r(x)$ approaches $q(x)$.

e. Students repeat the exploration using a quadratic function for $q(x)$. They should discover that a vertical asymptote occurs for $r(x)$ at $x = k$ when $g(k) = 0$. They also should observe that $r(x)$ takes on the behavior of $q(x)$ at points far from the point of discontinuity.

f. Students repeat the exploration using a cubic function for $q(x)$. As in Part **e**, they should discover that a vertical asymptote occurs for $r(x)$ at $x = k$ when $g(k) = 0$. They also should observe that $r(x)$ takes on the behavior of $q(x)$ at points far from the point of discontinuity.

Discussion 1

(page 78)

a. Sample response: Near the value excluded from its domain, $r(x)$ approaches $+\infty$ or $-\infty$, depending on the direction from which the excluded value is approached. This occurs because $g(x)$ approaches 0, which makes the absolute value of $1/g(x)$ increase without bound. At the excluded value, the denominator is 0 and the value of $r(x)$ is undefined.

Since $q(x)$ is a linear function with a domain of all real numbers, it is unaffected as x approaches the value excluded from the domain of $r(x)$.

b. Sample response: Adding the rational expression $1/g(x)$ to $q(x)$ creates a vertical asymptote at the point where $g(x) = 0$ and leaves the rest of the graph fairly similar to $q(x)$.

At points far from the asymptotic behavior, the rational expression does not add much to the range values. When $x = 100$, for example, the rational expression $1/(x - 4)$ adds approximately 0.0104 to $q(100)$

.

The end behavior of is not affected.

c. Sample response: If $1/g(x)$ is of the form shown below, there are vertical asymptotes at a and b.

$$\frac{1}{g(x)} = \frac{1}{(x-a)(x-b)}$$

d. Sample response: The graph would have been very similar to the graph of the original $r(x)$, except near the point(s) of discontinuity. Using the additive inverse of $1/g(x)$ would cause the graph to appear on the opposite sides of the asymptote when compared to the original graph. This occurs because the values of $1/g(x)$ that were originally positive would be negative, and vice versa.

e. Sample response: In order for the graph of the function to pass through all four gates, a vertical asymptote is needed at $x = -3$. The rational expression $1/(x + 3)$ should be added (or subtracted), as shown below:

$$f(x) = \frac{1}{3}x + 5 + \frac{1}{x+3} \text{ or } f(x) = \frac{1}{3}x + 5 - \frac{1}{x+3}$$

f.

1. The graph has a vertical asymptote at the point where $g(x) = 0$. As $|x|$ increases, the graph of $r(x)$ approaches the graph of $f(x)$.
2. The end behaviors are the same.

g. The form can be changed by multiplying $q(x)$ by $g(x)/g(x)$ as follows:

$$\begin{aligned} r(x) &= q(x) + \frac{h(x)}{g(x)} \\ &= q(x) \bullet \frac{g(x)}{g(x)} + \frac{h(x)}{g(x)} \\ &= \frac{q(x) \bullet g(x)}{g(x)} + \frac{h(x)}{g(x)} \\ &= \frac{q(x) \bullet g(x) + h(x)}{g(x)} \end{aligned}$$

If $f(x) = q(x) \bullet g(x) + h(x)$, then

$$r(x) = \frac{f(x)}{g(x)}$$

Note: Students investigate operations on functions in the Level 6 module "Ostriches Are Composed."

Exploration 2

(page 79)

In this exploration, students experiment with adding other types of rational expressions to a polynomial function.

a. Answers will vary. In the following sample responses, $h(x) = 3$ and $k(x) = x + 6$.

Sample response: When the expression $(x + 6)$ in

$$f(x) = 3 + \frac{1}{(x+6)^n}$$

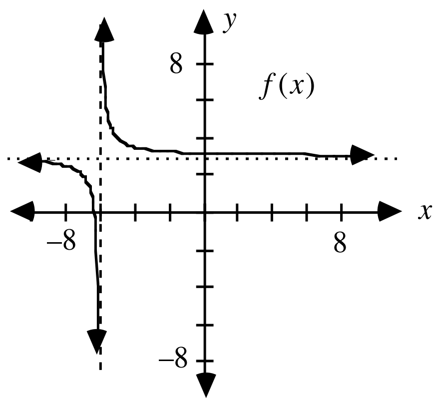
is raised to different odd powers, all the graphs have the same general shape.

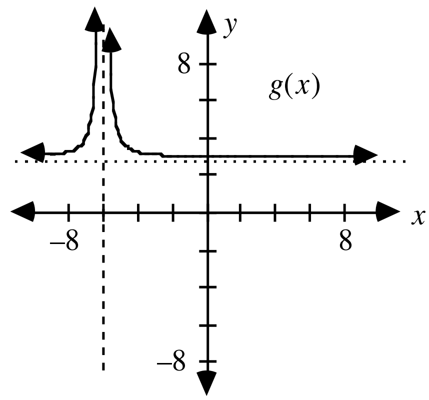
Similarly, when $(x + 6)$ in

$$g(x) = 3 + \frac{1}{(x+6)^n}$$

is raised to different even powers, all the graphs have the same general shape.

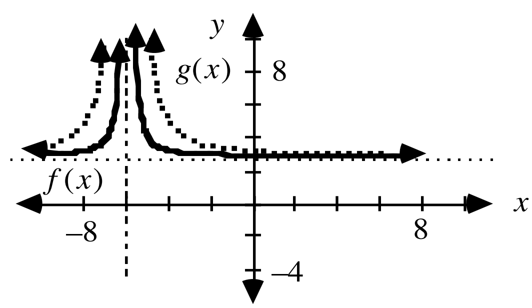
The following graphs illustrate these shapes.



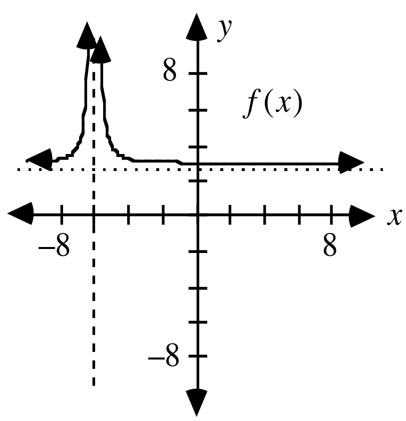


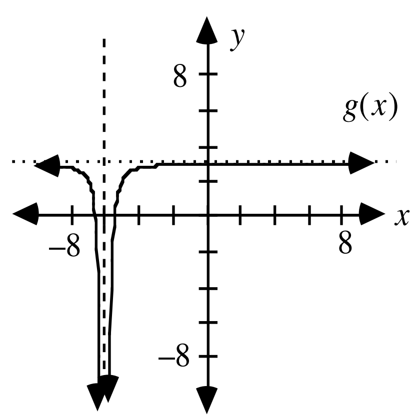
b. Students should observe that changing the magnitude of the numerator of the added rational expression affects the rate of increase in the absolute value of the function.

In the following sample graph, $h(x) = 3$, $k(x) = x + 6$, and $n = 8$. The range of both functions is the set of real numbers greater than 3 and both graphs have a vertical asymptote at the excluded value.



Students should discover that changing the sign of the numerator of the added rational expression changes the direction of the asymptotic behavior. In the following sample graphs, $h(x) = 3$, $k(x) = x + 6$, and $n = -1$.





Discussion 2 (page 80)

a. Sample response: Both graphs have an asymptote at $k(x) = 0$ and the same end behavior. As x approaches the value where $k(x) = 0$ from the right, both graphs increase without bound. The difference occurs as x approaches the value where $k(x) = 0$ from the left. In this case, the graph of $f(x)$ decreases without bound, while the graph of $g(x)$ increases without bound.

b.

1. Sample response: Both graphs have an asymptote at $k(x) = 0$ and the same end behavior. When n is positive, the graph of $g(x)$ increases faster than that of $f(x)$ as x approaches the value where $k(x) = 0$.
2. Sample response: Both graphs have an asymptote at $k(x) = 0$ and the same end behavior. When n is negative, the asymptotic behavior of the graph of $g(x)$ occurred in the opposite direction of that of $f(x)$.

c.

1. Sample response: As $|x|$ increases without bound, the value of the added rational expression approaches 0. Therefore, the value of the function approaches 3.
2. Sample response: As x approaches –6, the value of $1/(x+6)^2$ increases without bound, while the value of $-1/(x+6)^2$ decreases without bound. Therefore, $f(x)$ approaches ∞ while $g(x)$ approaches $-\infty$.

d.

1. Sample response: Raising the denominator of the added rational expression from an odd to an even power makes the function increase without bound in the same direction whether x approaches the excluded value from the left or the right.
2. Sample response: Changing the sign of the numerator of the added rational expression changes the direction of the asymptotic behavior as x approaches the value where $k(x) = 0$. Changing the magnitude of the numerator changes the rate of increase in the absolute value of the function as x approaches the excluded value.

e.

1. Sample response: A vertical asymptote is needed at $x = 2$, with the function increasing positively without bound as x approaches 2 from both the left and right. This can be accomplished by adding a rational expression to $f(x)$ as follows:

$$g(x) = 2 + \frac{1}{(x-2)^2}$$

2. Sample response: Yes. Raising the denominator to another even power also would accomplish the task.

f. Sample response: Divide the numerator of the rational function by the denominator. The quotient is equivalent to $q(x)$. When written over $g(x)$, the denominator, the remainder is equivalent to:

$$\frac{h(x)}{g(x)}$$

Assignment

(page 81)

3.1 The graph of any function of the form below, where $-3 \le k \le 3$ and $k \ne 0$, will pass through all four gates:

$$g(x) = -2 + \frac{k}{(x-2)^1}$$

The domain is the set of all real numbers except 2. The range is the set of all real numbers except –2. **Note:** Any rational function of the form of $g(x)$ in which $(x-2)$ is raised to an odd power will have a similar shape.

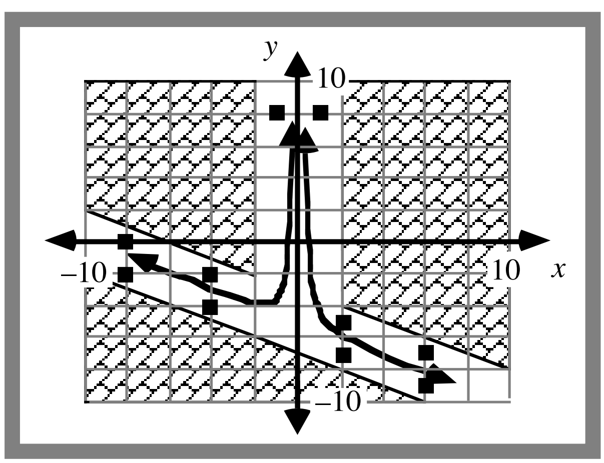
3.2 The graph of any function of the form below, where $-16 \le k < 0$, will pass through all three gates and miss the brick wall:

$$g(x) = \frac{1}{2}x^2 - 3x - \frac{7}{2} + \frac{k}{(x-3)^2}$$

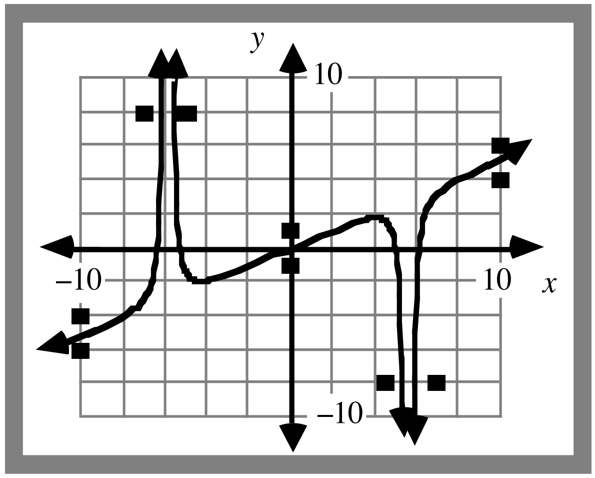
The domain is the set of all real numbers except 3. The range is the set of all real numbers. **Note:** Any rational function of the form of $g(x)$ in which $(x-3)$ is raised to an even power will have a similar shape.

***3.3** There are many functions that will pass through all the gates without touching the walls. The screen below shows a graph of the following sample function:

$$f(x) = -\frac{1}{2}x - 5 + \frac{1}{x^2}$$

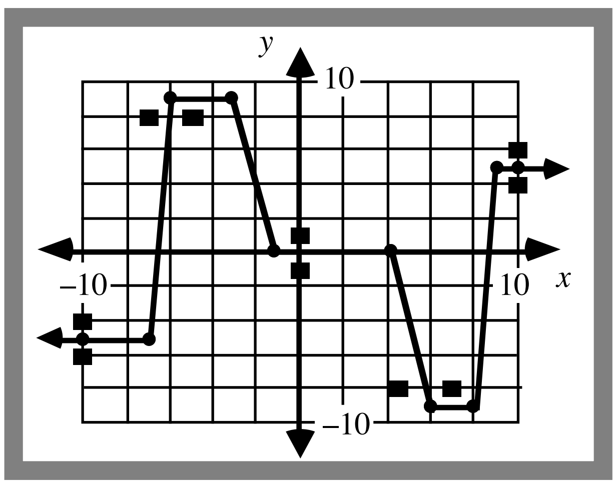


3.4 **a.** There are many functions that will pass through all five gates. The screen below shows a graph of the following sample function:



$$g(x) = \frac{1}{2}x + \frac{1}{(x+6)^2} + \frac{-1}{(x-6)^2}$$

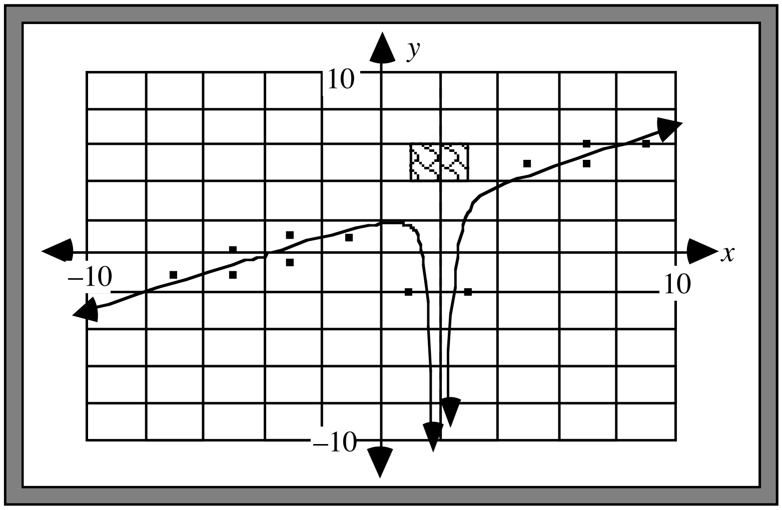
b. Answers will vary. The screen below shows a graph of the following sample function:



$$f(x) = \begin{cases} -5, & x \in (-\infty, -7] \\ 14x + 93, & x \in [-7, -6] \\ 9, & x \in [-6, -3] \\ -4.5x - 4.5, & x \in [-3, -1] \\ 0, & x \in [-1, 4] \\ -4.5x + 18, & x \in [4, 6] \\ -9, & x \in [6, 8] \\ 14x - 121, & x \in [8, 9] \\ 5, & x \in [9, \infty) \end{cases}$$

c. Sample response: The domain for $g(x)$ is the set of all real numbers except 6 and –6. Its range is the set of real numbers. The domain for $f(x)$ is all real numbers, while its range is the real-number interval $[-9, 9]$.

3.5 **a.** Students should graph the ordered pairs for the gates as points on a scatterplot. On many graphing utilities, the brick wall can be created using the "shade" feature. Sample graph:



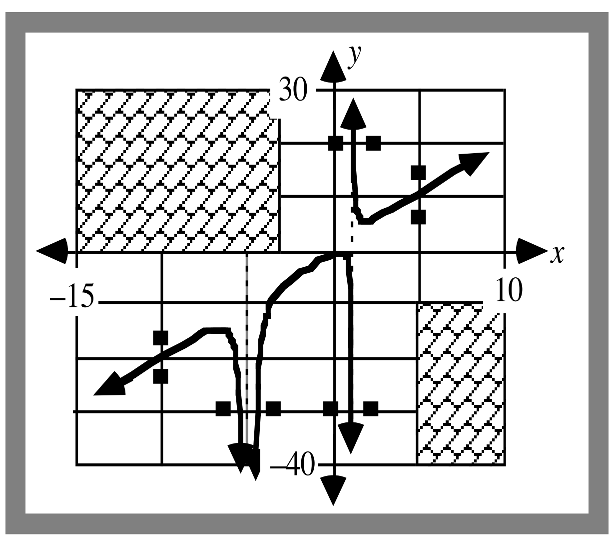
b. Answers will vary. The sample response above shows a graph of the rational function:

$$f(x) = \frac{1}{2}x + 2 - \frac{1}{(x-2)^2}$$

***3.6** Answers will vary. Sample response: The graph of the following function will pass through all five gates in the screen below while avoiding the two brick walls.

$$f(x) = 2x + 1 + \frac{1}{x-1} + \frac{-5}{(x+5)^2}$$

The domain is the set of all real numbers except 1 and –5. The range is the set of real numbers less than 0 and greater than approximately 5.7.



3.7 **a.** $h(x) = \dfrac{5x^3 - 34x^2 - 17x + 73}{x - 7}$

b. $g(x) = \dfrac{-33x^2 - 10x - 3}{3x + 2}$

c. $q(x) = \dfrac{x^4 + 2x^3 + x^2 + 3}{x + 1}$

* * * * *

3.8 **a.** This function may be rewritten as follows:

$$h(x) = 3x - 4 + \frac{5}{x + 11}$$

A vertical asymptote occurs at $x = -11$. As *x* approaches –11 from the negative side, $h(x)$ approaches $-\infty$. As *x* approaches –11 from the positive side, $h(x)$ approaches ∞. As $|x|$ approaches infinity, the graph of $h(x)$ has an oblique asymptote at $y = 3x - 4$.

b. This function may be rewritten as follows:

$$g(x) = -8x^2 - 4 + \frac{-2}{2x - 5}$$

There is a vertical asymptote at $x = 5/2$. As *x* approaches $5/2$ from the negative side, $g(x)$ approaches ∞. As *x* approaches $5/2$ from the positive side, $g(x)$ approaches $-\infty$. As $|x|$ approaches infinity, the graph of $g(x)$ is asymptotic to $y = -8x^2 - 4$.

c. This function may be rewritten as follows:

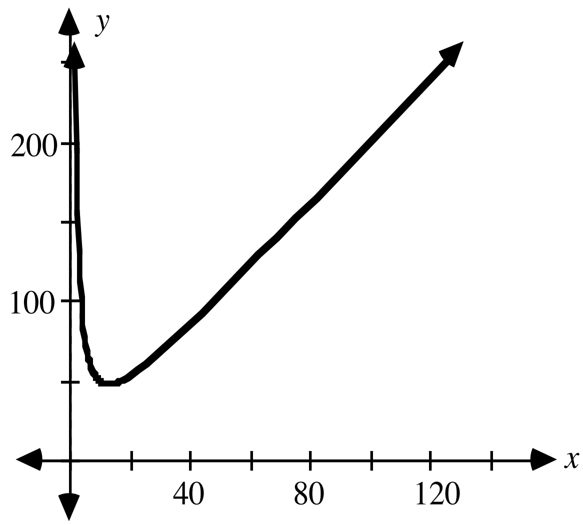
$$q(x) = x^4 - x^2 - 3x + 4 + \frac{-3}{x - 7}$$

There is a vertical asymptote at $x = 7$. As *x* approaches 7 from the negative side, $q(x)$ approaches ∞. As *x* approaches 7 from the positive side, $q(x)$ approaches $-\infty$. As $|x|$ approaches infinity, the graph of $q(x)$ is asymptotic to $y = x^4 - x^2 - 3x + 4$.

3.9 **a.** If the width of a rectangle with an area of 150 m^2 is *x*, then its length is $150/x$. The perimeter of the rectangle can be described by the following rational function:

$$p(x) = 2x + 2\left(\frac{150}{x}\right) = 2x + \frac{300}{x}$$

b. Sample response: A vertical asymptote occurs at $x = 0$. This indicates that the rancher cannot create a corral with a width of 0.



c. Sample response: Assuming that the rancher wants the length and width to be at least 1 m, a reasonable domain is the set of numbers greater than or equal to 1 and less than or equal to 150. The corresponding range is the set of numbers greater than or equal to approximately 49 but less than or equal to 302.

d. From the graph, the minimum perimeter is approximately 49 m. The corresponding values for the width and length are approximately 12.25 m.

e. The shape of the rectangle that minimizes the perimeter for a given area is a square. To test this conjecture, students should repeat Parts **a–d** for several values of *n* in the following function:

$$p(x) = 2x + \frac{2n}{x}$$

* * * * * * * * * *

Answers to Summary Assessment (page 85)

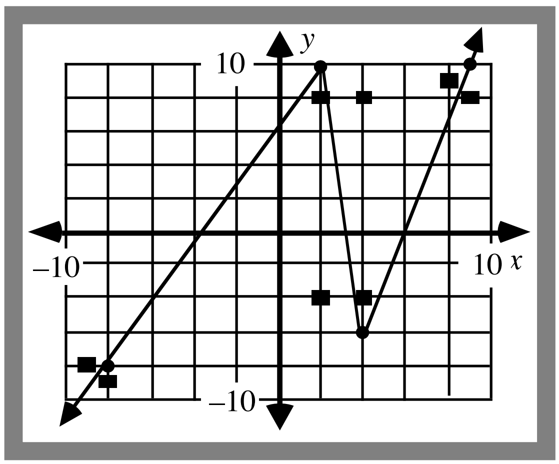
1. **a.** Sample response:

$$g(x) = x + \frac{1}{x-3}$$

b. Sample response:

$$g(x) = \frac{x^2 - 3x + 1}{x - 3}$$

c. Sample response:



$$f(x) = \begin{cases} 1.8x + 6.4, & x \in [-\infty, 2] \\ -8x + 26, & x \in [2, 4] \\ 3.2x - 18.8, & x \in [4, +\infty] \end{cases}$$

2. **a.** Sample response: This is a quadratic function with a negative leading coefficient. There is an absolute maximum at the vertex of the parabola, the point with coordinates (0,8). There is no absolute minimum for this function.

b. Sample response:

$$g(x) = -\frac{1}{5}x^2 + 8 - \frac{1}{x^2}$$

c. The sample function given in Part **b** can be written as follows:

$$g(x) = \frac{-\frac{1}{5}x^4 + 8x^2 - 1}{x^2}$$

The domain is the set of real numbers except 0. The range is the set of real numbers less than 8.

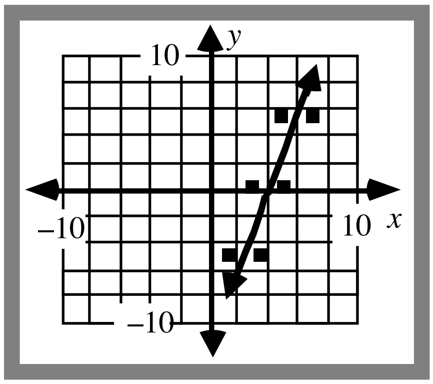
3. Sample response:

$$f(x) = -2 + \frac{-1}{(x+2)^2} + \frac{1}{(x-4)^2} = \frac{-2x^4 + 8x^3 + 24x^2 - 52x - 140}{x^4 - 4x^3 - 12x^2 + 32x + 64}$$

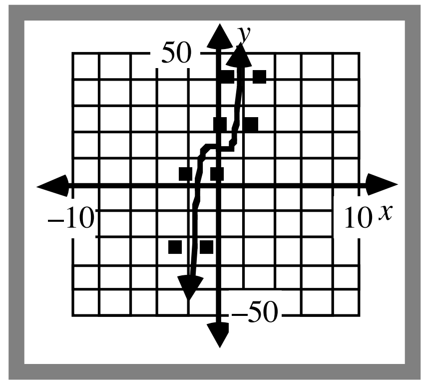
Module Assessment

1. Parts **a–c** show three screens in the first level of Gates. Determine a possible degree for each polynomial shown and describe its characteristics, including its domain and range.

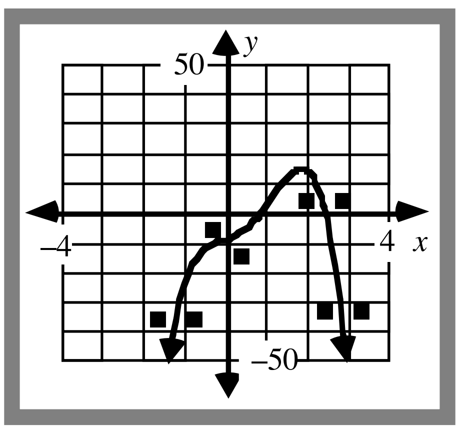
a.



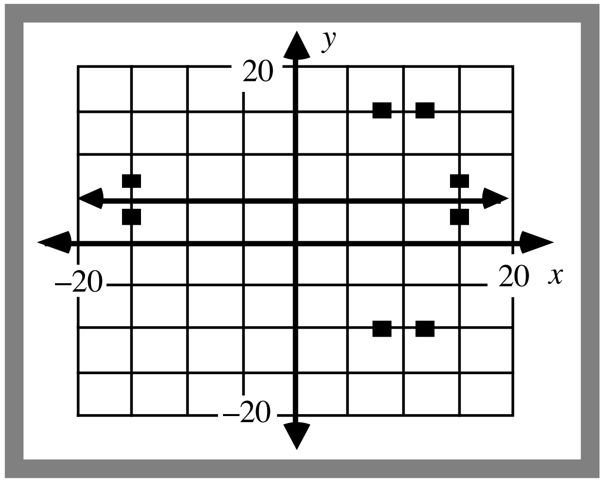
b.



c.



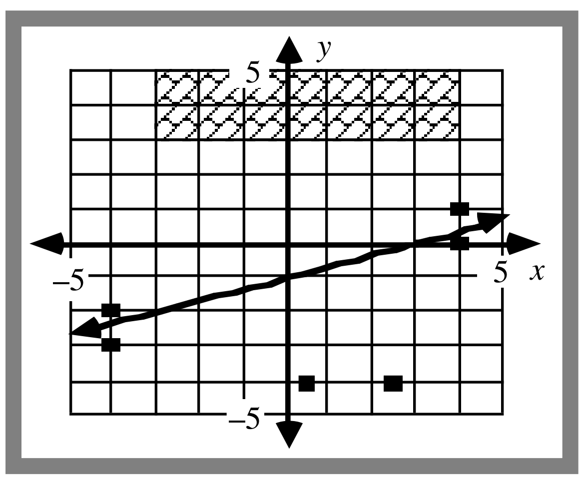
2. **a.** The figure below shows a screen in the third level of gates. The line represents the graph of the polynomial function $f(x) = 5$. Add a rational expression to $f(x) = 5$ so that the graph of the resulting function will pass through all four gates.



b. Write the new function in the general form of a rational function and identify its domain and range.

c. Determine a piecewise function that will pass through all the gates on the screen in Part **a**.

3. The figure below shows another screen in the third level of gates.

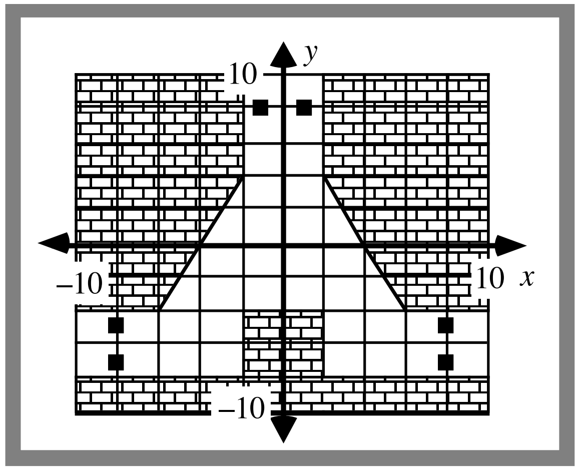


The line represents the graph of the polynomial function

$$f(x) = \frac{1}{3}x - 1$$

Add a rational expression to $f(x)$ so that the graph of the resulting function will pass through all three gates without hitting the brick wall.

4. The figure below shows a screen in the third level of Gates.



a. Determine a rational function whose graph passes through all three gates without touching the brick walls.

b. Write this function in the general form of a rational function.

5. Write each of the following rational functions as a polynomial function plus a rational expression. Describe where asymptotic behavior may occur in the graph of each function.

a. $h(x) = \dfrac{8x - 7}{x}$

b. $q(x) = \dfrac{4x^3 + 48x^2 + 44x + 3}{x + 11}$

Answers to Module Assessment

1. **a.** Sample response: This polynomial appears to be linear (of degree 1). The domain and range appear to be the set of real numbers.

b. Sample response: This polynomial function may have an odd degree because it appears to have opposite end behaviors. Since it is not a line, the degree must be 3 or more. The domain and range appear to be the set of real numbers.

c. Sample response: This polynomial function may have an even degree because it appears to have the same end behaviors. Since it is not a parabola, the degree must be greater than 2. The domain appears to be the set of real numbers and the range appears to be $y \le 15$.

2. **a.** Answers will vary. Sample response:

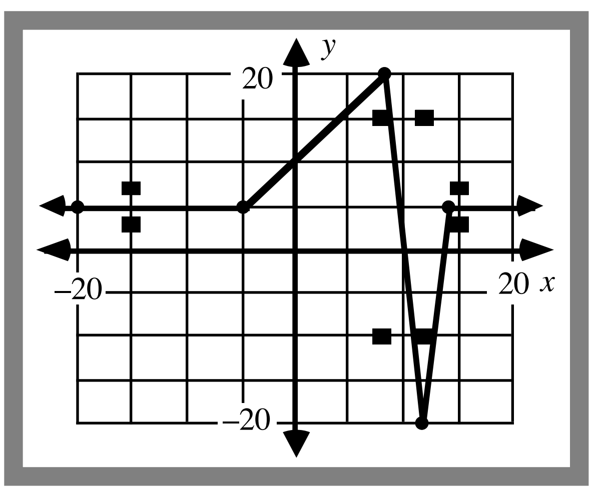
$$g(x) = 5 + \frac{1}{x - 10}$$

b. Sample response:

$$g(x) = \frac{5x - 49}{x - 10}$$

The domain is all real numbers except 10. The range is all real numbers.

c. Sample response:



$$f(x) = \begin{cases} 5, & x \in (-\infty, -5] \\ \frac{15}{13}x + \frac{140}{13}, & x \in [-5, 8] \\ -10x + 100, & x \in [8, 12] \\ 12.5x - 170, & x \in [12, 14] \\ 5, & x \in [14, \infty] \end{cases}$$

3. Answers will vary. Sample response:

$$g(x) = \frac{1}{3}x - 1 - \frac{1}{\left(x - (3/2)\right)^2}$$

4. **a.** Answers will vary. Sample response:

$$g(x) = -6 + \frac{15}{x^2}$$

b. The sample function given in Part **a** can be rewritten as follows:

$$g(x) = \frac{-6x^2 + 15}{x^2}$$

5. **a.** This function may be rewritten as follows:

$$h(x) = 8 - \frac{7}{x}$$

Asymptotic behavior occurs near $x = 0$.

b. This function may be rewritten as follows:

$$q(x) = 4x^2 + 4x + \frac{3}{x+11}$$

Asymptotic behavior occurs near $x = -11$.

Selected References

Demana, F., B. K. Waits, and S. R. Clemens. *College Algebra & Trigonometry*. New York: Addison-Wesley, 1992.

Faires, J. D., and B. T. Faires. *Calculus and Analytic Geometry*. Boston: Prindle, Weber & Schmidt, 1983.

Flashbacks

Activity 1

1.1 Given that $f(x) = -x^2 + 4x - 2$, find $f(-3)$.

1.2 Determine the value(s) of x for which $p(x) = 0$ for each of the following:

a. $p(x) = x(x-8)(x+5)(x-16)$

b. $p(x) = x^3 - 4x^2 - 11x + 30$

1.3 Solve the following system of equations and describe how you obtained your solution.

$$\begin{cases} x + y = 3 \\ 2x + z = 9 \\ 3x - 4y + z = 20 \end{cases}$$

Activity 2

2.1 Describe how to determine the equation of a line that passes through the points (4,6) and (–2,1).

2.2 What is the equation of a circle with center at the origin and radius r?

Activity 3

3.1 Determine the value(s) of x for which each of the following expressions equals 0.

a. $4x - 1$

b. $(x+3)(2-x)$

3.2 Identify the domain and range of the following function:

$$f(x) = \frac{x+3}{x-4}$$

Answers to Flashbacks

Activity 1

1.1 $f(-3) = -9 - 12 - 2 = -23$

1.2 **a.** 0, 8, –5, and 16

b. –3, 2, and 5

1.3 Students may solve this system using the substitution method or by using matrices. The solutions are:

$$\begin{cases} x = 23/5 = 4.6 \\ y = -8/5 = -1.6 \\ z = -1/5 = -0.2 \end{cases}$$

Note: Students may recall the use of matrices for solving systems of equations from the Level 2 module "Making Concessions."

Activity 2

2.1 Answers may vary. Sample response: The slope of the line though these points is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{4 - (-2)} = \frac{5}{6}$$

Using the point-slope form of the equation of a line,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{5}{6}(x - (-2)) \\ y &= \frac{5}{6}x + \frac{10}{6} + 1 \\ &= \frac{5}{6}x + \frac{16}{6} \end{aligned}$$

2.2 $x^2 + y^2 = r^2$

Activity 3

3.1 **a.** $x = 1/4$

b. $x = -3$ or 2

3.2 The domain is the set of real numbers except 4; the range is the set of real numbers except 1.

Changing the Rules Changes the Game

In the past century, women's basketball has changed from a game of six-person teams and half-court play to a game of five-person teams and full-court play. In this module, you investigate how changing the rules of geometry affects the game of mathematics.

Bill Chalgren • Janet Kuchenbrod • Tom Teegarden

Teacher Edition

Changing the Rules Changes the Game

Overview

In this module, students investigate finite arithmetic systems and finite geometries. They then use proofs to extend their knowledge of axiomatic systems.

Objectives

In this module, students will:

- be introduced to modular arithmetic systems
- coordinatize a finite geometry using modulo 3
- reconceptualize many terms of Euclidean geometry in a finite geometry
- explore geometries both analytically and synthetically
- construct direct and indirect proofs and proofs by exhaustion.

Prerequisites

For this module, students should know:

- how to count using combinations
- how to determine the equation of a line
- how to solve linear equations
- how to perform proofs by exhaustion.

Time Line

Activity	**1**	**2**	**3**	**4**	**Summary Assessment**	**Total**
Days	2	4	3	2	1	12

Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
tin can	X				
paper strips	X				
graph paper		X			
template A		X			
template B		X			
template C		X			

Teacher Note

Blackline masters for the templates appear at the end of the teacher edition FOR THIS MODULE.

Changing the Rules Changes the Game

Introduction (page 93)

As students will observe in later activities, when axioms of traditional Euclidean geometry are changed, the new geometries formed may be dramatically different.

Teacher Note

In the mid-19th century, Hungarian mathematician Janos Bolyai and Russian mathematician Nicolai Lobachevsky independently developed a different geometry by changing one axiom of Euclidean geometry. Each assumed that, in a plane, there is more than one parallel line to a given line through a point not on the line. The result was a geometry, often called "hyperbolic geometry," in which lines are not "straight" and the sum of the measures of the angles in a triangle is less than 180˚.

Shortly after Bolyai and Lobachevsky's discoveries, the German mathematician Bernard Riemann developed yet another geometry. Riemann assumed that there was no such thing as parallel lines in a plane and that all lines intersect. In Riemann's geometry, lines are not straight and the sum of the measures of the angles in a triangle is greater than 180˚.

Exploration (page 93)

Students examine arithmetic on a 12-hour clock face. (This is a preview of the modular arithmetic they will work with in Activity **1**.)

a. Students draw a 12-hour clock face.

b. **1.** Methods may vary. Sample response: On a 12-hour clock face, 12 can correspond to 0. A positive integer can be represented by counting that number of units clockwise from 12. The clock value on which you end is the corresponding value for the integer.

2. Sample response: A negative integer can be represented by counting that number of units counterclockwise from 12. The clock value on which you end is the corresponding value for the integer.

3. Using the sample methods described above, the integer 15 corresponds with the clock value 3, and the integer –5 corresponds with the clock value 7.

c. **1–2.** Sample response: Adding (or subtracting) integers on a 12-hour clock can be accomplished by first adding (or subtracting) the integers as real numbers, then representing the answer as a 12-hour clock value using the methods described in Part **b**.

3. Using the sample method described above, the sum $8 \oplus 7$ is 3 on a 12-hour clock. The difference $5 \ominus 10$ is 7 on a 12-hour clock.

d. The additive identity on a 12-hour clock is 12.

e.
1. The product $4 \otimes 2$ is 8 in 12-hour arithmetic.
2. The product $4 \otimes 5$ is 8 in 12-hour arithmetic.

Discussion

(page 94)

a. Students compare their methods for representing positive and negative integers on a 12-hour clock.

b. Sample response: The additive identity for a 12-hour arithmetic is 12. Any time 12 is added to a number on the clock face, the result is the same as the original number.

c.
1. The following pairs are additive inverses of each other on a 12-hour clock: (12,12), (1,11), (2,10), (3,9), (4,8), (5,7), and (6,6).
2. Sample response: Yes. Each of the numbers on a 12-hour clock is paired with only one value in Part **c1** above.

d. Sample response: The multiplicative identity for 12-hour arithmetic also is 1, since every number on the clock times 1 is that number.

e. Sample response: Each number on a 12-hour clock does not have a multiplicative inverse. For example, there is no number that when multiplied by 2 results in a product of 1. Multiplying by 2 will always result in an even product. Converting this even product to a 12-hour clock value will still result in an even value. Therefore, the product cannot be 1. **Note:** The numbers 1, 5, 7, and 11 are the only 12-hour clock values that have multiplicative inverses. Each is its own multiplicative inverse.

(page 95)

Activity 1

This activity introduces students to modular arithmetic, including additive and multiplicative identities and inverses. They construct basic addition and multiplication fact tables and use these tables to solve congruences.

Materials List

- tin can (1 per group)
- paper strips approximately 2.5 times the circumference of the can in length (1 per group)

Teacher Note

The mathematics note that precedes Discussion **1** defines two congruent numbers (mod n) as having the same remainder when divided by n. This also may be stated as follows: In modulo n, two numbers are congruent if their difference is exactly divisible by n.

You may want to explore this notion in more depth with your students. For example, 18 and 10 are both congruent to 2 (mod 8), because both $18 - 2 = 16$ and $10 - 2 = 8$ are exactly divisible by 8. Given the equation $18 \equiv x \pmod{8}$, therefore, many solutions are possible. Because this module does not discuss equivalence classes, however, students are expected only to give answers that are on the mod n clock. In this case, $18 \equiv 2 \pmod{8}$.

Note: In mod n arithmetic where n is not prime, traditional methods for solving equations may not produce solutions (even though they exist) because of the absence of multiplicative inverses. Students may solve such equations by substitution.

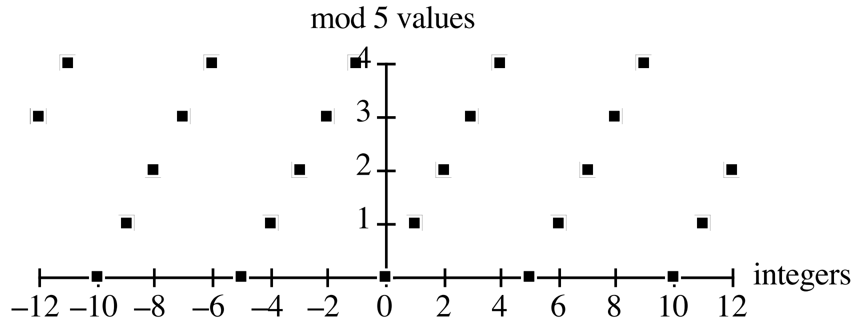
Exploration 1 (page 95)

Students use the notion of a wrapping function to determine the mod 5 value that corresponds with each integer.

a–b. Sample table:

Integer	Mod 5 Value	Integer	Mod 5 Value
–12	3	1	1
–11	4	2	2
–10	0	3	3
–9	1	4	4
–8	2	5	0
–7	3	6	1
–6	4	7	2
–5	0	8	3
–4	1	9	4
–3	2	10	0
–2	3	11	1
–1	4	12	2
0	0		

c. **1.** Sample graph:



2. Sample response: The integers that correspond with the same mod 5 value can be found by identifying the *x*-coordinates of all the points that lie on the same horizontal line. In mod 5, 0 corresponds with $\{\ldots,-10,-5,0,5,10,\ldots\}$; 1 corresponds with $\{\ldots,-9,-4,1,6,11,\ldots\}$; 2 corresponds with $\{\ldots,-8,-3,2,7,12,\ldots\}$; 3 corresponds with $\{\ldots,-12,-7,-2,3,8,\ldots\}$; 4 corresponds with $\{\ldots,-11,-6,-1,4,9,\ldots\}$.

d. Sample response: Since both 638 and –727 have a remainder of 3, they both correspond with 3 (mod 5).

$$\begin{array}{r} 127 \;\text{R3} \\ 5\overline{)\,638} \\ \underline{-635} \\ 3 \end{array} \qquad \begin{array}{r} -146 \;\text{R3} \\ 5\overline{)\,-727} \\ \underline{-(-730)} \\ 3 \end{array}$$

Discussion 1

(page 97)

a. Sample response: Points on the same horizontal line represent integers that are congruent in modulo 5.

b. Sample response: Determine the greatest negative integer that when multiplied by *n* results in a value less than the given number. Subtract this value from the given number. The remainder is the congruent mod *n* value.

c. The domain is the set of integers. The range is {0, 1, 2, 3, 4}.

Exploration 2 (page 97)

In this exploration, students investigate operations and properties in a modulo 5 arithmetic system.

a–b. Sample tables:

+	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

c. Sample table:

–	**0**	**1**	**2**	**3**	**4**
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

d. The additive identity element for mod 5 is 0 because, for $a \in \{0,1,2,3,4\}$, $a + 0 \equiv 0 + a \equiv a \pmod 5$.

e. The additive inverse of 0 (mod 5) is 0 because $0 + 0 \equiv 0 + 0 \equiv 0 \pmod 5$.

The additive inverse of 1 (mod 5) is 4 and the additive inverse of 4 (mod 5) is 1 because $1 + 4 \equiv 4 + 1 \equiv 0 \pmod 5$.

The additive inverse of 2 (mod 5) is 3 and the additive inverse of 3 (mod 5) is 2 because $3 + 2 \equiv 2 + 3 \equiv 0 \pmod 5$.

f. The multiplicative identity in mod 5 is 1 because, for $a \in \{0,1,2,3,4\}$, $a \bullet 1 \equiv 1 \bullet a \equiv a \pmod 5$.

g. There is no multiplicative inverse for 0 in mod 5.

The multiplicative inverse for 1 in mod 5 is 1 because $1 \bullet 1 \equiv 1 \bullet 1 \equiv 1 \pmod 5$.

The multiplicative inverse for 2 (mod 5) is 3 and the multiplicative inverse of 3 (mod 5) is 2 because $2 \bullet 3 \equiv 3 \bullet 2 \equiv 1 \pmod 5$.

The multiplicative inverse for 4 (mod 5) is 4 because $4 \bullet 4 \equiv 4 \bullet 4 \equiv 1 \pmod 5$.

h. Sample response:

$3x+1 \equiv 2 \pmod 4$	given
$3x+1+3 \equiv 2+3 \pmod 4$	addition property of congruence
$3x+0 \equiv 2+3 \pmod 4$	1 and 3 are additive inverses in mod 4
$3x+0 \equiv 1 \pmod 4$	definition of congruence
$3x \equiv 1 \pmod 4$	0 is the additive identity in mod 4
$3 \bullet 3x \equiv 3 \bullet 1 \pmod 4$	multiplication property of congruence
$1x \equiv 3 \bullet 1 \pmod 4$	3 and 3 are multiplicative inverses in mod 4
$x \equiv 3 \pmod 4$	1 is the multiplicative identity in mod 4

Discussion 2

(page 99)

a.

1. Addition in mod *n* is commutative: $a+b \equiv b+a$.

2. Subtraction in mod *n* is not commutative. However, there are some cases where $a-b \equiv b-a$. For example, $1-3 \equiv 2 \pmod 4$ and
$3-1 \equiv 2 \pmod 4$.

3. Multiplication in mod *n* is commutative: $a \bullet b \equiv b \bullet a$.

b.

1. Division can be defined as multiplication by the multiplicative inverse if it exists.

2. In modulo 5, division is defined since every member of the mod 5 set has a multiplicative inverse. However, division is not commutative. For example,

$$3 \div 1 \equiv 3 \bullet 1^{-1} \pmod 5 \qquad 1 \div 3 \equiv 1 \bullet 3^{-1} \pmod 5$$
$$\equiv 3 \bullet 1 \pmod 5 \qquad \equiv 1 \bullet 2 \pmod 5$$
$$\equiv 3 \pmod 5 \qquad \equiv 2 \pmod 5$$

Since 3 (mod 5) is not congruent to 2 (mod 5), division is not commutative.

Note: You may wish to discuss the fact that only in a prime modulo does every element have a multiplicative inverse. This is necessary for division to be defined.

c. Sample response: No. In this case, $16 \bullet 6 = 96$ and $96 \equiv 1 \pmod 5$. If both factors are converted to mod 5 before multiplying, $16 \bullet 6 \equiv 1 \bullet 1 \pmod 5 \equiv 1 \pmod 5$. **Note:** Although students are not expected to prove this property, it is always true.

d. Both addition and multiplication are associative in modular arithmetic.

e. Multiplication is distributive over addition mod n, for any natural number value of n. For example,

$$2(3+4) \equiv 2(2) \equiv 4 \equiv 1+3 \equiv 2 \bullet 3 + 2 \bullet 4 \pmod 5$$

f. **1.** In mod 6, the multiplicative identity is 1. There is no number in mod 6 which, when multiplied by 2, results in a product of 1.

$$2 \bullet 0 \equiv 0 \pmod 6$$
$$2 \bullet 1 \equiv 2 \pmod 6$$
$$2 \bullet 2 \equiv 4 \pmod 6$$
$$2 \bullet 3 \equiv 0 \pmod 6$$
$$2 \bullet 4 \equiv 2 \pmod 6$$
$$2 \bullet 5 \equiv 4 \pmod 6$$

2. Substituting all the numbers from mod 6 into the equation shows that none of the solutions are congruent to 4.

$$2 \bullet 0 - 5 = -5 \equiv 1 \pmod 6$$
$$2 \bullet 1 - 5 = -3 \equiv 3 \pmod 6$$
$$2 \bullet 2 - 5 = -1 \equiv 5 \pmod 6$$
$$2 \bullet 3 - 5 \equiv 1 \pmod 6$$
$$2 \bullet 4 - 5 \equiv 3 \pmod 6$$
$$2 \bullet 5 - 5 \equiv 5 \pmod 6$$

g. **1.** In mod 6, the multiplicative identity is 1. There is no number in mod 6 which, when multiplied by 3, results in a product of 1.

$$3 \bullet 0 \equiv 0 \pmod 6$$
$$3 \bullet 1 \equiv 3 \pmod 6$$
$$3 \bullet 2 \equiv 0 \pmod 6$$
$$3 \bullet 3 \equiv 3 \pmod 6$$
$$3 \bullet 4 \equiv 0 \pmod 6$$
$$3 \bullet 5 \equiv 3 \pmod 6$$

2. The solutions may be found by substituting all the numbers from mod 6 into the equation.

h. Modular arithmetic could be used when determining the value on an odometer or pedometer. It is also used in coloring patterns in some designs.

Assignment (page 100)

1.1 Sample response: When 33 is divided by 5, the remainder is 3. Therefore, $33 \equiv 3 \pmod{5}$.

1.2 **a.** $3 + 2 \equiv 0 \pmod{5}$

b. $12 \bullet 8 \equiv 1 \pmod{5}$

c. $13 - 20 \equiv 3 \pmod{5}$

1.3 Sample tables:

+	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

×	**0**	**1**	**2**
0	0	0	0
1	0	1	2
2	0	2	1

1.4 **a.** **1.** $2 + 1 \equiv 0 \pmod{3}$

2. $2 \bullet 2 \equiv 1 \pmod{3}$

3. $16 + 9 \equiv 1 \pmod{8}$

4. $11 \bullet 7 \equiv 7 \pmod{10}$

b. **1.** $-3 \equiv 2 \pmod{5}$

2. $-2 \equiv 1 \pmod{3}$

3. The multiplicative identity in mod 5 is 1. The multiplicative inverse of 3 in mod 5 is 2 because $3 \bullet 2 \equiv 1 \pmod{5}$.

4. The multiplicative inverse of 0 in mod 5 does not exist because there is no number that can be multiplied by 0 to obtain 1, the multiplicative identity.

***1.5** **a.** The additive identity is 0 because, for $a \in \{0,1,2\}$, $a + 0 \equiv 0 + a \equiv a \pmod{3}$.

b. The additive inverse for 0 is 0 because $0 + 0 \equiv 0 + 0 \equiv 0 \pmod{3}$.

The additive inverse for 1 is 2 and for 2 is 1 because $1 + 2 \equiv 2 + 1 \equiv 0$.

c. The multiplicative identity for mod 3 is 1 because $1 \bullet 0 \equiv 0 \bullet 1 \equiv 0 \pmod{3}$, $1 \bullet 1 \equiv 1 \bullet 1 \equiv 1 \pmod{3}$, and $1 \bullet 2 \equiv 2 \bullet 1 \equiv 2 \pmod{3}$.

d. Sample response: The number 0 in mod 3 has no multiplicative inverse because there is no number which can be multiplied by 0 to get 1, the multiplicative identity. This counterexample disproves the statement in Part **d**.

1.6 **a.** Because $2 \bullet 3 \equiv 1 \pmod 5$, $1 \div 2 \equiv 3 \pmod 5$.

b. Because $2 \bullet 4 \equiv 3 \pmod 5$, $3 \div 2 \equiv 4 \pmod 5$.

c. Because there is no number x in mod 5 for which $0 \bullet x \equiv 4 \pmod 5$, $4 \div 0$ is undefined.

1.7 The elements in modulo 4 are {0, 1, 2, 3}. The multiplicative identity in modulo 4 is 1. As shown in the multiplication table for mod 4 below, there is no element $x \in \{0,1,2,3\}$ such that $x \bullet 2 \equiv 2 \bullet x \equiv 1 \pmod 4$.

×	**0**	**1**	**2**	**3**
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

1.8 **a.** $x \equiv 5 \pmod 7$

b. $x \equiv 2 \pmod 7$

c. By substitution, $x \equiv 1 \pmod 6$, $x \equiv 3 \pmod 6$, or $x \equiv 5 \pmod 6$.

d. $x \equiv 1 \pmod 3$

e. $x \equiv 1$ or $x \equiv 2 \pmod 3$

* * * * *

1.9 **a.** $x \equiv 6 \pmod{13}$

b. $x \equiv 4 \pmod{11}$

1.10 At a rate of 0.5 m per hour, each 200-m roll lasts 400 hr. Using modular arithmetic, $400 \equiv 16 \pmod{24}$. Since the roll was installed at 9 A.M., $9 + 16 \equiv 1 \pmod{24}$. The device will run out of paper at 1 A.M.

Students may also solve this problem as follows:

$$200 \text{ m} \bullet \frac{1 \text{ hr}}{0.5 \text{ m}} \bullet \frac{1 \text{ day}}{24 \text{ hr}} = 16\tfrac{2}{3} \text{ days}$$

Since $16\frac{2}{3}$ days is equivalent to 16 days and 16 hr, the device will run out of paper 16 hr after 9 A.M. on the 16th day, or at 1 A.M.

* * * * * * * * * *

Research Project

(page 101)

To perform an internal check, bar-code readers use an algorithm which involves a mod 10 arithmetic system. For example, consider the UPC number in Figure **2**:

0 75520 57070 8

For this bar code to be valid, the following relationship must be true:

$$3 \times (\text{sum of digits in even positions}) + (\text{sum of digits in odd positions}) \equiv 0 (\text{mod } 10)$$

The first digit is considered to be in position 0, an even position. Thus,

$$3 \times (0 + 5 + 2 + 5 + 0 + 0) + (7 + 5 + 0 + 7 + 7 + 8) = 70 \equiv 0 (\text{mod } 10)$$

For more information, see Eric F. Wood's, "Self-Checking Codes—An Application of Modular Arithmetic," in the April 1987 issue of the *Mathematics Teacher* (pp. 312–316), Mary Blocksma's *Reading the Numbers* (New York: Penguin Books, 1989), or John Allen Paulos' *Innumeracy* (New York: Hill and Wang, 1988).

(page 102)

Activity 2

In this activity, students explore a finite geometry system developed using the solutions of linear equations (12 in all) in the field of integers modulo 3. Students discover that all triangles in this system are right triangles. The main objective of this activity is to challenge students' traditional concepts of points, lines, and other Euclidean terms, such as parallel and perpendicular lines and their slopes.

Materials List

- graph paper (several sheets per student)
- template A (one per student)
- template B (one per student)
- template C (for research project; optional)

Teacher Note

Template A is required for Part **c** of the exploration; template B for Problem **2.2**. Blackline masters of the templates appear at the end of the teacher edition for this module.

Discussion 1

(page 103)

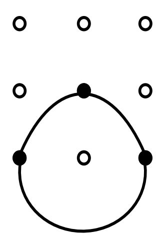
a. In both cases, a line is a set of points satisfying a given equation. In Euclidean geometry, a line consists of an infinite number of points. In the finite system, a line consists of exactly three points.

b. Students should recall the definition of a line in spherical geometry from the Level 6 module "What Shape Is Your World" and the Level 4 module "Having a Ball." Sample response: In spherical geometry, a line is a great circle consisting of an infinite number of points.

Note: Students also may recall that, in spherical geometry, perpendicular lines can exist but parallel lines do not. They examine these properties in the nine-point geometry in the following exploration and assignment.

c.

1. Sample response: A triangle could be determined by a set of three noncollinear points.
2. Sample response: Points (0,0), (1,1), and (2,0) form a triangle.



3. Sample response: This definition is comparable to the definition in Euclidean geometry, which states that a triangle is a three-sided polygon whose vertices can be determined by any three noncollinear points. It is unlike the Euclidean definition because it does not involve segments, polygons, or closed figures.

Exploration

(page 103)

Students coordinatize a lattice in mod 3 to obtain the distinct linear equations in a nine-point geometry.

a. Students should realize that, given equations of the form $Ax + By + C = 0$ in mod 3, there are 3 choices for A, 3 choices for B, and 3 choices for C. By the fundamental counting principle, there are $3 \bullet 3 \bullet 3 = 27$ possible equations. Because A and B cannot both be 0, that eliminates 3 possibilities, leaving 24.

Note: Students may not realize that the list of equations can be further reduced by the properties of modular arithmetic. For example, the equation $2x + 2y + 2 = 0$ can be expressed as $1x + 1y + 1 = 0$ by multiplying each side of the original equation by 2. As students will discover in Part **e**, there are actually only 12 distinct equations.

Students should list the following equations:

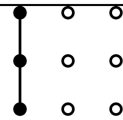
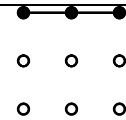
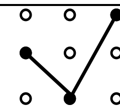
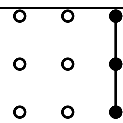
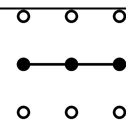
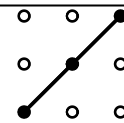
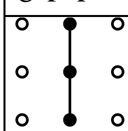
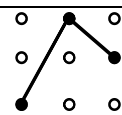
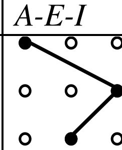
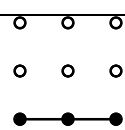
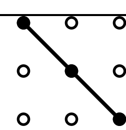
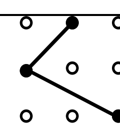
$0x + 0y + 0 = 0$ (not a line)	$1x + 0y + 0 = 0$	$2x + 0y + 0 = 0$
$0x + 1y + 0 = 0$	$1x + 1y + 0 = 0$	$2x + 1y + 0 = 0$
$0x + 2y + 0 = 0$	$1x + 2y + 0 = 0$	$2x + 2y + 0 = 0$
$0x + 0y + 1 = 0$ (not a line)	$1x + 0y + 1 = 0$	$2x + 0y + 1 = 0$
$0x + 1y + 1 = 0$	$1x + 1y + 1 = 0$	$2x + 1y + 1 = 0$
$0x + 2y + 1 = 0$	$1x + 2y + 1 = 0$	$2x + 2y + 1 = 0$
$0x + 0y + 2 = 0$ (not a line)	$1x + 0y + 2 = 0$	$2x + 0y + 2 = 0$
$0x + 1y + 2 = 0$	$1x + 1y + 2 = 0$	$2x + 1y + 2 = 0$
$0x + 2y + 2 = 0$	$1x + 2y + 2 = 0$	$2x + 2y + 2 = 0$

b. The coordinates of the points that satisfy each equation are given in the following table.

Equations	Points	Equations	Points
$x = 0$ $2x = 0$	(0,0), (0,1), (0,2)	$2x + 2y = 0$ $x + y = 0$	(0,0), (2,1), (1,2)
$x + 1 = 0$ $2x + 2 = 0$	(2,0), (2,1), (2,2)	$2x + 2y + 2 = 0$ $x + y + 1 = 0$	(2,0), (1,1), (0,2)
$x + 2 = 0$ $2x + 1 = 0$	(1,0), (1,1), (1,2)	$2x + 2y + 1 = 0$ $x + y + 2 = 0$	(1,0), (0,1), (2,2)
$y = 0$ $2y = 0$	(0,0), (1,0), (2,0)	$x + 2y = 0$ $2x + y = 0$	(0,0), (1,1), (2,2)
$2y + 2 = 0$ $y + 1 = 0$	(0,2), (1,2), (2,2)	$x + 2y + 2 = 0$ $2x + y + 1 = 0$	(1,0), (2,1), (0,2)
$2y + 1 = 0$ $y + 2 = 0$	(0,1), (1,1), (2,1)	$x + 2y + 1 = 0$ $2x + y + 2 = 0$	(0,1), (1,2), (2,0)

c. Each student will require a copy of template A to complete Part **c** of the exploration. **Note:** Students may draw segments or arcs to indicate the points contained in the same line. Although these segments or arcs may appear to cross, the only points in the geometry are the lattice points. Each line contains exactly three points. See sample graphs given in Part **d**.

d. The following table shows a graph of each line, its equation, the coordinates of the points it contains, and the corresponding letters using the labeling system given in Figure **5**.

 $x = 0$ $2x = 0$ (0,0), (0,1), (0,2) *A-D-G*	 $y + 1 = 0$ $2y + 2 = 0$ (0,2), (1,2), (2,2) *G-H-I*	 $x + y + 2 = 0$ $2x + 2y + 1 = 0$ (0,1), (1,0), (2,2) *D-B-I*
 $x + 1 = 0$ $2x + 2 = 0$ (2,0), (2,1), (2,2) *C-F-I*	 $y + 2 = 0$ $2y + 1 = 0$ ((0,1), (1,1), (2,1) *D-E-F*	 $x + 2y = 0$ $2x + y = 0$ (0,0), (1,1), (2,2) *A-E-I*
 $x + 2 = 0$ $2x + 1 = 0$ (1,0), (1,1), (1,2) *B-E-H* **g.** $y = 2x$	 $x + y = 0$ $2x + 2y = 0$ (0,0), (1,2), (2,1) *A-H-F* **h.** $y = 2x + 2$	 $x + 2y + 2 = 0$ $2x + y + 1 = 0$ (1,0), (2,1), (0,2) *B-F-G* **i.** (1,1), (1,0), (2,2) $y = 2x + 1$
 $y = 0$ $2y = 0$ (0,0), (1,0), (2,0) *A-B-C*	 $x + Y + 1 = 0$ $2x + 2Y + 2 = 0$ (0,2), (1,1), (2,0) *C-E-G*	 $x + 2y + 1 = 0$ $2x + y + 2 = 0$ (0,1), (1,2), (2,0) *C-D-H*

e. Both of these properties of lines also are true in the nine-point geometry.

f. The following lines in the nine-point geometry are parallel to each other: *A–B–C*, *D–E–F*, and *G–H–I*; *A–D–G*, *B–E–H*, and *C–F–I*; *A–E–I*, *C–D–H*, and *B–F–G*; *C–E–G*, *A–H–F*, and *D–B–I*. **Note:** The definition given in the student edition allows a line to be parallel to itself.

g. By considering the list of lines given in Part **f**, the Parallel Postulate is true in the nine-point geometry. (Because the Parallel Postulate is true for this geometry, it might be called a finite Euclidean geometry.)

Discussion 2 (page 104)

a. Sample response: Sets of parallel lines have the same values for A and B in their equations.

b.
1. Sample response: The line is horizontal with a slope of 0.
2. Sample response: The line is vertical. The value of the slope is undefined.

c. Sample response: Yes. The value of the slope found using two points is the same as the value of m in an equation of the form $y = mx + b$.

d. Sample response: In both geometries, the slope can be found in the same way:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This means that the slope of a line in both geometries represents the change in y with respect to the change in x.

Teacher Note

You may want to remind students that in the finite geometry, all calculations use mod 3 arithmetic.

Students will require a copy of template B to complete Problem **2.2**. A blackline master appears at the end of the teacher edition for this module.

Assignment (page 105)

2.1 The distinct linear equations are listed in the table given in Part **b** of the exploration. By converting each one into the form $y = mx + b$ or $x = a$, the following 12 equations are obtained.

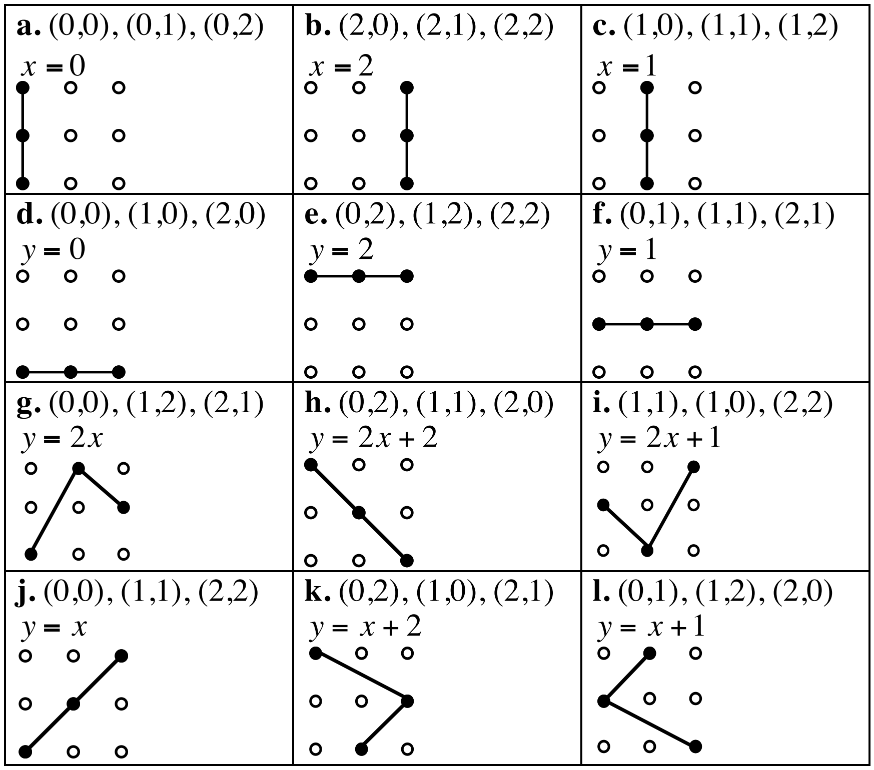
$x = 0$	$y = 0$	$y = x$	$y = 2x$
$x = 1$	$y = 1$	$y = x + 1$	$y = 2x + 1$
$x = 2$	$y = 2$	$y = x + 2$	$y = 2x + 2$

These equations may be derived as shown in the following two examples:

$$1x + 0y + 2 = 0$$
$$1x = -2$$
$$x = 1 \text{ since } -2 \equiv 1 \pmod 3$$

$$1x + 2y + 2 = 0$$
$$2y = -1x - 2$$
$$2(2y) = 2(-1x - 2)$$
$$1y = -2x - 2(2) \text{ since } 2 \bullet 2 \equiv 1 \pmod 3$$
$$y = 1x + 1(2) \text{ since } -2 \equiv 1 \pmod 3$$
$$y = x + 2$$

2.2 Sample chart:



a. (0,0), (0,1), (0,2) $x = 0$	**b.** (2,0), (2,1), (2,2) $x = 2$	**c.** (1,0), (1,1), (1,2) $x = 1$
d. (0,0), (1,0), (2,0) $y = 0$	**e.** (0,2), (1,2), (2,2) $y = 2$	**f.** (0,1), (1,1), (2,1) $y = 1$
g. (0,0), (1,2), (2,1) $y = 2x$	**h.** (0,2), (1,1), (2,0) $y = 2x + 2$	**i.** (1,1), (1,0), (2,2) $y = 2x + 1$
j. (0,0), (1,1), (2,2) $y = x$	**k.** (0,2), (1,0), (2,1) $y = x + 2$	**l.** (0,1), (1,2), (2,0) $y = x + 1$

2.3 **a.** $y = x + 1$

b. The line $y = x + 1$ contains the points (0,1), (1,2), and (2,0).

***2.4** Since there are 9 points, the number of possible sets of 3 points is the combination of 9 points taken 3 at a time, or 84. Since 12 of those sets are lines, this leaves 72 triangles.

2.5 The slopes of parallel lines are comparable in both systems. The slopes of all the lines in the first row of the chart are undefined. Therefore, these lines are parallel. The slopes of all the lines in the second row are 0. Therefore, these lines are parallel. The lines in third row are parallel because their slopes are all 2. Similarly, the lines in the fourth row are parallel because their slopes are all 1. In addition, no two lines in a row contain any common points.

***2.6** **a.** For two lines with non-zero slopes to be perpendicular, the product of their slopes must be 2, since $-1 \equiv 2 \pmod 3$.

b. In the nine-point geometry, all lines with undefined slopes are perpendicular to lines with slopes of 0. All lines with slopes of 1 are perpendicular to lines with slopes of 2.

c. Using proof by exhaustion, these three properties are true in the nine-point geometry.

2.7 **a.** The lines both contain (1,2).

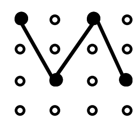
b. The lines both contain (0,0).

2.8 Sample response: No. Two intersecting lines may or may not be perpendicular. For example, using the letters that designate the lines in Problem **2.2**, **a** and **d** intersect at (0,0). Since **a** is vertical and **d** is horizontal, they are perpendicular.

However, **d** and **g** intersect at (0,0). The slope of **d** is 0; while the slope of **g** is 2. Since $0 \bullet 2 \equiv 0 \pmod 3$, and 0 is not congruent to 2 in mod 3, **d** and **g** are not perpendicular.

* * * * *

2.9 **a.** Sample graph:



$y = 2x + 3 \pmod 4$

b. $y = 2x$ or $y + 2x = 0$

c. In this system, there is no line perpendicular to $y + 2x = 0$ since there is no number in mod 4 which can be multiplied by 2 to give a product of -1.

2.10 The following proof by exhaustion uses the definition of perpendicular lines in Problem **2.7**. It is divided into two parts. The first part shows the intersections of horizontal and vertical lines. The second part shows the intersections of non-horizontal and non-vertical lines. The letters designating the lines correspond with the template used in Problem **2.2**.

Horizontal and Vertical Lines			
lines	**a**	**b**	**c**
d	(0,0)	(2,0)	(1,0)
e	(0,2)	(2,2)	(1,2)
f	(0,1)	(2,1)	(1,1)

Non-Horizontal and Non-Vertical Lines			
lines	**j**	**k**	**l**
g	(0,0)	(2,1)	(1,2)
h	(1,1)	(0,2)	(2,0)
i	(2,2)	(1,0)	(0,1)

Since all possible pairs of perpendicular lines have a point of intersection, all perpendicular lines intersect.

* * * * * * * * * *

Research Project (page 107)

From Problem **2.4**, there are 72 possible triangles. By carefully considering the lines that determine the sides of each triangle, students should observe that all are right triangles.

Note: To complete this project, students may require copies of template A. As an alternative, you may wish to distribute copies of template C, which shows representations of all 72 possible triangles. Blackline masters appear at the end of the teacher edition for this module.

(page 107)

Activity 3

In this activity, students explore axiomatic systems based on a finite geometry.

Teacher Note

Students may need help modeling the finite geometries. For this reason, you may wish to use a more directed, whole-class format for the following exploration.

Exploration (page 108)

Students examine Fano's and Young's finite geometries.

a. Sample response: Axiom **1** guarantees that a line exists and Axiom **2** guarantees that the line contains exactly three points. The line can be designated l_1 and the points labeled *A*, *B*, and *C*. Together, the axioms say that there are at least three points in the geometry.

b. Sample response: Considering Axioms **1** and **2**, at least three points exist. Axiom **3** guarantees the existence of at least one more point, *D*, that is not on l_1. The first three axioms guarantee that at least four points are in the geometry.

c. By Axioms **1** and **2**, there is a line and it contains three points. From Axiom **3**, there is at least one other point not on that line. This situation can be represented as shown in the table below.

l_1	l_2					
A	*D*					
B						
C						

From Axiom **4**, lines must exist that contain *A* and *D*, *B* and *D*, and *C* and *D*.

l_1	l_2	l_3	l_4			
A	*D*	*D*	*D*			
B	*A*	*B*	*C*			
C						

But according to Axiom **2**, each line must have three points. These new points cannot be duplicates of those already listed without violating Axiom **4**.

l_1	l_2	l_3	l_4			
A	*D*	*D*	*D*			
B	*A*	*B*	*C*			
C	*E*	*F*	*G*			

From Axiom **4**, each pairing of point *E* with one of the other points determines a line. Since *E* is already paired with *A* and *D* on l_2, that leaves points *B*, *C*, *F*, and *G*. Since *B* and *F* and *C* and *G* have already been paired, this creates two more lines.

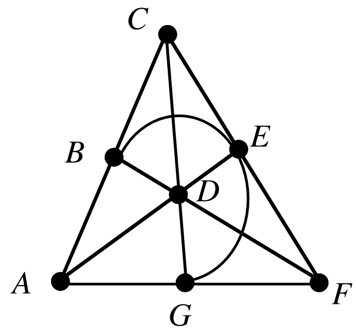
l_1	l_2	l_3	l_4	l_5	l_6	
A	*D*	*D*	*D*	*E*	*E*	
B	*A*	*B*	*C*	*B*	*C*	
C	*E*	*F*	*G*	*G*	*F*	

Similarly, point *F* has not been paired with either *A* or *G*. This creates one more line. Point *G* is now paired with every other point. Since every point introduced is paired with every other point, Axiom ***5f*** is satisfied.

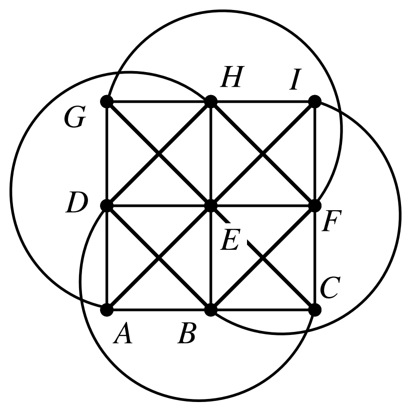
l_1	l_2	l_3	l_4	l_5	l_6	l_7
A	*D*	*D*	*D*	*E*	*E*	*F*
B	*A*	*B*	*C*	*B*	*C*	*G*
C	*E*	*F*	*G*	*G*	*F*	*A*

No more lines can be created without violating an axiom. This system has seven lines and a minimum of seven points. If there was an additional point, however, it could not be on any line because each contains three points already. Therefore, there also cannot be more than seven points.

d. The seven lines in the sample model shown below are: *A–B–C*, *A–D–E*, *B–D–F*, *C–D–G*, *B–E–G*, *C–E–F*, and *A–G–F*. Some students may argue that *B–E–G* is not a line. They should be reminded that a line contains only three points. Arcs and segments are shown here only to help students visualize the model.



e. Students may recognize this as the nine-point geometry with 12 lines discussed in Activity **2**. Sample model:



The twelve lines in the model are *ABC*, *DEF*, *GHI*, *ADG*, *BDH*, *CFI*, *AHF*, *GFB*, *IBD*, *CDH*, *AEI*, and *CEG*.

Discussion

(page 109)

a. Axiom **1** states that a line exists. The axiom says nothing about a line containing any points and does not describe what a line looks like. At this stage in a finite geometry, a line may even look like what is normally considered a point.

b. Sample response: This means that Fano's statements are assumed to be true without proof.

c. Sample response: No. By Axiom **4**, there is only one line that contains any two points.

d. Fano's geometry contains 7 points and 7 lines, while Young's geometry contains 9 points and 12 lines. Some students may observe that parallel lines appear to exist in Young's geometry but not in Fano's. This is proved in Problems **3.2** and **3.4**.

e. A theorem is a statement that must be proven true using logic, other theorems, definitions, and axioms. An axiom is accepted as true without proof.

f. Because the points and lines used in the proof are arbitrary, they can represent any of the points and lines in the geometry. Proving the statement true for arbitrary points allows one to avoid proof by exhaustion.

Assignment (page 110)

3.1 **a.** Sample response: This theorem can be restated as, "If there is a point D, then there is a line not containing point D." The hypothesis is that there is a point D. The conclusion is that there is a line not containing it.

b. **1**. This is the hypothesis and is assumed true.

2. Axiom **3**: Not all points are on the same line.

3. Axiom **5*y***: For each line l and point B not on l, point B is on one line that does not contain any other points from line l.

4. For point D, there is a line l not containing D.

3.2 Lines l_1 and l_2 are lines in Fano's geometry. By Axiom **5*f***, l_1 and l_2 have at least one point in common. By the definition of parallel lines, line l_1 is not parallel to l_2.

3.3 By hypothesis, point D is any point. From Problem **3.1**, there is a line l_1 that does not contain D. By Axiom **2**, l_1 contains exactly three points. By Axiom **4**, point D and each point on l_1 must determine a distinct line. So, there are at least three lines containing point D.

By Axiom **5*y***, there must be a line that contains D but contains no points on l_1. Therefore there are at least four lines containing point D.

Sample sketch:

3.4** By hypothesis, there is a line l. By Axiom **3**, there is a point B not on line l. By Axiom **5*y, there is a line m that contains point B and does not contain any other points of l. Therefore, from the definition of parallel lines, line m is parallel to line l.

3.5 Sample response: Having four points on a line would require that there be more lines and more points. **Note:** Some students may correctly determine that there would have to be at least 16 points.

3.6 Sample response: No. When using real numbers and the Pythagorean theorem, the distance between points with coordinates (0,0) and (1,1) is $\sqrt{2}$. However, this number does not exist in mod 3.

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3.7 Since all even numbers are multiples of 2 and n is even, it must be of the form $n = 2x$ where x is an integer. Squaring both sides of this equation, $n^2 = (2x)^2$. This can be simplified as follows: $n^2 = 4x^2 = 2\left(2x^2\right)$. Because x is an integer, x^2 is an integer and $2x^2$ is an integer. Because n^2 is 2 times an integer, n^2 is even.

3.8 If n is even, then it is of the form $n = 2x$, where x is an integer. Substituting into the expression $n^3 - n$, results in the following:

$$\begin{aligned} n^3 - n &= (2x)^3 - 2x \\ &= 8x^3 - 2x \\ &= 2(4x^3 - x) \end{aligned}$$

Because x is an integer, x^3 is an integer and $4x^3 - x$ is an integer. Therefore, $n^3 - n$ is divisible by 2 and is even.

If n is odd, then it is of the form $n = 2x + 1$, where x is an integer. Substituting into the expression $n^3 - n$, results in the following:

$$\begin{aligned} n^3 - n &= (2x + 1)^3 - 2x + 1 \\ &= 8x^3 + 12x^2 + 6x + 1 - 2x - 1 \\ &= 2(4x^3 + 6x^2 + 2x) \end{aligned}$$

Because x is an integer, x^3 and x^2 are integers and $4x^3 + 6x^2 + 2x$ is an integer. Therefore, $n^3 - n$ is divisible by 2 and is even.

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(page 111)

Activity 4

In this activity, students examine the technique of indirect proof. **Note:** Students may recall this method from the Level 4 module "Believe It or Not."

Exploration

(page 111)

a. If m is not even (or is odd), then it can be written as $m = 2a + 1$, where a is a natural number.

b. Squaring the equation in Part **a** yields the following:

$$\begin{aligned} m^2 &= (2a+1)^2 \\ &= 4a^2 + 4a + 1 \end{aligned}$$

The square $m^2 = n$ is an odd number because it is an even number plus 1.

c.

1. Sample response: No. The result in Part **b** is the negation of the original statement. A statement and its negation cannot both be true.
2. The assumption that m is odd leads to the conclusion that its square is odd. The hypothesis is contradicted since it states that the square is even. Because no mathematical truths were violated in the proof, and because both a statement and its negation cannot be true, the assumption must be false and its negation (the original statement) must be true.

Discussion

(page 112)

a. "The square root of 2 is not an irrational number (or is a rational number)."

b. The truth tables below demonstrate that, when the conditional is true, the negation is false, and vice versa.

p	q	$\sim q$	$p \rightarrow q$	$p \wedge \sim q$
T	T	F	T	F
T	F	T	F	T
F	T	F	T	F
F	F	T	T	F

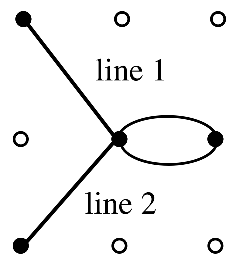
Assignment

(page 113)

4.1 **a.** The hypothesis is "two lines intersect." The conclusion is "two lines intersect in exactly one point."

b. To prove the statement using an indirect proof, assume that the negation is true: "Two lines intersect in more than one point or in no points."

c. Assuming that the lines intersect in no points means that they are parallel. This violates the hypothesis immediately. Assuming that the lines intersect in more than one point might yield the following sketch:



d. The assumption contradicts Axiom **4**: For any two points, there exists exactly one line that contains both of them.

e. Because the assumption contradicts a known fact, the assumption is false.

f. Because its negation is false, the theorem is true.

***4.2** This statement may be written symbolically as follows: If l_1 is parallel to l_2 and l_2 is parallel to l_3, then l_1 is parallel to l_3.

To prove this statement using an indirect proof, assume that l_1 is parallel to l_2 and l_2 is parallel to l_3, but that l_1 is not parallel to l_3.

Since l_1 is parallel to l_2 by hypothesis, l_1 and l_2 have no points in common. Since l_2 is parallel to l_3 by hypothesis, l_2 and l_3 have no points in common. By the assumption and the definition of nonparallel lines, l_1 and l_3 intersect at a common point.

Since l_2 is parallel to l_3 and l_1 intersects l_3, then l_1 must also intersect l_2. (This theorem—"If a line intersects one of two parallel lines, then it intersects the other"—is proven in the mathematics note in Activity **4**.) This leads to a contradiction of the hypothesis, which states that l_1 is parallel to l_2.

The assumption that l_1 is not parallel to l_3 must be false. Therefore, l_1 is parallel to l_3 and the theorem is proven.

4.3 Sample response: Given that l_1 and l_2 are different lines in Fano's geometry, assume that l_1 and l_2 intersect in two points (more than one). Call these points *A* and *B*. This means that *A* and *B* are on l_1 and on l_2. This is a contradiction of Axiom **4**, which states that for any two points, there is exactly one line that contains both of them. Therefore l_1 and l_2 cannot have more than one point in common. Since Axiom **5f** states that any two lines have at least one point in common, then l_1 and l_2 must have exactly one point in common.

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4.4 Sample response: The contradiction in the proof occurs after it has been proven that both *p* and *q* are even. Since they are both even, they can both be divided by 2. This means that p/q is not in lowest terms as was assumed in the proof. The assumption that $\sqrt{2}$ is rational must be false. Therefore, $\sqrt{2}$ is an irrational number.

4.5 Sample response: Assume that the cash register contains \$1.45 in nickels and dimes, and an even number of nickels.

Let *n* represent the number of nickels. If *n* is even, then $n = 2k$, where *k* is a whole number. If *d* is a whole number representing the number of dimes, then the total value of nickels and dimes is:

$$\begin{aligned} 2k(\$0.05) + d(\$0.10) &= \$1.45 \\ k(2 \bullet \$0.05) + d(\$0.10) &= \$1.45 \\ k(\$0.10) + d(\$0.10) &= \$1.45 \\ (k + d)(\$0.10) &= \$1.45 \end{aligned}$$

Since *k* and *d* are both whole numbers, $(k + d)$ must be a whole number and \$1.45 would have to be a multiple of \$0.10. This leads to a contradiction, since \$1.45 is not a multiple of \$0.10. The assumption that there was an even number of nickels must be false. Therefore, its negation must be true: "If the cash register contains \$1.45 in nickels and dimes, there is an odd number of nickels."

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Answers to Summary Assessment (page 115)

1. **a.** Sample tables:

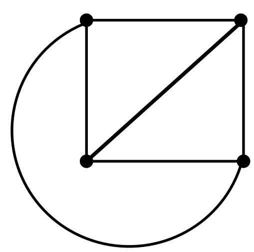
+	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	**0**	**1**	**2**	**3**
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

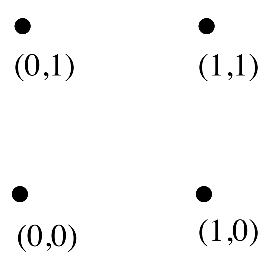
b. The additive identity in mod 4 is 0, because $x + 0 \equiv x \pmod 4$ for all x. Additive inverses are elements whose sum is congruent to the additive identity (0). For example, $3 + 1 \equiv 0 \pmod 4$, so 3 and 1 are additive inverses of each other. Each element in the system has an additive inverse.

The multiplicative identity for the system is 1, for reasons comparable to those for the additive identity. However, not every number in the system has a multiplicative inverse. For example, 2 does not have a multiplicative inverse in modulo 4.

2. **a.** Sample model:



b. Sample graph:



c. Sample tables:

+	**0**	**1**
0	0	1
1	1	0

×	**0**	**1**
0	0	0
1	0	1

d. The equations of the six lines in this four-point geometry, given in the form $Ax + By + C = 0$, are as follows: $1x + 0y + 0 = 0$, $1x + 0y + 1 = 0$, $1x + 1y = 0$, $1x + 1y + 1 = 0$, $0x + 1y + 0 = 0$, and $0x + 1y + 1 = 0$. **Note:** Because *A* and *B* cannot both be 0, two other possible equations, $0x + 0y + 0 = 0$ and $0x + 0y + 1 = 0$, do not apply in this situation.

e. To prove that the system has at least three pairs of parallel lines, students may identify three pairs of lines with no points in common. For example, the lines described by the following equations are parallel: $x = 0$ and $x = 1$; $y = 0$ and $y = 1$; and $y = x$ and $y = x + 1$.

f. Sample response: Let *A* represent an arbitrary point in the system. By Axiom **1**, there are three other points in the system: *B*, *C*, and *D*. Axioms **2** and **3** guarantee that there is exactly one line that contains point *A* and point *B*, exactly one line that contains point *A* and point *C*, and exactly one line that contains point *A* and point *D*. Therefore, point *A* is contained in at least three lines.

3. Sample response: Assume *n* is an integer, n^2 is odd, and *n* is even. If *n* is even, then $n = 2k$, where *k* is an integer. Squaring both sides of the equation, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

This implies that n^2 is even, which contradicts the assumption. Therefore, if *n* is an integer and n^2 is odd, then *n* is also odd.

Module Assessment

1. Imagine that you are the director of the Twirlers, a children's jump rope club. You will be taking a select number of teams from the Twirlers to a tournament in a neighboring state. The entry rules for the tournament are described below. Each club must satisfy these rules.

1. There exists at least one team.
2. Every team has exactly three jumpers on it.
3. Not all jumpers are on the same team.
4. For two jumpers, there exists exactly one team that contains both of them.
5. Every two teams have at least one jumper in common.

a. Make a diagram that shows your club's requirements for entry in the tournament.

b. Determine the minimum number of jumpers you can enter in the tournament.

c. Prove that your response to Part **b** also represents the maximum number of jumpers.

d. Determine the minimum number of teams you can enter in the tournament.

e. Prove the following statement: "Every two distinct teams have exactly one jumper in common."

2. Prove or disprove the following statement: "In mod 3, $x^2 \equiv 1$ or $x^2 \equiv 0$."

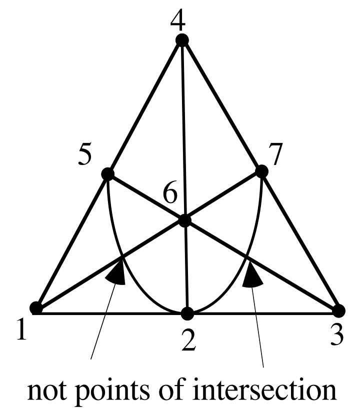
3. Consider the nine-point geometry defined on the set {0, 1, 2} and its coordinate system modulo 3. In this geometry, a circle is defined as the set of all points (x,y), where both *x* and *y* are elements of {0, 1, 2}, that satisfy the standard equation of a circle, $(x-h)^2 + (y-k)^2 = r^2$, where *h*, *k*, and *r* are elements of {0, 1, 2}. The point (h,k) is the center of the circle and *r* is the radius of the circle.

a. Draw all possible circles in this finite geometry.

b. Determine the intersection of the circle $x^2 + y^2 = 1^2$ and the line $y = 2x$. Describe your results.

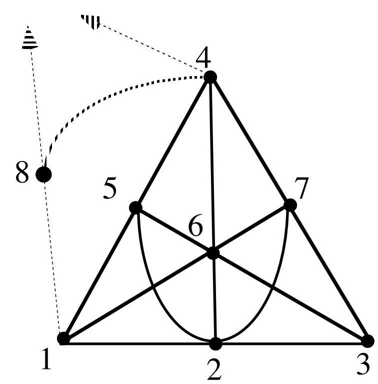
Answers to Module Assessment

1. **a.** In the following sample diagram, each point represents a jumper and each line represents a team:



b. The club must enter seven jumpers in the tournament. The proof is essentially the same as that for the minimum number of points in Fano's geometry. (See Part **c** of the exploration in Activity **3**.)

c. Sample response: Suppose that a coach decides to bring an eighth jumper. Is this more than the maximum number? Consider, as shown in the following figure, the team containing jumpers 1 and 8, and the team consisting of jumpers 3, 7, and 4.



Rule **5** requires that the team containing jumpers 1 and 8 and the team containing jumpers 3, 7, and 4 have a jumper in common. The common jumper required by Rule **5** cannot be jumper 3, 7, or 4, since that would violate Rule **4**. Thus, it must be a ninth jumper, but that violates Rule **2**. The original assumption that there is an eighth jumper has led to a contradiction; therefore, the assumption is false. The tournament team must have exactly seven jumpers.

d. The minimum number of teams that can be entered in the tournament is 7. (This relates to the seven lines of Fano's geometry).

e. By the hypothesis, there are two distinct teams l_1 and l_2. By Rule **5**, teams l_1 and l_2 share one or more common jumpers. There are four possibilities:

1. Teams l_1 and l_2 share three common jumpers. By Rule **2**, every team has exactly three jumpers. Therefore, teams l_1 and l_2 are the same team. This is false, since it is given that teams l_1 and l_2 are distinct.

2. Teams l_1 and l_2 share two common jumpers. By Rule **4**, any two jumpers determine exactly one team. Therefore, teams l_1 and l_2 are the same team. This is false since it is given that teams l_1 and l_2 are distinct.

3. Teams l_1 and l_2 share no common jumper. This violates Rule **5** because every two teams must share one jumper.

4. Teams l_1 and l_2 share one common jumper. Since possibilities **1–3** are all false, this must be true.

Therefore, two distinct teams have exactly one jumper in common.

2. Using proof by exhaustion, this statement is true:

$$0 \bullet 0 \equiv 0 \pmod 3$$
$$1 \bullet 1 \equiv 1 \pmod 3$$
$$2 \bullet 2 \equiv 1 \pmod 3$$

Note: You may also wish to point out that $-2 \equiv 1 \pmod 3$ and $-1 \equiv 2 \pmod 3$. Therefore, $(-2)^2 \equiv (-1)^2 \equiv 1 \pmod 3$.

3. **a.** The general equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$. In mod 3, there are three different cases to consider, $r = 0,\ 1,\ \text{or } 2$. From Problem **2**, however, the cases when $r = 1 \text{ or } 2$ reduce to the same case because $2^2 \equiv 1^2 \equiv 1 \pmod 3$.

When $r = 0$, there are nine circles of radius 0. These circles are the nine points in the coordinate system given by (h,k) where $h,k \in \{0,1,2\}$.

When $r = 1$ or 2, $r^2 = 1$. The equations for these circles are of the form $(x-h)^2 + (y-k)^2 = 1$. The following table shows the nine possible circles, where the symbol • represents a point on the circle and the symbol ■ represents the center of the circle.

h	***k***	**Center (*h*,*k*)**	**Equation**	**Points on Circle**	**Graph**
0	0	(0,0)	$(x)^2 + (y)^2 = 1$	{(0,1), (0,2), (1,0), (2,0)}	● ○ ○ ● ○ ○ ■ ● ●
0	1	(0,1)	$(x)^2 + (y-1)^2 = 1$	{(0,0), (0,2), (1,1), (2,1)}	● ○ ○ ■ ● ● ● ○ ○
0	2	(0,2)	$(x)^2 + (y-2)^2 = 1$	{(0,1), (0,2), (2,1), (2,2)}	■ ● ● ● ○ ○ ● ○ ○
1	0	(1,0)	$(x-1)^2 + (y)^2 = 1$	{(0,0), (0,2), (1,1), (1,2)}	○ ● ○ ○ ● ○ ● ■ ●
1	1	(1,1)	$(x-1)^2 + (y-1)^2 = 1$	{(0,1), (1,0), (1,2), (2,1)}	○ ● ○ ● ■ ● ○ ● ○
1	2	(1,2)	$(x-1)^2 + (y-2)^2 = 1$	{(0,2), (1,0), (1,1), (2,2)}	● ■ ● ○ ● ○ ○ ● ○
2	0	(2,0)	$(x-2)^2 + (y)^2 = 1$	{(0,0), (0,1), (2,1), (2,2)}	○ ○ ● ○ ○ ● ● ● ■
2	1	(2,1)	$(x-2)^2 + (y-1)^2 = 1$	{(0,1), (1,1), (2,0), (2,2)}	○ ○ ● ● ● ■ ○ ○ ●
2	2	(2,2)	$(x-2)^2 + (y-2)^2 = 1$	{(2,0), (2,1), (2,0), (2,1)}	● ● ■ ○ ○ ● ○ ○ ●

b. The circle $x^2 + y^2 = 1$ has center (0,0) and is the set $\{(0,1),(0,2),(1,0),(2,0)\}$. The line $y = 2x$ is the set $\{(0,0),(1,2),(2,1)\}$. The line does not intersect the circle.

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Flashbacks

Activity 1

1.1 Determine the time at each of the following:

a. 6 hours after 11:00 A.M.

b. 7 hours before 3:00 P.M.

c. 14 hours after 6:00 A.M.

1.2 What are the additive and multiplicative identities for the set of real numbers?

1.3 Find the additive and multiplicative inverses of each number below.

a. 4

b. –3

Activity 2

2.1 Determine the equation of the line containing the point (5,0) with a slope of –2.

2.2 Graph a set of parallel lines and write the equations of the lines.

2.3 Solve for x in each of the following:

a. $13 \equiv x \pmod 3$

b. $4x + 1 \equiv 2 \pmod 7$

c. $x \bullet 2 \equiv 4 \pmod 5$

Activity 3

3.1 Consider the following theorem: "All integers divisible by 2 are even."

a. Write the theorem as a conditional statement.

b. Identify the hypothesis and the conclusion.

3.2 Prove that every element of set S below is divisible by 3.

$$S = \{3,\ 9,\ 27,\ 81,\ 405\}$$

Activity 4

4.1 Consider the following statement: "If two triangles have equal perimeters, then they have congruent sides." Write the negation of this statement.

4.2 Consider the following statement: "All diagonals of an equilateral convex polygon are congruent."

a. Write this statement as a conditional.

b. Write the negation of the conditional.

4.3 Consider the following statement: "If n is an odd integer, then it can be written in the form $n = 2x + 1$, where x is an integer." Write the negation of this statement.

Answers to Flashbacks

Activity 1

1.1 **a.** 5:00 P.M.

b. 8:00 A.M.

c. 8:00 P.M.

1.2 The additive identity is 0. The multiplicative identity is 1.

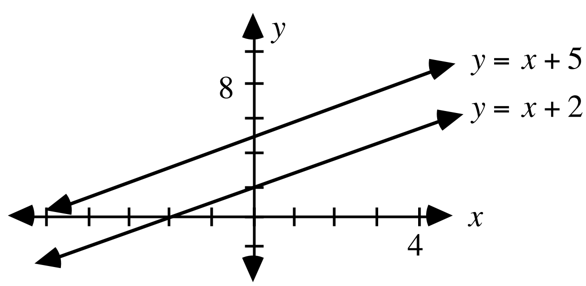
1.3 **a.** –4; 1/4

b. 3; –1/3

Activity 2

2.1 $y = -2x + 10$

2.2 Sample response:



2.3 **a.** 1 (mod 3)

b. 2 (mod 7)

c. 2 (mod 5)

Activity 3

3.1 **a.** Sample response: "If an integer is divisible by 2, then it is even."

b. The hypothesis is "an integer is divisible by 2." The conclusion is "the integer is even."

3.2 Using proof by exhaustion:

$$\frac{3}{3} = 1,\ \frac{9}{3} = 3,\ \frac{27}{3} = 9,\ \frac{81}{3} = 27,\ \frac{405}{3} = 135$$

Activity 4

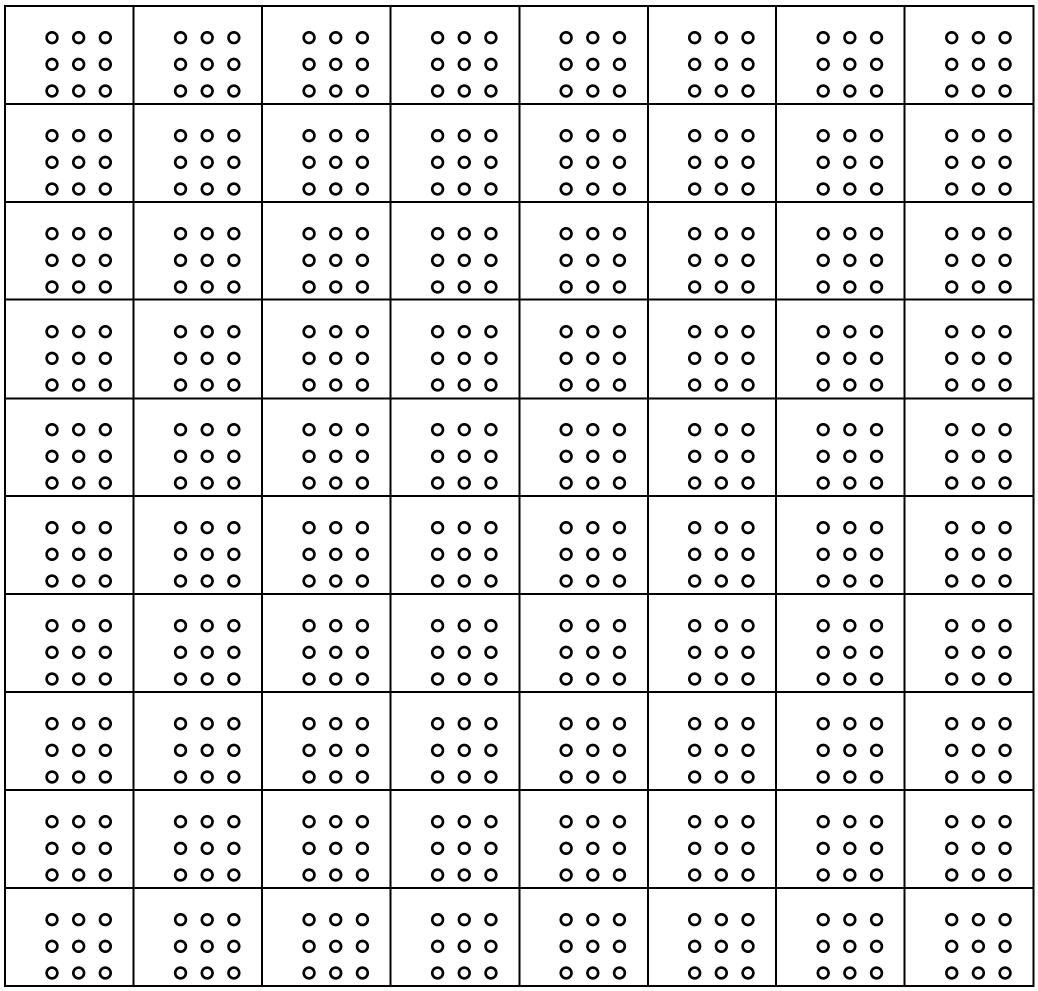
4.1 Sample response: "Two triangles have equal perimeters and they don't have congruent sides."

4.2 **a.** Sample response: "If a convex polygon is equilateral, then its diagonals are congruent."

b. Sample response: "A convex polygon is equilateral and its diagonals are not congruent."

4.3 Sample response: "n is an odd integer and it cannot be written in the form $n = 2x + 1$, where x is an integer."

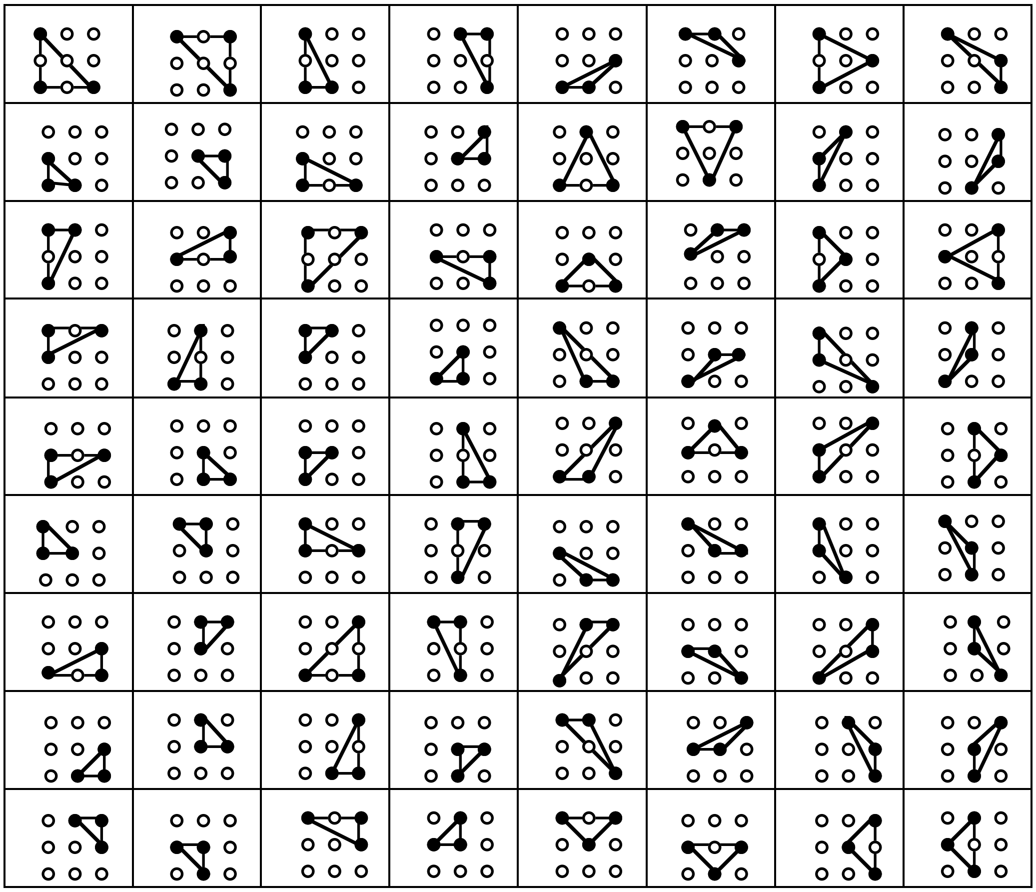
Template A



Template B

a. (0,0), (0,1), (0,2) Equation: Graph: o o o o o o o o o	**b.** (2,0), (2,1), (2,2) Equation: Graph: o o o o o o o o o	**c.** (1,0), (1,1), (1,2) Equation: Graph: o o o o o o o o o
d. (0,0), (1,0), (2,0) Equation: Graph: o o o o o o o o o	**e.** (0,2), (1,2), (2,2) Equation: Graph: o o o o o o o o o	**f.** (0,1), (1,1), (2,1) Equation: Graph: o o o o o o o o o
g. (0,0), (2,1), (1,2) Equation: Graph: o o o o o o o o o	**h.** (2,0), (1,1), (0,2) Equation: Graph: o o o o o o o o o	**i.** (1,0), (0,1), (2,2) Equation: Graph: o o o o o o o o o
j. (0,0), (1,1), (2,2) Equation: Graph: o o o o o o o o o	**k.** (1,0), (2,1), (0,2) Equation: Graph: o o o o o o o o o	**l.** (0,1), (1,2), (2,0) Equation: Graph: o o o o o o o o o

Template C



To Null or Not to Null

Imagine that you are a member of a jury in a criminal trial. What kinds of mistakes are possible in your verdict? In this module, you explore how statistics can help you analyze this situation.

Cliff Bara • Pat Mauch • Lisa Wood

Teacher Edition

To Null or Not to Null

Overview

This module reviews properties of the normal curve and introduces z-scores. Students formulate null and alternative hypotheses and investigate the process of hypothesis testing.

Objectives

In this module, students will:

- review the differences between statistics and parameters
- express null and alternative hypotheses
- use contrapositive logic
- explore characteristics of a normal curve
- examine the 68–95–99.7 rule
- compare individual observations to the mean in terms of standard deviations
- use the central limit theorem to evaluate sample means
- interpret and compare statistics using z-scores
- test null hypotheses using various levels of significance.

Prerequisites

For this module, students should know:

- how to calculate mean and standard deviation
- the definition of the conditional, its negation, and the contrapositive
- properties of normal distributions
- the basic ideas of the central limit theorem.

Time Line

Activity	**1**	**2**	**3**	**Summary Assessment**	**Total**
Days	2	2	3	2	9

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
pennies	X			
other coins (nickels and quarters)	X			
manila coin envelopes	X			
balance	X			
templates A and B (optional)		X		
transparencies (optional)		X		
water-soluble marking pens (optional)		X		

Teacher Note

Blackline masters of templates A and B appear at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
geometry utility		X		
graphing utility		X	X	X
statistics package			X	X

To Null or Not to Null

Introduction (page 121)

Students discuss uncertainty, the use of statistical analysis, and null and alternative hypotheses.

Discussion (page 121)

a.
1. Sample response: The consequences of making a wrong decision are dire. You could set a murderer free or send an innocent person to jail.
2. Sample response: The consequences for the president of the company are not life or death, but may affect corporate profits. A wrong decision could mean monetary losses for the company.
3. Sample response: The potential consequences for the newspaper editor are not necessarily as disastrous as in the first two examples. No one's future or job depends on this decision. However, it would be unethical to publish something that is not supported by facts.

b. Sample response: The juror scenario probably does not involve numbers and therefore cannot be analyzed in this manner. The tire scenario could be analyzed by conducting marketing surveys to decide whether or not increased tread life would result in more sales and more profit. The newspaper editor could decide if the class was significantly above the national mean by considering the standard deviation of the national scores.

c. Sample response: Although there is a chance that the test results are incorrect, this is usually small. Since the risks of not treating the illness can be severe, doctors usually assume that the tests are accurate. If no improvement occurs or other symptoms appear, they may decide to test the patient again.

d.
1. Sample response: No. Tests for many illegal substances are not 100% accurate. For example, use of some legal nonprescription drugs can lead to a positive test for a banned substance.
2. Answers may vary. Sample response: I would expect the committee to behave as if the results were accurate and impose a punishment or fine. It would then be up to the athlete to appeal the ruling.

e.
1. The negation is $\mu \neq 1000$.
2. Answers may vary. Sample response: In this case, consumers would not be concerned if the mean life of the bulbs was greater than 1000 hr. They are only interested in determining if the mean life is less than the advertised value.

f.

1. Answers may vary. One possible null hypothesis is: "There are no illegal concentrations of banned substances in the athlete's blood."
2. Sample response: If the null hypothesis is false, then the test should show positive results for at least one banned substance.

(page 122)

Activity 1

Students model hypothesis testing in a situation in which a population contains some variability.

Materials List

- pennies (18 per group)
- nickels and quarters (one each per group)
- manila coin envelopes (at least 20 per group)
- unknown coin in envelope (penny, nickel, or quarter; at least 1 per group)
- balance (one per group)

Teacher Note

Manila coin envelopes (#1) are typically available at office supply stores. Although other types of envelopes may be used, they should be opaque and of consistent size and weight.

Envelopes containing unknown coins (penny, nickel, or quarter) should be prepared before students begin the exploration. **Note:** Dimes should not be used since their masses may be too close to that of pennies for a difference to be observed using a simple balance.

Exploration

(page 123)

a. Students create 20 "test envelopes" using 18 pennies, 1 nickel, and 1 quarter. **Note:** In order for the pennies to have consistent masses, they should be relatively new and clean.

b. The envelope containing the unknown coin may contain a penny, nickel, or quarter.

1. Sample response: If the unknown coin is a penny, then the two envelopes will balance. The test envelope should contain a penny 90% of the time.
2. Sample response: If the unknown coin is not a penny, then the two envelopes will not balance. The test envelope should contain a nickel or quarter 10% of the time.

c. Students use the balance to test the null hypothesis.

d. Sample response: You can either reject the null hypothesis or fail to reject it. If the unknown coin is a penny, there is a 90% probability that the coins will balance. If they don't balance, I would reject the null hypothesis. If they do balance, I would fail to reject the null hypothesis.

Discussion

(page 123)

a. Answers will vary. Students should use their observations to support their decisions.

b. The probability of choosing a test envelope that contains a penny is $18/20 = 0.9$.

c.
1. Sample response: No. Since all of the test envelopes do not contain pennies, it is possible that the unknown coin is a quarter or nickel.
2. Sample response: No. If the test envelope contained a nickel or quarter, it would not balance with a penny.

d. Sample response: No. Since the population contains variability, any sample of one envelope from that population contains uncertainty.

Teacher Note

Students may recall from previous modules that, in logic, the following argument is accepted: "If A is true, then B is false. B is true. Therefore, A is false." In statistical analysis, however, their reasoning should proceed as follows: "If A is true, then B is unlikely. B occurs. Therefore, I doubt A (in other words, the null hypothesis is rejected)."

Similarly, it can be reasoned logically that "If A is true, then B is false. B is false. Nothing can be said about A." The comparable reasoning in statistical analysis is: "If A is true, then B is unlikely. B does not occur. I have no information about A (in other words, I fail to reject the null hypothesis)."

e.
1. The null hypothesis and the alternative hypothesis have opposite truth values. Therefore, only one of them can be true. Rejecting the null hypothesis results in the acceptance of the alternative hypothesis.
2. Sample response: No. It only means that enough evidence was found to reject the null hypothesis. Suppose, for example, that the two coins in the exploration did not balance. In this case, you would reject the null hypothesis and accept the alternative: "The unknown coin is not a penny." However, there was a 10% chance that the test envelope did not contain a penny.

f. **1.** Sample response: No. As shown in the tree diagram, there are two possible outcomes when failing to reject a null hypothesis: a correct decision or an error. As long as the probabilities on the branches leading to an error are greater than 0, failing to reject the null hypothesis does not prove that it is true.

2. Sample response: No. In the exploration, you would reject the null hypothesis if the two envelopes did not balance. This does not prove that the unknown coin was not a penny, since the test envelope had a 10% chance of containing some other coin.

Assignment

(page 125)

1.1 Sample response: The alternative hypothesis is that the mean salary is less than $10,000.

1.2 Sample response: The null hypothesis is that every customer is satisfied. The alternative hypothesis is that at least one customer is not satisfied.

1.3 Sample response: The only way to guarantee a correct decision would be to observe an entire population. Most decisions are based on statistics taken from samples. Even when the null hypothesis cannot be rejected, there is still some chance that it might be false.

***1.4** Sample response: Since the probability of drawing a test envelope that contains a penny is 0.95, I am reasonably sure that the unknown coin is a penny. However, I cannot be absolutely certain of this conclusion.

* * * * *

1.5 **a.** Sample response: The null hypothesis is that there is no difference in the mean amounts spent on tourism before and after the tax was put in place. The alternative hypothesis is that there is a difference between the mean amounts spent on tourism before and after the tax.

b. Sample response: An error could occur if a true null hypothesis is rejected. In this case, the governor would think that there is a difference in spending on tourism when there actually is none. Another error could occur if a false null hypothesis is not rejected. In this case, the governor would think that there was no difference in spending when there actually was.

1.6 **a.** The alternative hypothesis is: "The proportion of red cards is not 0.5."

b. Sample response: I would reject the null hypothesis if all the cards in the sample were the same color.

c. The probability that a random sample of 6 cards from a standard deck is all red (or black) can be found as follows:

$$2\left(\frac{26}{52}\bullet\frac{25}{51}\bullet\frac{24}{50}\bullet\frac{23}{49}\bullet\frac{22}{48}\bullet\frac{21}{47}\right) \approx 0.022$$

d. Sample response: Since all of the cards in the sample are the same color, it does not seem likely that the null hypothesis is true. Therefore, you should reject H_0. You can be relatively sure of your conclusion, but not completely sure. As shown in Part **c**, there is still about a 2% chance of obtaining such a sample from a standard deck.

* * * * * * * * * *

(page 126)

Activity 2

In this activity, students review properties of normal curves and interpret the areas under normal curves as probabilities. They also calculate *z*-scores and use them to compare individual scores to population means.

Materials List

- templates A and B (optional; one copy per group)
- transparencies (optional; two per group)
- water-soluble marking pens (optional, one per group))

Technology

- graphing utility
- geometry utility

Teacher Note

To simplify the examination of normal curves, only populations that are assumed to be normally distributed are considered in this activity. Other populations will be considered in later activities.

In the following exploration, students use a graphing utility to determine areas under a normal curve. If your technology does not have this capability, you may wish to use one of the following alternative methods.

1. Distribute copies of templates A and B to each group. Ask them to trace the outline of each curve onto a transparency, place the transparency on the screen of a geometry utility, and draw a polygon that approximates the outline on the template. The geometry utility can then be used to determine the area of the polygon.

2. Program a graphing calculator to determine the appropriate area. For example, the following program for the TI-92 calculator determines the area under the curve in the interval defined by the number of standard deviations above and below the mean. It requires the input of the mean (m), standard deviation (s) of the population, the number (a) of standard deviations above the mean, and the number (b) of standard deviations below the mean.

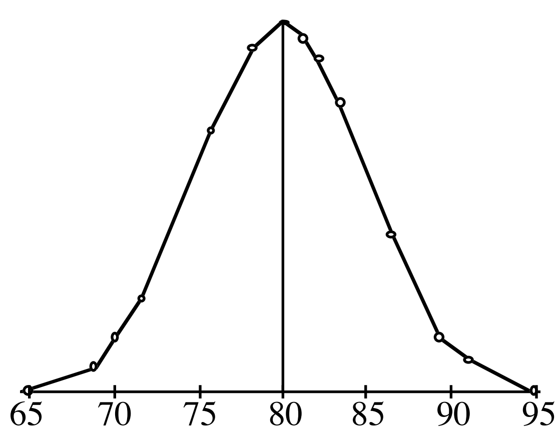
```
:area()
:Prgm

:ClrIO
:Input "Enter mu: ",m
:Input "Enter sigma: ",s
:Input "Enter # of standard deviations above mean: ",a
:Input "Enter # of standard deviations below mean: ",b
:1/(s*(2*π)^.5)*e ^(–.5((x-m)/s)^2)∅y1(x)
:ClrIO
:Disp "The area between ",m-b*s,"and ",m+a*s,"is ",
:nInt(y1(x),x,m-b*s,m+a*s)
:EndPrgm
```

Exploration

(page 127)

a. Students substitute $\mu = 80$ and $\sigma = 5$ into the general equation for a normal curve.

b.
1. Students should draw a vertical segment at $x = 80$.
2. The segment drawing feature of a geometry utility can be used to construct a polygon along the edge of the normal curve. A sample polygon with 15 vertices is shown below. The percentage of area to the left of the mean is approximately 50%. (In template A, the area to the left of the mean is about 13 cm^2, while the area under the entire curve is about 26 cm^2.)



c. **1.** Students construct vertical segments at $x = 75$ and $x = 85$.

2. Approximately 68% of the area falls between these segments. (In template A, this area is about 17.7 cm^2.)

d. **1.** Students construct vertical segments at $x = 70$ and $x = 90$.

2. Approximately 95% of the area falls between these segments. (In template A, this area is about 24.7 cm^2.)

e. Students repeat Parts **a–d** using two different values for σ and compare the results. They should find that the percentage of the area under the curve remains the same in each case, regardless of the value of the standard deviation. The shape of the graph, however, does change: the greater the standard deviation, the flatter the curve.

f. Students repeat Parts **a–d** using two different values for μ and compare the results. They should find that, for a given value of σ, the shape of the curve remains the same regardless of the value of the mean. The location of the graph, however, does change: the greater the mean, the farther the curve is located to the right.

g. Students substitute $\mu = 72$ and $\sigma = 3$ into the general equation for a normal curve. The percentage of the area to the left of the segment at $x = 72$ should be approximately 50%; the percentage of the area between the segments at $x = 69$ and $x = 75$ should be approximately 68%; and the percentage of the area between the segments at $x = 66$ and $x = 78$ should be approximately 95%. (In template B, the area of the entire polygon is about 28 cm^2. The area to the left of $x = 72$ is about 14 cm^2, the area between $[\mu - 1\sigma, \mu + 1\sigma]$ is about 19 cm^2, and the area between $[\mu - 2\sigma, \mu + 2\sigma]$ is about 26.6 cm^2.)

Discussion

(page 127)

a. The use of a continuous curve assumes that every real number along the *x*-axis is a possible score on the test. In reality, this is not true. In this case, a continuous curve is being used to model a discrete set of data.

b. Sample response: For greater values of σ, the normal curve is wider. For lesser values of σ, the normal curve is narrower.

c. **1.** About 68% of the area lies in the interval $[\mu - 1\sigma, \mu + 1\sigma]$, regardless of the value of σ.

2. About 95% of the area lies in the interval $[\mu - 2\sigma, \mu + 2\sigma]$, regardless of the value of σ.

d. Sample response: The shape of the normal curve remains the same, but its location along the *x*-axis changes as the value of μ changes. For greater values of μ, the graph is located farther from the origin.

e. Sample response: Each percentage represents the probability that an individual score from the population of English tests falls in that particular interval.

f. Each percentage given in the 68–95–99.7 rule represents the probability that a score falls in a particular interval. Therefore, the probabilities can be estimated as follows.

1. Since this describes half the area within 1 standard deviation of the mean, $0.68/2 = 0.34$.
2. Since this describes half the area within 2 standard deviations of the mean, $0.95/2 = 0.475$.
3. Since this describes the area that is not within 2 standard deviations of the mean, $1 - 0.95 = 0.05$.

g. Since the sum of the probabilities of all possible outcomes is 1 and all possible outcomes are represented on the x-axis, the area under the curve also must be 1.

h. Sample response: Curve **3** has the greatest standard deviation because it has the greatest spread.

i. The number of standard deviations above or below the mean can be determined by subtracting μ from x, then dividing by σ. (This is the z-score described in the mathematics note.)

j. Sample response: Yes. The value of z would be negative when the score is below the mean.

k. Dena's score of 610 is approximately 0.65 standard deviations above the mean. From the table at the back of the student edition, this corresponds to an area under the normal curve of 0.7422. This means that Dena outscored about 74.2% of the 32 seniors, or approximately 24 seniors.

l.

1. The score of 76 in set B is better because it is 1.33 standard deviations above the mean. The score of 76 in set A is only 1.2 standard deviations above the mean.
2. In set A, about 88.5% of the scores are less than 76. In set B, about 90.8% of the scores are less than 76. This means that it was more difficult to get a 76 in set B than in set A. Therefore, the score in set B is better.

m.

1. In this case, the z-score is –1.75. From the table, this corresponds with a probability of approximately 0.0401.
2. The z-score is –2.20. From the table, this corresponds with a probability of approximately 0.0139.
3. The z-score is 3.2. This corresponds with a probability of about 0.9993.

Assignment

(page 131)

2.1 **a.** $(100-99.7)/2 = 0.15\%$

b. $(100-95)/2 = 2.5\%$

c. $(100-68)/2 = 16\%$

d. 50%

e. $50 + 68/2 = 84\%$

f. $50 + 95/2 = 97.5\%$

g. $100-(50+34) = 16\%$

h. $100-95 = 5\%$

2.2 Using the formula for z-score,

$$z = \frac{(\mu - 1.25\sigma) - \mu}{\sigma}$$
$$= \frac{-1.25\sigma}{\sigma}$$
$$= -1.25$$

From the table, the area between the x-axis and a normal curve for $x < \mu - 1.25\sigma$ is about 0.1056.

2.3 **a.** **1.** 0.0013

2. 0.1587

3. 0.9772

b. **1.** $1-0.0228 = 0.9772$

2. 0.5

3. $1-0.8413 = 0.1587$

4. $1-0.9987 = 0.0013$

***2.4** Sample response: Using the mean and standard deviation for each exam, Dena's score is better because she scored farther above the mean than Rolf did. For Rolf's score, $z = 0.4$. For Dena's score, $z \approx 1.67$. Using the percentage of students who scored lower than each student, Dena's score is also better since she scored better than about 95.3% of the students, while Rolf scored better than 65.5% of the students.

***2.5** **a.** Using the table of z-scores, the probability is 0.0359.

b. Using the table of z-scores, the probability is 0.8849.

c. Since these values represent scores within 1 standard deviation of the mean, the probability is 0.68.

d. Using the table of *z*-scores, the probability is $0.9861 - 0.2119 = 0.7742$.

e. Using the table of *z*-scores, the probability is $1 - 0.7881 = 0.2119$.

* * * * *

***2.6** **a.** Approximately 57.1% of students scored lower than 500 on the exam.

b. About 448,075 (or 42.9% of 1,044,465) scored better than 500.

c. A candidate must score at least 639 to be considered for the scholarship. This can be determined by finding the table value that approximates 0.900 (or 90%). The closest table value is 0.9015, which corresponds to a *z*-score of 1.29. The test score that corresponds to a *z*-score of 1.29 is found by solving for *x* in the following equation: $1.29 = (x - 478)/125$.

2.7 Answers may vary. In any case in which a score of 90 is below the mean, while a score of 47 is above the mean, the score of 47 might be considered better.

***2.8** **a.** **1.** $1 - 0.7486 = 0.2514$

2. $1 - 0.9996 = 0.0004$

3. $1 - 0.0918 = 0.9082$

4. $1 - 0.2514 = 0.7486$

b. Answers will vary, depending on students' interpretation of the term *significant*. Sample response: Maeve scored better than approximately 75% of the students. Since Maeve did this, I believe she performed significantly better than others who took the exam.

2.9 Assuming that the temperatures are normally distributed, the range of values is $[37 - 2(0.23),\ 37 + 2(0.23)]$, or [36.54, 37.46].

* * * * * * * * * *

(page 133)

Activity 3

In this activity, students review the central limit theorem, use *z*-scores to compare a sample mean to a population mean, then conduct hypothesis tests.

Materials List

- none

Technology

- statistics package (optional)

Teacher Note

To simplify the discussion, small samples from populations that are not normally distributed are not considered in this activity. If σ is not known, the sample standard deviation s is used as an estimate of σ. When the sample size is small ($n < 30$) and σ is not known, z-scores should not be used.

Discussion 1

(page 134)

a. Since $\bar{x}$ is the mean of one sample, it is a statistic. Since μ is the mean of the population, it is a parameter. Since $\mu_{\bar{x}}$ is the mean of all the sample means, it is also a parameter. When the population is normally distributed, then $\mu_{\bar{x}} = \mu$. When the sample size is greater than or equal to 30, then $\bar{x}$ may be used to approximate μ.

b. Since σ is the standard deviation of the population, it is a parameter. Since $\sigma_{\bar{x}}$ is the standard deviation of all the sample means, it is also a parameter.

When the population is normally distributed, then $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. In this case, σ is $\sqrt{n}$ greater than $\sigma_{\bar{x}}$. As sample size increases, the standard deviation of all the sample means decreases.

c.
1. Yes. The 68–95–99.7 rule can be used to analyze the sample mean because the sample size is greater than 30.
2. The 68–95–99.7 rule should be used to analyze the sample mean of 20 tests only if this population is normally distributed.

d.
1. The probability that $\bar{x}$ is more than 1.6 standard deviations below the mean (the area to the left of $z = -1.6$) is 0.0548 or 5.48%
2. The probability that $\bar{x}$ is more than 2.3 standard deviations above the mean (the area to the right of $z = 2.3$) is $1 - 0.9893 = 0.0107$ or 1.07%.

Exploration

(page 134)

a. The null hypothesis is that this year's seniors did no better than last year's seniors, or H_0: $\mu \le 478$.

The alternative hypothesis is that this year's seniors did better than last year's seniors, or H_a: $\mu > 478$.

b. The corresponding z-score can be determined as follows:

$$z_{\bar{x}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{529 - 478}{125/\sqrt{32}} \approx 2.31$$

c. At 0.05 significance level, the critical region is defined by $z > 1.64$.

d. Since the z-score of 2.31 falls in the critical region, Dena should reject the null hypothesis and accept the alternative hypothesis.

e. Based on the results of the hypothesis test, Dena can claim, with 95% certainty, that this year's seniors performed significantly better than last year's seniors.

Discussion 2

(page 136)

a. Sample response: The formulas are similar. Each requires a comparison with μ in the numerator. For a single observation, x is compared to μ, while for a sample, $\bar{x}$ is compared to μ. Each formula also requires a standard deviation in the denominator. For a single observation, the formula uses σ, while for a sample mean, the formula uses $\sigma/\sqrt{n}$.

b. Sample response: Dena wanted to make a claim that the seniors did significantly better. In this case, the null hypothesis would be that they did no better. Rejection of the null hypothesis would lead to the acceptance of the alternative hypothesis (Dena's claim).

c. Sample response: No. Failing to reject the null hypothesis does not necessarily mean that it is true. It only means that there is not enough evidence to reject it.

d. Since the z-score of the observation is positive, the critical region is $z > 1.64$. If the z-score had been negative, the critical region would have been $z < -1.64$.

e.

1. The graph of the critical region would show two shaded areas: $z < -2$ and $z > 2$.
2. If the z-score falls in either shaded area, the null hypothesis should be rejected at the 0.05 significance level.

f. Sample response: Whenever an error could lead to very serious consequences, the researcher would want a very small margin for error, or significance level.

g. Sample response: The decision to reject the null hypothesis was based on the fact that the z-score is in the critical region.

h. Sample response: Rejecting the null hypothesis means that the alternative hypothesis is accepted. Therefore, Dena can claim, with 95% certainty, that this year's seniors performed better on the SAT than last year's seniors.

Assignment (page 137)

3.1 **a.** In this situation, H_0: $\mu \leq 478$ and H_a: $\mu > 478$.

b. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{516 - 478}{125 / \sqrt{100}} = 3.04$$

c. For a 0.10 significance level, the critical region is $z > 1.28$. Since the z-score is in the critical region, the null hypothesis should be rejected.

d. Sample response: The governor can claim, with 90% certainty, that the Montana students who took the mathematics SAT scored significantly better than students nationally.

***3.2** **a.** In this situation, H_0: $\mu \leq 516$ and H_a: $\mu > 516$.

b. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{529 - 516}{110 / \sqrt{32}} \approx 0.67$$

c. For a 0.05 significance level, the critical region is $z > 1.64$. Since the z-score is not in the critical region, the null hypothesis should not be rejected.

d. Sample response: The parent cannot claim, with 95% certainty, that students in this class did significantly better on the SAT than students statewide.

3.3 **a.** $\bar{x} = 528$; $s = 68$

b. For this situation, H_0: $\mu \leq 478$ and H_a: $\mu > 478$. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{528 - 478}{68 / \sqrt{40}} \approx 4.65$$

For a 0.01 significance level, the critical region is $z > 2.33$. Therefore, the null hypothesis should be rejected. A claim can be made, with 99% certainty, that this class did significantly better than the rest of the nation.

***3.4** **a.** For this situation, H_0: $\mu = 72$ and H_a: $\mu \neq 72$.

b. The z-score can be calculated as follows:

$$z = \frac{77 - 72}{3} \approx 1.67$$

From the table, the probability that a sample mean is more than 1.67 standard deviations above the population mean is 0.047. Since this probability is very low, it is unlikely that these four tests came from the physics class with a mean of 72.

3.5 **a.** For the reader's claim, H_0: $\mu \geq 168$ and H_a: $\mu < 168$.

b. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{164.5 - 168}{16.2/\sqrt{100}} \approx -2.16$$

c. For a 0.10 significance level, the critical region is $z < -1.28$. Since the z-score falls in the critical region, the null hypothesis should be rejected.

d. The reader can make the statement, with 90% certainty, that the actual mean height of females between the ages of 19 and 32 is less than 168 cm.

* * * * *

***3.6** In this situation, H_0: $\mu \geq 1500$ and H_a: $\mu < 1500$. The z-score of the statistic can be calculated as follows:

$$z_{\bar{x}} = \frac{1488 - 1500}{47.5/\sqrt{48}} \approx -1.75$$

For a 0.10 significance level, the critical region is $z < -1.28$. Since the z-score falls in the critical region, the null hypothesis should be rejected. This implies that the company may be cheating its customers.

3.7 **a.** For this situation, H_0: $\mu \geq 397$ and H_a: $\mu < 397$.

b. The critical region for this hypothesis test, at the 0.05 significance level, is $z < -1.64$.

c. For the sample data, $\bar{x} \approx 394$ with a standard deviation of 9. The z-score of the statistic can be calculated as follows:

$$z_{\bar{x}} = \frac{394 - 397}{9/\sqrt{40}} \approx -2.11$$

d. Since the z-score falls in the critical region at the 0.05 significance level, the null hypothesis should be rejected.

e. The students can conclude, with 95% certainty, that the mean mass of cereal in each box is less than 397 g.

3.8 **a.** For the population, $\mu = 75$ and $\sigma = 10$. Since the load limit is 1200 kg, this means that for any sample of 15 people to ride the elevator: $\bar{x} \leq 1200/15$, or 80 kg. This value can be converted to a z-score as follows:

$$z_{\bar{x}} = \frac{80 - 75}{10/\sqrt{15}} \approx 1.94$$

From the table, the probability that $\bar{x} \leq 80$ is 0.9738. Therefore, the probability that a group of 15 will exceed the load limit is $1 - 0.9738 = 0.0262$.

b. For 16 people, $\bar{x} \leq 1200/16$, or 75 kg. This value can be converted to a z-score as follows:

$$z_{\bar{x}} = \frac{75 - 75}{10/\sqrt{16}} = 0$$

From the table, the probability that $\bar{x} \leq 75$ is 0.5000. There is a 50% chance that the elevator will be overloaded.

* * * * * * * * * *

Answers to Summary Assessment (page 140)

Students should conduct a complete hypothesis test. Responses should include a sample statistic, null and alternative hypotheses, the selected significance level, the critical region, a decision to reject or fail to reject the null hypothesis, and an interpretation of the results.

For the sample data, $\bar{x} \approx 1.76$ with a standard deviation of 0.31.

The null and alternative hypotheses are $H_0: \mu = 1.618$ and $H_a: \mu \neq 1.618$. Students may select a significance level of 0.01, 0.05, or 0.10. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{1.76 - 1.618}{0.31/\sqrt{30}} \approx 2.51$$

For a 0.10 significance level, the critical region is $z < -1.64$ or $z > 1.64$; for a 0.05 significance level, the critical region is $z < -1.96$ or $z > 1.96$; and for a 0.01 significance level, the critical region is $z < -2.57$ or $z > 2.57$. Since the z-score is in the critical region at both the 0.10 and 0.05 levels of significance, the null hypothesis should be rejected at these levels. However, the null hypothesis should not be rejected at the 0.01 significance level. This implies, with 90% and 95% (but not 99%) certainty, that the beaded rectangles are not golden rectangles. However, this does not prove that the proportions of the Crow Indian designs are not in the golden ratio.

Module Assessment

1. The heights of plants of a certain species are normally distributed with a mean of 50 cm and a standard deviation of 5 cm. In a sample of 1000 plants, how many would you expect to be shorter than 45 cm?

2. In the United States, the average heights of adult males are normally distributed with a mean of 154 cm and a standard deviation of 6 cm. In a random sample of 500,000 adult males, about how many men would you expect to be:

a. shorter than 145 cm?

b. taller than 162 cm?

c. between 145 and 162 cm tall?

3. A motorcycle manufacturer claims that its motorcycles consume fuel at a mean rate of 20 km/L, with a standard deviation of 1.6 km/L. The marketing department, however, is concerned that the actual mean may lie below this figure.

Before using the claim in a new advertising campaign, the marketing department asks the quality control manager to collect data on the fuel consumption of a random sample of 50 motorcycles. The mean for this sample is 19.6 km/L.

a. State the null and alternative hypotheses for this situation.

b. Find the z-score for the sample statistic.

c. Decide whether to reject or fail to reject the null hypothesis. Justify your reasoning.

d. Explain what your decision means to the marketing department.

Answers to Module Assessment

1. In a sample of 1000 plants, approximately 16% (or 160), can be expected to be shorter than 45 cm.

2. **a.** The z-score for 145 is –1.5. Approximately 6.7% of the male population, or 33,500 men, should be shorter than 145 cm.

b. The z-score for 162 cm is 1.33. Approximately 9.2% of the male population, or 46,000 men, should be taller than 162 cm.

c. Approximately $100 - (6.7 + 9.2) = 84.1\%$ of the male population, or 420,500 men, should be between 145 and 162 cm in height.

3. **a.** For this situation, $H_0 : \mu \geq 20$ and $H_a : \mu < 20$.

b. The z-score can be calculated as follows:

$$z_{\bar{x}} = \frac{19.6 - 20}{1.6/\sqrt{50}} = -1.77$$

c. At a 0.10 significance level, the critical region is $z < -1.28$; at a 0.05 significance level, the critical region is $z < -1.64$; at a 0.01 significance level, the critical region is $z < -2.33$. Since the z-score is in the critical regions for the 0.05 and 0.10 levels of significance, the null hypothesis should be rejected at these levels. However, the null hypothesis should not be rejected at the 0.01 level.

d. Using the 0.05 or 0.10 levels of significance, students should report that the marketing department has sufficient evidence to reject the claim that the rate of fuel consumption is greater than or equal to 20 km/L. Therefore, they should not advertise that the motorcycles get 20 km/L.

Students who used a 0.01 significance level should report that the marketing department has insufficient evidence to reject the claim that the rate of fuel consumption is greater than or equal to 20 km/L. Therefore, the marketing department could advertise that the motorcycles get 20 km/L.

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Sincich, T. *Statistics by Example*. San Francisco: Dellen Publishing Co., 1987.

Sokal, R. R., and R. F. James. *Biometry*. San Francisco: W. H. Freeman and Co., 1969.

Travers, K. J., W. F. Stout, J. H. Swift, and J. Sextro. *Using Statistics*. Menlo Park, CA: Addison-Wesley, 1985.

Triola, M. F. *Elementary Statistics*. Reading, MA: Addison-Wesley, 1992.

Flashbacks

Activity 1

1.1 Write the negation of each of the following statements.

a. The sky is blue.

b. The statement is true.

c. $a = b$

d. $a > b$

e. $a \leq b$

1.2 Which of the following statement(s) has the same meaning as the conditional, "If it is raining, then I wear a raincoat"?

a. If it is not raining, then I do not wear a raincoat.

b. If I do not wear a raincoat, then it is not raining.

1.3 Determine whether each situation described below involves statistics or parameters. Explain your responses.

a. A consumer magazine asks randomly selected readers to complete a questionnaire.

b. A teacher finds the mean score of a test given in her class.

c. The Bureau of the Census reports the population density of the United States.

Activity 2

2.1 Consider the following set of numbers: {1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 8, 8, 8, 9}.

a. Determine $\bar{x}$ for the set.

b. Determine σ for the set.

c. Identify the relative frequency of each number in the set.

d. Determine the sum of all the relative frequencies from Part **c**.

e. Draw a relative frequency histogram for the set.

f. Explain whether the histogram is modeled better by a normal distribution or a uniform distribution.

g. Determine the probability that a number selected at random from this set is less than 5.

2.2 **a.** If $P(a \geq b) = 0.24$, then what is $P(a < b)$?

b. If P(not A) is 0.6, then what is P(A)?

Activity 3

3.1 Consider a sack containing 4 red marbles, 5 blue marbles, and 1 green marble. If one marble is chosen at random from the sack, determine the probability of each of the following outcomes.

a. The marble is red.

b. The marble is blue or red.

c. The marble is not red.

3.2 If A and B are complementary events and P(A) = 0.75, what is P(B)?

3.3 In a normal distribution, what is the probability that an individual observation x is:

a. more than 1.6 standard deviations above the mean?

b. more than 2.4 standard deviations below the mean?

c. between –2.4 and 1.6 standard deviations from the mean?

Answers to Flashbacks

Activity 1

1.1 **a.** The sky is not blue.

b. The statement is not true (or, the statement is false).

c. $a \neq b$

d. $a \leq b$

e. $a > b$

1.2 Statement **b**, "If I do not wear a raincoat, then it is not raining," has the same meaning as the conditional.

1.3 **a.** Since only a portion of readers would respond, the results of this study would be statistics.

b. Because every test in the class population is included in the calculation, the mean score is a parameter.

c. Because the calculation is based on a census of the entire United States, the population density is a parameter.

Activity 2

2.1 **a.** The mean, $\overline{x}$, is 5.

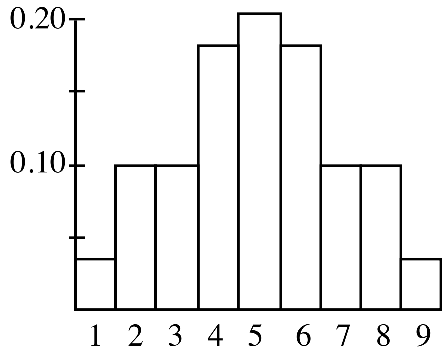
b. The standard deviation, σ, is 2.

c. A relative frequency table is shown below.

Number	Relative Frequency
1	0.03
2	0.10
3	0.10
4	0.17
5	0.20
6	0.17
7	0.10
8	0.10
9	0.03

d. The sum of the relative frequencies is 1.

e. Sample histogram:



f. Sample response: This graph is modeled better by a normal distribution because it more closely resembles the bell shape of a normal curve than a horizontal line.

g. 0.4

2.3 **a.** 1 – 0.24 = 0.76

b. 1 – 0.6 = 0.4

Activity 3

3.1 **a.** 0.4

b. 0.9

c. 0.6

3.2 P(B) = 0.25

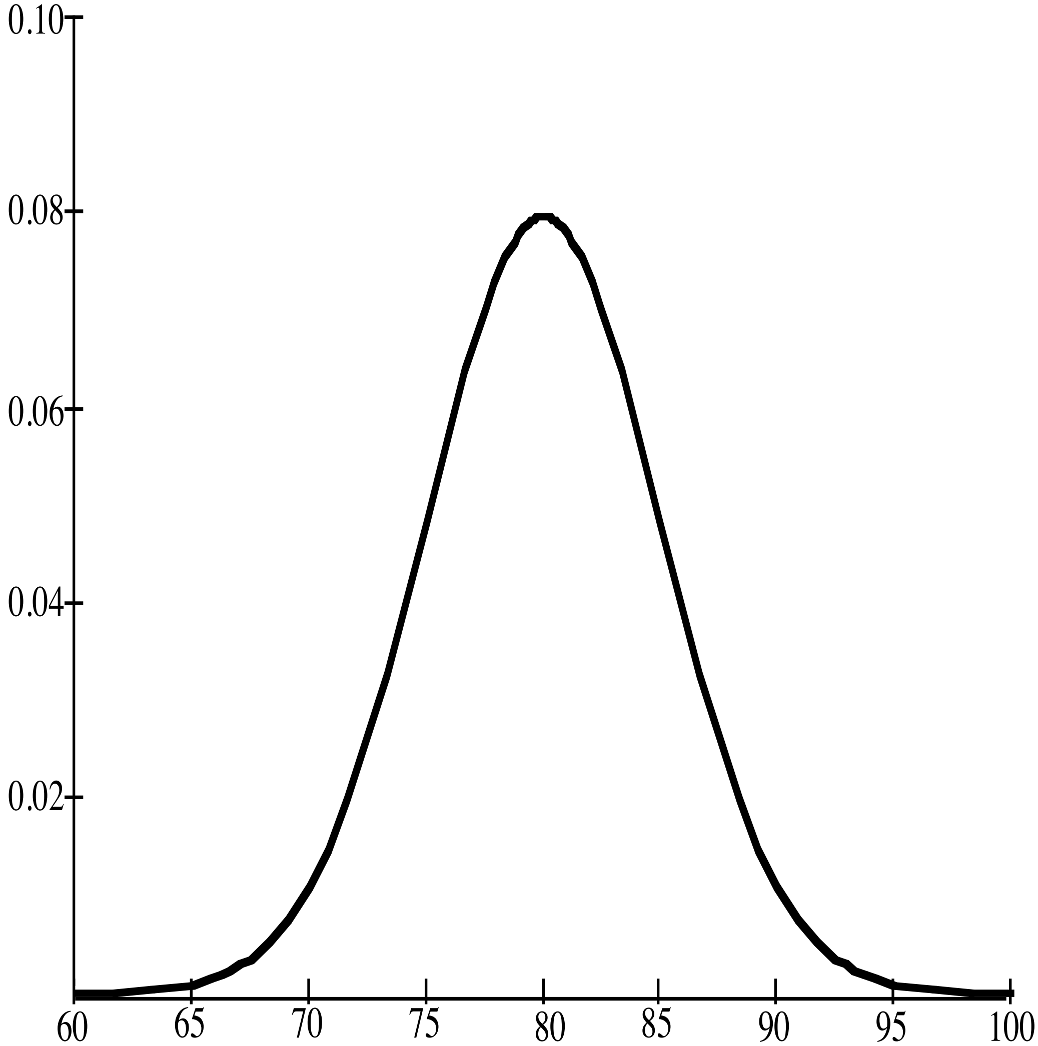
3.3 **a.** 1 – 0.945 = 0.055

b. 0.008

c. 1 – (0.055 + 0.008) = 0.937

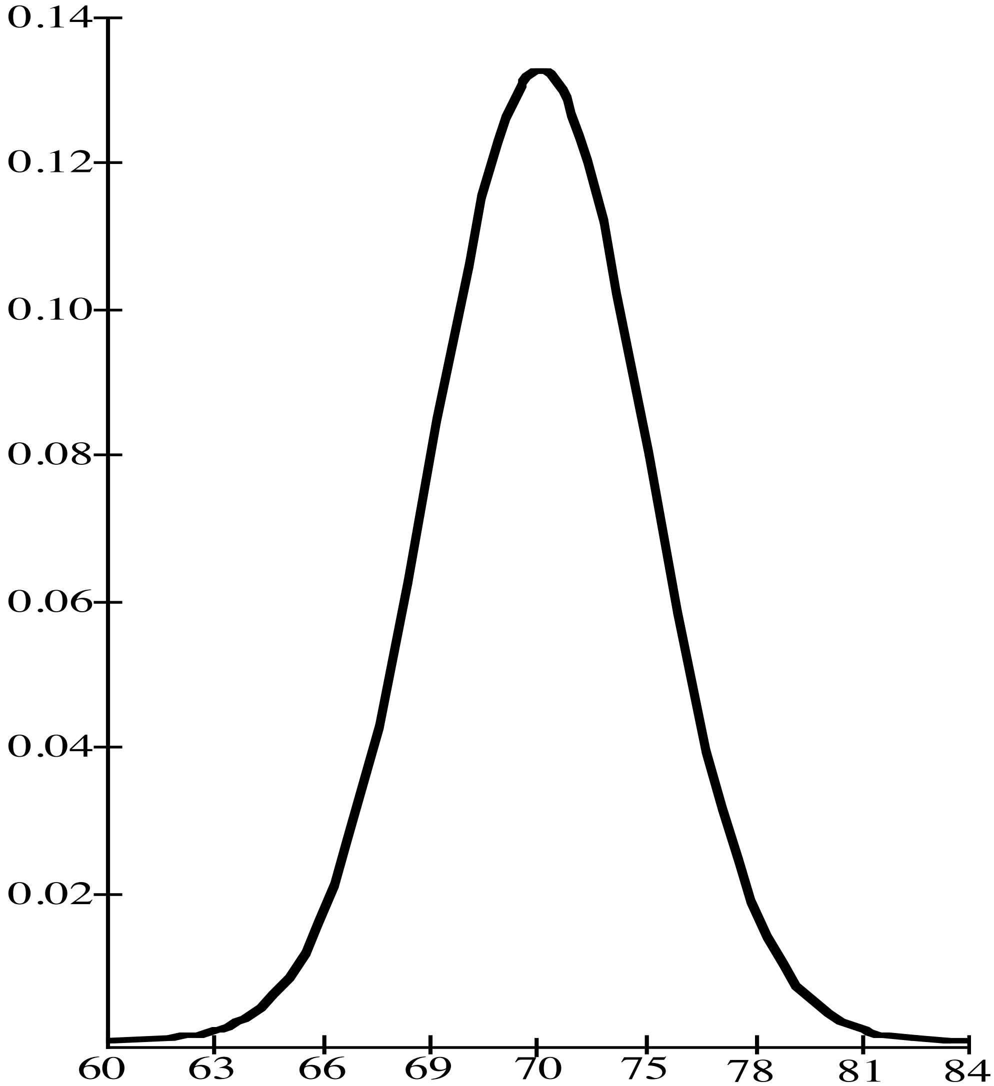
Template A

Normal curve for English exams with a mean of 80 and a standard deviation of 5



Template B

Normal curve for physics exams with a mean of 72 and a standard deviation of 3



Ostriches Are Composed

Although ostrich ranching has flourished in South Africa since the mid-1800s, it has only recently become popular in North America. In this module, you use compositions of functions to investigate the business of raising big birds.

Todd Fife • John Gebert • Sue Moore

Teacher Edition

Ostriches Are Composed

Overview

Through the context of ostrich ranching, students investigate operations on functions, composition of functions, and inverse functions.

Objectives

In this module, students will:

- represent functions using set diagrams and mapping diagrams
- identify domains and ranges of polynomial, logarithmic, and trigonometric functions, and their inverses
- explore composite functions algebraically
- examine inverse functions graphically and algebraically.

Prerequisites

For this module, students should know:

- the definition of a relation and a function
- how to read and write functional notation
- general shapes of polynomial and exponential functions
- general shapes and behaviors of sine and cosine functions
- the definition of a logarithmic function.

Time Line

Activity	Intro.	1	2	3	4	Summary Assessment	Total
Days	1	1	1	2	2	1	8

Materials Required

Materials	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
mapping template				X		

Teacher Note

A blackline master of the template appears at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity					
	Intro.	**1**	**2**	**3**	**4**	**Summary Assessment**
graphing utility		X	X		X	X
symbolic manipulator			X	X	X	X
spreadsheet			X		X	

Ostriches Are Composed

Introduction

(page 149)

Students determine the relationship between the set of digits {0, 1, 2, …, 9} and the letters on a telephone keypad as a review of relations, functions, domains, and ranges.

Discussion

(page 149)

a. The word OST-RICH correspond to the number 678-7424; BIG-BIRD corresponds to the number 244-2473.

b. Sample response: 1-800-TAL-CHIK.

c. Sample response: In Part **a**, a letter was paired with a unique number. In Part **b**, a number could be paired with any one of three letters.

d.
1. The set of ordered pairs is {(A,2), (B,2), (C,2), (D,3), (E,3), (F,3), …}.
2. The set of ordered pairs is {(2,A), (2,B), (2,C), …}.
3. In the first pairing, each letter (except Q and Z) is paired with exactly one number. In the second pairing, each number (except 1 and 0) is paired with three letters.

(page 150)

Activity 1

In this activity, students use a graphing utility to review relations, functions, domains, and ranges.

Materials List

- none

Technology

- graphing utility

Exploration (page 150)

a. Sample response: Because the square root of a negative number is undefined in the set of real numbers, $x - 1 \geq 0$ or $x \geq 1$. Thus, the domain is $[1, \infty)$. Because the square root of a non-negative real number is greater than or equal to 0, the range is $[0, \infty)$.

b. Sample response: Because the absolute value of any real number can be found, the domain is $(-\infty, \infty)$. The absolute value of any real number is a distance, and a distance is always greater than or equal to 0. Since the function is the opposite of the absolute value, the range is $(-\infty, 0]$.

c. Sample response: Because the square root of a negative number is undefined in the set of real numbers, $4 - x^2 \geq 0$ or $(2 - x)(2 + x) \geq 0$. Thus, the domain is $[-2, 2]$. Because $\sqrt{4 - x^2}$ is greatest when x^2 is least, $x = 0$ yields the greatest value of 2. Thus, the range is $[0, 2]$.

d. Sample response: Since division by 0 is undefined, the domain is all real numbers not equal to 5. The range is $(-\infty, \infty)$.

Discussion (page 151)

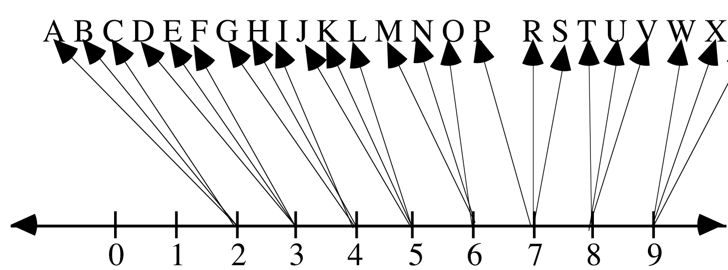
a. See sample responses given in Parts **a**, **c**, and **d** above.

b. Sample response: The graphing utility did not graph any ordered pairs for values of x where the function was undefined. **Note**: Some graphing utilities may not show discontinuities such as asymptotes and holes in the graphs. You may wish to address this issue with your students.

c. All of the relations in the exploration are functions.

d. Sample response: If a vertical line placed anywhere on the graph of a relation passes through more than one point of the graph, the relation is not a function.

e. One possible relation is defined by the following diagram:



1. The natural "relation" between the numbers and letters on the keypad is not a mathematical relation unless 0 and 1 are excluded from the domain. Therefore, the domain of the relation is the set of digits $\{2, 3, 4, \ldots, 9\}$.

2. The range consists of all the letters in the English alphabet except Q and Z. **Note**: Some telephone keypads may include Q and Z. If so, the range would consist of all the letters of the English alphabet.

3. Since each number in the domain is paired with more than one letter, the relation (as previously defined) is not a function.

f. 1. The domain is the set of letters in the English alphabet except Q and Z.

2. The range is $\{2, 3, 4, \ldots, 9\}$.

3. The relation is a function since each letter on a telephone keypad is paired with exactly one digit. Since some keypads feature the letters "OPER" above the number 0, some students might suggest that O, P, E, and R are paired with 0. In this case, the relation is not a function, because P, for example, is paired with both 7 and 0.

g. Sample response: Since the number of ostriches must be greater than 0 and an integer, the domain is the set of non-negative integers not exceeding the carrying capacity of the farm. Since the carrying capacity is unknown, the set of non-negative integers could be used as the domain. The corresponding range is contained in the set of numbers of the form $m/100$, where m is a non-negative integer and profit is expressed in dollars and cents.

Assignment

(page 151)

1.1 **a.** 1. The domain is $\{1, 4, 9, 16\}$ and the range is $\{-4, -3, -2, -1, 1, 2, 3, 4\}$. The rule may be $h(x) = \sqrt{x}$ or $-\sqrt{x}$. (Some students may use curve-fitting to guess the rule.)

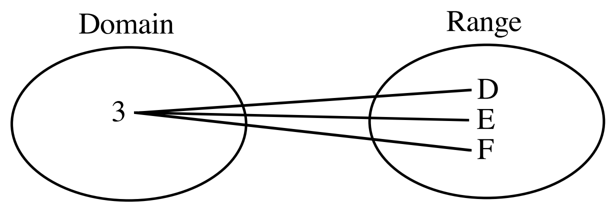
2. The domain is $\{-4, -2, 1, 3\}$ and the range is $\{-64, -8, 1, 27\}$. The rule may be $r(x) = x^3$.

3. The domain is the set of real numbers and the range is $\{2\}$. The rule is $f(x) = 2$.

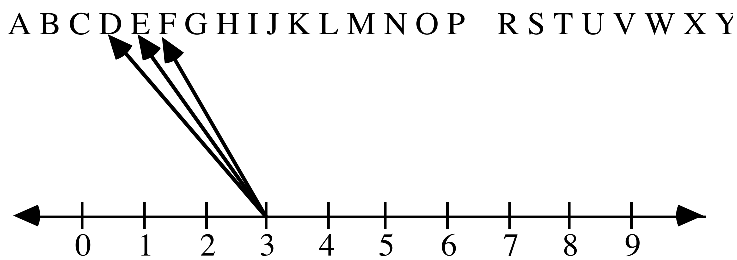
b. Sample response: The relations in Parts **a2** and **a3** are functions since each element of the domain is paired with only one element of the range.

1.2 **a.** Answers will vary.

1. Sample set diagram:

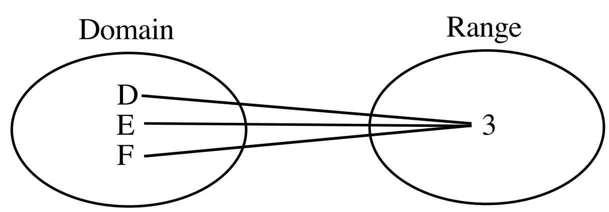


2. Sample mapping diagram:

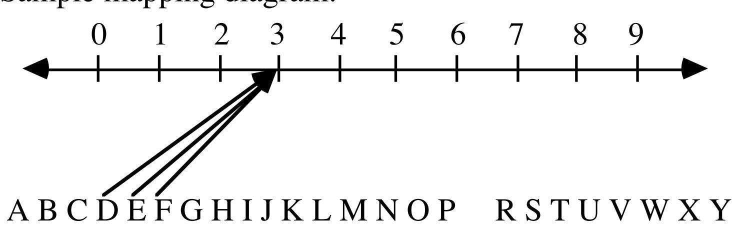


3. Sample response: The relation maps the integer 3 to three different letters of the alphabet, D, E, and F.

b. **1.** Sample set diagram:



2. Sample mapping diagram:



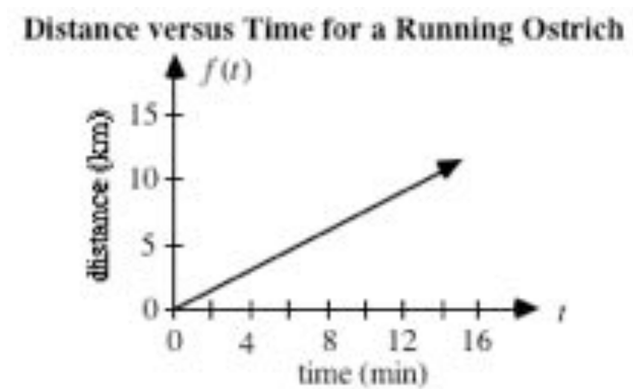
3. Sample response: The relation maps three letters of the alphabet, D, E, and F, to one integer, 3.

1.3 **a.** Sample response, where $f(t)$ describes distance in kilometers:

$$f(t) = \frac{50 \text{ km}}{1 \text{ hr}} \bullet \frac{1 \text{ hr}}{60 \text{ min}} \bullet t \text{ min} \approx 0.83t$$

b. Sample response: Since a negative value for time makes no sense in this setting, one possible domain is $[0,15]$. The corresponding range is $\{[0,12.5]$. **Note:** Disregarding the setting, the function $f(t) = 0.83t$ imposes no restrictions on *t*. Thus, the domain would be the set of real numbers.

c. Sample graph:



1.4 **a.** Sample response: Given any real number a, $3a$ is a real number. Also, if a is a real number, then $a/3$ is a real number. Thus $f(a/3) = 3(a/3) = a$ implies that every real number has a preimage in the domain. Thus, the range is the set of real numbers.

b. The range is $[-3,3]$.

c. The domain is $[-1/3,1/3]$.

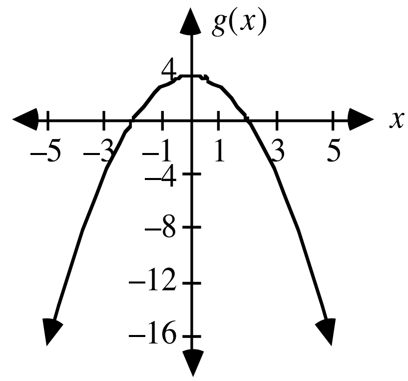
1.5 Sample response: To find the range, multiply the values in the domain by 3, then subtract 2 as shown below.

$$-5 \le x \le 10$$
$$-15 \le 3x \le 30$$
$$-17 \le 3x - 2 \le 28$$
$$-17 \le f(x) \le 28$$

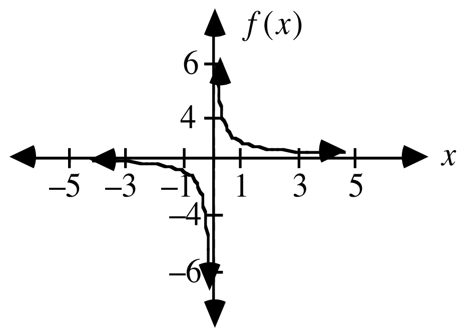
Therefore, the range is $[-17,28]$.

***1.6** Sample response: In the function $f(x) = x^2$, where the domain is $(-\infty,0) \cup (0,\infty)$, the range is $(0,\infty)$. Each element of the range is paired with two elements of the domain: x and $-x$. Since $0^2 = 0$ and 0 has only one square root, 0 is not in the domain.

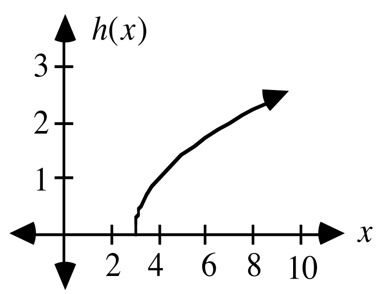
***1.7** **a.** Sample response: $g(x) = -x^2 + 4$.



b. Sample response: $f(x) = 1/x$.



c. Sample response: $h(x) = \sqrt{x - 3}$.



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1.8 **a.** To find a function, students may use the ordered pairs (0,32) and (100,212), representing the freezing point and boiling point of water, respectively. The slope of the line containing these points is $9/5$ and the y-intercept is 32. With this information, students should be able to write the function $f(c) = (9/5)c + 32$ where c represents the temperature in degrees Celsius.

b. Sample response: Both the domain and the range for the function given in Part **a** are the set of real numbers. **Note:** Some students may argue that temperatures less than absolute zero should be excluded from the domain, with a corresponding effect on the range.

1.9 **a.** Sample response: The graph of $x + 5y = 11$ is in general a line. To find the coordinates of integer pairs, consider the following equation: $5y = 11 - x$. This implies that $11 - x$ is a multiple of 5. For $11 - x$ to be a multiple of 5, x could be a positive integer with 1 or 6 in its units place, yielding such values of x as seen in the set $\{1, 6, 11, 16, 21, \cdots\}$. These values for x yield the following ordered pairs: $\{(1,2), (6, 1), (11,0), (16,-1), (21,-2), \cdots\}$

Negative numbers with 4 or 9 in their units place also yield multiples of 5 when substituted in $11 - x$. Thus, the set of x–values $\{-4, -9, -14, -19, \cdots\}$ produces the following ordered pairs:
$\{(-4,3), (-9,4), (-14,5), (-19,6), (-24,7), \cdots\}$.

The entire solution set is $\{\cdots, (-24,7), (-19,6), (-14,5), (-9,4), (-4,3), (1,2), (6, 1), (11,0), (16,-1), (21,-2), \cdots\}$.

b. There are no solutions. Using the distributive property, $3x + 6y = 3(x + 2y)$, which implies that the left-hand side of the original equation is a multiple of 3. The right-hand side of the original equation (71) is not. Using only integers, there are no solutions.

1.10 **a.** Sample response: Since producing a negative number of shirts is impossible, an appropriate domain of the function is the set of non-negative integers. The corresponding range for this function is the set of odd integers greater than or equal to 23.

b. An appropriate domain for the function is the set of non-negative real numbers. The corresponding range is also the set of non-negative real numbers. **Note:** Students may impose other restrictions by specifying a height from which the object is dropped.

c. An appropriate domain for the function is the set of non-negative real numbers. The corresponding range is the set of non-negative real numbers less than or equal to $q_\circ$.

* * * * * * * * * *

(page 154)

Activity 2

Students investigate arithmetic operations involving polynomial and trigonometric functions.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator

Exploration 1

(page 154)

Students write equations for two profit functions, then add the functions together and determine a rule for the new function.

Note: You may wish to point out that these models have many limitations, given the number of assumptions involved. For example, costs are assumed to be constant for any number of ostriches in the flock, while prices are assumed to be stable, regardless of supply and demand.

a. **1.** $e(x) = (40 - 4)(500)x - 60,000 = 18,000x - 60,000$

2. $p(x) = \frac{4}{2}(10,000)x - 80,000 = 20,000x - 80,000$

b. **1-2.** Sample response:

Domain Value (*x*)	***e*(*x*)**	***p*(*x*)**
5	30,000	20,000
6	48,000	40,000
7	66,000	60,000
8	84,000	80,000
9	102,000	100,000
10	120,000	120,000
11	138,000	140,000
12	156,000	160,000
13	174,000	180,000
14	192,000	200,000
15	210,000	220,000

c. $(e + p)(x) = (20,000x - 80,000) + (18,000x - 60,000)$

d. Sample response: The domain is the set of integers in the interval [5, 15]. The following values for the range of $(e+p)(x)$ were found by adding the corresponding values of $e(x)$ and $p(x)$.

Domain Value (x)	$e(x)$	$p(x)$	$(e+p)(x)$
5	30,000	20,000	50,000
6	48,000	40,000	88,000
7	66,000	60,000	126,000
8	84,000	80,000	164,000
9	102,000	100,000	202,000
10	120,000	120,000	240,000
11	138,000	140,000	278,000
12	156,000	160,000	316,000
13	174,000	180,000	354,000
14	192,000	200,000	392,000
15	210,000	220,000	430,000

Discussion 1

(page 155)

a. Students may have determined several different forms for the functions $e(x) = 18{,}000x - 60{,}000$ and $p(x) = 20{,}000x - 80{,}000$.

b. Students should have found some form of the function $(e+p)(x) = 38{,}000x - 140{,}000$.

c. Sample response: Each element of the range of $(e+p)$ is the sum of the corresponding elements of the ranges of e and p.

d. Sample response: The answer depends on whether the equation $38{,}000x - 140{,}000 = 200$ has an integer solution. It does not. Thus, the profit cannot be exactly this amount. In other words, there is no ordered pair $(x, 200)$, where x is an integer, that fits this function.

e.

1. Because their exponents are different, x^2 and x^3 are not like terms. Therefore, they cannot be combined.
2. Since $3\sqrt{x}$ and $12\sqrt{x}$ are like terms, they can be combined as follows:

   $$12\sqrt{x} + 3\sqrt{x} = (12+3)\sqrt{x} = 15\sqrt{x}$$

3. Since $\cos x$ and $5\cos x$ are like terms, they can be combined as follows:

   $$\cos x + 5\cos x = (1+5)\cos x = 6\cos x$$

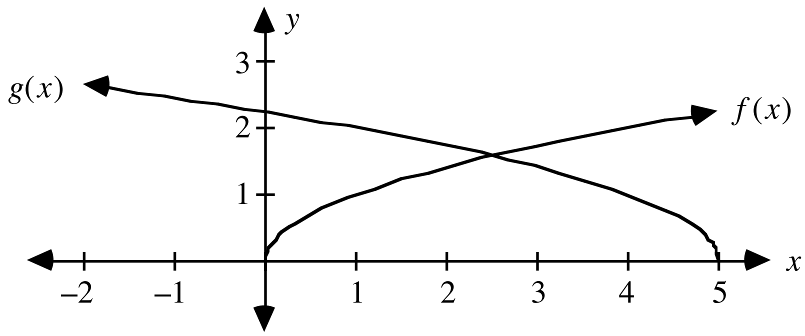
4. The terms $\sin 2x$ and $\sin x$ are not like terms. Therefore, they cannot be combined.

Exploration 2

(page 156)

Students investigate addition, subtraction, multiplication, and division of functions and examine the domains and ranges of the resulting functions.

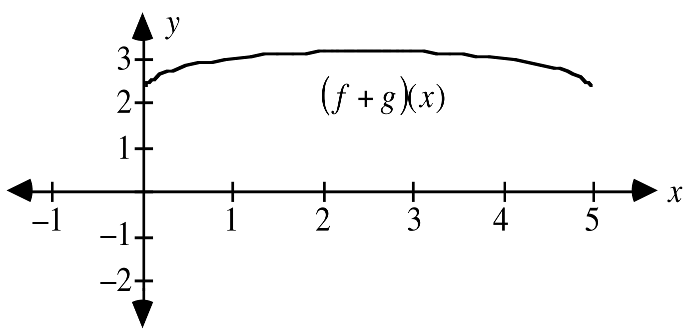
a. **1.** Sample graph:



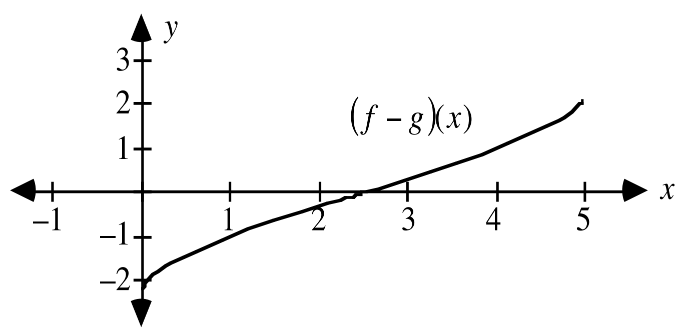
2. The domain of *f* is $[0,\infty)$. The range is $[0,\infty)$. The domain of *g* is $(-\infty,5]$. The range is $[0,\infty)$.

3. The intersection of the domains is [0,5].

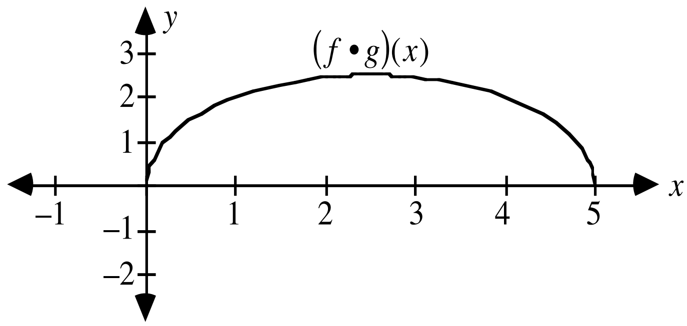
b. **1.** The domain is $[0,5]$. The range is $\left[\sqrt{5},2\sqrt{2.5}\right]$.



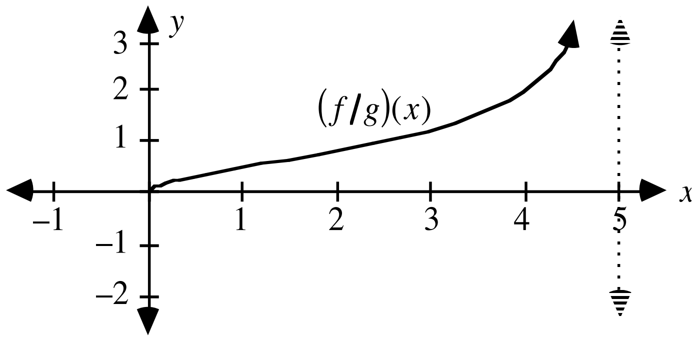
2. The domain is $[0,5]$. The range is $\left[-\sqrt{5},\sqrt{5}\right]$.



3. The domain is $[0,5]$. The range is $\left[0,5/2\right]$.

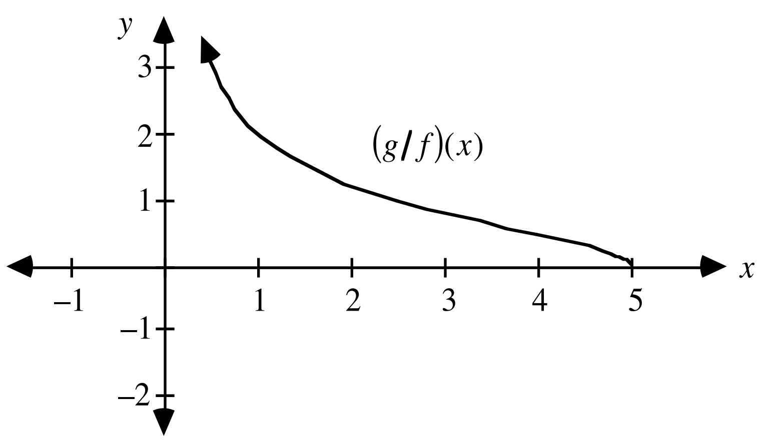


c. **1.** Sample graph:



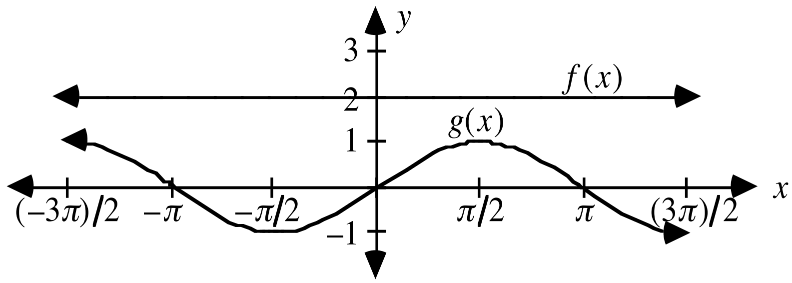
The domain of $(f/g)(x)$ is $[0,5)$. Note that 5 is not included because $g(5)=0$. The range is $[0,\infty)$.

2. Sample graph:

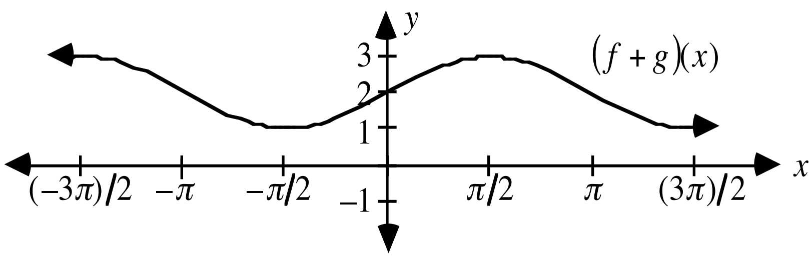


The domain of $(g/f)(x)$ is $(0,5]$. The range is $(0,\infty)$.

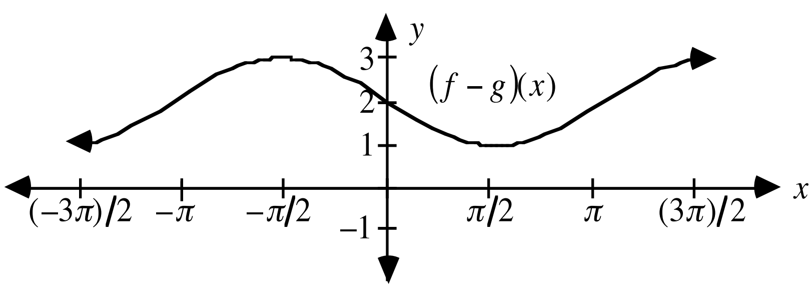
d. The domains of $f(x)$ and $g(x)$ are both $(-\infty,\infty)$. The intersection of the two sets is $(-\infty,\infty)$. Therefore, each of the resulting functions has a domain of $(-\infty,\infty)$, except for $(f/g)(x)$.



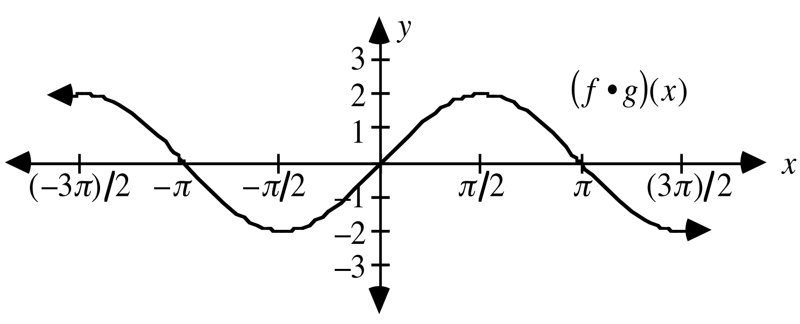
The function $(f+g)(x)$ has a range of $[1,3]$.



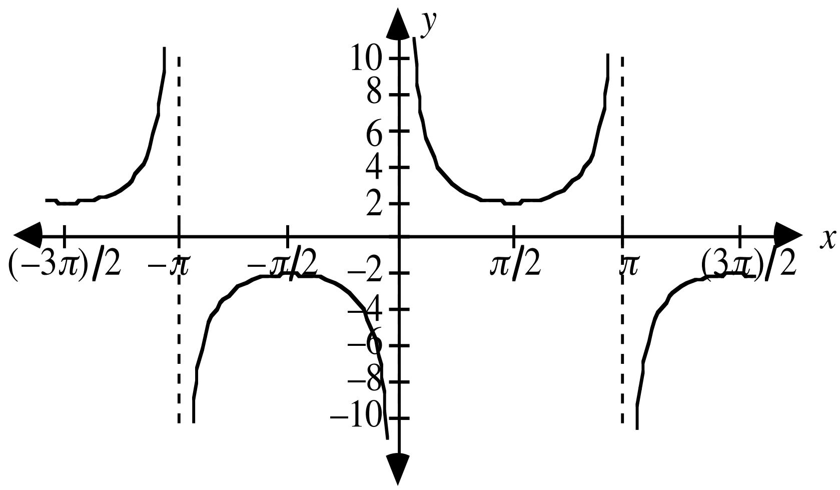
The function $(f-g)(x)$ has a range of $[1,3]$.



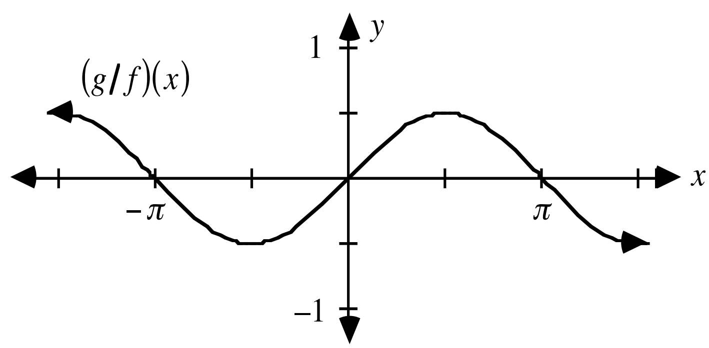
The function $(f \bullet g)(x)$ has a range of $[-2,2]$.



Since $g(x) = 0$ at values of $n\pi$ where *n* is an integer, the domain for $(f/g)(x)$ is the set of all real numbers except those of the form $n\pi$ where *n* is an integer. The range of $(f/g)(x)$ is the set of all real numbers excluding those in the interval $(-2, 2)$.



Since $f(x) \neq 0$ for any real number, the domain for $(g/f)(x)$ is $(-\infty, \infty)$. The range is [–0.5,0.5].



Discussion 2

(page 157)

a. Sample response: The domain would be $\{1, 2, 3, 4, 5\}$. When the expression $f(x) + g(x)$ is evaluated, the values for *x* must be the same. Therefore, only values that are in the intersection of the two original domains are included in the domain of $(f + g)$.

b. Sample response: In the function defined by $(f/g)(x) = f(x)/g(x)$, the value of the quotient does not exist if $g(x) = 0$. Therefore, the function is undefined at the values of *x* that result in $g(x) = 0$.

c. The domain for $(f/g)(x)$ is $[0, 5)$, while the domain for $(g/f)(x)$ is (0,5]. The range for both functions is $[0, \infty)$.

Assignment (page 157)

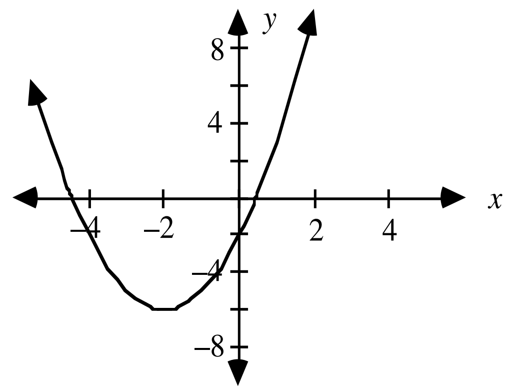
2.1 **a.** **1.** $(f+h)(x) = x^2 + 4x - 2$

2. $(f-h)(x) = x^2 + 2x + 2$

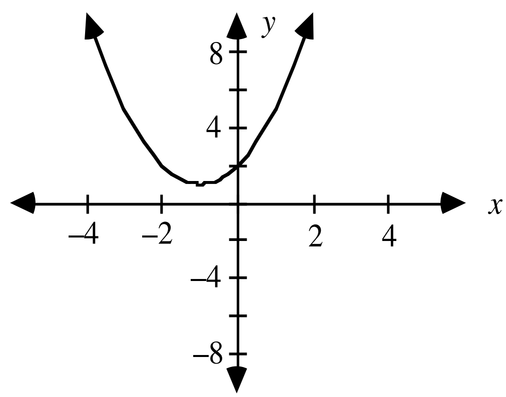
3. $(f \bullet h)(x) = x^3 + x^2 - 6x$

4. $(f/h)(x) = \dfrac{x^2 + 3x}{x - 2}$

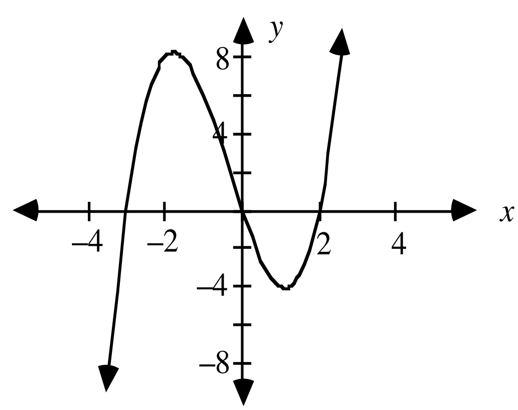
b. **1.** The domain is $(-\infty,\infty)$; the range is $[-6,\infty)$. Sample graph:



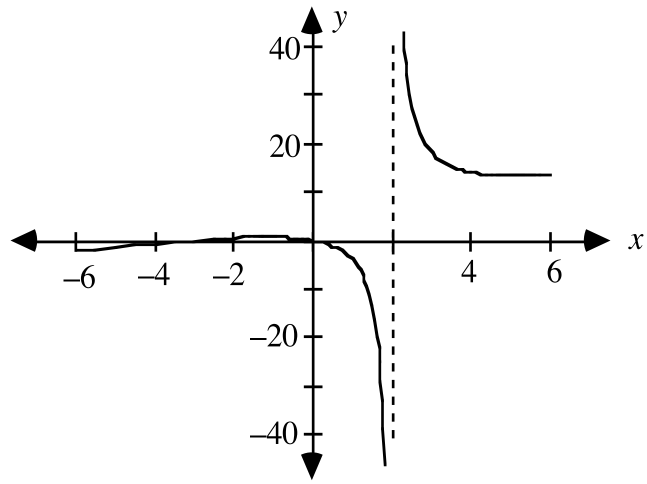
2. The domain is $(-\infty,\infty)$; the range is $[1,\infty)$. Sample graph:



3. The domain is $(-\infty,\infty)$; the range is $(-\infty,\infty)$. Sample graph:



4. The domain is $(-\infty,2)\cup(2,\infty)$; the approximate range is $(-\infty,1.53)\cup(13.32,\infty)$. Sample graph:

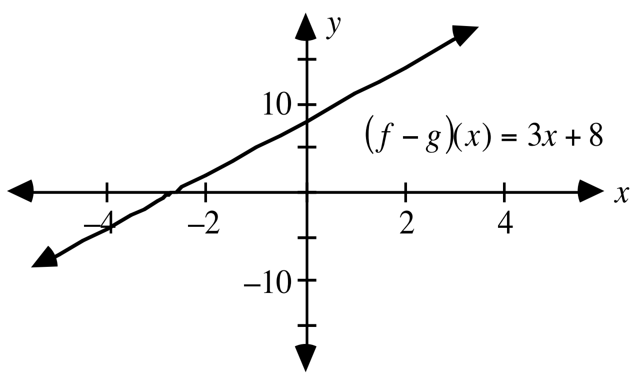


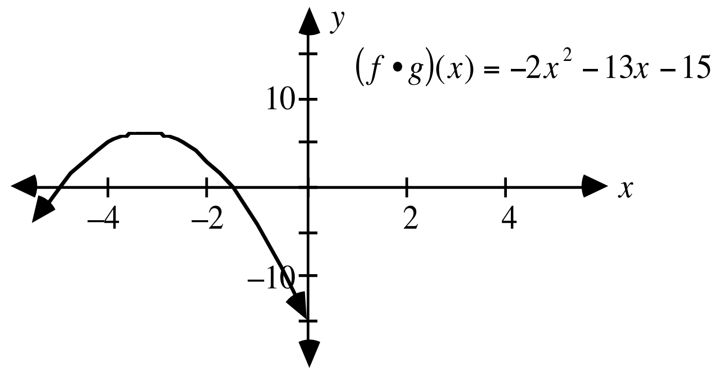
2.2 **a.** Sample response: The domains for addition, subtraction, and multiplication are $(-\infty,\infty)$. The domain for both $(f/g)(x)$ and $(g/f)(x)$ $(-\infty,0)\cup(0,\infty)$.

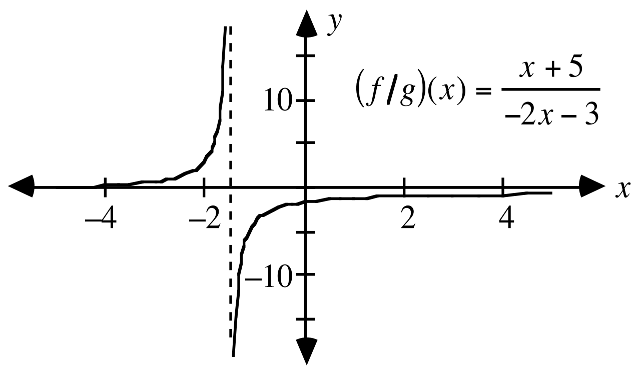
b. Sample response: The domains for addition, subtraction, and multiplication are $[0,\infty)$. The domain of $(g/f)(x)$ is $(0,\infty)$. The domain of $(f/g)(x)$ is $[0,1)\cup(1,\infty)$.

c. Sample response: The domains for addition, subtraction, and multiplication are $(-\infty,\infty)$. The domain of $(g/f)(x)$ is $(-\infty,\infty)$. The domain of $(f/g)(x)$ is the set of real numbers that are not multiples of π.

2.3 **a.** Sample graphs:





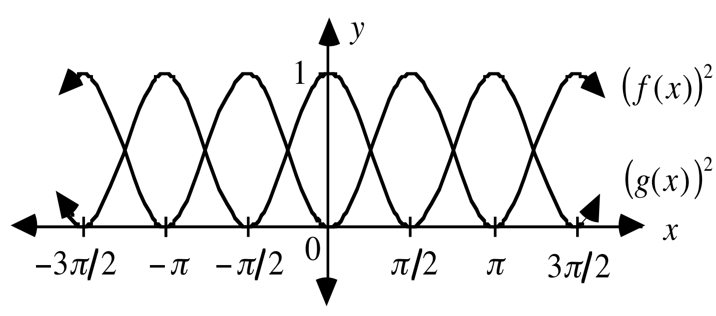


b. The domain and range for $(f-g)(x) = 3x+8$ is $(-\infty,\infty)$. The domain for $(f \bullet g)(x) = -2x^2 - 13x - 15$ is $(-\infty,\infty)$. The range is $(-\infty, 6.1]$. The domain for

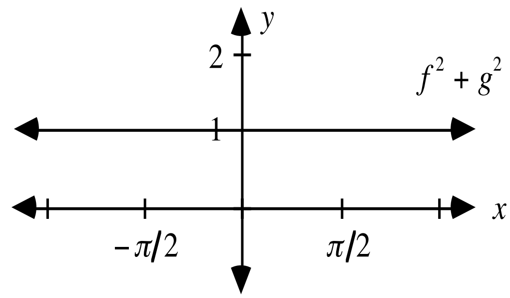
$$(f/g)(x) = \frac{x+5}{-2x-3}$$

is $(-\infty,-3/2) \cup (-3/2,\infty)$. The range is $(-\infty,-1/2) \cup (-1/2,\infty)$.

2.4 **a.** The domain of each function is $(-\infty,\infty)$. The range for each function is $[0,1]$. Sample graph:



b. The domain of the function $(f^2 + g^2)(x) = (\sin x)^2 + (\cos x)^2$ is $(-\infty,\infty)$. The range is $\{1\}$. Sample graph:



Note: You may wish to point out that $(\sin x)^2 + (\cos x)^2$ may also be written using the following notation: $\sin^2 x + \cos^2 x$.

c. $(f^2 + g^2)(x) = (\sin x)^2 + (\cos x)^2 = 1$

d. Sample response: Yes, because the relationship is true for all values of *x* in the domain.

e. The domain is $\{x \neq n\pi/2$, where *x* is a real number and *n* is an odd integer$\}$. The range is $(-\infty,\infty)$.

f. Sample response: Yes, since the relationship is true for all values of *x* in the domain.

***2.5** **a.** **1.** $c(x) = 700x - 36,000$

2. $p(y) = 10,000y - 40,000$

b. Sample response: The value of $c(5)$ is the profit from selling 5 chicks, while $p(5)$ is the profit from selling 5 breeding pairs.

c. Sample response: This is not reasonable because the variable represents chicks in one function and breeding pairs in the other. The function $(c + p)$ would only give the profit in situations where the number of chicks sold was equal to the number of breeding pairs sold.

d. **1.** Sample response: This represents profit from the sale of 10 chicks and 8 breeding pairs.

2. Sample response: The domain contains ordered pairs. Based on the information in Exploration **1**, a possible domain would be $\{(x,y)$, where *x* is an integer and $140 \leq x \leq 420$, *y* is an integer and $8 \leq y \leq 24\}$. The corresponding range would be $\{r(x,y)$, where $r(x,y)$ is an integer, $102,000 \leq r(x,y) \leq 458,000\}$.

3. Sample response: Yes. Every combination of number of chicks sold and number of breeding pairs sold can be used in the function. This removes the limitations of the function $c + p$ mentioned in Part **c**.

2.6 **a.** The domain of $(f+g)(x) = 3x + 1/(3x)$ is $(-\infty,0)\cup(0,\infty)$.

b. The domain of $(f-h)(x) = 2x+3$ is $(-\infty,\infty)$.

c. The domain of $(f \bullet g)(x) = 1$ is $(-\infty,0)\cup(0,\infty)$.

d. The domain of $(h/g)(x) = 3x^2 - 9x$ is $(-\infty,0)\cup(0,\infty)$.

e. The domain of

$$(f/h)(x) = \frac{3x}{x-3}$$

is $(-\infty,3)\cup(3,\infty)$.

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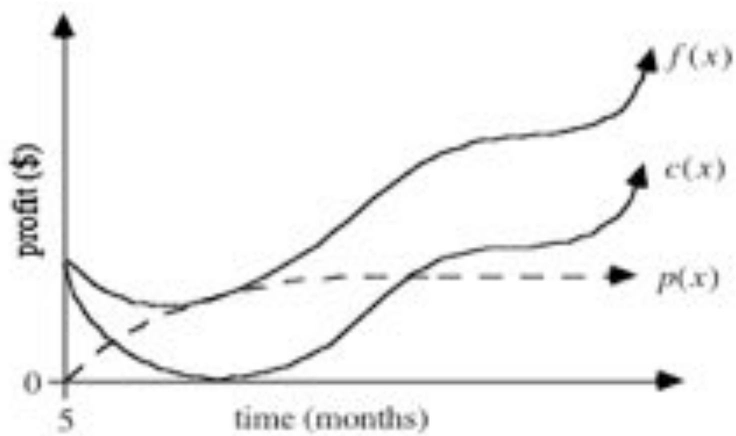
2.7 The following graph shows $(f-c)(x) = p(x)$.



2.8 **a.** $h(t) = 50 - 10t - 4.9t^2$

b. The function $h(t)$ in Part **b** has two roots: $t \approx 2.33$ and $t \approx -4.37$. The object will hit the ground after about 2.33 sec.

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(page 161)

Activity 3

Students investigate composite functions and examine the relationships that exist between the original functions and the composed function.

Materials List

- mapping template (one per student; a blackline master appears at the end of the teacher edition for this module)

Technology

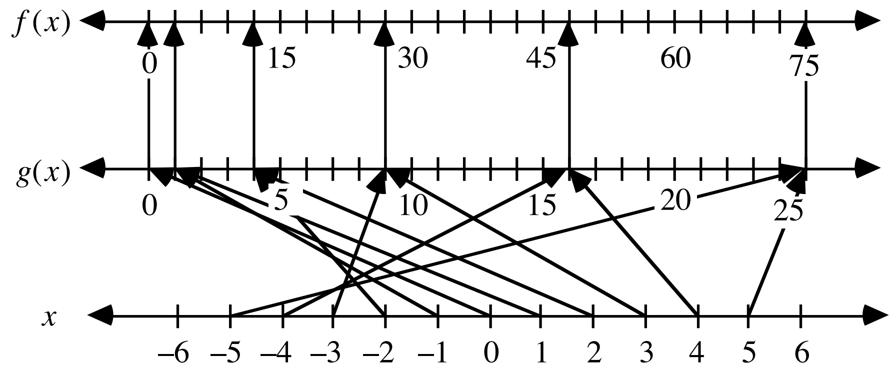
- symbolic manipulator

Exploration

(page 162)

a. **1.** The range for $g(x) = x^2$ is $\{0, 1, 4, 9, 16, 25\}$.

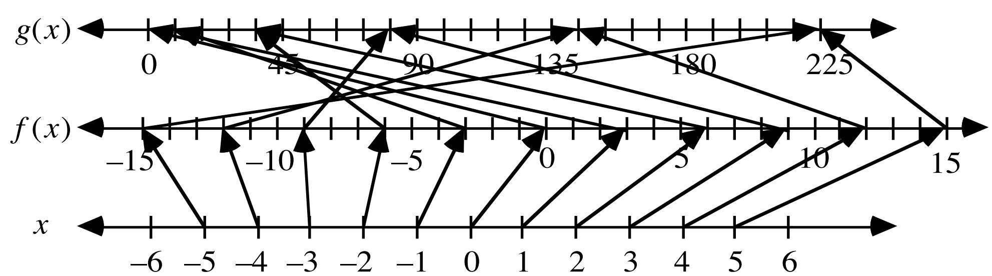
2. Sample mapping diagram:



3. $f\left(g(x)\right) = 3\left(x^2\right) = 3x^2$. The range for $f\left(g(x)\right)$ over the domain {–5, –4, –3, ..., 5} is{0, 3, 12, 27, 48, 75}.

b. **1.** The range for $f(x) = 3x$ is $\{-15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15\}$.

2. Sample mapping diagram:



3. $g\left(f(x)\right) = (3x)^2 = 9x^2$ The range for $g(f(x))$ over the domain {–5, –4, –3, ..., 5} is{0, 9, 36, 81, 144, 225}.

Discussion

(page 163)

a. **1.** $q(a) = 3(a) + 2 = 3a + 2$

2. $f(q(a)) = (3(a) + 2)^2 = 9a^2 + 12a + 4$

b. Sample response: They are different except for the number 0.

c. Sample response: Since $f(g(x))$ and $g(f(x))$ result in different outcomes for the same value of x, composition of functions is not commutative.

d. Sample response: The ordered pairs are $\{(3,10), (4,14)\}$. The domain of $g(f(x))$ is $\{3, 4\}$ since only these elements of the domain of f have corresponding range values that are in the domain of g.

e. **1.** To find an equation for $f(g(x))$, the expression $x^3 + 3$ is substituted for x in the function $f(x) = \sqrt{x + 5}$, resulting in $f(g(x)) = \sqrt{x^3 + 8}$. The domain is $\{x \geq -2$, where x is a real number$\}$. The range is the non-negative real numbers.

2. Sample response: The process for finding $g(f(2))$ substitutes 2 into $f(x)$ first, then substitutes the result into $g(x)$. The process for finding $f(g(2))$ substitutes 2 into $g(x)$, then substitutes the results into $f(x)$.

3. As shown in Part **e1** above, the domain of $f \circ g$ is $\{x \geq -2$, where x is a real number$\}$. Since the domain of $g(x)$ is the set of real numbers, the domain of $f \circ g$ does not contain all the domain values of $g(x)$.

4. The domain of $g \circ f$ is the same as the domain of $f(x)$ since $g \circ f = \left(\sqrt{x + 5}\right)^3 + 3$.

Assignment

(page 163)

3.1 **a.** Sample response: First determine the number of unbroken eggs. This is a percentage of all eggs, so it must be computed first. Then remove the 4 that the ranch will keep from the remaining eggs: $h(g(40)) = 28$.

b. Sample response: Since $g(h(40)) = 28.8$ and $h(g(40)) = 28$, composition of these functions is not commutative.

3.2 Sample response: The domain is $(-\infty, \infty)$. The range is $[0,1]$.

3.3 **a.** **1.** The domain is $[0,\infty)$. The range is $[0,\infty)$.

2. The domain is $(-\infty,\infty)$. The range is $[0,\infty)$.

b. **1.** $h(k(x)) = \left(\sqrt{x}\right)^2 = x$, where $x \geq 0$

2. $k(h(x)) = \sqrt{x^2} = |x|$

c. A function consists of a rule, its domain, and range. Since the square root is only defined when *x* is non-negative, the domain of $\left(\sqrt{x}\right)^2$ is the set of all non-negative real numbers. Since x^2 is always non-negative, however, the domain of $\sqrt{x^2}$ is the set of all real numbers. Since the domains are different, the functions are different.

3.4 The function can be simplified as follows:

$$\begin{aligned} c(x) &= \frac{f(x+4) - f(x)}{4} \\ &= \frac{4(x+4) - 4x}{4} \\ &= \frac{4x + 4^2 - 4x}{4} \\ &= \frac{16}{4} \\ &= 4 \end{aligned}$$

***3.5** **a.** Sample response: $s(x) = 0.8x/2 = 0.4x$

b. $s(n(7)) = s(126{,}000) = 50{,}400$

c. $(s \circ n)(x) = 15{,}200x - 56{,}000$

***3.6** Answers will vary. The following sample responses assume that the student earns $7.00 per hour and saves 40% of earnings for college.

a. $p(x) = 7x$

b. $c(x) = 0.4x$

c. $(c \circ p)(x) = 2.8$

* * * * *

3.7 **a.** The domain of $g(x) = \sqrt{16 - x^2}$ is $[-4, 4]$. The range is $[0, 4]$. Both the domain and range of $h(x) = 5x - 1$ are $(-\infty, \infty)$.

b. **1.** $g(h(x)) = \sqrt{15 + 10x - 25x^2}$

2. $h(g(x)) = 5\sqrt{16 - x^2} - 1$

c. **Note**: Students may graph $(g \circ h)(x)$ and $(h \circ g)(x)$ first to observe the domain and range graphically, then use a symbolic manipulator to identify the appropriate intervals more accurately.

1. The domain of $(g \circ h)(x)$ is $[-0.6, 1]$. The range is $[0, 4]$.

2. The domain of $(h \circ g)(x)$ is $[-4, 4]$. The range is $[-1, 19]$.

3.8 **a.** $g(f(h(7))) = \sqrt{40} + 5 \approx 11.32$; $f(h(g(7))) = 23 + 10\sqrt{7} \approx 49.46$; $g(h(f(7))) = 7$

b. $g(f(h(x))) = \sqrt{x^2 - 9} + 5$; domain: $x \le -3$ and $x \ge 3$

3.9 **a.** Sample response: The range is the set of all mothers.

b. Sample response: The range is the set of all mothers of mothers (in other words, the maternal grandmothers).

* * * * * * * * * *

(page 165)

Activity 4

Students explore various functions and their inverses, including polynomial, trigonometric, and logarithmic functions. They investigate domains and ranges of these functions and any restrictions needed to make their inverses functions.

Materials List

- none

Technology

- graphing utility

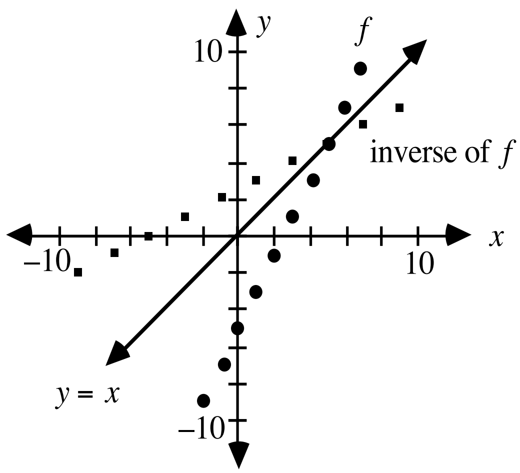
Exploration 1

(page 166)

Students explore the inverses of linear and quadratic functions, determine the equations of the inverse functions, and investigate the symmetry of the graphs of a function and its inverse. Note that $f^{-1}(x)$ indicates that the inverse of a function *f* is also a function; while $(f(x))^{-1}$ indicates

$$\frac{1}{f(x)}$$

a. Using technology, students enter two columns of data and create two scatterplots on the same set of axes: one scatterplot using the first column as the domain and the second column as the range, and another scatterplot using the first column as the range and the second column as the domain. Sample graph:



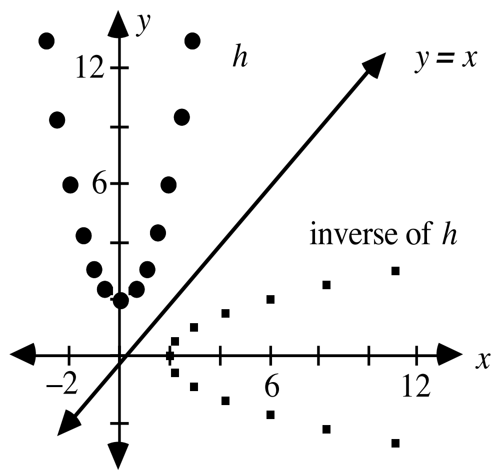
b. (5,5)

c. The equation of the line is $y = x$. (See sample graph given in Part **a**.)

d. Sample response: The two graphs are reflections in the line $y = x$.

e. Students repeat Parts **a–d** using their own linear functions.

f. Sample graph:



g. Sample response: The two graphs are reflections in the line $y = x$.

Discussion 1 (page 167)

a. Sample response: For f and g, the graph of the inverse is a line. The graph of the inverse of h is a parabola reflected in the line $y = x$.

b. Sample response: The inverse of f is a function because elements of the range do not repeat. If g is not a vertical or horizontal line, then its inverse is also a function. The inverse of h is not a function.

c. Sample response: To form the inverse, the values for x and y are interchanged. The ordered pair (a,b) becomes (b,a). The ordered pair (a,b) is the reflected image of (b,a) in the line $y = x$ .Therefore, the graph of a function and its inverse are reflections of each other with respect to the line $y = x$.

Exploration 2 (page 167)

Students perform inverse operations on the rule that defines a function in order to determine a rule for its inverse.

a.

1. $f^{-1}(x) = \dfrac{x+1}{3}$; there are no restrictions on x
2. $g^{-1}(x) = x^2 - 7,\ \ x \geq -7$
3. The inverse consists of the union of parts of $\sqrt{x}/3$ and $-\sqrt{x}/3$, where $x \geq 0$.
4. $k^{-1}(x) = \sqrt[3]{x+8}$; there are no restrictions on x

b. Students verify their results from Part **a** using a symbolic manipulator. **Note:** Some symbolic manipulators may or may not give the correct domain restrictions.

c. **1.** $\left(f \circ f^{-1}\right)(x) = f\left(\frac{x+1}{3}\right) = 3\left(\frac{x+1}{3}\right) - 1 = x$;

$\left(f^{-1} \circ f\right)(x) = f^{-1}(3x - 1) = \frac{(3x-1)+1}{3} = x$

2. $\left(g \circ g^{-1}\right)(x) = g\left(x^2 - 7\right) = \sqrt{x^2 - 7 + 7} = \sqrt{x^2} = x$, for $x \geq 0$;

$\left(g^{-1} \circ g\right)(x) = g^{-1}\left(\sqrt{x+7}\right) = \left(\sqrt{x+7}\right)^2 - 7 = x$, for $x \geq -7$

3. The inverse of $h(x) = 9x^2$ is not a function.

4. $\left(k \circ k^{-1}\right)(x) = k\left(\sqrt[3]{x+8}\right) = \left(\sqrt[3]{x+8}\right)^3 - 8 = x$;

$\left(k^{-1} \circ k\right)(x) = k^{-1}\left(x^3 - 8\right) = \sqrt[3]{x^3 - 8 + 8} = x$

Teacher Note

You may wish to discuss the following with your students in conjunction with the mathematics note that describes one-to-one functions.

- A function maps a set A *into* a set B. If each element of set B is an image, then B is the range. In this case, the function is sometimes referred to as an *onto* function.
- If each element in the range of a function (whether or not B is the range) is paired with only one element of the domain, the function is called *one-to-one*.
- If a function from set A to set B is both *one-to-one* and *onto*, the function is a *one-to-one correspondence* between sets A and B.
- If a function f is a one-to-one correspondence between sets A and B, then f^{-1} is a function from B to A.
- If f is a one-to-one function from A to the range, then f^{-1} is a function from the range to A. (This statement is the one presented in the mathematics note.)

Discussion 2

(page 168)

a. Sample response: The inverse of $h(x) = 9x^2$ is not a function since two values are paired with each element of the domain except 0.

b.
1. The graph of each relation and its inverse is symmetrical with respect to the line $y = x$.
2. Sample response: In the graph in Figure **8b**, the relation $x = \sqrt[4]{y}$ has the domain restriction $y \geq 0$. Solving this equation for y gives $y = x^4$. This equation is identical to one of the equations in the graph in Figure **8a**, but the domain restriction limits the relation to the first quadrant. So, the graphs of $x = \sqrt[4]{y}$ and $y = x^4$ are identical only in the first quadrant because of domain restrictions. The same argument applies to the graphs of $y = \sqrt[4]{x}$ and $x = y^4$.
3. Sample response: Since a horizontal line can intersect $y = x^4$ in two places; its inverse is not a function. This is confirmed because $x = y^4$ fails a vertical line test.

   It appears that a horizontal line will contact the graph of $x = \sqrt[4]{y}$ in only one place. Since the graph of $y = \sqrt[4]{x}$ appears to pass the vertical line test, it could be a function. (The rule $y = \sqrt[4]{x}$ confirms that it is a function.)

c. Sample response: The ordered pairs (1,9) and (–1,9) both satisfy $h(x) = 9x^2$. Since the same element of the range is paired with two different elements of the domain, it is not a one-to-one function. Thus, the inverse of $h(x)$ is not a function.

d. Sample response: If a horizontal line intersects the graph of a function in more than one point, then those points have a common range value but different domain values. This indicates that the function is not one-to-one.

Assignment

(page 169)

4.1 **a.**
1. Sample response: In 5 days, an ostrich will use about 40 L of water.
2. Sample response: An ostrich will use 20 L of water in about 2.5 days.
3. $f(x) \approx 8x$
4. Sample response: $f(60) \approx 8 \bullet 60 = 480$ L. This is the amount required for each bird for 60 days. For 40 ostriches, the ranch would need approximately $40 \bullet 480$ or 19,200 L.

b. **1.** Sample response: It would take an ostrich about 9 days to use 70 L of water.

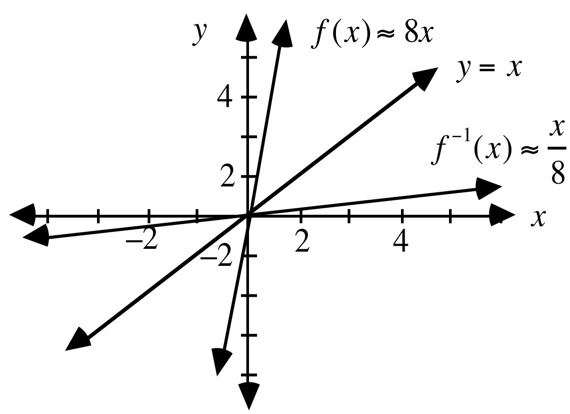
2. Sample response: In 2 days, an ostrich would use about 16 L of water.

3. $g(x) \approx x/8$

4. Sample response: $g(16,000) = 16,000/8 = 2000$. Thus, 16,000 L will carry 1 ostrich for 2000 days or approximately $2000/60 \approx 33$ ostriches for 60 days.

c. Sample response: The functions are inverses of each other.

d. The graph of $g(x) \approx x/8$ is a reflection of $f(x) \approx 8x$ in the line $y = x$, indicating the relations are inverses of each other. Sample graph:



4.2 **a.** Students may analyze the rules or graph the functions to determine if the functions are one-to-one. **Note:** To analyze whether or not a rule is one-to-one algebraically, show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

1. one-to-one

2. one-to-one

3. one-to-one

4. not one-to-one

b. **1.** $f^{-1}(x) = \dfrac{x-3}{2}$

2. $f^{-1}(x) = \sqrt[3]{x}$

3. $f^{-1}(x) = 5x + 4$

c. All three one-to-one functions and their inverses have identical domains and ranges. For both $f(x)$ and $f^{-1}(x)$, the domain and range are $(-\infty, \infty)$.

4.3 **a.** The domain is $(-\infty,\infty)$. The range is $(0,\infty)$.

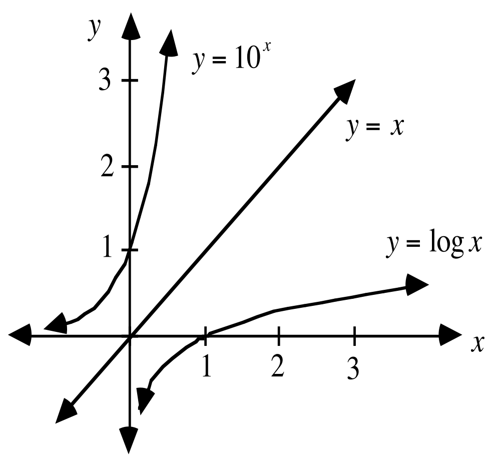
b. $y = \log x$ or $y = \log_{10} x$

c. Sample table:

$f(x)=10^x$		inverse of $f(x)$	
x	10^x	**x**	**log x**
–4	0.0001	0.0001	–4
–2	0.01	0.01	–2
–1	0.1	0.1	–1
0	1	1	0
1	10	10	1
2	100	100	2
4	10,000	10,000	4

d. Sample response: The coordinates in the table are reversed, showing an inverse relationship.

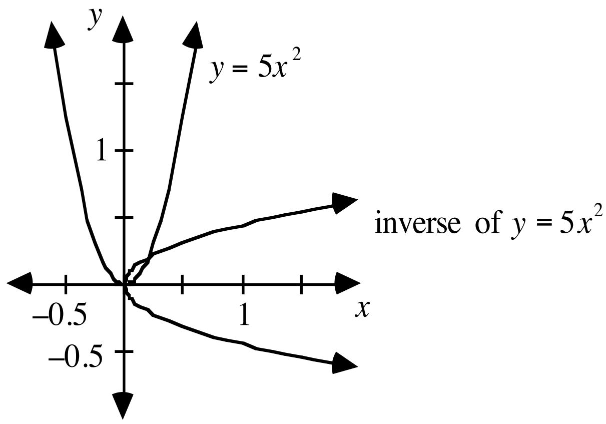
e. The graphs are reflections of each other in the line $y = x$. Sample graph:



f. The domain is $(0,\infty)$; the range is $(-\infty,\infty)$.

g. Sample response: Yes. The function $f(x)=10^x$ is one-to-one, so its inverse is also a function.

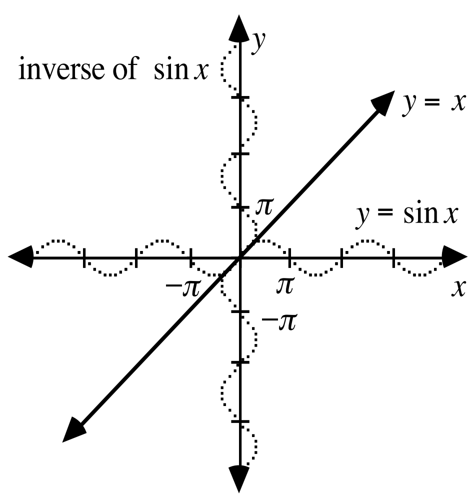
4.4 **a.** Sample graph:



b. Sample response: Graph the line $y = x$ parametrically (by graphing $y = t$, $x = t$), then observe that $y = x$ is the line of reflection for the other two graphs.

c. Sample response: Interchanging the parametric definitions of x and y interchanges the variables. When the values for x and y are interchanged, the inverse results.

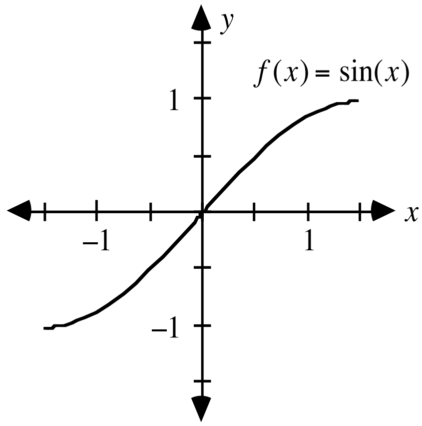
***4.5** **a–b.** Sample graph of $y = \sin x$ and its inverse:

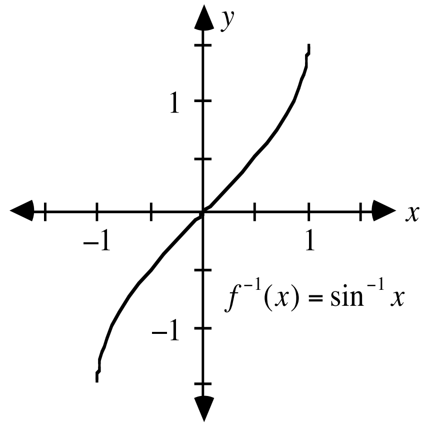


c. Sample response: The inverse of $y = \sin x$ is not a function since elements of the domain are paired with more than one element of the range.

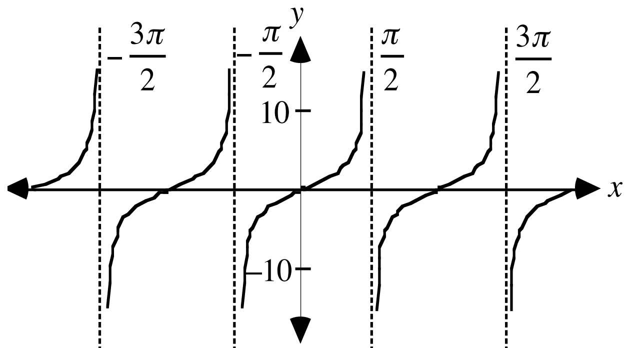
d. Sample response: Since $y = \sin x$ is a periodic function, any interval longer than one period will include repeated values in the range. To make the inverse a function, restrict the domain to $[-\pi/2, \pi/2]$, or any interval π units long starting at $n\pi/2$ where n is an odd integer. Over this domain, every possible range value for $\sin x$ is used as a domain value in the inverse. For other intervals, this is not true.

e. Sample graphs:





4.6 **a.** Sample graph:

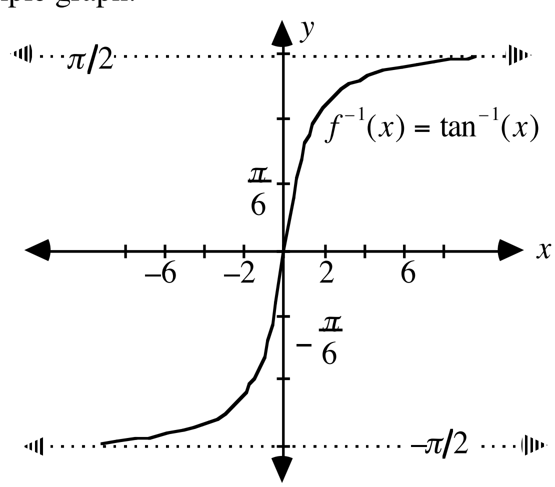


b. Sample response: $(-\pi/2, \pi/2)$

c. The range is $(-\infty, \infty)$.

d. The domain is $(-\infty, \infty)$; the range is $(-\pi/2, \pi/2)$.

e. Sample graph:



4.7 Sample response: The result of a composition of a function with its inverse is the identity function, $f(x) = x$.

* * * * * * * * * *

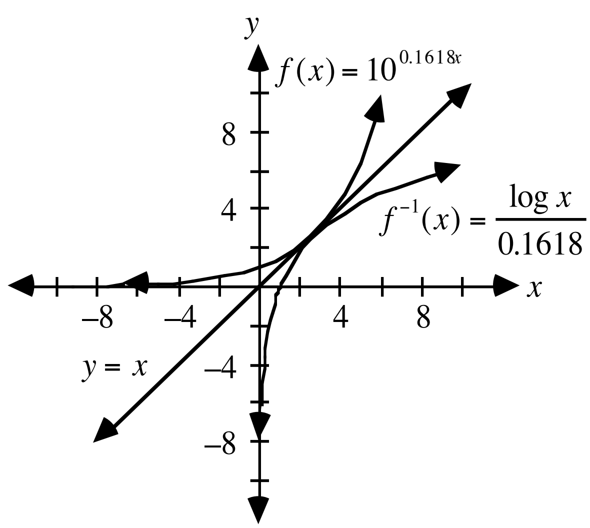
Answers to Summary Assessment (page 173)

1. In 7 years, Terry will have $f(x) = 10^{0.1618(7)} \approx 14$ ostrich pairs.

2. Sample response: The new function is the inverse of $f(x) = 10^{0.1618x}$. Switching the variables only results in one form of the inverse. This equation is $x = 10^{0.1618y}$. Using the laws of logarithms to solve for y results in the following:

$$\log x = \log 10^{0.1618y}$$
$$\log x = 0.1618y$$
$$f^{-1}(x) = y = \frac{\log x}{0.1618}$$

3. **a.** Sample graph:



b. Since the functions are inverses, their graphs are reflections of each other in the line $y = x$.

c. Sample response: It appears that any horizontal line passing through the graph of f will touch it in only one place, which suggests it could be one-to-one. The rule $f(x) = 10^{0.1618x}$ confirms it is one-to-one: any given range value is assigned to only one domain value.

It also appears that any horizontal line passing through the graph of f^{-1} will touch it in only one place, which suggests it could also be one-to-one. The rule

$$f^{-1}(x) = \frac{\log x}{0.1618}$$

confirms it as one-to-one.

4. **a.** $g(x) = 4\sqrt{1400x}$

b. $h(x) = 30x$

c. $k(x) = 10x$

5. **a.** The domain is the set of non-negative integers and the range is $(-\infty,\infty)$.

b. The domain is $(0,\infty)$ and the range is $(-\infty,\infty)$.

c. The domain is $(0,\infty)$ and the range is $(-\infty,\infty)$.

6. $(h+k)x = m(x) = 30x + 10x = 40x$

7. **a.**

$$\begin{aligned}(m \circ g \circ f)(x) &= 40\left(4\sqrt{1400\left(10^{0.1618}\right)^{x}}\right) \\ &\approx 1600\sqrt{14(1.4514)^{x}} \circ\end{aligned}$$

b. The domain of $(m \circ g \circ f)(x)$ is $(0,\infty)$ and the range is approximately $(5986.65,\infty)$.

Module Assessment

1. Consider two functions $h(x)$ and $f(x)$. The domain of $h(x)$ is $[0,\infty)$ and its range is $(-\infty,\infty)$. The domain of $f(x)$ is $(-\infty,\infty)$ and its range is $(-\infty,0]$.

a. Find the domain and range of $(f \circ h)(x)$.

b. Explain why the domain of $h \circ f$ has only one element.

2. For Parts **a–c** below, consider the following function:

$$h(x) = \frac{2x^3 - 5}{3}$$

a. Write an equation for $h^{-1}(x)$.

b. Write an equation for the inverse of $h^{-1}(x)$. Describe the result.

3. Singh works as a salesperson for the Music Box Instrument Company. He receives a salary of $20,000 a year, and earns a 30% commission on the instruments he sells.

a. Write an equation that represents Singh's annual income as a function of the value of the instruments he sells.

b. Find the inverse of the function in Part **a**.

c. What information does the inverse provide?

4. For Parts **a–d** below, consider the functions $f(x) = \sin x$ and $g(x) = 2$. Graph each of the following:

a. $(f + g)(x)$

b. $(f \bullet g)(x)$

c. $(f \circ g)(x)$

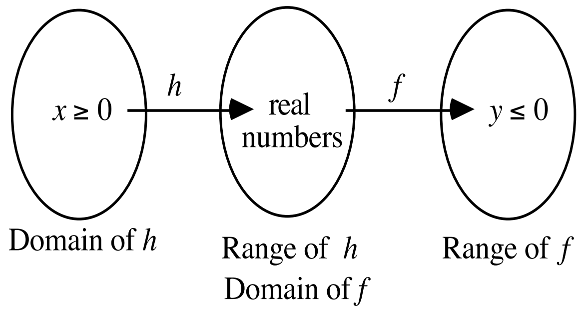
d. $(g \circ f)(x)$

5. **a.** Determine a domain for which the function $f(x) = \sin(2x)$ is a one-to-one function.

b. Describe the domain and range of the inverse of *f* over the domain in Part **a**.

Answers to Module Assessment

1. **a.** Sample response: In this composition, the domain of h is $[0,\infty)$ and the range of h is $(-\infty,\infty)$, which is the domain of f. Therefore, the domain is $[0,\infty)$. The range of $f \circ h$ is $(-\infty,0]$ since every element of the domain of f is in the domain of $f \circ h$.



b. Sample response: There is only one number in the domain of $h \circ f$ since the range of f and the domain of h share only a single real number, 0.

2. **a.** Sample response:

$$h^{-1}(x) = \sqrt[3]{\frac{3x+5}{2}}$$

b. The inverse of $h^{-1}(x)$ is $h(x)$.

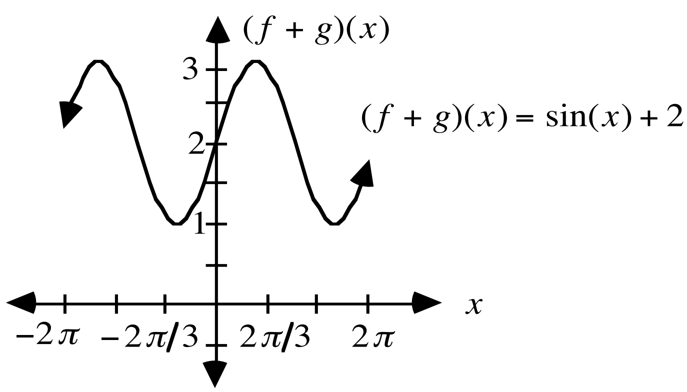
3. **a.** Sample response: $f(x) = 20{,}000 + 0.30x$

b. Sample response:

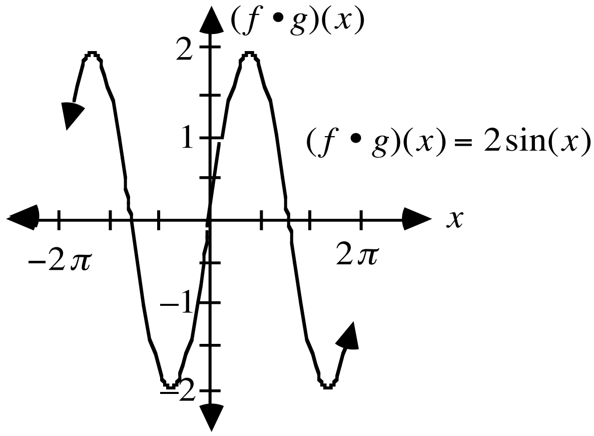
$$f^{-1}(x) = \frac{x - 20{,}000}{0.30}$$

c. Sample response: The inverse finds Singh's annual sales in terms of his annual income.

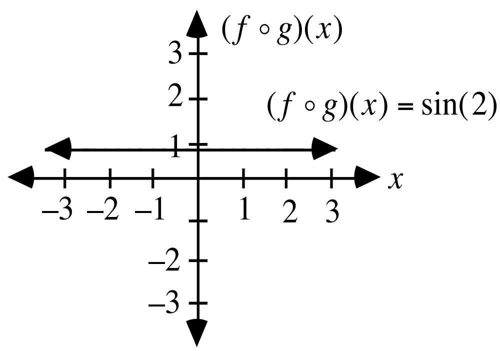
4. **a.** Sample graph:



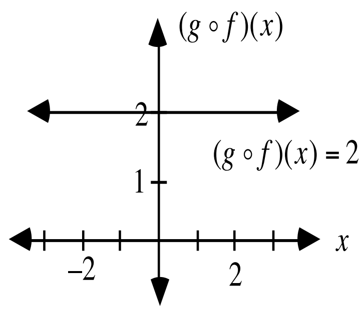
b. Sample graph:



c. Sample graph:



d. Sample graph:



5. **a.** Sample response: $[-\pi/4, \pi/4]$

b. Sample response: The domain of the inverse would be $[-1,1]$. The range would be $[-\pi/4, \pi/4]$.

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Pulfer, W. "Make Up a Story to Explain the Graph." *Mathematics Teacher* 77(January 1984): 32–35.

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Flashbacks

Activity 1

1.1 Describe the subset of real numbers for which each of the following expressions can be evaluated.

a. $\sin x$

b. $1/x$

1.2 For each of the following equations, determine the value of $f(x)$ when $x = 3$.

a. $f(x) = 3x + 2$

b. $f(x) = 4\pi \sin(\pi x)$

Activity 2

2.1 Describe the domain and range of each of the following functions:

a. $h(x) = x^2$

b. $g(x) = \sqrt{x}$

2.2 Find a possible rule for the following function:

$$(-3,8), (-2,3), (-1,0), (0,-1), (1,0), (2,3), (3,8)$$

2.3 Use the distributive property to find an equivalent expression for each of the following:

a. $16x + 18x$

b. $ax + bx$

Activity 3

3.1 For what values of x does $\sqrt{x}$ exist?

3.2 Identify the domain and range of each of the following functions:

a. $h(x) = \dfrac{\sin x}{x - 5}$

b. $g(x) = \dfrac{x^2 + 2}{5 - x}$

c. $f(x) = \sin x$

3.3 Determine the set of possible values for $2x + 3$ if $-2 < x \leq 7$.

Activity 4

4.1 **a.** Sketch the graph of a relation on a coordinate system.

b. On the same coordinate system, sketch the line $y = x$.

c. Sketch the reflection of the graph from Part **a** in the line $y = x$.

4.2 **a.** Express $2^y = 16$ as a logarithmic equation.

b. Evaluate the logarithmic equation from Part **a**.

Answers to Flashbacks

Activity 1

1.1 **a.** $(-\infty,\infty)$

b. $(-\infty,0)\cup(0,\infty)$

1.2 **a.** 11

b. 0

Activity 2

2.1 **a.** The domain is $(-\infty,\infty)$ and the range is $[0,\infty)$.

b. The domain is $[0,\infty)$ and the range is $[0,\infty)$.

2.2 Sample rule: $f(x)=x^2-1$.

2.3 **a.** $x(16+18)=34x$

b. $x(a+b)$

Activity 3

3.1 Sample response: The value of x must be greater than or equal to 0.

3.2 **a.** The domain is $(-\infty,5)\cup(5,\infty)$ and the range is $(-\infty,\infty)$.

b. The domain is $(-\infty,5)\cup(5,\infty)$ and the range is $(-\infty,\infty)$.

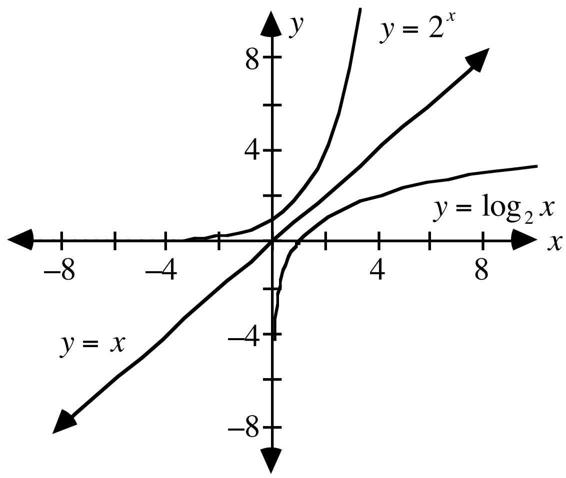
c. The domain is $(-\infty,\infty)$ and the range is $[-1,\ 1]$.

3.3 Students should solve as follows:

$$-2<x\le 7$$
$$2(-2)<2x\le 2(7)$$
$$-4<2x\le 14$$
$$-4+3<2x+3\le 14+3$$
$$-1<2x+3\le 17$$

Activity 4

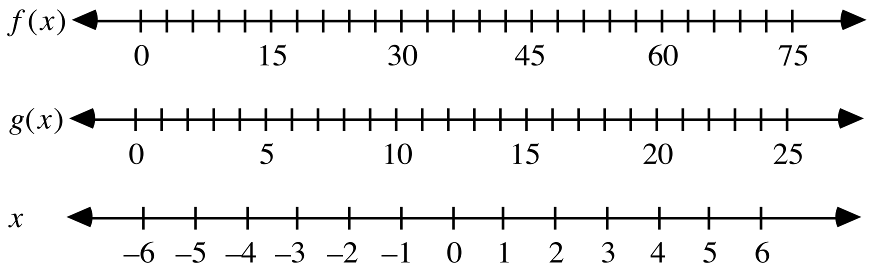
4.1 **a–c.** Answers will vary. Sample sketch:

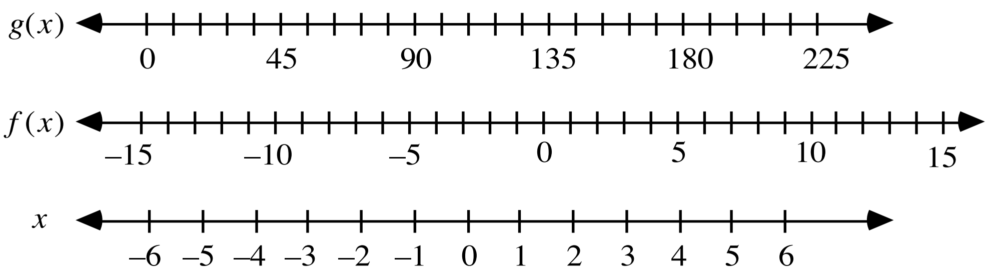


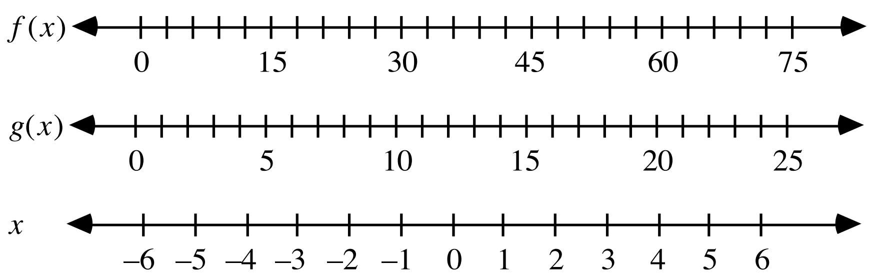
4.2 **a.** $y = \log_2 16$

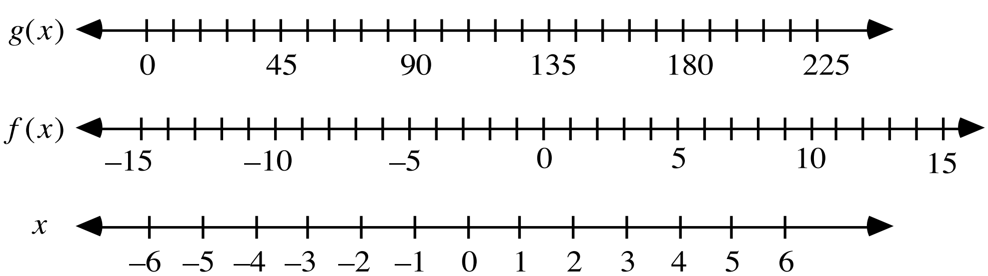
b. $y = 4$ since $2^4 = 16$

Mapping Template









Mathematics in Motion

As an arrow glides toward a target, could you describe its exact location at any time during the flight? In this module, you examine methods for modeling the paths of moving objects.

Ruth Brocklebank • Phil Lieske • Mike Trudnowski

Teacher Edition

Mathematics in Motion

Overview

In this module, students model paths of projectiles with parametric equations. They also use parametric equations to graph circles and ellipses.

Objectives

In this module, students will:

- use parametric equations to investigate the paths of moving objects
- develop parametric equations for a parabola
- review and extend parametric equations for a circle
- develop parametric equations for an ellipse
- use vectors and trigonometry to determine parametric relationships
- examine the difference between speed and angular speed.

Prerequisites

For this module, students should know:

- how to use parametric equations to model linear paths
- the general shapes of parabolas and ellipses
- the standard forms of the equations of circles and ellipses
- how to draw vector diagrams
- how to resolve a vector into its horizontal and vertical components
- the sine, cosine, and tangent functions
- the inverse tangent relationship
- how to solve systems of equations.

Time Line

Activity	Intro.	1	2	3	Summary Assessment	Total
Days	1	2	2	3	2	10

Materials Required

Materials	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
metersticks or rulers	X				
coins or washers	X				
index cards	X				
binder clips	X				

Technology

Materials	Activity				
	Intro.	**1**	**2**	**3**	**Summary Assessment**
graphing utility		X	X	X	X
spreadsheet		X	X		
symbolic manipulator		X	X	X	X
geometry utility				X	

Mathematics in Motion

Introduction

(page 179)

The introduction describes a plan to airlift a crate of equipment to firefighters. This context initiates a discussion of two-dimensional motion.

Materials List

- meterstick or ruler (one per group)
- identical coins or metal washers (two per group)
- coins or washers of various sizes and masses (two per group)
- index cards (one per group)
- binder clips (one per group)

Discussion 1

(page 179)

a.

1. Sample response: The path of the crate is influenced by its initial velocity, the force of gravity, friction (air resistance), and wind speed and direction.
2. Sample response: The crate's initial velocity is the same as the plane's velocity. This affects the horizontal distance it will travel. The force of gravity causes the crate to accelerate toward Earth. The force of friction slows the crate's horizontal and vertical velocities. Wind speed and direction also affect the crate's velocity.

b. At this point in the module, it is not critical that students identify the path as parabolic. However, they should realize that the crate's motion has both horizontal and vertical components.

c. Sample response: No. While in the plane (and instantaneously after leaving the plane) the crate is moving in the same direction and at the same speed as the plane. If the crate is dropped when the plane is directly over the landing area, its forward motion will cause it to miss the target.

Teacher Note

Because students often fail to recognize the independence of the horizontal and vertical components of projectile motion, you may wish to consult with a physics teacher at your school before conducting the following exploration. Many physics departments can provide an apparatus for demonstrating two-dimensional motion (typically involving one steel ball projected horizontally and another dropped simultaneously).

Your physics department also may have a feather-and-coin apparatus to demonstrate the effects of air resistance on falling objects. This device consists of a vacuum tube containing a feather and a coin. When the tube is inverted, the coin and feather drop from one end to the other in the same amount of time.

Exploration (page 179)

Students investigate two-dimensional motion.

a. Students create a device for simultaneously projecting and dropping objects from the same height at the same time. Both wooden and plastic metersticks will work. Students may want to fasten a second index card beneath the horizontal platforms for reinforcement.

b–c. Students observe and sketch the paths of the falling objects and compare the time required for them to hit the ground. They may obtain better results by placed the objects near the outer edges of the platforms. **Note:** You also may wish to ask students to vary the initial height.

d. Students repeat the experiment using two unlike objects, as in Galileo’s experiment at Pisa.

Discussion 2 (page 180)

a. Sample response: The path of the object projected horizontally is curved. The path of the object that falls straight down is linear.

b. Sample response: Yes. Both objects fell from the same height.

c. Students should observe that both objects hit the floor at the same time.

d. Sample response: No. The objects hit the ground at the same time, although their paths were very different.

e. Students should obtain similar results even when using objects of different sizes or weights.

f. Due to air resistance, the feather and the coin would not hit the floor at the same time. However, if they were placed in a vacuum, they would hit the ground at the same time.

(page 181)

Activity 1

In this activity, students use parametric equations to model the paths of falling objects that are either dropped from rest or projected horizontally.

Materials List

- none

Technology

- graphing utility
- spreadsheet
- symbolic manipulator

Exploration

(page 181)

a–b. Sample table:

	Object Dropped from Rest (□)		**Object Projected Horizontally** (■)	
Time (sec)	Horizontal Distance (m)	Vertical Distance (m)	Horizontal Distance (m)	Vertical Distance (m)
0.0	0	10.0	0	10.0
0.2	0	9.8	0.2	9.8
0.4	0	9.2	0.4	9.2
0.6	0	8.2	0.6	8.2
0.8	0	6.9	0.8	6.9
1.0	0	5.1	1.0	5.1
1.2	0	2.9	1.2	2.9
1.4	0	0.4	1.4	0.4

c.

1. The change in horizontal position between consecutive points for both objects is constant. For the object dropped from rest, the change is 0 m. For the object projected horizontally, the change is 0.2 m.

2. The average horizontal velocity between consecutive points is $0\ \text{m/sec}$ for the object dropped from rest, and $1\ \text{m/sec}$ for the object projected horizontally.

3. For the object dropped from rest, the equation is $x(t) = 0 \bullet t = 0$. For the object projected with horizontal velocity, the equation is $x(t) = 1 \bullet t = t$.

d. **1.** Because of the acceleration due to gravity, the change in vertical position between consecutive points for each object is not constant. (Their heights, however, are the same at any given time. This indicates that both objects are experiencing the same increase in velocity as they continue to fall.)

2. Sample table:

Time Interval (sec)	**Object Dropped from Rest (m/sec)**	**Object Projected Horizontally (m/sec)**
$[0, 0.2)$	0.98	0.98
$[0.2, 0.4)$	2.94	2.94
$[0.4, 0.6)$	4.90	4.90
$[0.6, 0.8)$	6.86	6.86
$[0.8, 1.0)$	8.82	8.82
$[1.0, 1.2)$	10.78	10.78
$[1.2, 1.4)$	12.74	12.74

e–f. The average acceleration between consecutive intervals for each object is 9.8 m/sec^2. This equals the value given for acceleration due to gravity.

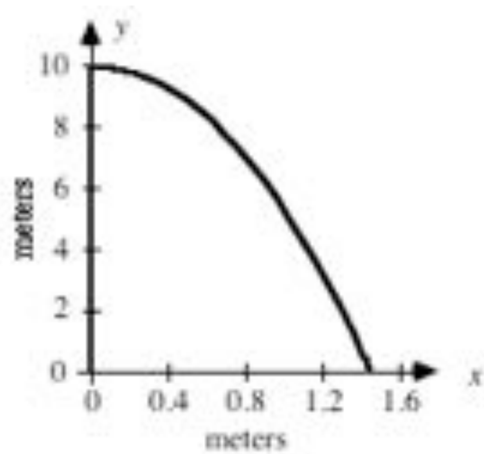
g. **1.** For both objects, the equation is $y(t) = -4.9t^2 + 10$.

2. The resulting y-values should be reasonably close to those estimated from the graph: $y(0.2) = 9.8$, $y(0.8) = 6.9$, and $y(1.2) = 2.9$.

h. The parametric equations for the object dropped from rest are $x(t) = 0$ and $y(t) = -4.9t^2 + 10$. The parametric equations for the object projected with horizontal velocity are $x(t) = 1t$ and $y(t) = -4.9t^2 + 10$.

i. **Note:** On some graphing utilities, the graph of the object falling straight down may not be visible, since it is located along the y-axis. If possible, students should turn off the axes.

1. Sample graph:



2. Sample response: Changing the increment for t affects the speed at which the graph is drawn. It also changes the increments of the trace feature. **Note:** Students should recognize that the utility plots the graph more slowly as the increment decreases because the number of ordered pairs to be calculated increases.

3. Students should observe that the height (y) is the same for both objects at any time t.

Discussion (page 184)

a. Sample response: The graph of the object falling straight down is a vertical line segment. The graph of the object projected horizontally appears to be part of a parabolic curve. When the two are graphed simultaneously, both reach the x-axis at the same time.

b. Sample response: No. The speed at which the graphs are drawn depends on the increment chosen for the parameter t.

c. 1. Students may determine this time either by identifying the value of t that corresponds with the point where the graph crosses the x-axis, or by solving the following equation for t:

$$-4.9t^2 + 10 = 0$$
$$t^2 = 10/4.9$$
$$t \approx 1.43$$

2. Sample response: It takes 1.43 sec for the object to fall to the ground. Half of 1.43 sec is 0.715 sec. Substituting 0.715 sec for t into the parametric equations gives the coordinates for each object halfway through its fall.

d. Sample response: To determine this distance, find the approximate time for the object to hit the ground and substitute this value into the equation for horizontal distance. For example, the time required for the objects to hit the ground is about 1.43 sec. Using the equation $x(t) = 1t$, the horizontal distance traveled is $x(1.43) = 1 \bullet 1.43 = 1.43$ m.

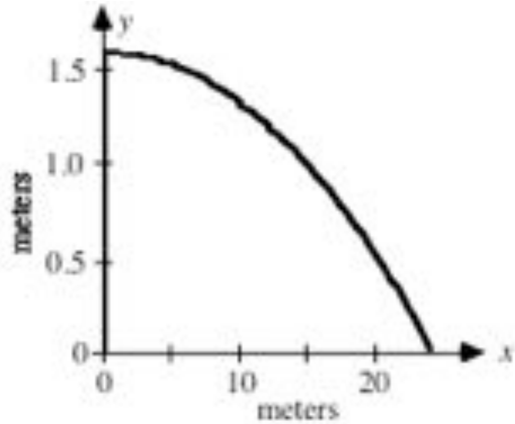
e. In this context, a negative y-value would indicate that the object is below ground level.

Assignment (page 184)

1.1 **a.** 1. The parametric equations for the motion of the watch are $x(t) = 0$ and $y(t) = -4.9t^2 + 1.6$. The motion is linear and downward.

2. The parametric equations for the motion of the arrow are $x(t) = 42t$ and $y(t) = -4.9t^2 + 1.6$. The motion is parabolic.

b. Sample graph:



c. After 0.25 sec, both the watch and the arrow are 1.29 m above the ground.

d. The total time required for each object to hit the ground is 0.57 sec.

e. The horizontal distance traveled by the arrow is $42 \bullet 0.57 \approx 24$ m .

1.2 **a.** Students should convert 250 km/hr to 69.4 m/sec . The parametric equations that model the path of the crate are:

$$x(t) = 69.4t$$

$$y(t) = -4.9t^2 + 100$$

b. Solving $y(t) = 0$ for t:

$$-4.9t^2 + 100 = 0$$

$$t \approx 4.5 \text{ sec}$$

c. Substituting 4.5 sec into $x(t)$ yields the following:

$$x(4.5) = 69.4 \bullet 4.5$$

$$= 312.3 \text{ m}$$

d. Sample response: Since the plane maintains the same horizontal velocity as the crate, it also will travel 312.3 m. Therefore, it will be directly over the spot where the crate hits the ground.

***1.3** The difference in the elevations of the plane and the climbers is 270 m. The parametric equations that model the motion of the package are $x(t) = 90t$ and $y(t) = -4.9t^2 + 270$.

a. Solving $0 = -4.9t^2 + 270$ for t, the package reaches the ground about 7.4 sec after it is released from the airplane.

b. Since it will take 7.4 sec for the package to hit the ground and the plane is flying at 90 m/sec, the rescuers should release the package when the horizontal distance from the target is $7.4 \bullet 90 = 666$ m .

* * * * *

1.4 **a.** Sample response: The path of the skydiver is part of a parabola that opens downward from its vertex.

b. $x(t) = 65t$ and $y(t) = -4.9t^2 + 1300$

c. At $t = 10$ sec, the skydiver is 810 m above the ground. The vertical distance fallen is $1300 - 810 = 490$ m.

d. At $t = 10$ sec, the skydiver has traveled 650 m horizontally.

e. Since the horizontal velocity of both the skydiver and the airplane is 65 m/sec, both will have traveled 650 m after 10 sec. Ignoring air resistance, the airplane will be 490 m directly above the skydiver.

***1.5** Sample response: The motion of the marble can be modeled by the parametric equations $x(t) = v_x t$ and $y(t) = -4.9t^2 + 0.8$. Solving the equation $y(t) = -4.9t^2 + 0.8$ when $y(t) = 0.1$ yields $t \approx 0.378$ sec, the time when the object reaches falls to the height of the cup.

Substituting $t \approx 0.378$ into the equation $x(t) = v_x t$ and solving for v_x when $x(t) = 0.75$, the distance to the front of the cup, yields $v_x = 1.98$ m/sec. Solving when $x(t) = 0.80$, the distance to the back of the cup, yields $v_x = 2.12$ m/sec. So, the horizontal velocity necessary to roll the marble off the table and into the cup is between 1.98 m/sec and 2.12 m/sec.

* * * * * * * * * *

(page 186)

Activity 2

In this activity, students study two-dimensional motion in which an object is projected with an initial velocity at a given angle of elevation. **Note:** The physics department at your school may be able to provide a device for demonstrating this type of motion.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator
- spreadsheet (optional)

Discussion 1

(page 186)

a. Sample response: There are at least three forces acting on the ball when it is hit. One is gravity; another is friction. The force exerted by the bat on the ball is what gives it an initial velocity away from the bat.

b. Sample response: Many factors influence the distance traveled by a hit ball, including the force with which it was hit, the velocity of the pitch, the angle of elevation, the amount of rotation on the ball, the direction in which the ball leaves the bat, and the wind velocity and direction.

For example, a ground ball or pop fly will not travel as far horizontally as a line drive, while a foul ball may even land behind the hitter.

Teacher Note

In Part **f** of the exploration, students are asked to simulate the motion of a softball hit at various angles of elevation. A graphing utility's list feature may be helpful here. For example, if using a TI-92 graphing calculator, define L_1 by the formula seq(A,A,0,90,5). This enters angle measures that are multiples of 5 between 0° and 90°, inclusive, into L_1. Next, define the horizontal component L_2 as $40\cos L_1$, and the vertical component L_3 as $40\sin L_1$. Finally, enter and graph the parametric equations $X_{1T} = L_2T$ and $Y_{1T} = -4.9T^2 + L_3T + 1$ with $0 \le T \le 10$, $-10 \le X \le 170$, and $-20 \le Y \le 80$.

Exploration

(page 186)

a. **1.** $v_x = 40\cos\theta$

2. $v_y = 40\sin\theta$

b. **Table 1: Component velocities of a softball**

Initial Velocity	Angle (degrees)	Horizontal Component	Vertical Component
40	0	40	0
40	5	39.85	3.49
40	10	39.39	6.95
40	15	38.64	10.35
40	20	37.59	13.68
40	25	36.25	16.91
40	30	34.64	20
40	35	32.77	22.94
40	40	30.64	25.71
40	45	28.28	28.28
40	50	25.71	30.64
40	55	22.94	32.77
40	60	20	34.64
40	65	16.91	36.25
40	70	13.68	37.59
40	75	10.35	38.64
40	80	6.95	39.39
40	85	3.49	39.85
40	90	0	40

c. **1.** $y(t) = -4.9t^2 + 20t + 1$

2. The height of the softball after 4 sec can be found as follows:

$$\begin{aligned} y(t) &= -4.9t^2 + 20t + 1 \\ &= \left(-4.9 \text{ m/sec}^2\right)(4 \text{ sec})^2 + (20 \text{ m/sec})(4 \text{ sec}) + 1 \text{ m} \\ &= -78.4 \text{ m} + 80 \text{ m} + 1 \text{ m} \\ &= 2.6 \text{ m} \end{aligned}$$

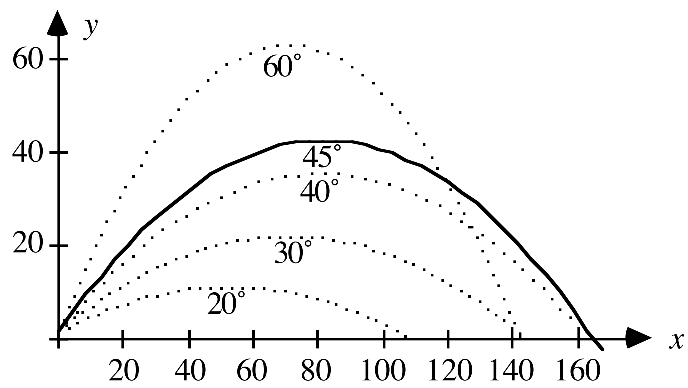
d. **1.** $x(t) = 34.6t$

2. The horizontal distance traveled by the softball in 4 sec can be found as follows:

$$\begin{aligned} x(t) &= 34.6t \\ &= (34.6 \text{ m/sec})(4 \text{ sec}) \\ &= 138.4 \text{ m} \end{aligned}$$

e. When hit at an angle of 30°, the ball travels a horizontal distance of approximately 143 m. (See sample graph in Part **f** below.)

f. An angle of 45° gives a hit ball the maximum horizontal distance. Sample graph:



Discussion 2

(page 188)

a. The paths of the softball are parabolic.

b. **1.** Sample response: Set $y(t)$ equal to 0 and solve for t to determine the time it takes for the softball to hit the ground. Then substitute that value into $x(t)$ to find the maximum horizontal distance.

Note: You may wish to discuss the use of the quadratic formula for finding the roots of the general quadratic equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Sample response: The softball will be at its maximum height about halfway through the path, which corresponds with about half the time it takes for the ball to hit the ground. You could find the exact time when the ball is at its highest point by solving $y(t) = 0$ for t, then taking the mean of the two roots.

3. Sample response: Substitute your answer from Step **2** into $y(t)$.

c. A 45° angle provides the maximum distance.

d. **1.** Sample response: Since the wind typically blows parallel to the ground, it affects the horizontal velocity. If the wind is blowing against the flight of the ball, it will slow the ball down. If the wind is blowing with the flight of the ball, it will speed the ball up.

2. If a wind is blowing toward Kami, its velocity must be subtracted from the horizontal component of the velocity of the ball.

3. If a wind is blowing away from Kami, its velocity must be added to the horizontal component of the velocity of the ball.

Assignment

(page 189)

2.1 **a.** The following parametric equations can be used to model the motion of the softball: $x(t) = 40\cos 20° \bullet t$ and $y(t) = -4.9t^2 + 40\sin 20° \bullet t + 1$. Using this parabolic model, the ball's maximum height occurs at the vertex of the parabola. Solving $y(t) = 0$ for t yields approximately –0.07 and 2.86. The x-value of the vertex may be calculated by finding the mean of these roots: $(-0.07 + 2.86)/2 \approx 1.4$ sec.

b. Substituting 1.4 sec into $y(t)$, the maximum height of the softball is approximately 10.5 m.

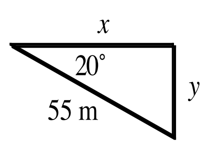
c. Solving $x(t) = 80$ for t gives $t = 2.13$ sec, the time when the softball is 80 m from home plate. Substituting $t = 2.13$ sec into $y(t)$ gives 7.9 m, the height of the softball at that time. Since the fence is 2 m tall, the softball will clear the fence by 5.9 m.

2.2 The parametric equations for the flight of the softball with a 20° angle of elevation and a headwind of 8.5 m/sec are:

$$x(t) = 40\cos 20° \bullet t - 8.5t \qquad y(t) = -4.9t^2 + 40\sin 20° \bullet t + 1$$
$$\approx 37.6t - 8.5t \qquad \approx -4.9t^2 + 13.7t + 1$$
$$\approx 29.1t$$

Solving $x(t) = 80$ for t gives a time of 2.74 sec. Substituting 2.74 sec into $y(t)$ gives a height of approximately 1.7 m. The ball will not clear the 2-m fence.

***2.3** Sample response: Assuming that the initial velocity of the skier has no vertical component, the horizontal and vertical distances traveled can be determined as follows: $y = 55\sin 20 \approx 18.81$ m and $x = 55\cos 20 \approx 51.7$ m.



The following parametric equations model the path of the skier, where v_x is the horizontal velocity and 4 is the distance above the x-axis:

$$x(t) = v_x t$$
$$y(t) = -4.9t^2 + 4$$

To find the horizontal velocity, first solve $y(t) = -18.81$ for t.

$$y(t) = -4.9t^2 + 4$$
$$-18.81 = -4.9t^2 + 4$$
$$t \approx 2.16 \text{ sec}$$

Substituting 2.16 for t and solving $x(t) = 51.7$ for v_x, the horizontal velocity is:

$$x(t) = v_x t$$
$$51.7 = v_x(2.16)$$
$$v_x \approx 23.9 \text{ m/sec}$$

2.4 Sample response: By analyzing different angles of elevation, the maximum horizontal distance debris can travel is approximately 308 m. This occurs for angle measures of approximately 45°. The parametric equations modeling the motion of the rocks and debris are $x(t) = 55(\cos\theta)t$ and $y(t) = -4.9t^2 + 55(\sin\theta)t$. These equations assume that all debris and rocks begin at ground level and that air resistance is negligible. **Note:** Some students may examine different initial heights.

* * * * *

2.5 Answers will vary. Sample response: Using a 40° angle of elevation, the parametric equations for Rowdy's flight are $x(t) = v_0 \cos 40° \bullet t$ and $y(t) = -4.9t^2 + v_0 \sin 40° \bullet t + 2$. The following range of values for the initial velocity will produce paths allowing Rowdy to land in the net.

$$17.6 \text{ m/sec} \le v_0 < 19.1 \text{ m/sec}$$

These solutions were found by solving the following systems of equations:

$$\begin{cases} 32 = v_0 \cos 40 \bullet t \\ 1 = 4.9t^2 + v_0 \sin 40 \bullet t + 2 \end{cases} \text{ and } \begin{cases} 38 = v_0 \cos 40 \bullet t \\ 1 = 4.9t^2 + v_0 \sin 40 \bullet t + 2 \end{cases}$$

***2.6** Students should assume that the initial height of the golf ball is 0.

The parametric equations for the flight of the golf ball using a 6-iron are $x(t) = 44.5\cos 32° \bullet t$ and $y(t) = -4.9t^2 + 44.5\sin 32° \bullet t$. The golf ball will hit the ground approximately 181 m away from Lief.

The parametric equations for the flight of the golf ball using a 7-iron are $x(t) = 41.5\cos 36° \bullet t$ and $y(t) = -4.9t^2 + 41.5\sin 36° \bullet t$. The golf ball will hit the ground approximately 167 m away from Lief.

The parametric equations for the flight of the golf ball using an 8-iron are $x(t) = 38.5\cos 40° \bullet t$ and $y(t) = -4.9t^2 + 38.5\sin 40° \bullet t$. The golf ball will hit the ground approximately 149 m away from Lief.

The parametric equations for the flight of the golf ball using a 9-iron are $x(t) = 36.5\cos 44° \bullet t$ and $y(t) = -4.9t^2 + 36.5\sin 44° \bullet t$. The golf ball will hit the ground approximately 135 m away from Lief.

Lief should use the 7-iron.

* * * * * * * * * *

(page 191)

Activity 3

In this activity, students review the parametric equations for a circle, extend them to the modeling of circular motion, and develop the parametric equations for an ellipse.

Materials List

- none

Technology

- graphing utility
- geometry utility
- symbolic manipulator

Teacher Note

On some graphing utilities, graphs of circles may appear to be elliptical. To eliminate this misrepresentation, the ratio of the range of values on the x-axis to the range of values on the y-axis should be set equal to the ratio of the number of horizontal pixels to the number of vertical pixels. Students may also find that decreasing the increment for the parameter will produce more “rounded” graphs.

Exploration 1

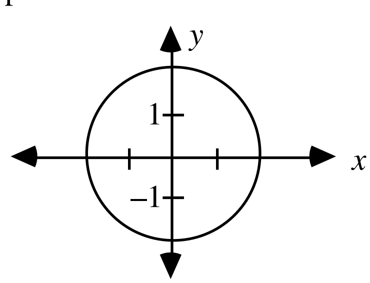
(page 191)

Students should set their technology to use angle measures in radians.

a. **1.** The following parametric equations model the path of the airplane:

$$x(t) = 2\cos t$$
$$y(t) = 2\sin t$$

2. Sample graph:



b. Sample response: The circumference of the circle is $2\pi(2) \approx 12.6$ m . Since the plane travels this distance in 1 sec, its speed is 12.6 m/sec .

c. Sample response: In 1 revolution, the airplane passes through an angle of 2π radians. Since the plane completes 1 revolution in 1 sec, its angular speed is 2π radians/sec.

d. To determine the appropriate time, some students may graph each pair of parametric equations and trace a point as it travels around the circle. Others may reason that since *c* equals angular speed in radians per second, the time required to complete 1 revolution is $2\pi/c$.

1. 2 sec

2. 1 sec

3. 3 sec

e. **1.** $x(t) = 2\cos\left(\frac{2\pi}{2.5}t\right) \approx 2\cos(2.51t)$

$y(t) = 2\sin\left(\frac{2\pi}{2.5}t\right) \approx 2\sin(2.51t)$

2. $x(t) = 2\cos\left(\frac{2\pi}{0.8}t\right) \approx 2\cos(7.85t)$

$y(t) = 2\sin\left(\frac{2\pi}{0.8}t\right) \approx 2\sin(7.85t)$

3. $x(t) = 2\cos\left(\frac{2\pi}{a}t\right)$

$y(t) = 2\sin\left(\frac{2\pi}{a}t\right)$

f. Since the circumference of the circle is $2\pi(2) = 4\pi$ m, the plane's speed is $4\pi/a$ m/sec, where *a* is the time to complete 1 revolution.

1. $4\pi/2.5 \approx 5$ m/sec

2. $4\pi/0.8 \approx 15$ m/sec

3. $4\pi/a$ m/sec

Discussion 1

(page 194)

a. Sample response: No. The plotting speed depends on the increments of *t* chosen when graphing the equations.

b. Sample response: Since the initial position corresponds with $t = 0$, it can be found by substituting 0 into the parametric equations. The result is the ordered pair $(h + r, k)$.

c. Sample response: If the object's initial position was $(r, 0)$, its position with respect to time could be modeled by the equations $x(t) = r\cos(ct)$ and $y(t) = r\sin(ct)$, where c is the angular speed.

If the initial position was A, you could model the object's movement with the equations below:

$$x(t) = r\cos\left(ct + \frac{\pi}{2}\right)$$

$$y(t) = r\sin\left(ct + \frac{\pi}{2}\right)$$

If the initial position was B, you could model the object's movement with the equations below:

$$x(t) = r\cos(ct + \pi)$$

$$y(t) = r\sin(ct + \pi)$$

If the initial position was C, you could model the object's movement with the equations below:

$$x(t) = r\cos\left(ct + \frac{3\pi}{2}\right)$$

$$y(t) = r\sin\left(ct + \frac{3\pi}{2}\right)$$

d. Sample response: If the object completes 1 revolution in 5 sec, and the initial position is $(-4, 3)$, then the parametric equations that model its movement are:

$$x(t) = -4 + 7\cos\left(\frac{2\pi}{5}t\right)$$

$$y(t) = 3 + 7\sin\left(\frac{2\pi}{5}t\right)$$

e. Sample response: Since $c = 6$ radians/hr, it takes $\pi/3$ hr for the object to complete 1 revolution around the circle. The circumference of the circle is $2\pi(9) = 18\pi$ units. Thus, the object's speed is:

$$\frac{18\pi}{\pi/3} \approx 54 \text{ units/hr}$$

f.

1. Sample response: The speed of P is greater than the speed of Q since an arc traveled on the larger circle is longer than the corresponding arc on the smaller circle.
2. The angular speeds of P and Q are equal.

Teacher Note

In Exploration **2**, students develop a set of parametric equations for an ellipse with center at the origin and foci on the *x*-axis. You may wish to review the geometric properties of an ellipse, including its major and minor axes, as well as its equation in standard form (also known as its rectangular equation):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

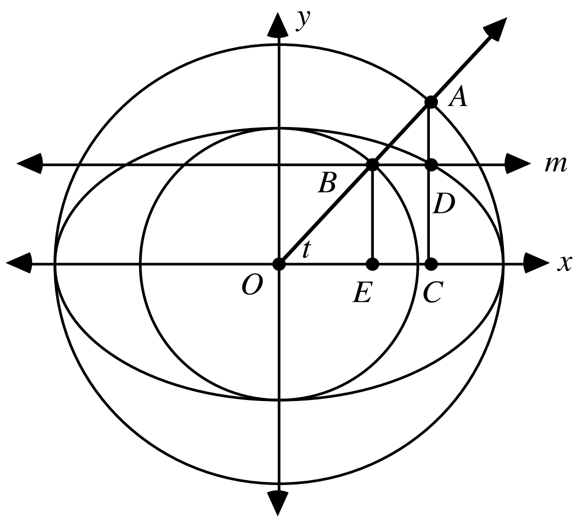
Students should recall the standard form of the equation of an ellipse from the Level 4 module, "Transmitting Through Conics."

Exploration 2

(page 195)

Note: The development of the graph of an ellipse in this exploration depends on Euclidean constructions, rather than on the classic definition as a locus of points *P* such that the sum of the distances from two fixed points to *P* is a constant.

a–b. You may wish to demonstrate this construction to the class. As shown in the diagram below, point *D* is a point on the ellipse.



c. Students should note that *OA* and *OB* are the radii of the larger and smaller circles, respectively.

1. Sample response:

$$\cos t = \frac{OC}{OA}$$

$$= \frac{x}{OA}$$

$$x = OA \bullet \cos t$$

2. Sample response:

$$\sin t = \frac{BE}{OB} = \frac{DC}{OB}$$
$$= \frac{y}{OB}$$
$$y = OB \bullet \sin t$$

d. The following parametric equations model the path of point A:

$$x(t) = 4\cos t$$
$$y(t) = 4\sin t$$

These parametric equations model the path of point B:

$$x(t) = 2\cos t$$
$$y(t) = 2\sin t$$

The parametric equations below model the path of point D:

$$x(t) = 4\cos t$$
$$y(t) = 2\sin t$$

Discussion 2

(page 196)

a. Sample response: All the constructions form ellipses. The size of the ellipse appears to depend on the sizes of the circles. The shape of the ellipse seems to be determined by the difference in the sizes of the circles.

b. Students should come to the conclusion that for any ordered pair (x, y) on the ellipse in the construction, the x-coordinate can be expressed as $x(t) = a\cos t$, where a is the radius of the larger circle, while the y-coordinate can be expressed as $y(t) = b\sin t$, where b is the radius of the smaller circle. These are the parametric equations for an ellipse with center at the origin and foci on the x-axis.

c. Sample response: The graph of the ellipse does not represent a function, since there are points on the ellipse with the same x-values, but different y-values.

d.

1. Sample response: The domain is all real numbers t. The range is the set of ordered pairs (x, y), where $x = a\cos t$ and $y = b\sin t$.

2. Sample response: This relationship is a function because for each value of t, there is exactly one ordered pair (x, y) that describes a point on the ellipse.

e. Sample response: If $a > b$, the value of a is half the length of the major axis, while the value of b is half the length of the minor axis. If $b > a$, the value of b is half the length of the major axis, while the value of a is half the length of the minor axis.

f. If $a = b$, then a circle is formed.

g. Sample response: On many graphing utilities, the equation of an ellipse in standard form must be solved for y then graphed in two parts, where each part represents a function. Using parametric equations, an ellipse can be graphed in its entirety with no algebraic manipulation. Using parametric equations also allows you to control the accuracy of the graph by setting the increment size for the parameter.

h. The result is the standard form of the equation of an ellipse with center at the origin and foci on the x-axis:

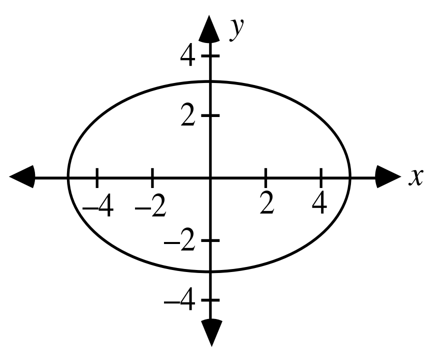
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Assignment

(page 197)

3.1 **a.** The parametric equations for the ellipse are $x(t) = 5\cos t$ and $y(t) = 3\sin t$.

b. The graph of these equations is an ellipse with foci on the x-axis. Since $c^2 = a^2 - b^2$, where c is the distance from the center to the foci, a is half the length of the major axis, and b is half the length of the minor axis, the locations of the foci are (–4,0) and (4,0).



c. The graph of these equations is an ellipse with foci on the y-axis. Since $c^2 = a^2 - b^2$, where c is the distance from the center to the foci, a is half the length of the major axis, and b is half the length of the minor axis, the locations of the foci are (0,–4) and (0,4).

***3.2** **a.** Since the circumference of the circle is $2\pi(8) = 16\pi$ m, the speed of a chair on the Ferris wheel is:

$$\frac{16\pi}{20} \approx 2.5 \text{ m/sec}$$

The angular speed of the chair is $2\pi/20 \approx 0.31$ radians/sec.

b. **1.** Sample response:

$$x(t) = 8\cos\left(\frac{2\pi}{20}t\right)$$

$$y(t) = 10 + 8\sin\left(\frac{2\pi}{20}t\right)$$

2. Sample response: After 10 sec, the chair has traveled halfway around the wheel and will be at the point directly opposite A. Its height will be $8 + 2 = 10$ m.

3. Answers may vary. The height of the chair will be greater than 16 m for times between approximately $20(n + 0.135)$ and $20(n + 0.365)$, where n is a non-negative integer. For example, the height will be greater than 16 m when t is between approximately 2.7 sec and 7.3 sec. These solutions can be found by solving the following equation for t:

$$16 = 10 + 8\sin\left(\frac{2\pi}{20}t\right)$$

c. Sample response: To model the movement of point B, start the parameter t at 15 sec, or use the following equations:

$$x(t) = 8\cos\left(\frac{2\pi}{20}t - \frac{\pi}{2}\right)$$

$$y(t) = 10 + 8\sin\left(\frac{2\pi}{20}t - \frac{\pi}{2}\right)$$

3.3 **a.** **1.** The angular speed of both trains is $2\pi/15 \approx 0.42$ radians/sec.

2. Since the circumference of the outer track is $2\pi(1) \approx 6.28$ m, the speed of that train is:

$$\frac{6.28}{15} \approx 0.42 \text{ m/sec}$$

Since the circumference of the inner track is $2\pi(0.5) \approx 3.14$ m, the speed of that train is:

$$\frac{3.14}{15} \approx 0.21 \text{ m/sec}$$

3. The parametric equations that model the motion of the train on the outer track are:

$$x(t) = \cos\left(\frac{2\pi}{15}t\right)$$

$$y(t) = \sin\left(\frac{2\pi}{15}t\right)$$

The parametric equations that model the motion of the train on the inner track are:

$$x(t) = 0.5\cos\left(\frac{2\pi}{15}t\right)$$

$$x(t) = 0.5\sin\left(\frac{2\pi}{15}t\right)$$

b. 1. The time required for the train on the outer track to complete one lap can be found as follows:

$$\frac{6.14 \text{ m}}{0.25 \text{ m/sec}} \approx 25.13 \text{ sec}$$

Its angular speed, therefore, is $2\pi/25.13 \approx 0.25$ radians/sec. The parametric equations for this train are:

$$x(t) = \cos(0.25t)$$
$$y(t) = \sin(0.25t)$$

The time required for the train on the inner track to complete one lap is:

$$\frac{3.14 \text{ m}}{0.25 \text{ m/sec}} \approx 12.57 \text{ sec}$$

Its angular speed, therefore, is $2\pi/12.57 \approx 0.50$ radians/sec. The parametric equations for this train are:

$$x(t) = 0.5\cos(0.5t)$$
$$y(t) = 0.5\sin(0.5t)$$

2. Sample response: It will take about 25.13 sec. This can be found by solving the equation below for *t*.

$$\frac{0.5t}{2\pi} = \frac{0.25t}{2\pi} + 1$$

The ratios represent the proportions of the circular tracks covered in time *t*. For the inner train to lap the outer train, it must complete one lap more than the outer train, so 1 is added to the ratio on the right-hand side of the equation.

3.4 **a.** $\frac{x^2}{12^2} + \frac{y^2}{7^2} = 1$ or $\frac{x^2}{144} + \frac{y^2}{49} = 1$

b. $x(t) = 6\cos t$ and $y(t) = 2\sin t$

3.5 **a.** Answers may vary. The equations $x(t) = 2 + 3.5\cos t$ and $y(t) = 3 + 1.5\sin t$ describe an ellipse with foci on the line $y = 3$.

The equations $x(t) = 2 + 1.5\cos t$ and $y(t) = 3 + 3.5\sin t$ describe an ellipse with foci on the line $x = 2$.

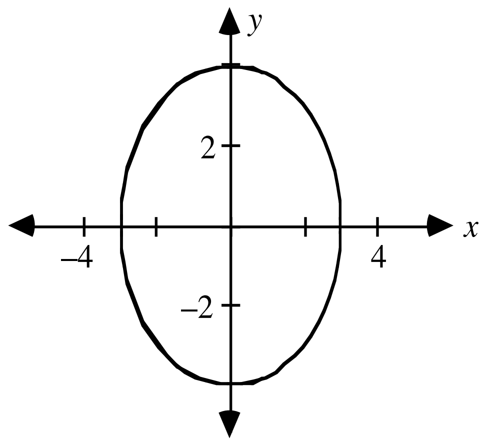
b. The points of intersection for an ellipse with foci on $y = 3$ are $(2,1.5)$, $(2,4.5)$, $(-1.5,3)$, and $(5.5,3)$.

The points of intersection for an ellipse with foci on $x = 2$ are $(2,-0.5)$, $(2,6.5)$, $(0.5,3)$, and $(3.5,3)$.

c. The equations $x(t) = h + a\cos t$ and $y(t) = k + b\sin t$ describe an ellipse with foci on a line parallel to the *x*-axis.

The equations $x(t) = h + b\cos t$ and $y(t) = k + a\sin t$ describe an ellipse with foci on a line parallel to the *y*-axis.

3.6 **a.** In the following sample graph, $a = 3$ and $b = 4$:



b. The area of the sample ellipse in Part **a** is $\pi(3)4 = 12\pi$ units^2.

c. Sample response: The area of an ellipse with parametric equations $x(t) = a\cos t$ and $y(t) = b\sin t$ is πab. As the value of a approaches the value of b, the expression πab approaches πa^2, the area of a circle with radius a.

* * * * *

3.7 **Note:** The expression below provides a better approximation of the perimeter of an ellipse than $\pi(a + b)$:

$$2\pi\sqrt{\frac{a^2 + b^2}{2}}$$

A formula for the exact perimeter of an ellipse requires calculus.

a. Sample response: Since Earth's elliptical orbit is nearly circular, $a \approx b$. As the value of a approaches the value of b, the expression $\pi(a + b)$ approaches $2\pi a$, the circumference of a circle with radius a.

b. $\pi\left(1.4958 \bullet 10^8 + 1.4955 \bullet 10^8\right) \approx 9.3974 \bullet 10^8$ km

3.8 The total area enclosed by the elliptical path of Earth's orbit is:

$$\pi ab = \pi\left(1.4958 \bullet 10^8\right)1.4955 \bullet 10^8$$
$$\approx 7.0276 \bullet 10^{16} \text{ km}^2$$

According to Kepler's second law, equal time periods sweep equal areas. Multiplying the total area by $30/365$ gives an area of about $5.7722 \bullet 10^{15} \text{ km}^2$.

3.9 **a.** Since the circumference of pulley A is $2\pi(10) \approx 63$ cm, the speed of a point on its circumference is:

$$\frac{63}{0.1} = 630 \text{ cm/sec}$$

b. The angular speed of pulley A is $2\pi/0.1 \approx 62.8$ radians/sec. The following parametric equations model the movement of a point on its circumference with respect to time:

$$x(t) = 10\cos(62.8t)$$
$$y(t) = 10\sin(62.8t)$$

c. Sample response: Because the belt connects the two pulleys, points on the circumference of each pulley must be traveling the same speed.

d. Answers may vary. Sample response: Since the circumference of pulley B is $2\pi(6) \approx 38$ cm and its speed is equal to the speed of pulley A, pulley B makes 1 revolution in:

$$\frac{38 \text{ cm}}{630 \text{ cm/sec}} \approx 0.06 \text{ sec}$$

Thus, the angular speed of pulley B is $2\pi/0.06 \approx 104.7$ radians/sec. The following parametric equations model the movement of a point on its circumference with respect to time:

$$x(t) = 40 + 6\cos(104.7t)$$
$$y(t) = 6\sin(104.7t)$$

* * * * * * * * * *

Answers to Summary Assessment (page 201)

1. Sample response: The angle of the ramp is $\theta = \sin^{-1}(2.5/14.4) \approx 10°$. The velocity of 130 km/hr is approximately 36 m/sec. The parametric equations that model the path of the motorcycle in flight are:

$$x(t) = (36\cos 10°)t$$
$$y(t) = -4.9t^2 + (36\sin 10°)t + 2.5$$

The motorcycle must be at least 1.5 m above the ground to clear the last car. Setting $y(t) = 1.5$ and solving for t gives $t = 1.4$ sec. Substituting 1.4 sec into $x(t)$ gives the maximum horizontal distance, 49 m. Since each car is approximately 1.7 m wide and $49/1.7 \approx 28.8$ cars, the maximum number of cars that could be cleared is 28.

2. **a.** Sample response: The speed of point S is 4.5 km/hr or 1.25 m/sec. Since the circumference of the circle described by the path of S is $2\pi(1.8) \approx 11.3$ m, the time required to complete 1 revolution is:

$$\frac{11.3 \text{ m}}{1.25 \text{ m/sec}} \approx 9 \text{ sec}$$

The angular speed is $2\pi/9 \approx 0.7$ radians/sec. Locating the origin at the surface of the water, directly below the center of the wheel, the following parametric equations model the movement of point S.

$$x(t) = 1.8\cos(0.7t)$$
$$y(t) = 1.2 + 1.8\sin(0.7t)$$

b. Sample response: Point S enters the water at approximately 5.54 sec and leaves the water at approximately 7.96 sec. So, the time under water is 2.42 sec. This can be found by tracing the graph or solving the following equation for t:

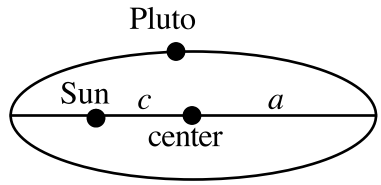
$$0 = 1.2 + 1.8\sin(0.7t)$$

c. Sample response: The paddles are evenly spaced, so the wheel would rotate $\pi/4$ radians from the time the first paddle touched the water until the next paddle entered the water. The time the first paddle is in the water alone is:

$$\frac{\pi/4 \text{ radians}}{0.7 \text{ radians/sec}} \approx 1.12 \text{ sec}$$

From Part **b**, each paddle is in the water for 2.42 sec. So, two consecutive paddles would be in the water for about $2.42 - 1.12 = 1.3$ sec.

3. Sample response: As shown below, the aphelion is the sum of a and c.



Therefore, $a + c = 7.3812 \bullet 10^9$. The orbital eccentricity is the ratio $c/a = 0.2482$. Solving this system of equations results in $a = 5.9135 \bullet 10^9$ km and $c = 1.4677 \bullet 10^9$ km.

For any ellipse, $a^2 = b^2 + c^2$, where b is half the length of the minor axis. So $b = 5.7285 \bullet 10^9$ km.

The parametric equations for Pluto's orbit are:

$$x(t) = \left(5.9135 \bullet 10^9\right)\cos t$$

$$y(t) = \left(5.7285 \bullet 10^9\right)\sin t$$

Pluto's orbit is nearly circular, even though it has the most eccentric orbit of any planet in the solar system.

Note: The closer the eccentricity is to 1, the more elongated the ellipse. The closer the eccentricity is to 0, the more circular the ellipse.

Module Assessment

1. Mikal throws a ball with an initial velocity of 5 m/sec and an angle of elevation of 40° from an initial height of 2 m. Her friend Azra simultaneously drops an identical ball from the same initial height.

a. Write parametric equations that describe the path of each ball.

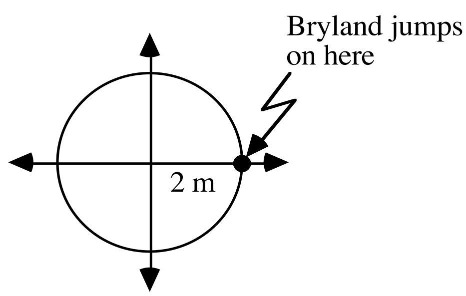
b. Will the two balls hit the ground at the same time? Justify your response.

2. Mikal and Azra are riding a merry-go-round that turns counterclockwise at 1 revolution every 5 sec. Mikal is 0.5 m from the center and Azra is 1.5 m from the center.

a. Determine the speed of each person on the merry-go-round.

b. Write parametric equations that model each person's position over time.

c. The following diagram shows the point at which their friend Bryland jumped onto the merry-go-round. He jumped off 12 sec later. Assuming that the speed of the merry-go-round remains constant, locate the point at which Bryland jumped off.



3. Graph the parametric equations $x(t) = 4\cos t$ and $y(t) = 7\sin t$.

a. Describe the effect that the coefficients 4 and 7 have on the graph.

b. Determine the area of the ellipse.

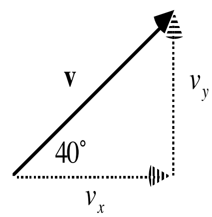
c. Write the equation of the ellipse in standard form.

Answers to Module Assessment

1. **a.** Since Azra's ball has no initial velocity, its parametric equations are:

$$x(t) = 0$$
$$y(t) = -4.9t^2 + 2$$

The vector diagram for Mikal's ball shows both a horizontal and a vertical component of the initial velocity:



The parametric equations for Mikal's ball are:

$$x(t) = 5\cos 40^\circ \bullet t$$
$$y(t) = -4.9t^2 + 5\sin 40^\circ \bullet t + 2$$

b. Sample response: No, during the time Mikal's ball is reaching its maximum height, Azra's ball has already begun its downward flight. Azra's ball reaches the ground first.

2. **a.** The circumference of Mikal's path is $2\pi(0.5) \approx 3.1$ m. Her speed is:

$$\frac{3.1}{5} = 0.62 \text{ m/sec}$$

The circumference of Azra's path is $2\pi(1.5) \approx 9.4$ m. Her speed is:

$$\frac{9.4}{5} \approx 1.9 \text{ m/sec}$$

b. The angular speed of both students is $2\pi/5 \approx 1.26$ radians/sec. The following parametric equations model Mikal's movement:

$$x(t) = 0.5\cos(1.26t)$$
$$y(t) = 0.5\sin(1.26t)$$

The parametric equations below model Azra's movement:

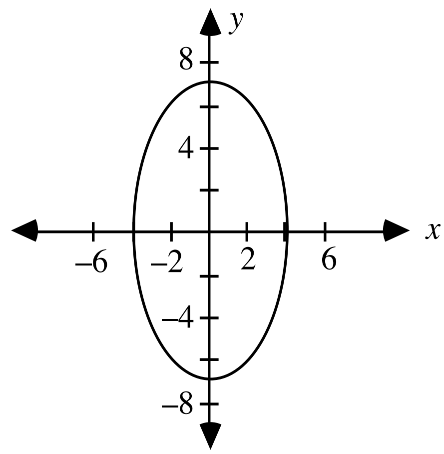
$$x(t) = 1.5\cos(1.26t)$$
$$y(t) = 1.5\sin(1.26t)$$

c. The following parametric equations model Bryland's movement:

$$x(t) = 2\cos(1.26t)$$
$$y(t) = 2\sin(1.26t)$$

Substituting 12 sec for *t* shows that he jumped off at the point $(-1.6, 1.2)$.

3. Sample graph:



a. Sample response: Since the two coefficients are not equal, the graph is an ellipse. Half the length of the minor axis is 4. Half the length of the major axis is 7.

b. The area of the ellipse is $ab\pi = 4 \bullet 7 \bullet \pi = 28\pi \approx 88$ units^2.

c. Sample response:

$$\frac{x^2}{16} + \frac{y^2}{49} = 1$$

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Flashbacks

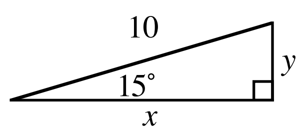
Activity 1

1.1 Write parametric equations to model the data in the table below.

t	*x*	*y*
0	2000	0
1	1850	100
2	1700	200
3	1550	300

1.2 Solve the following equation for *x*: $4x^2 - 2x + 1 = 7$.

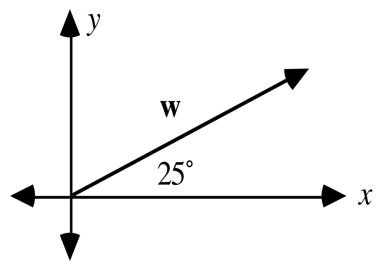
1.3 Determine the values of *x* and *y* in the diagram below.



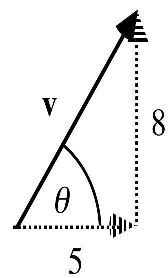
1.4 What type of conic section is described by the equation $y = x^2 - 5x + 4$?

Activity 2

2.1 Vector **w** in the diagram below has a magnitude of 5 units. Determine its horizontal and vertical components.



2.2 The following diagram shows vector **v** and its components.



a. Determine the magnitude of vector **v.**

b. Determine the value of θ.

2.3 Use the distributive property to rewrite each of the following expressions in the form $a(x^2 + bx)$.

a. $4x^2 + 8x$

b. $3x^2 - 4x$

c. $\frac{1}{2}x^2 + 2x$

Activity 3

3.1 The following equation defines a circle.

$$(x-7)^2 + (y+6)^2 = 36$$

a. Determine the radius of the circle.

b. Write the ordered pair for the center of the circle.

c. What is the circumference of the circle?

3.2 The following parametric equations define a circle.

$$x(t) = 3 + 4\cos(t)$$
$$y(t) = -5 + 4\sin(t)$$

a. Determine the radius of the circle.

b. Write the ordered pair for the center of the circle.

c. What is the circumference of the circle?

3.3 The following equation defines an ellipse.

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$$

a. Determine the lengths of the major and minor axes.

b. Write the ordered pair for the center of the ellipse.

c. Write the ordered pairs for the positions of the foci.

Answers to Flashbacks

Activity 1

1.1 $x(t) = 2000 - 150t$ and $y(t) = 100t$

1.2 $x = 1.5$ or $x = -1$

1.3 $x = 10\cos 15^\circ \approx 9.7$; $y = 10\sin 15^\circ \approx 2.6$

1.4 This equation describes a parabola.

Activity 2

2.1 $w_x = 5\cos 25^\circ \approx 4.5$; $w_y = 5\sin 25^\circ \approx 2.1$

2.2
- **a.** $|\mathbf{v}| = \sqrt{8^2 + 5^2} \approx 9.4$
- **b.** $\theta = \tan^{-1}(8/5) \approx 58^\circ$

2.3
- **a.** $4(x^2 + 2x)$
- **b.** $3\left(x^2 - \frac{4}{3}x\right)$
- **c.** $\frac{1}{2}\left(x^2 + 4x\right)$

Activity 3

3.1
- **a.** 6
- **b.** (7,–6)
- **c.** $2\pi(6) \approx 37.7$

3.2
- **a.** 4
- **b.** (3,–5)
- **c.** $2\pi(4) \approx 25.1$

3.3
- **a.** The length of the major axis is 12; the length of the minor axis is 6.
- **b.** (0,0)
- **c.** The distance from the center of the ellipse to each focus is $\sqrt{6^2 - 3^2}$. The coordinates of the foci are $\left(-\sqrt{27},0\right)$ and $\left(\sqrt{27},0\right)$.

Here We Go Again!

What do teeter-totters, pendulums, and trumpet notes have in common? These real-life phenomena can all be modeled by the same type of functions.

Byron Anderson • Glenn Blake • Anne Merrifield

Teacher Edition

Here We Go Again!

Overview

In this module, students model periodic events, investigate the effects of changing the parameters of a periodic function on its graph, examine reciprocals of periodic functions, and explore trigonometric identities. They also explore addition of periodic functions, and examine some non-sinusoidal periodic functions.

Objectives

In this module, students will:

- explore the effects of parameters *a*, *b*, *h*, and *k* on trigonometric functions of the form $g(x) = af\big(b(x-h)\big) + k$
- make predictions based on periodic models
- investigate functions created by adding periodic functions
- examine some fundamental trigonometric identities
- explore the cosecant, secant, and cotangent functions as other examples of periodic functions.

Prerequisites

For this module, students should know:

- properties of trigonometric functions
- how to determine the domains and ranges of functions
- how to perform operations on functions
- how to identify asymptotes.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	3	2	11

Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
centimeter graph paper	X			X
tape	X			
string	X			X
paper cups	X			
support stand	X			
sand	X			
stop watch	X			
rope	X			

Technology

Software	Activity			
	1	2	3	Summary Assessment
graphing utility	X	X	X	X
symbolic manipulator	X	X	X	
spreadsheet		X		X
science interface device (optional)	X			
motion detector (optional)	X			

Here We Go Again!

Introduction

(page 209)

Students discuss real-world events that repeat over time.

Discussion

(page 209)

a. Sample responses might include ocean tides, geyser eruptions, class bells, pendulums, and biorhythms.

b. Sample response: Each event occurs in cycles which have definite beginnings and endings.

(page 209)

Activity 1

Students explore periodic functions based on the sine function, determine the amplitudes and periods of graphs, and investigate vertical and horizontal translations of functions.

Materials List

- centimeter graph paper (8 sheets per group)
- tape (several pieces per group)
- string (2 pieces per group)
- paper cup (1 per group)
- support stand (1 per group)
- sand
- stopwatch (1 per group)
- rope (1 per group)

Teacher Note

The overhead support for the pendulum must be stable and secure. Suitable materials may be available from your school's science department. The sand for creating graphs must be fine, since pebbles may plug the hole in the paper cup. You may wish to use table salt or mixed tempera paint instead. A funnel may be used instead of a paper cup. The bottom of the cup should not be more than 3 cm above the graph paper.

Technology

- graphing utility
- symbolic manipulator
- science interface device (optional)
- motion detector (optional)

Teacher Note

If a motion detector is available, students may wish to use it to analyze the pendulum. In this case, you may want to provide an object larger than a paper cup.

If a motion detector is not available, you may want to use the following data for Parts **f–h** of the exploration. One equation that models this data is $f(x) = 1.04\sin[2.16(x - 0.85)] + 3.24$. **Note:** The point (4.5,4.887) is an outlier.

Time (sec)	**Distance (cm)**	**Time (sec)**	**Distance (cm)**	**Time (sec)**	**Distance (cm)**
0.1	2.121	2.1	3.709	4.1	3.860
0.2	2.135	2.2	3.497	4.2	3.994
0.3	2.186	2.3	3.273	4.3	4.116
0.4	2.287	2.4	3.054	4.4	4.220
0.5	2.424	2.5	2.845	4.5	4.887
0.6	2.618	2.6	2.665	4.6	4.120
0.7	2.841	2.7	2.500	4.7	4.026
0.8	3.083	2.8	2.362	4.8	3.914
0.9	3.313	2.9	2.280	4.9	3.774
1.0	3.543	3.0	2.247	5.0	3.590
1.1	3.742	3.1	2.261	5.1	3.392
1.2	3.925	3.2	2.326	5.2	3.183
1.3	4.080	3.3	2.406	5.3	3.003
1.4	4.184	3.4	2.332	5.4	2.823
1.5	4.249	3.5	2.704	5.5	2.658
1.6	4.260	3.6	2.903	5.6	2.528
1.7	4.231	3.7	3.100	5.7	2.431
1.8	4.159	3.8	3.302	5.8	2.362
1.9	4.040	3.9	3.490	5.9	2.341
2.0	3.896	4.0	3.670	6.0	2.366

Exploration (page 209)

Students construct a pendulum and use it to create a sinusoidal curve. They then determine an equation for a periodic function that models the data.

a. Students construct a pendulum using materials suitable for your classroom. **Note:** Remind students to make sure that the support stand is stable.

b–d. By pulling the long sheet of graph paper underneath the moving pendulum as the sand pours out, students create a graph of periodic motion. Students may need to practice this procedure.

Note: The amplitude should remain constant over the relatively short time interval that students record the motion of the pendulum.

e. Answers will vary. Students should recognize that the curve is periodic and determine its period and amplitude.

f–h. Students experiment with functions of the form $f(x) = a\sin[b(x-h)] + k$ to find a periodic function that models their data points. Sample response: $f(x) = 2\sin(2(x-1)) + 3$.

i. Sample response: $f(x) = 4\sin(2(x-(3+\pi))) + 7$.

Teacher Note

As part of the discussion, you may wish to ask groups to share their sand graphs and model functions with the class.

Discussion (page 213)

a. Student graphs should be a transformation of $g(x) = \sin x$. Students may notice differences in amplitude and period, and identify horizontal and vertical translations.

b.
1. Sample response: The maximum amplitude corresponds to the farthest distance that you could pull the pendulum back before releasing it.
2. The minimum amplitude is 0, which occurs when the pendulum is stationary.

c. Answers may vary. The period P of a pendulum is related to the length l of the string by the following equation, where g is the acceleration due to gravity:

$$P = 2\pi\sqrt{l/g}$$

Note that the period does not depend either on the mass of the swinging object or the amplitude of the swing.

d. Answers will vary. In their equations of the form $f(x) = a\sin[b(x-h)] + k$, students should identify the horizontal translation as $-h$ and the vertical translation as k.

e. Sample responses: cosine, tangent. **Note:** Later in this module, students also will examine cotangent, secant, and cosecant.

f.
1. A horizontal line is a periodic function. It has an amplitude of 0 and an infinitely small period.
2. The greatest integer function is not a periodic function. The range values do not repeat over equal intervals on the domain.

g. Sample response: The amplitude increases to 6, the period increases to 4π, there is a horizontal shift of –5, and a vertical shift of –2.

h. Sample response: Suppose the function repeats once every d units. Then $b = 2\pi/d$.

i. Sample response: Locate a point on the translated function that corresponds to the y-intercept of $f(x) = \sin x$. The distance from that point to the y-axis is one possible amount for the horizontal translation.

j. There are infinitely many equations that model its graph. Given one equation for the function, another can be found by adding or subtracting multiples of the period from h (the horizontal translation) in a function of the form $f(x) = a\sin[b(x-h)] + k$.

Note: Because of this fact, answers may vary in all problems in which students are asked to write trigonometric equations from either data or graphs. Once one equation is determined, horizontal translations may lead to equations that produce the same graph. Students may appreciate this notion more once identities are discussed in Activity **3**.

Teacher Note

In Problems **1.6**, you may wish to allow students to use a motion detector to collect their own data.

Assignment

(page 214)

1.1 **a.**
1. Sample response: To find the period, choose a point where the maximum value occurs. Then locate the closest point where the maximum occurs again. The difference between the x-coordinates of these two points is the period, which appears to be about 3.
2. Sample response: The amplitude is defined as

$$\left|\frac{M-m}{2}\right|$$

Since M is approximately –3 and m is approximately –11, the amplitude is about 4.

3. Sample response: To determine the vertical translation, find the average of the maximum and minimum y-values. The vertical translation for this curve is

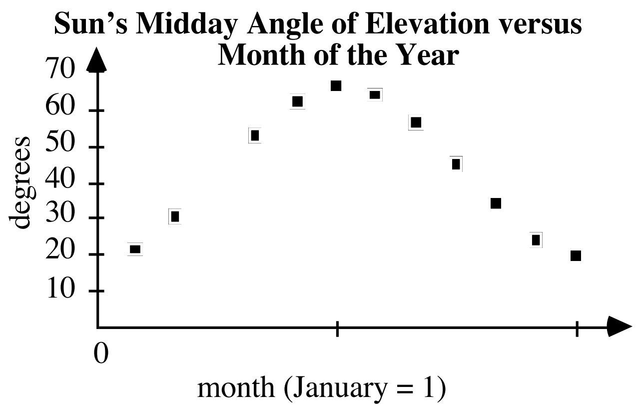
$$\frac{(-3)+(-11)}{2} = -7$$

4. Sample response: To determine the horizontal translation, locate a point that corresponds to the y-intercept of the parent function. The distance from that point to the y-axis is the amount of horizontal translation, which is about 3 units for the curve shown.

b. To determine an equation that models the curve, students should substitute the appropriate constants into the general equation $f(x) = a\sin[b(x-h)] + k$. A second equation can be found by translating $f(x)$ horizontally by a multiple of the period.

Sample response: $f(x) = 4\sin 2(x-3) - 7$ and $g(x) = 4\sin 2(x-3-\pi) - 7$.

1.2 **a.** Sample scatterplot:



b. To determine the equation of a periodic model, students must find the amplitude, period, vertical shift, and horizontal shift.

Sample response: The amplitude is $(65.83 - 19.13)/2 = 23.35$. If the period is measured in months, then $b = 2\pi/12 \approx 0.524$. If the period is measured in days, then $b = 2\pi/365 \approx 0.017$.

The vertical shift from $f(x) = \sin x$ is $(65.83 + 19.13)/2 = 42.48$. Since the point (3,40.87) corresponds to (0,1) on $f(x) = \sin x$, the horizontal translation is 3 months (or 90 days).

One possible periodic model, where x represents number of months, is $f(x) = 23.35\sin(0.524(x-3)) + 42.8$. If x is the number of days, the equation is $f(x) = 23.35\sin(0.017(x-90)) + 42.8$.

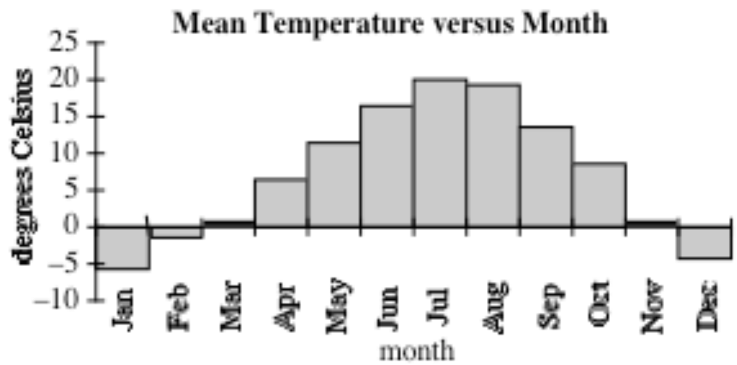
c. **1.** If students assign January 15 as 1, then June 1 corresponds to 5.5. Using the sample function given in Part **b**, the estimated angle of elevation is $f(5.5) = 23.5\sin(0.524(5.5 - 3)) + 42.48 \approx 65.2^\circ$.

2. If students assign January 15, 1994, as 1, then October 31, 1995, corresponds to 22.5. The predicted angle of elevation is approximately 25.7˚.

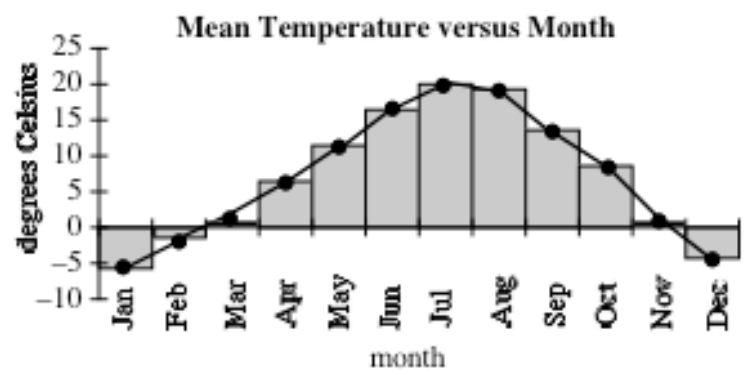
d. Sample response: To predict the angle for a specific day of the year, estimate the decimal equivalent of the date in months and substitute this value for *x* in the equation. For example, to find the sun's approximate angle of elevation on September 3, 1994, convert September 3 to 8.6 (since January 15, 1994, equals 1), and substitute it in the model $f(x) = 23.35\sin(0.524(x - 3)) + 42.8$.

1.3 **a.** Sample response: As the sun's angle of elevation increases, the average monthly temperature increases.

b. Sample graph:



c. **1.** Sample graph:



2. Sample response: The general shape of the dots on the temperature graph is similar to the graph for the sun's angle of elevation. The temperature graph has a different amplitude, and a smaller maximum. So, it has a smaller vertical translation. The horizontal translation appears to be greater for the temperature graph.

3. Answers may vary, depending on students' predictions.

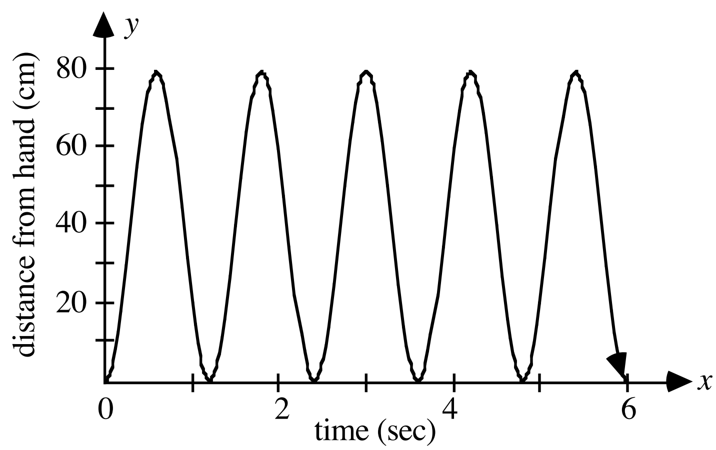
1.4 Answers will vary, including any cosine function with an amplitude of 4, a period of 2, a vertical shift of 3, and a horizontal shift of $2n - 1$, where *n* is an integer. Sample response: $y = 4\cos(\pi(x - 1)) + 3$ or $y = 4\cos(\pi(x - 3)) + 3$.

* * * * *

***1.5** **a.** **1.** Sample response: The amplitude is half the difference between the maximum and minimum distances from the hand, or half the length of the string.

2. Sample response: The period is the time it takes for the yo-yo to unwind to the full length of the string, then rewind.

b. **1.** Sample graph:



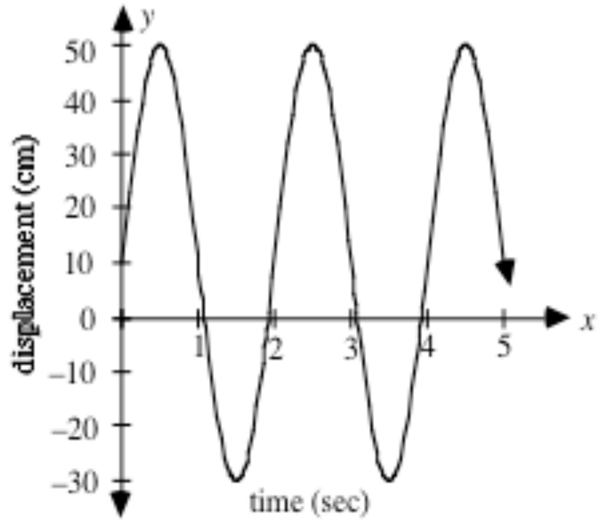
2. One function that models the curve is:

$$f(x) = 39.5\sin\left(\frac{2\pi}{1.2}(x - 0.3)\right) + 39.5$$

c. Students should substitute $x = 2$ into their model equation from Part **b**. Using the sample equation, the estimate is 59 cm.

d. Using the sample equation given in Part **b**, the yo-yo will be 60 cm from the player's hand at about 0.4, 0.8, 1.6, 2.0, and 2.8 sec. These values can be found by considering the intersection of the line $y = 60$ and the graph of the model equation.

1.6 **a.** Sample graph:



b. Sample response: $d(t) = 40\cos(\pi(t - 0.5)) + 10$

c. After 3.2 sec, the object is about 14 cm below the table.

* * * * * * * * * *

(page 217)

Activity 2

Students investigate the amplitudes and periods of more complex periodic functions by adding simple functions of the form $f(x) = a\sin(bx)$ and $g(x) = c\cos(dx)$.

Exploration

(page 217)

a.

1. The period of $f(x)$ is $2\pi/3$; its amplitude is 1. The period of $g(x)$ is $2\pi/3$; its amplitude is 2.
2. The period of $h(x)$ is the same as the period of $f(x)$ and $g(x)$.
3. The amplitude is 3, the sum of the amplitudes of $f(x)$ and $g(x)$.

b. Students should discover that the period of $h(x)$ is always $2\pi/b$, while its amplitude is always $a + c$.

c. Students should discover that the period of $h(x)$ is always $2\pi/b$, while its amplitude is always less than $a + c$.

d.
1. The period of $f(x)$ is π; its amplitude is 1. The period of $g(x)$ is $2\pi/3$; its amplitude is 2.
2. Sample response: Cycles of $f(x)$ begin at 0, π, 2π, 3π, and 4π. Cycles of $g(x)$ begin at 0, $2\pi/3$, $4\pi/3$, 2π, and $8\pi/3$. The smallest positive value common to both lists is 2π.
3. The period of $h(x)$ is 2π, the length of the interval between common starting points for cycles of $f(x)$ and $g(x)$.
4. The amplitude of $h(x)$ is approximately 2.9, which is less than the sum of the amplitudes of $f(x)$ and $g(x)$.

e. Students should discover that the period is always the least common multiple of the periods of $f(x)$ and $g(x)$. The amplitude is always less than or equal to the sum of the amplitudes of $f(x)$ and $g(x)$. **Note:** One combination of a, b, c, and d for which the amplitude is $a + c$ is $a = 1$, $b = 1$, $c = 1$, and $d = 5$.

f. Students should discover that the period is always the length of the interval between common starting points for cycles of $f(x)$ and $g(x)$. The amplitude is always less than or equal to the sum of the amplitudes of $f(x)$ and $g(x)$. **Note:** One combination of a, b, c, and d for which the amplitude is $a + c$ is $a = 1$, $b = 0.5$, $c = 1$, and $d = 2$.

Discussion

(page 218)

a. Sample response: The period of $f(x) + g(x)$ equals the period of $f(x)$ when the periods of $f(x)$ and $g(x)$ are the same.

b. The period of $f(x) + g(x)$ is always equal to the least common multiple of the periods of $f(x)$ and $g(x)$.

c. Sample response: The function $g(x)$ has cycles starting at π, 2π, and 3π. The function $f(x)$ has cycles starting at $2\pi/3$, $4\pi/3$, and 2π. The period is 2π, the least common value.

d. Sample response: No. The period of $\sin 2x$ is π, while the period of $\sin \pi x$ is 2. Although 2π is a multiple of π, it is not a multiple of 2 (since π is not an integer). Therefore, the period cannot be 2π.

e. The amplitude of the sum of $f(x)$ and $g(x)$ is always less than or equal to the sum of the amplitude of $f(x)$ and the amplitude of $g(x)$.

f. Sample response: The functions $f(x) = x$ and $g(x) = -x$ are not periodic, but $f(x) + g(x) = 0$ is periodic.

Assignment (page 219)

2.1 **a.** The amplitudes and periods for $f(x)$, $g(x)$, and $f(x)+g(x)$ are shown in the table below.

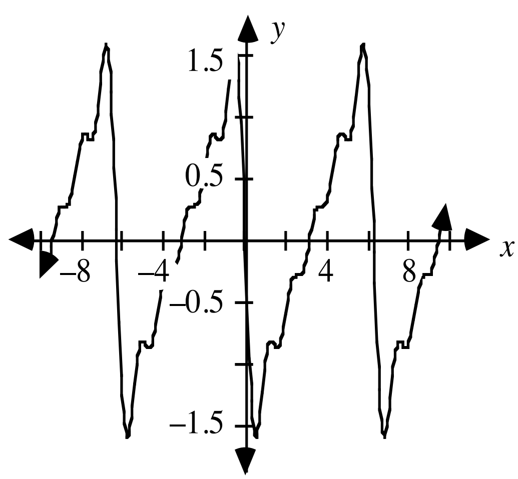
Function	**Amplitude**	**Period**
$f(x)=3\cos(4\pi x)$	3	0.5
$g(x)=5\cos(4\pi x)$	5	0.5
$f(x)+g(x)$	8	0.5

b. The amplitudes and periods for $f(x)$, $g(x)$, and $f(x)+g(x)$ are shown in the table below.

Function	**Amplitude**	**Period**
$f(x)=2\cos(3x)$	2	$2\pi/3$
$g(x)=4\cos(5x)$	4	$2\pi/5$
$f(x)+g(x)$	6	2π

c. Answers will vary. Sample response: Yes, since the cosine can be expressed as horizontal translations of the sine function, the effects on the amplitude and period are the same when cosine curves are added. **Note:** Students will determine explicit identities of sine and cosine in Activity **3**.

2.2 **a.** Sample response: The shape of the graph is a "sawtooth" wave.



b. Sample response: The amplitude should be less than or equal to the sum below.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\approx 2.28$$

By tracing the graph, the amplitude appears to be approximately 1.6.

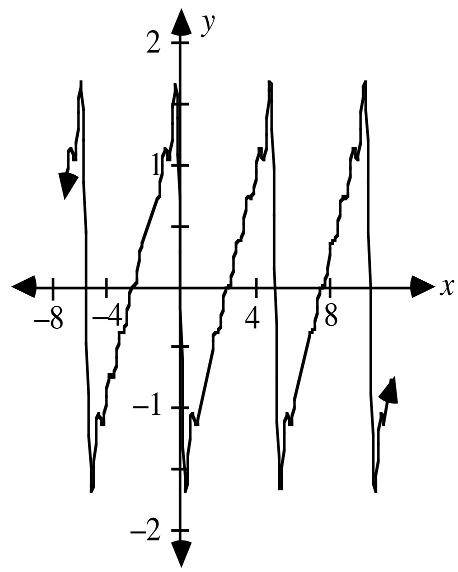
The period should be the least common multiple of the following:

$$2\pi,\ \pi,\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{5} \text{ which is } 2\pi.$$

c. The next three terms of the series are:

$$f_6(x) = -\frac{1}{6}\sin(6x),\ f_7(x) = -\frac{1}{7}\sin(7x),\ f_8(x) = -\frac{1}{8}\sin(8x)$$

d. Sample graph:



e. Sample response: The amplitude should be less than or equal to the sum below.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \approx 2.72$$

By tracing the graph, the amplitude appears to be approximately 1.7.

The period should be the least common multiple of the following:

$$2\pi,\ \pi,\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{5},\ \frac{\pi}{3},\ \frac{2\pi}{7},\ \frac{\pi}{4} \text{ which is } 2\pi.$$

2.3 **a.** By substituting 0.25 for $d(t)$ and 0 for t, the following equation results:

$$0.25 = A\cos\left(0 \bullet \sqrt{1.6/10}\right) + B\sin\left(0 \bullet \sqrt{1.6/10}\right)$$
$$0.25 = A(1) + B(0)$$
$$0.25 = A$$

Similarly, by substituting 2 for $v(t)$ and 0 for t, the following equation results:

$$2 = -A\left(\sqrt{1.6/10}\right)\sin\left(0 \bullet \sqrt{1.6/10}\right) + B\left(\sqrt{1.6/10}\right)\cos\left(0 \bullet \sqrt{1.6/10}\right)$$

$$2 = -A(0.4)(0) + B(0.4)(1)$$

$$2 = B(0.4)$$

$$5 = B$$

b. The displacement after 3.6 sec is approximately 5.0 m. The velocity after 3.6 sec is approximately 0.16 m/sec.

c. Students may use the graph of the sum to determine the function. Sample response:

$$d(t) \approx 5\sin(0.4t + 0.05)$$

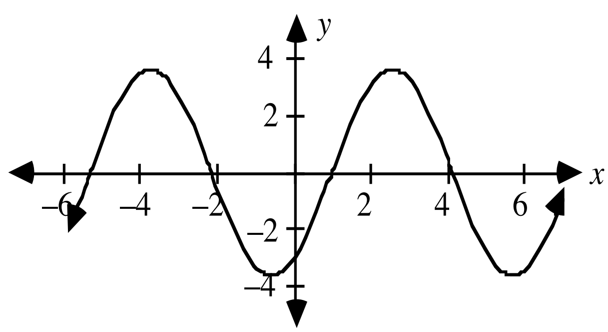
d. Students may use the graph of the sum to determine the function. Sample response:

$$v(t) \approx 2\sin[0.4(t + 4.1)] = 2\sin(0.4t + 1.64)$$

e. The displacement after 3.6 sec using the equation from Part **c** is approximately 4.98 m. The velocity after 3.6 sec using the equation from Part **d** is approximately 0.12 m/sec. These are reasonably close to the corresponding values determined in Part **b**.

2.4 Students may select any values for a and b not equal to 0. In the following sample responses $a = 2$ and $b = -3$.

a. Sample graph of $2\sin x - 3\cos x$:

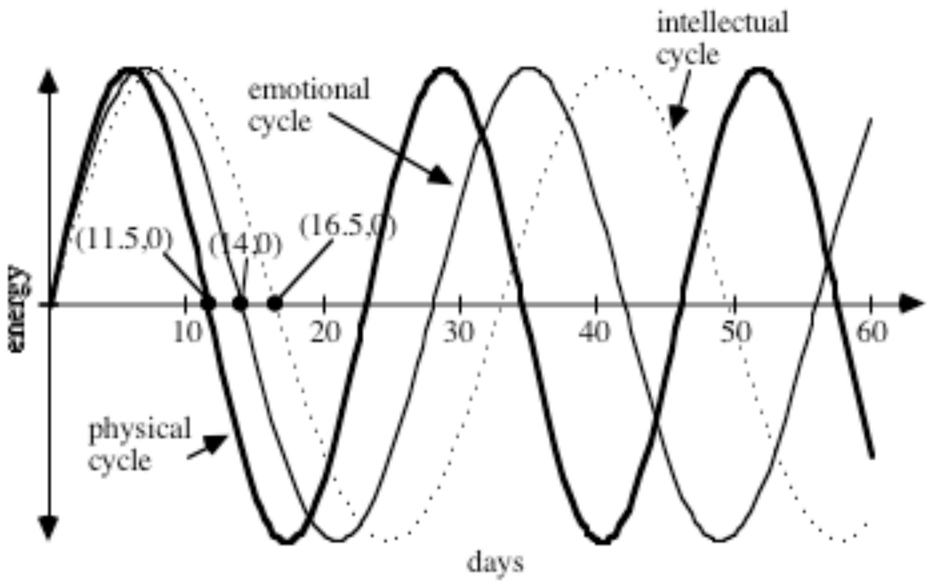


b. For the sample function given in Part **a**, the domain of $h(x)$ is $(-\infty, \infty)$; its range is approximately (–3.61, 3.61).

c. The sample function given in Part **a** can be represented as $h(x) \approx 3.61\sin(x - 0.98)$.

2.5 **Note:** The sample responses given for Parts **c** and **d** are based on 365.25 days per year. This takes into account a leap year every 4 yr.

a. Sample graph:



b. See sample graph above.

c. Students should calculate their age in days, then divide this figure by the period of each cycle. A student who is exactly 16 years old, for example, has lived 5844 days. The physical cycle has gone through approximately 254.1 cycles. Since $0.1 \bullet 23 = 2.3$ days, the physical cycle is on the positive side and increasing daily. The emotional cycle has gone through approximately 208.7 cycles. Since $0.7 \bullet 28 \approx 20$ days, the emotional cycle is on the negative side and nearing its lowest point. The intellectual cycle has gone through about 177 cycles. Since $0.1 \bullet 33 \approx 3$ days, the intellectual cycle is on the positive side and increasing daily.

d. The three cycles coincide initially at birth and then again about 21,252 days (or 58 years and 66 days) later. This is calculated by finding the least common multiple of 23, 28, and 33.

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Research Project (page 221)

A physics or music teacher may be a useful source of information for this project. For example, a graph of the sound produced by a french horn consists of the sum of the graphs of the fundamental frequency and the second harmonic.

(page 222)

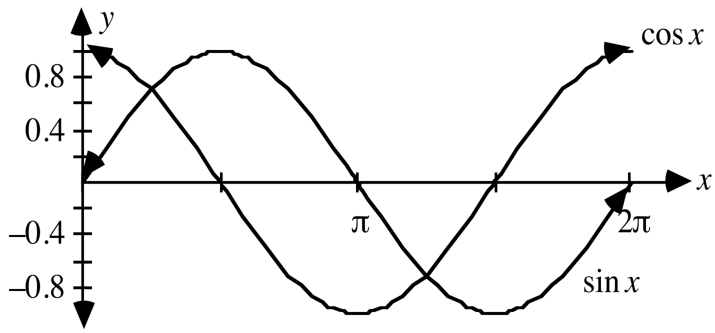
Activity 3

Students investigate some fundamental trigonometric identities, then examine the cosecant, secant, and cotangent functions.

Exploration 1

(page 222)

a. **1.** Sample graph:



2. There are several such transformations. One is a translation of the graph of the $\cos x$ by $\pi/2$ units to the right.

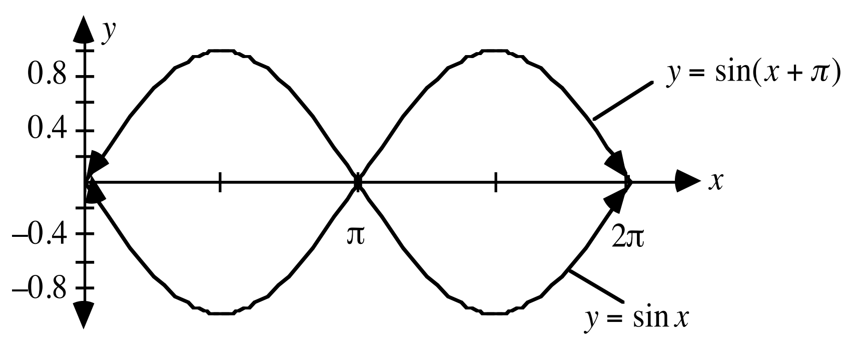
3. Sample response: $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$.

4. The graph verifies that the two expressions are equivalent.

5. Sample response: $h(x) = 3\cos\left(2x - \dfrac{\pi}{2}\right)$.

b. **1 –3.** Several transformations result in an identity between $\sin x$ and $\cos x$. One transformation is a translation of the graph of the $\sin x$ by $\pi/2$ units to the left. The resulting identity is: $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$.

c. **1.** Sample graph:



2 – 3. Since corresponding y-values are additive inverses, one identity is $\sin(x + \pi) = -\sin x$.

Discussion 1

(page 223)

a. **1.** Sample responses:

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{6}\right)$$

or

$$\sin\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3} + \pi\right) = -\sin\left(\frac{4\pi}{3}\right)$$

2. Sample response:

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{4}\right)$$

b. **1.** Sample response:

$$f(x) = \sin\left(x + \frac{\pi}{2}\right) - \sin x$$

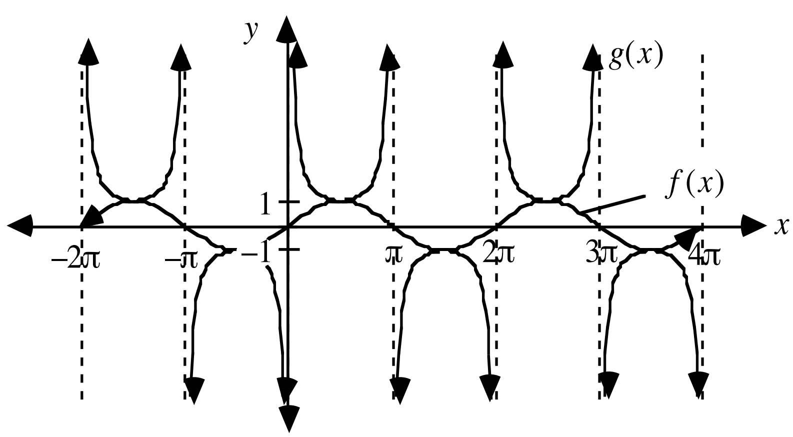
2. Sample response:

$$f(x) = \cos x + \cos\left(x + \frac{3\pi}{2}\right)$$

Exploration 2

(page 223)

a–b. Students graph $f(x) = \sin x$ and $g(x) = \csc x$ on the same set of coordinate axes. Sample graph:



c. **1.** The points of intersection for the two graphs are all at maximum and minimum values for the function $f(x) = \sin x$. The coordinates of these points are:

$$\left(\frac{\pi}{2} + 2\pi n, 1\right) \text{ and } \left(\frac{3\pi}{2} + 2\pi n, -1\right)$$

where n is an integer.

2. Since $\sin(-3\pi/2) = 1$, it follows that

$$\csc\left(\frac{-3\pi}{2}\right) = \frac{1}{\sin(-3\pi/2)} = \frac{1}{1} = 1$$

Also, since $\sin(-\pi/2) = -1$, it follows that

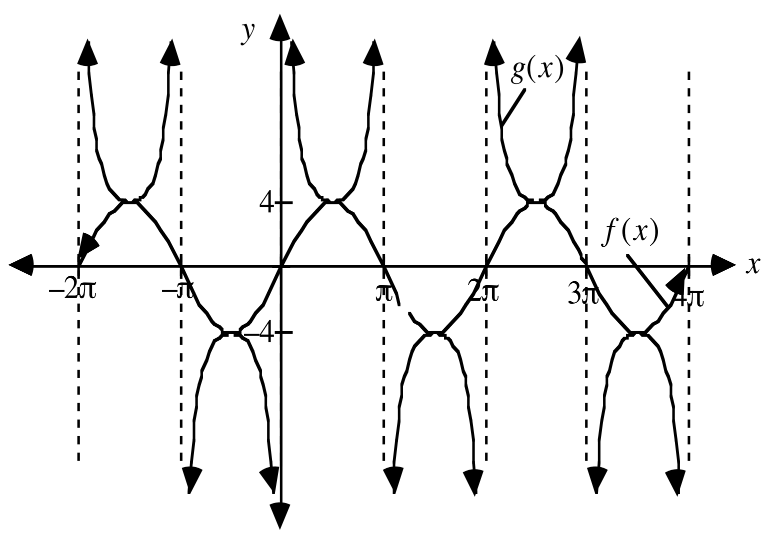
$$\csc\left(\frac{-\pi}{2}\right) = \frac{1}{\sin(-\pi/2)} = \frac{1}{-1} = -1$$

For all other points of intersection of the two functions, since the reciprocal of 1 is 1, and the reciprocal of –1 is –1, it follows that the reciprocal identity from the mathematics note holds true.

d. Since the value of the function $f(x) = \sin x$ is 0 for all multiples of π, it follows that its reciprocal function $g(x) = \csc x$ will be undefined at those values. Furthermore, as values for the sine function approach 0, the corresponding values for the cosecant function increase without bound. Thus, $g(x)$ has vertical asymptotes at $x = n\pi$, where n is an integer.

e. Answers will vary. In the following sample responses, $a = 4$ and $b = \pi$.

1–2. Sample graph:



3. The points of intersection of the graphs are all at maximum and minimum values for $f(x) = a\sin bx$. For the sample graph, the points of intersection are, from left to right, $(-3\pi/2, 4)$, $(-\pi/2, 4)$, $(\pi/2, 4)$, $(3\pi/2, -4)$, $(5\pi/2, 4)$, and $(7\pi/2, -4)$.

Other solutions are of the form:

$$\left(\frac{(4n+1)\pi}{2}, 4\right) \text{ or } \left(\frac{(4n-1)\pi}{2}, -4\right)$$

where n is an integer.

In general, the points of intersection for $f(x) = a\sin(bx)$ and $g(x) = a\csc(bx)$ are of the form

$$\left(\frac{(4n+1)b}{2}, a\right) \text{ or } \left(\frac{(4n-1)b}{2}, -a\right)$$

where n is an integer.

4. Sample response: Since $4\sin(-3\pi/2) = 4 \bullet 1 = 4$,

$$4\csc(-3\pi/2) = 4\left(\frac{1}{\sin(-3\pi/2)}\right) = 4 \bullet \frac{1}{1} = 4$$

Likewise, since $4\sin(-\pi/2) = 4 \bullet -1 = -4$, then

$$4\csc(-\pi/2) = 4\left(\frac{1}{\sin(-\pi/2)}\right) = 4 \bullet \frac{1}{-1} = -4$$

5. Sample response: Since the values of $f(x) = 4\sin \pi x$ are 0 for all integer values of x, it follows that $g(x) = 4\csc \pi x$ will be undefined at those values. Furthermore, as values for the sine function approach 0 near all integer values of x, the corresponding absolute values for the cosecant function increase without bound. Thus, $g(x)$ has vertical asymptotes at $x = n\pi$, where n is any integer.

Discussion 2

(page 224)

a. Sample response: Since $\sin(n\pi) = 0$, where n is an integer, it follows that $\csc(n\pi)$ cannot be defined because

$$\csc(n\pi) = \frac{1}{\sin(n\pi)} = \frac{1}{0}$$

which is undefined. Thus, $g(x) = \csc x$ is undefined for $x = n\pi$.

b. The domain of the function $f(x) = \sin x$ is the set of all real numbers.

The domain of the function $g(x) = \csc x$ is the set of all real numbers except multiples of π.

c. Since the sine function is continuous and has a minimum value of –1 and a maximum value of 1, the range is [–1, 1]. The range of $g(x) = \csc x$ is $(-\infty, -1] \cup [1, +\infty)$.

d. Sample response: The period of the function $g(x) = \csc x$ is 2π, because the graph repeats itself every 2π units.

e. Sample response: Yes. Since range values of $f(x) = a \sin bx$ repeat in cycles, the reciprocals of those values will also repeat in cycles.

Assignment

(page 224)

3.1 Sample response: The points of intersection are the maximum or minimum values for the sine function.

3.2 Sample responses:

a. $\sin(x - \pi) = \sin x$

b. $\sin(\pi - x) = \sin x$

c. $\sin\left(x - \frac{3\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$

d. $\sin\left(x + \frac{3\pi}{2}\right) = \sin\left(x - \frac{\pi}{2}\right)$

e. $\sin(-x) = -\sin x$

3.3 Sample responses:

a. $\sin(x - \pi) = -\cos\left(x - \frac{\pi}{2}\right)$

b. $\sin(\pi - x) = \cos\left(x - \frac{\pi}{2}\right)$

c. $\sin\left(x - \frac{3\pi}{2}\right) = \cos x$

d. $\sin\left(x + \frac{3\pi}{2}\right) = \cos(x - \pi)$

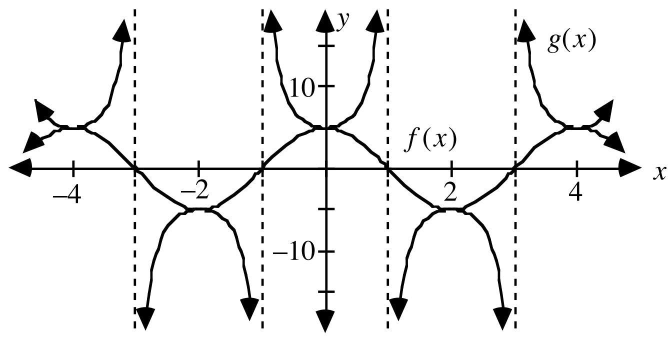
e. $\sin(-x) = -\cos\left(x - \frac{\pi}{2}\right)$

3.4 **a.** $\cos^2 x + \sin^2 x = 1$

b. $\frac{1 - \cos^2 x}{\sin x} = \sin x;\ \sin x \neq 0$

3.5 Sample response: Since the secant function is the reciprocal of the cosine function, and $\cos\ 0 = 1$, it follows that $\sec\ 0 = 1$. The only graph containing the ordered pair (0,1) is graph **c.**

***3.6** **a.** Sample graph:

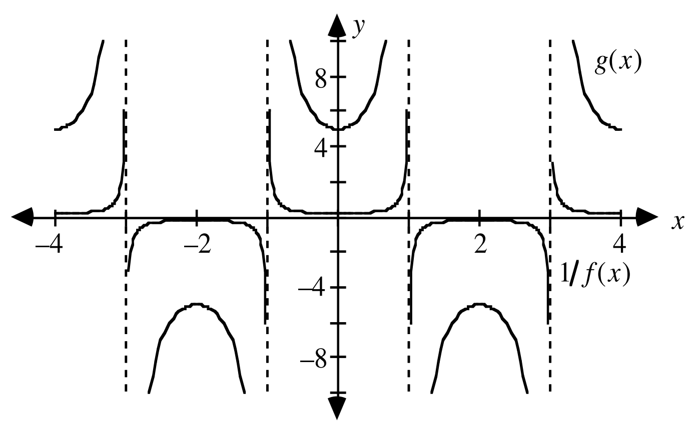


b. The domain for $f(x)$ is the set of all real numbers; its range is $[-5,5]$.

The domain for $g(x)$ is the set of all real numbers except for values where $x = n$, and n is an odd integer; its range is $(-\infty,-5]\cup[5,+\infty)$.

c. Sample response: Yes. The secant function is periodic because its graph, like the graph of the cosine function, repeats itself.

d. Sample graph and response:



Both graphs have the same asymptotes. The graph of $1/f(x)$ is wider and has a minimum value of $1/5$. The graph of $g(x)$ has a minimum of 5.

e. The function $1/f(x)$ is equal to the following:

$$\frac{1}{f(x)} = \frac{1}{a\cos(bx)} = \frac{1}{a}\sec(bx)$$

Since $g(x) = a\sec(bx)$, the ratio of corresponding function values of $g(x)$ to $1/f(x)$ is a^2.

3.7 **a.** The equation is an identity. Students may compare the graph of the function $f(x) = \tan x$ with the graph of the function $g(x) = \sin x / \cos x$. The graphs are everywhere identical.

b. Sample response: From the mathematics note in Activity **3**, $\cot x = 1/\tan x$. Multiplication of both sides of this equation by $\tan x$ yields the identity $\tan x \bullet \cot x = 1$. Division of both sides of this identity by $\cot x$ provides the desired solution, $\tan x = 1/\cot x$.

c. For $f(x) = \tan x$, the domain is the set of all real numbers except multiples of $\pi/2$; its range is all real numbers. For $g(x) = \cot x$, the domain is the set of all real numbers except multiples of π; its range is all real numbers.

d. Sample response: Yes. The cotangent function is periodic because its graph, like the graph of the tangent function, repeats itself. The period of both the tangent and cotangent functions is π.

* * * * *

3.8 **a.** $\cos(x + \pi) = -\cos x$

b. $\cos(x + \pi) = \cos(x - \pi)$

3.9 Because $\cot \pi x = \cos \pi x / \sin \pi x$, the function is undefined when $\sin \pi x = 0$. In graphs **b** and **c**, the asymptotes occur where $\sin \pi x = 0$. From $[0, 0.5)$, $\sin \pi x$ and $\cos \pi x$ are both positive. Therefore, graph **b** is the graph of the curve.

3.10 **a.** $\sin x \bullet \sec x \bullet \cot x = \sin x \bullet \dfrac{1}{\cos x} \bullet \dfrac{\cos x}{\sin x} = 1$

b. $\csc x \bullet \tan x \bullet \cos x = \dfrac{1}{\sin x} \bullet \dfrac{\sin x}{\cos x} \bullet \cos x = 1$

* * * * * * * * * *

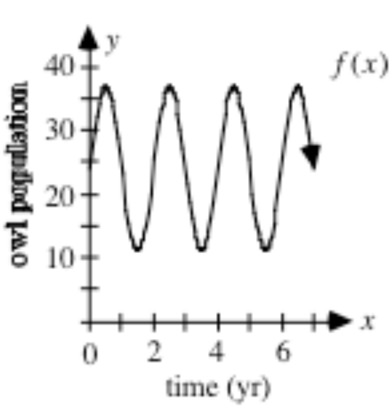
Teacher Note

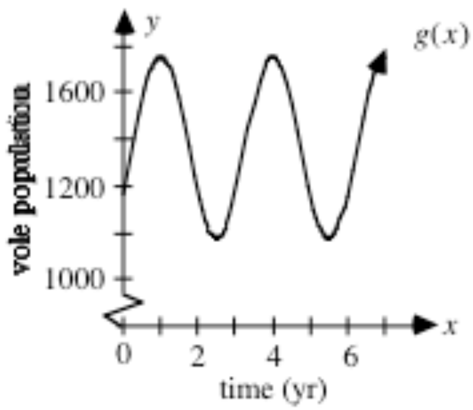
To complete Problem **2**, students will need to know the time of sunset on June 21 and December 21 for your town. This information typically can be found in local newspapers or almanacs.

Answers to Summary Assessment (page 227)

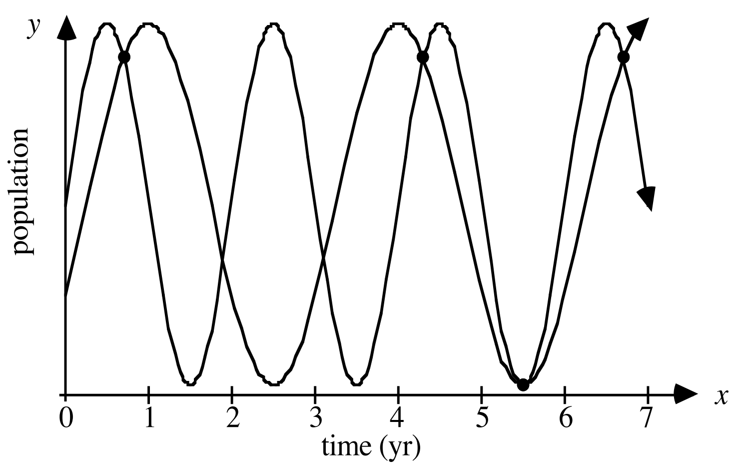
1. Sample response: Using the data in the table, one possible model for the owl population is $f(x) = 13\cos[\pi(x - 0.5)] + 24$. One possible model for the vole population is $g(x) = 388\cos[(2\pi/3)(x - 1)] + 1364$.

If both equations are graphed on the same coordinate system, the difference in the ranges can make the result difficult to interpret. Graphing them on separate coordinate systems—with different scales on the *y*-axis but the same scale on the *x*-axis—makes comparison somewhat easier. In the following graphs, January 1, 1992, represents 0.





To make it easier to determine the best and worst times for reintroduction of the prairie dogs, one of the graphs can be rescaled and super-imposed on the other graph. Changing the amplitude and vertical shift of $f(x)$ to match those of $g(x)$ results in the following graph.

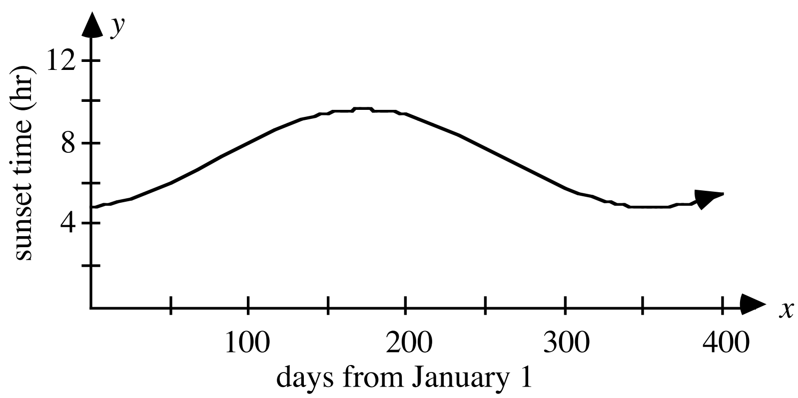


After 5.5 yr, both the owl and vole populations are at a minimum. Since 0 represents January 1, 1992, this time corresponds to July 1, 1997. This is the best time to introduce the prairie dogs because both predation and competition for food will be low.

The worst times to introduce the prairie dog population are when both the owl and vole populations are high—after approximately 0.7, 4.3, and 6.7 yr. Therefore, the middle of August 1992, the middle of March 1996, and the middle of August 1998 are the worst times to introduce prairie dogs.

2. Answers will vary. In the following sample responses, sunset on June 21 occurs at 9:34 P.M., while sunset on December 21 occurs at 4:50 P.M. (disregarding daylight savings time).

a. In the following sample graph, January 1 represents 0.



b. Some possible models for the sample graph above are:

$$y = 2.4\cos\left[(2\pi/365)(x - 172)\right] + 7.2$$

$$y = 2.4\sin\left[(2\pi/365)(x - 81)\right] + 7.2$$

c. Using the sample models given in Part **b**, the sun will set at 6:30 P.M. on February 28. (The time reported by a local newspaper for this date was 6:20 P.M.)

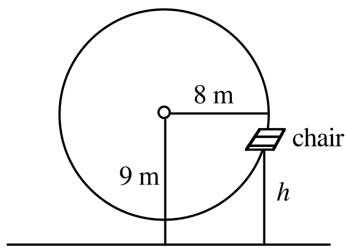
d. Sample response: The model appears to be useful for estimating the time of sunset.

Module Assessment

1. **a.** Identify the amplitude, period, and frequency of the functions $f(x) = a\sin(bx)$ and $g(x) = 2\sin(25\pi x)$.

b. Describe the amplitude and period of $f(x) + g(x)$.

2. Imagine that you are the last person to board the Ferris wheel at the fair. This Ferris wheel travels at a constant rate of 2 revolutions per minute. Its dimensions are shown in the diagram below.



a. Determine several data points for the height of your chair over time.

b. Create a scatterplot of these points.

c. Find an equation that models these points. Identify the amplitude, period, and any horizontal or vertical translations from the parent function.

3. A ferry to Wild Horse Island, a popular tourist destination, departs from the mainland every 75 min. The first ferry of the day leaves promptly at midnight.

a. Create a graph of the minutes spent waiting for the next ferry versus the time of arrival at the dock.

b. Does your graph in Part **a** represent a periodic function? Explain your response.

c. At what times between midnight and 6:00 A.M. would a tourist wait 13 min for the next ferry?

d. If you arrive at the dock at 11:15 A.M., how long must you wait for the next ferry?

4. Another periodic function used as an electronic test signal is known as a "square wave." The square wave is formed by adding the terms of the following series:

$$\sin(x),\ \frac{1}{3}\sin(3x),\ \frac{1}{5}\sin(5x),\ldots$$

a. To demonstrate the formation of a square wave, add the first six terms of the series and graph the resulting sum.

b. What is the period of this function?

5. The ratio for the cotangent is

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0$$

a. Verify that this ratio is an identity.

b. Determine the domain and range of $\cot x$.

6. The identity $\cos(\pi/2 - x) = \sin x$ is true for all values of x. What transformation of $\sin x$ is equal to $\cos x$?

Answers to Module Assessment

1. **a.** For $f(x) = a\sin(bx)$, the amplitude is $|a|$ the period is $2\pi/b$, and the frequency is $b/2\pi$. For $g(x) = 2\sin(25\pi x)$, the amplitude is 2, the period is $2/25 = 0.08$, and the frequency is $25/2 = 12.5$.

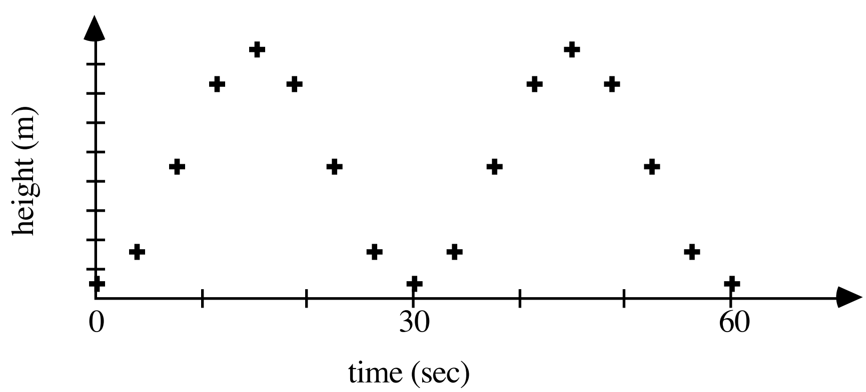
b. For the function $f(x) + g(x)$, the amplitude must have a positive value less than or equal to $|a| + 2$.

The period of $f(x) + g(x)$ must be the smallest possible real number that both $2\pi/b$ and $2/25$ divide evenly. This value cannot be determined since b is unknown.

2. **a.** Sample data points:

Time (sec)	Height (m)	Time (sec)	Height (m)
0	1	33.75	3.34
3.75	3.34	37.5	9
7.5	9	41.25	14.66
11.25	14.66	45	17
15	17	48.75	14.66
18.75	14.66	52.5	9
22.5	9	56.25	3.34
26.25	3.34	60	1
30	1		

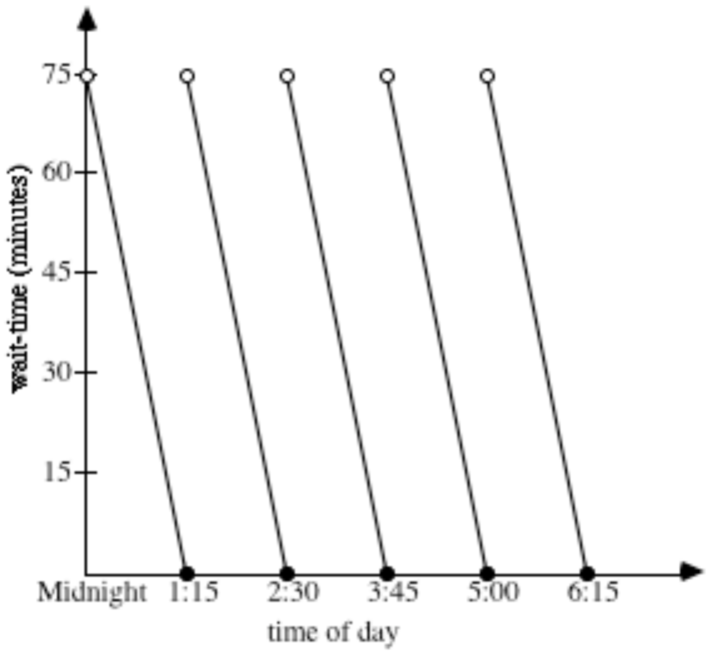
b. Sample scatterplot:



c. Sample response: This data can be modeled by an equation based on the sine function. The amplitude is 8, the period is 30, the horizontal shift is 7.5, and the vertical shift is 9.

$$f(x) = 8\sin\left(\frac{2\pi}{30}(x - 7.5)\right) + 9$$

3. **a.** Sample graph:

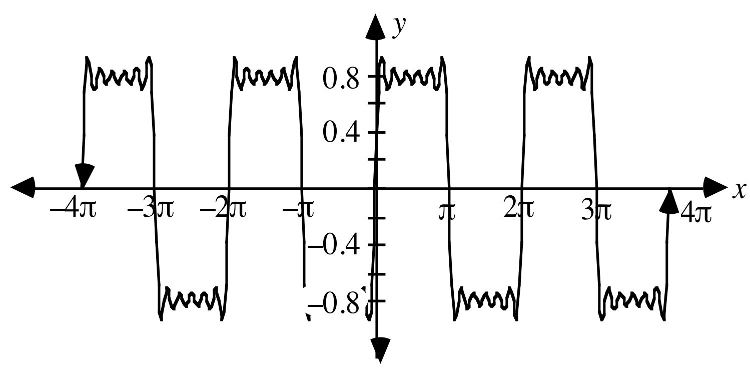


b. This function is periodic. The graph repeats every 75 min.

c. Tourists who arrive at 1:02 A.M., 2:17 A.M., 3:32 A.M., and 4:47 A.M. will wait 13 min for the next ferry.

d. Sample response: Since a ferry leaves for the island at 11:15 A.M., you would not have to wait at all.

4. **a.** Sample graph:



b. The period of the square wave is 2π.

5. **a.** Students may verify that this is an identity by comparing the graphs of $f(x) = \cot x$ and $g(x) = \cos x / \sin x$. The graphs are identical.

b. The domain of $\cot x$ is all real numbers x such that $x \neq 0 + n\pi$, where n is an integer. The range is all real numbers.

6. $\sin((\pi/2) - x) = \cos x$

Selected References

Giancoli, D. *Physics*. Englewood Cliffs, NJ: Prentice-Hall, 1991.

Hewitt, P. *Conceptual Physics*. Menlo Park, CA: Addison-Wesley, 1987

Flashbacks

Activity 1

1.1 Describe how the function $y = 2(x-4)^2 + 5$ is transformed from the parent function $y = x^2$.

1.2 Sketch a graph of each of the following functions.

a. $y = \sin x$

b. $y = \cos x$

1.3 Convert the time 3:50 P.M. to a decimal on a 24-hour clock.

Activity 2

2.1 Solve $4 = 4 + 2\sin 5(x+3)$ over the interval $[0, \pi]$.

2.2 For what values of x, where $0 \le x \le 2\pi$, is $\sin x = \cos x$?

2.3 **a.** Given the functions $f(x) = x^2$ and $g(x) = x^2 + 4$, find $f(x) + g(x)$ and describe its domain.

b. Given the functions $f(x) = x^2$ and $g(x) = (x+4)^2$, find $f(x) + g(x)$ and describe its domain.

Activity 3

3.1 Solve $\sin\theta = 0.5$ over each of the following intervals:

a. $0 \le \theta \le 2\pi$

b. $(-\infty, +\infty)$

3.2 Identify the domain and range of $f(x) = 1/(x-2)$.

3.3 Sketch a graph of the following function, including its asymptotes.

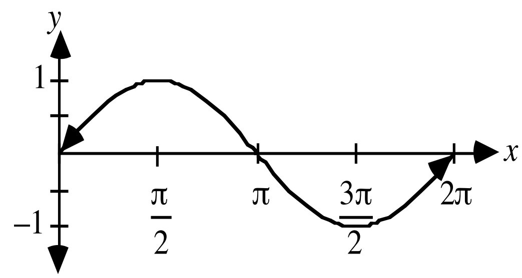
$$g(x) = \frac{1}{(x-2)(x-4)}$$

Answers to Flashbacks

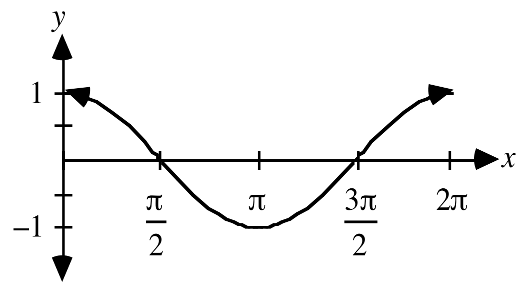
Activity 1

1.1 The function is a vertical "stretch" of the function $y = x^2$. It has been translated 5 units up and 4 units to the right of the parent.

1.2 **a.** Sample graph:



b. Sample graph:



1.3 The time 3:50 P.M. is equivalent to 15.83 hr.

Activity 2

2.1 $x \approx 0.14,\ 0.77,\ 1.40,\ 2.03,\ 2.65$

2.2 $x = \frac{\pi}{4},\ \frac{5\pi}{4}$

2.3 **a.** $f(x) + g(x) = 2x^2 + 4$; its domain is all real numbers

b. $f(x) + g(x) = 2x^2 + 8x + 16$; its domain is all real numbers

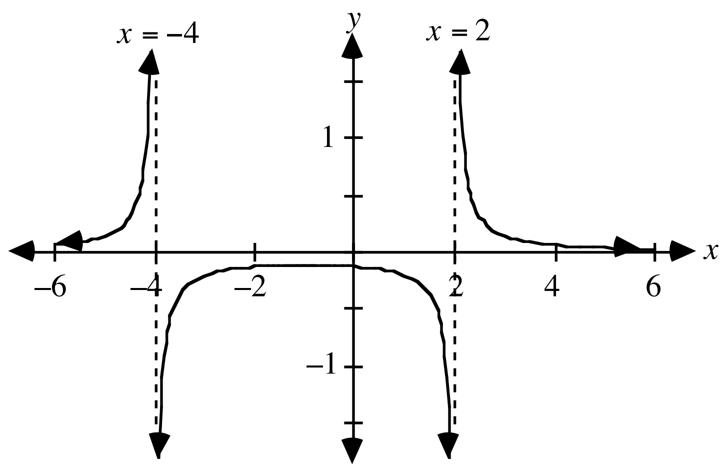
Activity 3

3.1 **a.** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

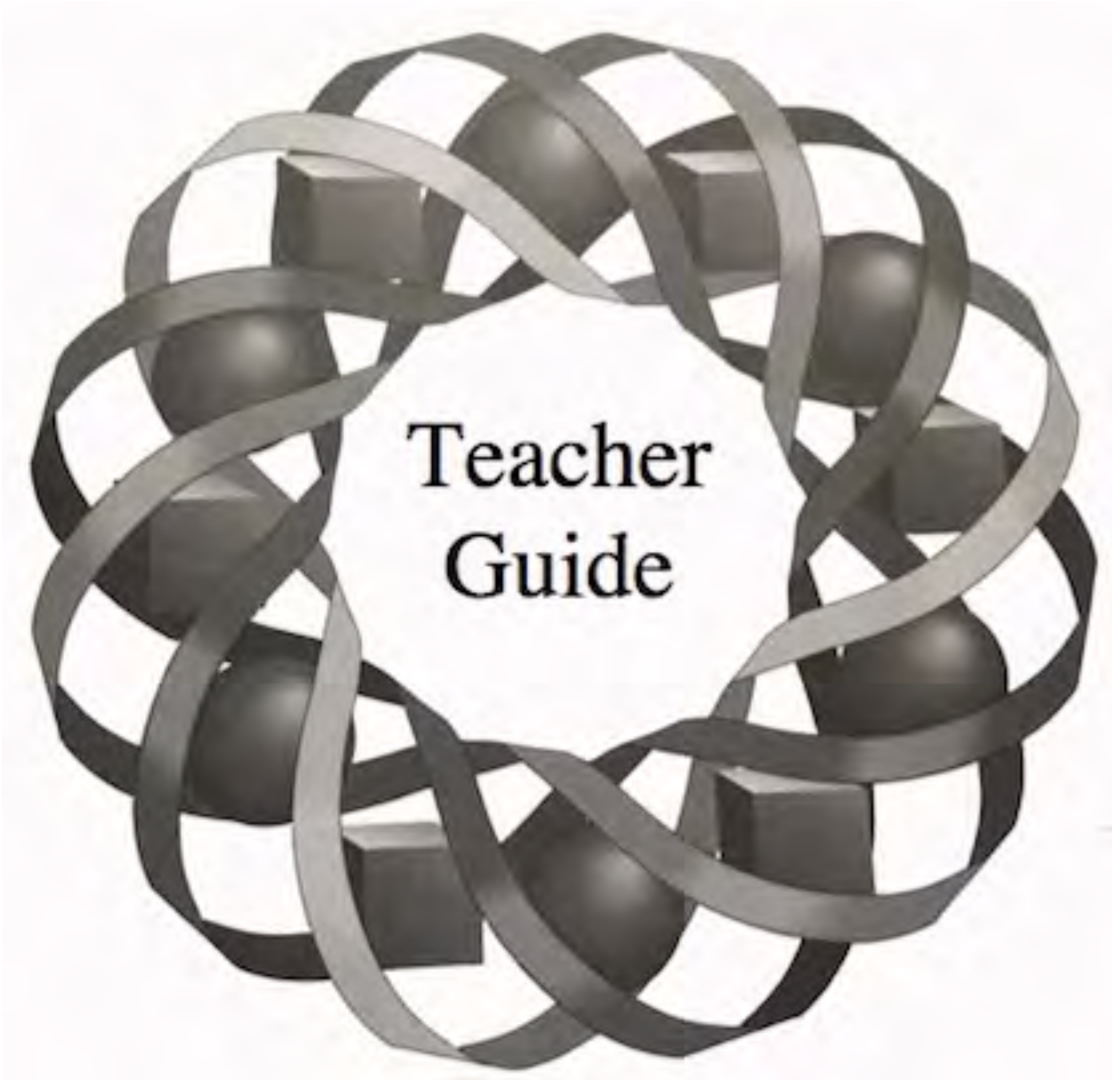
b. $\theta = \frac{\pi}{6} \pm 2\pi n$ or $\frac{5\pi}{6} \pm 2\pi n$ where n is a whole number

3.2 The domain is all real numbers x, such that $x \neq 2$. The range is all real numbers y, such that $y \neq 0$.

3.3 Sample graph:



The Sequence Makes the Difference



Some number patterns are easy to recognize, while others are more difficult. In this module, you'll explore one strategy for recognizing number patterns described by polynomials.

Darlene Pugh • Tom Teegarden

Teacher Edition

The Sequence Makes the Difference

Overview

This module deals with the recognition and mathematical representation of sequences. Students use the finite-difference process to recognize polynomial sequences.

Objectives

In this module, students will:

- generate sequences using polynomial functions
- determine explicit and recursive formulas for sequences
- use the finite-difference process to determine the least degree of a polynomial that generates a polynomial sequence
- determine a polynomial function that generates a given sequence.

Prerequisites

For this module, students should know:

- how to determine explicit and recursive formulas for arithmetic and geometric sequences
- the general forms of polynomial functions
- methods for solving systems of equations
- how to determine regression equations for data sets.

Time Line

Activity	**1**	**2**	**Summary Assessment**	**Total**
Days	3	3	1	7

Materials Required

Materials	Activity		
	1	2	Summary Assessment
stacking blocks		X	

Technology

Software	Activity		
	1	2	Summary Assessment
graphing utility	X	X	X
symbolic manipulator	X	X	X
geometry utility		X	
statistics package		X	X

The Sequence Makes the Difference

Introduction

(page 233)

Students should recall the characteristics of arithmetic and geometric sequences from previous modules, including “From Rock Bands to Recursion” (Level 1) and “Take It to the Limit” (Level 2). The introductory discussion is designed to make students aware of several other types of sequences.

Teacher Note

You may want to discuss with your students that polynomial functions of the form $f(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \cdots + a_1 n^1 + a_0$, where the domain of n is a subset of natural numbers, also can be thought of as sequences of the form $t_n = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \cdots + a_1 n^1 + a_0$.

Discussion

(page 233)

a. The degree of the polynomial is k, provided that $a_k \neq 0$.

b. Using a function $f(n)$ and the domain of natural numbers, $n = 1$ produces the first term of the sequence, while $n = k$ produces the kth term.

c. Answers will vary. Sample response: 5, 5, 5, ...

d. Sample response: Since the difference between each pair of consecutive terms in a linear sequence is the same (a common difference), a linear sequence also is an arithmetic sequence.

e.

1. Answers will vary. Sample response: The following finite sequence can be generated by $f(n) = n^2$ using a domain of 5, 6, 7, 8, 9:

   $$25, 36, 49, 64, 81$$

2. Sample response: The following finite sequence can be generated by $f(n) = n^3$ using a domain of 5, 6, 7, 8, 9:

   $$125, 216, 343, 512, 729$$

3. Sample response: The following finite sequence can be generated by $f(n) = n^4$ using a domain of 5, 6, 7, 8, 9:

   $$625, 1296, 2401, 4096, 6561$$

f. Sample response: Since a polynomial function can be used to generate a sequence, then it is an explicit formula for the sequence. It allows you to determine the value of any term without knowing the previous term.

However, not all sequences that can be generated by an explicit formula can be generated by a polynomial. For example, consider the sequence generated by $t_k = \sin k$.

(page 234)

Activity 1

In this activity, students use the finite-difference process to determine the least degree of a polynomial function that generates a given sequence.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator

Exploration 1

(page 234)

In this exploration, students discover a relationship between the degree of a polynomial that generates a sequence and the corresponding sequences of differences.

a. Answers will vary. The table below shows sample responses:

Type	Function
linear	$f(n) = 2n$
quadratic	$f(n) = 2n^2$
cubic	$f(n) = n^3$
quartic	$f(n) = n^4$

b. The sample response shown below uses the cubic function $f(n) = n^3$.

1. 1, 8, 27, 64, 125, 216

2. 7, 19, 37, 61, 91

3. 12, 18, 24, 30

4. 6, 6, 6

5. In this case, the third sequence of differences is the first constant sequence.

c. Students repeat Part **b** for each polynomial in Part **a**. They should obtain the following results:

Type of Function	First Constant Sequence of Differences
linear	first
quadratic	second
cubic	third
quartic	fourth

d. Students repeat the finite-difference process using geometric and Fibonacci-type sequences. **Note:** Some students may argue that a sequence with one term is a constant sequence. It should be pointed out that a single term is not a sequence. Therefore, if the process does not result in a constant sequence with more than one term, it is not considered to end with a constant sequence.

1. Answers will vary. For example, when $t_1 = 2$ and $r = 2$, the recursive formula generates the sequence 2, 4, 8, 16, 32, 64. The first sequence of differences is 2, 4, 8, 16, 32. The second sequence of differences is 2, 4, 8, 16. The third sequence of differences is 2, 4, 8.

   Students should recognize that unless $r = 1$, the finite-difference process will not result in a constant sequence.

2. Answers will vary. For example, when $t_1 = 2$ and $t_2 = 2$, the recursive formula generates the sequence 2, 2, 4, 6, 10, 16, 26, 42. The first sequence of differences is 0, 2, 2, 4, 6, 10, 16. The second sequence of differences is –2, 0, 2, 2, 4, 6. The third sequence of differences is –2, –2, 0, 2, 2.

   Students should recognize that unless $t_1 = 0$ and $t_2 = 0$, the finite-difference process will not result in a constant sequence.

Discussion 1

(page 235)

a. Sample response: The only types of sequences that resulted in a constant sequence of differences were those generated by polynomial functions.

b. Sample response: The degree of the polynomial function corresponds to the number of sequences of differences required to generate the first constant sequence of differences.

c. Sample response: No. Using the finite-difference process, it would not be possible to obtain a constant sequence of differences. Each successive sequence of differences also would be a geometric sequence, with the same common ratio.

d. Given any finite sequence, there is not a unique polynomial function that will generate it. However, there is a unique polynomial function of degree n that will generate it, where n is the number of sequences of differences required to reach the first constant sequence of differences.

e. Students should recall from previous modules, including the Level 6 module "Functioning on a Path," that it is possible to determine a polynomial function that contains any number of given points by solving a system of equations. Typically, the degree of the function will be one less than the number of points.

Teacher Note

Exploration **2** provides the opportunity for students to use a symbolic manipulator to generate $f(n+1) - f(n)$ in both tabular and functional form.

Exploration 2

(page 236)

a. The first 10 terms of the sequence generated by the function $f(n) = n^4 - 2n^2 + n$ are:

$$0, 10, 66, 228, 580, 1230, 2310, 3976, 6408, 9810.$$

b. **1.** The first nine terms of the first sequence of differences are:

$$10, 56, 162, 352, 650, 1080, 1666, 2432, 3402$$

2. The function may be written as $f_1(n) = 4n^3 + 6n^2$. It is a polynomial of degree 3.

c. **1.** The first eight terms of the second sequence of differences are:

$$46, 106, 190, 298, 430, 586, 766, 970$$

2. The function may be written as $f_2(n) = 12n^2 + 24n + 10$. It is a polynomial of degree 2.

d. **1.** The first seven terms of the third sequence of differences are:

$$60, 84, 108, 132, 156, 180, 204$$

2. The function may be written as $f_3(n) = 24n + 36$. It is a polynomial of degree 1.

e. **1.** The first six terms of the fourth sequence of differences are:

$$24, 24, 24, 24, 24, 24$$

2. The function may be written as $f_4(n) = 24$. It is a polynomial of degree 0.

f. Using their results in Parts **a–e**, students should have completed Table **2** as follows:

Function	Degree
$f_1(n)$	3
$f_2(n)$	2
$f_3(n)$	1
$f_4(n)$	0

In general, if the degree of a polynomial function that generates an infinite sequence is d, then the degree of the polynomial function that generates the nth sequence of differences is $(d - n)$, where $n \leq d$.

In other words, if the degree of the polynomial function is d, then the degree of the polynomial function that generates the first sequence of differences is $(d - 1)$, the degree of the polynomial function that generates the second sequence of differences is $(d - 2)$, and so on, until a constant sequence of differences is generated.

Note: This process also works for finite sequences. However, different polynomials of various degrees can generate the same finite sequence. The finite-difference process only determines the polynomial of least degree that can be used to generate a finite sequence.

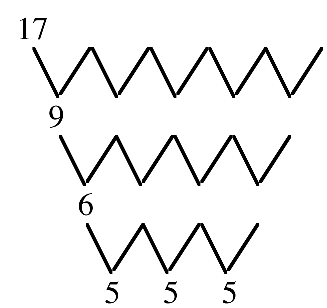
Discussion 2

(page 237)

a.
1. 6
2. 4
3. 2
4. 1

b. See response given in Part **f** of Exploration **2**.

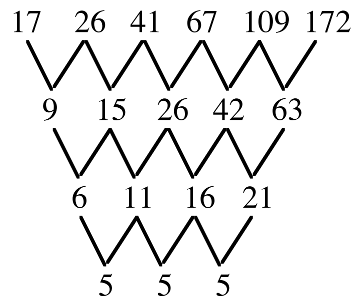
c. Sample response: Using the given information, it is possible to draw the following diagram.



Each term in the second sequence of differences can be found by adding 5 to each of the previous terms, starting with 6. This results in the sequence 6, 11, 16, 21.

Adding 6 to the first term of the first sequence of differences (9), gives the second term. The third term is the second term plus 11, and so on. The first sequence of differences is therefore 9, 15, 26, 42, 63.

Adding 9 to the first term of the original sequence (17) gives the second term. The third term is the second term plus 15, and so on. The original sequence is therefore 17, 26, 41, 67, 109, 172, as shown in the diagram below.



Assignment

(page 238)

1.1 **a.** A completed table is shown below:

n	**Sequence**	**Sequences of Differences**		
1	4	**First**		
2	29	25	**Second**	
3	90	61	36	**Third**
4	205	115	54	18
5	392	187	72	18
6	669	277	90	18
7	1054	385	108	18
8	1565	511	126	18
9	2220	655	144	18
10	3037	817	162	18

b. The least degree of a polynomial that can generate the original sequence is 3. The least degree of a polynomial that can generate the first sequence of differences is 2. The least degree of a polynomial that can generate the second sequence of differences is 1. The least degree of a polynomial that can generate the third sequence of differences is 0.

1.2 Answers will vary. Students should create a fourth-degree (quartic) polynomial of the form $f(n) = a_4n^4 + a_3n^3 + a_2n^2 + a_1n + a_0$.

***1.3** **a.** Sample response: The missing term is 28. It can be found using the table below. Once the constant sequence of differences is determined, you can work backwards to find the numbers in the unknown cells (shown in parentheses).

m	**Sequence**	**Sequences of Differences**		
1	2	**First**		
2	9	7	**Second**	
3	(28)	(19)	(12)	**Third**
4	65	(37)	(18)	(6)
5	126	61	(24)	(6)
6	217	91	30	(6)
7	344	127	36	6
8	513	169	42	6

b. The least degree of a polynomial function that generates the sequence in Part **a** is 3.

1.4 Sample response: The first sequence of differences for a sequence generated by a fifth-degree polynomial is a sequence generated by a fourth-degree polynomial. The first sequence of differences of a sequence generated by a fourth-degree polynomial is a sequence generated by a third-degree function. This process continues until a constant sequence is generated.

***1.5** Students may identify the missing terms using the method described in Problem **1.3** or by finding the corresponding polynomial.

a. 370

b. –720

* * * * *

***1.6** **a.** The least degree of polynomial function that could generate this sequence is 3 ($t_n = n^3 + n + 1$).

b. The least degree of polynomial function that could generate this sequence is 2 ($t_n = n^2$).

***1.7** **a.** The first eight terms of sequence p are 1, 3, 6, 10, 15, 21, 28, 36.

The first eight terms of sequence q are 1, 5, 14, 30, 55, 91, 140, 204.

The first eight terms of sequence r are 1, 9, 36, 100, 225, 441, 784, 1296.

b. Sample response: All three sequences could be polynomial sequences because each seems to generate a constant sequence of differences.

c. The least degree of a polynomial function that could have generated sequence *p* is 2.

The least degree of a polynomial function that could have generated sequence *q* is 3.

The least degree of a polynomial function that could have generated sequence *r* is 4.

1.8 **a.** Sample response: –490.

b. The least degree of polynomial function that could generate this sequence is 4 ($t_n = 2n^4 + n^2 + 6$).

* * * * * * * * * *

(page 239)

Activity 2

In this activity, students use the finite-difference process to help determine recursive and explicit formulas for sequences generated by polynomials. In all recursive and explicit formulas, the domain is restricted to a subset of the natural numbers.

Materials List

- stacking blocks or sugar cubes (55 per group)

Technology

- graphing utility
- symbolic manipulator
- statistics package
- geometry utility

Exploration

(page 239)

Students use stacks of blocks to model a sequence generated by a cubic function. They then analyze the resulting pattern to find the polynomial that generates the sequence.

a. Students create a pyramid that contains 55 blocks.

b. **1.** The sequence that models the number of blocks after each level is:

$$1, 5, 14, 30, 55$$

2. A recursive formula for this sequence is:

$$\begin{cases} t_1 = 1 \\ t_n = t_{n-1} + n^2,\ n > 1 \end{cases}$$

3. The next three terms of the sequence are 91, 140, and 204.

c. Using the finite-difference process, students determine that the least degree of a polynomial that generates the sequence is 3.

d. Students may use matrices or substitution to solve their system of equations. Sample response: The solutions to the following system are $a_3 = 1/3$, $a_2 = 1/2$, $a_1 = 1/6$, and $a_0 = 0$.

$$\begin{cases} 1 = a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 \\ 5 = a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 \\ 14 = a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 \\ 30 = a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 \end{cases}$$

Therefore, the cubic function that generates the sequence is:

$$f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

e. Student use a cubic regression to identify the third-degree polynomial function that generates the sequence.

Discussion

(page 241)

a. Sample response: Except for the differences caused by rounding in the regression equation, the two polynomials are the same.

b. **1.** Sample response: To find the 100th term of a sequence using the recursive formula, you would have to calculate the first 99 terms, then use the formula to find the 100th term.

2. Sample response: To find the 100th term of a sequence using the explicit formula, you would substitute 100 into the equation for *n* and find the value.

c. Sample response: You must assume the pattern among the known terms of the sequence continues through the 100th term.

d. **1.** Answers may vary. Sample response: Since each term appears to be twice the previous term, I predict that the next term is 32.

2. Using the given function, the next term is 72.

e. Sample response: Unless you also know the recursive definition of a sequence or can safely assume that the observed pattern continues, predicting terms using an explicit formula can be unreliable.

f. Sample response: No. Since the sequence can be described by a recursive formula, the pattern must continue in all subsequent terms. I would expect only one possible explicit formula.

Assignment

(page 242)

2.1 **a.** **1.** One possible explicit formula is $f(n) = 0.5n^3 + 0.5n^2$. Using this function, the next three terms are 288, 405, and 550.

2. One possible explicit formula is $f(n) = 4.5n^2 - 3n + 5$. Using this function, the next three terms are 149, 204.5, and 269.

3. One possible explicit formula is $f(n) = -2.1n^4 - 7n^2 + 18$. Using this function, the next three terms are –5367.1, –9031.6, and –14,327.1.

4. One possible explicit formula is $f(n) = -n^2 + 6n - 6$. Using this function, the next three terms are –33, –46, and –61.

b. Sample response: No. There is an infinite number of possible "next terms" that could result in a different polynomials. There are also an infinite number of polynomials that could generate the same first terms of the given sequence.

2.2 **a.** Sample response: Yes. The total number of pipes after any row, starting with the first row, is the sequence 1, 3, 6, 10, 15, … . This sequence could be generated by a second-degree polynomial, since the second sequence of differences is a constant sequence.

b. Sample response: The number of pipes in the bottom row is also the number of rows. One formula that describes this sequence, where *n* is the number of pipes in the bottom row is $t_n = 0.5n^2 + 0.5n$.

c. Sample response: By substituting 40 for *n* in the formula $t_n = 0.5n^2 + 0.5n$, there are 820 pipes in the pile.

2.3 **a.** Answers may vary. Sample response: The fourth term is 4.

b. Sample response: One polynomial function that produces the first four terms of the sequence is $f(x) = x$, where *x* is a natural number.

c. Answers will vary. Sample response: The new fourth term is 8.

d. One polynomial function that produces the terms 1, 2, 3, 8 is:

$$f(x) = \frac{2}{3}x^3 - 4x^2 + \frac{25}{3}x - 4$$

where the domain of *x* is 1, 2, 3, 4.

e. Using the sample response given in Part **b**, the fifth term of the sequence is 5. Using the sample response given in Part **d**, the fifth term of the sequence is 21.

***2.4** **a.** Rows 4 and 5 are shown below.

1 — row 1
2 3 — row 2
4 5 6 — row 3
7 8 9 10 — row 4
11 12 13 14 15 — row 5

b. Sample response: The first number of each row forms a sequence whose explicit formula is the polynomial $t_n = 0.5n^2 - 0.5n + 1$, where n is the row number.

Using this equation, you can determine that $t_{45} = 991$ and $t_{46} = 1036$. This means the 45th row starts with the number 991 and the 46th row starts with the number 1036. Therefore, 1000 is in the 45th row.

c. Sample response: The sums of the numbers in each row form a sequence whose explicit formula is the polynomial $t_n = 0.5n^3 + 0.5n$. The sum of the 100th row is $t_{100} = 500{,}050$.

2.5 **a.** Sample response: A recursive formula for the sequence is:

$$\begin{cases} t_1 = -5 \\ t_n = t_{n-1}(-1.5), \quad n > 1 \end{cases}$$

b. Using the sample response in Part **a**, the next three terms are –25.3125, 37.96875, and –56.953125.

c. This is a geometric sequence with a common ratio of –1.5. The explicit formula is $t_n = -5(-1.5)^{n-1}$.

d. The 50th term is $t_{50} = -5(-1.5)^{49} \approx 2{,}125{,}405{,}000.71$.

2.6 **a.** One polynomial that models the sequence is:

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$$

If the display contain 10 levels, it would contain 220 oranges.

b. The seven cases of oranges would be enough to complete 11 levels of the display.

***2.7** **a.** The first five terms of this sequence are 3996, 3984, 3964, 3936, 3900.

b. The first five terms of this sequence are 3744, 6992, 9768, 12096, 14000.

c. Sample response: If m is the length of the sides of the square tabs, then the surface area of the box is:

$$\begin{aligned} A &= (80 - 2m)(50 - 2m) + (2)(m)(50 - 2m) + (2)(m)(80 - 2m) \\ &= 4000 - 4m^2 \end{aligned}$$

The maximum term of the sequence generated by this polynomial expression is $t_1 = 3996 \text{ cm}^2$. This corresponds to a square tab with a side length of 1 cm.

Note: Students may also reason that the greatest surface area corresponds with the least amount removed from the corners.

d. Sample response: If m is the length of the sides of the square tabs, then the volume of the box is:

$$\begin{aligned} V &= m(80 - 2m)(50 - 2m) \\ &= 4m^3 - 260m^2 + 4000m \end{aligned}$$

The maximum term of the sequence generated by this polynomial is $t_{10} = 18{,}000 \text{ cm}^3$. This corresponds to square tabs with side lengths of 10 cm. **Note:** There are greater values for t_n after the 45th term, but the side of a square tab cannot exceed 25 cm.

* * * * *

2.8 **a.** Sample response: The first six terms of the sequence of gifts are 1, 3, 6, 10, 15. Since the second sequence of differences is a constant sequence, a second-degree polynomial can describe the sequence. One quadratic function that models the sequence is:

$$f(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

Using this formula, 78 gifts are given on day 12.

Note: The table below shows the gifts added on each day.

Day	Gifts	Day	Gifts
1	1 partridge	7	7 swans a' swimming
2	2 turtle doves	8	8 maids a' milking
3	3 French hens	9	9 ladies dancing
4	4 calling birds	10	10 pipers piping
5	5 golden rings	11	11 drummers drumming
6	6 geese a' laying	12	12 lords a' leaping

b. Sample response: The first six terms of the sequence that describes the total number of gifts received after each day are 1, 4, 10, 20, 35, 56. Since the third sequence of differences is a constant sequence, a third-degree polynomial can describe the sequence. One cubic function that models the sequence is:

$$f(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$$

Using this formula, a total of 364 gifts are received during the 12 days.

2.9 **a.** The sequence for the first 10 rings of the quilt is 6, 12, 18, 24, 30, 36, 42, 48, 54, 60. The sequence is arithmetic, with the following recursive formula:

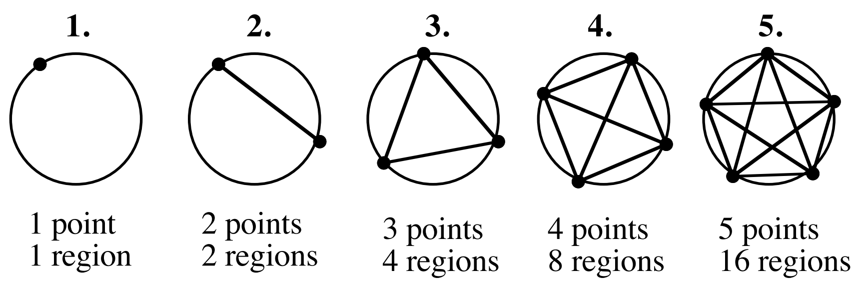
$$\begin{cases} t_1 = 6 \\ t_n = t_{n-1} + 6, \ n > 1 \end{cases}$$

b. An explicit formula for this sequence is $t_n = 6n$. There are 126 hexagons in the 21st ring.

2.10 **a.** Sample response: The first five terms of the sequence for the amount deposited each month are 10, 12, 14, 16, and 18. This sequence has the explicit formula $t_n = 2n + 8$. The amount deposited during the last month of the fifth year is $t_{60} = \$128$.

b. Sample response: The first five terms of the sequence for the total money deposited after each month are 10, 22, 36, 52, and 70. This sequence has the explicit formula $t_n = n^2 + 9n$. The total amount deposited after 5 years is $t_{60} = \$4140$.

2.11 **a.** Students should complete these constructions using a geometry utility.

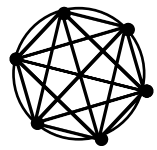


b. Sample response: The number of regions seems to form a geometric sequence with a common ratio of 2.

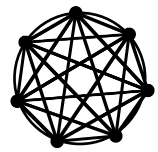
c. **1.** If the pattern continues, six points should divide the circle into 32 regions.

2. If the pattern continues, seven points should divide the circle into 64 regions.

d. Sample response: The constructions below show that six points divide the circle into 30 regions and seven points divide the circle into 51 regions. The sequence is not a geometric sequence. This example illustrates the dangers of assuming that a pattern exists after examining only a few terms of a sequence.



6 points
30 regions



7 points
51 regions

* * * * * * * * * *

Research Project (page 246)

a. The minimum number of moves necessary to move the rings from one ring to another while maintaining the same order is shown below.

No. of Rings in Stack	Minimum No. of Moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
$\vdots$	$\vdots$
n	$2^n - 1$

b. **1.** A recursive formula is shown below:

$$\begin{cases} t_1 = 1 \\ t_n = t_{n-1} + 2^{n-1},\ n > 1 \end{cases}$$

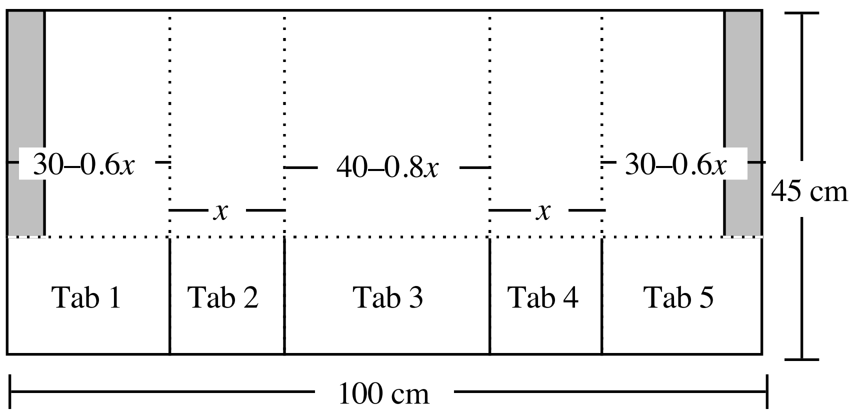
2. The explicit formula is $t_n = 2^n - 1$.

Answers to Summary Assessment

(page 247)

Note: You may wish to use this as a group assessment.

In the diagram below, the dimensions of the square tabs (Tab **2** and Tab **4** in the template) are represented as x by x. The widths of the remaining tabs are given in terms of x.



The volume of the bag can now be expressed in terms of x as follows:

$$
\begin{aligned}
V &= x(40 - 0.8x)(45 - x) \\
&= 0.8x^3 - 76x^2 + 1800x
\end{aligned}
$$

Students can use this polynomial to generate a sequence, then determine the term number that corresponds to a volume greater than 12,500 cm^3.

The surface area of the bag can be expressed in terms of x as follows:

$$
\begin{aligned}
A &= x(40 - 0.8x) + 2x(45 - x) + 2(45 - x)(40 - 0.8x) \\
&= -1.2x^2 + 43.6x + 320
\end{aligned}
$$

By substituting values of x that produce volumes greater than 12,500 cm^3 in this polynomial, students can identify the value of x that generates the maximum surface area. (Students also may choose to graph the polynomials to help determine these same values.)

The dimensions of the square tabs that create a bag with a volume of at least 12,500 cm^3 and the maximum surface area are $15 \text{ cm} \times 15 \text{ cm}$. The dimensions of the bag are $28 \text{ cm} \times 15 \text{ cm} \times 30 \text{ cm}$.

Some students may wish to ignore the area of the bottom of the bag, arguing that no advertising should be placed there. In this case, the polynomial representing the lateral surface area is $-0.4x^2 - 6.2x + 3600$. The dimensions of the desired bag are the same.

Module Assessment

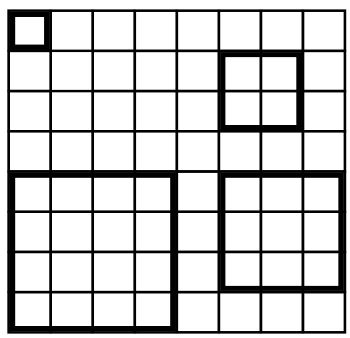
1. **a.** Determine the least possible degree of a polynomial function that could generate the following sequence.

$$11, 21, 41, 71, 111, 161, ...$$

b. Identify a function that generates the sequence.

2. A standard checkerboard has 8 rows and 8 columns of squares. How many squares are the board? One quick response might be 64. If you consider the question more thoroughly, however, there are many more.

As shown in the diagram below, there are 64 squares that measure 1×1, but the board also contains 2×2 squares, 3×3 squares, and so on.



a. Complete the table below for checkerboards of various sizes.

Size of Checkerboard	No. of 1×1 Squares	Total No. of Squares
1×1	1	1
2×2		
3×3		
4×4		
5×5	25	

b. Suggest a formula that could be used to calculate the total number of squares on an $n \times n$ checkerboard.

c. Use your formula to predict the total number of squares on an 8×8 checkerboard.

3. Create as many sequences as possible of at least four terms by drawing straight lines—horizontal or diagonal—through sets of numbers in the diagram below.

$$
\begin{array}{c}
1 \\
2 \quad 3 \\
4 \quad 5 \quad 6 \\
7 \quad 8 \quad 9 \quad 10 \\
11 \quad 12 \quad 13 \quad 14 \quad 15 \\
16 \quad 17 \quad 18 \quad 19 \quad 20 \quad 21
\end{array}
$$

Identify the polynomial function of least degree that generates each sequence.

Answers to Module Assessment

1. **a.** Sample response: Since the second sequence of differences appears to be a constant sequence, the least degree of a polynomial that generates the sequence appears to be 2.

b. Sample response: If the pattern in the first six terms continues, a possible explicit formula is $f(n) = 5n^2 - 5n + 11$, where n is a natural number.

2. **a.** Sample table:

Size of Checkerboard	**No. of 1 × 1 Squares**	**Total No. of Squares**
1×1	1	1
2×2	4	5
3×3	9	14
4×4	16	30
5×5	25	55

b. Sample response: If the pattern in the first five terms continues, a possible explicit formula is:

$$t_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

where the dimensions of the checkerboard are $n \times n$.

c. By substituting 8 for n in the formula found in Part **b**, the total number of squares on an 8×8 checkerboard is 204.

3. Answers may vary. Those sequences found by drawing horizontal lines are arithmetic and can be generated by linear functions. Those found by drawing diagonal lines through numbers in consecutive rows can be generated by quadratic functions.

The arithmetic sequences (and their corresponding formulas) are shown in the following table:

Sequence	**Formula**
7, 8, 9, 10	$f(n) = n + 6$
11, 12, 13, 14, 15	$f(n) = n + 10$
16, 17, 18, 19, 20, 21	$f(n) = n + 15$

The table below shows sequences that can be generated by quadratic functions (and their corresponding formulas):

Sequence	Formula
1, 2, 4, 7, 11, 16	$f(n) = 0.5n^2 - 0.5n + 1$
3, 5, 8, 12, 17	$f(n) = 0.5n^2 + 0.5n + 2$
6, 9, 13, 18	$f(n) = 0.5n^2 + 1.5n + 4$
1, 3, 6, 10, 15, 21	$f(n) = 0.5n^2 + 0.5n$
2, 5, 9, 14, 20	$f(n) = 0.5n^2 + 1.5n$
4, 8, 13, 19	$f(n) = 0.5n^2 + 2.5n + 1$

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Flashbacks

Activity 1

1.1 Simplify each of the following polynomial expressions.

a. $2x(x-3)(2x+4)$

b. $(12-2x)(15-2x)(x)$

c. $(10-x)(15+x)+2(x)(23-3x)$

1.2 Determine the range of the function $f(x)=x^4+3x-2$, given each of the following domains for *x*:

a. $\{-3, 1.5, 0, 2\}$

b. the set of natural numbers

1.3 Determine a quadratic equation that models the data points (–5,–70), (–2,–16), (0,0), (2,0), and (7,–70).

1.4 **a.** Give an example of an arithmetic sequence.

b. Give an example of a geometric sequence.

Activity 2

2.1 List the first five terms of the arithmetic sequence defined by the recursive formula below:

$$\begin{cases} t_1 = 4 \\ t_n = t_{n-1} + 3,\ n > 1 \end{cases}$$

2.2 Write a recursive formula for the geometric sequence 3, 6, 12, 24, 48, … .

2.3 Use matrices to solve each of the following systems of equations.

a. $\begin{cases} 2a + 3b = -1 \\ -3a + 7b = -33 \end{cases}$

b. $\begin{cases} 4x - 3y + 2z = 5/2 \\ 5x + 2y + z = 14/3 \\ -3x - 2z = 1 \end{cases}$

c. $\begin{cases} r + s + t - u = 4.4 \\ 2r - 4t + u = 4.2 \\ 2r + 3s - 4t + u = 8 \\ 5r - 4s + 5t = 1.2 \end{cases}$

Answers to Flashbacks

Activity 1

1.1 **a.** $2x(x-3)(2x+4) = 4x^3 - 4x^2 - 24x$

b. $(12-2x)(15-2x)(x) = 180x - 54x^2 + 4x^3$

c. $(10-x)(15+x) + 2(x)(23-3x) = 150 + 41x - 7x^2$

1.2 **a.** $\{70, 7.5625, -2, 20\}$

b. $\{2, 20, 88, 266 \ldots\}$

1.3 $f(x) = -2x^2 + 4x$

1.4 **a.** Sample arithmetic sequence: 2, 6, 10, 14, ...

b. Sample geometric sequence: –3, 9, –27, 81, ...

Activity 2

2.1 The first five terms are 4, 7, 10, 13, 16.

2.2 A recursive formula for the sequence is:

$$\begin{cases} t_1 = 3 \\ t_n = t_{n-1} \bullet 2, \ n > 1 \end{cases}$$

2.3 **a.** $a = 4$ and $b = -3$

b. $x = 1$, $y = 5/6$, and $z = -2$

c. $r = 1$, $s = 0.2$, $t = -0.6$, and $u = 3$

Brilliant Induction

In this module, you use the principle of mathematical induction to establish the validity of mathematical statements.

Staci Auck • Monty Brekke • Tim Skinner

Teacher Edition

Brilliant Induction

Overview

Students investigate the principle of mathematical induction as a method of proof.

Objectives

In this module, students will:

- write proofs using the principle of mathematical induction.

Prerequisites

For this module, students should know:

- how to find recursive formulas for sequences and series
- how to find explicit formulas for sequences and series
- the definition of proof by exhaustion
- how to use counterexamples to disprove statements
- how to use finite differences to determine explicit formulas for sequences.

Time Line

Activity	1	2	Summary Assessment	Total
Days	3	3	1	7

Materials Required

Materials	Activity		
	1	**2**	**Summary Assessment**
dominos (optional)	X		

Technology

Software	Activity		
	1	**2**	**Summary Assessment**
symbolic manipulator	X	X	
spreadsheet	X	X	
geometry utility		X	

Brilliant Induction

Introduction

(page 253)

The process of proof by mathematical induction is similar to the process of lining up dominoes and pushing one to make them all fall over. If the dominos are arranged properly, all the dominos will fall in succession after the first one falls. **Note:** You may wish to demonstrate this to your students.

Materials List

- dominos (optional)

Discussion

(page 253)

a. Sample response: As each domino falls, it must knock over the next domino.

b. Sample response: The 50th domino falls because it is knocked over by the 49th domino. The 5th falls because it is knocked over by 4th. Both are knocked over by the one immediately before it.

c. Sample response: Suppose the first domino falls. Then, if it can be shown that every other domino will fall after the preceding domino falls, all dominos will fall.

d. Sample response: The relay team will not finish the race if one of the runners does not complete 100 m, or if one of the baton exchanges does not occur, or if the exchange happens outside the designated boundaries. They may also be disqualified if the first runner commits a false start.

e. Sample response: Once you've reached a particular rung on a ladder, you can climb to the next rung. Reaching each rung depends on reaching the previous rung. With dominos, each one's fall depends on the fall of the previous one.

(page 254)

Activity 1

In this activity, students investigate what might be called "pre-induction." Using geometric models, they encounter an almost complete proof by mathematical induction. They also begin to investigate some of the conditions necessary for the use of induction.

Materials List

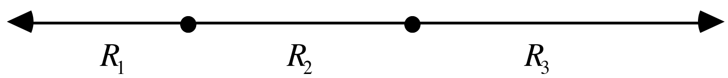
- none

Technology

- spreadsheet (optional)
- symbolic manipulator (optional)

Exploration 1 (page 254)

a. As shown below, adding a second distinct point forms three regions.



b. Sample response: If the point is placed in R_1, then that region is separated into two parts. Similarly, if it is placed in R_2, then that region is separated into two parts. No matter where the new point is placed, one more region is formed.

c. Sample table:

Number of Distinct Points (*n*)	Number of Regions Added with Each Additional Point	Total Number of Regions
1		2
2	1	3
3	1	4
4	1	5
5	1	6
6	1	7

Discussion 1 (page 254)

a. Sample response: An additional region is added to the total.

b. Sample response: It makes no difference where the point is placed, as long as it is distinct. An existing region is divided into two regions in each case.

c. Sample response: Yes. The same appears to be true regardless of the number of points already placed on the line.

d. Sample response: Only one region is affected when an additional point is placed on the line. That region will be separated into two parts. No other region is affected and thus only 1 is added to the total number of regions when there were *k* points on the line.

Exploration 2 (page 255)

a. **1.** Sample response: To build the second rectangle from the first, one can think of the toothpick on the right-hand side being moved farther to the right and two more added (one on top and one on the bottom).

2. $a_2 = a_1 + 2$

b. Sample response: The third rectangle can be built from the second in exactly the same way. One can think of the toothpick on the right-hand side being moved farther to the right and two more added (one on top and one on the bottom). In this case, $a_3 = a_2 + 2$.

c. Sample response: Yes, it appears that one can build the next successive rectangle from the current one by following the same process. If this is true, $a_{k+1} = a_k + 2$.

d. The first five terms of the series are 4, 10, 18, 28, 40.

e. Students may use systems of equations or a quadratic regression to show that a possible formula is $S_n = n^2 + 3n = n(n+3)$.

f. The following steps validate the formula for S_1:

$$S_1 = a_1 = 2(1) + 2 = 4 = 1(1+3)$$

g. **1.** The following steps validate that the formula is true for S_3, given that it is true for S_2.

$$S_3 = S_2 + a_3 = 2(2+3) + 2(3) + 2 = 18 = 3(3+3)$$

2. The following steps validate that the formula is true for S_4, given that it is true for S_3:

$$S_4 = S_3 + a_4 = 3(3+3) + 2(4) + 2 = 28 = 4(4+3)$$

h. **1.** The following steps validate that the formula is true for S_{101}, given that it is true for S_{100}:

$$S_{101} = S_{100} + a_{101} = 100(100+3) + 2(101) + 2 = 10{,}504 = 101(101+3)$$

2. The following steps validate that the formula is true for S_{753}, given that it is true for S_{752}:

$$\begin{aligned} S_{753} &= S_{752} + a_{753} \\ &= 752(752+3) + \left(2(753) + 2\right) \\ &= 569{,}268 \\ &= 753(753+3) \end{aligned}$$

Discussion 2

(page 256)

a. No. The formula cannot be proven true by checking it for every value of n since there are an infinite number of possibilities.

b. Sample response: To verify that the formula is true for S_{k+1}, given that it is true for S_k, you must show that $S_k + a_{k+1}$ is the same as the value of S_{k+1} given by the formula.

Given that the formula is true for S_k, $S_k = k(k+3)$. Using the explicit formula for each term of the sequence, $a_{k+1} = 2(k+1)+2$.

Thus, to show that $S_k + a_{k+1}$ is the same as the value of S_{k+1} given by the formula $S_n = n(n+3)$, you would have to verify that $S_k + a_{k+1} = k(k+3) + [2(k+1)+2]$ can be written in the form $(k+1)([k+1]+3)$.

Note: Students are asked to complete this verification in the exploration in Activity **2**. You may wish to guide students through the following steps:

$$\begin{aligned}
S_k + a_{k+1} &= k(k+3) + [2(k+1)+2] \\
&= k(k+1+2) + [2(k+1)+2] \\
&= k(k+1) + 2k + 2(k+1) + 2 \\
&= k(k+1) + 2(k+1) + 2k + 2 \\
&= k(k+1) + 2(k+1) + 2(k+1) \\
&= (k+1)(k+2+2) \\
&= (k+1)([k+1]+3)
\end{aligned}$$

Because $(k+1)([k+1]+3)$ is in the desired form for S_{k+1}, the process is complete.

Assignment

(page 257)

***1.1** **Note:** Some students may recall this setting from the Level 2 module, "Take It to the Limit."

a. Sample response:

$$1 + 2 + 3 + \cdots + n = \left(\frac{1}{2}\right)n^2 + \left(\frac{1}{2}\right)n = \frac{n(n+1)}{2}$$

b. **1.** Sample response:

$$\begin{aligned} S_{101} &= S_{100} + a_{101} \\ &= \frac{100(100+1)}{2} + 101 \\ &= 101\left(\frac{100}{2} + 1\right) \\ &= 101\left(\frac{100}{2} + \frac{2}{2}\right) \\ &= \frac{101(101+1)}{2} \end{aligned}$$

2–3. The arguments for 102 and 103 are similar to the case for 101.

c. Sample response: Assuming that the statement is true for $n = 100$ allowed us to show that the statement is also true for $n = 101$, $n = 102$, and $n = 103$. This is like saying that if the 100th domino falls, it will start a chain reaction that knocks over the 101st, 102nd, and 103rd dominoes.

***1.2** **a.** Sample response: If the statement is true for the *k*th case, then one can add $k+1$ to both sides of the equation and write the result in the same form as the original formula:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k &= \frac{k(k+1)}{2} + 1 \\ 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + 1 + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} + 1 \\ &= \frac{(k+1)(k+2)}{2} + 1 \\ &= \frac{(k+1)([k+1]+1)}{2} + 1 \end{aligned}$$

Because the result can be written in the same form as the original, the conjecture is true for $k+1$, if it is true for k.

b. Sample response: If $n = 1$, then the formula produces the following equation, which is untrue.

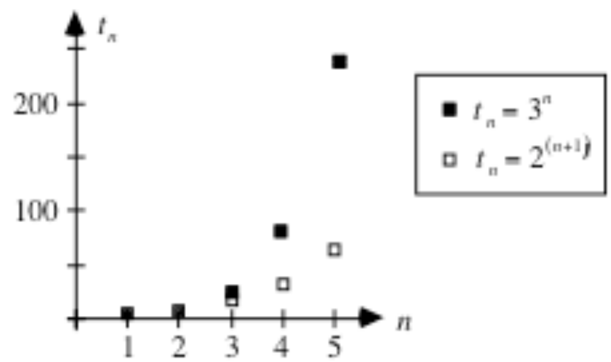
$$1 = \frac{1(1+1)}{2} + 1 = 2$$

c. Sample response: In logic, if the hypothesis of an if-then statement is false, then the statement is true, regardless of the truth or falsity of the conclusion. The truth values for any conditional are illustrated in the table below.

p	***q***	***p → q***
T	T	T
T	F	F
F	T	T
F	F	T

d. No type of argument will prove that the conjecture is true because the counterexample found in Part **b** proves that it is false.

1.3 **a.** Sample graph:



b. From the graph, $t_n = 2^{(n+1)}$ appears to be less than $t_n = 3^n$ for $n \geq 2$.

c. By the laws of exponents, $2^{(578+1)} = 2^{(577+1)} \bullet 2^1$. Since it is assumed that the inequality is true for $n = 577$, $2^{(577+1)} < 3^{577}$. Substituting 3^{577} for $2^{(577+1)}$ on the right-hand side does not affect the inequality. Substituting 3^1 for 2^1 on the right-hand side also does not affect the inequality. By the laws of exponents, $3^{577} \bullet 3^1 = 3^{578}$. Therefore, $2^{(578+1)} < 3^{578}$.

d. Sample response: For all natural numbers $n > 1$, $2^{n+1} < 3^n$.

1.4 **a.** One possible formula is $2 + 4 + 6 + \cdots + 2n = n(n+1)$, where n is a natural number.

b. $S_{50} = 2 + 4 + 6 + \cdots + 2(50) = 50(50 + 1)$

c. Sample response:

$$\begin{aligned} S_{51} &= S_{50} + a_{51} \\ &= 50(50+1) + 2(51) \\ &= 50(51) + 2(51) \\ &= 51(50+2) \\ &= 51(51+1) \end{aligned}$$

1.5 One possible formula, where n represents the number of people in the room, is shown below:

$$h(n) = \left(\frac{1}{2}\right)n^2 + \left(\frac{1}{2}\right)n = \frac{n(n+1)}{2}$$

1.6 **a.** Sample response:

$$\begin{aligned}3^{10,004} + 1 &= 3^{10,003} \bullet 3^1 + 1 \\ &= 3^{10,003}(2+1) + 1 \\ &= 3^{10,003} \bullet 2 + 3^{10,003} \bullet 1 + 1 \\ &= 3^{10,003} \bullet 2 + (3^{10,003} + 1)\end{aligned}$$

In the last equation, both parts are divisible by 2 and thus the sum is divisible by 2.

b. Sample response: No. Proving that something is true for one case does not prove it in general. Also in Part **a**, it is not known if the assumption is actually true, only that we have assumed it to be true.

1.7 **a.** Sample response: The statement is false when $n = 3$, since $1 \bullet 2 \bullet 3 = 6 \neq 5 = 3^3 - 2^{3-1}$.

b. Sample response: The statement is false when $n = 2$, since $5^2 = 25 < 32 = 2^5$.

c. Sample response: The statement is false when $n = 1$, since $12^1 - 8^1 = 4$ and 8 is not a factor of 4.

d. Sample response: The statement is false when $n = 4$, since $2(4) + 1 = 9$ and 9 is not a prime number.

* * * * *

1.8 **a.** 3, 7, 11, 15, 19

b. Sample formula: $a_n = 4n - 1$.

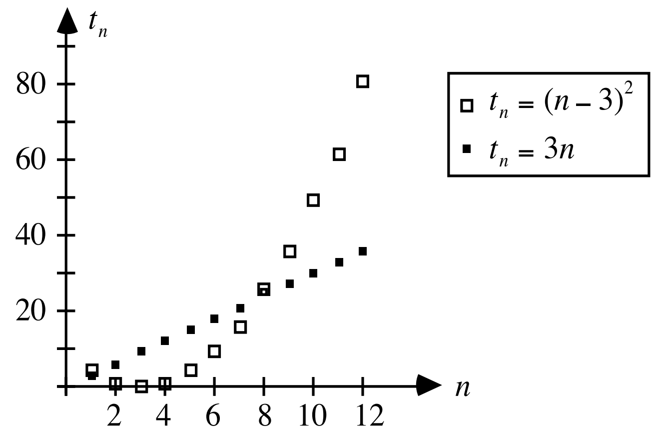
c. If the explicit formula is true for $n = 35$, then $a_{35} = 4(35) - 1$. Using the recursive formula, $a_{36} = a_{35} + 4$. Substituting the implied value for a_{35} into this equation:

$$\begin{aligned}a_{36} &= [4(35) - 1] + 4 \\ &= 4(35) + 4 - 1 \\ &= 4(35 + 1) - 1 \\ &= 4(36) - 1\end{aligned}$$

Therefore, the explicit formula will also be true for $n = 36$.

1.9 **a.** It is not possible to show that the formula is true for $n = 8$ because $25 = (8-3)^2 > 3 \bullet 8 = 24$.

b. Sample graph:



c. The inequality is true for $2 \le n \le 7$.

1.10 **a.** Sample response: The conjecture is true for $n = 7$, since $2^7 - 2 = 126$ and 126 is divisible by 7.

b. Sample response: No. This evidence is not a proof. To prove the conjecture, you would have to consider all possible values for *n*.

1.11 **a.** Sample response: Each new term is formed by adding three toothpicks to the right-hand side of the previous term to form another congruent square.

b. Sample formula: $a_n = 3n + 1$.

c. If the formula is true for $n = 50$, then

$$\begin{aligned} a_{51} &= a_{50} + 3 \\ &= 3(50) + 1 + 3 \\ &= [3(50) + 3] + 1 \\ &= 3(50 + 1) + 1 \\ &= 3(51) + 1 \end{aligned}$$

* * * * * * * * * *

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Activity 2

Students continue their investigations of mathematical induction and use this principle to prove some of the conjectures made in Activity **1**.

Materials List

- none

Technology

- symbolic manipulator (optional)
- spreadsheet (optional)
- geometry utility (optional)

Teacher Note

As noted in the mathematics note, proving that $P(2)$ is true given that $P(1)$ is true may suggest a process for showing that $P(k+1)$ is true given that $P(k)$ is true.

Discussion 1

(page 260)

a. **1.** Sample response: If the first domino is tipped over, then it tips the second over, which tips the third over, and so on, until the 999,999th domino tips the millionth domino over.

2. Sample response: Any domino in the row will fall as long as the one immediately in front of it falls. Since the first one fell, all the following ones will fall as well.

b. Sample response: The first four dominoes remain standing while the rest fall over.

c. Sample response: Since T is a non-empty subset of natural numbers, then it has a least number. This number acts as the first domino. Then, just as the second domino will be tipped over by the first, the next number is also in the set. This reasoning can be continued for each consecutive number after the first.

d. **1.** Sample response: Yes, it should work for the integers for the same reasons that it works for the natural numbers.

2. Sample response: No, it will not work for the set of real numbers because there is no “next” real number after any given real number.

e. Sample response: No, the principle of mathematical induction cannot be used to prove this conjecture. For $n = 1$, the conjecture is false:

$$2^{1+1} = 2^2 = 4 > 3 = 3^1$$

f. Sample response: This conjecture could be proved by showing that it holds true for $n = 2$. Next, you could show it is true for $n = 3$, given that it is true $n = 2$. Finally, you must show that whenever the conjecture holds true for $n = k$, it is also true for $n = k + 1$. This is the same process as the one illustrated in the mathematics note except that the proof begins at 2 instead of 1.

Exploration (page 263)

In this exploration, students work through the steps necessary to prove a conjecture by mathematical induction.

a. $S_k = 4 + 6 + 8 + \cdots + (2k + 2) = k(k + 3)$

b. Sample response:

$$\begin{aligned}
S_k + \big(2(k+1) + 2\big) &= k(k + 3) + \big(2(k + 1) + 2\big) \\
&= k\big((k + 1) + 2\big) + \big(2(k + 1) + 2\big) \\
&= k(k + 1) + 2k + 2(k + 1) + 2 \\
&= k(k + 1) + 2(k + 1) + 2k + 2 \\
&= k(k + 1) + 2(k + 1) + 2(k + 1) \\
&= (k + 1)(k + 2 + 2) \\
&= (k + 1)\big((k + 1) + 3\big)
\end{aligned}$$

Note: If students use a symbolic manipulator, the expression $k(k + 3) + \big(2(k + 1) + 2\big)$ may be expanded to obtain $k^2 + 5k + 4$. The factors of $k^2 + 5k + 4$ are $(k + 1)(k + 4)$, which may be rewritten as $(k + 1)\big((k + 1) + 3\big)$.

Discussion 2 (page 263)

a. By substitution into the conjectured formula, $S_k = (k + 1)\big((k + 1) + 3\big)$. If the right-hand side of the equation in Part **b** can be manipulated to be equal to $(k + 1)\big((k + 1) + 3\big)$, then $P(k + 1)$ is true whenever $P(k)$ is true.

b. Sample response: Yes. By the principle of mathematical induction, the steps in this exploration, along with the verification in Activity **1**, constitute a proof that the conjecture is true for all natural numbers.

c. Sample response: The conjecture could be disproved by providing one counterexample. For instance, the inequality is false for $n = 1$, because $(1+1)! < 2^{1+3}$. **Note:** This inequality is true for all natural numbers except 1, 2, 3, and 4.

d. The mathematics note describes three steps for an inductive proof. First, show that $P(1)$ is true. Then, use $P(1)$ to show that $P(2)$ is true. Finally, show that whenever the conjecture is true for *k*, where *k* is an arbitrary natural number, $P(k+1)$ is true.

e. Sample response: If the conjecture is true for $n = 1$, and it can be shown to be true using the "domino effect" thereafter, then the conjecture is true for all natural numbers.

Assignment

(page 264)

2.1 **a.** Sample response: $S_1 = 2 \bullet 1 = 1(1+1)$.

b. Sample response:

$$\begin{aligned} S_2 &= S_1 + (2 \bullet 2) \\ &= 1(1+1) + (2 \bullet 2) \\ &= 2 \bullet 1 + 2 \bullet 2 \\ &= 2(1+2) \\ &= 2(2+1) \end{aligned}$$

Thus, since $P(1)$ is true, $P(2)$ also is true.

c. $S_k = k(k+1)$

d. $S_{k+1} = (k+1)\big((k+1)+1\big)$

e. Sample response:

$$\begin{aligned} S_{k+1} &= S_k + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)\big((k+1)+1\big) \end{aligned}$$

Thus, if $P(k)$ is true, then $P(k+1)$ is true. Therefore, $P(n)$ is true for all natural numbers.

2.2 **a.** Sample response: $P(1)$ is true because $S_1 = 3^1 + 1 = 2(2)$.

b. Using $P(1)$, it can be shown that $P(2)$ is true:

$$\begin{aligned} S_2 &= 3^{1+1} + 1 \\ &= 3^1\left(3^1\right) + 1 \\ &= 3(2 + 1) + 1 \\ &= 3(2) + 3(1) + 1 \\ &= 3(2) + \left(3^1 + 1\right) \end{aligned}$$

Since 3(2) and $\left(3^1 + 1\right)$ are both even, $3(2) + \left(3^1 + 1\right)$ is even.

c. Given that $P(k)$ is true, $3^k + 1 = 2p$ where p is an integer. Students then must show that $P(k + 1)$ is true.

$$\begin{aligned} 3^{k+1} + 1 &= 3^k(3) + 1 \\ &= 3^k(2 + 1) + 1 \\ &= 3^k(2) + 3^k + 1 \\ &= 3^k(2) + 2p \\ &= 2\left(3^k + p\right) \end{aligned}$$

Thus, $3^{n+1} + 1$ is even for all natural numbers.

***2.3** **a.** The number of unit squares in the *n*th term can be described by the series $S_n = 1 + 3 + 5 + \cdots + (2n - 1)$. Sample conjecture: $S_n = n^2$.

b. Sample response: $P(1)$ is true, as shown below.

$$S_1 = 2(1) - 1 = 1^2$$

Using $P(1)$, it can be shown that $P(2)$ also is true.

$$\begin{aligned} S_2 &= S_1 + 2(2) - 1 \\ &= 1 + 2(2) - 1 \\ &= 4 \\ &= 2^2 \end{aligned}$$

Assuming that $P(k)$ is true:

$$S_k = 1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Given that $P(k)$ is true, $P(k+1)$ also is true:

$$
\begin{aligned}
S_{k+1} &= S_k + \left(2(k+1) - 1\right) \\
&= k^2 + \left(2(k+1) - 1\right) \\
&= k^2 + 2k + 2 - 1 \\
&= k^2 - 1 + 2k + 2 \\
&= (k+1)(k-1) + 2(k+1) \\
&= (k+1)(k-1+2) \\
&= (k+1)(k+1) \\
&= (k+1)^2
\end{aligned}
$$

Therefore, by the principle of mathematical induction, the conjecture is true for all natural numbers.

***2.4** Sample response: The assumption that $P(k+1)$ is true was used to show that $P(k)$ is true. Instead, it should have been shown that whenever $P(k)$ is true, then $P(k+1)$ is true.

2.5 Sample response: As shown below, $P(1)$ is true.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^1 = \begin{bmatrix} a^1 & 0 \\ 0 & b^1 \end{bmatrix}$$

Given this fact, it can be shown that $P(2)$ is true.

$$
\begin{aligned}
\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^1 \bullet \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\
&= \begin{bmatrix} a^1 & 0 \\ 0 & b^1 \end{bmatrix} \bullet \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\
&= \begin{bmatrix} a^1 \bullet a & 0 \\ 0 & b^1 \bullet b \end{bmatrix} \\
&= \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}
\end{aligned}
$$

Assuming that $P(k)$ is true implies the following,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

Given that $P(k)$ is true, it can be shown that $P(k+1)$ is also true.

$$\begin{aligned}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{k+1} &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{k} \bullet \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \bullet \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} a^k \bullet a & 0 \\ 0 & b^k \bullet b \end{bmatrix} \\ &= \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}\end{aligned}$$

Therefore, by the principle of mathematical induction, the conjecture is true for all natural numbers.

2.6 It can be shown that $P(1)$ is true as follows: $1^3 + 5(1) + 6 = 12$. Since 3 is a factor of 12, $P(1)$ is true.

Given that $P(1)$ is true, then it can be shown that $P(2)$ is true.

$$\begin{aligned}2^3 + 5(2) + 6 &= (1+1)^3 + 5(1+1) + 6 \\ &= 1^3 + 3 \bullet 1 + 3 \bullet 1 + 1^3 + 5 \bullet 1 + 5 \bullet 1 + 6 \\ &= (1^3 + 5 \bullet 1 + 6) + (1^3 + 5 \bullet 1 + 3 + 3) \\ &= (1^3 + 5 \bullet 1 + 6) + 12\end{aligned}$$

Since 3 is a factor of $1^3 + 5 \bullet 1 + 6$ and 3 is a factor of 12, 3 is a factor of $(1^3 + 5 \bullet 1 + 6) + 12$. So, $P(2)$ is true.

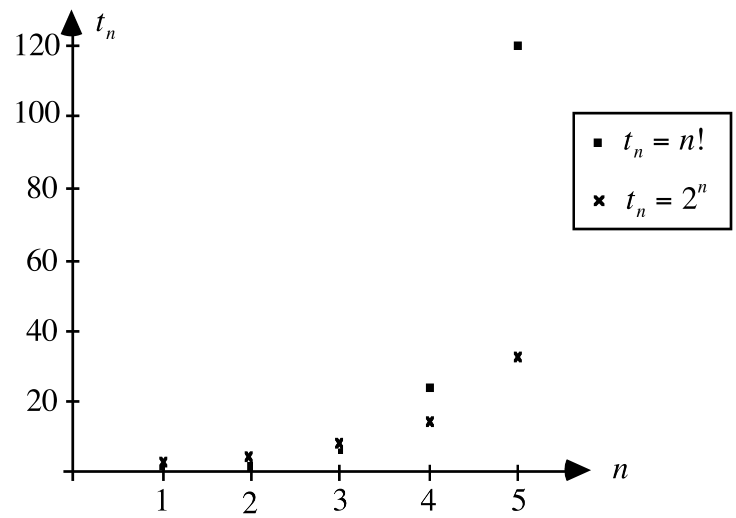
Assuming that $P(k)$ is true, 3 is a factor of $k^3 + 5k + 6$. Given that this is true, it can be shown that $P(k+1)$ also is true as follows:

$$\begin{aligned}(k+1)^3 + 5(k+1) + 6 &= k^3 + 3k^2 + 3k + 1^3 + 5k + 5 \bullet 1 + 6 \\ &= (k^3 + 5k + 6) + (3k^2 + 3k + 6) \\ &= (k^3 + 5k + 6) + 3(k^2 + k + 2)\end{aligned}$$

Since 3 is a factor of both $k^3 + 5k + 6$ and $3(k^2 + k + 2)$, 3 is a factor of $(k+1)^3 + 5(k+1) + 6$. So, $P(k+1)$ is true.

Therefore, the conjecture is true for all natural numbers.

2.7 **a.** **1.** Sample graph:



The conjecture is first true when $n = 4$.

2. $P(4)$ can be shown to be true as follows: $4! = 24 > 16 = 2^4$.

3. Sample response: Judging from the graph, the conjecture appears to be true for $n \geq 4$.

b. Assuming that $P(k)$ is true, $k! > 2^k$. Given that this is true, $P(k+1)$ can be shown to be true as follows:

$$(k+1)! = (k+1) \bullet k! > 2 \bullet k! > 2^1 \bullet 2^k = 2^{k+1}$$

* * * * *

2.8 It can be shown that $P(1)$ is true as follows.

$$4(1) + 3 = 7 = 2(1)^2 + 5(1)$$

Given this fact, $P(2)$ is also true.

$$\begin{aligned} 7 + 4(2) + 3 &= (2 \bullet 1^2 + 5 \bullet 1) + 4(2) + 3 \\ &= 18 \\ &= 8 + 10 \\ &= 2(2)^2 + 5(2) \end{aligned}$$

Assuming that $P(k)$ is true, $7 + 11 + 15 + \cdots + (4k + 3) = 2k^2 + 5k$. Given that this is true, $P(k+1)$ also is true.

$$\begin{aligned} (2k^2 + 5k) + 4(k+1) + 3 &= 2k^2 + 5k + 4k + 4 + 3 \\ &= (2k^2 + 4k + 2) + (5k + 5) \\ &= 2(k+1)^2 + 5(k+1) \end{aligned}$$

2.9 **a.** Substituting 2 for n in the inequality yields: $2^{(2+1)} = 2^3 = 8 < 9 = 3^2$. Thus, $2^{(2+1)} < 3^2$.

b. Sample response:

$$\begin{aligned} 2^{(3+1)} &= 2^{2+1} \bullet 2^1 \\ &< 3^2 \bullet 2^1 \\ &< 3^2 \bullet 3^1 = 3^3 \end{aligned}$$

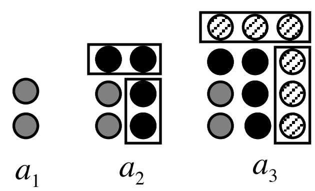
c. $2^{(k+1)} < 3^k$

d. $2^{((k+1)+1)} < 3^{k+1}$

e. Sample response:

$$\begin{aligned} 2^{((k+1)+1)} &= 2^{k+1} \bullet 2^1 \\ &< 3^k \bullet 2^1 \\ &< 3^k \bullet 3^1 = 3^{k+1} \end{aligned}$$

2.10 **a.** Sample response: As shown in the diagram below, the second term can be generated by adding 2 sets of 2 dots to the first term. The third term can be generated by adding 2 sets of 3 dots to the second term. In general, the nth term can be generated by adding 2 sets of n dots to the previous term.



The following table shows the number of dots in each array.

Term No. (n)	**No. of Dots (a_n)**
1	2
2	6
3	12
4	20
5	30
⋮	⋮
n	$a_n = a_{n-1} + 2n$

A recursive formula for the sequence is:

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 2n, \ n > 1 \end{cases}$$

An explicit formula for the sequence is $a_n = n(n+1)$.

b. Sample response: It can be shown that $P(1)$ is true because $a_1 = 1(1+1) = 2$.

Given this fact, then $P(2)$ is also true.

$$\begin{aligned} a_2 &= a_1 + 2(1+1) \\ &= 1(1+1) + 2(1+1) \\ &= (1+1)(1+2) \\ &= 2(2+1) \end{aligned}$$

Assuming that $P(k)$ is true, $a_k = k(k+1)$. Given that this is true, it can be shown that $P(k+1)$ is also true.

$$\begin{aligned} a_{k+1} &= a_k + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)\big((k+1)+1\big) \end{aligned}$$

Therefore, the conjecture is true for all natural numbers.

2.11 **a.** Students may use a geometry utility or sketch figures by hand. Sample table:

Term No. (n)	No. of Sides	Additional Diagonals	Total No. of Diagonals (a_n)
1	3		0
2	4	2	2
3	5	3	5
4	6	4	9
5	7	5	14

b. Sample recursive formula:

$$\begin{cases} a_1 = 0 \\ a_n = a_{n-1} + n;\ n > 1 \end{cases}$$

c. Sample response: It can be shown that $P(1)$ is true as follows:

$$\frac{(1+2)(1-1)}{2} = 0$$

Given this fact, $P(2)$ is also true.

$$\begin{aligned} a_2 &= a_1 + 2 \\ &= \frac{(1+2)(1-1)}{2} + 2 \\ &= 2 \\ &= \frac{(2+2)(2-1)}{2} \end{aligned}$$

Assuming that $P(k)$ is true, then:

$$a_k = \frac{(k+2)(k-1)}{2}$$

Given that this is true, it can be shown that $P(k+1)$ is also true.

$$\begin{aligned}
a_{k+1} &= a_k + (k+1) \\
&= \frac{(k+2)(k-1)}{2} + (k+1) \\
&= \frac{(k+2)(k-1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2 + k - 2 + 2k + 2}{2} \\
&= \frac{k^2 + 3k}{2} \\
&= \frac{k(k+3)}{2} \\
&= \frac{\big((k+1)-1\big)\big((k+1)+2\big)}{2} \\
&= \frac{\big((k+1)+2\big)\big((k+1)-1\big)}{2}
\end{aligned}$$

Therefore, by the principle of mathematical induction, the conjecture is true.

2.12 Sample response: The proof isn't valid because it is not true for $P(1)$. The expression $4(1)+3=7$ is not divisible by 4.

* * * * * * * * * *

Answers to Summary Assessment (page 268)

1. **a.** Sample table:

No. of Lines (n)	Additional Pairs of Supplementary Angles	Total Pairs of Supplementary Angles (a_n)
1		0
2	4	4
3	8	12
4	12	24

b. Sample response: Each new line forms 4 pairs of supplementary angles with each existing line. At each step, therefore, $4(n-1)$ pairs of supplementary angles are added.

c. Sample response: When $n = 4$, there are 24 pairs of angles. Adding another line creates $4(5-1) = 16$ additional pairs of supplementary angles. Therefore, the total is $24 + 16 = 40$.

d. Sample recursive formula:

$$\begin{cases} a_1 = 0 \\ a_n = a_{n-1} + 4(n-1);\ n > 1 \end{cases}$$

2. Sample response: The explicit formula is true for $n = 1$ because $a_1 = 2(1)(1-1) = 2(0) = 0$.

Given this fact, the formula is true for $n = 2$ as follows:

$$\begin{aligned} a_2 &= a_1 + 4(2-1) \\ &= 2(1)(1-1) + 4(2-1) \\ &= 4(2-1) \\ &= 2(2)(2-1) \end{aligned}$$

Assuming that the formula is true for $n = k$, $a_k = 2k(k-1)$. Given that this is true, it can be shown that the formula is true for $n = k + 1$:

$$\begin{aligned} a_{k+1} &= a_k + 4\big((k+1)-1\big) \\ &= 2k(k-1) + 4\big((k+1)-1\big) \\ &= 2k(k-1) + 4(k+1) - 4 \\ &= 2k^2 - 2k + 4k + 4 - 4 \\ &= 2k^2 + 2k \\ &= 2(k+1)k \\ &= 2(k+1)\big((k+1)-1\big) \end{aligned}$$

Therefore, by the principle of mathematical induction, the conjecture is true for all natural numbers.

Module Assessment

1. Consider the following sequence:

$$\frac{2}{3}, \frac{3}{8}, \frac{4}{15}, \frac{5}{24}, \ldots$$

a. Determine an explicit formula for the *n*th term of this sequence. Hint: Notice that consecutive denominators follow the pattern $n(n+2)$. Determine a pattern for the numerators.

b. Write an expression for the product P_n of the first *n* terms of the sequence. The first three products are given below:

$$P_1 = \frac{2}{3}$$

$$P_2 = P_1 \bullet \frac{3}{8} = \frac{2}{3} \bullet \frac{3}{8} = \frac{2}{8} = \frac{1}{4}$$

$$P_3 = P_2 \bullet \frac{4}{15} = \frac{1}{4} \bullet \frac{4}{15} = \frac{1}{15}$$

c. The following explicit formula has been proposed for finding the product of the first *n* terms of the sequence:

$$P_n = \frac{2}{n!(n+2)}$$

Make a prediction about the validity of the proposed formula.

d. Either prove or disprove your prediction in Part **c**.

2. **a.** Predict which values of *n* make the following statement true:

$$n! > 4^n$$

b. Use a form of mathematical induction to prove that the statement is true for values of *n* you predicted in Part **a**.

Answers to Module Assessment

1. **a.** Sample response:

$$a_n = \frac{n+1}{n(n+2)}$$

b. $P_n = a_1 \bullet a_2 \bullet a_3 \bullet \cdots \bullet a_n$

c. Sample response: The formula is true for all natural numbers *n*.

d. Sample proof of response given in Part **c** above: It can be shown that the formula is true for $n = 1$ is true as follows,

$$a_1 = \frac{1+1}{1(1+2)} = \frac{2}{3}$$
$$= \frac{2}{1!(1+2)}$$

Given this fact, the formula is also true for $n = 2$:

$$P_1 \bullet a_2 = \frac{2}{1!(1+2)} \bullet \frac{2+1}{2(2+2)}$$
$$= \frac{2}{2!(2+2)}$$

Assuming that it is true for $n = k$,

$$P_k = \frac{2}{k!(k+2)}$$

Given that this is true, it can be shown that the formula also is true for $n = k + 1$ as follows:

$$P_{k+1} = P_k \bullet a_{k+1}$$
$$= \frac{2}{k!(k+2)} \bullet \frac{(k+1)+1}{(k+1)((k+1)+2)}$$
$$= \frac{2(k+2)}{k!(k+1)(k+2)(k+3)}$$
$$= \frac{2}{k!(k+1)(k+3)}$$
$$= \frac{2}{(k+1)!(k+3)}$$
$$= \frac{2}{(k+1)!((k+1)+2)}$$

By the principle of mathematical induction, the conjecture is true for all natural numbers.

2. **a.** Sample response: The statement is true for $n \geq 9$.

b. Sample response: Since $9! = 362,880 > 262,144 = 4^9$ then the formula is true for $n = 9$. Given this fact, the formula can be shown to be true for $n = 10$:

$$\begin{aligned} 10! &= 10 \bullet 9! \\ &> 10 \bullet 4^9 \\ &> 4 \bullet 4^9 \\ &> 4^{10} \end{aligned}$$

Assuming the formula is true for k, where $k > 9$:

$$k! > 4^k$$

Given that this is true, the formula can be shown to be true for $k + 1$:

$$\begin{aligned} (k+1)! &= (k+1) \bullet k! \\ &> (k+1) \bullet 4^k \\ &> 4 \bullet 4^k \\ &> 4^{k+1} \end{aligned}$$

By the principle of mathematical induction, the formula is true for all natural numbers greater than or equal to 9.

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Flashbacks

Activity 1

1.1 Write a possible explicit formula for each sequence below.

a. $1, 3, 5, 7, \ldots$

b. $a+5, a+11, a+17, \ldots$

c. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

d. $t^{-2}, t^{-1}, t^{0}, \ldots$

1.2 Determine the value of x, in terms of n, in each of the following equations.

a. $6 \bullet 6^n = 6^x$

b. $3^n = 3^1 \bullet 3^x$

1.3 Show that the following equation holds true for $n = 11$:

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

1.4 **a.** What does it mean for one number to be divisible by another number?

b. How can you use multiplication to show that 30 is divisible by c?

Activity 2

2.1 Consider the sequence $1, 4, 7, 10, \ldots$.

a. Is this sequence arithmetic or geometric?

b. Write a recursive formula for this sequence.

c. Write an explicit formula for this sequence.

2.2 Consider the sequence $1, 4, 16, 64, \ldots$.

a. Is this sequence arithmetic or geometric?

b. Write a recursive formula for this sequence.

c. Write an explicit formula for this sequence.

2.3 For each term a_n listed below, write the next possible consecutive term in the sequence, a_{n+1}.

a. $a_n = n$

b. $a_n = n + 3$

c. $a_n = 2n$

2.4 Show that the equation below holds true for $n = 21$, if it is true for $n = 20$:

$$1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Answers to Flashbacks

Activity 1

1.1 **a.** $a_n = 2(n-1)+1$

b. $a_n = a + 6n - 1$

c. $a_n = \frac{1}{2^n}$

d. $a_n = t^{n-3}$

1.2 **a.** $x = n + 1$

b. $x = n - 1$

1.3 Sample response:

$2+4+6+8+10+12+14+16+18+20+2(11) = 132 = 11(11+1)$

1.4 **a.** Sample response: A number *n* is divisible by a second number *p* if there is an integer *c* such that $pc = n$.

b. Sample response: There is an integer *p* which can be multiplied by *c* to get the product of 30.

Activity 2

2.1 **a.** Sample response: This appears to be an arithmetic sequence with a common difference of 3.

b. Sample response:

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 3; \ n > 1 \end{cases}$$

c. Sample response: $a_n = 3n - 2$.

2.2 **a.** Sample response: This appears to be a geometric sequence with a common ratio of 4.

b. Sample response:

$$\begin{cases} g_1 = 1 \\ g_n = 4 \bullet g_{n-1}; \ n > 1 \end{cases}$$

c. Sample response: $g_n = 4^{n-1}$.

2.3 **a.** $a_{n+1} = n + 1$

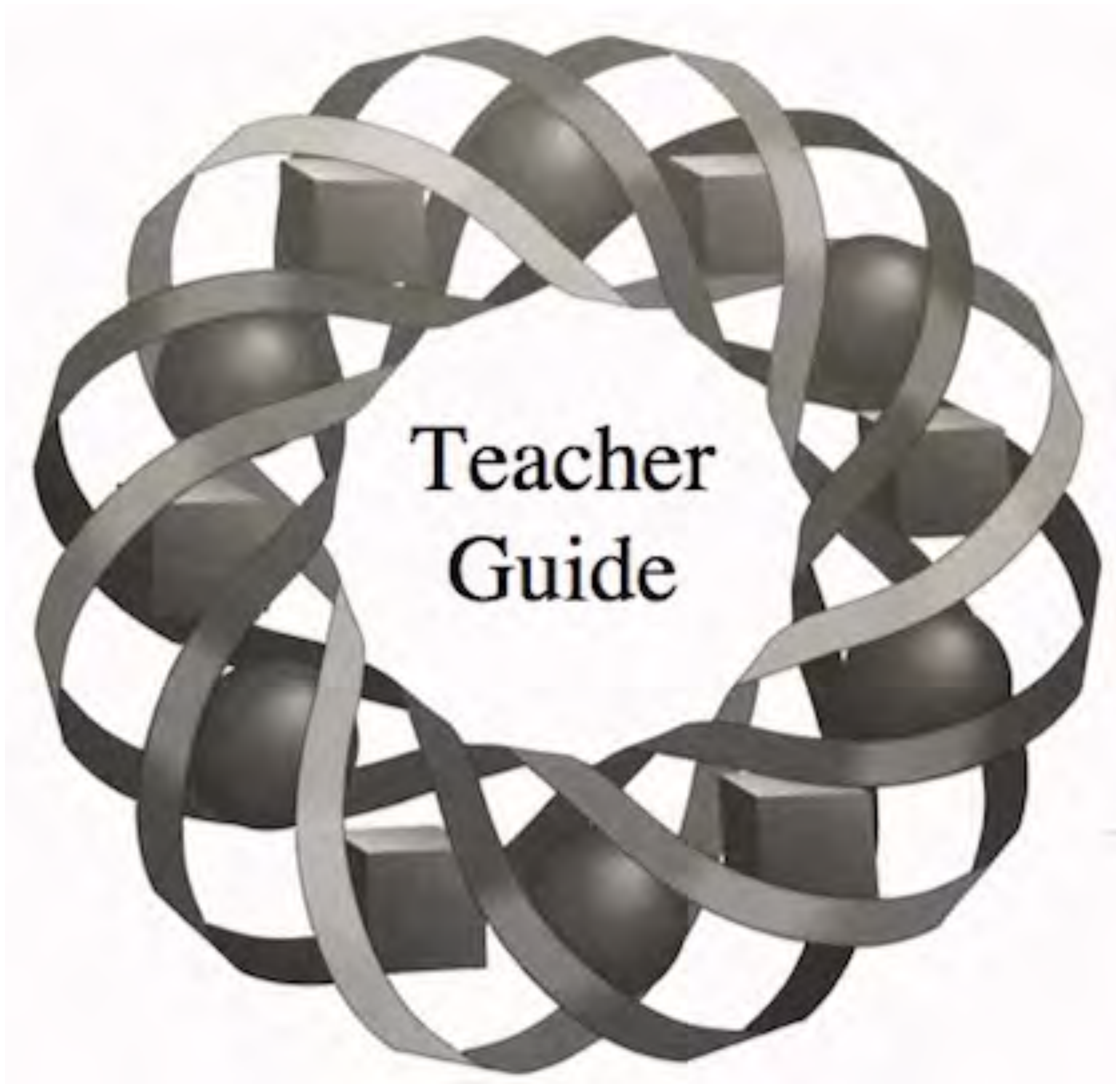
b. $a_{n+1} = (n + 1) + 3$

c. $a_{n+1} = 2(n + 1)$

2.4 Sample response:

$$\begin{aligned} S_{21} &= S_{20} + 21^2 \\ &= \frac{20(20+1)(2 \bullet 20+1)}{6} + 21^2 \\ &= \frac{20(20+1)(2 \bullet 20+1)}{6} + \frac{6 \bullet 21^2}{6} \\ &= \frac{20(20+1)(2 \bullet 20+1) + 6 \bullet 21^2}{6} \\ &= \frac{19,866}{6} \\ &= \frac{21 \bullet 22 \bullet 43}{6} \\ &= \frac{21(21+1)(2 \bullet 21+1)}{6} \end{aligned}$$

Cards and Binos and Reels, Oh My!



The turn of a card or the spin of a reel can add suspense to a game. In this module, you use your knowledge of randomness, probability, and combinations to explore how these games challenge players.

Sherry Horyna • Satinee Lightbourne • Teri Willard

Teacher Edition

Cards and Binos and Reels, Oh My!

Overview

Students use random variables, binomial probability, and expected value to investigate several different types of games.

Objectives

In this module, students will:

- design simulations
- determine conditional probabilities
- perform binomial experiments
- represent elements in Pascal's triangle using combinations
- develop a formula for the binomial distribution
- determine expected values.

Prerequisites

For this module, students should know:

- the definitions of theoretical and experimental probabilities
- the definition of independent events
- how to create tree diagrams
- how to calculate combinations
- the definition of a fair game
- how to determine expected value.

Time Line

Activity	Intro.	1	2	3	4	Summary Assessment	Total
Days	1	2	2	2	2	1	10

Materials Required

Materials	Activity					
	Intro.	1	2	3	4	Summary Assessment
playing cards	X	X				
binostat template			X	X		
coins						X

Teacher Note

A blackline master of the template appears at the end of the teacher edition FOR THIS MODULE

Technology

Software	Activity					
	Intro.	1	2	3	4	Summary Assessment
random number generator		X	X			
spreadsheet		X			X	
graphing utility		X				

Cards and Binos and Reels, Oh My!

Introduction (page 273)

Students use the game Cards of Chance to review conditional probabilities. A player wins only if all cards turned face up are the same color. Play progresses from the two-card level to the four-card level. Players may continue to the next level only if a winning combination has been obtained at the previous level.

Materials List

- 16 playing cards, 4 from each suit (one set per group)

Teacher Note

Throughout this module, probabilities will be reported as fractions and their corresponding decimal approximations. Because these approximations are often rounded, the sum in a probability distribution may not be exactly 1.

Exploration (page 274)

Students simulate Cards of Chance using playing cards and calculate experimental probabilities for the two-card level.

a. Each group of students should shuffle 16 playing cards (8 black and 8 red), then draw 2 cards, without replacement. **Note:** This is faster than laying out the cards as in Figure **1**, then turning over 2 of them.

b. Sample data:

Game	Outcome	Game	Outcome
1	loss	6	win
2	win	7	loss
3	win	8	loss
4	win	9	win
5	loss	10	win

c. Answers will vary. Using the sample data, $P(\text{win}) = 6/10$, or 60%.

d. The experimental probability of winning for a large number of games should approximate the theoretical probability of $7/15$, or about 47%. (See Part **d** of the discussion.) This value is referred to again in Part **c** of the exploration in Activity **1**.

e. If R represents a red card and B represents a black card, the 4 possible outcomes are RR, BR, RB, and BB. **Note:** Some students may need to be reminded that color, not suit, is the important characteristic in this game.

f. $P(\text{RR}) = \frac{8}{16} \bullet \frac{7}{15} = \frac{7}{30} \approx 23\%$ $\quad$ $P(\text{RB}) = \frac{8}{16} \bullet \frac{8}{15} = \frac{4}{15} \approx 27\%$

$P(\text{BR}) = \frac{8}{16} \bullet \frac{8}{15} = \frac{4}{15} \approx 27\%$ $\quad$ $P(\text{BB}) = \frac{8}{16} \bullet \frac{7}{15} = \frac{7}{30} \approx 23\%$

g. Sample response: To win, players must obtain either two red cards or two black cards. Since these are mutually exclusive events, the theoretical probability of winning at the two-card level is:

$$P(\text{win}) = P(\text{RR}) + P(\text{BB}) = \left(\frac{8}{16} \bullet \frac{7}{15}\right) + \left(\frac{8}{16} \bullet \frac{7}{15}\right) = \frac{7}{15} \approx 47\%$$

Discussion

(page 275)

a. Sample response: For each outcome, multiply the probability of drawing the first card by the probability of drawing the second.

b. As noted in Part **g** of the exploration, the theoretical probability of winning at the two-card level is:

$$P(\text{win}) = P(\text{RR}) + P(\text{BB}) = \left(\frac{8}{16} \bullet \frac{7}{15}\right) + \left(\frac{8}{16} \bullet \frac{7}{15}\right) = \frac{7}{15} \approx 47\%$$

Another way to analyze this situation is to assign a probability of 1 to the first card. The probability that the color of the second card will match the first is $7/15$, so the probability of winning is $1 \bullet 7/15 \approx 47\%$.

c. Sample response: As the number of games increases, the experimental probability should get closer and closer to the theoretical probability.

(page 275)

Activity 1

Students design their own simulations for the three-card level of Cards of Chance and complete a tree diagram for the game. They then determine the theoretical probabilities of winning at the three-card level and create probability distribution tables.

Materials List

- 16 playing cards, 4 from each suit (one set per group)

Technology

- random number generator (optional)
- spreadsheet (optional)
- graphing utility (optional)

Teacher Note

Before students combine data from their simulations, you should make sure that all simulations have the same theoretical probabilities for corresponding outcomes.

Exploration

(page 275)

a. Some students may use cards and design simulations like the one in the introduction. Others may use a random number generator, spreadsheet, or graphing utility to create their models.

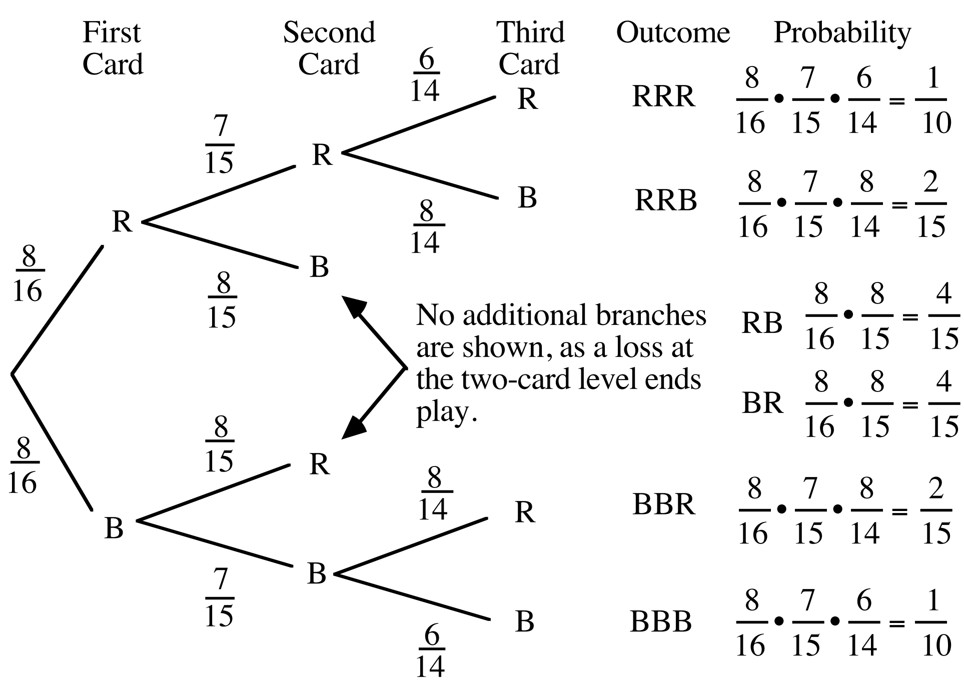
b. For the sample data listed below, the experimental probability is $3/10 = 30\%$.

Game	Outcome	Game	Outcome
1	loss	6	loss
2	loss	7	loss
3	win	8	win
4	loss	9	loss
5	win	10	loss

c.

1. For a large number of games, the experimental probability should approach the theoretical probability of $1/5 = 20\%$. (See calculations in Part **f**.)
2. Answers may vary. The theoretical probability of winning is greater at the two-card level than at the three-card level.

d. Sample diagram:



e. Sample table:

RRR	RRB	RB	BR	BBR	BBB
1/10	2/15	4/15	4/15	2/15	1/10

f. The theoretical probability of winning at the three-card level is:

$$P(\text{RRR}) + P(\text{BBB}) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5} = 20\%$$

Discussion

(page 277)

a. Sample response: The probability of winning at the two-card level is not the same as that of winning at the three-card level. At the two-card level, the probability of winning is $14/30 \approx 47\%$. At the three-card level, the probability of winning is $1/5 = 20\%$. The probabilities differ because, as the level increases, the probability of drawing a card of the same color as the previous cards decreases.

b. Sample response: The sum of all the probabilities is 1, or 100%. This means that all the possible outcomes are accounted for.

c. Sample response: To win at Cards of Chance, the two cards must have the same color. The probability of drawing a red card (or a black card) on the first draw is $26/52$. If a red card is drawn first, the probability of drawing a red card on the second draw is reduced to $25/51$, while the probability of drawing a black card increases to $26/51$. If the draws were independent events, these probabilities would not have been affected.

d. Sample response: If the cards were replaced after each draw, the game would involve independent events.

e.
1. The probability of drawing a third black card, given that the first two cards are black, is $P(\text{B}|\text{BB}) = 6/14 \approx 43\%$.
2. The probability of drawing a black card on the third draw, given that the first two cards are red, is $P(\text{B}|\text{RR}) = 8/14 \approx 57\%$.
3. The probability of drawing a black card on the third draw, given that the first two cards are different colors is 0, since play does not continue at the three-card level unless the player has won at the two-card level.

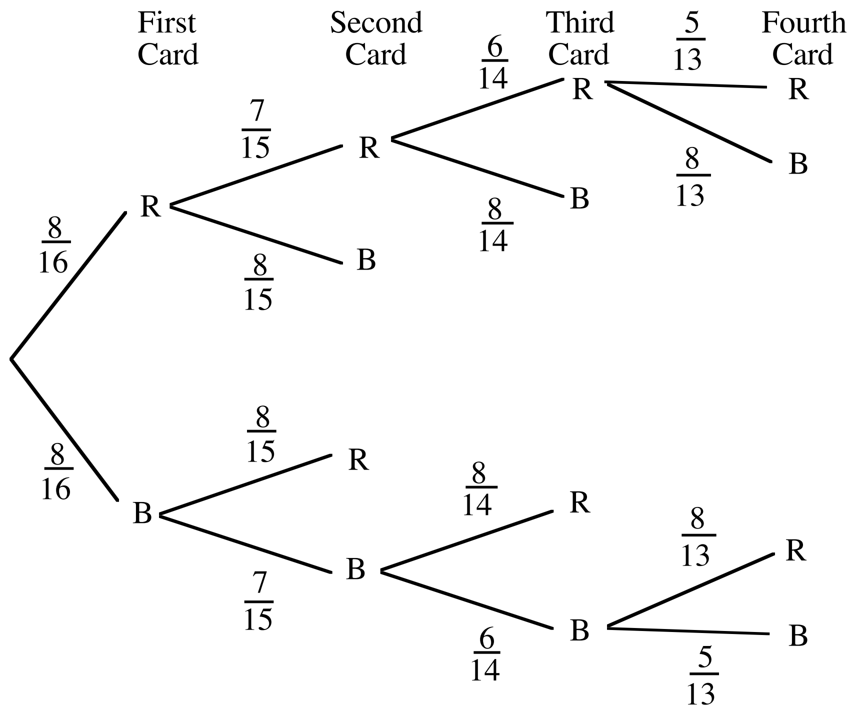
Assignment

(page 278)

1.1
a. The probability of drawing three red cards at the three-card level is $8/16 \bullet 7/15 \bullet 6/14 = 336/3360 = 10\%$.

b. Sample response: Multiply the probabilities assigned to the branches that correspond with RRR.

***1.2** **a.** Sample tree diagram:



b. Sample table:

RRRR	RRRB	RRB	RB	BR	BBR	BBBR	BBBB
1/26	4/65	4/15	2/15	2/15	4/15	4/65	1/26

***1.3** **a.** The probability of drawing four cards of the same color, given that three cards of the same color were drawn previously, is $5/13 \approx 38\%$.

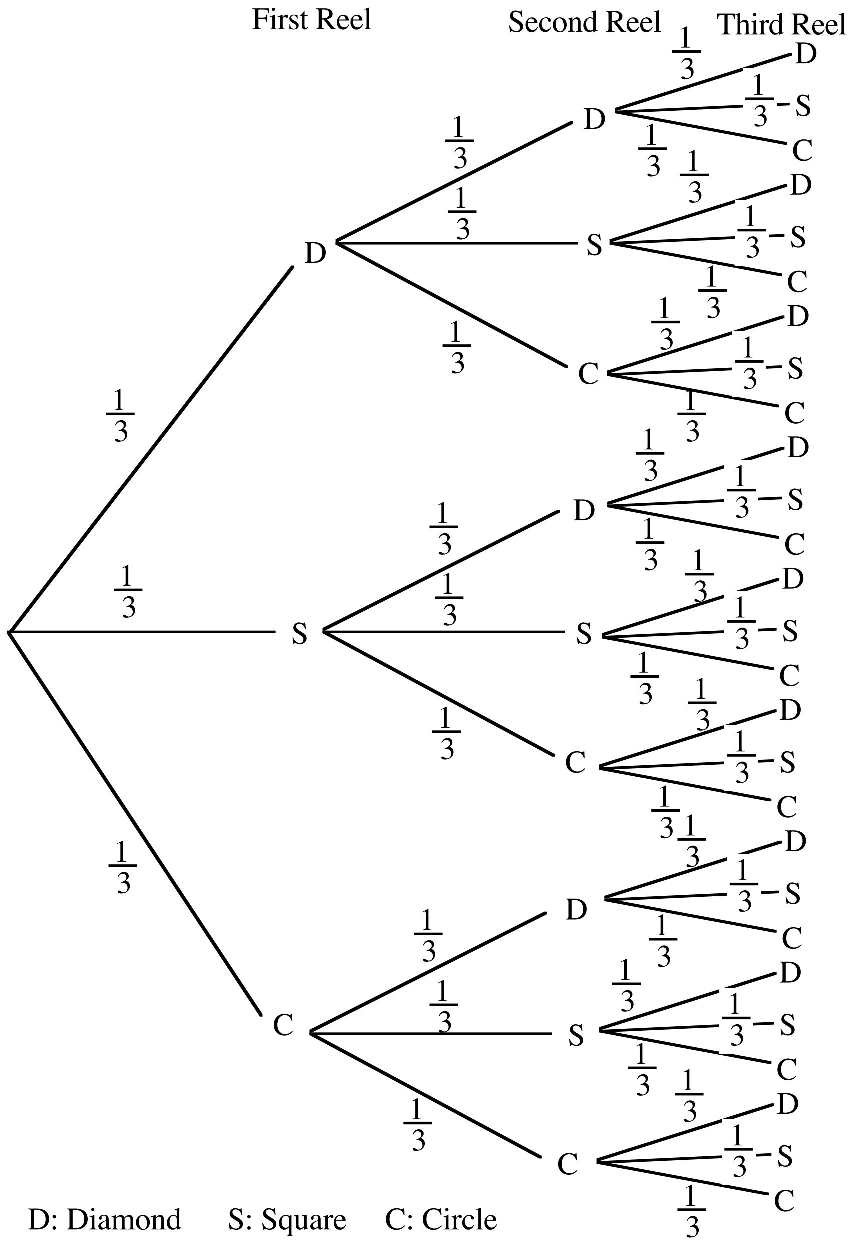
b. The probability of losing at the four-card level, given that three cards of the same color were drawn previously, is $1 - (5/13) = 8/13 \approx 62\%$.

c. The probability of winning at the four-card level is:

$$P(\text{RRRR}) + P(\text{BBBB}) = \frac{8}{16} \bullet \frac{7}{15} \bullet \frac{6}{14} \bullet \frac{5}{13} + \frac{8}{16} \bullet \frac{7}{15} \bullet \frac{6}{14} \bullet \frac{5}{13}$$
$$= \frac{1680}{43,680} + \frac{1680}{43,680}$$
$$= \frac{3360}{43,680} = \frac{1}{13} \approx 8\%$$

d. Sample response: The probability of losing at the four-card level is $1 - (1/13) = 12/13 \approx 92\%$.

1.4 **a.** Sample tree diagram:



b. The probability of getting a diamond on the 2nd reel is $1/3 \approx 33\%$.

c. The probability of getting a diamond on the 3rd reel is $1/3 \approx 33\%$.

d. Sample response: The events of the Reel Game are independent, because the result of the spin of any reel does not affect the results of the other reels. The probability of obtaining a diamond on both the second and third reels is $1/9$. This equals the product of the probabilities of obtaining a diamond on each individual reel.

e. The probability of winning the Reel Game with three diamonds, given that two diamonds have already appeared, is $1/3 \approx 33\%$.

f. Sample response: The probabilities are both $1/3 \approx 33\%$, because the reels represent independent events. No matter what happens on the first two reels, the probability of a diamond on the third reel is $1/3 \approx 33\%$.

1.5 **a.** Sample response: Yes, they are independent events. The probability of obtaining an ace on the first draw is $4/52$. The probability of obtaining an ace on the second draw also is $4/52$. In this case, $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B})$.

b. Sample response: In this case, $P(\text{B}|\text{A}) = P(\text{B}) = 4/52$.

c. Sample response: If the first card is not replaced, then the probability of obtaining an ace on the second draw is either $3/51$ or $4/51$. In this case, $P(\text{A and B}) \neq P(\text{A}) \bullet P(\text{B})$.

1.6 **a.** The probability of winning a game with two reels and two symbols on each reel is:

$$P(\text{diamond}, \text{diamond}) + P(\text{triangle}, \text{triangle}) =$$
$$1/2 \bullet 1/2 + 1/2 \bullet 1/2 = (1/4) + (1/4) = 2/4 = 50\%$$

b. The probability of winning a game with three reels and three symbols on each reel, where s_1, s_2, and s_3 represent the three different symbols, can be determined as follows:

$$P(s_1 s_1 s_1) + P(s_2 s_2 s_2) + P(s_3 s_3 s_3) = (1/27) + (1/27) + (1/27)$$
$$= 3/27 \approx 11\%$$

1.7 The table below shows the probability distribution for *S:*

s_i	0	5	10
p_i	3/4	1/6	1/12

* * * * *

1.8 **a.** The probability of answering all five questions correctly is $0.2^5 = 0.00032$.

b. The probability of answering none of the questions correctly is $0.8^5 = 0.32768$.

c. The probability of answering exactly four questions correctly is $5 \bullet 0.2^4 \bullet 0.8 = 0.0064$.

d. The probability of passing the test is $0.00032 + 0.0064 = 0.00672$.

e. The probability of not passing the test is $1 - 0.00672 = 0.99328$.

1.9 The following sample response assumes that there are 5 vowels: a, e, i, o, and u.

a. $\frac{5}{26} \bullet \frac{4}{25} \bullet \frac{3}{24} = \frac{1}{260} \approx 0.004$

b. $\frac{21}{26} \bullet \frac{20}{25} \bullet \frac{19}{24} \bullet \frac{18}{23} \bullet \frac{17}{22} = \frac{20,349}{65,780} \approx 0.31$

c. $\frac{3}{24} = 0.125$

d. $\frac{2}{21} \approx 0.095$

e. $\frac{7}{26} \bullet \frac{6}{25} \bullet \frac{5}{24} \bullet \frac{4}{23} \bullet \frac{3}{22} \bullet \frac{2}{21} \bullet \frac{1}{20} = \frac{5040}{3,315,312,000} \approx 0.000001$

1.10 In order to draw a total of $15 from his pocket, Louis must draw a $10.00 bill then a $5.00 bill, or a $5.00 bill followed by a $10.00 bill. The probability can be calculated as follows:

$$P(\$5,\ \$10) \text{ or } P(\$10,\ \$5) = \frac{2}{9} \bullet \frac{3}{8} + \frac{3}{9} \bullet \frac{2}{8} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} \approx 0.17$$

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Activity 2

Students use random variables and probability distributions to explore a binostat game. They also apply the law of large numbers.

Materials List

- binostat template (one copy per student; a blackline master appears at the end of the teacher edition for this module)

Technology

- random number generator (optional)

Exploration (page 281)

Students investigate binostat games with various levels and determine the probability associated with each path. **Note:** Each student will require a copy of the binostat template to complete Part **d**.

a. **1.** The table below shows the paths in a three-level binostat.

Slot	Paths
1	LLL
2	LLR, LRL, RLL
3	LRR, RLR, RRL
4	RRR

2. The following table shows the paths in a four-level binostat.

Slot	Paths
1	LLLL
2	LLLR, LLRL, LRLL, RLLL
3	LLRR, LRLR, LRRL, RLLR, RLRL, RRLL
4	LRRR, RLRR, RRLR, RRRL
5	RRRR

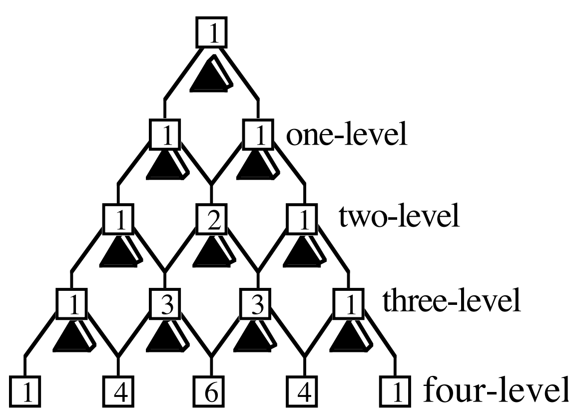
b. The probability of the ball landing in slot 1 is $1/2$. The probability of the ball landing in slot 2 is also $1/2$.

c. For a two-level binostat, there are three slots and four paths: $P(1) = 1/4$, $P(2) = 2/4$, and $P(3) = 1/4$.

For a three-level binostat, there are four slots and eight paths: $P(1) = 1/8$, $P(2) = 3/8$, $P(3) = 3/8$, $P(4) = 1/8$.

For a four-level binostat, there are five slots and 16 paths: $P(1) = 1/16$, $P(2) = 4/16$, $P(3) = 6/16$, $P(4) = 4/16$, $P(5) = 1/16$.

d. **1.** Sample diagram:



2. Sample response: Each row begins and ends with 1. Any other number in the figure is the sum of the two numbers immediately above it at the preceding level.

e. 1. Sample response:

Level	Number of Paths
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$

2. The total number of paths for an n-level binostat is 2^n.

f. The following table shows the entries for the 5th through 10th levels of the template:

Level	Entries
5	1, 5, 10, 10, 5, 1
6	1, 6, 15, 20, 15, 6, 1
7	1, 7, 21, 35, 35, 21, 7, 1
8	1, 8, 28, 56, 70, 56, 28, 8, 1
9	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
10	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

Teacher Note

When referring to Pascal's triangle in the remainder of this module, rows are designated by number as shown below:

```
         1              Row 0
      1     1           Row 1
   1     2     1        Row 2
1     3     3     1     Row 3
```

Discussion

(page 283)

a. Sample response: The first and last numbers in each row are 1. Each of the remaining elements is the sum of the two numbers above it in the previous row. In row 4, for example, the second number from the left can be found by adding 1 and 3.

b. Sample response: The numbers in Pascal's triangle correspond to the number of paths that lead to a particular slot for a given level.

c. Sample response: The total number of paths for an n-level binostat is the sum of all the numbers in the nth row of Pascal's triangle.

d. Sample response: Since there are 2 paths at each peg, 2 was chosen as the base.

e. Sample response: For an n-level binostat, the denominator of the probability is the sum of all the numbers in row n of Pascal's triangle, while the numerator is the xth element of row n.

f. Sample response: They both have two equally likely outcomes.

Assignment (page 283)

2.1 **a.** A three-level binostat has 4 slots.

b. The ball could follow $2^3 = 8$ different paths.

c. **1.** There are 3 different paths that lead to slot 2.

2. The probability of the ball landing in slot 2 is $3/8 \approx 38\%$.

2.2 **a.** **1.** 1

2. 1

3. 1

b. Sample response: The ball must fall in one of the slots. When the probabilities of all possible outcomes are considered, the sum is 1.

2.3 **a.** The probability can be expressed as $1/2^n$, where n is the level of the binostat. Therefore, the probabilities of a ball landing in slot 1 are $1/2^1$, $1/2^2$, and $1/2^3$ for one-, two-, and three-level binostat games, respectively.

b. $1/2^{10}$

2.4 **a.** A completed table is shown below.

Slot	1	2	3	4
Probability	1/8	3/8	3/8	1/8

b. Sample response: The numerators of the probabilities in the probability distribution table for an n-level binostat game correspond to the numbers in the nth row of Pascal's triangle, assuming the first row is row 0. The denominator is the sum of all the numbers in the row.

2.5 Sample table:

Slot	1	2	3	4	5	6	7
Probability	1/64	3/32	15/64	5/16	15/64	3/32	1/64

* * * * *

2.6 **a.** 2^8

b. This corresponds to the eighth row of Pascal's triangle.

c. 28 (this corresponds to the seventh element of the eighth row)

d. $28 + 8 + 1 = 37$ (this corresponds to the sum of the seventh, eighth, and ninth elements of the eighth row)

2.7 **a.** The total number of outcomes for the experiment is $2^{10} = 1024$. This corresponds to the 10th row of Pascal's triangle.

b. The number of ways that exactly 8 out of 10 customers would choose a specific brand corresponds to the 9th element of the 10th row, or 45.

c. To find the number of ways that at least 8 out of 10 customers would choose a specific brand, add the 9th, 10th, and 11th terms of the 10th row of Pascal's triangle, or $45 + 10 + 1 = 56$.

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Activity 3

Students use the binomial distribution to calculate the probabilities for a binostat game.

Materials List

- binostat template (optional)

Discussion 1

(page 286)

a. **1.** This represents a binomial experiment because there are a fixed number of flips, each flip is independent of the others, there are only two possible outcomes (heads or tails), and the probabilities of the outcomes remains constant from flip to flip.

2. This represents a binomial experiment because there are a fixed number of cards drawn, each draw is independent of the other draw (because of replacement), there are only two possible outcomes (diamonds or not diamonds), and the probability of drawing a diamond remains constant ($13/52 = 1/4$) from draw to draw.

3. This does not represent a binomial experiment. The probability of drawing a second diamond depends on the first card drawn. If the first card was a diamond, the probability of the second card also being a diamond is $12/51 \approx 24\%$. If the first card was not a diamond, the probability of the second card being a diamond is $13/51 \approx 25\%$. Since the probability of drawing a diamond does not remain constant from draw to draw, this is not a binomial experiment.

b. Sample response: The sum of the probabilities of all outcomes in an experiment must be 1. In a binomial experiment, there are only two possible outcomes. Therefore, $P(\text{failure}) = 1 - P(\text{success})$.

Teacher Note

Students may wish to use a copy of the binostat template from Activity **2** for reference in the exploration.

Exploration (page 286)

a. The number of paths the ball can take to land in slot 2 is $C(4,1)$, or 4. This corresponds with the second element of the fourth row of Pascal's triangle.

b. To land in slot 1, the ball must take the path LLLL. There are no deflections to the right and four deflections to the left.

To land in slot 3, the ball can take six different paths: RRLL, RLRL, RLLR, LRRL, LRLR, and LLRR. Each path contains two deflections to the right and two deflections to the left.

To land in slot 4, the ball can take four different paths: RRRL, RRLR, RLRR, and LRRR. Each path contains three deflections to the right and one deflection to the left.

To land in slot 5, the ball must take the path RRRR. There are four deflections to the right and no deflections to the left.

c.

1. $C(4,0)$ or 1 (this corresponds with the first element of the fourth row of Pascal's triangle)
2. $C(4,2)$ or 6 (this corresponds with the third element of the fourth row of Pascal's triangle)
3. $C(4,3)$ or 4 (this corresponds with the fourth element of the fourth row of Pascal's triangle)
4. $C(4,4)$ or 1 (this corresponds with the fifth element of the fourth row of Pascal's triangle)

d. The probability for slot 1 can be expressed as:

$${\frac{1}{2}}^{0} \bullet {\frac{1}{2}}^{4} = \frac{1}{16} \approx 6\%$$

The probability for slot 3 can be expressed as:

$$6 \bullet {\frac{1}{2}}^{2} \bullet {\frac{1}{2}}^{2} = \frac{6}{16} \approx 38\%$$

The probability for slot 4 can be expressed as:

$$4 \bullet {\frac{1}{2}}^{3} \bullet {\frac{1}{2}}^{1} = \frac{4}{16} = 25\%$$

The probability for slot 5 can be expressed as:

$${\frac{1}{2}}^{4} \bullet {\frac{1}{2}}^{0} = \frac{1}{16} \approx 6\%$$

e. The probability for slot 1 can be expressed as:

$$C(4,0) \bullet [P(\mathrm{R})]^0 \bullet [P(\mathrm{L})]^4$$

The probability for slot 2 can be expressed as:

$$C(4,1) \bullet [P(\mathrm{R})]^1 \bullet [P(\mathrm{L})]^3$$

The probability for slot 2 can be expressed as:

$$C(4,2) \bullet [P(\mathrm{R})]^2 \bullet [P(\mathrm{L})]^2$$

The probability for slot 4 can be expressed as:

$$C(4,3) \bullet [P(\mathrm{R})]^3 \bullet [P(\mathrm{L})]^1$$

The probability for slot 5 can be expressed as:

$$C(4,4) \bullet [P(\mathrm{R})]^4 \bullet [P(\mathrm{L})]^0$$

Discussion 2

(page 287)

a. Sample response: A ball that makes r deflections to the right in an n-level binostat would land in slot $r+1$. The number of paths for this slot corresponds to the $(r+1)$st element of the nth row of Pascal's triangle.

b. The notation $C(n,r)$ represents the $(r+1)$st element of the nth row of Pascal's triangle.

c. **1.** $C(n,r) \bullet P(\mathrm{R})^r \bullet P(\mathrm{L})^{n-r}$

2. $C(n,n-r) \bullet P(\mathrm{R})^{n-r} \bullet P(\mathrm{L})^r$

d. The probability of getting at least 3 twos with 5 rolls of a fair die can be calculated as follows:

$$P(\text{at least }3\text{ twos}) =$$
$$P(\text{exactly }3\text{ twos}) + P(\text{exactly }4\text{ twos}) + P(\text{exactly }5\text{ twos}) =$$
$$C(5,3)\bullet\left(\frac{1}{6}\right)^3\bullet\left(\frac{5}{6}\right)^2 + C(5,4)\bullet\left(\frac{1}{6}\right)^4\bullet\left(\frac{5}{6}\right)^1 + C(5,5)\bullet\left(\frac{1}{6}\right)^5\bullet\left(\frac{5}{6}\right)^0 =$$
$$10\bullet\frac{1}{216}\bullet\frac{25}{36} + 5\bullet\frac{1}{1296}\bullet\frac{5}{6} + 1\bullet\frac{1}{7776}\bullet 1 =$$
$$\frac{250}{7776} + \frac{25}{7776} + \frac{1}{7776} =$$
$$\frac{276}{7776} = \frac{23}{648} \approx 4\%$$

Assignment

(page 288)

3.1 **a.** Tossing a die three times and recording the number that appears on top is not a binomial experiment since there are six possible outcomes for each trial.

b. Drawing two cards one at a time, without replacement, is not a binomial experiment since the trials are not independent.

c. Flipping a coin until it lands heads up is not a binomial experiment because the number of trials is not fixed.

***3.2** The probabilities for each slot are shown in the table below. The numerators in the fractional answers correspond with the elements in row 6 of Pascal's triangle: 1, 5, 10, 10, 5, 1.

Slot	Probability
1	$C(5,0)\bullet(1/2)^0(1/2)^5 = 1/32 \approx 3\%$
2	$C(5,1)\bullet(1/2)^1(1/2)^4 = (5)(1/32) = 5/32 \approx 16\%$
3	$C(5,2)\bullet(1/2)^2(1/2)^3 = (10)(1/32) = 10/32 \approx 31\%$
4	$C(5,3)\bullet(1/2)^3(1/2)^2 = (10)(1/32) = 10/32 \approx 31\%$
5	$C(5,4)\bullet(1/2)^4(1/2)^1 = (5)(1/32) = 5/32 \approx 16\%$
6	$C(5,5)\bullet(1/2)^5(1/2)^0 = 1/32 \approx 3\%$

3.3 **a.** Sample response: Getting an automobile on a reel would be considered a "success."

b. Since there are 5 different symbols, the probability of a "success" on any one reel is $1/5$, or 20%.

c. The probability of a "failure" on any one reel is $4/5$, or 80%.

***3.4** **a.** The probability of getting a diamond on every reel is $C(5,5) \bullet (1/5)^5 (4/5)^0 = 1/3125 = 0.00032 = 0.032\%$.

b. The probability of getting a heart on three of the reels is $C(5,3) \bullet (1/5)^3 (4/5)^2 = 160/3125 \approx 5\%$.

c. The probability of getting a heart on at least three reels is:

$$P(3 \text{ hearts}) + P(4 \text{ hearts}) + P(5 \text{ hearts}) =$$

$$C(5,3) \bullet \left(\frac{1}{5}\right)^3 \bullet \left(\frac{4}{5}\right)^2 + C(5,4) \bullet \left(\frac{1}{5}\right)^4 \bullet \left(\frac{4}{5}\right)^1 + C(5,5) \bullet \left(\frac{1}{5}\right)^5 \bullet \left(\frac{4}{5}\right)^0 =$$

$$\frac{160}{3125} + \frac{20}{3125} + \frac{1}{3125} =$$

$$\frac{181}{3125} \approx 6\%$$

d. Since the middle reel always displays a diamond, the machine is reduced to a four-reel game. The probability of getting 2 diamonds in a four-reel game is $C(4,2) \bullet (1/5)^2 (4/5)^2 = 96/625 \approx 15\%$.

3.5 **a.** Sample response: In each expanded expression, the sum of the exponents on each term equals the exponent of the original expression.

b. Sample response: In each expanded expression, the coefficients of each term correspond to the entries of row *n* of Pascal's triangle, where *n* is the exponent of the original expression.

c. Sample response:

$$(x + y)^4 = C(4,4)x^4 + C(4,3)x^3y + C(4,2)x^2y^2 + C(4,1)xy^3 + C(4,0)y^4$$

d. Sample response:

$$(x + y)^n = C(n,n)x^n y^0 + C(n,n-1)x^{n-1}y^1 + C(n,n-2)x^{n-2}y^2 + \cdots + C(n,1)x^1y^{n-1} + C(n,0)y^n$$

* * * * *

3.6 **a.** The tack toss is a binomial experiment since each trial is independent, there are a fixed number of trials (1), the probability of success remains the same for each trial, and each trial consists of a success or a failure.

b. $C(10,6) \bullet 0.25^6 \bullet 0.75^4 \approx 0.01622 = 1.622\%$

3.7 **a.** $C(4,1) \bullet 0.5^1 \bullet 0.5^3 = 0.25 = 25\%$ or
$C(4,3) \bullet 0.5^3 \bullet 0.5^1 = 0.25 = 25\%$

b. $C(4,4) \bullet 0.5^4 \bullet 0.5^0 = 0.0625 = 6.25\%$ or
$C(4,0) \bullet 0.5^0 \bullet 0.5^4 = 0.0625 = 6.25\%$

c. $C(4,2) \bullet 0.5^2 \bullet 0.5^2 = 0.375 = 37.5\%$

3.8 The probability of obtaining at least 6 correct responses may be calculated as follows:

$$C(10,6) \bullet 0.25^6 \bullet 0.75^4 + C(10,7) \bullet 0.25^7 \bullet 0.75^3 +$$
$$C(10,8) \bullet 0.25^8 \bullet 0.75^2 + C(10,9) \bullet 0.25^9 \bullet 0.75^1 +$$
$$C(10,10) \bullet 0.25^{10} \bullet 0.75^0 \approx 0.020 = 2.0\%$$

3.9 **a.** $C(2,2) \bullet 0.999^2 \bullet 0.001^0 = 0.998001 \approx 99.8\%$

b. $C(2,0) \bullet 0.999^0 \bullet 0.001^2 = 0.000001 = 0.0001\%$

c. $C(2,1) \bullet 0.999^1 \bullet 0.001^1 + C(2,0) \bullet 0.999^0 \bullet 0.001^2 = 0.001999 = 0.1999\%$

3.10 To find the probability of at least 1 failure, students may add the probabilities of exactly 1 failure, exactly 2 failures, exactly 3 failures, and so on. Another method is to subtract the probability of 0 failures from 1: $1 - C(6,6) \bullet 0.977^6 \bullet 0.023^0 \approx 0.13 = 13\%$.

3.11 **Note:** A proof of the binomial theorem requires mathematical induction.

a. $(x - y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

b. $(x^2 + 3y)^3 = x^6 + 9x^4y + 27x^2y^2 + 27y^3$

c. $(2xy - 5)^4 = 16x^4y^4 - 160x^3y^3 + 600x^2y^2 - 1000xy + 125$

* * * * * * * * * *

(page 290)

Activity 4

In this activity, students use the concepts of expected value and fair games to assign payoff values for binostat games.

Materials List

- none

Technology

- spreadsheet

Exploration (page 291)

a. The expected payoff for the game is 25 tokens, as shown below:

$$\frac{1}{32}(200)+\frac{5}{32}(20)+\frac{10}{32}(10)+\frac{10}{32}(10)+\frac{5}{32}(20)+\frac{1}{32}(200)=25$$

b. Since the cost to play of 100 tokens is greater than the expected payoff, this is not a fair game. To make the game fair, some students may assign larger payoffs to the slots with smaller probabilities. Others may simply recommend changing the cost to 25 tokens and leaving the payoffs as stated in Part **a.** Sample response:

Slot	1	2	3	4	5	6
Payoff	800	128	16	16	128	800

This is a fair game because the expected payoff is:

$$\frac{1}{32}(800)+\frac{5}{32}(128)+\frac{10}{32}(16)+\frac{10}{32}(16)+\frac{5}{32}(128)+\frac{1}{32}(800)=100$$

c. Answers may vary. The expected payoff should be at least 70 tokens. Sample response:

Slot	1	2	3	4	5	6
Payoff	608	64	20	20	64	608

This game satisfies the 70% requirement because its expected payoff is:

$$\frac{1}{32}(608)+\frac{5}{32}(64)+\frac{10}{32}(20)+\frac{10}{32}(20)+\frac{5}{32}(64)+\frac{1}{32}(608)=70.5$$

The owner's profit is 29.5%, slightly less than the maximum of 30%.

Discussion (page 292)

a. Sample response: Slots with larger probabilities have smaller payoffs and slots with smaller probabilities have larger payoffs. There is an inverse relationship between the probability and the payoff for each outcome.

b. Students should realize that there are many different ways to set payoffs for a fair game.

c. Answers will vary. A fair game requires that the expected payoff equal the cost to play (in other words, a 100% return). To make the game profitable for owners, the payoffs to players must be reduced. To be legal, however, the return must be at least 70% of the cost to play.

d. Sample response: Players would be attracted to a game that had some payoffs considerably higher than the cost of playing. As long as the chances of winning these payoffs are low, the game can still be profitable.

Assignment (page 292)

4.1 Sample response: First, find the fractional part of the circle represented by the unshaded sector. Since $30°/360° = 1/12$, the probability of landing in the shaded sector is $1 - 1/12 = 11/12$. The expected value can be determined as follows:

$$5 \bullet \frac{11}{12} + 50 \bullet \frac{1}{12} = \frac{55}{12} + \frac{50}{12} = \frac{105}{12} \approx 9 \text{ points}$$

4.2 **a.** **Note:** This method provides only one set of payoffs. Many others are possible. Sample response: The payoffs should be set as shown below.

Slot	1	2	3	4	5	6	7
Probability	1/64	6/64	15/64	20/64	15/64	6/64	1/64
Payoff	457	76	30	23	30	76	457

b. Using the sample response given in Part **a**, the expected value for the game, in tokens, is:

$$457 \bullet \frac{1}{64} + 76 \bullet \frac{6}{64} + 30 \bullet \frac{15}{64} + 23 \bullet \frac{20}{64} + 30 \bullet \frac{15}{64} + 76 \bullet \frac{6}{64} + 457 \bullet \frac{1}{64} \approx 49.8$$

To make the game approximately fair, the cost to play should be 50 tokens.

4.3. Answers will vary. The expected payoff must be at least 70% of 50 tokens, or at least 35 tokens. Sample response: The payoffs should be set as shown in the table below.

Slot	1	2	3	4	5	6	7
Payoff	320	53	22	16	22	53	320

This game is legal and profitable because the expected payoff is:

$$\frac{1}{64}(320) + \frac{6}{64}(53) + \frac{15}{64}(22) + \frac{20}{64}(16) + \frac{15}{64}(22) + \frac{6}{64}(53) + \frac{1}{64}(320) \approx 35.2$$

Since $320/53 \approx 6$, $320/22 \approx 15$, and $320/16 = 20$, the payoffs show an inverse relationship with the probabilities in the game.

***4.4** Answers will vary. The expected payoff must be at least 70% of 50 tokens, or at least 35 tokens. Some students may use their tree diagrams from Problem **1.2a** to help assign payoffs. The probabilities of the winning outcomes at each level are shown in the table below:

Outcome	**Probability**
RR	56/240
RRR	336/3360
RRRR	1680/43,680
BB	56/240
BBB	336/3360
BBBB	1680/43,680

One possible set of payoffs is shown below:

Outcome	**Payoff**
RR or BB	26
RRR or BBB	60
RRRR or BBBB	150

This game is legal because the expected payoff is:

$$(26)\left(\frac{112}{240}\right)+(60)\left(\frac{672}{3360}\right)+(150)\left(\frac{3360}{43{,}680}\right)\approx 35.7$$

4.5 Answers will vary. The expected payoff must be at least 70% of 50 tokens, or at least 35 tokens. Sample response: The payoffs should be set as shown in the table below.

Slot	1	2	3	4
Payoff	70	25	25	70

This game is legal because the expected payoff is:

$$\left(\frac{1}{8}\right)70+\left(\frac{3}{8}\right)25+\left(\frac{3}{8}\right)25+\left(\frac{1}{8}\right)70 = 36.25$$

Since the probability of getting into slots 1 or 4 is $1/3$ the probability of getting into slots 2 or 3 and the payoff for slots 1 and 4 is approximately 3 times the payoff for slots 2 and 3, the payoffs are inversely related to the probabilities in the game.

4.6 Answers will vary. Depending on the payoffs assigned to each winning outcome, all of the games could appeal equally to players—and each could be equally profitable. In terms of complexity, it is more difficult to determine appropriate payoff values in Cards of Chance because it is not a binomial game.

* * * * *

4.7 Sample response: The probability of obtaining two of any one symbol is $C(3,2)(1/3)^2(2/3)^1 = 2/9 \approx 22\%$. Since there are three different symbols on the reels, the probability of matching two symbols is three times this value, or about 66%.

The probability of obtaining three of any one symbol is $C(3,3)(1/3)^3(2/3)^0 = 1/27 \approx 4\%$. Since there are three different symbols on the reels, the probability of matching three symbols is three times this value, or about 12%.

The expected payoff must be at least $70\% \bullet 50 = 35$ tokens. If matching two symbols pays 27 tokens and matching three symbols pays 162 tokens, then the expected payoff fits the requirements of a legal and profitable game:

$$\frac{2}{3}(27) + \frac{1}{9}(162) = 36 \text{ tokens}$$

4.8 **a.** **1.** $0.94 \bullet 0.96 = 0.9024$

2. $0.06 \bullet 0.96 = 0.0576$

3. $0.94 \bullet 0.04 = 0.0376$

4. $0.06 \bullet 0.04 = 0.0024$

b. The expected value for the policyholder can be calculated as follows:

$$\$0 \bullet (0.9024) + \$1000 \bullet (0.0576) + \$2000 \bullet (0.0376)$$
$$+\$7500 \bullet (0.0024) = \$150.80$$

Since the annual premium is \$200, the company's expected profit is \$49.20 per policyholder.

* * * * * * * * * *

Answers to Summary Assessment (page 294)

Some students may choose to investigate this question with a tree diagram. Others may recognize that this is a binomial experiment and use the binomial probability formula. In general, the probability of matching *n* coins is $(1/2)^{n-1}$.

1. Some students may use coins to simulate the game. Others may choose a random number generator.

2.
a. Answers will vary. Students should justify their decisions.

b. Answers will vary, depending on the levels chosen in Part **a**. The table below shows the probabilities of winning at levels 4–10.

Level	**Probability**	**Level**	**Probability**
4	1/8	8	1/128
5	1/16	9	1/256
6	1/32	10	1/512
7	1/64		

3. The sample payoff scheme shown below assumes a cost to play of 10 tokens.

Level	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Payoff	5	15	30	75	125	250	750

This is a legal game since the expected payoff is at least 70% of the cost to play:

$$\frac{1}{8}(5)+\frac{1}{16}(15)+\frac{1}{32}(30)+\frac{1}{64}(75)+\frac{1}{128}(125)+\frac{1}{256}(250)+\frac{1}{512}(750)\approx 7.1$$

4. Students incorporate their responses to Problems **1–3** in a report.

Module Assessment

1. High Tech Games has developed a new version of Cards of Chance called Second Chance. As in the original version, the game uses 16 cards, 4 from each suit. Players win (and can advance to the next level) only if the cards turned face up are the same color. However, Second Chance awards more credits for a combination of cards that is all the same suit than for a combination that is all the same color, but not the same suit.

a. Draw a tree diagram that shows all the possible outcomes for Second Chance up to the four-card level. Label each branch with the appropriate probabilities.

b. Determine the probability of getting three cards of the same suit at the three-card level.

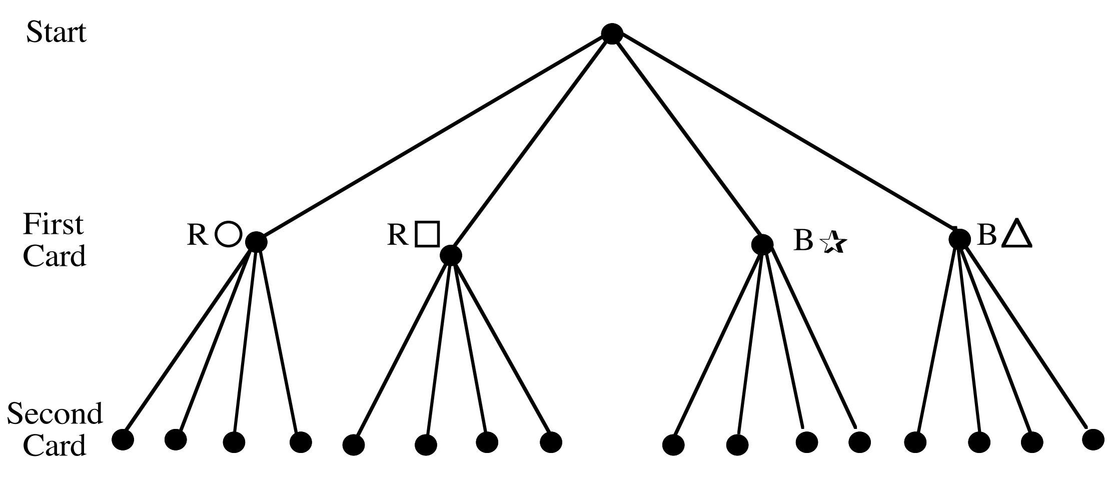
c. Determine the probability of getting three cards of the same color, but not the same suit, at the three-card level.

d. Determine the probability of drawing a fourth card that is a club, given that the first three cards drawn are also clubs.

2. Consider a video gaming machine that uses eight cards. Each card displays one colored symbol. There are two cards with red circles, two with red squares, two with blue stars, and two with blue triangles.

The video machine electronically "turns" one card over, then a second. Players win if both cards have the same symbol or both cards have the same color.

a. Complete the following tree diagram for this game. List each outcome and label each branch with the corresponding probability.



b. What is the probability of getting two red cards?

c. What is the probability of getting one red card and one blue card?

d. What is the probability of getting two red circles?

e. What is the probability of getting two cards with the same symbol?

f. What is the probability of getting two cards with the same color?

g. This game costs 50 tokens to play. Design a payoff scheme that provides at least a 70% return to players and makes the game profitable for owners.

3. If a married couple plans to have three children, what is the probability that they have at least one girl? (Assume an equal probability of having a boy or a girl.)

4. Consider a game that involves rolling a die four times. If the die shows a 6 exactly three of the four times, you win a prize worth $10.00. Any other roll and you get nothing back. What is the expected value for this game?

5. A 6-sided die is painted in the following manner: 2 faces are red, 1 face is blue, and 3 faces are white.

a. Does rolling this die 10 times and counting the number of times a red face appears represent a binomial experiment? Explain your response.

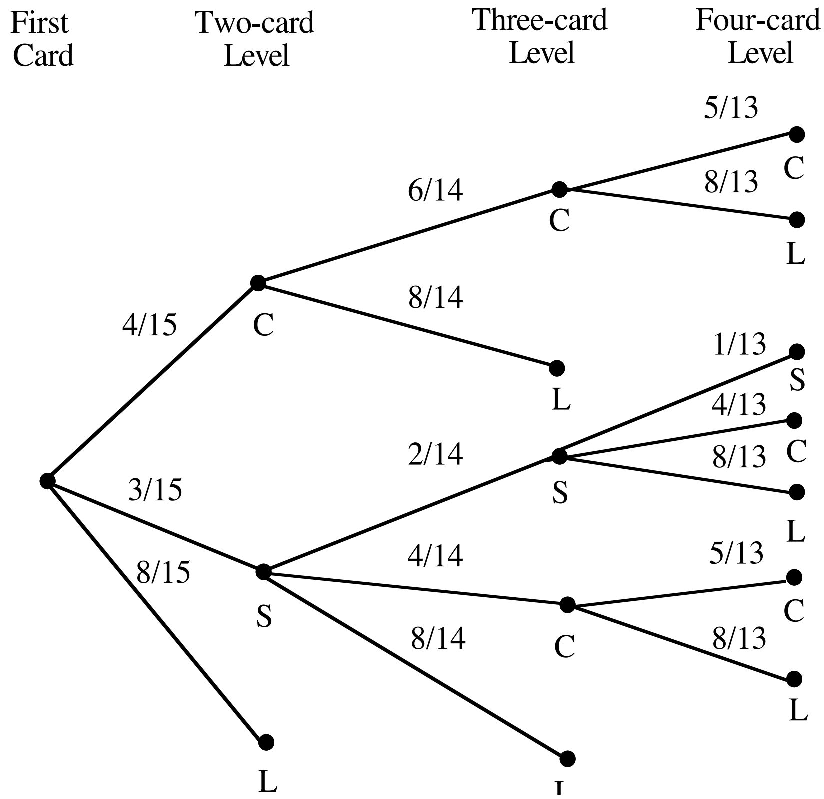
b. What is the probability of rolling a red face?

c. What is the probability of getting exactly 3 red faces on 4 rolls of the die?

d. What is the probability of getting at least 3 red faces on 4 rolls of the die?

Answers to Module Assessment

1. **a.** In the following sample tree diagram, S represents an outcome in which all cards are the same suit, C represents an outcome in which all cards are the same color, but not the same suit, and L represents an outcome in which the player loses.



b. The probability of getting three cards of the same suit at the three-card level can be calculated by multiplying the probabilities on the corresponding branches of the tree diagram:

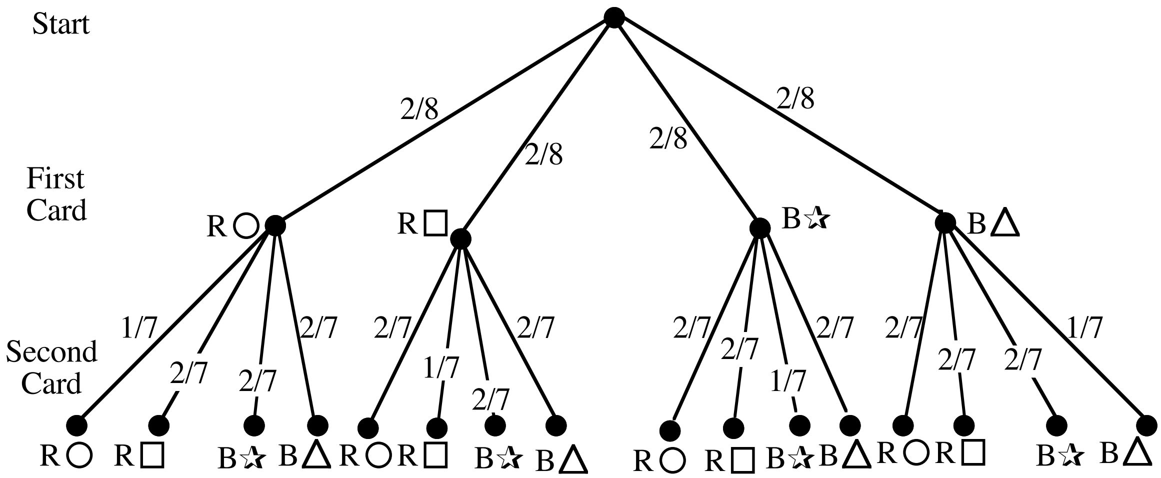
$$P(\text{3 same suit}) = \frac{3}{15} \bullet \frac{2}{14} = \frac{6}{210} \approx 3\%$$

c. The probability of getting three cards of the same color, but not the same suit, at the three-card level is the sum of the probabilities of two outcomes:

$$P(\text{CC}) + P(\text{SC}) = \frac{4}{15} \bullet \frac{6}{14} + \frac{3}{15} \bullet \frac{4}{14} = \frac{36}{210} \approx 17\%$$

d. Since there is only 1 club left in the 13 remaining cards, the probability of drawing a fourth card that is a club, given that the first three cards are also clubs, is $1/13 \approx 8\%$.

2. **a.** Sample tree diagram:



b. $P(\text{two red cards}) = 2\left(\frac{2}{8} \bullet \frac{1}{7}\right) + 2\left(\frac{2}{8} \bullet \frac{2}{7}\right) = \frac{12}{56} \approx 21\%$

c. $P(\text{one red and one blue card}) = 8\left(\frac{2}{8} \bullet \frac{2}{7}\right) = \frac{32}{56} \approx 57\%$

d. $P(\text{two red circles}) = \frac{2}{8} \bullet \frac{1}{7} = \frac{2}{56} \approx 4\%$

e. $P(\text{same symbol}) = 4\left(\frac{2}{8} \bullet \frac{1}{7}\right) = \frac{8}{56} \approx 14\%$

f. $P(\text{same color}) = 4\left[\left(\frac{2}{8} \bullet \frac{1}{7}\right) + \left(\frac{2}{8} \bullet \frac{2}{7}\right)\right] = \frac{24}{56} \approx 43\%$

g. Answers will vary. The expected payoff must be at least 70% of 50 tokens, or at least 35 tokens. Sample response: If the cards match symbols, the payoff is 122 tokens. If the cards match color, the payoff is 41 tokens. This game is legal and profitable for the company because the expected payoff is:

$$\left(\frac{8}{56}\right)(122) + \left(\frac{24}{56}\right)(41) = 35$$

In addition, the ratios of the payoffs are very close to the ratios of the probabilities.

3. The probability that the family will have at least one girl is:

$$C(3,1) \bullet (0.5)^1 \bullet (0.5)^2 + C(3,2) \bullet (0.5)^2 \bullet (0.5)^1 + C(3,3) \bullet (0.5)^3 \bullet (0.5)^0 = 0.875$$

4. The expected value for this game is:

$$\$10\left[C(4,3) \bullet (1/6)^3 \bullet (5/6)^1\right] \approx \$0.15$$

5. **a.** Sample response: This is a binomial experiment because there are a fixed number of rolls, each roll is independent of any other roll, each roll has only two possible outcomes (red or not red), and the probability of a success remains constant from roll to roll.

b. The probability of rolling a red face is $2/6 = 1/3 \approx 33\%$.

c. The probability of obtaining exactly 3 red faces on 4 rolls is:

$$C(4,3) \bullet (1/3)^3 (2/3)^1 = (4)(2/81) = 8/81 \approx 10\%$$

d. The probability of obtaining at least 3 red faces on 4 rolls is:

$$P(3 \text{ red faces}) + P(4 \text{ red faces}) =$$

$$C(4,3) \bullet \left(\frac{1}{3}\right)^3 \bullet \left(\frac{2}{3}\right)^1 + C(4,4) \bullet \left(\frac{1}{3}\right)^4 \bullet \left(\frac{2}{3}\right)^0 =$$

$$4 \bullet \left(\frac{2}{81}\right) + 1 \bullet \left(\frac{1}{81}\right) =$$

$$\frac{9}{81} \approx 11\%$$

Selected References

Consortium for Mathematics and Its Applications (COMAP). *For all Practical Purposes*. New York: W. H. Freeman and Co., 1988.

Pitman, J. *Probability*. New York: Springer-Verlag, 1993.

The School Mathematics Project. *Living with Uncertainty*. Cambridge: Cambridge University Press, 1991.

Flashbacks

Activity 1

1.1 Explain whether or not each of the following situations involves independent events.

a. Two playing cards are drawn from a standard deck, one at a time and without replacement. The number of red cards that appear is recorded.

b. Two playing cards are drawn from a standard deck, one at a time with replacement. The number of red cards that appear is recorded.

1.2 Consider an experiment that involves flipping a penny, a nickel, and a quarter. What is the theoretical probability of having only one coin land heads up?

1.3 Consider a fair four-sided die in the shape of a triangular pyramid. Three of the faces are black and one face is white. Construct a tree diagram that shows all the possible outcomes of rolling the die two times. Label each branch with the corresponding probability.

Activity 2

2.1 Predict the next three terms in each of the following sequences and describe the method you used to find them.

a. 1, 1, 2, 3, 5, 8, 13, …

b. 1, 4, 9, 16, 25, …

2.2 Consider an experiment that involves drawing two playing cards from a standard deck, one at a time and without replacement. What is the probability of drawing two cards of the same color?

Activity 3

3.1 Evaluate each of the following expressions:

a. $C(6,2)$

b. $C(4,0)$

c. $C(8,8)$

3.2 Determine the probability of each of the following outcomes when using an ordinary six-sided die.

a. rolling a 5

b. not rolling a 5

c. rolling a number greater than or equal to 5

d. not rolling a number less than 5

3.3 Use the properties of exponents to expand each of the following expressions.

a. $\left(6m^3n\right)^2$

b. $\left(-4xy^5\right)^3$

c. $(2xy)^2 + \left(3xy^3\right)^2$

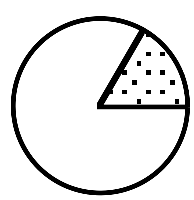
Activity 4

4.1 Determine the expected value for each of the following experiments.

a. An ordinary six-sided die is rolled and the number shown is recorded.

b. Two ordinary dice are rolled and the sum of the numbers shown is recorded.

4.2 What fractional part of a circle is represented by a sector whose central angle measures 60°?



Answers to Flashbacks

Activity 1

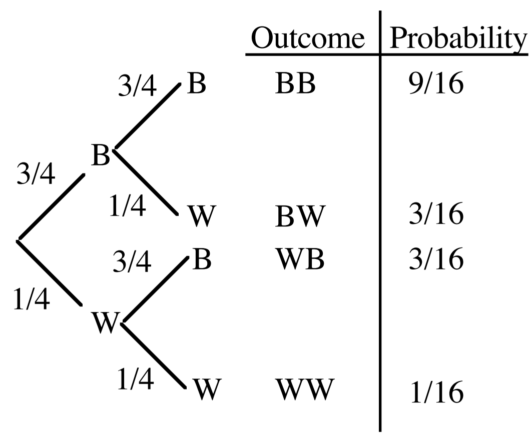
1.1 **a.** These events are not independent: $P(\text{R and R}) \neq P(\text{R}) \bullet P(\text{R})$.

b. These events are independent: $P(\text{R and R}) = P(\text{R}) \bullet P(\text{R})$.

1.2 The theoretical probability of obtaining only one head when flipping three coins can be calculated as follows:

$$P(\text{HTT}) + P(\text{THT}) + P(\text{TTH}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \approx 38\%$$

1.3 In the following sample tree diagram, W represents white and B represents black.



Activity 2

2.1 **a.** Sample response: The next three terms are 21, 34, and 55. Each of these was found by adding the two previous terms.

b. Sample response: The next three terms are 36, 49, and 64. They were found by adding 11, 13, and 15, respectively, to the preceding terms. They can also be found by squaring the term number.

2.2 $P(\text{BB}) + P(\text{RR}) = 2\left[\frac{26}{52} \bullet \frac{25}{51}\right] = \frac{25}{51}$

Activity 3

3.1 **a.** ${}_6C_2 = 15$

b. ${}_4C_0 = 1$

c. ${}_8C_8 = 1$

3.2 **a.** $1/6 \approx 17\%$

b. $5/6 \approx 83\%$

c. $2/6 \approx 33\%$

d. $2/6 \approx 33\%$

3.3 **a.** $36m^6n^2$

b. $-64x^3y^{15}$

c. $4x^2y^2 + 9x^2y^6$

Activity 4

4.1 **a.** The expected value for rolling an ordinary die is:

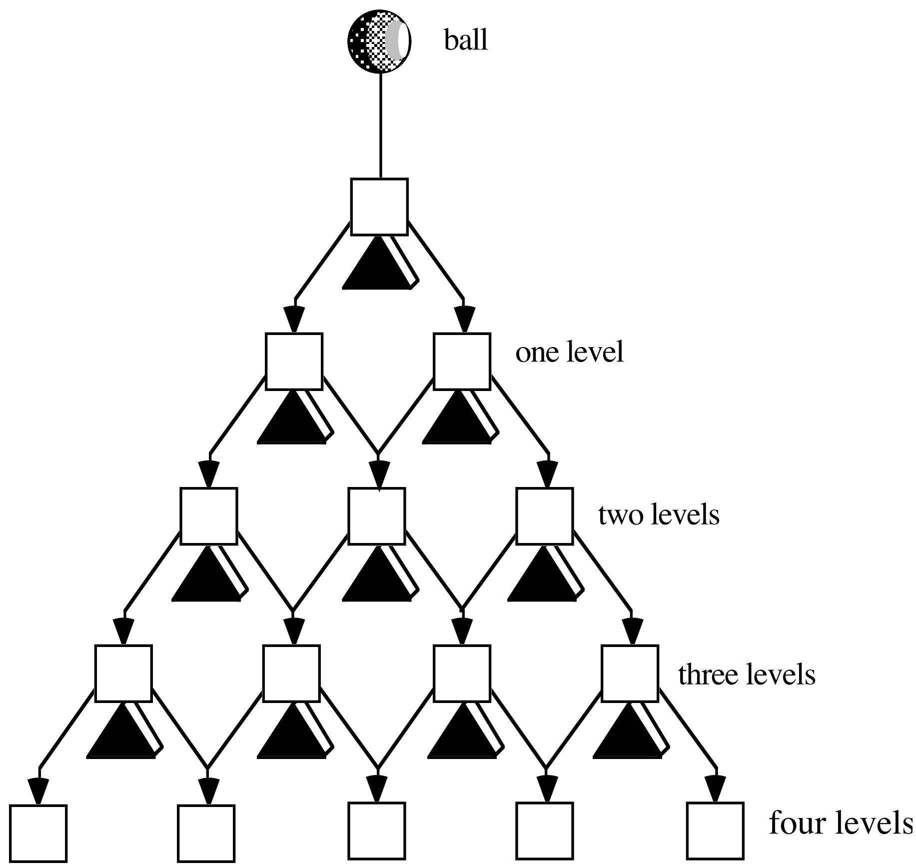
$$\frac{1}{6} \bullet 1 + \frac{1}{6} \bullet 2 + \frac{1}{6} \bullet 3 + \frac{1}{6} \bullet 4 + \frac{1}{6} \bullet 5 + \frac{1}{6} \bullet 6 = \frac{7}{2} = 3\frac{1}{2}$$

b. The expected value for the sum of a pair of dice is:

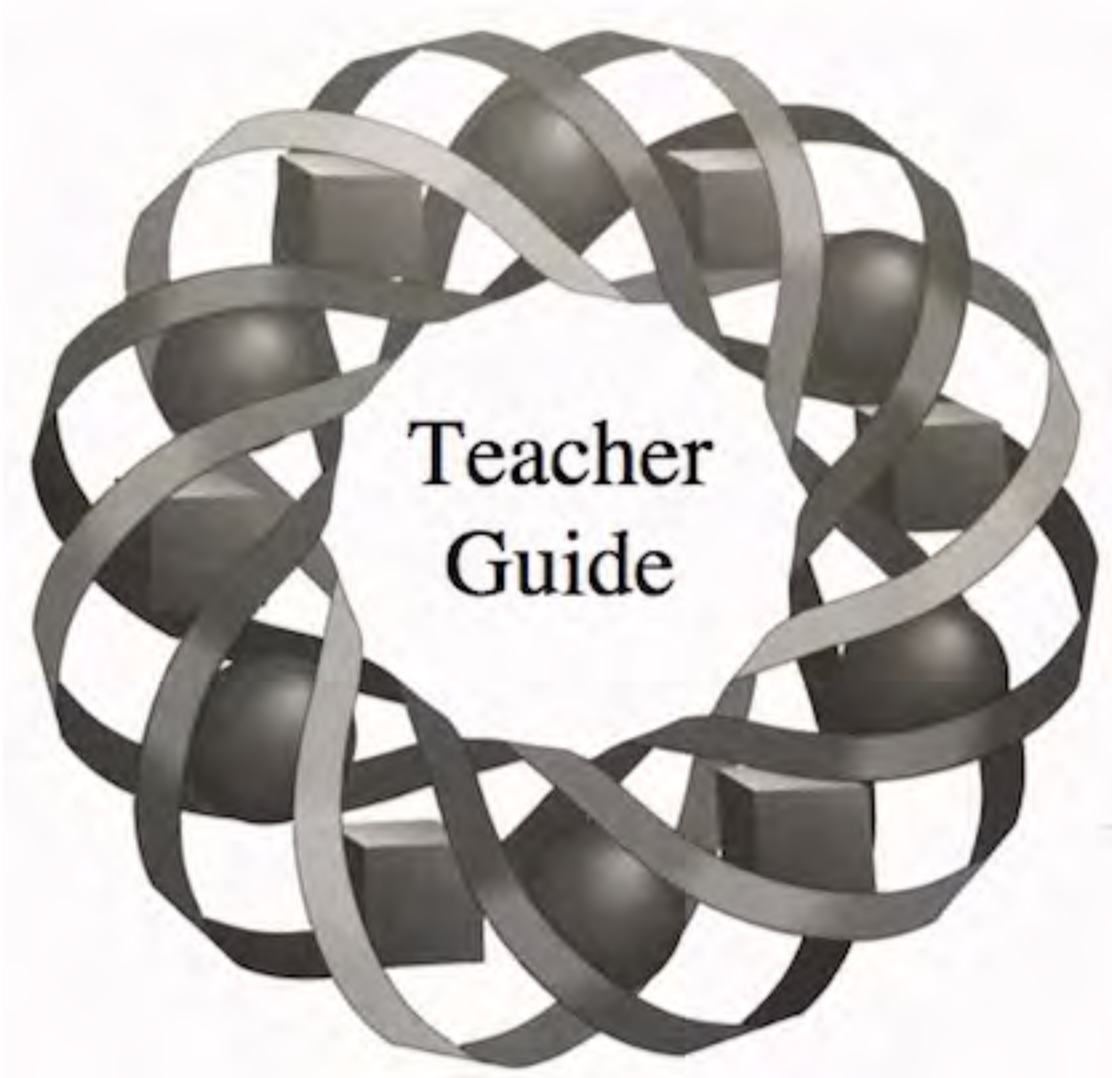
$$\frac{1}{36} \bullet 2 + \frac{2}{36} \bullet 3 + \frac{3}{36} \bullet 4 + \frac{4}{36} \bullet 5 + \frac{5}{36} \bullet 6 + \frac{6}{36} \bullet 7$$
$$+ \frac{5}{36} \bullet 8 + \frac{4}{36} \bullet 9 + \frac{3}{36} \bullet 10 + \frac{2}{36} \bullet 11 + \frac{1}{36} \bullet 12 = 7$$

4.2 $1/6$

Template of Binostat Paths



An Imaginary Journey Through the Real World



Can you find the square root of a negative number? In this module, you discover that you can!

Lee Brown • John Freal • Anne Watkins

Teacher Edition

An Imaginary Journey Through the Real World

Overview

Students are introduced to the set of complex numbers as an extension of the set of real numbers.

Objectives

In this module, students will:

- represent complex numbers in multiple forms
- perform operations on complex numbers using multiple representations
- determine complex roots of polynomials
- evaluate roots and powers of complex numbers.

Prerequisites

For this module, students should be able to:

- find real roots of polynomials
- solve systems of equations
- add and multiply matrices
- recognize geometric transformations
- use trigonometric functions.

Time Line

Activity	1	2	3	4	5	6	Summary Assessment	Total
Days	1	2	2	1	2	2	1	11

Materials Required

Materials	Activity						
	1	2	3	4	5	6	Summary Assessment
graph paper			X	X	X	X	X

Technology

Software	Activity						
	1	**2**	**3**	**4**	**5**	**6**	**Summary Assessment**
graphing utility	X	X	X	X	X	X	
symbolic manipulator	X	X	X	X	X	X	X
geometry utility						X	
spreadsheet				X			

An Imaginary Journey Through the Real World

Introduction

(page 301)

Our present number system evolved as changes became necessary in existing number systems. In this module, students explore some of the needs for complex numbers.

(page 302)

Activity 1

Students are introduced to complex numbers in the context of finding roots of quadratic equations. **Note**: Complex numbers were first used in conjunction with determining the real roots of cubic functions.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator

Teacher Note

You may wish to point out to students that one reason for creating *i* was to give solutions to equations such as $x^2 + 1 = 0$, which has no real roots. (This equation is examined in Part **c** of the discussion.)

For this module, students may reasonably assume that complex arithmetic behaves like real arithmetic, with the additional simplifying fact that $i^2 = -1$.

Discussion

(page 302)

a. **1.** The solutions are $\sqrt{2}$ and $-\sqrt{2}$.

2. The factors of the polynomial $x^2 - 2$ are $\left(x - \sqrt{2}\right)$ and $\left(x - \left(-\sqrt{2}\right)\right)$, or $\left(x + \sqrt{2}\right)$.

b. The solutions to $x^2 - a^2 = 0$ are a and $-a$.

c. Sample response: No. This equation is equivalent to $x^2 = -1$. Since no real number is a square root of a negative number, there are no real-number solutions.

d. **1.** The factors are $(x-2i)$ and $(x+2i)$.

2. The roots are $2i$ and $-2i$.

e. An expression of the form $x^2 + a^2$ can be factored as $(x-ai)(x+ai)$. Its zeros are ai and $-ai$.

f. **1.** Sample response: The length of the side of the third square can always be found because lengths are positive and the square root of a positive number can always be found in the set of real numbers.

2. Sample response: It is not always possible to find b in the set of real numbers. If $c < a$, the solution does not exist in the set of real numbers.

Assignment (page 304)

1.1 In the responses to Parts **a–f** below, the sum is given first, followed by the difference.

a. $0+10i$; $0-8i$

b. $11+3i$; $-3-3i$

c. $21+9i$; $21-21i$

d. $-10+3i$; $-16+5i$

e. $24+0i$; $0+10i$

f. $(a+c)+(b+d)i$; $(a-c)+(b-d)i$

1.2 **a.** $3i$

b. -2

c. $42-77i$

d. $33-9i$

e. $(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$

1.3 Sample response: To multiply two complex numbers, the result is still comparable to the following:

$$(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$$

1.4 **a.** Sample response: $2-5i$ and $2+5i$.

b. For the sample response given in Part **a**, the sum is 4 and the product is 29.

c. For complex conjugates $a+bi$ and $a-bi$, the sum $2a$ and the product a^2+b^2 are real numbers.

1.5 The multiplicative identity for the set of complex numbers is $1+0i$. An example of the use of the multiplicative identity is seen in the following:

$$(3+4i)(1+0i)=3+4i$$

* * * * *

1.6 One possible response is to substitute the values into the equation and check as follows:

$$\left(3i\sqrt{3}\right)^2+27=-27+27=0$$

$$\left(-3i\sqrt{3}\right)^2+27=-27+27=0$$

Another possibility is to solve as shown below:

$$x^2+27=0$$
$$x^2=-27$$
$$x=\sqrt{-27} \text{ and } x=-\sqrt{-27}$$
$$x=\sqrt{-1 \bullet 27} \text{ and } x=-\sqrt{-1 \bullet 27}$$
$$x=\sqrt{-1}\sqrt{9 \bullet 3} \text{ and } x=-\sqrt{-1}\sqrt{9 \bullet 3}$$
$$x=i\sqrt{9} \bullet \sqrt{3} \text{ and } x=-i\sqrt{9} \bullet \sqrt{3}$$
$$x=3i\sqrt{3} \text{ and } x=-3i\sqrt{3}$$

1.7 **a.** There are no real-number solutions to this equation.

b. There are two complex-number solutions: $2i\sqrt{3}$ and $-2i\sqrt{3}$.

* * * * * * * * * * *

(page 305)

Activity 2

In this activity, students explore reciprocals of complex numbers and properties of the complex number system.

Materials List

- none

Technology

- graphing utility
- symbolic manipulator

Exploration (page 305)

Students represent reciprocals of complex numbers in the form $a + bi$. They also investigate division of complex numbers.

a. **1.** $3a + 4ai + 3bi - 4b = 1 + 0i$

2. $(3a - 4b) + (4a + 3b)i = 1 + 0i$

b. **1.** $3a - 4b = 1$; $4a + 3b = 0$

2. Solving the system of equations gives $a = 3/25$ and $b = -4/25$. Thus,

$$a + bi = \frac{3}{25} - \frac{4}{25}i$$

c. Sample response:

$$(3+4i)\left(\frac{3}{25} - \frac{4}{25}i\right) = \frac{9}{25} - \frac{12}{25}i + \frac{12}{25}i - \frac{16}{25}i^2 = \frac{9}{25} + \frac{16}{25} = 1$$

d. Students should obtain the following expression.

$$\frac{3}{25} - \frac{4}{25}i$$

Note: On some technology, students may see $0.12 - 0.16i$.

e. **1.** The product is shown below:

$$\frac{3 - 4i}{25}$$

2. The results are equal.

3. Sample response: To find the reciprocal of $a + bi$, multiply the numerator and denominator of the fraction $1/(a + bi)$ by the conjugate of $a + bi$ as follows:

$$\left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

f. To complete the division, the reciprocal of $3+4i$ is found using the procedure from Part **e**.

$$\begin{aligned}\left(\frac{7-5i}{3+4i}\right) &= (7-5i)\left(\frac{1}{3+4i}\right)\\ &= (7-5i)\left(\left(\frac{1}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right)\right)\\ &= (7-5i)\left(\frac{3}{25}-\frac{4}{25}i\right)\\ &= \frac{21}{25}-\frac{28}{25}i-\frac{15}{25}i-\frac{20}{25}\\ &= \frac{1}{25}-\frac{43}{25}i\end{aligned}$$

This can also be expressed as follows:

$$\begin{aligned}\left(\frac{7-5i}{3+4i}\right) &= \left(\frac{7-5i}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right)\\ &= \left(\frac{21-28i-15i+20i^2}{9-12i+12i-16i^2}\right)\\ &= \left(\frac{1-43i}{25}\right)\\ &= \frac{1}{25}-\frac{43}{25}i\end{aligned}$$

Discussion

(page 306)

a. Sample response: The set of natural numbers is not closed under subtraction. For example, $3-5$ is not a natural number.

b. It can be shown that the commutative law of addition is preserved in the complex-number system as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(c+di)+(a+bi)=(c+a)+(d+b)i=(a+c)+(b+d)i$$

c. It can be shown that the associative law of addition is preserved in the complex-number system as follows:

$$\begin{aligned}(a+bi)+((c+di)+(e+fi)) &= (a+bi)+((c+e)+(d+f)i)\\ &= (a+c+e)+(b+d+f)i\\ ((a+bi)+(c+di))+(e+fi) &= ((a+c)+(b+d)i)+(e+fi)\\ &= (a+c+e)+(b+d+f)i\end{aligned}$$

d. It can be shown that the commutative law of multiplication is preserved in the complex numbers as follows:

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$
$$(c+di)(a+bi) = (ca-db)+(cb+da)i = (ac-bd)+(ad+bc)i$$

e. It can be shown that the associative law of multiplication is preserved in the complex numbers as follows:

$$\begin{aligned}(a+bi)((c+di)(e+fi)) &= (a+bi)((ce-df)+(cf+de)i)\\ &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i\end{aligned}$$

$$\begin{aligned}((a+bi)(c+di))(e+fi) &= ((ac-bd)+(ad+bc)i)(e+fi)\\ &= (ace-bde-adf-bcf)+(acf-bdf+ade+bce)i\\ &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i\end{aligned}$$

Assignment

(page 307)

2.1
a. $0+11i$
b. $0-10i$
c. $-5+2i$
d. $6-13i$

2.2
a. $i^1 = i^5 = i^9 = i$

$i^2 = i^6 = i^{10} = -1$

$i^3 = i^7 = -i$

$i^4 = i^8 = 1$

b. Sample response: The pattern cycles through four values: $i, -1, -i$, and 1.

c. $i^{90} = i^{88+2} = i^{88} \bullet i^2 = (i^4)^{22} \bullet i^2 = 1^{22} \bullet i^2 = 1 \bullet i^2 = i^2 = -1$

d. $i^n = i$ when $n = 4k+1$ for any integer $k \geq 0$

$i^n = -1$ when $n = 4k+2$ for any integer $k \geq 0$

$i^n = -i$ when $n = 4k+3$ for any integer $k \geq 0$

$i^n = 1$ when $n = 4k+4$ for any integer $k \geq 0$

2.3
a. $198+0i$
b. $30+i$
c. $1-43i$

2.4 **a.** Sample response:

$$\frac{3}{5-6i} \bullet \frac{5+6i}{5+6i} = \frac{15+18i}{61} = \frac{15}{61} + \frac{18}{61}i$$

b. Sample response:

$$\frac{-8-i}{-3-9i} \bullet \frac{-3+9i}{-3+9i} = \frac{33-69i}{90} = \frac{33}{90} - \frac{69}{90}i = \frac{11}{30} - \frac{23}{30}i$$

2.5 **a.** $r_1 = 2\sqrt{7}$ and $r_2 = -2\sqrt{7}$; $y = \left(x - 2\sqrt{7}\right)\left(x + 2\sqrt{7}\right)$

b. $r_1 = 2i\sqrt{7}$ and $r_2 = -2i\sqrt{7}$; $y = \left(x - 2i\sqrt{7}\right)\left(x + 2i\sqrt{7}\right)$

2.6 **a.** Sample response: From the graph, one can see no real roots. However, from the discussion in Activity **1**, one could expect that there are two complex-number solutions.

b. There are two real-number solutions for this quadratic equation.

c. There is one real root (a double root) for this quadratic equation.

2.7 Answers will vary. Sample responses are given below.

a. $x^2 + 25 = 0$ (this equation has both $5i$ and $-5i$ as solutions)

b. $x^2 + 28 = 0$

c. $x^2 - 12x + 85 = 0$

* * * * *

***2.8** **a.** Finding the roots for $y = x^2 + x - 6$ is equivalent to solving $0 = x^2 + x - 6$, which is equivalent to the equation $-x + 6 = x^2$.

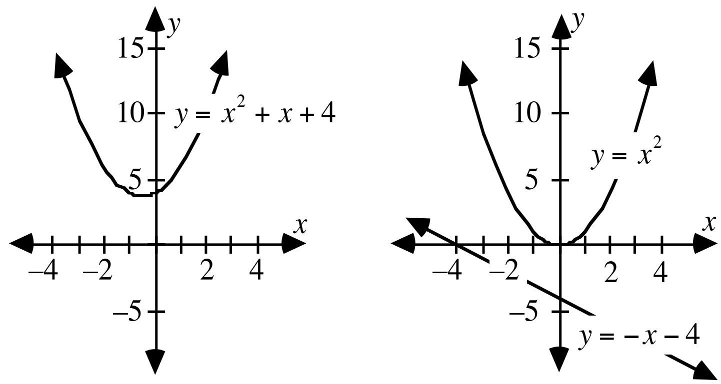
b. **1.** This graph shows the roots as the intersections of the graph of $y = x^2 + x - 6$ and the line $y = 0$ (the x-axis).

2. This graph shows the roots as the x-coordinates of the intersections of $y = -x + 6$ and $y = x^2$. This intersection shows where $x^2 = -x + 6$.

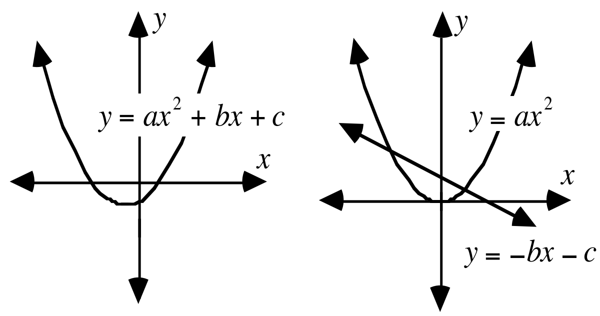
c. **1.** In this graph, the x-coordinate of the point where the graph of $y = x^2 - 4x + 4$ is tangent to the x-axis is the root.

2. In this graph, the x-coordinate of the intersection of $y = x^2$ and $y = 4x - 4$ is the root. This intersection shows where $x^2 = 4x - 4$.

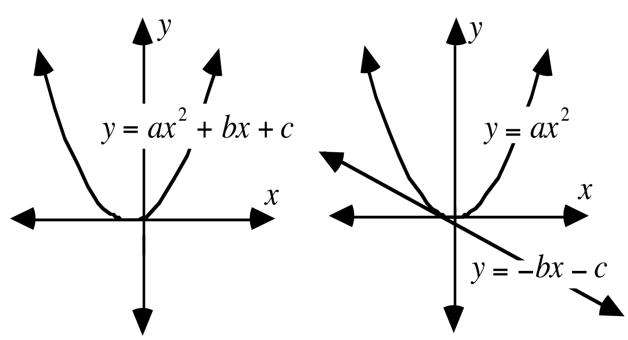
d. The following sample graphs indicate that no real-number roots exist.



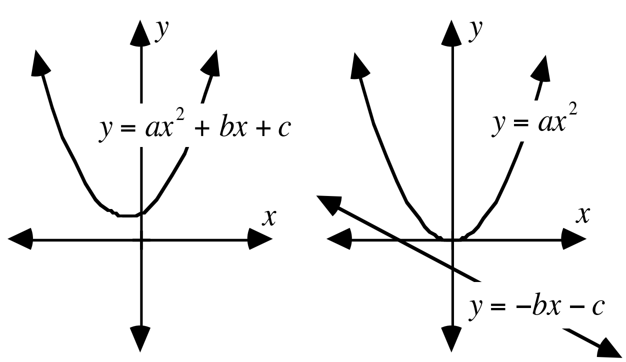
e. 1. The following sample graphs show two real roots.



2. The following sample graphs show one real root (a double root).



3. The following sample graphs show two complex roots that are not real numbers.



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(page 309)

Activity 3

Students investigate roots of polynomial equations of degree 2, 3, and 4. The fundamental theorem of algebra is introduced, along with a corollary stating that *n*th-degree polynomial equations have exactly *n* roots in the set of complex numbers.

Materials List

- graph paper

Technology

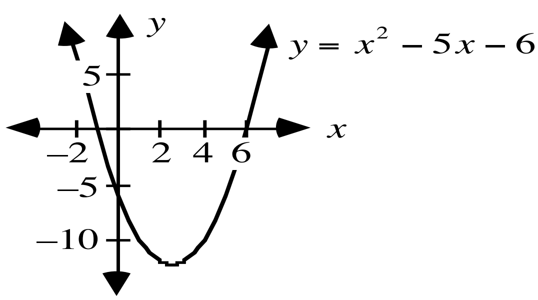
- graphing utility
- symbolic manipulator

Exploration 1

(page 309)

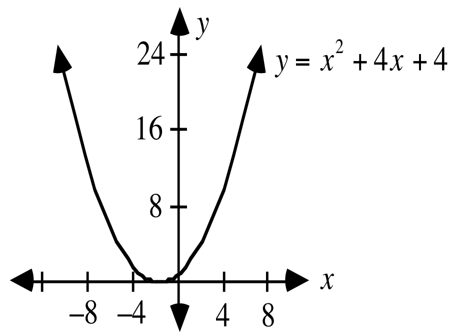
a. Students may use trial and error to determine values for a, b, and c. They can check their responses by graphing the equations and observing the number of zeros.

1. Sample response: $a = 1, b = -5, c = -6$ or $y = x^2 - 5x - 6$. In this case, $r_1 = 6$ and $r_2 = -1$. The equation may be written in factored form as $y = (x - 6)(x + 1)$.
2. The following graph shows that the equation has two real roots.



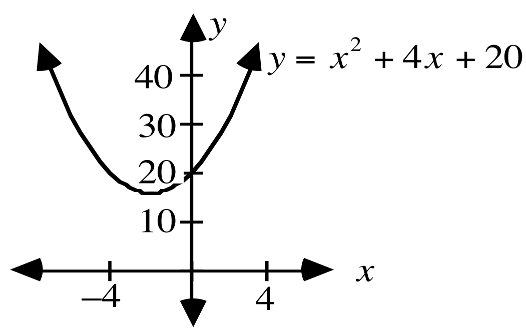
b.

1. Sample response: $a = 1, b = 4, c = 4$ or $y = x^2 + 4x + 4$. In this case, $r = -2$. The equation may be written in factored form as $y = (x + 2)^2$.
2. The following graph shows that the equation has one real root.



c.

1. Sample response: $a = 1, b = 4, c = 20$ or $y = x^2 + 4x + 20$. In this case, $r_1 = -2 - 4i$ and $r_2 = -2 + 4i$. The equation may be written in factored form as $y = (x - (-2 - 4i))(x + (-2 + 4i))$.
2. The following graph shows that the sample equation has no real roots.



d. The solutions are:

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Students may obtain other equivalent forms, depending on the technology used. The TI-92 calculator, for example, reports these solutions in the following form:

$$\frac{\left(\sqrt{-\left(ac - 0.25b^2\right)} + 0.5b\right)}{a} \text{ and } \frac{-\left(\sqrt{-\left(ac - 0.25b^2\right)} + 0.5b\right)}{a}$$

e. The solutions should agree with the roots determined in Parts **a–c**.

f. Student values will depend on their equations. When a quadratic equation has two real roots, $b^2 - 4ac > 0$. When there is one double root, $b^2 - 4ac = 0$. When there are no real roots, $b^2 - 4ac < 0$.

Discussion 1

(page 310)

a. **1.** Sample response: The graph intersects the x–axis in two points.

2. Sample response: The graph intersects the x–axis in only one point.

3. Sample response: The graph does not intersect the x–axis.

b. The two complex solutions are conjugates.

c. Sample response: Since $b^2 - 4ac$ is under the radical, if it is 0 or positive, then the square root is a real number and the roots of the equation are real numbers. If it is negative, the square root is not a real number and the roots of the equation are complex numbers.

d. Sample response: No. There will not always be real roots. **Note:** A polynomial is reducible over the complex numbers if it can be expressed as the product of two or more polynomials of degree 1 and with complex numbers as coefficients. Every second-degree polynomial is reducible over the complex numbers.

Teacher Note

As an alternative to the directions given in the student edition for Exploration **2**, you may wish to allow students to use a tool such as the TI-92. In this case, the command "cSolve(randPoly(x, 3)=0, x)" can be used to generate random polynomials of degree 3 and to solve them in the set of complex numbers. This allows for a wider variety of cubics to be solved.

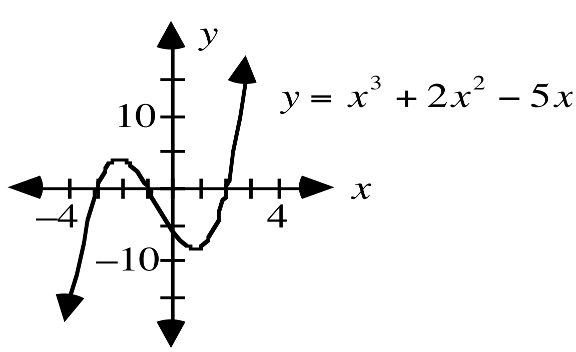
Exploration 2

(page 311)

Students use technology to investigate the number and types of solutions to third- and fourth-degree polynomials.

a. Sample response: $r_1 = 2$, $r_2 = -1$, and $r_3 = -3$. The equation may be written in factored form as $y = (x-2)(x+1)(x+3)$. In this case, $y = x^3 + 2x^2 - 5x - 6$ and $a = 1, b = 2, c = -5, d = -6$.

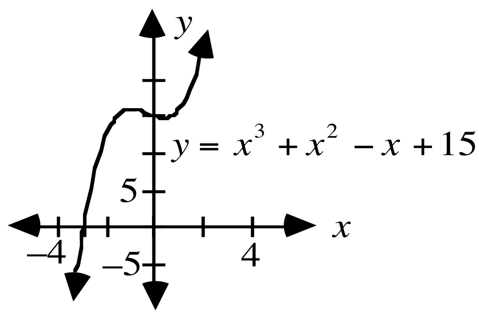
The following graph of the equation shows the three real roots: 2, –1, and –3.



b. Students should recognize that unless the two non-real roots are complex conjugates, the resulting equation will not have real coefficients.

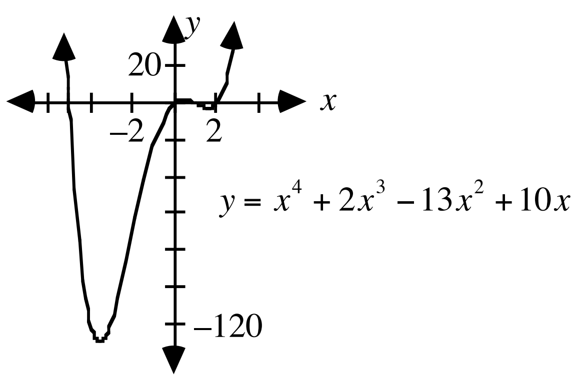
Sample response: $r_1 = 1 - 2i$, $r_2 = 1 + 2i$, and $r_3 = -3$. The equation may be written in factored form as $y = (x - (1 - 2i))(x - (1 + 2i))(x + 3)$. In this case, $y = x^3 + x^2 - x + 15$ and $a = 1, b = 1, c = -1, d = 15$.

The following graph of the equation shows that it has one real root: –3.



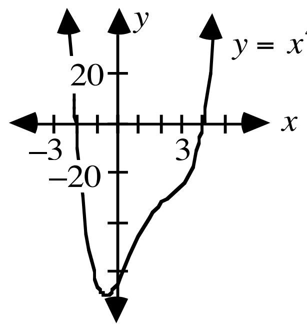
c. Sample response: $r_1 = 2$, $r_2 = 1$, $r_3 = -5$, and $r_4 = 0$. The equation may be written in factored form as $y = (x - 2)(x - 1)(x + 5)(x - 0)$. In this case, $y = x^4 + 2x^3 - 13x^2 + 10x$ and $a = 1, b = 2, c = -13, d = 10, e = 0$.

The following graph of the equation shows that it has four real roots: –5, 0, 1, and 2.



d. Sample response: $r_1 = -2$, $r_2 = 4$, $r_3 = 2 - 2i$, and $r_4 = 2 + 2i$. The equation may be written in factored form as $y = (x + 2)(x - 4)(x - (2 - 2i))(x - (2 + 2i))$. In this case, $y = x^4 - 6x^3 + 8x^2 + 16x - 64$ and $a = 1, b = -6, c = 8, d = 16, e = -64$.

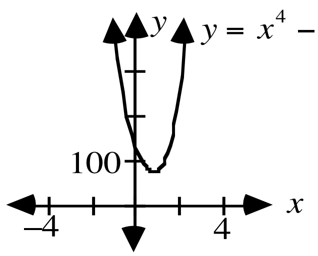
The following graph of the equation shows that it has two real roots: –2 and 4.



Note: As in Part **b**, students should recognize that unless the two non-real roots are complex conjugates, the resulting equation will not have real coefficients.

e. Sample response: $r_1 = -4i$, $r_2 = 4i$, $r_3 = 2 - 2i$, and $r_4 = 2 + 2i$. The equation may be written in factored form as $y = (x + 4i)(x - 4i)(x - (2 - 2i))(x - (2 + 2i))$. In this case, $y = x^4 - 4x^3 + 24x^2 - 64x + 128$ and $a = 1, b = -4, c = 24, d = -64, e = 128$.

The following graph of the equation shows that it has no real roots.



Teacher Note

You may wish to remind students that some or all of the roots of a polynomial may be identical.

Discussion 2

(page 311)

a.
1. By graphing, students should be able to predict the number of real-number solutions to a cubic equation—one intersection with the *x*-axis implies one real root and so on.
2. There are either three real roots or one real root.
3. There are always three complex-number solutions.

b.
1. There are either four real roots, two real roots, or no real roots.
2. There are always four complex-number solutions.

c. The complex solutions always occur in conjugate pairs.

d. Sample response: Because the complex solutions of the form $a + bi$, where $b \neq 0$, occur in conjugate pairs, there can be 0, 2, or 4 of these solutions. Since there are a total of 5 solutions for a fifth-degree equation, the corresponding numbers of real solutions possible are 5, 3, or 1.

e. Sample response: In general, an *n*th-degree polynomial with real coefficients can have *k* conjugate pairs of complex roots where *k* is an integer and $0 \leq k \leq n/2$.

f. Since complex numbers appear in conjugate pairs, there must be at least one real root when the degree of the polynomial is odd and the coefficients are real.

Assignment

(page 312)

3.1 **a.** The solution is a double root: $x = -2/3$.

$$9\left(x + \frac{2}{3}\right)\left(x + \frac{2}{3}\right) = 9x^2 + 12x + 4$$

b. The solutions are $x = 1/9$ and $x = -4$.

$$9\left(x - \frac{1}{9}\right)(x + 4) = 9x^2 + 35x - 4$$

c. The solutions are $x = -2 + i\sqrt{5}$ and $x = -2 - i\sqrt{5}$.

$$\left(x - \left[-2 + i\sqrt{5}\right]\right)\left(x - \left[-2 - i\sqrt{5}\right]\right) = x^2 + 4x + 9$$

d. The solutions are $x = -2i$, $x = 2i$, and $x = 4$.

$$3(x - 2i)(x - (-2i))(x - 4) = 3x^3 - 12x^2 + 12x - 48$$

e. The solutions are $x = 3$, $x = -2$, $x = 1 - 3i$, and $x = 1 + 3i$.

$$2(x - 3)(x + 2)(x - (1 - 3i))(x - (1 + 3i)) = 2x^4 - 6x^3 + 12x^2 + 4x - 120$$

3.2 **a.** $x^2 - 4x + 5 = 0$

b. The solutions are $x = 3 + 2i$ and $x = 3 - 2i$. They are complex conjugates.

c. Answers will vary. Sample response: The four solutions $\{1 - i,\ 1 + i,\ 2 - i,\ 2 + i\}$ result in the equation below.

$$(x - [1 - i])(x - [1 + i])(x - [2 - i])(x - [2 + i]) = 0$$
$$x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$$

***3.3** Sample response: Because complex solutions occur in conjugate pairs, the fourth solution must be $2 - 3i$. Using these four roots,

$$(x - 2)(x + 5)(x - [2 - 3i])(x - [2 + 3i]) = x^4 - x^3 - 9x^2 + 79x - 130$$

Thus, $c = -9$.

***3.4** Sample response: An equation of this form always has six complex roots. Since complex roots of the form $a + bi$, where $b \neq 0$, occur in conjugate pairs, there can be either 0, 2, 4, or 6 roots of this type. The corresponding numbers of real solutions are 6, 4, 2, or 0. Once the roots are determined, the equation can be written as:

$$a(x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)(x - r_6) = 0$$

* * * * *

3.5

a. Finding the roots of $y = x^3 - 12x + 16$ is equivalent to solving the equation $0 = x^3 - 12x + 16$, which is equivalent to the equation $x^3 = 12x - 16$.

b. **1.** Sample response: In this graph, the roots are shown where the graph of $y = x^3 - 12x + 16$ intersects the *x*-axis. The point where the graph of $y = x^3 - 12x + 16$ is tangent to the *x*-axis indicates a double root.

2. Sample response: In this graph, the *x*-coordinate of the point at which $y = x^3$ is tangent to $y = 12x - 16$ is a double root. The other root is an intersection that occurs in the third quadrant beyond the borders of the given graph.

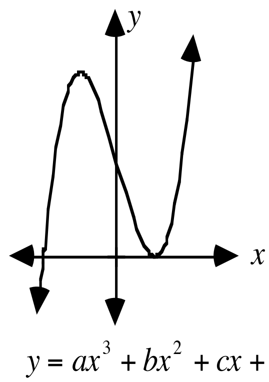
c. **1.** Sample response: The graphs will intersect three times.

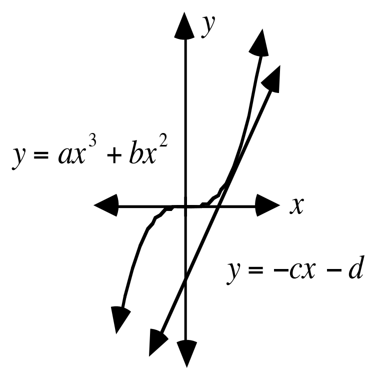
2. Sample response: Shifting the graph of $y = x^3 - 12x + 16$ down 6 units will result in three intersections with the *x*-axis. This indicates three distinct real roots for the equation $y = x^3 - 12x + 10$, compared to two distinct real roots for the equation $y = x^3 - 12x + 16$.

3. The zeros for $y = x^3 - 12x + 10$ are approximately 2.9, 0.9, and –3.8. This confirms a prediction of three distinct zeros.

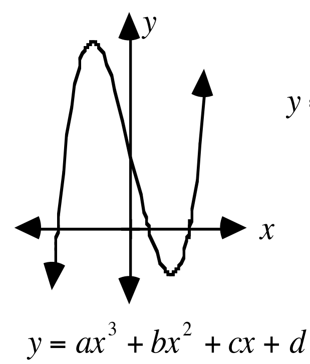
d. Sample response: Increasing the constant 16 would shift the graph of $y = x^3 - 12x + 16$ upward, reducing the number of intersections with the *x*-axis to 1. This indicates that there will be two non-real solutions and one real solution. This result is also shown in the graph of $y = x^3$ and $y = 12x - 16$, where increasing the constant 16 in the graph of $y = 12x - 16$ will cause a downward shift of the line, reducing the number of intersections to 1.

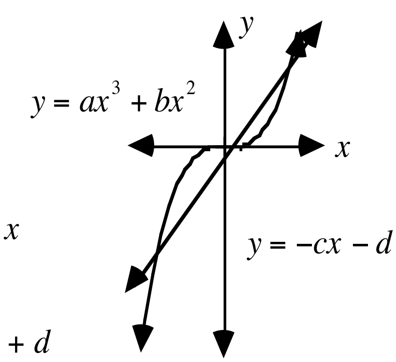
e. There are three possible cases. In one case, the equation has two real zeros, one of which is a double root, as illustrated in the following graphs. **Note:** In the second graph, the graphs of the two equations also intersect in the third quadrant, beyond the portion of the graph shown.



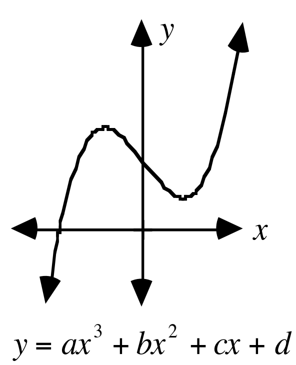


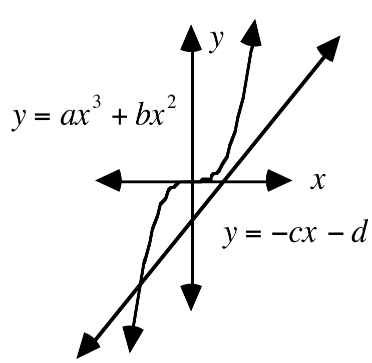
The equation also could have three distinct real roots, as illustrated in the following graphs:





In the third case, the equation has one real root, as illustrated below:





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(page 314)

Activity 4

Students represent complex numbers of the form $a+bi$ as ordered pairs (a,b), then graph these numbers on the complex plane.

Materials List

- graph paper

Technology

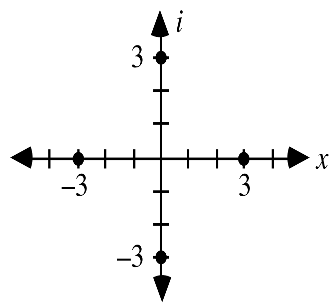
- graphing utility
- symbolic manipulator
- spreadsheet

Exploration

(page 315)

Students explore how multiplying a complex number by i or $-i$ rotates its graph about the origin in the complex plane.

a–b. The following sample graph shows the point $3 + 0i$ and its images.



c. Sample response: Multiplication by i rotates a point $\pi/2$ radians counterclockwise about the origin.

d. **1.** Multiplication by $-i$ rotates the point $\pi/2$ radians clockwise about the origin.

2. Students check their predictions graphically.

e. **1.** $(a + bi)i = ai + bi^2 = -b + ai$

2. Sample response: If (a, b) is rotated counterclockwise $\pi/2$ radians about the origin, the resulting position is $(-b, a)$. Yes, the conjecture appears to apply to all complex numbers.

Discussion

(page 315)

a. **1.** Multiplying by i performs a counterclockwise rotation of $\pi/2$ radians about the origin.

2. Multiplying by $-i$ performs a clockwise rotation of $\pi/2$ radians about the origin.

b. Sample response: When multiplying by i, the x-coordinate of the image is the opposite of the i-coordinate of the preimage. The i-coordinate of the image is the x-coordinate of the preimage. This results in the ordered pair $(1, a)$.

When multiplying by $-i$, the x-coordinate of the image is the i-coordinate of the preimage and the i-coordinate of the image is the opposite of the x-coordinate of the preimage. This results in the ordered pair $(1, -a)$.

c. Sample response: Multiplying 0 by any number yields 0, and (0,0) is its own image as the center of rotation.

Assignment
(page 315)

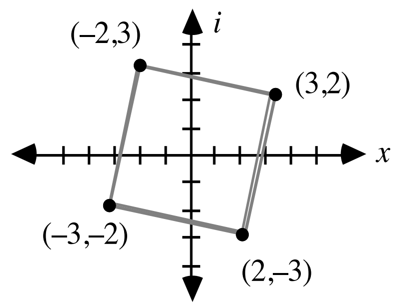
4.1 **a.** **1.** (–2,3)

2. (–3,–2)

3. (2,–3)

4. (3,2)

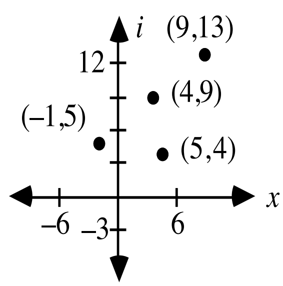
b. These points are equally spaced on a circle with center at the origin. Some students also may observe that the points are the vertices of a square. **Note:** If the axes are not scaled equally, students may not recognize either a circle or a square.



4.2 Complex conjugates are reflections of each other in the x-axis.

4.3 **a.** $u + v = 9 + 13i$ or (9,13); $u - v = -1 + 5i$ or (–1,5)

b. Sample graph:



c. The addition of complex numbers using ordered pairs is defined as $(a,b) + (c,d) = (a + c, b + d)$. The subtraction of complex numbers using ordered pairs is defined as $(a,b) - (c,d) = (a - c, b - d)$.

***4.4** **a.** Sample response: The solutions are 1, i, and $-i$ since $1^3 - 1^2 + 1 - 1 = 0$, $i^3 - i^2 + i - 1 = 0$, and $(-i)^3 - (-i)^2 + (-i) - 1 = 0$. The number –1 is not a solution since $(-1)^3 - (-1)^2 + (-1) - 1 = -4$.

b. $(x - 1)(x - i)(x + i) = 0$

c. Students should multiply the factors to verify their solutions.

* * * * *

Teacher Note

Problem **4.5** involves graphing Julia sets on the complex plane. Students may complete the problem using a spreadsheet. They also may use other tools. For example, the following program can be used to graph Julia sets on a TI-92.

```
:Comp()
:Prgm
:ClrGraph
:FnOff
:Plots Off
:Clr10
:DelVar r,i,c,n,l1,l2
:Input "Real part",r
:Input "Imaginary Part",i
:Input "Number of Points",c
:r→L1[1]
:i→l2[1]
:For n,1,c–1
:2*l1[n]*l2[n]+i→l1[n+1]
:l1[n]^2–l2[2]^2+r→l1[n+1]
:Endfor
:NewPlot 2,1,l1,l2,,,,4
:ZoomData
:EndPrgm
```

4.5 **a.** The third term is:

$$\begin{aligned} a_3 + b_3 i &= (a_2 + b_2)^2 + a_1 + b_1 \\ &= (-3 + 15i)^2 + 2 + 3i \\ &= -216 - 90i + 2 + 3i \\ &= -214 - 87i \end{aligned}$$

b. $(-214, -87)$

c. **1.** Student responses should be equivalent to the expression below:

$$\begin{aligned} (a_{n-1} + b_{n-1}i)^2 + (a_1 + b_1 i) &= (a_{n-1}^2 + 2a_{n-1}b_{n-1} + b_{n-1}^2 i^2) + (a_1 + b_1 i) \\ &= (a_{n-1}^2 - b_{n-1}^2 + a_1) + (2a_{n-1}b_{n-1} + b_1)i \end{aligned}$$

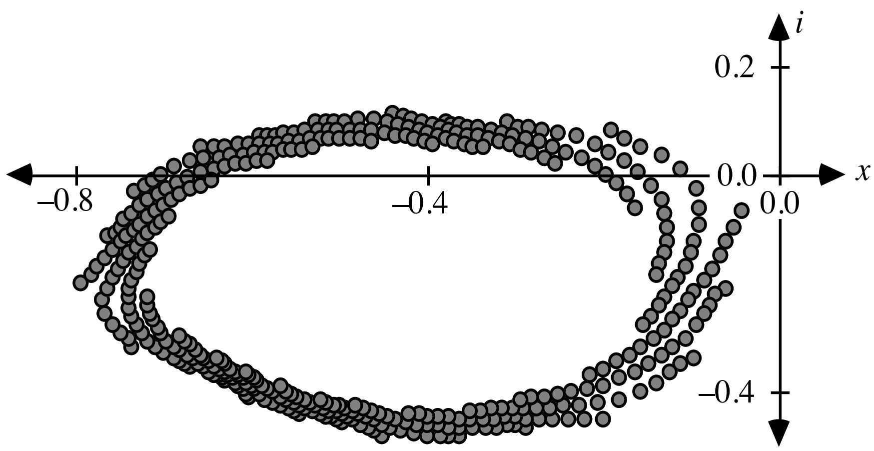
2. $\left((a_{n-1}^2 - b_{n-1}^2 + a_1), (2a_{n-1}b_{n-1} + b_1)\right)$

d. 1. The following sample spreadsheet shows formulas written for Microsoft Excel.

	A	B	C
1	**Term (n)**	**Real (a_n)**	**Imaginary (b_n)**
2	1	–0.63	–0.37
3	=A2+1	=B2^2–C3^2+$B$2	=2*B2*C2+$C$2
4	=A3+1	=B3^2–C3^2+$B$2	=2*B3*C3+$C$2
⋮	⋮	⋮	⋮

2. Student graphs should resemble the scatterplot given in the student edition.

e. 1. Sample scatterplot:



2. Students should observe that small changes can lead to very different scatterplots.

f. Students will observe a variety of patterns in the scatterplots of the Julia sets.

4.6 **a.** Sample response:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -b+(-d) & a+c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix}$$

b. Sample response:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \bullet \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -(bc+ad) & ac-bd \end{bmatrix}$$

c. The additive identity is:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The multiplicative identity is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d. The multiplicative inverse is:

$$\begin{bmatrix} \dfrac{a}{a^2+b^2} & \dfrac{-b}{a^2+b^2} \\ \dfrac{b}{a^2+b^2} & \dfrac{a}{a^2+b^2} \end{bmatrix}$$

e. The square of the matrix yields:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This matrix represents the additive inverse of the multiplicative identity. **Note:** In this set of matrices, the square of an element is the additive inverse of the multiplicative identity. This is not the case in the set of real numbers.

f. The complex number $0 + i$ can be represented as:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

g. Multiplication by the following matrix produces a 90˚ rotation about the origin:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

h. The matrices are equal.

* * * * * * * * * *

(page 319)

Activity 5

In this activity, students represent complex numbers in trigonometric form, and continue their investigations of operations on complex numbers.

Materials List

- graph paper

Technology

- graphing utility
- symbolic manipulator

Exploration 1 (page 320)

a.

1. The angle $\tan^{-1}(\sqrt{3}/1) = \pi/3 \approx 1.05$ is a first-quadrant angle with $(1, \sqrt{3})$ on its terminal ray. Therefore, $\pi/3$ is an argument of $1 + \sqrt{3}i$.

2. The angle $\tan^{-1}(\sqrt{3}/-1) = -\pi/3 \approx -1.05$ is a fourth-quadrant angle. The point $(-1, \sqrt{3})$ is in the second quadrant, so $-\pi/3$ is not an argument of $-1 + \sqrt{3}i$.

The angle $\pi + (-\pi/3) = 2\pi/3 \approx 2.09$ is an argument of $-1 + \sqrt{3}i$.

3. The angle $\tan^{-1}(-\sqrt{3}/-1) = \pi/3 \approx 1.05$ is a first-quadrant angle. The point $(-1, -\sqrt{3})$ is in the third quadrant, so $\pi/3$ is not an argument of $-1 - \sqrt{3}i$.

The angle $\pi + \pi/3 = 4\pi/3 \approx 4.19$ is an argument of $-1 - \sqrt{3}i$.

4. The angle $\tan^{-1}(-\sqrt{3}/1) = -\pi/3 \approx -1.05$ is a fourth-quadrant angle with $(1, -\sqrt{3})$ on its terminal ray. Therefore $-\pi/3$ is an argument of $1 - \sqrt{3}i$.

b. Sample response:

Complex Number	Positive Argument	Positive Argument	Negative Argument	Negative Argument
$1 + \sqrt{3}i$	$7\pi/3$	$13\pi/3$	$-5\pi/3$	$-11\pi/3$
$-1 + \sqrt{3}i$	$8\pi/3$	$14\pi/3$	$-4\pi/3$	$-10\pi/3$
$-1 - \sqrt{3}i$	$10\pi/3$	$16\pi/3$	$-2\pi/3$	$-8\pi/3$
$1 - \sqrt{3}i$	$5\pi/3$	$11\pi/3$	$-7\pi/3$	$-13\pi/3$

Discussion 1 (page 320)

a. Sample response: The point has to be in quadrant I or quadrant IV.

b. Sample response: An argument can be determined by adding π to $\tan^{-1}(b/a)$.

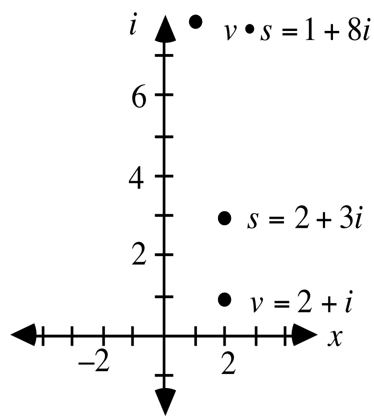
c. Sample response: Other positive arguments can be found by adding positive multiples of 2π. Other negative arguments are found by adding negative multiples of 2π.

Exploration 2

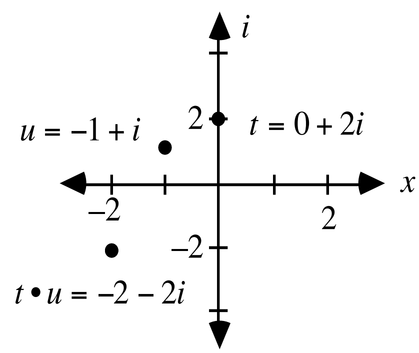
(page 321)

Students should use the Pythagorean theorem and trigonometry to find the absolute values and arguments when necessary.

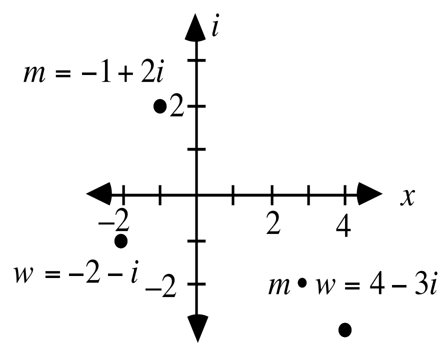
a. **1.** Sample graph:



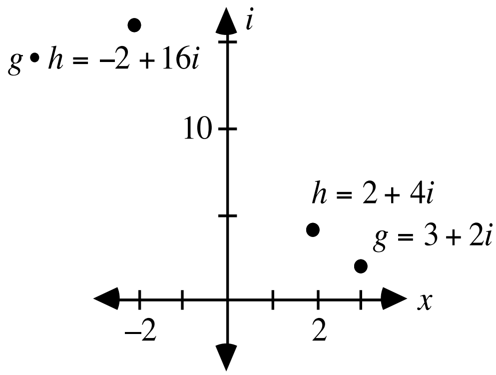
2. Sample graph:



3. Sample graph:



4. Sample graph:



b. Sample table:

Number	a + bi	Absolute Value	Argument
v	$2 + i$	$\sqrt{5}$	0.46
s	$2 + 3i$	$\sqrt{13}$	0.98
$v \bullet s$	$1 + 8i$	$\sqrt{65}$	1.45
t	$0 + 2i$	$\sqrt{4}$	1.57
u	$-1 + i$	$\sqrt{2}$	2.36
$t \bullet u$	$-2 - 2i$	$\sqrt{8}$	3.93
m	$-1 + 2i$	$\sqrt{5}$	2.03
w	$-2 - i$	$\sqrt{5}$	3.61
$m \bullet w$	$4 - 3i$	$\sqrt{25}$	5.64
g	$3 + 2i$	$\sqrt{13}$	0.59
h	$2 + 4i$	$\sqrt{20}$	1.11
$g \bullet h$	$-2 + 16i$	$\sqrt{260}$	1.70

c. Students should observe that the absolute values of conjugates are equal. The arguments of conjugates are additive inverses. The product of conjugates is $a^2 + b^2$.

d.
1. The steps needed to simplify depend on the technology used. For example, the TI-92 calculator requires the use of the "tCollect" command in rectangular mode and approximate mode. Students should recognize that the absolute values are multiplied and the arguments are added.
2. Sample response: When multiplying two complex numbers in trigonometric form, the absolute values of the numbers are multiplied and the arguments of the numbers are added.
3. The rule described above should be illustrated in student tables.

e. **1.** $1 - i \approx 1.41[\cos(-0.79) + i\sin(-0.79)]$;
$2 + i \approx 2.24[\cos(2.68) + i\sin(2.68)]$

2. $3.16[\cos(1.89) + i\sin(1.89)]$

3. $3.16[\cos(1.89) + i\sin(1.89)] \approx -0.99 + 3.00i$

4. $(1 - i)(-2 + i) = -2 + i + 2i - i^2 = -1 + 3i$

5. Sample response: The results from Steps **2** and **4** should be equal. The difference is due to rounding error.

Discussion 2

(page 322)

a. Their absolute values are equal and their arguments are additive inverses.

b. The argument of the product is 0. The resulting argument also can be any multiple of 2π, since any such angle would have the same terminal ray as 0.

c. Sample response: Yes. The multiplication of complex numbers in trigonometric form involves multiplication of two positive real numbers and the addition of two angles in radian measure. Since both the multiplication of real numbers and the addition of real numbers are commutative, the multiplication of complex numbers is commutative.

d. Sample response: When multiplying numbers in $a + bi$ form, it is necessary to use the distributive property, combine similar terms, and simplify i^2 terms. When multiplying numbers in trigonometric form, you find one product (the product of the absolute values) and one sum (the sum of the arguments).

Assignment

(page 322)

5.1 **a.** $9(\cos 0 + i\sin 0)$

b. $121(\cos 0 + i\sin 0)$

c. $7(\cos 0 + i\sin 0)$

5.2. **a.** $6(\cos 9.64 + i\sin 9.64) \approx 6(\cos 3.36 + i\sin 3.36)$

b. $r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

5.3 **a.** Sample response: Using trigonometric form, (–1,2) can be approximated as $2.24(\cos 2.03 + i\sin 2.03)$ and (3,–2) can be approximated as $3.61(\cos(-0.59) + i\sin(-0.59))$. The product is $8.09(\cos 1.44 + i\sin 1.44)$ and the corresponding ordered pair in the complex plane is approximately $(1.06, 8.02)$.

Using the form $a + bi$, (–1,2) can be written as $-1 + 2i$ and (3,–2) can be written as $3 - 2i$. Their product in $a + bi$ form is $(-1 + 2i)(3 - 2i) = -3 + 8i - 4i^2 = 1 + 8i$, which is (1,8) in the complex plane.

b. **Note:** Students must compare products in the same forms. Sample response: The values are slightly different, due to rounding when converting.

5.4 Multiplication by $2(\cos(-\pi/6) + i\sin(-\pi/6))$ dilates by 2 and rotates by $-\pi/6$ radians about the origin. The two numbers are conjugates. (Note that multiplication by $2(\cos(11\pi/6) + i\sin(11\pi/6))$ gives the same result.)

5.5 **a.** The trigonometric form of $(3 - 4i)$ is approximately $5(\cos(-0.93) + i\sin(-0.93))$; the trigonometric form of $(3 - 4i)^2$ is approximately $25(\cos(-1.86) + i\sin(-1.86))$.

b. The trigonometric form of $(3 - 4i)^3$ is approximately $125(\cos(-2.79) + i\sin(-2.79))$.

c. $(3 - 4i)^5 = (3 - 4i)^2(3 - 4i)^3 \approx 3125(\cos(-4.65) + i\sin(-4.65))$

d. $(3 - 4i)^n = 5^n\left(\cos[n(-0.93)] + i\sin[n(-0.93)]\right)$

***5.6** $(a + bi)^2 = r^2(\cos(2\theta) + i\sin(2\theta))$

$(a + bi)^3 = r^3(\cos(3\theta) + i\sin(3\theta))$

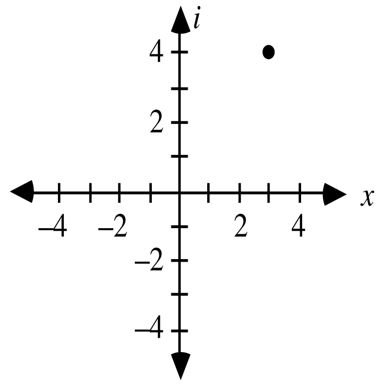
$(a + bi)^4 = r^4(\cos(4\theta) + i\sin(4\theta))$

$(a + bi)^n = r^n(\cos(n\theta) + i\sin(n\theta))$

* * * * *

5.7 **a.** The modulus is 5.

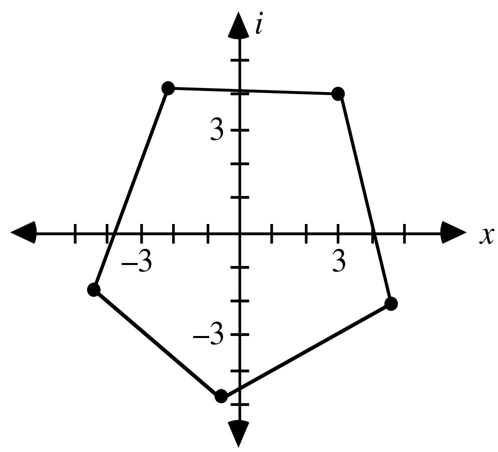
b. Sample response:



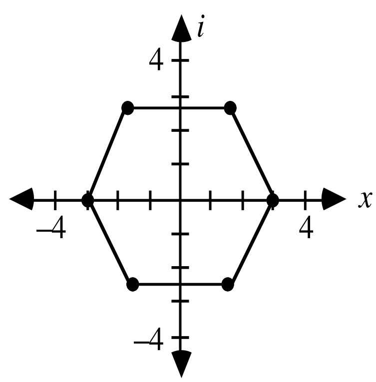
c. The radius is 5.

d. The measure of the angle is 72°.

e. The resulting pentagon may be skewed due to errors introduced by approximations. Sample graph:



5.8 In this case, $r = 3$ and $\theta = 60°$. Sample graph:



5.9 **a.** The solution requires the multiplication of two complex numbers in trigonometric form.

$$V = I \bullet Z = 4(\cos(\pi/18) + i\sin(\pi/18)) \bullet 29(\cos(\pi/9) + i\sin(\pi/9))$$
$$= 116(\cos(\pi/6) + i\sin(\pi/6))$$

b. The division in this problem requires students to multiply the numerator and denominator by the conjugate of $44(\cos(11\pi/6) + i\sin(11\pi/6)$.

$$I = \frac{V}{Z} = \frac{120(\cos 0 + i\sin 0)}{44\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)} \bullet \frac{44\left(\cos\left(-\frac{11\pi}{6}\right) + i\sin\left(-\frac{11\pi}{6}\right)\right)}{44\left(\cos\left(-\frac{11\pi}{6}\right) + i\sin\left(-\frac{11\pi}{6}\right)\right)}$$

$$= \frac{5280\left(\cos\left(-\frac{11\pi}{6}\right) + i\sin\left(-\frac{11\pi}{6}\right)\right)}{1936}$$

$$\approx 2.73\left(\cos\left(-\frac{11\pi}{6}\right) + i\sin\left(-\frac{11\pi}{6}\right)\right)$$

c. $Z = \dfrac{77 + 77i}{2.9 - 0.35i} \bullet \dfrac{2.9 + 0.35i}{2.9 + 0.35i} \approx 23.01 + 29.33i$

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(page 324)

Activity 6

In this activity, students write complex numbers in trigonometric form and examine roots of complex numbers geometrically and algebraically. They are introduced to De Moivre's theorem and use it to raise complex numbers to powers.

Materials List

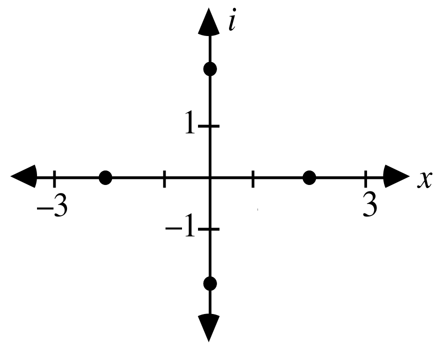
- graph paper

Technology

- graphing utility
- symbolic manipulator
- geometry utility (optional)

Exploration (page 325)

a. **1–2.** Sample graph:



3. The coordinates of the four distinct points are (2,0), (0,2), (–2,0), and (0,–2).

b. **1.** Sample response: The points are equally spaced around a circle of radius 2 with center at the origin. The figure formed by connecting the points in order is a square.

2. The numbers are $2(\cos 0 + i\sin 0)$, $2(\cos(\pi/2) + i\sin(\pi/2))$, $2(\cos\pi + i\sin\pi)$, and $2(\cos(3\pi/2) + i\sin(3\pi/2))$.

c.

$$2^4(\cos(4 \bullet 0) + i\sin(4 \bullet 0)) = 16 + 0i$$
$$2^4(\cos(4 \bullet (\pi/2)) + i\sin(4 \bullet (\pi/2))) = 16 + 0i$$
$$2^4(\cos(4 \bullet \pi) + i\sin(4 \bullet \pi)) = 16 + 0i$$
$$2^4(\cos(4 \bullet (3\pi/2)) + i\sin(4 \bullet (3\pi/2))) = 16 + 0i$$

d. **1.** The point must be rotated $2\pi/5$ radians.

2. The vertices correspond to $2(\cos 0 + i\sin 0)$, $2(\cos(2\pi/5) + i\sin(2\pi/5))$, $2(\cos(4\pi/5) + i\sin(4\pi/5))$, $2(\cos(6\pi/5) + i\sin(6\pi/5))$, and $2(\cos(8\pi/5) + i\sin(8\pi/5))$.

e.

$$2^5(\cos(5 \bullet 0) + i\sin(5 \bullet 0)) = 32 + 0i$$
$$2^5(\cos(5 \bullet 2\pi/5) + i\sin(5 \bullet 2\pi/5)) = 32 + 0i$$
$$2^5(\cos(5 \bullet 4\pi/5) + i\sin(5 \bullet 4\pi/5)) = 32 + 0i$$
$$2^5(\cos(5 \bullet 6\pi/5) + i\sin(5 \bullet 6\pi/5)) = 32 + 0i$$
$$2^5(\cos(5 \bullet 8\pi/5) + i\sin(5 \bullet 8\pi/5)) = 32 + 0i$$

f. Students select any complex number and find its cube roots in trigonometric form.

g. The points are equally spaced around a circle with center at the origin. The radius of the circle is the modulus of the complex number in Part **f**. Connecting the points in order forms an equilateral triangle.

Discussion

(page 326)

a. Sample response: Raising each of the numbers to the fourth power results in 16. This implies that $2(\cos 0 + i\sin 0)$ or $2 + 0i$, $2(\cos(\pi/2) + i\sin(\pi/2))$ or $(0 + 2i)$, $2(\cos\pi + i\sin\pi)$ or $-2 + 0i$, and $2(\cos(3\pi/2) + i\sin(3\pi/2))$ or $0 - 2i$, are all fourth roots of 16. Therefore, 16 has 4 fourth roots.

b. Sample response: Each of these numbers raised to the fifth power is 32. This implies that $2(\cos 0 + i\sin 0)$ or $2 + 0i$, $2(\cos(2\pi/5) + i\sin(2\pi/5))$ or approximately $0.6 + 1.9i$, $2(\cos(4\pi/5) + i\sin(4\pi/5))$ or approximately $-1.6 + 1.2i$, $2(\cos(6\pi/5) + i\sin(6\pi/5))$ or approximately $-1.6 - 1.2i$, and $2(\cos(8\pi/5) + i\sin(8\pi/5))$ or approximately $0.6 - 1.9i$, are all fifth roots of 32. Therefore, 32 has 5 fifth roots.

c. Sample response: In trigonometric form, each *n*th root of the complex number $r(\cos\theta + i\sin\theta)$ has a modulus of $\sqrt[n]{r}$.

d. Sample response: In trigonometric form, each *n*th root of the complex number $r(\cos\theta + i\sin\theta)$ has an argument of θ/n.

e.
1. $r(\cos 0 + i\sin 0) = r$
2. $r(\cos(\pi/2) + i\sin(\pi/2)) = ri$

f. Sample response: The value of θ determines the polygon's orientation.

Assignment

(page 327)

6.1

a. $(2\sqrt{3} + 2i)^3 = (2\sqrt{3} + 2i)^2(2\sqrt{3} + 2i) = (8 + 8i\sqrt{3})(2\sqrt{3} + 2i) = 64i$

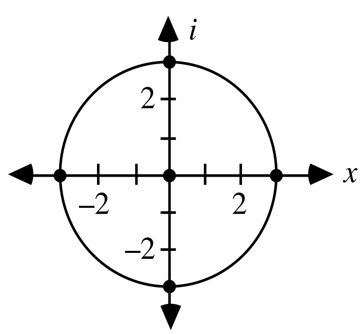
b. $(2\sqrt{3} + 2i)^3 \approx 64(\cos 1.572 + i\sin 1.572) \approx -0.07 + 64i$ (the error is due to rounding)

6.2

a. 3

b. $\pi/2$

c. Sample graph:



d. The fourth roots of 81 are $3 + 0i$, $0 + 3i$, $-3 + 0i$, and $0 - 3i$.

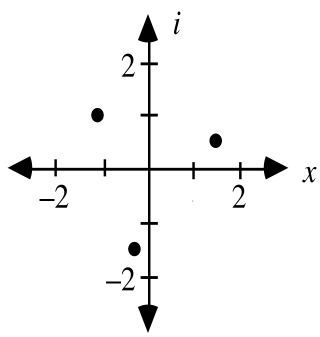
***6.3** **a.** Answers will vary. The following sample responses are based on $z = a + bi$, where $a = 2$ and $b = 3$.

b. Solving the equation $x^3 = z$ yields the three cube roots of z. Sample response: Solving $x^3 = 2 + 3i$ results in

$$x = \begin{cases} 1.452 + 0.493i \\ -1.153 + 1.011i \\ -0.299 - 1.504i \end{cases}$$

c. Sample response: $r_1 = 1.53(\cos 0.33 + i \sin 0.33)$, $r_2 = 1.53(\cos 2.42 + i \sin 2.42)$, $r_3 = 1.53(\cos 4.52 + i \sin 4.52)$.

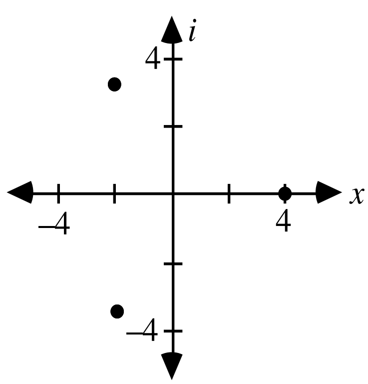
d. Student sketches should show the vertices of an equilateral triangle. The orientation is determined by the original values of *a* and *b* selected in Part **a**. Sample graph:



6.4 The 8 distinct eighth roots of $-2 + 3i$ are represented by the ordered pairs (0.58,1.02), (–0.31,1.13), (–1.02,0.58), (–1.13,–0.31), (–0.58,–1.02), (0.31,–1.13), (1.02,–0.58), and (1.13,0.31).

6.5 **a.** $2\pi/3$

b. The two additional cube roots are $-2 + \left(2\sqrt{3}\right)i \approx -2 + 3.46i$ and $-2 - \left(2\sqrt{3}\right)i \approx -2 - 3.46i$. Sample graph:



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6.6 **a.** Sample response:

$$\sqrt[3]{1}=\sqrt[3]{1}(\cos 0+i\sin 0)=1$$

$$\sqrt[3]{1}=\sqrt[3]{1}\left(\cos\left[0+\frac{2\pi}{3}\right]+i\sin\left[0+\frac{2\pi}{3}\right]\right)=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$\sqrt[3]{1}=\sqrt[3]{1}\left(\cos\left[0+\frac{4\pi}{3}\right]+i\sin\left[0+\frac{4\pi}{3}\right]\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

b. Students may use the distributive property or a symbolic manipulator to cube their responses to Part **a**.

6.7 **a.** The other vertices are:

$3(\cos\pi+i\sin\pi)$, $3(\cos(5\pi/4)+i\sin(5\pi/4))$,

$3(\cos(3\pi/2)+i\sin(3\pi/2))$, $3(\cos(7\pi/4)+i\sin(7\pi/4))$,

$3(\cos 0+i\sin 0)$, $3(\cos(\pi/4)+i\sin(\pi/4))$,

$3(\cos(\pi/2)+i\sin(\pi/2))$

b. These complex numbers are the 8 distinct eighth roots of 6561. **Note:** $3^8=6561$.

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Research Project (page 328)

Leonhard Euler's contributions to modern-day notation include e, the base of natural logarithms, called "Euler's number" in his honor. He also introduced the notation $f(x)$ for functions and Σ for summation, and was the first to use a, b, and c for the sides of triangle ABC.

a. Euler's formula states that $e^{i\theta}=\cos\theta+i\sin\theta$ where θ is measured in radians. This formula simplifies the calculations necessary for differentiation and integration of complex numbers.

b. Sample explanation: For the complex number $\sqrt{3}+i$,
$r=\sqrt{1^2+(\sqrt{3})^2}=2$ and $\tan^{-1}(1/\sqrt{3})=\pi/6$. Therefore,
$\sqrt{3}+i=2(\cos(\pi/6)+i\sin(\pi/6))=2e^{i\pi/6}$.

c. To determine a natural logarithm of -1, π is substituted for θ:

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$e^{i\pi} = -1 + i(0) = -1$$

This equation relates three of the most important numbers in mathematics.

$$\ln e^{i\pi} = \ln(-1)$$
$$i\pi = \ln(-1)$$

d. The natural logarithm of $-n$, a negative real number when n is a positive real number, can be found in the following manner:

$$\ln(-n) = \ln(-1 \bullet n) = \ln(-1) + \ln n$$

Since n is positive, $\ln n$ exists in the real numbers. From Part **b**, $\ln(-1)$ is a complex number. Thus, $\ln(-n)$ is a complex number.

Answers to Summary Assessment (page 329)

1. Sample response: Since complex roots of the form $a + bi$, where $b \neq 0$, occur in conjugate pairs, then if n is even, there can be $n, n-2, \ldots, 0$ of these roots. The corresponding numbers of real roots are $0, 2, \ldots, n$. So if n is even, the number of real roots possible is also even.

If n is odd, the possible numbers of complex roots of the form $a + bi$, where $b \neq 0$, are $n-1,\ n-3,\ \ldots,\ 2$. The corresponding numbers of real roots are $1,\ 3,\ \ldots,\ n-2$. If n is odd, the number of real roots possible also is odd, and there will be at least one real root.

2. **a.** The roots are $1 + 1.41i$, $1 - 1.41i$, 1.41, and -1.41.

b. **1.** $\left(x - [1 + i\sqrt{2}]\right)\left(x - [1 - i\sqrt{2}]\right)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$

2. $\left(x^2 - 2x + 3\right)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$

3. $\left(x^2 - 2x + 3\right)\left(x^2 - 2\right)$

3. **a.** Using the binomial theorem:

$$(1+i)^8 = C(8,8)1^8 i^0 + C(8,7)1^7 i^1 + C(8,6)1^6 i^2 + C(8,5)1^5 i^3 + C(8,4)1^4 i^4 + C(8,3)1^3 i^5 + C(8,2)1^2 i^6 + C(8,1)1^1 i^7 + C(8,0)1^0 i^8$$

b. $(1+i)^8 = 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1 = 16$

c. $(1+i) = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$

d. $(1+i)^8 = \left(\sqrt{2}\right)^8\left(\cos(2\pi) + i\sin(2\pi)\right) = 16(1+0) = 16$

e. Sample response: It is much easier to evaluate $(1+i)^8$ using the trigonometric form of $(1+i)$ than to use the binomial theorem, which requires computing combinations and powers of i.

4. **a.** The image is produced by a rotation, dilation, and translation of ΔABC. Both ΔABC and $\Delta A'B'C'$ are isosceles right triangles.

b. The ratio is $\sqrt{5} \approx 2.236$. This is the modulus of z.

c. Using ordered pairs, $B - A$ is $(4,1) - (2,1) = (2,0)$. Converting to trigonometric form: $(2,0) = 2 + 0i = 2(\cos 0 + i\sin 0)$.

Using ordered pairs, $B' - A'$ is $(-7,9) - (-5,5) = (-2,4)$. Converting to trigonometric form: $(-2,4) = -2 + 4i \approx 4.47(\cos 2.03 + i\sin 2.03)$. The value 2.03 radians is the argument of z.

d. Sample response: $z = 2.236(\cos 2.03 + i\sin 2.03) \approx (-1,2)$. Using C:

$$\begin{aligned} Cz + z &= (3,2)(-1,2) + (-1,2) \\ &= (-7,4) + (-1,2) \\ &= (-8,6) = C' \end{aligned}$$

Module Assessment

1. In electrical circuits, resistors are often connected in series. As a result, the current travels through each resistor separately and in sequence.

Since the current is the same at all points along the circuit, the total measure of the resistance is the sum of the impedance from each resistor.

a. Two resistors with impedance $z_1 = 1 + 2i$ and $z_2 = 3 - 4i$ are connected in series. Find the total impedance, *Z*.

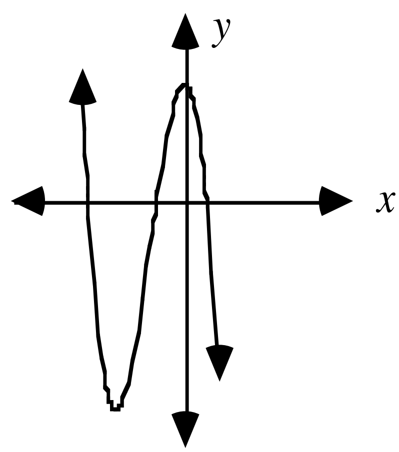
b. The effective current *I* is related to the effective voltage *V* and the impedance *Z* by the following equation:

$$I = \frac{V}{Z}$$

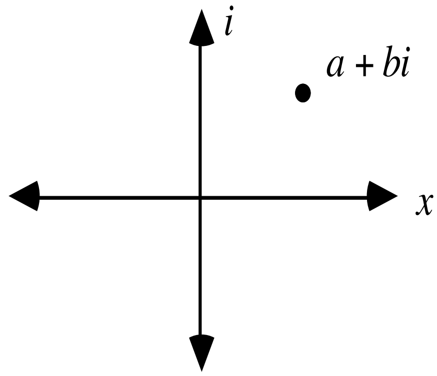
If the effective current is $2 + 3i$, determine the effective voltage needed for the circuit from Part **a**.

c. Find the effective current in the circuit from Part **a** if the effective voltage needed is $6 + 2i$.

2. The figure below shows the graph of a fifth-degree polynomial function *f* with real coefficients. Describe the solutions of the equation $f(x) = 0$ over the set of complex numbers.



3. On a copy of the following graph, plot three points representing three complex numbers that, when raised to the fourth power, each have a value of $(a + bi)^4$.



4. **a.** Write $\sqrt{3} + i$ in trigonometric form.

b. Determine the trigonometric form of $\left(\sqrt{3} + i\right)^4$

c. Write the value of $\left(\sqrt{3} + i\right)^4$ in the form $a + bi$.

5. **a.** Solve $x^5 = 32i$. Write your answers in the form $r(\cos\theta + i\sin\theta)$.

b. Convert your answers to ordered pairs and graph them on a complex plane.

Answers to Module Assessment

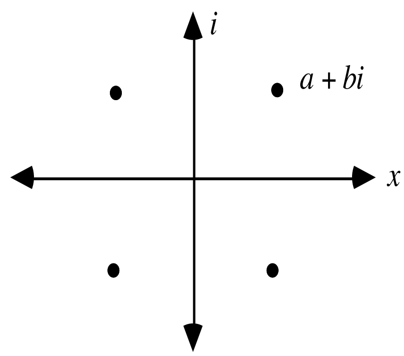
1. **a.** $Z = z_1 + z_2 = (1 + 2i) + (3 - 4i) = 4 - 2i$

b. $V = (2 + 3i)(4 - 2i) = 14 + 8i$

c. $I = \frac{6 + 2i}{4 - 2i} = \left(\frac{6 + 2i}{4 - 2i}\right)\left(\frac{4 + 2i}{4 + 2i}\right) = \frac{20 + 20i}{20} = 1 + i$

2. Sample response: The equation has three real roots, two negative and one positive. It also has two complex conjugate roots.

3. Sample graph:



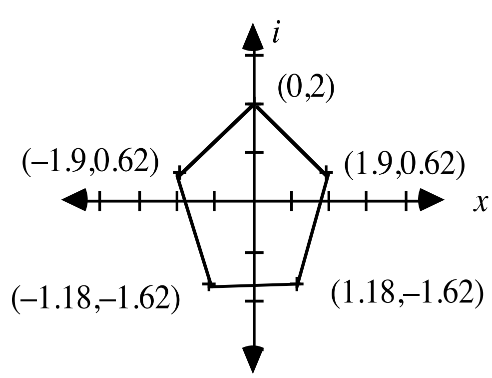
4. **a.** $\sqrt{3} + i = 2\left(\cos(\pi/6) + i\sin(\pi/6)\right)$

b. $\left(\sqrt{3} + i\right)^4 = 16\left(\cos(2\pi/3) + i\sin(2\pi/3)\right)$

c. $\left(\sqrt{3} + i\right)^4 = -8 + 8i\sqrt{3} \approx -8 + 13.86i$

5. **a.** The solutions are $2\left(\cos(\pi/10) + i\sin(\pi/10)\right)$, $2\left(\cos(\pi/2) + i\sin(\pi/2)\right)$, $2\left(\cos(9\pi/10) + i\sin(9\pi/10)\right)$, $2\left(\cos(13\pi/10) + i\sin(13\pi/10)\right)$, and $2\left(\cos(17\pi/10) + i\sin(17\pi/10)\right)$.

b. Sample graph:



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Flashbacks

Activity 1

1.1 Simplify each of the following expressions:

a. $(x + 7) + (-3x - 11)$

b. $(x + 7)(-3x - 11)$

1.2 Determine the roots of the following functions:

a. $f(x) = x^2 - 5$

b. $f(x) = x^2 - 6x - 7$

1.3 Solve each of the following expressions for *b*:

a. $25 = 9 + b^2$

b. $9 = 25 + b^2$

Activity 2

2.1 Identify the reciprocal of each number below:

a. 5

b. $-9/2$

2.2 Solve the following system of equations:

$$\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$$

2.3 Describe the solutions to the equation $0 = x^2 + 4$ using:

a. the set of real numbers

b. the set of complex numbers.

Activity 3

3.1 Multiply $2 + 7i$ by its complex conjugate.

3.2 Write each of the following numbers in the form $a + bi$:

a. $\dfrac{9 - i}{4 + 2i}$

b. i^{23}

c. $i^{28}(5 - i)(2 + 2i)$

Activity 4

4.1 Multiply the complex number $5 - 8i$ by each of the following:

a. i^5

b. i^7

4.2 What 2×2 matrix, when multiplied on the left, produces a rotation of angle θ about the origin?

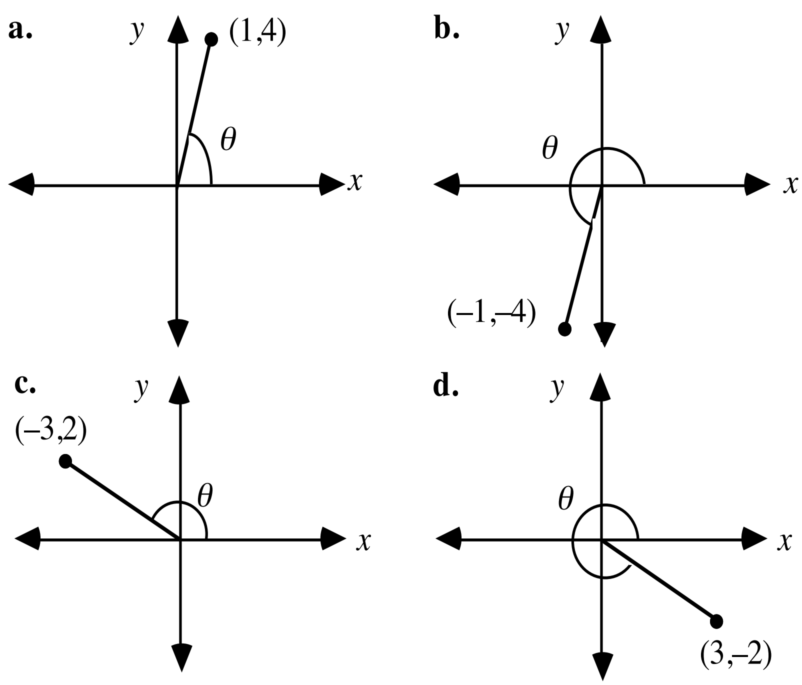
Activity 5

5.1 Determine the following angles to the nearest 0.1 radians:

a. $\tan^{-1}\left(1/\sqrt{2}\right)$

b. $\tan^{-1}\left(-1/\sqrt{2}\right)$

5.2 Determine the measure of θ in each of the following graphs. Round your answers to the nearest 0.1 radians.



Activity 6

6.1 If the coordinates of the vertices of triangle *ABC* are $(0,0)$, $(3,0)$, and $(0,2)$, respectively, determine the coordinates of the image resulting from a counterclockwise rotation of triangle *ABC* by $3\pi/2$ radians with center at $(0,0)$.

6.2 Describe the geometric properties of a regular hexagon.

Answers to Flashbacks

Activity 1

1.1 **a.** $-2x-4$

b. $-3x^2-32x-77$

1.2 **a.** $\sqrt{5}$ and $-\sqrt{5}$

b. 7 and -1

1.3 **a.** $b=4$ or -4

b. There are no real-number solutions.

Activity 2

2.1 **a.** $1/5$

b. $-2/9$

2.2 The solution is $(-1,2)$.

2.3 **a.** There are no solutions to $0=x^2+4$ in the set of real numbers.

b. The solutions are $2i$ and $-2i$.

Activity 3

3.1 $(2+7i)(2-7i)=53$

3.2 **a.** $\left(\dfrac{9-i}{4+2i}\right)\left(\dfrac{4-2i}{4-2i}\right)=\dfrac{34-22i}{20}=\dfrac{17-11i}{10}$

b. $-i$

c. $12+8i$

Activity 4

4.1 **a.** $8+5i$

b. $-8-5i$

4.2 $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Activity 5

5.1 **a.** $\tan^{-1}(1/\sqrt{2}) = 0.6$

b. $\tan^{-1}(-1/\sqrt{2}) = -0.6$

5.2 **a.** $\theta = \tan^{-1}(4/1) \approx 1.3$

b. To determine the measure of angle θ, it is necessary to add π to $\tan^{-1}(-4/-1)$: $\pi + \tan^{-1}(-4/-1) \approx 4.4$.

c. To determine the measure of angle θ, it is necessary to add π to $\tan^{-1}(2/-3)$: $\pi + \tan^{-1}(2/-3) \approx 2.5$.

d. To determine the measure of angle θ, it is necessary to add 2π to $\tan^{-1}(-2/3)$: $2\pi + \tan^{-1}(-2/3) \approx 5.7$.

Activity 6

6.1 The coordinates are $A'(0,0)$, $B'(0,-3)$, and $C'(2,0)$.

6.2 Sample response: A regular hexagon has six congruent sides and the interior angles are congruent. The six interior angles each have a measure of 120°.

What Did You Expect, Big Chi?

How did the U.S. Surgeon General determine that cigarette smoking is hazardous to your health? In this module, you investigate the statistical tests that medical researchers—and others—use to assess the information they collect.

Bill Chalgren • Robbie Korin • Pete Stabio

Teacher Edition

What Did You Expect, Big Chi?

Overview

In this module, students use chi-square statistics (χ^2) and chi-square probability distributions to test hypotheses.

Objectives

In this module, students will:

- calculate chi-square values
- use chi-square values to test observed frequencies versus expected frequencies
- use the chi-square distribution to determine probabilities
- determine and use degrees of freedom when conducting tests on hypotheses
- use chi-square tests to determine whether two variables are independent.

Prerequisites

For this module, students should know:

- how to calculate expected value
- how to determine theoretical and experimental probabilities
- how to formulate null and alternative hypotheses
- how to test a null hypothesis
- the properties of a normal curve
- the law of large numbers
- how to create and interpret stem-and-leaf plots
- how to determine critical regions and critical values.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	3	3	2	1	9

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
fair dice	X	X		
altered dice		X		
cup		X		

Teacher Note

One easy way to alter a die is to file three edges at one vertex.

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
spreadsheet	X	X	X	X
statistics package	X		X	

Teacher Note

The exploration in Activity **3** includes a survey of color preferences. Students should start collecting data at the beginning of the module. See the student edition for a description of the survey.

What Did You Expect, Big Chi?

Introduction
(page 337)

When analyzing data, researchers must decide if sample statistics support or fail to support a null hypothesis. In order to do this, they must determine a sample statistic and select the significance level to use in testing the hypothesis. The introduction presents one of the difficulties that researchers face—by how much should observed frequencies differ from expected frequencies to be considered significant?

Discussion
(page 337)

a. Sample response: $0.40 \bullet 1000 = 400$ smokers.

b.
1. In Table **1**, 650 of the 1000 subjects had been smokers.
2. This is 250 more than the expected number.
3. Sample response: This large difference is probably not due to chance. It is more likely that smoking increases the chances of dying for males between the ages 45–64.

c. Sample response: The probability of winning a lottery is very small, although some people do win. Likewise, there is a very small chance of being struck by lightning, yet some people are struck by lightning. The probability of a meteor reaching the earth's surface is also very small, and yet some do strike the earth.

(page 338)

Activity 1

Students use expected and observed frequencies to calculate chi-square values. They also compare chi-square probability distributions.

Materials List
- none

Technology
- spreadsheet
- statistics package

Teacher Note

In Explorations **1** and **2**, students use simulations to generate sample data. If an appropriate statistics package is not available, you may use a programmable calculator to create a simulation.

For example, the following program was written for the TI-92 calculator. It simulates rolling an *n*-sided die for any given number of rolls. The chi-square values are stored in list2. **Note:** The simulation takes approximately 1 min per side for 100 experiments.

```
:DICE ( )
:Prgm
:ClrIO: DelVar list2
:Input “Number of sides?”, sides
:Input “How many rolls?”, rolls
:rolls/sides→expect
:Input “How many experiments?”, exper
:For m,1,exper
:    seq(0,x,1,sides)→list1
:    For i,1,rolls
:    rand(sides)→outcome
:    list1[outcome]+1→list1[outcome]
:    EndFor
:    sum((list1-expect)^2/expect)→list2[m]
:EndFor
:ClrIO
:Disp list2
:Pause
:FnOff
:PlotOff
:NewPlot 1,4,list2,,,,,,1
:DispG
:EndPrgm
```

As an alternative to these simulations, tables of sample chi-square values are included at the end of this teacher edition. Students may enter each set of data on a spreadsheet to create the corresponding histogram.

Discussion 1

(page 338)

a. Sample response: Because the probability of heads on each flip is 0.5, you would expect 100 heads in 200 flips.

b. Sample response: No. Since the frequency is a random variable, it will not always equal its expected value.

c. Sample response: Yes, it is possible to get 185 heads when tossing a coin 200 times since the occurrence of heads or tails is random.

d. Answers will vary. Some students might think that obtaining 150 or more heads is not reasonable with a fair coin. Others may disagree. Students should realize that no exact point exists at which the results can be said with certainty not to be due to chance.

e. Answers will vary. Some students may suggest finding the sum of the absolute values of the differences between the observed and the expected frequencies for all six possible outcomes. (They should realize that, unless the absolute values of these differences are used, the sum will always be 0.)

Exploration 1

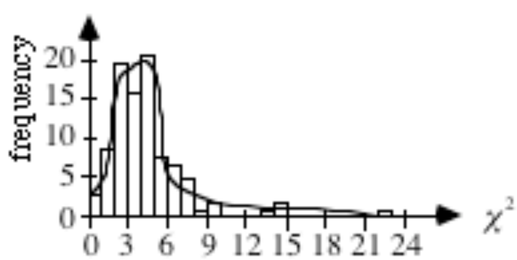
(page 339)

Students investigate chi-square distributions using simulations and histograms.

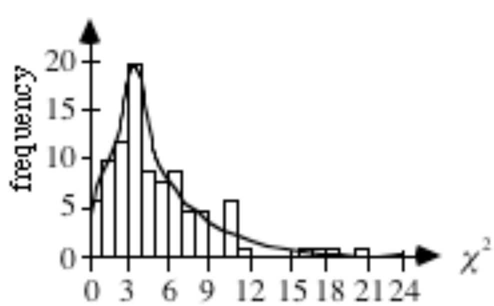
a–c. Sample table:

Outcome	**1**	**2**	**3**	**4**	**5**	**6**	**Sum of Row**
Expected Frequency (E_i)	5	5	5	5	5	5	30
Observed Frequency (O_i)	6	7	4	5	2	6	30
$O_i - E_i$	1	2	–1	0	–3	1	0
$(O_i - E_i)^2$	1	4	1	0	9	1	16
$(O_i - E_i)^2/E_i$	0.2	0.8	0.2	0	1.8	0.2	$\chi^2 = 3.2$

d–e. Sample response: The smooth curve has a "hump" for small values of χ^2, then tapers off as χ^2 increases.



f. Sample response: The smooth curve has a "hump" for small values of χ^2, then tapers off as χ^2 increases.



Teacher Note

The data in Table **4** was generated using a simulation written for the TI-92 calculator. The program simulates flipping *n* coins a given number of times, then stores chi-square values in list3. (This simulation required approximately 90 min for 100 experiments.)

```
:coin( )
:Prgm
:ClrIO :DelVar list1,list2, list3
:Input "How many coins?", coins
:Input "How many times?", times
:Input "How many experiments?", exper
:For m,1,coins + 1, 1
:  times * nCr(coins, m–1) * (0.5^coins)→list2[m]
:EndFor
:For i,1,exper,1
:  0→a
:  seq(0,x,1,coins+1)→list1
:  For j,1,times,1
:     0→c
:     For k,1,coins,1
:         rand(2)→b
:         If b = 1 Then
:             c + 1→c
:         EndIf
:     EndFor
:     For s,1,coins + 1,1
:         If c =(s–1) Then
:             list1[s]+1→list1[s]
:         EndIf
:     EndFor
:  EndFor
:  sum((list1-list2)^2/list2)→list3[i]
:EndFor
:ClrIO
:Display "Chi Square values",list3
:Pause
:FnOff
:PlotOff
:NewPlot 1,4,list3,,,,,,1
:DispG
:EndPrgm
```

Discussion 2

(page 341)

a. Since the sum of both the expected frequencies and the observed frequencies is 30, their difference is 0. This will always be true since both the sum of the expected frequencies and the sum of the observed frequencies equal the number of trials in the experiment.

b. **1.** There are 6 equally likely outcomes.

2. The probability for each outcome is $1/6$.

c. **1.** When counting the number of heads, there are 6 possible outcomes, 0 heads to 5 heads.

2. The probability distribution for the number of heads is given in the table below:

Number of Heads	0	1	2	3	4	5
Probability	$1/32$	$5/32$	$10/32$	$10/32$	$5/32$	$1/32$

d. The shapes of the curves should be similar. You may wish to ask students to compare their sketches with those of others in the class.

e. Sample response: No. The shapes of the two curves are similar and the difference in probabilities appears to have no effect on shape.

Exploration 2

(page 341)

a. The expected frequencies for each outcome are shown in the table below.

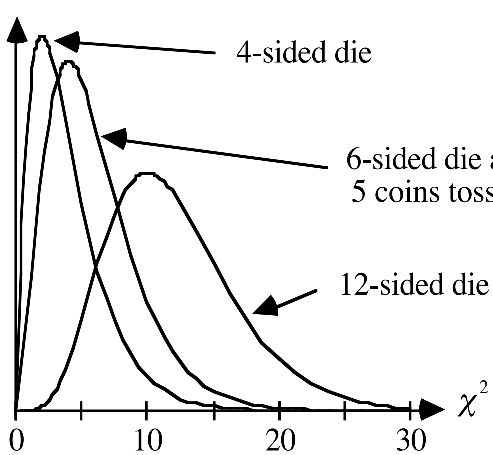
Outcome	1	2	3	4
Probability	$1/4$	$1/4$	$1/4$	$1/4$
Expected Frequency	5	5	5	5

b. See teacher note at the beginning of this activity.

c. See sample graph in Part **e** below.

d. Since the probability of each outcome is $1/12$, the expected frequency for each outcome in an experiment consisting of 60 rolls is 5.

e. Sample response: The hump in the curve for the 4-sided die is the highest, narrowest, and farthest to the left. The hump in the curve for the 12-sided die is the lowest, widest, and farthest to the right. The curves for the coins and the 6-sided die are located between the other two and more closely resemble the curve for the 4-sided die. However, its hump is slightly lower, wider, and farther to the right than that of the 4-sided die.



Discussion 3 (page 341)

a. Sample response: The number of outcomes for tossing the 6-sided die and tossing 5 coins is the same, 6. The number of outcomes for tossing the 4-sided die is 4. For the 12-sided die, the number of outcomes is 12.

b. Sample response: The number of outcomes appears to affect the shape and characteristics of the graphs. The coin toss and the 6-sided die roll had similar curves and the same number of outcomes, while the other two experiments had a different number of outcomes and different curves.

c. Sample response: The events are independent because the probability of any one event is not affected by the previous events.

d. Sample response: Each experiment consists of independent events for which the probability of each outcome is the same every time the experiment is run. Therefore, the number of degrees of freedom for each experiment is 1 less than the number of outcomes. The degrees of freedom for the 4-sided die is 3. For the 6-sided die and the 5 coins, it is 5. For the 12-sided die, it is 11. **Note:** This type of experiment is referred to as a *multinomial* experiment.

e. Sample response: As the number of degrees of freedom increases, the hump in the curve moves closer to the center of the curve. The shape of the curve will approach the normal curve.

Assignment (page 343)

1.1 **a.** Since there are 2 possible outcomes, the degrees of freedom is 1.

b. There are 4 possible outcomes; hearts, clubs, spades, and diamonds. The degrees of freedom is $4 - 1 = 3$.

c. Sample response: Since there are 13 different faces, there would be 13 outcomes. Therefore, the degrees of freedom would be $13 - 1 = 12$.

***1.2** **a.** Sample response: Curve B involves more degrees of freedom because the hump is lower, wider, and farther to the right.

b. Sample response: The value for curve A is approximately 4 and the value for curve B is approximately 10. The chi-square value that divides the area under curve B in half is greater.

c. Sample response: The greater the degrees of freedom, the greater the chi-square value required to divide the area under the curve in half.

d. Answers may vary. The generalization given in Part **c** is true for any percentage of the area.

1.3 **a.** Sample data for a class of 24 students:

	Males	**Females**
Expected	12	12
Observed	8	16
$O - E$	–4	4
$(O - E)^2$	16	16
$(O_i - E_i)^2 / E_i$	1.333	1.333

b. For the sample data, $\chi^2 = 2.67$.

c. The degrees of freedom for this experiment is 1.

1.4 **a.** Students may use a calculator program, spreadsheet, statistics package, or table of random digits to simulate the coin-flipping experiment.

b. For the sample data below, $\chi^2 = 0.72$.

	Heads	**Tails**
Expected	25	25
Observed	28	22
$O - E$	3	–3
$(O - E)^2$	9	9
$(O_i - E_i)^2 / E_i$	0.36	0.36

c–d. Answers will vary. In the sample stem-and-leaf plot below, the χ^2 that best estimates the value that divides the upper 10% of the data from the lower 90% is 3.92.

χ^2	
0.0	00000000
0.0	8888888
0.3	222222
0.7	22
1.2	888
2.0	00000
2.8	88
3.9	2222
5.1	2
6.4	88

3.9 | 2 = 3.92

e. Sample response: The estimate could be improved by increasing the number of times the simulation is repeated.

* * * * *

1.5 **a.** There are 2 degrees of freedom in this situation.

b. In this case, $\chi^2 = 18.017$. Sample table:

	Small	**Medium**	**Large**
Expected	66.67	66.67	66.66
Observed	42	67	91
$O - E$	24.67	–0.33	–24.34
$(O - E)^2$	608.61	0.11	592.44
$(O_i - E_i)^2 / E_i$	9.129	0.001	8.887

1.6 **a.** There is 1 degree of freedom in this situation.

b. In this case, $\chi^2 = 2.88$. Sample table:

	Job	**No Job**
Expected	9.6	22.4
Observed	14	18
$O - E$	4.4	–4.4
$(O - E)^2$	19.36	19.36
$(O_i - E_i)^2 / E_i$	2.02	0.86

* * * * * * * * * *

(page 345)

Activity 2

In this activity, students develop an intuitive understanding of the chi-square statistic. They observe that smaller values indicate good matches between observed and expected frequencies, while larger values indicate poor matches.

The explorations take a classical approach to hypothesis testing: write a null hypothesis, set the significance level, calculate the value of χ^2, compare the calculated value to the appropriate value from a chi-square distribution table, and then either reject or fail to reject the null hypothesis based on this comparison.

Materials List

- altered dice (one per group)
- fair dice (one per group)
- cups (one per group)

Teacher Note

One way to alter a die is to file three edges at one vertex.

Technology

- spreadsheet

Discussion 1

(page 346)

a.

1. Sample response: Using a chi-square distribution table like Table **6**, if the chi-square value is greater than the value from the table (in this case, 9.24), the null hypothesis is rejected.

2. Sample response: Using a graph of a chi-square distribution, if less than 10% of the area under the curve lies to the right of the chi-square value, then the null hypothesis is rejected.

b. Sample response: At the 0.10 level, 10% of the values are to the right of the table value. At the 0.05 level, only 5% of the values are to the right of the table value. Therefore, a true null hypothesis is rejected more often at the 0.10 level.

c.

1. Sample response: 95%. If the null hypothesis is not rejected, the chi-square values obtained are smaller than the table value corresponding to the 0.05 significance level. This corresponds with 95% of the area under the curve.

2. Sample response: 99%. If the null hypothesis is not rejected, the chi-square values obtained are smaller than the table value corresponding to the 0.01 significance level. This corresponds with 99% of the area under the curve.

d. Sample response: If the value of χ^2 falls between the values for two significance levels in the table, the null hypothesis would be rejected at the greater level of significance but not at the lesser level.

e. Sample response: From Table **6**, a value larger than 12.83 with 5 degrees of freedom would occur by chance less than 2.5% of the time.

f. Sample response: The significance level is the probability that a chi-square value greater than the table value can occur by chance when the null hypothesis is true. Therefore, this is also the probability that a valid null hypothesis will be incorrectly rejected.

g.

1. Sample response: On weekends, more people are likely to be on the road, they may be more likely to drive while intoxicated, drive long distances, drive at night, or drive at unsafe speeds.

2. One possible null hypothesis is H_0: P(fatal accident each day) = 1/7. In this case, the alternative hypothesis might be H_a: P(fatal accident on at least one day of the week) ≠ 1/7. **Note:** The desired response for this question is the null hypothesis. Students may require extra guidance or discussion to state good statistical hypotheses.
3. Sample response: No. This would not indicate a cause-and-effect relationship. Using the information from Table **7**, you cannot tell why there are fewer fatalities on Sunday, Monday, Tuesday, and Wednesday, and more fatalities on Friday and Saturday.

Teacher Note

In the following exploration, students collect data simultaneously for a fair die and an altered die. Make sure that they record their data separately for each, and do not lose track of which data goes with each die.

In order to gather enough data, each member of the class should perform the experiment in Part **a** at least once, or each group should repeat the experiment three or four times. Students are asked to use 60 rolls to allow any abnormalities in the outcomes to develop. Because each expected frequency must be 5 or greater to use the chi-square statistic, the experiment cannot include less than 30 rolls.

Exploration

(page 347)

a–b. To ensure randomness of outcomes, students should roll their dice as described in Part **a** of the student edition. The following sample data was obtained using a fair die. The numbers 1, 2, 3, 4, 5, and 6 appeared 13, 11, 8, 8, 8, and 12 times, respectively. The chi-square statistic for this data is 2.6 determined by using the method outlined in Activity **1**.

c. The sample stem-and-leaf plot below shows 22 chi-square values for a fair die. The χ^2 that best estimates the value that divides the upper 10% of the data from the lower 90% is between 8.6 and 9.8. **Note:** 10% of 22 is 2.2. Since 2.2 is not a whole number, the dividing point is between 8.6 and 9.8, the second and third highest numbers.

0	4
1	224
2	00246
3	08
4	02
5	06
6	0
7	26
8	06
9	8
10	6

5 | 6 = 5.6

d. The following sample data was obtained using a die which had three edges of one face filed off. The numbers 1, 2, 3, 4, 5, and 6 appeared 21, 6, 5, 6, 8, and 14 times, respectively. The chi-square statistic for this data is 19.8. Students are to plot their chi-square value for the altered die on the stem-and-leaf plot.

e. Students should not simply reject or fail to reject the null hypothesis. Their summary statements should include the chi-square value obtained, the significance level used, the corresponding table value, a statement of rejection or failure to reject, and an explanation of how they made their decisions.

Using the sample data, the null hypothesis should be rejected. In this case, students should conclude that the altered die is not fair, since a chi-square value of 19.8 is much larger than the majority of the chi-square values for a fair die.

Discussion 2

(page 348)

a. Sample response: If the altered die is still fair, you would expect the expected frequencies and observed frequencies to be close. Small differences result in small values of χ^2.

b. The class will probably obtain large differences in chi-square values. A range of 6–20 would not be uncommon.

c. If all the differences are 0, the value of χ^2 would be 0.

Assignment

(page 348)

2.1 Answers will vary. For the sample data given in the exploration, $\chi^2 = 19.8$. The probability of obtaining a value at least this great is less than 0.5% for an experiment with 5 degrees of freedom.

2.2
- **a.** Sample response: Yes. In determining expected frequencies, we assumed that males and females were equally likely to enroll.
- **b.** Answers will vary. The degrees of freedom in the experiment is 1. For the sample data given in Problem **1.3**, $\chi^2 = 2.66$. The probability of obtaining a value at least this great is approximately 0.10.

2.3
- **a.** As the degrees of freedom increase, the value of χ^2 also increases. As shown in the table of χ^2 values, this relationship appears to hold true for all significance levels.
- **b.** Sample response: As the degrees of freedom increase, the corresponding number of outcomes also increases. So, when the chi-square statistic is calculated, there are more numbers to add, resulting in a greater sum.

2.4 **a.** The total number of fatal accidents is 166. The expected frequency for each day is $(1/7) \bullet 166 \approx 23.7$.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Observed (O)	15	16	15	23	34	38	25
Expected (E)	23.7	23.7	23.7	23.7	23.7	23.7	23.7
$(O-E)^2/E$	3.20	2.50	3.19	0.02	4.48	8.63	0.07

b. $\chi^2 = 22.09$

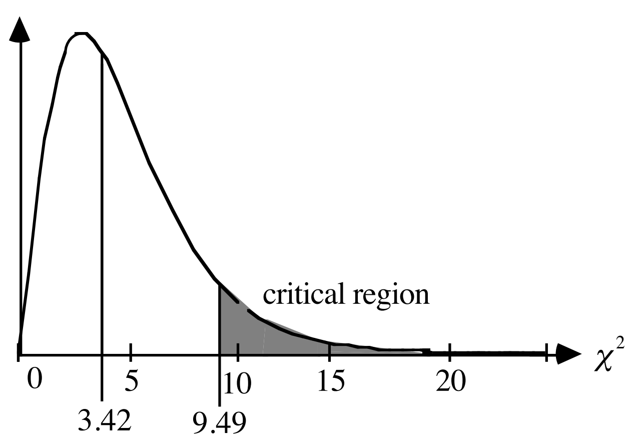
c. Sample response: At a 0.05 significance level, the chi-square value for 6 degrees of freedom is 12.59. This means that the probability of obtaining a chi-square value greater than 12.59 is 0.05. The value of χ^2 from Part **b** is 22.09. Therefore, at a 0.05 significance level, it appears that the numbers of fatal accidents that occur on each day of the week are not due to chance. The null hypothesis should be rejected.

2.5 **a.** One possible null hypothesis is H_0: The percentages of greeting card sales are 25% love, 30% birthday, 20% wedding, 10% sympathy, and 15% other. Another way to state this is H_0: $p_1 = 0.25$, $p_2 = 0.30$, $p_3 = 0.20$, $p_4 = 0.10$, $p_5 = 0.15$. The alternative hypothesis is H_a: At least one of the probabilities in the null hypothesis is incorrect.

b. The expected frequencies can be found by multiplying the probability of each event by the total number of trials. For example, the expected sales of love/friendship cards is $0.25 \bullet 229 = 57.25$. The value of χ^2 is 3.42.

Type	Love	Birthday	Wedding	Sympathy	Other
Expected	57.25	68.7	45.8	22.9	34.35
Sales	54	71	42	19	43
$(O-E)^2/E$	0.18	0.08	0.32	0.66	2.18

c. Sample response: The value for χ^2 found in Part **b** was 3.42. Using a significance level of 0.05 and 4 degrees of freedom, the table value is 9.49, which indicates that there is not enough evidence to reject the null hypothesis.



2.6 **a.** Sample response: As shown in the following table, the value of χ^2 is 3.42.

Outcome	Red	Pink	White
Observed	19	59	22
Expected	25	50	25
$(O - E)^2/E$	1.44	1.62	0.36

At the 0.05 significance level and 2 degrees of freedom, the value in the chi-square distribution table is 5.99. This indicates that the null hypothesis should not be rejected. The ratio of colors is not significantly different from 1:2:1.

b. Sample response: No. When testing a hypothesis, it is possible to make the error of failing to reject a false null hypothesis.

* * * * *

***2.7** Sample response: As shown in the following table, $\chi^2 \approx 5.7$.

Color	Red	Green	Yellow	Brown
Observed	50	54	46	50
Expected	60	60	40	40
$(O - E)^2/E$	1.7	0.6	0.9	2.5

At the 0.05 significance level and 3 degrees of freedom, the value in the chi-square distribution table is 7.81. The null hypothesis should not be rejected. Based on this sample, the company's claim may be accurate.

2.8 Sample response: In this case, the null hypothesis is H_0: The spinner is equally likely to land in any sector. This means that the probability of the spinner landing in any one of the five sectors is $1/5$. The expected frequency for each is $1/5 \bullet 100 = 20$.

Outcome	1	2	3	4	5
Observed	19	20	25	26	10
Expected	20	20	20	20	20
$(O - E)^2/E$	0.05	0.0	1.25	1.8	5

In this case, $\chi^2 = 8.1$. At the 0.10 significance level and 4 degrees of freedom, the value in the chi-square distribution table is 7.78. Therefore, the null hypothesis should be rejected. The spinner does not appear to be fair.

2.9 Answers will vary. Students should formulate a null hypothesis and select an appropriate significance level for their tests.

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Research Project (page 351)

Before allowing students to collect data, you may wish to check their hypotheses to make sure that they are suitable for chi-square testing. Students should use unbiased sampling techniques and display data using appropriate graphs or tables.

(page 351)

Activity 3

In this activity, students use the chi-square statistic to determine whether two variables represent independent events.

Materials List

- none

Technology

- spreadsheet
- statistics package (optional)

Teacher Note

Since the expected frequencies for each outcome must be at least 5, you may wish to ask students to compile their results and analyze a single set of data.

Exploration (page 351)

a. Student predictions will vary.

b. Students may select colors other than those listed, but should maintain a total of 4 different choices. Sample data:

	Blue	**Green**	**Purple**	**Red**	**Total**
Female	6	9	10	9	34
Male	16	6	6	8	36
Total	22	15	16	17	70

c. Students create a blank table with headings like the one in Part **b**.

	Blue	**Green**	**Purple**	**Red**	**Total**
Female					34
Male					36
Total	22	15	16	17	70

d. Answers will vary. The following probabilities were calculated using the sample data from Part **b**.

1. The probability that a person in the sample is female is $34/70$.
2. The probability that a person in the sample prefers blue is $22/70$.
3. Assuming that the two events are independent, the probability that a female in the sample prefers blue is $(34/70) \bullet (22/70) \approx 0.15$.
4. The expected number of females who prefer blue is $(34/70) \bullet (22/70) \bullet 70 \approx 10.7$.

e. The expected frequencies shown in the table below were calculated using the sample data given in Part **b**.

Expected	**Blue**	**Green**	**Purple**	**Red**	**Total**
Female	10.7	7.3	7.8	8.3	34
Male	11.3	7.7	8.2	8.7	36
Total	22	15	16	17	70

f. Once three values are arbitrarily assigned without completely filling any one row or column (as shown below), each of the remaining cells is forced to contain a specific value to reach the required totals. Therefore, there are 3 degrees of freedom.

	Blue	**Green**	**Purple**	**Red**	**Total**
Female			10		34
Male	16			8	36
Total	22	15	16	17	70

g. The degrees of freedom in this experiment are $(2-1) \bullet (4-1) = 3$.

h. Answers will vary. Using the sample data, $\chi^2 = 6.12$.

	Blue	**Green**	**Purple**	**Red**	**Total**
Female $(O-E)^2/E$	2.06	0.40	0.62	0.06	3.14
Male $(O-E)^2/E$	1.95	0.38	0.59	0.06	2.98

Discussion

(page 354)

a. Sample response: Since the hypothesis being tested states that color preference does not depend on gender, the expected frequencies are calculated assuming independence.

b. Sample response: If the distributions of color preferences are the same for both genders, then gender had no bearing on color preference.

c. Students should have found 3 degrees of freedom in both Parts **f** and **g** of the exploration.

d. Sample response: No, this is not true. Statistical dependence does not imply a cause-and-effect relationship.

Assignment

(page 354)

3.1 **a.** You may wish to remind students that the null hypothesis states that the variables are independent. Using the sample data, $\chi^2 = 6.12$. From the table, the chi-square value that corresponds to a 0.05 significance level and 3 degrees of freedom is 7.81. In this case, the null hypothesis is not rejected. Student responses may vary.

b. Sample response: The hypothesis that color preference is independent of gender could not be rejected as a result of the chi-square test conducted on the survey data. The test was conducted at the 0.05 significance level and the results of the test indicate that the differences between the expected values and the observed values are likely to be due to chance.

c. Depending on the data collected, the null hypothesis may or may not be rejected at the different significance levels. In the case of the sample data, the null hypothesis would not be rejected at either the 0.025 or the 0.005 significance levels.

3.2 The table below shows one way to calculate χ^2; the same process can be used in the remainder of the problems in this activity.

	Expected	**Observed**	$(O-E)^2/E$
Nonsmoker Cancer	$\frac{350}{1000} \bullet \frac{192}{1000} \bullet 1000 = 67.2$	56	1.87
Nonsmoker Heart Disease	161.35	153	0.43
Nonsmoker Other	121.45	141	3.15
Smoker Cancer	124.8	136	1.01
Smoker Heart Disease	299.65	308	0.23
Smoker Other	225.55	206	1.69
			$\chi^2 = 8.38$

a. Sample response: The calculated χ^2 of 8.38 is greater than the table value of 5.99 at a significance level of 0.05 with 2 degrees of freedom. Therefore, the null hypothesis is rejected, and cause of death is judged to be dependent on smoking habits.

b. Sample response: Based on the information given in the table and a chi-square test, there is enough evidence to judge that cause of death for males 45–64 is dependent on smoking habits. At the 0.05 significance level with 2 degrees of freedom, the chi-square value of 8.38 is greater than the table value of 5.99. This indicates that the null hypothesis should be rejected.

3.3 **a.** The calculated χ^2 of 10.4 is less than the table value of 12.59 at the 0.05 significance level with 6 degrees of freedom. Therefore, the null hypothesis is not rejected. In the following table, the expected frequencies appear in parentheses. The expected frequency in the first row of the first column is calculated as follows:

$$\frac{44}{200} \bullet \frac{84}{200} \bullet 200 = 18.48$$

The rest of the expected frequencies are calculated in a similar manner.

	Magazine A	Magazine B	No Preference	Total
Northeast	16 (18.48)	23 (20.46)	5 (5.506)	44
South	33 (23.94)	18 (26.505)	6 (6.55)	57
Midwest	15 (17.64)	20 (19.53)	7 (4.83)	42
West	20 (23.94)	32 (26.505)	5 (6.55)	57
Total	84	93	23	200

b. Sample response: Based on the information given in the table and the chi-square statistic, there is not enough evidence to reject the null hypothesis. At the 5% significance level with 6 degrees of freedom, the chi-square statistic 10.4 is less than the critical value of 12.59. Therefore, the null hypothesis that magazine preference is independent of region is not rejected.

***3.4** **a.** The numbers in each cell have been multiplied by 10. Therefore, the sample size has increased by a factor of 10.

b. Sample response: No, the probability that a person will favor magazine A is approximately 0.47 in each case.

c. Sample response: No, the probabilities are the same because the proportions in the two data sets are the same.

d. In this case, the calculated χ^2 of 104 is greater than the table value 12.59. Therefore, the null hypothesis is rejected. In the following table, expected frequencies are shown in parentheses.

The rest of the expected frequencies are calculated in a similar manner.

	Magazine A	Magazine B	No Preference	Total
Northeast	160 (184.8)	230 (204.6)	50 (55.06)	440
South	330 (239.4)	180 (265.05)	60 (65.5)	570
Midwest	150 (176.4)	200 (195.3)	70 (48.3)	420
West	200 (239.4)	320 (265.05)	50 (65.5)	570
Total	840	930	230	2000

e. Sample response: In this case, the larger sample size increased the chi-square value greatly because the observed frequencies differed from the expected frequencies by greater margins. This results in a rejected null hypothesis.

f. Sample response: No. The size of the chi-square statistic depends on the difference between the observed and expected frequencies. If the differences are small, the chi-square statistic will be small regardless of the size of the sample.

* * * * *

3.5 Sample response: In this case, the null hypothesis is H_0: Product quality is independent of the day of the week. The calculated χ^2 is 8.87. The table value at the 0.05 significance level with 4 degrees of freedom is 9.49. This indicates that the null hypothesis should not be rejected at the 0.05 significance level.

No matter what the results of this chi-square test, however, no conclusions should be drawn regarding a cause-and-effect relationship between product quality and day of the week.

The following tables show the calculation of χ^2.

	Mon.	Tues.	Wed.	Thurs.	Fri.	Total
Observed Acceptible	182	210	190	186	175	943
Observed Unacceptable	19	9	15	20	23	85
Total	200	219	205	206	198	1028

Expected Acceptable	183.46	200.89	188.05	188.97	181.63	
Expected Unacceptable	16.54	18.11	16.95	17.03	16.37	
Acceptable $(O-E)^2/E$	0.012	0.413	0.020	0.047	0.242	
Unacceptable $(O-E)^2/E$	0.129	4.581	0.224	0.517	2.684	

3.6 Sample response: The following table shows the probabilities and expected frequencies calculated using the observed totals.

	Enlisted	**Officer**
Female	$0.122 \bullet 0.843 \approx 0.103$ 103	$0.122 \bullet 0.157 \approx 0.019$ 19
Male	$0.878 \bullet 0.843 \approx 0.740$ 740	$0.878 \bullet 0.157 \approx 0.138$ 138

The table below was used to organize the data needed to calculate χ^2.

	Observed	**Expected**	$(O-E)^2/E$
Female Officer	21	19	0.21
Female Enlisted	101	103	0.04
Male Officer	136	138	0.03
Male Enlisted	742	740	0.01

In this case, $\chi^2 = 0.29$. There is 1 degree of freedom. Since the calculated χ^2 value is less than the table value at 0.10 level of significance, the null hypothesis is not rejected.

* * * * * * * * * *

Answers to Summary Assessment (page 357)

1. **a.** The expected probability of each digit is $1/10 = 0.1$.

b. One possible null hypothesis is H_0: In this random number generator, each digit is equally likely to occur.

c. The following sample data were generated using Microsoft Excel.

0	6	9	2	7	0	0	0	0	3
1	7	7	4	7	0	8	4	5	6
0	5	7	9	5	2	6	2	2	6
6	5	4	5	1	6	1	1	3	5
9	4	4	7	4	4	5	4	6	9
7	2	4	4	5	9	4	0	9	3
3	6	8	7	9	1	3	0	6	1
4	0	0	7	3	4	7	6	8	3
7	5	7	6	8	4	3	0	4	8
5	3	0	6	8	9	5	4	3	1

d. A frequency table for the sample data is shown below.

Digit	Frequency
0	13
1	7
2	5
3	10
4	16
5	11
6	12
7	12
8	6
9	8

e. Using the sample data given in Part **c**, $\chi^2 = 10.8$.

Digit	Observed	Expected	$(O - E)^2/E$
0	13	10	0.9
1	7	10	0.9
2	5	10	2.5
3	10	10	0
4	16	10	3.6
5	11	10	0.1
6	12	10	0.4
7	12	10	0.4
8	6	10	1.6
9	8	10	0.4
			$\chi^2 = 10.8$

f. Sample response: At the 0.10 significance level with 9 degrees of freedom, the calculated χ^2 of 10.8 is smaller than the table value of 14.68. Therefore, we fail to reject the null hypothesis.

2. Answers will vary. Students may wish to conduct surveys on social or political issues. When testing the null hypothesis, students should select and defend an appropriate significance level.

Module Assessment

1. The Red Star Shoe Company recommends that its retail outlets stock the following mix of shoes: 25% running shoes, 20% cross-training shoes, 15% hiking boots, 10% cleats, and 30% basketball shoes. The manager of a new store is concerned that these percentages will not be appropriate for the local market.

Imagine that you have been hired to analyze this situation. A survey of 283 local residents reveals the following information: 75 plan to purchase running shoes in the coming year, 47 plan to buy cross-training shoes, 51 plan to buy hiking boots, 17 plan to buy cleats, and 93 plan to buy basketball shoes.

a. State the null hypothesis in this situation.

b. Using the company's suggested mix, determine the expected number of sales for each type of shoe.

c. Calculate χ^2 for this data.

d. How many degrees of freedom are there in this situation? Describe how you determined your response.

e. Test the hypothesis at the 0.05 significance level.

f. Is the company's suggested mix appropriate for the new store? Explain your response.

2. As a result of your report, the Red Star Shoe Company has decided to determine if athletic shoe preference is dependent on geographical region. After selecting a sample from the national population, they classify shoe wearers by preference and by region. The results of the company's survey are shown in the following table.

	Running	**Cross-training**	**Hiking**	**Cleats**	**Basketball**
West	44	10	43	41	62
East	55	16	31	29	69

a. Formulate the null hypothesis for this situation.

b. Calculate χ^2 and test the hypothesis at the 0.05 significance level.

c. Does shoe preference appear to depend on geographic region? Explain how you determined your response.

Answers to Module Assessment

1. **a.** One possible null hypothesis is H_0: The percentages of shoe sales in each category equal the suggested percentages.

b. The expected numbers are 71 running shoes, 57 cross-training shoes, 42 hiking boots, 28 cleats, and 85 basketball shoes.

c. In this case, $\chi^2 = 8.98$.

d. In this experiment, the degrees of freedom is the number of possible outcomes minus 1, or 4.

e. The calculated value for χ^2 is less than the table value of 9.49; therefore, the null hypothesis is not rejected.

f. Sample response: Since the null hypothesis is not rejected, the company may have suggested an appropriate mix of shoes. However, this test does not prove that they did. Failure to reject a null hypothesis does not imply that it is absolutely true.

2. **a.** One possible null hypothesis is H_0: Athletic shoe preference is independent of geographic region.

b. The chi-square statistic is $\chi^2 = 6.98$. In the table below, expected frequencies are shown in parentheses.

	Running	**Cross-training**	**Hiking**	**Cleats**	**Basketball**
West	44 (49.5)	10 (13.0)	43 (37.0)	41 (35.0)	62 (65.5)
East	55 (49.5)	16 (13.0)	31 (37.0)	29 (35.0)	69 (65.5)
West $(O-E)^2/E$	0.61	0.69	0.97	1.03	0.19
East $(O-E)^2/E$	0.61	0.69	0.97	1.03	0.19

c. Since $\chi^2 = 6.98$ is less than the critical value of 9.49 at a significance level of 0.05 with 4 degrees of freedom the null hypothesis, H_0: shoe preference is independent of region, is not rejected.

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Flashbacks

Activity 1

1.1 Consider an experiment in which a fair coin is flipped 10 times and the number of heads recorded. What is the expected number of heads?

1.2 A county has a population of 65,652 registered voters. In the last election, only 20,103 people voted. Determine the probability that a registered voter selected at random from this county voted in the last election.

1.3 Assume that 30% of a population displays a certain characteristic. In a random sample of 1500 individuals from this population, how many would you expect to display this characteristic?

1.4 Evaluate the following sum:

$$\sum_{n=1}^{5}(5n)$$

Activity 2

2.1 Consider a population in which heights are normally distributed.

a. One researcher claims that the mean height is 44 cm. Based on a sample you collected, you feel that the mean height is considerably less than 44 cm. Write null and alternative hypotheses for this situation.

b. To test the null hypothesis, a random sample was taken from the population. The corresponding *z*-score for the sample mean was –1.2. At the 10% significance level, the critical region is $z < -1.28$. Should the null hypothesis be rejected?

c. Sketch the normal curve for the *z*-scores and identify the critical region for the 10% significance level.

d. Based on the results in Part **b**, what can you conclude about the mean height of the population?

2.2 Create a stem-and-leaf plot of the following data: 23, 35, 34, 40, 23, 25,41, 34, 33, 24, 25, 34, 34, 56, 23, 26, 28, 37, 39, 45, and 46.

2.3 How many degrees of freedom are involved in a survey in which 32 students are asked to respond either yes or no to the following question: "Do you plan to attend college?"

Activity 3

3.1 Consider a population that is 51% female. Of this population, 8% are under the age of 20.

a. If one individual is selected at random from this population, what is $P(\text{female})$? $P(\text{under } 20)$?

b. Assuming that the events "selecting a female" and "selecting someone under 20" are independent, calculate $P(\text{female and under } 20)$.

3.2 Consider an experiment with 12 degrees of freedom. In the test of a null hypothesis, a chi-square value of 20.05 was calculated.

a. Would you reject the null hypothesis at the 0.05 significance level?

b. Would you reject the null hypothesis at the 0.10 significance level?

Answers to Flashbacks

Activity 1

1.1 The expected number of heads is 5.

1.2 $20{,}103/65{,}652 \approx 31\%$

1.3 $0.3 \bullet 1500 = 450$ individuals

1.4 The sum can be found as follows:

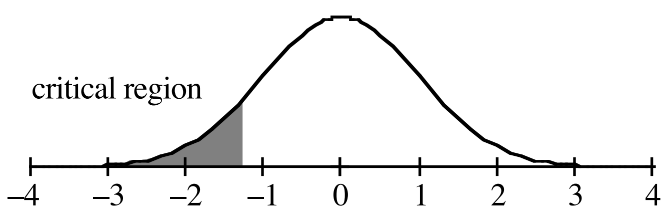
$$\sum_{n=1}^{5}(5n) = 5(1) + 5(2) + 5(3) + 5(4) + 5(5)$$
$$= 75$$

Activity 2

2.1 **a.** H_0: $\mu \geq 44$; H_a: $\mu < 44$

b. In this case, the null hypothesis should not be rejected since the z-score does not fall within the critical region.

c. Sample sketch:



d. Sample response: You can only conclude that there is not enough evidence to reject the null hypothesis.

2.2 Sample stem-and-leaf plot:

2	3 3 3 4 5 5 6 8
3	3 4 4 4 4 5 7 9
4	0 1 5 6
5	6

4 | 2 = 42

2.3 Since there are 2 possible outcomes, there is 1 degree of freedom.

Activity 3

3.1 **a.** $P(\text{female}) = 0.51$; $P(\text{under } 20) = 0.08$

b. $P(\text{female and under } 20) = 0.51 \bullet 0.08 \approx 0.04$

3.2 **a.** At the 0.05 significance level, students should fail to reject the hypothesis since the critical value (21.03) is greater than the obtained value.

b. At the 0.10 significance level, students should reject the hypothesis since the critical value (18.55) is less than the obtained value.

Data Tables
for Explorations 1 and 2 in Activity 1

99 chi-square values for tossing a 6-sided die 30 times

1.2	3.2	2.4	0.4	3.6	1.2	4.4	3.6	6.4	2.0	1.6
2.4	1.6	5.2	7.6	3.2	2.8	3.6	6.4	3.6	1.6	7.2
2.8	4.8	2.8	2.0	4.4	2.4	6.0	3.2	2.8	3.6	3.2
6.4	2.8	5.2	4.0	4.8	1.2	6.0	4.8	5.2	4.8	2.4
4.0	3.2	3.2	3.6	2.8	4.0	6.8	2.0	2.8	4.4	9.2
2.4	5.6	2.8	3.2	4.4	13.2	0.8	4.4	5.2	4.8	1.2
6.0	1.6	7.2	14.8	5.6	4.8	3.2	4.8	2.8	3.6	4.4
1.2	4.8	3.6	2.0	4.0	4.4	7.6	4.4	9.2	2.8	22.0
5.2	4.8	5.2	2.8	7.6	0.4	14.4	3.2	5.2	2.0	8.0

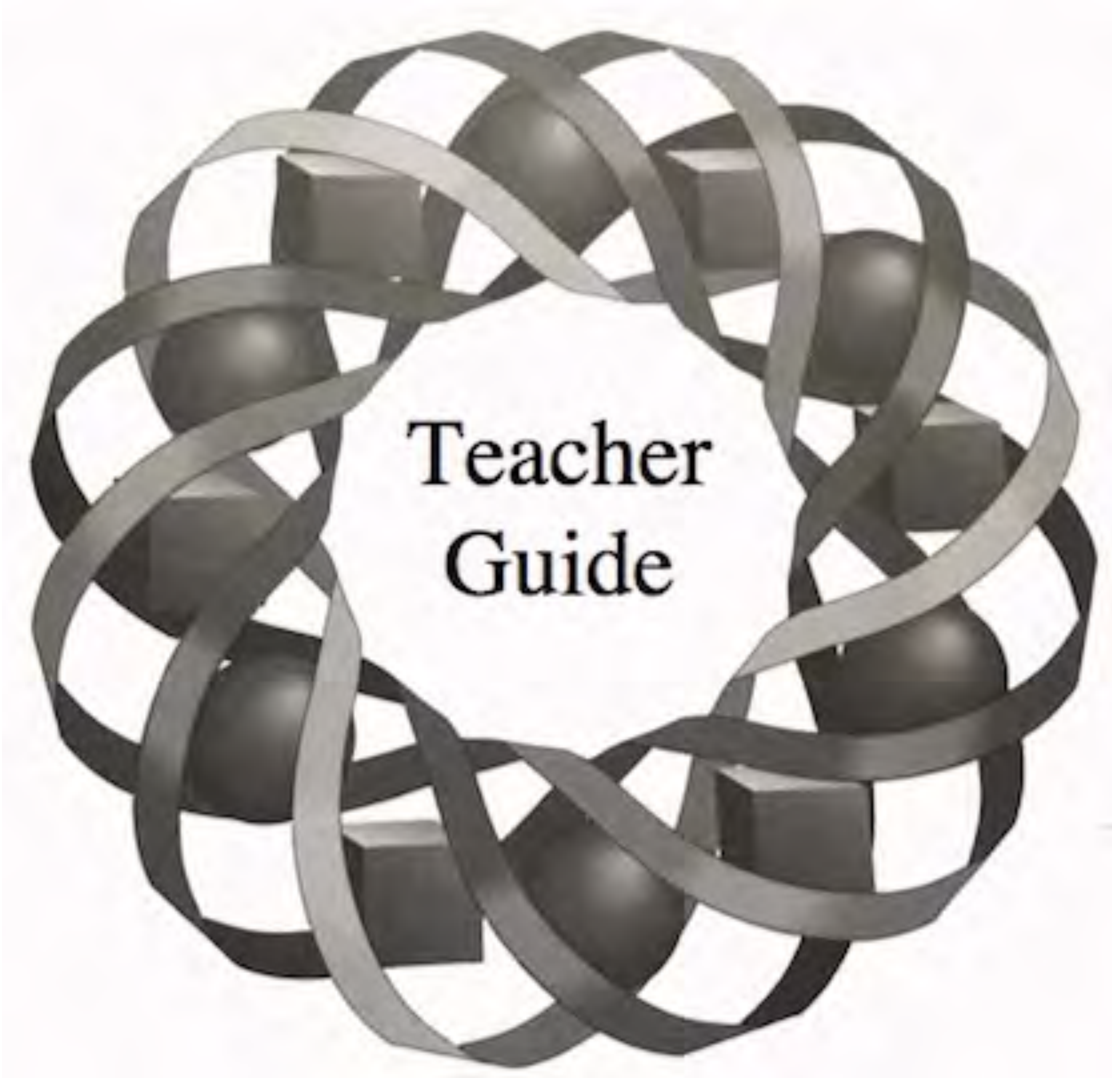
99 chi-square values for tossing a 4-sided die 20 times

1.2	4.0	2.8	2.8	15.2	2.8	3.6	4.4	3.6	4.4	6.0
0.4	5.2	1.2	6.0	2.4	2.8	6.0	2.8	2.0	10.0	1.2
1.2	0.4	2.0	0.4	2.8	0.4	9.2	6.0	0.8	6.0	3.6
4.4	7.6	2.4	0.4	2.8	3.6	2.0	0.4	6.0	7.6	4.0
0.8	0.4	0.4	7.6	5.2	1.2	0.4	1.6	6.0	2.8	4.8
2.8	2.0	2.8	7.6	1.2	0.4	3.6	2.0	4.8	2.8	1.2
5.2	1.2	1.2	4.8	0.4	1.2	0.8	3.6	2.0	0.4	13.6
1.2	2.8	2.8	4.4	2.8	0.4	2.0	5.2	2.8	4.0	1.6
1.2	4.4	1.6	8.4	0.4	2.8	2.8	2.8	0.4	6.0	4.0

99 chi-square values for tossing a 12-sided die 60 times

6.4	5.8	6.0	9.6	20.8	11.4	8.0	9.6	14.0	8.0	9.6
7.8	6.0	8.0	6.0	12.4	22.8	12.8	18.8	10.8	13.2	5.0
14.0	8.4	5.8	13.0	18.4	6.0	14.4	10.8	11.2	11.6	7.6
20.8	9.2	4.0	8.4	16.0	6.8	11.2	11.2	21.2	14.0	9.2
9.6	14.8	15.6	17.2	9.2	10.0	13.2	19.6	8.0	12.0	4.4
9.2	12.0	6.8	24.8	17.0	7.2	3.6	4.8	13.6	19.2	5.2
14.8	6.4	5.2	7.2	14.0	18.0	12.8	35.6	12.0	8.8	13.6
7.2	15.2	4.4	7.6	8.4	11.8	15.2	7.6	6.4	4.0	6.0
6.8	12.4	11.6	10.0	21.2	16.4	12.4	10.0	10.4	12.4	9.6

Slow Down! You're Deriving over the Limit



How does the velocity of a falling object change over time? And exactly how fast is it traveling at any instant during its fall? In this module, you discover how to answer these questions.

Byron Anderson • Ruth Brocklebank • Mike Trudnowski

Teacher Edition

Slow Down! You're Deriving over the Limit

Overview

Students explore rates of change prior to the study of derivatives. **Note:** The principal context in the activities is the physics of motion. You may wish to consult with a physics teacher at your school while preparing for this module.

Objectives

In this module, students will:

- investigate the relationship between average rate of change and the slope of a line
- investigate the relationship between instantaneous rate of change and the slope of a tangent line
- explore graphical interpretations of derivatives
- develop a definition for derivative
- examine the derivatives of specific functions.

Prerequisites

For this module, students should know:

- the definitions of displacement, velocity, and acceleration
- how to determine the slope of a line
- the definition of a tangent line
- how to interpret limit notation.

Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	2	3	2	9

Materials Required

Materials	Activity			
	1	**2**	**3**	**Summary Assessment**
ball	X			
graph paper	X			X
falling-ball template		X		

Teacher Note

Use of the sample data given on the falling-ball template is optional. A blackline master appears at the end of the teacher edition FOR THIS MODULE.

Technology

Software	Activity			
	1	**2**	**3**	**Summary Assessment**
graphing utility	X	X	X	X
symbolic manipulator	X		X	X
science interface device	X			
sonar range finder	X			
spreadsheet		X		X
geometry utility			X	

Slow Down! You're Deriving over the Limit

Introduction

(page 363)

Students discuss the velocity of a freely falling object. **Note:** As in previous modules, the terms *speed* and *velocity* are not used interchangeably in this context. Speed is a scalar and has no direction. Velocity is a vector, with both magnitude and direction.

Discussion

(page 363)

a. Students should recall that the graph is not linear. Sample response: The graph is a curve shaped like part of a parabola.

b. **1.** The average velocity can be calculated as follows:

$$\frac{95.1 - 100}{1 - 0} = -4.9 \text{ m/sec}$$

2. Sample response: In this case, the negative sign on the velocity indicates that the ball's height above the ground is decreasing.

c. Sample response: No. The acceleration due to gravity is 9.8 m/sec^2 in the direction toward Earth's center. Since the ball's speed will increase over time, it will not travel the same distance over equal time intervals.

(page 363)

Activity 1

In this activity, students use a quadratic equation to model the motion of a freely falling object.

Materials List

- ball (one per group)

Technology

- graphing utility
- symbolic manipulator
- science interface device
- sonar range finder

Teacher Note

In the following exploration, students use a range finder to collect data on the height of a falling ball over time. To accomplish this task, the range finder can be placed in a protective cage and a soft rubber ball dropped onto it. As an alternative, the range finder can be placed above the falling ball and the data transformed appropriately.

You may wish to conduct this exploration as a class demonstration.

Exploration

(page 363)

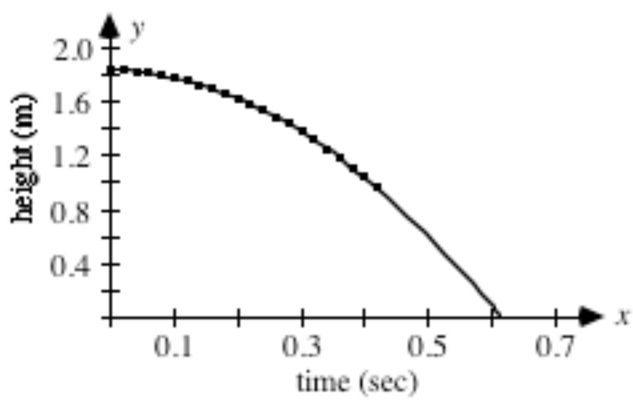
a. Sample data:

Time (sec)	Height (m)	Time (sec)	Height (m)
0.00	1.822	0.22	1.566
0.02	1.817	0.24	1.520
0.04	1.809	0.26	1.471
0.06	1.798	0.28	1.417
0.08	1.781	0.30	1.360
0.10	1.762	0.32	1.300
0.12	1.739	0.34	1.236
0.14	1.711	0.36	1.168
0.16	1.682	0.38	1.097
0.18	1.646	0.40	1.023
0.20	1.608	0.42	0.942

b. See sample graph given in Part **d**.

c. Students should realize that a quadratic polynomial models this situation. Using the quadratic regression on a graphing utility, one equation for the sample data is $y = -4.65x^2 - 0.14x + 1.82$.

d. Sample graph:



e. Students should set their model equation equal to 0 and find the positive root. Using the sample equation $y = -4.65x^2 - 0.14x + 1.82$, the object reaches a height of 0 m after approximately 0.61 sec.

f. Sample response: Since the ball is in the air for 0.61 sec and its initial velocity is 0, its velocity just before it hits the ground should be:

$$-9.8 \ \frac{\text{m}}{\text{sec}^2} \bullet 0.61 \text{ sec} \approx -6.0 \ \text{m/sec}$$

Discussion (page 364)

a. Sample response: The graph representing the height of the ball at any time during its fall is curved and appears to be part of a parabola.

b. Sample response: The slope of the line containing any two points in the scatterplot represents the change in position divided by the change in time. This is the average velocity over the interval defined by the two points.

c.
1. The equation should be a second-degree polynomial.
2. Sample response: The graph of the equation fits the scatterplot very well. The equation appears to be a good model.

d.
1. In the sample data given in the exploration, the initial height was 1.82 m. In this case, an appropriate model would be:

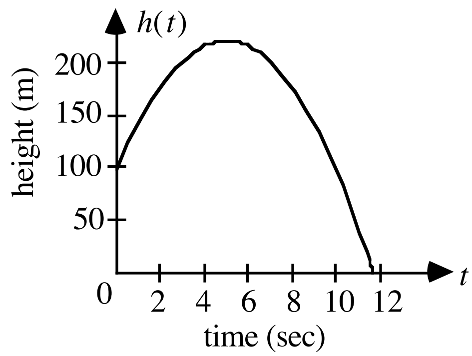
   $$h(t) = -0.5(9.8)t^2 + 0t + 1.82 = -4.9t^2 + 1.82$$

2. Sample response: The coefficients of x^2 are reasonably close and the constant terms are equal. There is not much difference in the two equations.

e.
1. $[0,1)$
2. $(1,2]$
3. At the instant when the ball reaches its highest point, its velocity is 0.

Assignment (page 365)

1.1 **a.** The following sample graph shows the function $h(t) = -4.9t^2 + 49t + 98$.



b. The velocity is positive for $[0, 5)$.

c. Sample response: The velocity is negative for $(5, 11.7]$, where 11.7 sec is an approximation of the time when the object strikes the ground. **Note:** Some students may argue that the velocity is negative for the interval (5,10], since it will land back on the tower after 10 sec.

d. **1.** The greatest height reached is approximately 220.5 m.

2. It takes the object 5 sec to reach its maximum height.

3. Sample response: The velocity is 0 at this point, because gravity has overcome the initial upward velocity.

e. It will take the ball 10 sec to return to its original height.

1.2 **a.** Since the ball's acceleration is -9.8 m/sec^2, its initial height can found by solving the following equation:

$$\begin{aligned} 0 &= -4.9(3)^2 + h_0 \\ 44.1 \text{ m} &= h_0 \end{aligned}$$

b. Sample response: $h(t) = -4.9t^2 + 44.1$.

1.3 **a.** Sample response: The object's height over time can be modeled by $h(t) = -4.9t^2 + 34.3t$. Solving the equation for t when $h(t) = 0$ results in $t \approx 7$ sec.

Note: Some students may observe that the object rises until gravity overcomes the initial upward velocity at time $t \approx 34.3/9.8 = 3.5$ sec. It then falls for the same amount of time. Therefore, the object stays in the air for 7 sec.

b. Sample response: Since the object was under the influence of gravity for 3.5 sec after reaching its maximum height, its velocity just before hitting the ground is
$-9.8 \text{ m/sec}^2 \bullet 3.5 \text{ sec} = -34.3 \text{ m/sec}$.

c. Sample response: The initial velocity can be found by substituting $h_0 = 0$, $h(t) = 0$, and $t = 6$ into the equation $h(t) = -4.9t^2 + v_0 t + h_0$, and solving for v_0. The result is 29.4 m/sec.

1.4 **a.** $h(t) = -4.9t^2 + 49t + 0 = -4.9t^2 + 49t$

b. Sample response: Solving $0 = -4.9t^2 + 49t$ for t results in $t = 0$ and $t = 10$. This implies that the flight lasted 10 sec.

c. The height after each second is given in the following table:

Time (sec)	Height (m)
1	44.1
2	78.4
3	102.9
4	117.6
5	122.5
6	117.6
7	102.9
8	78.4
9	44.1
10	0

d. Sample table:

Interval (sec)	Average Velocity (m/sec)
$[0, 2)$	$(78.4 - 0)/2 = 39.2$
$[2, 4)$	$(117.6 - 78.4)/2 = 19.6$
$[4, 6)$	$(117.6 - 117.6)/2 = 0$
$[6, 8)$	$(78.4 - 117.6)/2 = -19.6$
$[8, 10)$	$(0 - 78.4)/2 = -39.2$

e. Sample response: The ball's average velocity decreases constantly over time. The magnitudes of the velocities are symmetrical with respect to the interval which contains the maximum height reached.

* * * * *

1.5 **a.** Sample response: Velocity is change in displacement divided by the change in time. Since both the change in displacement and the change in time are positive over the interval $[0, 2)$, the velocity is positive. The velocity is negative for the interval $(2, 8]$.

b. average velocity $\approx 3\text{ m}/2\text{ sec} = 1.5\text{ m}/\text{sec}$

c. Sample response: The toy changed direction at 2 sec. This is indicated by the point where the displacement began to decrease.

d. Sample response: After 5 sec, the toy is back at its starting point. This is indicated by a displacement of 0, which occurs where the curve intersects the *x*-axis.

e. Sample response: The velocity has the greatest magnitude in the interval [0,1]. This is indicated by the steepness of the curve.

f. Sample response: The toy's velocity was greatest during the first second and then decreased, to $0\text{ m}/\text{sec}$ after 2 sec. The velocity then became negative and remained nearly constant over the next 5 sec. The toy returned to the starting point after 5 sec, continued to move past it, although more slowly, and then stopped after about 8 sec.

1.6 **a.** $h(t) = -4.9t^2 + 0t + 381 = -4.9t^2 + 381$

b. Solving $0 = -4.9t^2 + 381$ for *t* results in $t \approx 8.82\text{ sec}$.

c. Sample response: Since gravity is acting on the object for approximately 8.82 sec, its velocity should be about $-9.8 \bullet 8.82 \approx -86.4\text{ m}/\text{sec}$.

d. Sample response: Objects dropped from the top of the Empire State Building pose a serious threat to pedestrians on the sidewalks below, since the objects will be traveling very fast when they strike the ground. A speed of 86.4 m/sec is approximately 310 km/hr.

1.7 **a.** 24.5 m

b. $19.6\text{ m}/\text{sec}$

c. Sample response: Solving the equation $0 = 24.5 + 19.6t - 4.9t^2$ for *t* gives a positive root of 5. This means that the ball was in the air for 5 sec.

d. The ball reaches its maximum height at the time which corresponds to half the distance between the two roots from Part **c**, or $(-1+5)/2 = 2\text{ sec}$. Students can determine the maximum height by substituting this value into the function $h(t) = 24.5 + 19.6t - 4.9t^2$ and evaluating $h(t)$. This results in a maximum height of 44.1 m.

e. Sample response: The ball will fall under the influence of gravity for 3 sec. Therefore, its velocity is $3 \bullet -9.8 = -29.4\text{ m}/\text{sec}$.

* * * * * * * * * *

(page 367)

Activity 2

Students interpret the slope of a line passing through two of the data points from the experiment in Activity **1** as the average velocity over a particular time interval. The slopes of secant lines are introduced as a method for approximating rates of change.

Materials List

- falling-ball template (optional: a blackline master appears at the end of the teacher edition for this module)

Teacher Note

As an alternative to using student data in the exploration, you may wish to distribute copies of the sample data given on the falling ball template. (This is the same data shown in Figure **2** and Table **2** in the student edition.)

Technology

- spreadsheet
- graphing utility

Exploration

(page 367)

Students choose a data point near the middle of the data set from the exploration in Activity **1** and use the slopes of lines passing through this point and various other points to approximate the instantaneous velocity of the ball at the chosen point.

a–b. Student graphs should resemble the one shown in Figure **2**. For the sample data,

$$m = \frac{1.52 - 1.80}{0.24 - 0.06} \approx -1.56$$

This slope represents the ball's average velocity in meters per second during the interval $[0.06, 0.24]$.

c. Answers will vary. For the sample data in Figure **2**,

$$m = \frac{0.94 - 1.52}{0.42 - 0.24} \approx -3.22$$

This slope represents the ball's average velocity in meters per second during the interval $[0.24, 0.42]$.

d. Sample response: The slopes of the two lines describe average velocities for two intervals that include A. The instantaneous velocity of the ball at A is somewhere between these values.

e–f. Sample table:

Value of *d*	**(*x* – *d*)**	***y*-coord. of *P***	**Slope of $\overleftrightarrow{PA}$**	**(*x* + *d*)**	***y*-coord. of *Q***	**Slope of $\overleftrightarrow{AQ}$**
0.18	0.06	1.798	–1.54	0.42	0.942	–3.21
0.16	0.08	1.781	–1.63	0.40	1.023	–3.11
0.14	0.10	1.762	–1.73	0.38	1.097	–3.02
0.12	0.12	1.739	–1.83	0.36	1.168	–2.93
0.10	0.14	1.711	–1.91	0.34	1.236	–2.84
0.08	0.16	1.682	–2.03	0.32	1.3	–2.75
0.06	0.18	1.646	–2.10	0.30	1.36	–2.67
0.04	0.20	1.608	–2.20	0.28	1.417	–2.58
0.02	0.22	1.566	–2.30	0.26	1.471	–2.45

g. Using the sample data, the instantaneous velocity at *A* appears to be between –2.30 m/sec and –2.45 m/sec.

Discussion

(page 368)

a. Sample response: This means that for this 0.36-sec interval, the ball's average velocity was –2.39 m/sec.

b. Answers will vary. Sample response: By looking at the values in Table **1**, the slopes of the lines appear to be converging on a value somewhere between –2.30 m/sec and –2.45 m/sec.

c.
1. Sample response: As the coordinates of *B* approach the coordinates of *A*, the slope of the secant line that passes through these points approaches the velocity of the ball at 0.24 sec.
2. Sample response: Since the approximation improves as *B* approaches *A*, the smaller the value of *d*, the better the approximation.
3. Sample response: The slope of the tangent line to the curve at *A* represents the instantaneous velocity of the falling ball at 0.24 sec.

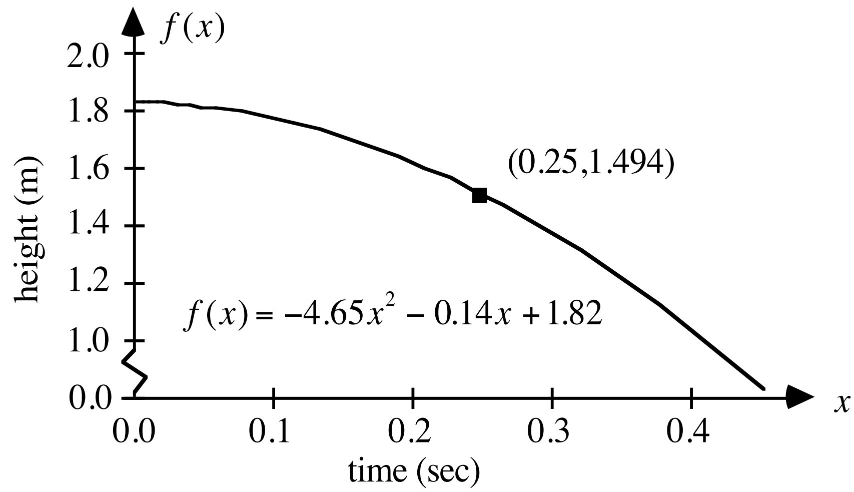
d. Sample response: As the value of *d* decreases, the average velocity gets closer and closer to the instantaneous velocity at *A*. This value is somewhere between –2.30 m/sec and –2.45 m/sec.

e. Sample response: A better approximation could be found if you knew the slopes of secant lines that pass through *A* and points closer to *A* than the ones shown in the table.

Assignment

(page 370)

2.1 **a–b.** Answers will vary. In the following sample graph, the regression equation is $f(x) = -4.65x^2 - 0.14x + 1.82$ and the coordinates of the selected point are $(0.25, 1.494)$.



c. Using the sample spreadsheet below, the values for slope appear to be approaching –2.5 m/sec.

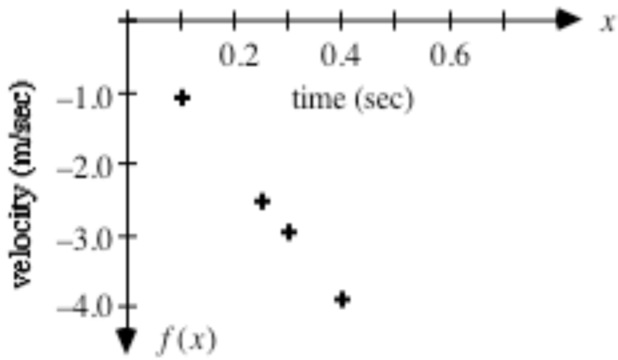
d	$(x-d)$	$f(x-d)$	Slope	$(x+d)$	$f(x+d)$	Slope
0.03	0.22	1.564	–2.333	0.28	1.416	–2.605
0.02	0.23	1.542	–2.400	0.27	1.443	–2.550
0.01	0.24	1.519	–2.500	0.26	1.469	–2.500

d. Sample response: The ball's instantaneous velocity at 0.25 sec is the slope of the tangent to the curve at the point (0.25, 1.494).

2.2 **a.** Answers will vary. The table below shows the approximate slopes of the tangents at three other points.

Time (sec)	Velocity (m/sec)
0.10	–1.07
0.25	–2.50
0.30	–2.93
0.40	–3.86

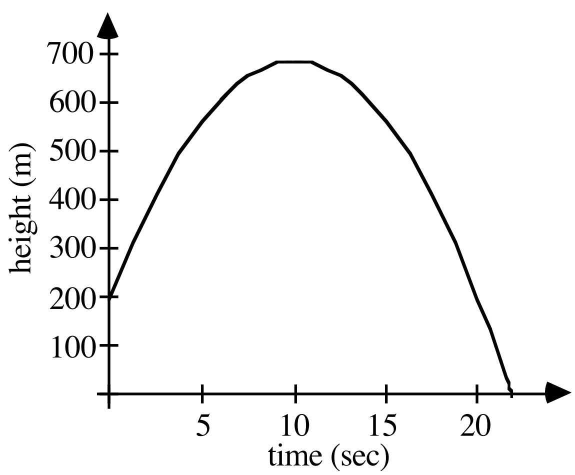
b. Sample scatterplot:



c. Sample response: The set of points appears to have a linear relationship. One possible model is the linear regression $y = -9.3x - 0.15$.

d. Sample response: The change in velocity with respect to time appears to be constant. This makes sense, since the acceleration due to gravity is constant.

2.3 **a.** The height with respect to time can be modeled by the function $h(t) = -4.9t^2 + 98t + 196$. Sample graph:



b. **1.** $\dfrac{h(2) - h(0)}{2 - 0} = \dfrac{372.4 - 196}{2} = 88.2 \text{ m/sec}$

2. $\dfrac{h(20) - h(18)}{20 - 18} = \dfrac{196 - 372.4}{2} = -88.2 \text{ m/sec}$

3. $\dfrac{h(18) - h(2)}{18 - 2} = \dfrac{372.4 - 372.4}{16} = 0 \text{ m/sec}$

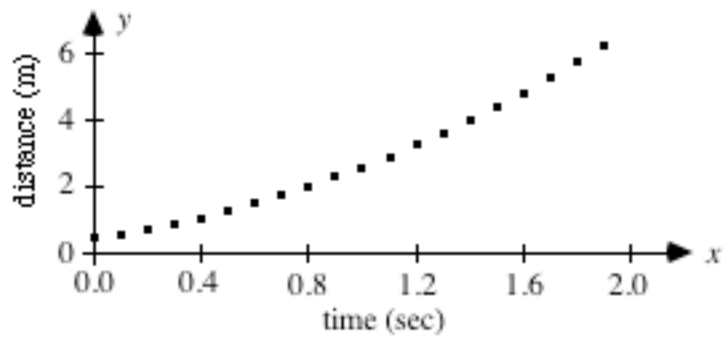
Note: You may wish to ask students to discuss how a moving object can have an average velocity of 0.

c. Sample response: Average velocity takes the direction of travel into account. In this interval, the object travels an equal distance upward and downward. Using the total distance traveled gives the average speed of the object, not the average velocity.

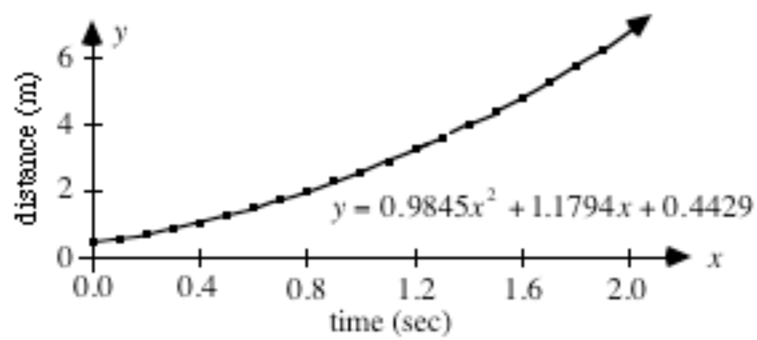
d. Sample response: Using the slopes of secant lines that pass through (16,509.6) and the two points 0.01 sec on either side, the average velocity is somewhere between –58.751 and –58.849 m/sec. Using the average of these two values, it can be approximated as $(-58.751 + (-58.849))/2 = -58.8$ m/sec.

2.4 **a.** Sample response: The change in distance from the range finder increases with each unit of time. Since the ball is under the influence of gravity, the graph should be quadratic.

b. Sample scatterplot:



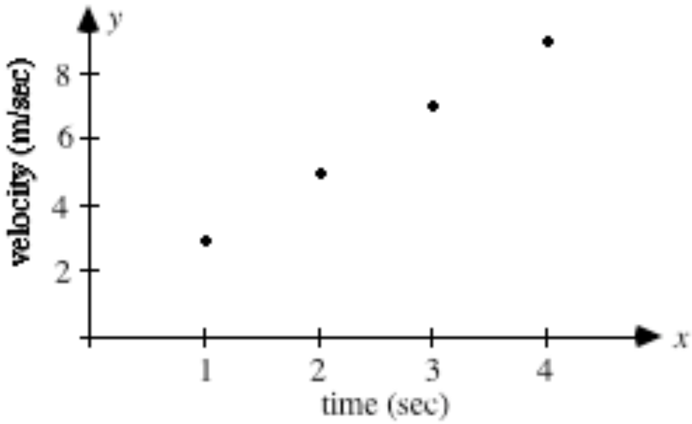
c. A quadratic regression model is $y = 0.9845x^2 + 1.1794x + 0.4429$. Sample graph:



d. The following table shows the approximate distances predicted by the sample model given in Part **c**, as well as some estimated velocities.

Time (sec)	Distance (m)	Velocity (m/sec)
1	2.61	3
2	6.74	5
3	12.84	7
4	20.91	9

e. Sample scatterplot:



f. The scatterplot appears to be linear. An appropriate model for the sample data is $y = 2x + 1$.

g. Sample response: Since the graph of velocity versus time is linear and increasing, this shows that the acceleration is constant and positive. This makes sense, since the only force acting on the ball is gravity and displacement is being measured from the top of the ramp.

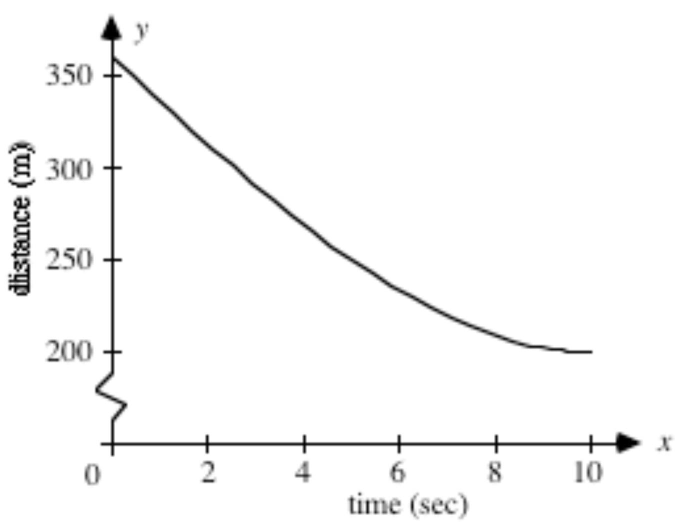
2.5 **a.** The distance is $\sqrt{200^2 + 300^2} \approx 361$ m.

b. The train will be closest to the observer in 10 sec, since it will have traveled 300 m along the track, placing it 200 m away.

c. $f(x) = \sqrt{200^2 + (300 - 30x)^2}$

d. Sample response: The rate is not constant. As the train approaches the point on the tracks where it is closest to the observer, the rate of change in the distance between the observer and the train decreases.

e. The following sample graph confirms that the rate at which the distance is changing is not constant:



f. Sample response: As shown in the following table, the rate of change at 5 sec is between –18.012 and –17.988 m/sec. The average of these two values is –18 m/sec.

d	$(x-d)$	$f(x-d)$	Slope	$(x+d)$	$f(x+d)$	Slope
0.01	4.99	250.18	–18.012	5.01	249.82	–17.988

At 10 sec, the rate appears to be very close to 0 m/sec.

d	$(x-d)$	$f(x-d)$	Slope	$(x+d)$	$f(x+d)$	Slope
0.01	9.99	200.00	0	10.01	200.00	0

* * * * * * * * * *

(page 373)

Activity 3

In this activity, students investigate the derivative as the limiting value of the slope of a secant line as it approaches the tangent line.

Materials List

- none

Technology

- graphing utility
- geometry utility
- symbolic manipulator

Exploration

(page 373)

Students use a geometry utility to approximate instantaneous rates of change on a portion of a circle.

a–d. Students create the construction shown in Figure **4** on a geometry utility.

e. As the points move closer together, the slopes of the secant lines become approximately equal.

f. Student responses will vary. The following sample table shows the slope of a tangent line at three different points.

x-coordinate of Point	Slope of Tangent
1.5	–0.31
2.5	–0.58
3.5	–0.98

Discussion (page 374)

a. Sample response: In this case, the slope of the secant line would equal the average velocity between two points on the graph of $f(x)$.

b. Sample response: The derivative $h'(t)$ represents the object's velocity with respect to time.

c. Sample response: The derivative describes the slope of the curve at any point. Since this function is a line, its slope is always the same. Therefore, the derivative of a line is equal to the slope of the line.

d. Sample response: The graph of the function is a parabola that is concave down. The object will be at its highest point at the vertex of the parabola. The tangent line to the vertex is a horizontal line with a slope of 0. Therefore, the value of the derivative at this point is 0. This makes sense, since the instantaneous velocity is 0 at the highest point.

Assignment (page 377)

3.1 **a.** Sample response:

$$\begin{aligned}\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) &= \lim_{h\to 0}\left(\frac{3(x+h)-5-(3x-5)}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{3x+3h-5-3x+5}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{3h}{h}\right)\\ &= \lim_{h\to 0} 3\\ &= 3\end{aligned}$$

b. Sample response: The derivative of a linear function is the slope of the line.

c. Sample response: The degree of the derivative is 1 less than the degree of the original function.

3.2 **a.** The derivative can be found as follows:

$$h'(t) = \lim_{a \to 0}\left(\frac{-4.65(t+a)^2 - 0.14(t+a) + 1.82 - \left(-4.65t^2 - 0.14t + 1.82\right)}{a}\right)$$

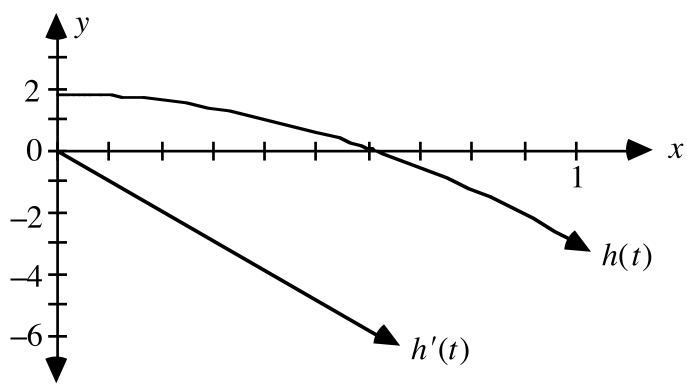
$$= \lim_{a \to 0}\left(\frac{-4.65t^2 - 9.3at - 4.65a^2 - 0.14t - 0.14a + 1.82 + 4.65t^2 + 0.14t - 1.82}{a}\right)$$

$$= \lim_{a \to 0}\left(\frac{-9.3ta - 4.65a^2 - 0.14a}{a}\right)$$

$$= \lim_{a \to 0}\left(-9.3t - 4.65a - 0.14\right)$$

$$= -9.3t - 0.14$$

b. Sample graph:



c. Sample response: The value of $h(t)$ indicates the height at time t, while the value of $h'(t)$ indicates the instantaneous velocity at t. For example, the coordinates of the point (0.4,1.07) on $h(t)$ show that the ball is approximately 1.07 m from the ground at 0.4 sec. The corresponding coordinates on $h'(t)$ show the ball has a velocity of approximately –3.86 m/sec at 0.4 sec.

d. Sample response: The degree of the derivative is 1 less than the degree of the original function.

3.3 **a.** Since the slope of the function is 0 when the derivative is 0, the slope is 0 when $0 = 2x - 7$, or when $x = 3.5$.

b. The function is increasing over the interval $(3.5, \infty)$ for x:

c. The function is decreasing over the interval $(-\infty, 3.5)$ for x:

3.4 **a.** Sample response:

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^2 + 2hx + h^2 - x^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2hx + h^2}{h}\right)$$

$$= \lim_{h\to 0}(2x + h)$$

$$= 2x$$

b. Students confirm their responses to Part **a** using a symbolic manipulator.

c. The slope of the graph at $x = -3$ is –6; the slope of the graph at $x = 15$ is 30.

d. Sample response: The slope is 0 when $2x = 0$ or when $x = 0$. Since the graph of $y = x^2$ is a parabola, this is its vertex. Since the graph opens upwards, this is the minimum point of the graph.

3.5 **a.** Sample response:

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{2(x+h)^3 - 3(x+h)^2 + 4(x+h) + 2 - (2x^3 - 3x^2 + 4x + 2)}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 3(x^2 + 2xh + h^2) + 4(x+h) + 2 - (2x^3 - 3x^2 + 4x + 2)}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 3x^2 - 6xh - 3h^2) + 4x + 4h + 2 - 2x^3 + 3x^2 - 4x - 2}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{6x^2h + 6xh^2 + 2h^3 - 6xh - 3h^2 + 4h}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{h(6x^2 + 6xh + 2h^2 - 6x - 3h + 4)}{h}\right) =$$

$$\lim_{h\to 0}(6x^2 - 6xh + 2h^2 - 6x - 3h + 4) = 6x^2 - 6x + 4$$

b. The derivative is $6x^2 - 6x + 4$.

c. The degree of the derivative is 1 less than the degree of the original function.

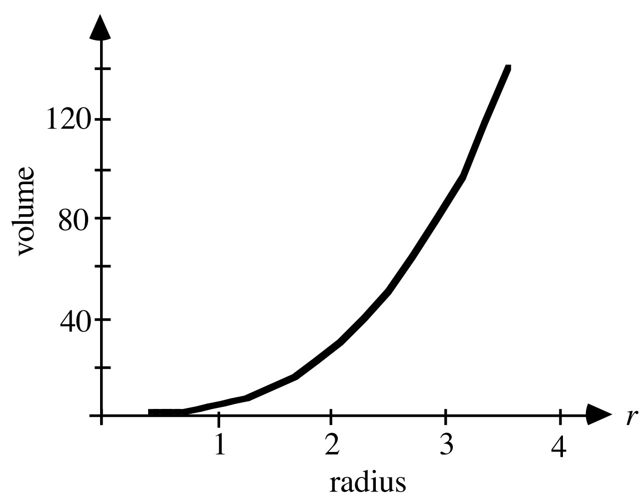
3.6 **a.** $f'(x) = \dfrac{-x}{\sqrt{-\left(x^2 - 25\right)}}$

b. $f'(1.5) \approx -0.3145$, $f'(2.5) \approx -0.5774$, and $f'(3.5) \approx -0.9802$.

c. Sample response: The slopes of the lines containing the radii to the three points in Part **b** are approximately 3.18, 1.732, and 1.02, respectively. The corresponding derivatives are approximately –0.3145, –0.5774, and –0.9802. The respective products are –1.00011, –1.0000, and –0.999804, each of which is close to –1.

3.7 **a.** Sample response: $f(r) = 2\pi r^3$.

b. Sample graph:



b. Sample response: The derivative represents the rate of change in the volume of the cylinder with respect to its radius.

3.8 **a.** Using the definition of a derivative,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left(m(x+h) + b\right) - (mx + b)}{h} \\ &= \lim_{h \to 0} \left(\frac{mh}{h}\right) \\ &= m \end{aligned}$$

b. Sample response: The derivative of a linear equation in the form $f(x) = mx + b$ is m, the slope of the line.

* * * * *

3.9 **a.** Students may draw a secant line through the points (15,150) and (35,350). The slope of the secant line is:

$$\frac{350-150}{35-15}=10$$

This value represents an average rate of change of 10 fruit flies/day.

b. Students should draw a line tangent to the curve through the point (25,300) and approximate its slope. The instantaneous rate of change on day 25 is approximately 10 fruit flies/day.

3.10 **a.** **1.** A tangent to the graph has positive slope in the intervals $(-\infty,-4)$, $(-1,2)$, and $(3,\infty)$.

2. A tangent to the graph has negative slope in the intervals $(-4,-1)$ and $(2,3)$.

b. Sample response: The derivative is 0 at the points where the slope of the tangent to the curve changes from negative to positive or from positive to negative. This occurs when x is –4, –1, 2, and 3.

c. The derivative is $f'(x)=x^4-15x^2+10x+24$. Some students may evaluate the derivative when x is –4, –1, 2, and 3. The resulting values are 0 in each case.

Others may graph the derivative on a graphing utility. This graph has x-intercepts at –4, –1, 2, and 3.

d. Sample response: The points for which the slope of the tangent to a curve is 0 are the points where the local maximums and minimums occur.

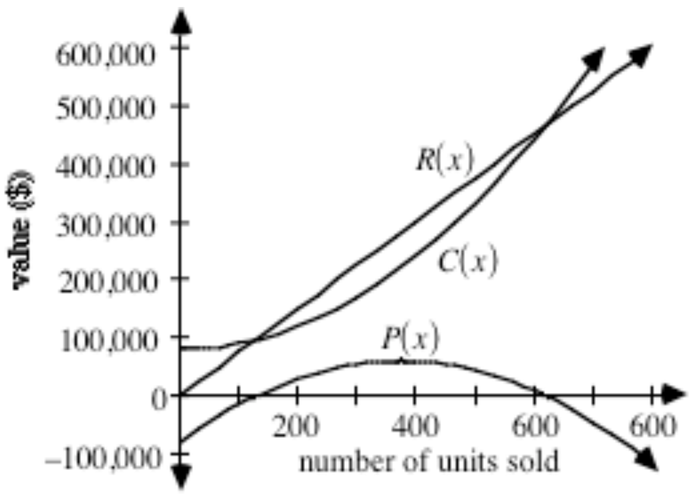
* * * * * * * * * *

Answers to Summary Assessment (page 380)

1. **a.** $R = 750x$

b. $P = R - C = 750x - x^2 - 80,000$

2. **a.** Sample graphs of $P(x)$, $R(x)$ and $C(x)$:



b. The graph of total profit, $P(x)$, is the difference between total revenue and total cost. Its two x-intercepts correspond to the intersections of the graphs of $R(x)$ and $C(x)$. The intersection at $x \approx 129$ indicates the number of items at which the company will break even. The intersection at $x \approx 626$ shows the point where costs overtake revenue and the company starts to lose money.

3. The marginal profit is the derivative of P, which is $750 - 2x$. The marginal revenue is the derivative of R, which is 750. The marginal cost is the derivative of C, which is $2x$.

4. Since $C'(200) = \$400$ and $C'(201) = \$402$, the difference in cost is \$2. This means that it costs \$2 more to make the 201st item.

5. Students can find the maximum profit (and the number of items produced and sold to reach this profit) by examining the graph, or by setting $P'(x) = 0$ and solving for x. The profit is \$60,625 when 375 items are produced and sold.

Substituting 375 into $C'(x)$ and $R'(x)$ results in 750 for both. Therefore, marginal cost and marginal revenue are equal for the number of items that results in maximum profit.

6. Answers may vary. Most students will recommend that the company produce 375 items. The total profit for 375 items is \$60,625. In order to increase this total profit, the company must either reduce the start-up cost of \$80,000, reduce the production cost per item, or increase the revenue per item.

Module Assessment

1. The function $f(t) = t^2 + 5$ describes the distance in meters traveled by an object with respect to time, where t represents time in seconds.

a. Graph this function.

b. Determine the object's average speed from $t = 2$ sec to $t = 8$ sec.

c. Determine the derivative of $f(t)$ and graph it.

d. What does the derivative of $f(t)$ represent in terms of the object's motion?

e. Determine the object's instantaneous speed at $t = 5$ sec.

2. Consider the function $f(x) = 4x^2 + 3$.

a. Use the definition of derivative to find $f'(x)$.

b. To which family of graphs does $f'(x)$ belong?

c. Graph $f(x)$ and $f'(x)$ on the same set of axes.

d. Find the value of the derivative at $x = 0.9$.

e. Explain what the value you determined in Part **d** means in terms of the graphs.

3. Consider an object projected from ground level with an initial velocity of 70 m/sec at an angle of elevation of 30°.

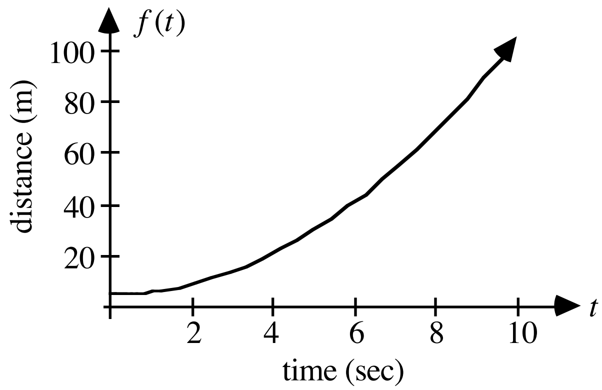
a. Determine a function that describes the object's height in meters with respect to time.

b. Write an expression for the vertical component of the velocity of the projectile with respect to time.

c. An object's acceleration with respect to time can be modeled by the derivative of the function that models its instantaneous velocity. Write an expression that describes the projectile's acceleration.

Answers to Module Assessment

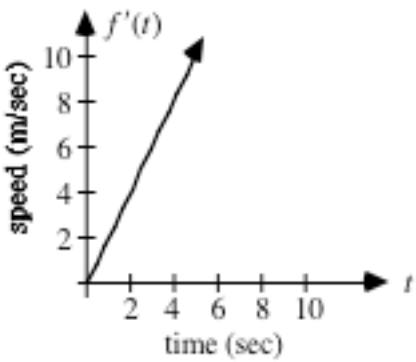
1. **a.** Sample graph:



b. Sample response: The average speed from $t = 2$ sec to $t = 8$ sec is the slope of the secant through the points (2,9) and (8,69).

$$\frac{69 - 9}{8 - 2} = 10 \text{ m/sec}$$

c. The derivative is $f'(t) = 2t$. Sample graph:



d. The derivative of $f(t)$ represents the speed of the object with respect to time.

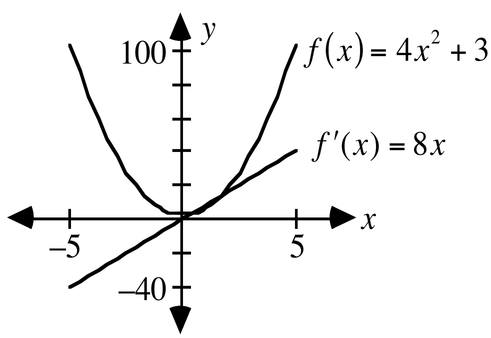
e. At $t = 5$, the instantaneous speed is $f'(5) = 2(5) = 10$ m/sec.

2. **a.** Using the definition of derivative,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left(4(x+h)^2 + 3\right) - \left(4x^2 + 3\right)}{h} \\ &= \lim_{h \to 0} \left(\frac{8xh + 4h^2}{h}\right) \\ &= \lim_{h \to 0} (8x + 4h) \\ &= 8x \end{aligned}$$

b. The derivative of $f(x) = 4x^2 + 3$ is a linear function.

c. Sample graph:



d. The value of the derivative at $x = 0.9$ is $8(0.9) = 7.2$.

e. Sample response: The slope of the tangent to the curve $y = 4x^2 + 3$ at $x = 0.9$ is 7.2.

3. **a.** Sample response: Since the initial velocity has an angle of elevation of 30°, its vertical component is $v_y = 70 \bullet \sin 30° = 35$ m/sec. The object's height, therefore, can be described by $h(t) = -4.9t^2 + 35t$.

b. The object's velocity with respect to time is the derivative of $h(t)$:

$$h'(t) = -9.8t + 35$$

c. The object's acceleration with respect to time is the derivative of $h'(t)$:

$$h''(t) = -9.8 \text{ m/sec}^2$$

This is the acceleration due to gravity.

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Hewitt, P. *Conceptual Physics*. Menlo Park, CA: Addison-Wesley, 1987.

Joint Matriculation Board. Shell Centre for Mathematical Education. *The Language of Functions and Graphs*. University of Nottingham. December 1985. Joint Matriculation Board. Manchester M156EU.

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Pearce, F. "Licensed to Thrill." *New Scientist* 135 (August 29, 1992): 23–25.

Flashbacks

Activity 1

1.1 **a.** Graph $f(x) = -x^2 + 9$ over the interval $[-3, 3]$.

b. What is the maximum value of $f(x)$ over this interval?

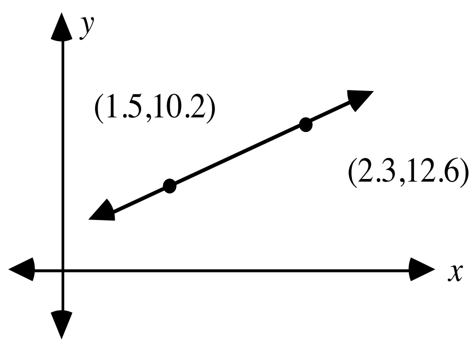
c. Do the values of $f(x)$ increase or decrease as x increases in value over the interval $[-3, 0)$?

d. Describe what happens to $f(x)$ as x increases in value over the interval $(0, 3]$.

1.2 Given the function $d(t) = t^3 - 2t^2 + 3t$, find $d(2)$ and $d(5)$.

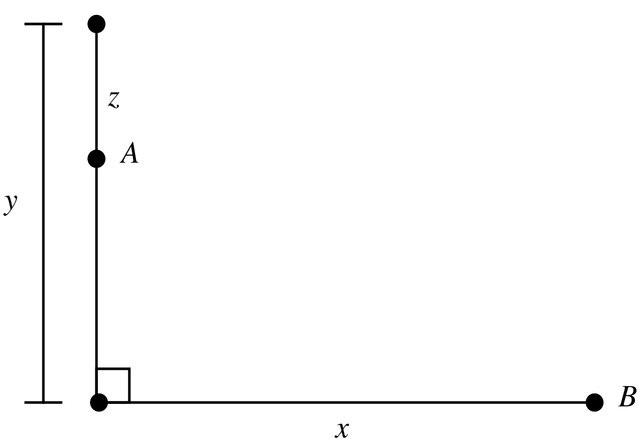
Activity 2

2.1 **a.** Determine the slope of the line in the following graph:



b. Consider the linear function $f(x)$ where $f(1) = -15$ and $f(-5) = 33$. Determine the slope of the graph of $f(x)$.

2.2 Find the shortest distance between points A and B below in terms of x, y, and z.



Activity 3

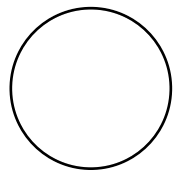
3.1 Write the equation of the circle with radius 4 and center at (0,0).

3.2 **a.** Graph the equation $x^2 + y^2 = 36$.

b. Explain whether or not the graph from Part **a** is a function.

c. Solve the equation in Part **a** for y.

3.3 On a copy of the circle below, draw a secant line and a tangent line.



3.4 Given $f(x) = 3x^2 + 5x + 2$, find and simplify $f(x+3)$.

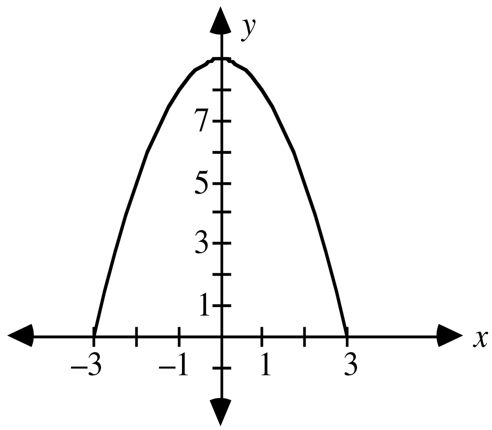
3.5 Evaluate the following expression:

$$\lim_{h \to 0} \left(\frac{2h(5+h)}{h} \right)$$

Answers to Flashbacks

Activity 1

1.1 **a.** Sample graph:



b. The maximum value of $f(x)$ over the interval is 9.

c. The values of $f(x)$ increase from 0 to 9.

d. The values of $f(x)$ decrease from 9 to 0.

1.2 $d(2) = 2^3 - 2 \bullet 2^2 + 3 \bullet 2 = 6$; $d(5) = 5^3 - 2 \bullet 5^2 + 3 \bullet 5 = 90$

Activity 2

2.1 **a.** $m = \dfrac{12.6 - 10.2}{2.3 - 1.5} = 3$

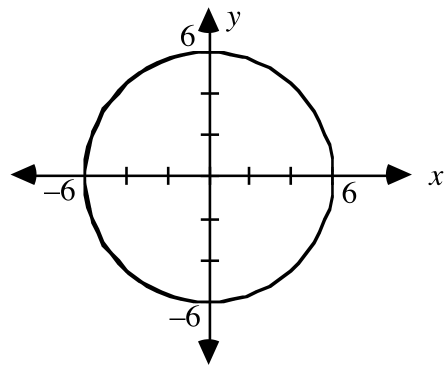
b. $m = \dfrac{33 - (-15)}{-5 - 1} = -8$

2.2 $\sqrt{(y - z)^2 + x^2}$

Activity 3

3.1 $x^2 + y^2 = 16$

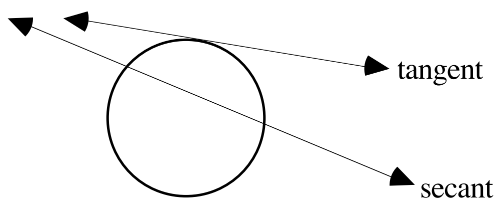
3.2 **a.** Sample graph:



b. Sample response: The graph is not a function because, for example, $x = 0$ is paired with both –6 and 6.

c. $y = \sqrt{36 - x^2}$ or $y = -\sqrt{36 - x^2}$

3.3 Sample response:



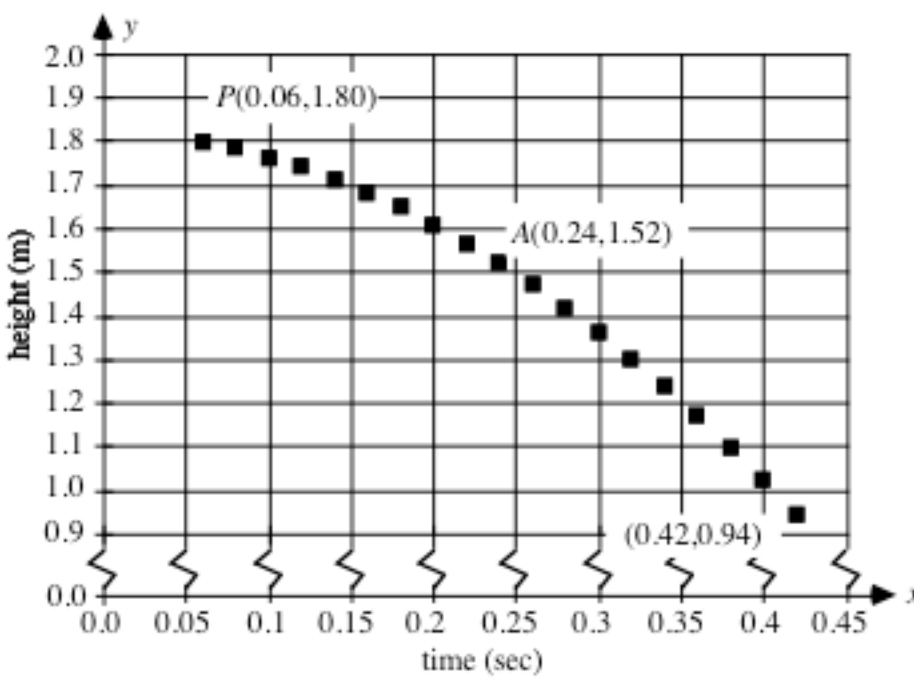
3.4 Sample response:

$$\begin{aligned} f(x+3) &= 3(x+3)^2 + 5(x+3) + 2 \\ &= 3(x^2 + 6x + 9) + 5x + 15 + 2 \\ &= 3x^2 + 18x + 27 + 5x + 15 + 2 \\ &= 3x^2 + 23x + 44 \end{aligned}$$

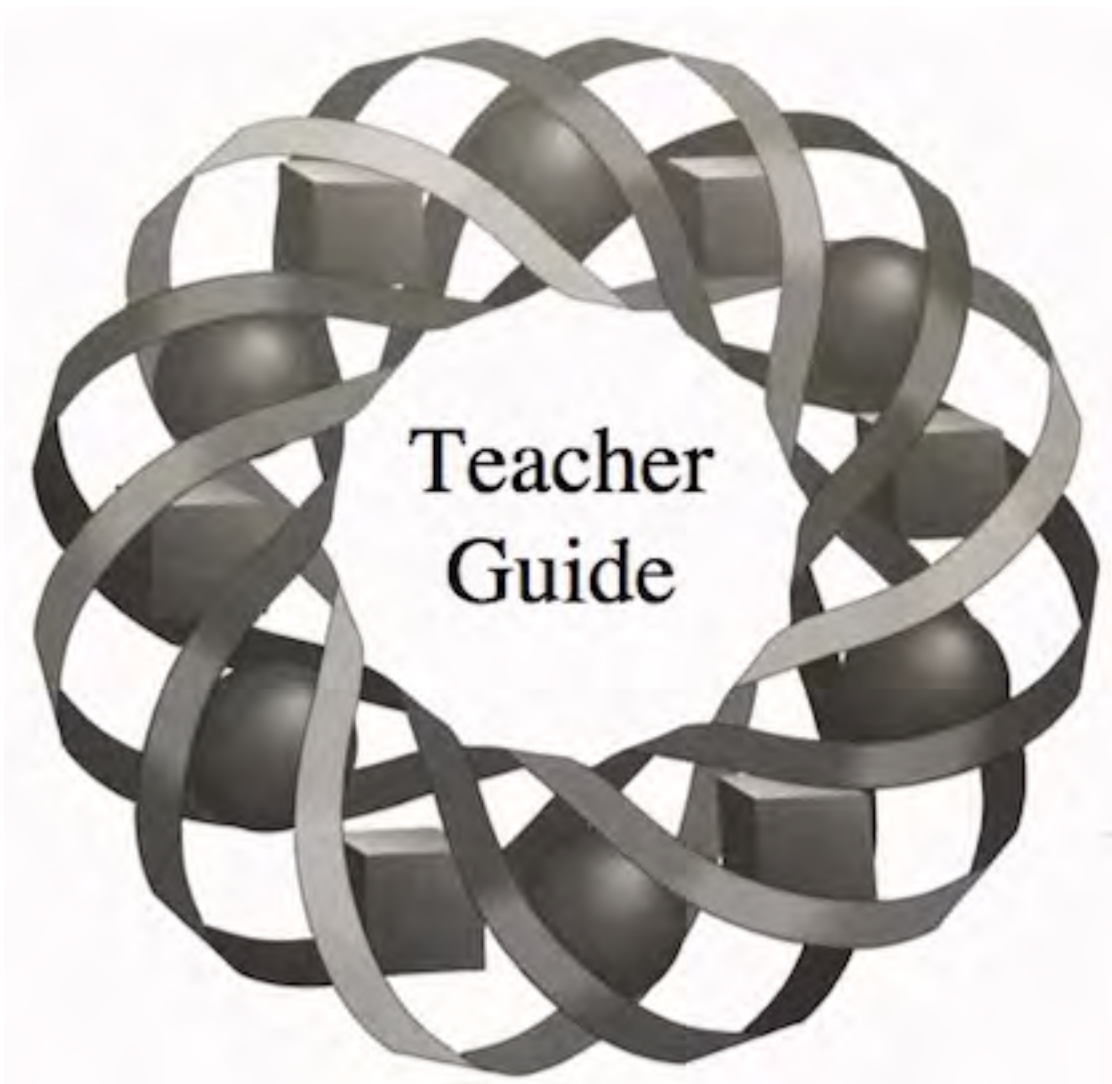
3.5 $\lim_{h \to 0} \left(\frac{2h(5+h)}{h} \right) = 10$

Data for Falling Ball Experiment

Time (sec)	Height (m)	Time (sec)	Height (m)
0.00	1.822	0.22	1.566
0.02	1.817	0.24	1.520
0.04	1.809	0.26	1.471
0.06	1.798	0.28	1.417
0.08	1.781	0.30	1.360
0.10	1.762	0.32	1.300
0.12	1.739	0.34	1.236
0.14	1.711	0.36	1.168
0.16	1.682	0.38	1.097
0.18	1.646	0.40	1.023
0.20	1.608	0.42	0.942



Total Chaos



The world is full of unpredictable and ever-changing systems, including the weather, the stock market, and animal populations. In this module, you investigate chaos theory, a relatively new branch of mathematics that studies the behavior of dynamical systems.

Byron Anderson • Sandy Johnson • Lisa Wood

Teacher Edition

Total Chaos

Overview

In this module, students explore dynamical systems using recursive and iterative processes with geometric shapes, linear functions, and nonlinear functions.

Objectives

In this module, students will:

- investigate recursive processes
- use recursive and iterative functions
- study the following terms: *orbit*, *fixed point*, *attractor*, *repeller*, *period*, and *cycle*.
- investigate functions using web plots
- classify orbits as fixed, periodic, or chaotic.

Prerequisites

For this module, students should know:

- how to perform recursive processes
- functions and functional notation
- the basic concepts of sequences
- the basic concepts of limits
- how to perform operations with complex numbers.

Time Line

Activity	**1**	**2**	**3**	**4**	**5**	**Summary Assessment**	**Total**
Days	3	2	2	2	1	1	11

Materials Required

Materials	Activity					
	1	**2**	**3**	**4**	**5**	**Summary Assessment**
video camera	X					
television	X					
mirrors	X					
isometric dot paper	X					
Pascal's triangle template	X					
colored pencils	X					
graph paper			X			

Teacher Note

Blackline masters for isometric dot paper and the Pascal's triangle template appear at the end of the teacher edition FOR THIS MODULE. The isometric dot paper may be helpful in constructing the Koch curve and Sierpinski's triangle in Activity **1**.

Technology

Software	Activity					
	1	**2**	**3**	**4**	**5**	**Summary Assessment**
graphing utility		X	X	X	X	X
geometry utility	X					X
spreadsheet		X	X	X	X	X
symbolic manipulator		X	X	X	X	X

Total Chaos

Teacher Note

As a preface to this module, you may wish to ask students to read an excerpt from Michael Crichton's novel *Jurassic Park*. The chapter titled "Malcolm" discusses chaos theory in a way that students may find interesting.

Introduction (page 385)

A video feedback demonstration can yield beautiful and unexpected results. If you have the equipment, it is strongly recommended that you conduct this demonstration for your class.

To perform the video feedback demonstration, a video camera must be connected to a television monitor. An adapter may be required for this connection. You are encouraged to set up and test this demonstration prior to using it in the classroom.

The video camera should be positioned so that only the screen of the television appears in the focused image. A distance of 2–3 m works well, although some adjustments may be necessary. For best results, place the camera directly in front of the screen and rotate it approximately 45˚ from vertical. If no tripod is available, the camera can be supported with a soft pillow on a desk or table.

The demonstration works best in very dim light. Adjust the brightness and contrast settings of the monitor to obtain a clear image. On most machines, best results occur when the contrast is turned up all the way and the brightness is turned down considerably. Peitgen, Jürgens, and Saupe (1992) suggest lighting a match in front of the television screen to start the iterative process.

Once the video feedback demonstration is operating, it is possible to generate different dramatic effects by tilting the camera slightly, by adjusting the focus and/or zoom on the camera, or by placing a finger, pencil, or other object between the camera lens and the television screen.

Discussion (page 386)

a. The input, like the output of the video feedback, is the image on the television screen.

b. The process proceeds as follows: the camera records a picture of the television screen; the signal for that picture is then fed back to the television, where it appears on the screen as an image, and the process is repeated.

c. Sample response: The images formed by video feedback and the reflections in a set of parallel mirrors are both generated by an iterative process, where the output of one stage becomes the input for the next stage.

(page 386)

Activity 1

Students use a recursive or iterative process to create several stages of fractal curves and other self-similar figures. Throughout this activity, students should be encouraged to predict the future outcomes of a specific system.

Note: A precise mathematical definition of *fractal* is beyond the scope of this module. If students have questions about *fractal dimensions*, you may wish to point out that lines and segments have dimension 1, while a plane has dimension 2. The dimension of a fractal such as the Koch curve is between 1 and 2.

Materials List

- isometric dot paper template (one sheet per student; optional)
- Pascal's triangle template (four copies per student; for research project)
- colored pencils (optional; for research project)

Teacher Note

Blackline masters for isometric dot paper and the Pascal's triangle template appear at the end of the teacher edition for this module.

Technology

- geometry utility
- spreadsheet (optional)

Teacher Note

In some versions of The Geometer's Sketchpad, a script is featured that produces the Koch curve. In those versions, it can be found in the "Fractals" folder, in the "Sample Scripts." To run the script, select any two points, then choose "play." Enter the number of recursions desired. You may wish to review the section of the user's manual that discusses scripts containing recursive loops. The script can also be used to create the hat curve, as well as many other fractals.

The TI-92 features a Macro tool in its geometry utility that also allows you to iterate a construction.

If students use isometric dot paper to construct stages of the Koch curve, make sure that the lengths of the segments for stage 0 are a power of 3, such as 27 units.

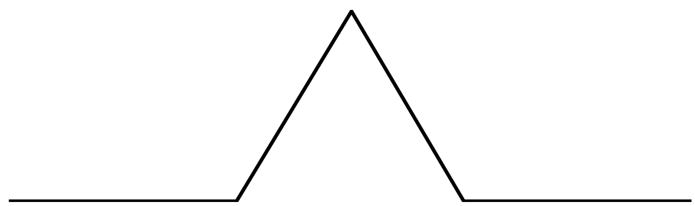
Exploration

(page 387)

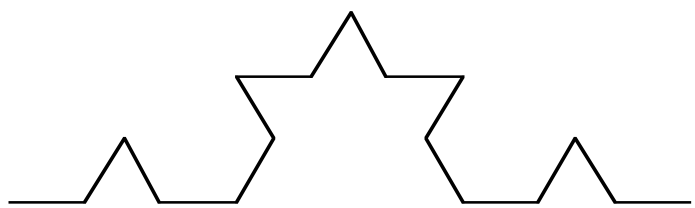
a. Students may use either isometric dot paper or a geometry utility to complete this part of the exploration.

1. Stage 0 of the Koch curve:

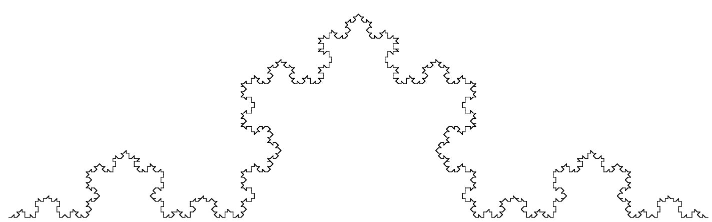
2–4. Stage 1 of the Koch curve:



5. Stage 2 of the Koch curve:

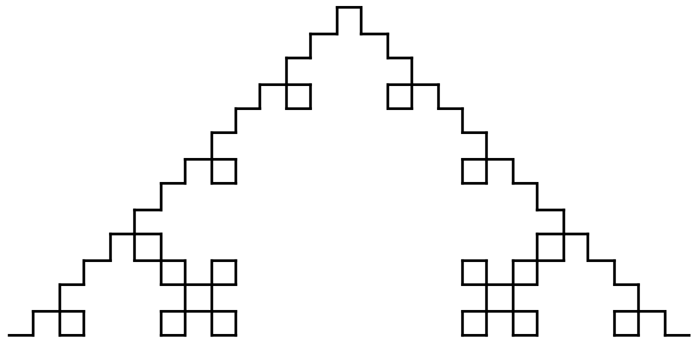


6. Sketch of stage 4 of the Koch curve, using a geometry utility:



b. **1.** Sample response: Divide each segment into thirds. On the middle third of each segment, construct a square with side lengths equal to one-third that of the segment. Remove the side of the square that lies along the original segment to create a shape like a hat.

2. Although students may construct stage 3 of the hat curve using graph paper and pencil, they are encouraged to use a geometry utility. Sample sketch:



Discussion

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a. Sample response: Yes. Each part of these sets would contain smaller replicas of the whole. Therefore, the sets would be self-similar. **Note:** Some fractals are not self-similar and some self-similar objects are not fractals.

b.

1. As the number of iterations increases without bound, the total length of the stages of the Koch curve increases without bound. **Note:** If the original segment is 1 unit long, then the length at stage n is $(4/3)^n$ units.

2. As the number of iterations increases without bound, the total length of the stages of the hat curve increases without bound. **Note:** If the original segment is 1 unit long, then the length at stage n is $(5/3)^n$ units.

c. Sample response: You repeat the process n times.

Assignment

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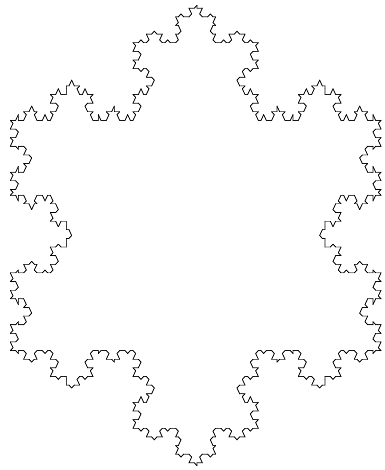
1.1

a. The following table shows the lengths for both a Koch curve and a hat curve.

Stage	Length of Koch Curve	Length of Hat Curve
0	1	1
1	4/3	5/3
2	16/9	25/9
3	64/27	125/27

b. For the Koch curve, the length at stage n is $(4/3)^n$. For the hat curve, the length is $(5/3)^n$.

1.2 **a.** Sample response:



b. As the number of iterations increases without bound, the perimeter of the Koch snowflake also increases without bound. At every stage, the length of each new portion is $1/3$ greater than the side being replaced. In other words, the total length of the new figure is $4/3$ the length of the original. If the perimeter of the original triangle is 1 unit, then the perimeter at stage *n* is $(4/3)^n$ units.

c. As the number of iterations increases without bound, the area of the Koch snowflake appears to approach a limit. This limit depends on the perimeter of the original equilateral triangle from stage 0. Some students may be surprised to encounter a geometric figure that has an infinite perimeter, but a finite area. If the area of the original triangle is 1 unit^2, then the area at stage *n* is:

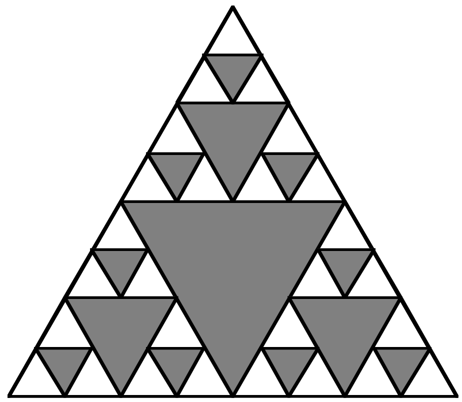
$$1+3\left(\frac{1}{9}\right)+12\left(\frac{1}{9}\right)^2+48\left(\frac{1}{9}\right)^3+\cdots+3\bullet 4^{n-1}\left(\frac{1}{9}\right)^n$$

This is a geometric series whose sum is the following:

$$1+\frac{1/3}{1-(4/9)}=\frac{8}{5}$$

1.3 Sample response: Stage 0 consists of a square topped by an isosceles triangle whose base is one side of the square. The isosceles triangle has a vertex angle that measures 120˚. Stage 1 is formed by constructing a square on each leg of the isosceles triangle, with sides equal to the length of the leg. Another isosceles triangle with a vertex angle of 120˚ is then created on the side of the square opposite the side that was the leg of the previous isosceles triangle. The process is then repeated.

***1.4** **a.** Stage 3 of Sierpinski's triangle:



b. Sample response: There is 1 unshaded triangle at stage 0. There are 3 such triangles at stage 1. There are 9 triangles at stage 2. There are 27 triangles at stage 3. There are 3^n triangles at stage *n*. At each successive stage, the number of triangles increases by a factor of 3. The sequence is 1, 3, 9, 27, … .

As the number of iterations increases without bound, the number of unshaded triangles also increases without bound.

c. If the perimeter of the original triangle is 1 unit, then the sequence for the total perimeter is:

$$1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \ldots, \left(\frac{3}{2}\right)^n$$

where *n* is the stage number.

As the number of iterations increases without bound, the perimeter also increases without bound.

d. The following sequence describes the total area at each stage:

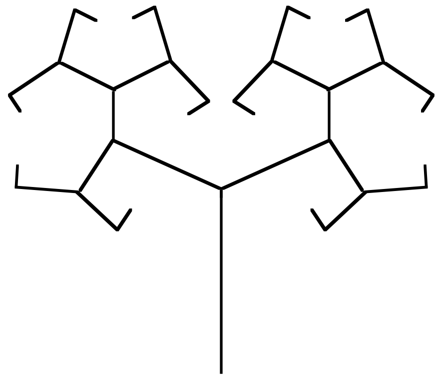
$$1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \ldots, \left(\frac{3}{4}\right)^n$$

As the number of iterations increases without bound, the sum of the areas of each of the remaining triangles approaches 0. Students may be intrigued by a figure with an infinite perimeter and zero area.

* * * * *

1.5 **a.** Answers will vary. Sample response: Stage 0 is a segment. To create stage 1, a segment 2/3 the length of the original segment is rotated 120° with a common endpoint as the center, and another segment 2/3 the length of the original is rotated –120° with the same endpoint as the center. The following stages are created by repeating the process for stage 1 on each new segment. **Note:** This type of fractal is often referred to as a binary tree.

b. Sample sketch:



c. The length of the fractal described in Part **a** changes as follows:

Stage	Length
0	1
1	$1+\frac{4}{3}=\frac{7}{3}$
2	$1+\frac{4}{3}+\frac{16}{9}=\frac{37}{9}$
3	$1+\frac{4}{3}+\frac{16}{9}+\frac{64}{27}=\frac{175}{27}$
n	$2^0\left(\frac{2}{3}\right)^0+2^1\left(\frac{2}{3}\right)^1+2^2\left(\frac{2}{3}\right)^2+\cdots+2^n\left(\frac{2}{3}\right)^n=\frac{1-(4/3)^{(n+1)}}{1-(4/3)}$

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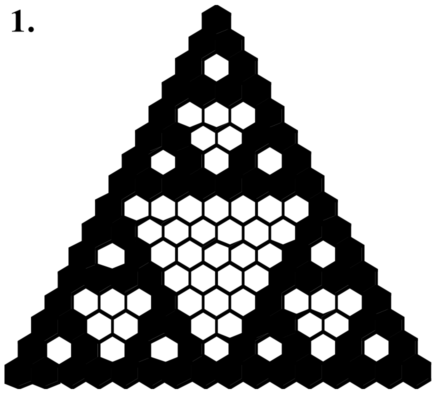
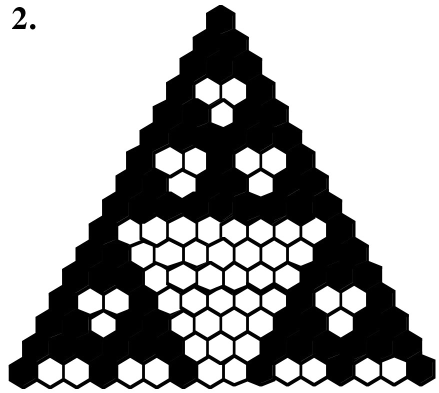
Research Project (page 391)

Students may use colored pencils to color their triangular arrays. A blackline master of the triangle template appears at the end of the teacher edition for this module.

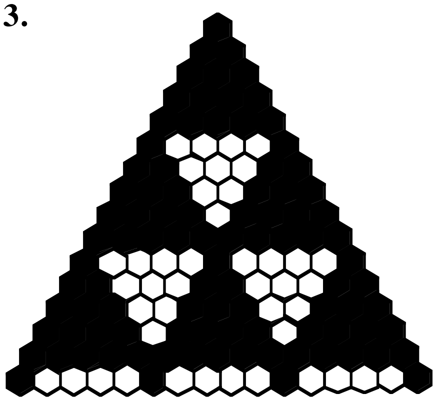
1.



2.



3.



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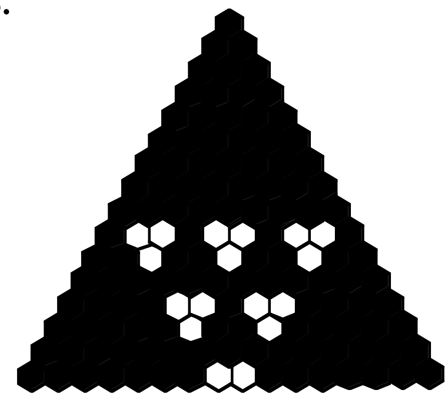

Students should discover patterns with regularities and self-similarities, much like those in Sierpinski's triangle. Several excellent illustrations of these patterns appear in Peitgen, Jürgens, and Saupe (1992), *Fractals for the Classroom. Part One: Introduction to Fractals and Chaos*, pp. 100–101.

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Activity 2

In this activity, students use iteration to explore orbits of functions.

Materials List

- none

Technology

- graphing utility
- spreadsheet (optional)
- symbolic manipulator (optional)

Exploration 1

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In the following exploration, students use iterations of the function $f(x) = \sqrt{x}$ to explore fixed points.

a. An initial value of 1 generates the sequence: 1, 1, 1, 1, … .

b. **1.** A fixed point for the function $f(x)$ is a value of x such that $f(x) = x$, $f\big(f(x)\big) = x$, $f\Big(f\big(f(x)\big)\Big) = x$, and so on. Because $f(1) = f\big(f(1)\big) = f\Big(\cdots f\big(f(1)\big)\Big) = 1$, 1 is a fixed point.

2. Students should determine that 0 is also a fixed point for $f(x) = \sqrt{x}$ by solving as follows:

$$\begin{aligned} f(x) = \sqrt{x} &= x \\ x &= x^2 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0, \text{ or } x = 1 \end{aligned}$$

Thus, the only fixed points are 0 and 1.

c. Students may wish to explore this problem with a spreadsheet. Each sequence decreases and approaches 1. For example, using an initial value of 16, the resulting sequence is:

16, 4, 2, 1.414213562, …, 1.0016924

d. Each sequence increases and approaches 1. For example, using an initial value of 0.0302, the resulting sequence is:

0.1738, 0.4169, 0.6457, …, 0.9998

e. Sample response: All six sequences appear to approach 1 as a limit.

Discussion 1

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a. The orbit is the sequence $p, \; p, \; p, \; \ldots$.

b. The solutions to the equation $-x^3 = x$ are $i, -i$, and 0, as shown below:

$$\begin{aligned} -x^3 &= x \\ x^3 + x &= 0 \\ x(x^2 + 1) &= 0 \\ x(x - i)(x + i) &= 0 \\ x &= 0, \; i, \; -i \end{aligned}$$

c.

1. The number 1 is an attractor for this function since for all initial values greater than 0, the sequence has a limit of 1.
2. Zero is not an attractor. There is no value of x in the domain, other than the fixed point 0, for which an orbit approaches 0. In other words, 0 is a fixed point but not an attractor.

Exploration 2

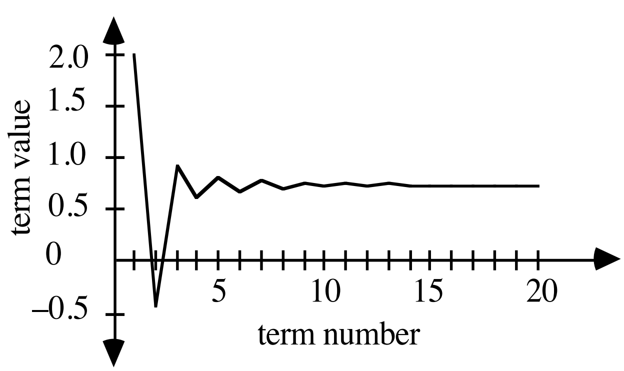
(page 393)

Students iterate functions and observe the behaviors of the orbits.

a. The following table shows 6 of the first 20 terms of the orbit:

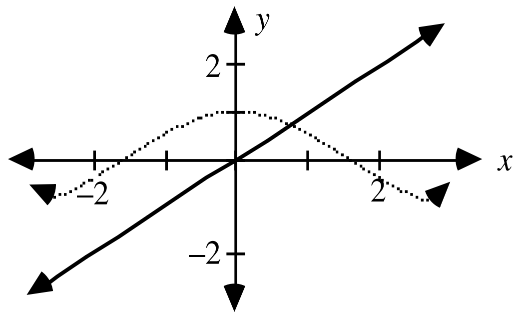
Term No.	Term Value of Orbit
1	2
2	–0.4161468
3	0.9146533
4	0.6100653
⋮	⋮
19	0.7394108
20	0.7388657

b. Sample graph:



c–d. For $f(x) = \cos x$, the terms of the orbit approach a value of about 0.739, regardless of the initial value.

e. The only fixed point for $f(x) = \cos x$ is approximately 0.739. Students should graph $y = \cos x$ and $y = x$ on the same coordinate system and observe that there is only one point of intersection.



f. For $f(x) = \sin x$, the terms of the orbit approach 0, regardless of the initial value. For $f(x) = \sin x$, 0 is a fixed point since, when 0 is the initial value, every term in the orbit is 0.

g. For $f(x) = 0.5x + 5$, the terms of the orbit approach 10, regardless of the initial value. When 10 is the initial value, every term in the orbit is 10.

h. For $f(x) = 2x + 3$, the terms of the orbit do not appear to approach a limit (unless the initial value is –3, the fixed point).

Discussion 2

(page 392)

a. The limit for the orbit generated by $f(x) = \cos x$ is approximately 0.739. The limit of the orbit generated by $f(x) = \sin x$ is 0. The limit of the orbit generated by $f(x) = 0.5x + 5$ is 10. All of these limits are attractor points.

b. Sample response: The function $f(x) = 2x + 3$ does not appear to approach a limit. The fixed point for $f(x) = 2x + 3$ is –3.

c. Sample response: The fixed point *p* satisfies the equation $f(x) = x$, which is also the solution to the system of equations:

$$\begin{cases} y = f(x) \\ y = x \end{cases}$$

This solution also is the intersection of the graphs.

d. Sample response: Yes. Any function whose graph does not intersect the line $y = x$ does not have a fixed point. It cannot satisfy the equation $f(x) = x$.

Assignment

(page 392)

2.1 **a.** The orbit has 9 as an attractor, regardless of the initial value.

b. Sample response:

$$f(x) = -\frac{1}{3}x + 12 = x$$

$$12 = \frac{4}{3}x$$

$$9 = x$$

***2.2** **a.** Sample table:

b	**Orbit**	**Behavior**
45	2, 45.67, 60.22, 65.07, …, 67.49, 67.5	appears to converge to 67.5
–12	2, –11.33, –15.78, –17.26, …, –17.99, –18	appears to converge to –18
2	2, 2.67, 2.89, 2.96, …, 2.99, 3	appears to converge to 3
–100	2, –99.33, –133.11, –144.37, …, –149.99, –150	appears to converge to –150

b. Sample response: All the orbits converge to a specific value.

c. The ratio of the limit to b is 1.5, except when $b = 0$.

d. Sample response: The fixed point occurs at $f(x) = x$. Solving for x:

$$x = \frac{1}{3}x + b$$

$$= \frac{3}{2}b$$

2.3 The fixed point for any linear function $f(x) = ax + b$, where $a \neq 1$ and $a \neq 0$, is:

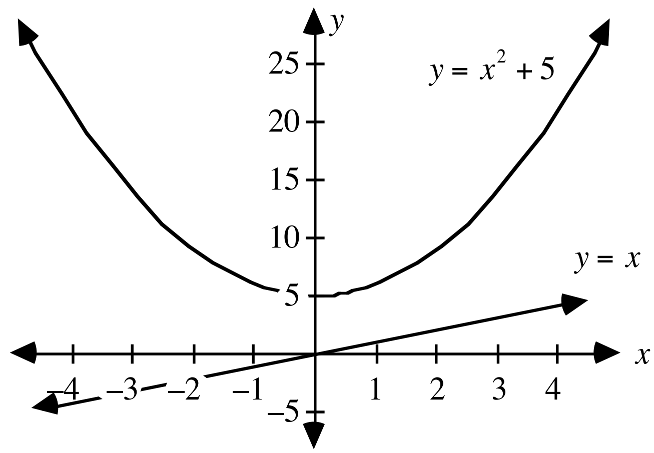
$$x = \frac{b}{1 - a}$$

***2.4** **a.** The fixed points, where $a \neq 0$, are:

$$\frac{1 - \sqrt{1 - 4ab}}{2a} \text{ and } \frac{1 + \sqrt{1 - 4ab}}{2a}$$

b. Sample response: No. If the value under the radical is negative, then the value of the fixed point is not real.

c. Sample response: The product of a and b must be less than or equal to $1/4$, where $a \neq 0$. If these restrictions are not met, a real fixed point will not exist, since $1 - 4ab$ will be negative. This can be shown in the graph of $y = x$ and $f(x) = x^2 + 5$. The line does not intersect the curve. In this case, $a = 1$ and $b = 5$. The value of $\sqrt{1 - 4ab} = \sqrt{-19}$ is not real.



2.5 The fixed point for the function $f(x) = 19x + 6$ is $x = -1/3$.

2.6 The fixed points are all non-negative real numbers.

2.7 The fixed points are $x = 1 - i$ and $x = i$. Some students may obtain the solution in the following equivalent form:

$$\frac{1 + \sqrt{-3 - 4i}}{2} \text{ or } \frac{1 - \sqrt{-3 - 4i}}{2}$$

Note: You may want to use DeMoivre's theorem to show the equivalence of the complex-number solutions.

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2.8 Sample response: The formula for account balance has an initial value of \$100. Substituting this value for x in the function $f(x) = 1.12x$ gives the account balance after 1 year. Multiplying the resulting account balance by 112% every year will produce a sequence of annual balances.

2.9 Sample recursive formula:

$$\begin{cases} g_1 = 4 \\ g_n = 1/g_{n-1}, \quad n > 1 \end{cases}$$

2.10 The fixed point is 1.

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Activity 3

In this activity, students use web plots as graphical models of orbits. They also use web plots to explore the notion of attractors and repellers.

Materials List

- graph paper

Technology

- graphing utility
- spreadsheet
- symbolic manipulator

Teacher Note

In the following exploration, students should graph functions over the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. They may find it helpful to magnify the graphs near the intersection of $f(x)$ and $y = x$.

Some graphing utilities may offer features that will allow students to obtain web plots very quickly. For example, the TI-82 and TI-92 calculators offer a WEBPLOT command, while the software program TEMATH features a fixed-point iteration tool.

Exploration

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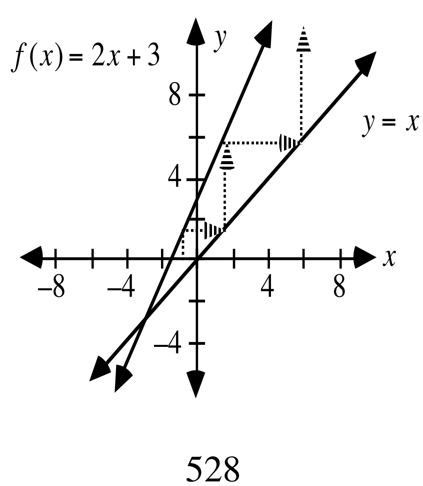
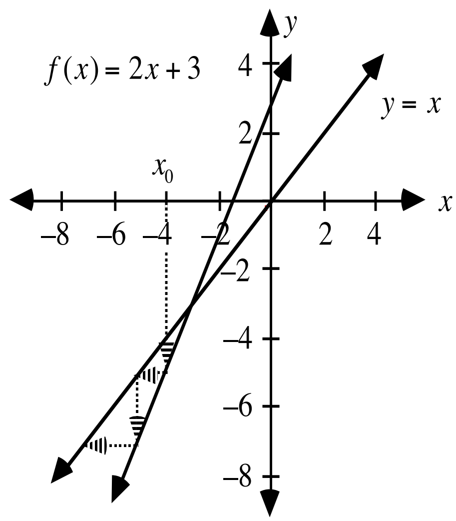
a. Since fixed points occur when $f(x) = x$, solving the equation $x = 2x + 3$ yields the fixed point $x = -3$.

The fixed points can be estimated graphically by examining the intersection of $f(x)$ and the line $y = x$. Students may confirm that –3 is the fixed point by substitution, $-3 = 2(-3) + 3$.

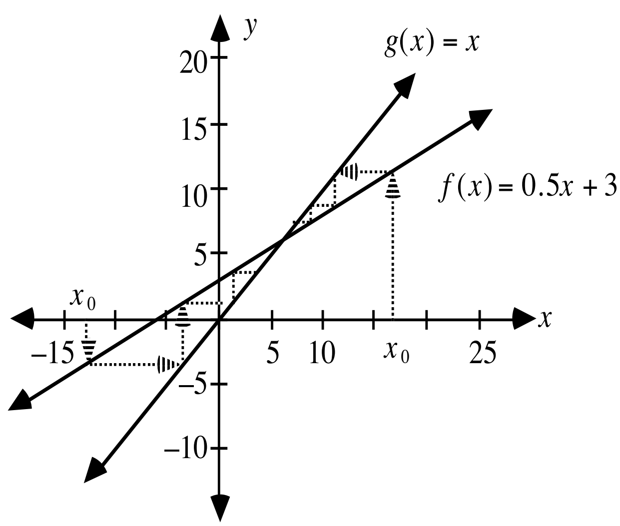
The points of the orbit move away from the fixed point in a staircase pattern. Sample graph using $x_0 = -1$:



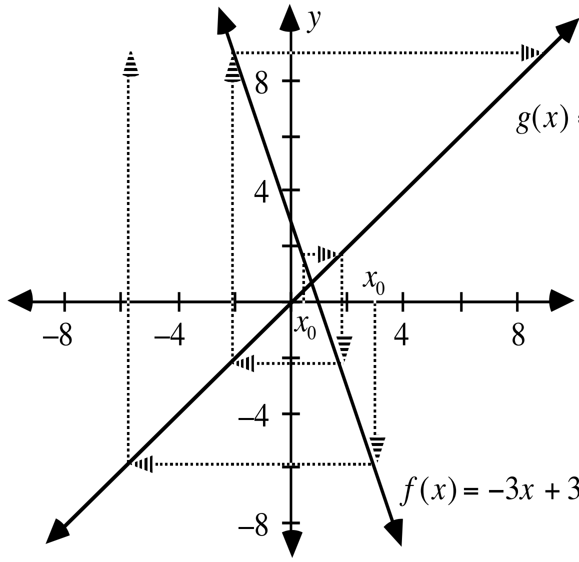
b. The points of this orbit also move away from the fixed point in a staircase pattern. Sample graph using $x_0 = -4$:



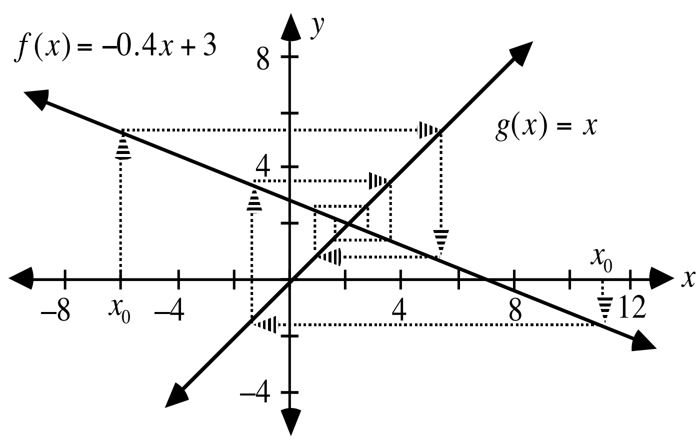
c. **1.** Solving $x = 0.5x + 3$ yields a fixed point at 6. The points of the orbits converge to the fixed point following a staircase path. Sample graph using $x_0 > 6$ and $x_0 < 6$:



2. Solving $x = -3x + 3$ yields a fixed point at 0.75. The points of the orbits follow a spiral path away from the fixed point. Sample graph using $x_0 < 0.75$ and $x_0 > 0.75$:



3. Solving $x = -0.4x + 3$ yields a fixed point at 15/7 . The points of the orbits approach the fixed point following a spiral path. Sample graph using $x_0 < 15/7$ and $x_0 > 15/7$:



Discussion

(page 396)

a. Sample response: Drawing a vertical segment to $f(x)$, followed by a horizontal segment to $g(x)$, illustrates the input and output, respectively, of the iterative process. The first vertical segment represents the input of the initial value x_0 in the function and results in an output $f(x_0)$. The horizontal line represents the output $f(x_0)$, which becomes the input of the next iteration. The next vertical line represents the new input $f(x_0)$, resulting in an output of $f\big(f(x_0)\big)$.

b. Sample response: The paths of the orbits for both $f(x) = 2x + 3$ and $f(x) = 0.5x + 3$ look like staircases.

c. Sample response: The paths of the orbits for both $f(x) = -3x + 3$ and $f(x) = -0.4x + 3$ look like spirals.

d.
1. The orbits generated by the functions $f(x) = 0.5x + 3$ and $f(x) = -0.4x + 3$ appear to move toward their fixed points.
2. Sample response: The path for $f(x) = 0.5x + 3$ is a staircase, while the path for $f(x) = -0.4x + 3$ is a spiral.
3. Both coefficients have absolute values less than 1. The coefficient with a negative value corresponds to a spiral path. The positive coefficient corresponds to a staircase path.

e.
1. The orbits generated by $f(x) = 2x + 3$ and $f(x) = -3x + 3$ move away from their fixed points.
2. The path for $f(x) = 2x + 3$ is a staircase, while the path for $f(x) = -3x + 3$ is a spiral.
3. Sample response: Both coefficients have absolute values greater than 1. The negative coefficient corresponds to a spiral, while the positive coefficient corresponds to a staircase.

f. The fixed points for $f(x) = 0.5x + 3$ and $f(x) = -0.4x + 3$ are attractors. The fixed points for $f(x) = 2x + 3$ and $f(x) = -3x + 3$ are repellers.

Note: Some fixed points are neither attractors nor repellers. For example, there are an infinite number of fixed points for the function $f(x) = -|x|$, yet none are repellers or attractors. A function also may have more than one attractor, as in a periodic orbit. Students encounter this situation in Problem **3.3**.

g. Sample response: In Exploration **2** in Activity **2**, the functions whose orbits approach limits have fixed points that are attractors. The function whose orbit did not approach a limit has a fixed point that is a repeller.

h. The fixed point can be found as follows:

$$\begin{aligned} ax + 3 &= x \\ 3 &= x - ax \\ 3 &= x(1 - a) \\ x &= \frac{3}{1 - a} \end{aligned}$$

i. **1.** When a linear function has a slope less than –1, the orbit has a path that moves away from the fixed point, the repeller.

2. When a linear function has a slope between –1 and 1, the orbit has a path which approaches the fixed point, the attractor.

3. When a linear function has a slope greater than 1, the orbit has a path that moves away from the fixed point, the repeller.

Assignment

(page 397)

3.1 **a.** The fixed point can be found as follows:

$$-\frac{1}{3}x + 12 = x$$

$$12 = \frac{4}{3}x$$

$$9 = x$$

b. Sample response: Judging from the web plot, the fixed point is an attractor. This makes sense because the slope of the linear function is between –1 and 1.

3.2 **a.** Sample response:

Value of *b*	**Initial value x_0**	**Orbit**
2	–3	–3, 5, –3, 5, …
	1	1, 1, 1, 1, …
	7	7, –5, 7, –5, …
–4	–2	–2, –2, –2, –2, …
	1	1, –5, 1, –5, …
	7	7, –11, 7, –11, …
1	1	1, 0, 1, 0, …
	–1	–1, 2, –1, 2, …
	0	0, 1, 0, 1, …

b. Sample response: Found algebraically, the fixed point of $f(x) = -1x + b$ is $x = b/2$. The orbits for various values of *b* cycle between two numbers. The cycling begins either immediately or after a few iterations. Changing the initial value changes the values that the numbers cycle between. The orbits of $f(x) = ax + b$ when $a \neq -1$ do not show this cyclic behavior.

3.3 Sample response: If *a* is greater than 1 or less than –1, the fixed point is a repeller. If *a* is between –1 and 1, the fixed point is an attractor. If $a = 1$, there is no fixed point. If $a = -1$, the orbit alternates between two values.

***3.4** **a.** In general, the orbits are cycles of period 2 oscillating between a and 1. Sample spreadsheet for $f(x) = a/x$ with a fixed initial value of 1 and varying values for a from –2 to 2:

a	**Orbit**				
–2	1	–2	1	–2	…
–1.5	1	–1.5	1	–1.5	…
–1	1	–1	1	–1	…
–0.5	1	–0.5	1	–0.5	…
0.5	1	0.5	1	0.5	…
1	1	1	1	1	…
1.5	1	1.5	1	1.5	…
2	1	2	1	2	…

b. Solving the equation $a/x = x$, the fixed points are $x = -\sqrt{a}$ and $x = \sqrt{a}$.

c. In order for the orbit to be fixed, the initial value must be one of the fixed points, either $x_0 = -\sqrt{a}$ or $x_0 = \sqrt{a}$.

d. The orbit is periodic and has period 2 for all non-negative values of a, except when the initial value is $x_0 = -\sqrt{a}$, or $x_0 = \sqrt{a}$. When $x_0 = -\sqrt{a}$ or $x_0 = \sqrt{a}$, the orbit is fixed. **Note:** If $x_0 = 0$, the orbit is undefined. If $a = 0$, the orbit becomes undefined after two iterations.

* * * * *

3.5 **a.** The solution may be found as follows:

$$\frac{16}{x^3} = x$$
$$x^4 = 16$$
$$x = 2, -2,\ 2i,\ -2i$$

b. Students may verify their solutions by substitution:

$$f(2) = 16/2^3 = 2$$
$$f(-2) = 16/(-2)^3 = -2$$
$$f(-2i) = 16/(-2i)^3 = -2i$$
$$f(2i) = 16/(2i)^3 = 2i$$

* * * * * * * * * *

(page 399)

Activity 4

In this activity, students examine the behavior of orbits of nonlinear functions. Although this behavior may look predictable in the early stages, it can suddenly become unpredictable and chaotic.

Materials List

- none

Technology

- spreadsheet
- graphing utility
- symbolic manipulator

Teacher Note

Some graphing utilities feature an iteration tool that may be used to observe the behavior of orbits for different values of a. The TI-92 calculator, for example, can create web plots for iterated functions (refer to the section on web formats in the manual).

If using a spreadsheet, students should identify at least 100 terms of each orbit. In order to more easily identify attractors, they should round values to no more than four places.

Note that the definition given for chaos is not precise. At best, students will only be able to suggest that a system is chaotic based on orbits that do not appear to be fixed or periodic. Systems with large periods may appear to be chaotic.

Exploration

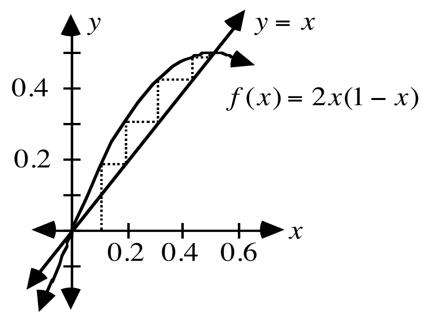
(page 399)

Students explore the effects of changing the parameter a in the quadratic function $f(x) = ax(1 - x)$, where $a \neq 0$. They should discover that what at first appears predictable can turn chaotic.

a. The following sample spreadsheet shows Microsoft Excel formulas.

	A	B	C
1	**a**	2	
2	**initial value**	0.1	
3	**Term No.**	**Orbit**	**Output**
4	1		=$B$1*B2*(1-B2)
5	=A4+1	=C4	=$B$1*B5*(1-B5)
6	=A5+1	=C5	=$B$1*B6*(1-B6)
7	=A6+1	=C6	=$B$1*B7*(1-B7)
8	=A7+1	=C7	=$B$1*B8*(1-B8)
9	=A8+1	=C8	=$B$1*B9*(1-B9)
10	=A9+1	=C9	=$B$1*B10*(1-B10)
11	=A10+1	=C10	=$B$1*B11*(1-B11)
12	=A11+1	=C11	=$B$1*B12*(1-B12)
13	=A12+1	=C12	=$B$1*B13*(1-B13)

b. **1.** Sample web plot for $a = 2$ and $x_0 = 0.1$.



2. Sample table:

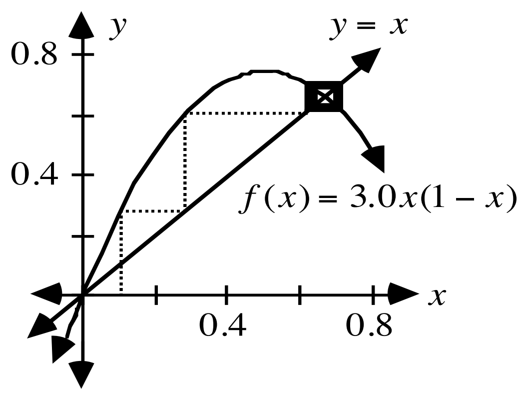
a	Fixed Point(s)	No. of Attractors	Value of Attractor(s)	Behavior of Orbit
2.0	0, 0.5000	1	0.5000	staircase in
2.2	0, 0.5454	1	0.5454	staircase in, then spiral around fixed point
2.4	0, 0.5833	1	0.5833	staircase in, then spiral around fixed point
2.6	0, 0.6154	1	0.6154	staircase in, then spiral around fixed point

c. Sample response: For $f(x) = ax(1 - x)$, where $a \neq 0$, as the value of a increases, there is one attractor, the value of the attractor increases, and the orbit staircases in and then spirals around the attractor.

d. Sample table:

a	**Fixed Point(s)**	**No. of Attractors**	**Value of Attractor(s)**	**Behavior of Orbit**
2.8	0, 0.6429	1	0.6429	staircase in, then spiral around fixed point
3.0	0, 0.6700	1	0.6700	staircase in, then spiral around fixed point
3.2	0, 0.6875	2	0.7994, 0.5130	staircase into intersection of function and diagonal, then cycle between attractors

Note: For $a = 3$, some students may observe that the orbit does not appear to approach the fixed point of $2/3$. Even after 2000 iterations, the orbit appears to cycle between two attractors: approximately 0.6614 and 0.6719. Sample web plot for $a = 3$ and $x_0 = 0.1$:



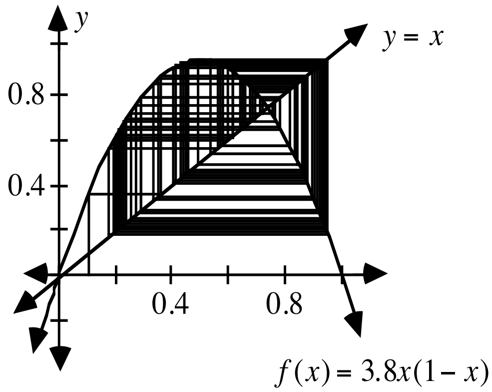
e. Sample response: The predictions in Part **c** did not hold true. Changing the value of *a* to 3 or a number greater than 3 resulted in two attractors instead of a single attractor.

f. Sample response: As the value of *a* increases past 3, there will be two attractors with values of about 0.8 and 0.5. The path of the orbit begins by "staircasing" in to the fixed point and then cycles between the two attractors.

g. Sample table:

a	**Fixed Point(s)**	**No. of Attractors**	**Value of Attractor(s)**	**Behavior of Orbit**
3.4	0, 0.7059	2	0.4520, 0.8422	staircase in, then spiral around fixed point, cycling between attractors
3.6	0, 0.7222	no obvious attractors		unpredictable
3.8	0, 0.7360	no obvious attractors		unpredictable

Sample web plot for $a = 3.8$ and $x_0 = 0.1$:



h. Sample response: As a increases, the behavior of the graph becomes more unpredictable.

i. Sample response: When $a = 4$, the behavior of the orbit is unpredictable.

j. Students should observe that small changes in the initial value produce orbits that exhibit unpredictable behavior.

Discussion

(page 401)

a. Sample response: Since the first predictions were not supported by the results of the tests, the next predictions were made with less confidence.

b. Sample response: As the value of a increased from 3.0 to 3.2, the number of attractors increased from 1 to 2. For values of a greater than 3.6, no attractors were identifiable.

c. Sample response: As the value of a increased, the paths of the orbits became more and more erratic and unpredictable.

d. Sample response: The web plots of the orbits became chaotic.

e. Sample responses may include the fluctuations of the stock market, the motion of dust particles in the air, or the dynamics of wildlife populations.

Assignment

(page 401)

4.1 Sample response:

c	**Behavior of Orbit**
0.1	attracted to 0.1127016654
–0.5	attracted to –0.3660254047
–0.8	cycle of period 2 {–0.2764, –0.7234}
–1.3	cycle of period 4 {–1.149, 0.019, –1.300, 0.389}
–1.37	cycle of period 8
–1.4	cycle of period 16
–1.76	cycle of period 3
–1.77	cycle of period 6
–1.95	no apparent pattern

***4.2** As in the exploration, students should use a spreadsheet or graphing utility to find values of *a* that produce an example of each type of behavior. Sample response:

Long-term Behavior	a
fixed point	2.95
cycle of period 2	3.05
cycle of period 3	3.84
cycle of period 4	3.5
cycle of period 5	3.74
chaos	3.8

* * * * *

4.3 The orbit cycles between the values –1 and 0.

4.4 The magnitudes of values in the orbit tend to increase without bound:

$$-1-i,\ -1+i,\ -1-3i,\ -9+5i,\ 55-91i,\ \ldots$$

* * * * * * * * * *

(page 402)

Activity 5

Students explore an iterative model for population dynamics.

Materials List

- none

Technology

- spreadsheet
- graphing utility
- symbolic manipulator (optional)

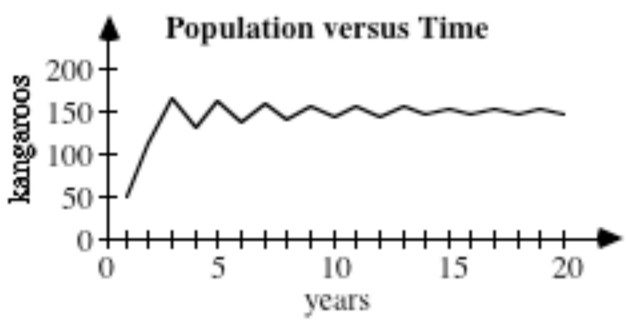
Exploration

(page 402)

a. Sample table:

Year	Population
1	50
2	113
3	166
⋮	⋮
17	153
18	147
19	153
20	148

b. Sample graph:



c. Using a spreadsheet, students can vary the initial values, then observe what happens to the population over time. If they substitute values for the initial population between 2 and 228, inclusive, the resulting populations approach the carrying capacity of 150 kangaroos. (Students should realize that an initial population less than 2 will not allow the kangaroos to reproduce.) Values greater than 228 cause the population to become extinct in year 2.

d. Sample response: The long-term behavior of the population settles into a cycle of period 2. The predicted size cycles between 124 and 169 kangaroos.

e. Values for the initial population between 2 and 221, inclusive, will result in a population that cycles between 124 and 169 kangaroos. Values greater than 221 cause the population to become extinct in year 2.

Discussion

(page 403)

a. Sample response: The population approaches the carrying capacity by alternating between values less than and greater than C.

b. Sample response: When a population is considerably less than the carrying capacity of its environment, there is enough food, water, and space to encourage high rates of reproduction and low mortality. This might cause the population to increase to a value greater than the carrying capacity in the next year. In the following year, when the population is too large for the available resources, many animals might die from starvation, predation, or disease. The new, smaller population might increase again the next year, and so on.

c. Sample response: When the initial population is less than 229, the population eventually approaches the carrying capacity. However, when the initial population is greater than or equal to 229, the model indicates that the entire population dies in the first year. **Note:** Students should discuss the model's applicability at this point. They should also question the wisdom of placing 229 animals in an environment with a carrying capacity of 150.

d. Sample response: In Part **c** of the exploration, the population eventually approached a single value (the carrying capacity). In Parts **d** and **e** of the exploration, however, the population size cycled yearly between 124 and 169. This suggests that a small change in the growth rate may result in very different long-term behavior of the population size.

Assignment

(page 403)

5.1 Sample response: The value of the fixed point may be found by solving the equation $x = f(x)$. In this case, $P_t = P_{t+1}$. Thus, the logistic equation

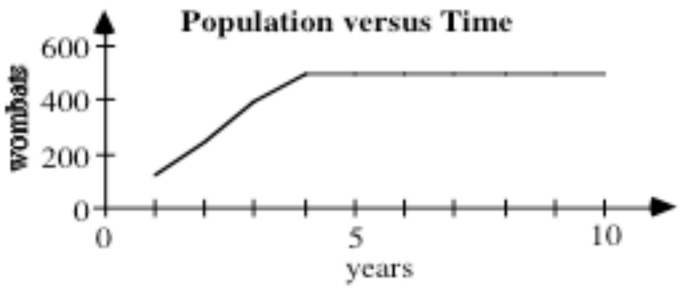
$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

becomes

$$P_t = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

Solving yields $P_t = 0$, $r = 0$, or $P_t = C$. So one of the fixed points for the initial population is *C*, the carrying capacity.

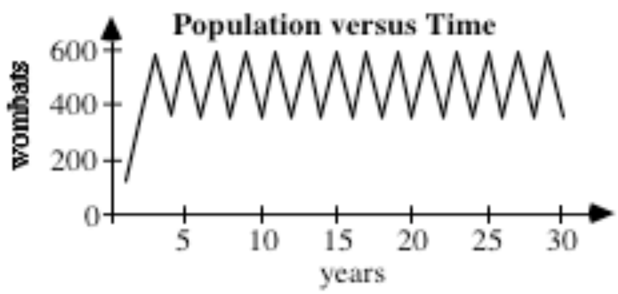
5.2 **a.** Sample graph:



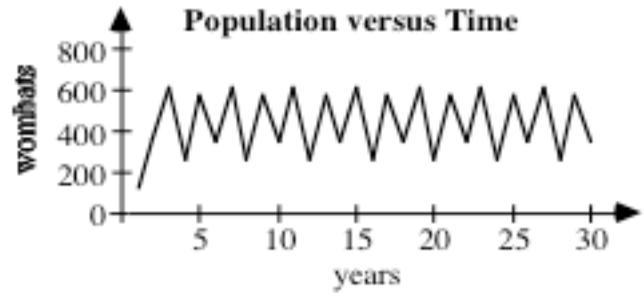
b. Sample response: Since the graph predicts that the population will approach the carrying capacity, this is fixed behavior.

***5.3** **a.** **Note:** For $r > 2.5$, students should extend their spreadsheets beyond 30 years to verify the existence either of a period or apparent chaotic behavior.

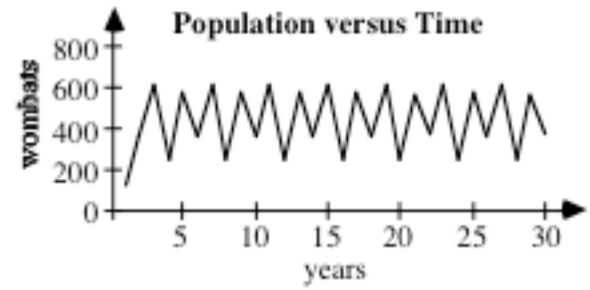
1. For $r = 2.25$, the long-term behavior is a cycle with period 2.



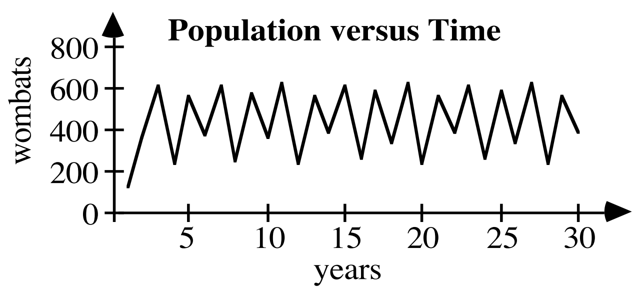
2. For $r = 2.5$, the long-term behavior is a cycle with period 4.



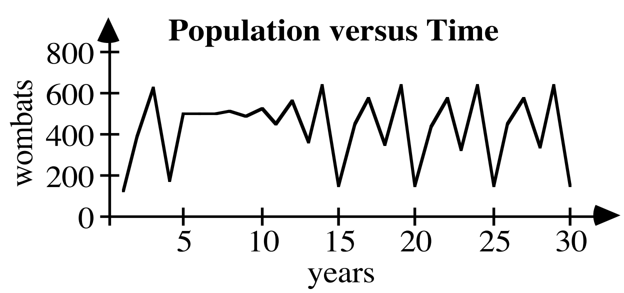
3. For $r = 2.55$, the long-term behavior is a cycle with period 8.



4. For $r = 2.57$, the long-term behavior is a cycle with period 16.



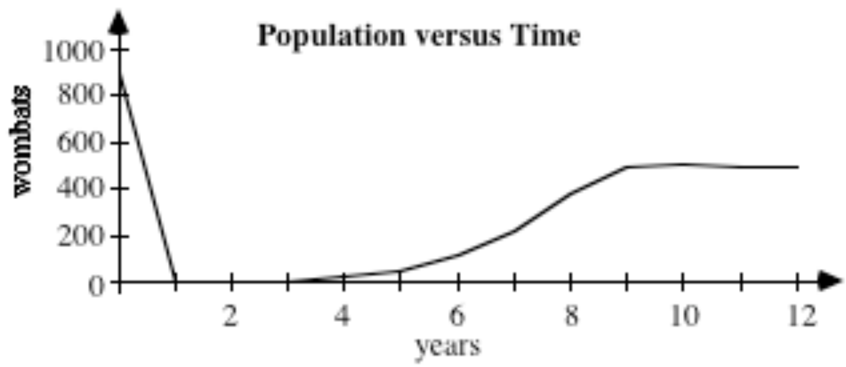
5. For $r = 2.75$, the long-term behavior of the population appears to be chaotic.



b. Sample response: If the model is accurate, then small changes in an initial condition like growth rate can have significant effects on the size of future populations. And because the behavior of the model changes dramatically for small changes in the growth rate, long-term predictions of population size would be risky at best.

5.4 Using a spreadsheet, students can vary the initial values, then observe what happens to the population over time. According to the model, if the initial population is 899, the wombat population would still approach the carrying capacity. However, most of the animals would die in the first year. Given an initial population is 900, the wombat population would fall to 0 after the first year. Sample data and graph:

Year	Population	Year	Population
0	899	7	227
1	2	8	382
2	5	9	495
3	11	10	501
4	25	11	500
5	55	12	500
6	116		



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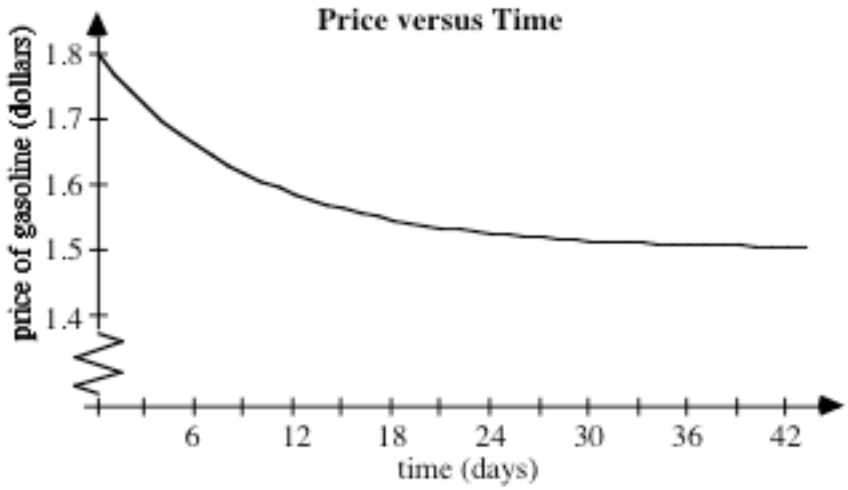
5.5 **a.** The equilibrium price can be found as follows:

$$900p - 500 = 1000 - 100p$$
$$1000p = 1500$$
$$p = \$1.50$$

b. Using $p_1 = \$1.80$ and $k = 0.0001$, the market price will approach the equilibrium price in 40 days. Sample graph:



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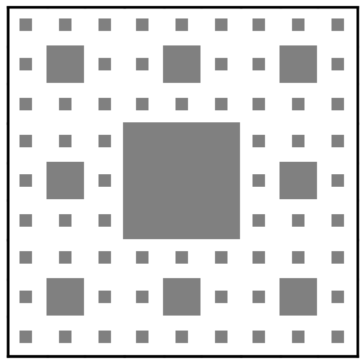
Answers to Summary Assessment (page 405)

1. **a.** Sample response: Start with a square. Trisect each side and create nine smaller squares. Shade (or remove) the middle square. Repeat the process on the remaining eight squares, and continue iterating.

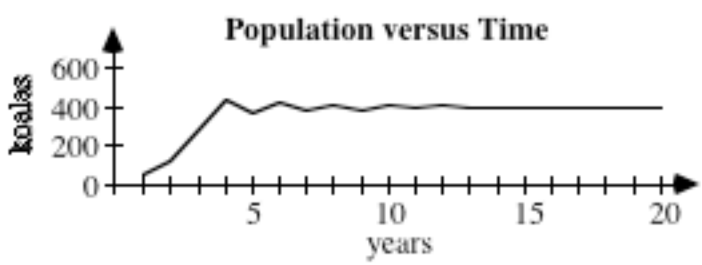
b. Sample sketch of stage 3 of Sierpinski's carpet:



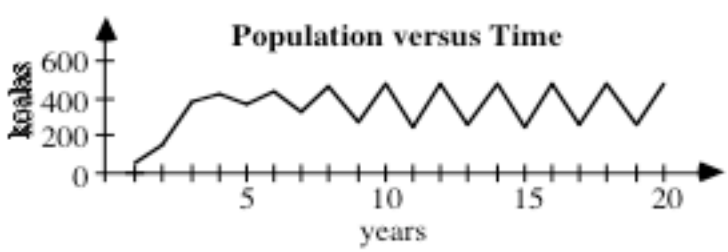
c. Assuming that the area of the original square is 1 unit^2, the sequence of areas is $0,\ 1/9,\ 17/81,\ 217/729,\ 2465/6561,\ \ldots$ or approximately $0,\ 0.1,\ 0.21,\ 0.29,\ 0.37,\ 0.44,\ \ldots$. The area of the shaded region approaches 1.

2. Answers will vary. Responses should include a description of the iterative process used to create the model, as well as drawings of at least the first three stages.

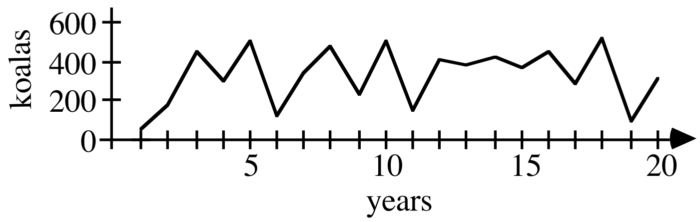
3. Answers will vary. Sample response: For $r = 1.8$, the orbit of the population becomes fixed at the carrying capacity, as shown below.



For $r = 2.4$, the long-term behavior of the population size appears to be periodic, as shown in the following graph:



For $r = 2.8$, the long-term behavior of the population size appears to be chaotic, as shown in the following graph:



4. **a.** The limit of each orbit is $\sqrt{a}$, $a \neq 0$.

b. Sample response: When used in an iterative process, the function approximates the square root of a.

c. Sample response: When using negative values for a, the result is a chaotic orbit with no identifiable limit. This is appropriate because no real value for $\sqrt{a}$ exists when a is negative.

d. Answers will vary. Sample response: The limit of the orbit generated by $g(x)$ will be the $\sqrt[3]{a}$.

e. Sample response: The limit of each orbit is $\sqrt[3]{a}$. When used in an iterative process, the function approximates the cube root of a.

f. Students may predict that $h(x)$ results in $\sqrt[4]{a}$. This is not the case.

g. Sample response: When $n = 1$, the limit of the orbit generated by $h(x)$ is $\sqrt{a}$. When $n = 2$, the limit of the orbit generated by $h(x)$ is $\sqrt[3]{a}$. Other values for n do not appear to result in useful approximations.

Module Assessment

1. In the late 19th century, the German mathematician Georg Cantor worked with the algorithm described below.

- Begin with a line segment (stage 0).
- Divide it into thirds and remove the middle segment (stage 1).
- Repeat the previous step on the remaining segments.

This algorithm generates a fractal known as the Cantor set.

a. Construct the Cantor set through stage 4.

b. Assuming that the original length of the segment at stage 0 is 1 unit, find the sum of the lengths of the segments at each stage. Show these sums as a sequence of numbers.

c. Use the sequence from Part **b** to determine the relationship between the stage number and the total length of all segments at that stage.

d. Use the relationship from Part **c** to complete the following.

1. Verify the total length of all segments at stage 2.
2. Determine the total length of all segments at stage 8.
3. Determine the total length of all segments at stage *n*.

e. Georg Cantor was interested in the concepts of limits and of infinity. Use limit notation to express the total length of all segments as the number of stages increases without bound.

2. Imagine that 30 crocodiles are currently living in a habitat with a carrying capacity of 150 crocodiles. Using the logistic model

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

and appropriate technology, create graphs of predicted population size versus time for annual growth rates of 1.8, 2.5, 2.7, 4.0, and 5.0. Classify the behavior of each graph as fixed, periodic, or chaotic.

3. **a.** Consider the function $g(x) = x^3 - 3x$. What are the fixed points of this function?

b. Show that when $x_0 = -\sqrt{2}$ or $x_0 = \sqrt{2}$, the orbit has a cycle of period 2.

c. Sketch the web plot illustrating the orbit from Part **b**.

4. Physicist and mathematician Isaac Newton (1642–1727) defined an iterative process still in use today. This process finds the roots of a function $f(x)$ after a relatively small number of iterations. Newton's formula is shown below, where x_n represents the initial guess for a root and $g(x_n)$ is the **derivative** of $f(x_n)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

a. Consider the function $f(x) = x^3 - x^2 - 14x + 24$ and its derivative $g(x) = 3x^2 - 2x - 14$. Perform iterations using Newton's formula with each of the following initial values:

1. $x_0 = 5$

2. $x_0 = -5$

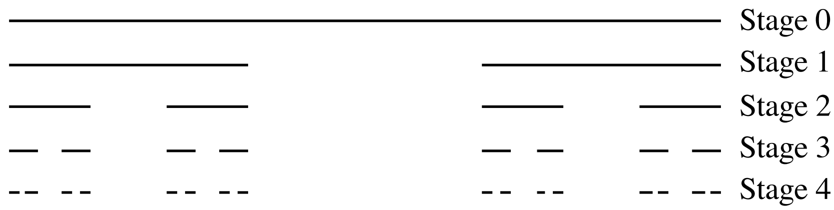
3. $x_0 = 1$.

b. Classify each orbit from Part **a** as fixed, periodic, or chaotic.

c. Verify that the results from Part **a** are the roots of the function *f*.

Answers to Module Assessment

1. **a.** The following figure shows the first 4 stages of the Cantor set:



b. At stage 0, the length of the segment is 1 unit. At stage 1, the sum of the lengths of the segments is $2/3$ units. At stage 2, the sum of the lengths of the segments is $4/9$ units. At stage 3, the sum of the lengths of the segments is $8/27$ units. At stage 4, the sum of the lengths of the segments is $16/81$ units. As a sequence, this can be written as follows:

$$\frac{1}{1}, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$$

c. The sum of the lengths of the segments, where n is the stage number, is $(2/3)^n$.

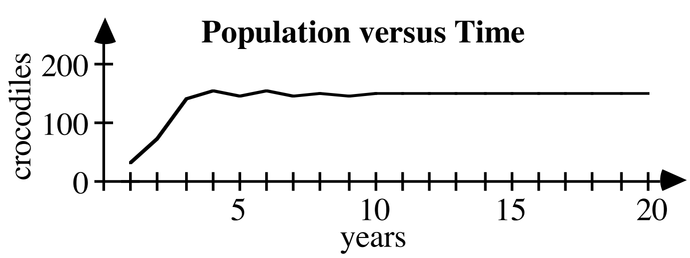
d. **1.** At stage 2, the sum of the lengths of the segments is $2^2/3^2 = 4/9$ units.

2. At stage 8, the sum of the lengths of the segments is $2^8/3^8 = 256/6561 \approx 0.039$ units.

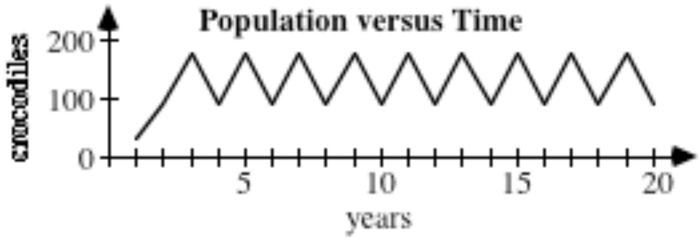
3. At stage n, the sum of the lengths of the segments is $(2/3)^n$ units.

e. As the number of iterations increases without bound, the total length of all segments approaches $\lim_{n \to \infty}(2/3)^n = 0$ units.

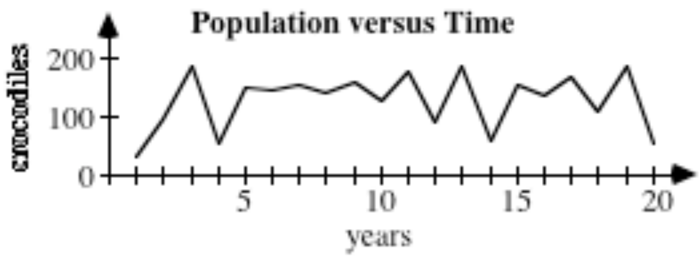
2. Sample response: For $r = 1.8$, the orbit becomes fixed, as shown in the graph below.



For $r = 2.5$, the orbit becomes periodic, as shown below:

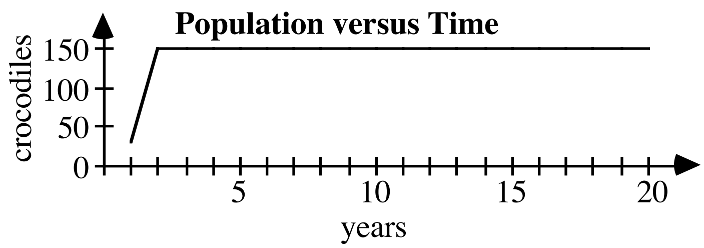


For $r = 2.7$, the orbit appears to be chaotic, as shown below:



For $r = 4$, the population becomes extinct in year 4.

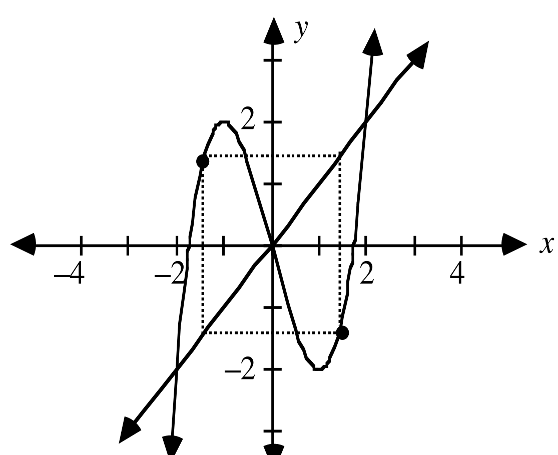
For $r = 5$, the orbit becomes fixed again, as shown below:



3. **a.** By solving $x = x^3 - 3x$, the fixed points are -2, 0, and 2.

b. Since $\left(\sqrt{2}\right)^3 - 3\sqrt{2} = -\sqrt{2}$ and $\left(-\sqrt{2}\right)^3 - 3\left(-\sqrt{2}\right) = \sqrt{2}$, the values of $-\sqrt{2}$ and $\sqrt{2}$ for the initial value generate an orbit with period 2.

c. Sample graph:



4. **a.** **1.** With an initial value of 5, the orbit converges quickly to 3, as illustrated in the following table.

Term No.	Orbit Member
0	5
1	3.94117647
2	3.35417645
3	3.08375158
4	3.00685194
5	3.00005292
6	3
7	3

2. With an initial value of –5, the orbit converges quickly to –4, as illustrated in the table below.

Term No.	Orbit Member
0	–5
1	–4.2112676
2	–4.0125791
3	–4.0000487
4	–4
5	–4

3. With an initial value of 1, the orbit converges quickly to 2, as illustrated in the table below.

Term No.	Orbit Member
0	1
1	1.76923077
2	1.97033685
3	1.99930929
4	1.9999996
5	2
6	2

b. Sample response: Since all three orbits converge to fixed values, they are fixed orbits.

c. Sample response: Since $f(3) = 0$, $f(-4) = 0$, and $f(2) = 0$, they are all roots of *f*.

Selected References

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Flashbacks

Activity 1

1.1 Write the first 10 terms of the sequence defined by the recursive formula below:

$$\begin{cases} t_1 = 1 \\ t_2 = 3 \\ t_n = t_{n-1} + 2t_{n-2} \end{cases}$$

1.2 What is the limit of the following sequence?

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

Activity 2

2.1 Given $f(x) = 3x^2$, find $f\big(f\big(f(-1)\big)\big)$.

2.2 Solve each of the following equations for x:

a. $3x^2 - 2 = x$

b. $\cos x = x$

Activity 3

3.1 Sketch a graph for equations of each of the following forms:

a. $y = ax + b$, where $a > 0$ and $b > 0$

b. $y = ax^2 + b$, where $a > 0$ and $b > 0$

c. $y = a/x$, where $a > 0$

d. $y = a/x^2$, where $a > 0$

Activity 4

4.1 Describe the role of c in the graph of the function $f(x) = x^2 + c$.

4.2 **a.** Given an initial value of 0, list the first 10 terms of the orbit for the equation below.

$$f(x) = -\frac{2}{3}x + 1$$

b. Find the fixed point of the orbit and classify it as an attractor, a repeller, or neither.

Activity 5

5.1 Consider the sequence described by the following recursive formula:

$$\begin{cases} p_1 = 1 \\ p_{n+1} = 0.8p_n + 3, \ n \geq 1 \end{cases}$$

a. Create a connected scatterplot of the first 30 terms of the sequence.

b. What is the limit of the sequence?

Answers to Flashbacks

Activity 1

1.1 1,3,5,11,21,43,85,171,341,683

1.2 The limit of the sequence is 1.

Activity 2

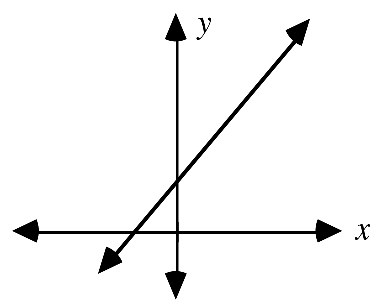
2.1 $f\left(f\left(f(-1)\right)\right) = 2187$

2.2 **a.** $x = -2/3$ and $x = 1$

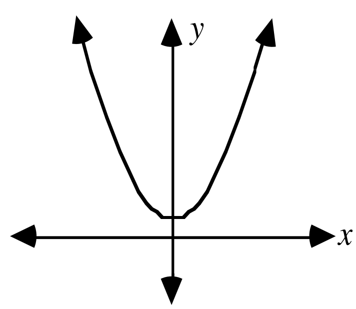
b. Sample response: One solution is $x \approx 0.74$.

Activity 3

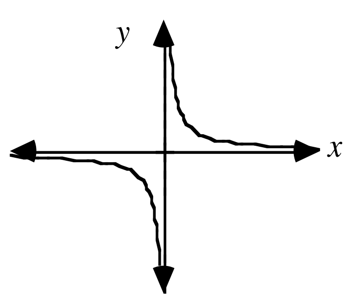
3.1 **a.** Sample sketch:



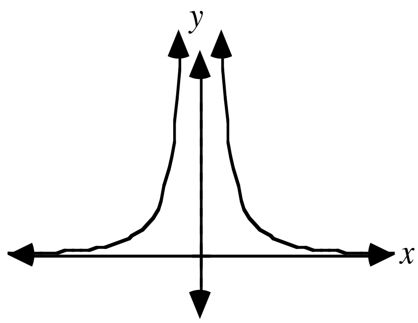
b. Sample sketch:



c. Sample sketch:



d. Sample sketch:



Activity 4

4.1 Sample response: The constant *c* determines the *y*-intercept of $f(x) = x^2 + c$.

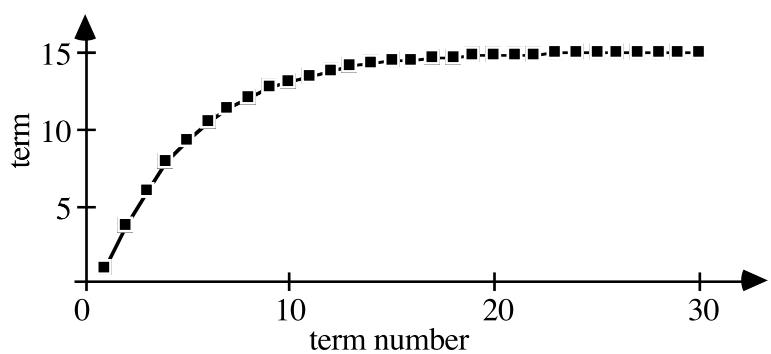
4.2 **a.** The first 10 terms of the orbit are:

$1,\ 0.33,\ 0.78,\ 0.48,\ 0.67,\ 0.55,\ 0.64,\ 0.58,\ 0.62,\ 0.59,\ \ldots$

b. The fixed point is $x = 3/5$; it is an attractor.

Activity 5

5.1 **a.** Sample graph:



b. The limit of the sequence is 15.

Pascal's Triangle Template