Reflect on This

What do you see when you look in the mirror? This module takes a peek at some concepts in both physics and geometry—through the looking glass.

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Introduction

A kaleidoscope creates fascinating designs that change as it is rotated. Two mirrors hinged together may be used to model the effects of a kaleidoscope. To make such a model, complete Parts **a–e** below.

a. Place the reflective sides of two mirrors face to face. Tape one set of edges together to make a hinge.

b. Cut small pieces of confetti from colored paper.

c. Overlap a sheet of white paper with a sheet of colored paper. Half of each sheet should remain visible.

d. Position the hinged mirrors across the two sheets of paper as shown in Figure **1**. The distance from *A* to *B* should be approximately equal to the distance from *A* to *C*.

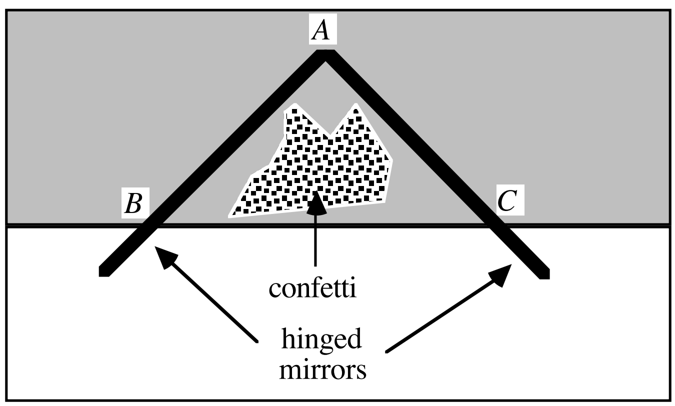

Figure 1: Kaleidoscope model

e.
1. Open and close the mirrors, keeping the distance from *A* to *B* approximately equal to the distance from *A* to *C*. Observe the patterns formed by the confetti and colored paper.
2. Make a record of the patterns formed by the colored paper.
2. Rearrange the confetti, then open and close the mirrors again. Observe the new patterns formed.

Discussion

a. Describe the patterns created by the colored paper in your kaleidoscope.

b. What happened as you opened and closed the hinged mirrors?

c. What is the relationship between the size of the hinge angle and the number of reflections seen

Activity 1

Opening and closing hinged mirrors causes multiple reflections. These reflections form patterns and shapes similar to those you created with a model kaleidoscope.

Mathematics Note

A **polygon** is a union of coplanar segments intersecting only at endpoints. At most two segments intersect at any one endpoint and each segment intersects exactly two other segments. Each segment in a polygon is a **side**; each endpoint is a **vertex** (plural **vertices**).

In Figure **2**, for example, points C, G, and O are vertices, while $\overline{AB}$, $\overline{EF}$, and $\overline{MN}$ are sides.

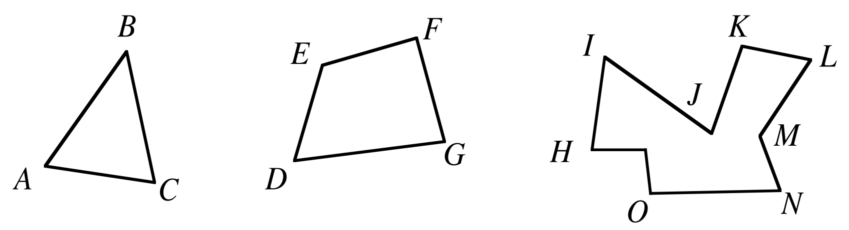

Figure 2: Three polygons

Exploration

In this exploration, you examine the polygons formed by reflections in hinged mirrors and investigate the relationship between these polygons and the hinge angle.

a. Reconstruct the kaleidoscope from the introduction (without the confetti).

b. Begin with the mirrors completely open. Slowly close them until a triangle is formed by the colored paper and its reflections. As shown in Figure **3**, the mirrors form a hinge angle at the center of the triangle. Draw this angle and determine its measure.

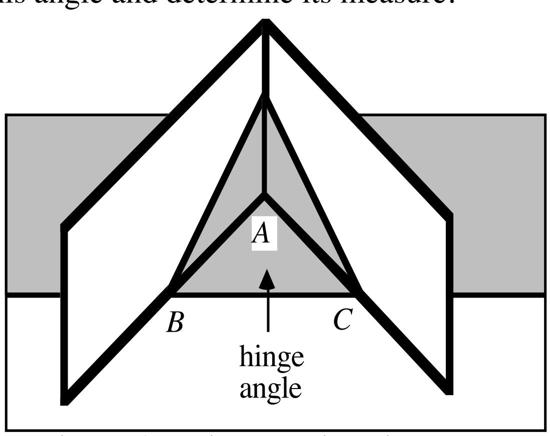

Figure 3: Triangle with hinge angle

c. Continue to close the mirrors, keeping the distance from A to B equal to the distance from *A* to *C*. Other polygons will appear, such as a quadrilateral, a pentagon, and a hexagon. Record the measure of the hinge angle for each polygon that appears, up to a decagon, in Table **1**.

Table 1: Polygon information

Polygon	No. of Sides	Measure of Hinge Angle
triangle	3	120°
quadrilateral		
pentagon		
hexagon		
heptagon		$51\frac{3}{7}$°
octagon		
nonagon		
decagon	10	

d. Use the patterns you observe in the table to describe a relationship between the number of sides of a polygon and the measure of the hinge angle.

Discussion

a. What appears to occur as the measure of the hinge angle becomes smaller?

Mathematics Note

A polygon is **equiangular** if its interior angles are congruent. A polygon is **equilateral** if its sides are congruent. A polygon is **regular** if its sides are congruent and its interior angles are congruent.

For example, Figure **4** shows two regular polygons, a square and an equilateral triangle.

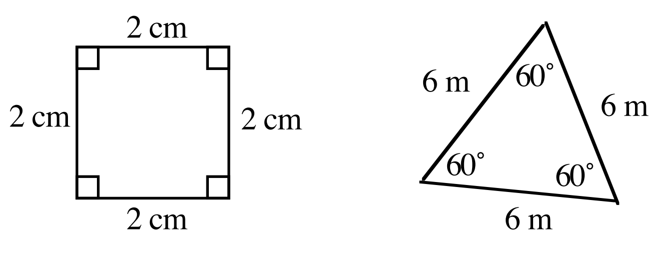

Figure 4: Two regular polygons

An angle formed by two rays drawn from the center of a circle is a **central angle**. If a regular polygon is inscribed in a circle (all vertices of the polygon lie on the circle), then the central angle formed by rays drawn from the center of the circle to two consecutive vertices of the polygon divide the polygon into congruent isosceles triangles.

For example, Figure **5** shows equilateral triangle ABC inscribed in a circle with center O. In this case, $\angle AOB$, $\angle AOC$, and $\angle BOC$ are central angles of ΔABC.

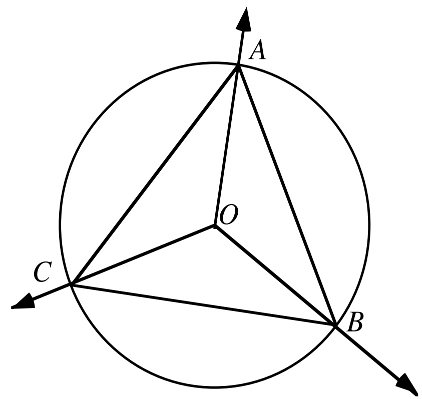

Figure 5: Inscribed equilateral triangle with central angles

b. Are the polygons formed by the reflections in the exploration always regular polygons? Why or why not?

c.
  1. What relationship exists between the number of sides of a regular polygon and the measure of its central angle?
  2. Express this relationship as an algebraic formula.

d. How many central angles are there in any regular polygon?

e. Identify the congruent isosceles triangles in Figure **5**.

f. What is the relationship between the vertex angles of the isosceles triangles and the central angles of the polygon?

g. What is the relationship between the base angles of the isosceles triangles and the interior angles of the polygon?

Assignment

1.1 Determine the measure of the central angle for a regular polygon with:

- **a.** 12 sides
- **b.** 18 sides
- **c.** 21 sides
- **d.** n sides.

1.2 Find the number of sides in a regular polygon with a central angle measure of:

- **a.** 40°
- **b.** 20°
- **c.** 24°
- **d.** 0°

1.3 Explain how to draw each of the following figures using only a ruler and two hinged mirrors.

a. two perpendicular lines

b. an angle with measure 120˚

c. a regular hexagon with sides 5 cm long

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1.4 Determine the measure of a central angle for each of the following:

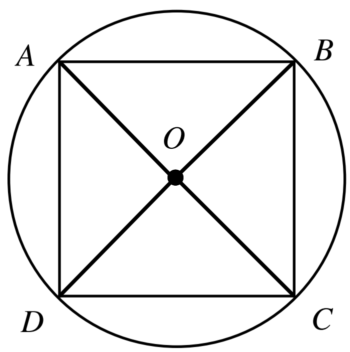
a. a regular pentagon

b. a regular octagon

c. a regular decagon

d. a regular *n*-gon

1.5 The following diagram shows square *ABCD* inscribed in a circle with center *O*. In this case, the measure of central angle *AOB* is 90˚, while the measure of angle *OAB* is 45˚.



What is the measure of angle *OAB* if the inscribed polygon is:

a. a regular pentagon?

b. a regular octagon?

c. a regular decagon?

d. a regular *n*-gon?

1.6 **a.** Find the measure of the interior angles in each of the following:

1. a regular pentagon

2. a regular octagon

3. a regular decagon

4. a regular *n*-gon

b. Determine the sum of the measures of the interior angles for each regular polygon in Part **a**.

1.7 As mentioned in the mathematics note, the central angles of a regular polygon divide the polygon into congruent isosceles triangles. Describe how you could use the area of these triangles to find the area of each of the following:

a. a regular pentagon

b. a regular octagon

c. a regular decagon

d. a regular *n*-gon

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Activity 2

When you look into your model kaleidoscope, the images you see are caused by light rays reflecting from the confetti, striking the hinged mirrors, then bouncing off the mirrors into your eyes. As you change the positions of the mirrors, the paths of the light rays also change. In this activity, you explore the paths that light rays follow in a reflection.

Exploration

a. Draw an *x*-axis and a *y*-axis on a sheet of graph paper with the origin (0,0) near the center of the sheet. Label the origin as point *A* and a point on the positive *x*-axis as point *B*.

b. As shown in Figure **6**, place a mirror tightly against one edge of a block of fiberboard and perpendicular to the broad face of the block.

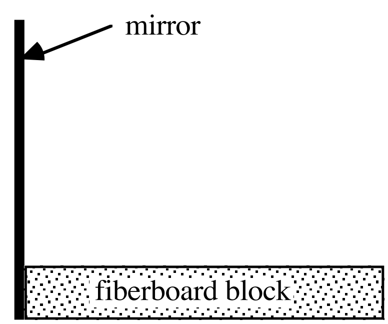

Figure 6: Light-ray experiment (side view)

c. Carefully fold the graph paper along the x-axis. Place the folded sheet on top of the fiberboard block with the x-axis tightly against the mirror, as shown in Figure **7**.

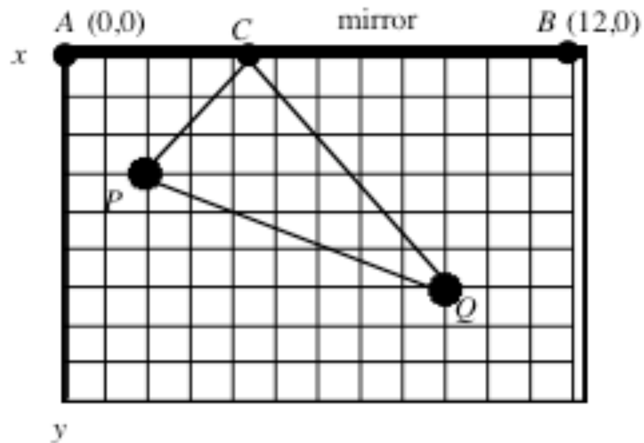

Figure 7: Light-ray experiment (top view)

d. Choose two points that fit the description below.

1. The x-coordinates are at least 3 units apart.

2. The y-coordinates are different and between -3 and -7.

e. **1.** Label the point with the lesser x-coordinate as P and the other point as Q.

2. Place one pushpin through P and another through Q.

3. Sight through P toward the mirror with one eye closed. Looking at the reflection of Q, move your head until P aligns with the image of Q.

4. Mark a point C on the x-axis where P, C, and the reflection of Q are collinear (on the same line).

f. Place a rubber band around the pushpins. Using the point of a pencil, pull the side of the rubber band closer to the mirror until the pencil is at C. Check the position of C again to make sure P, C, and the reflection of Q are collinear. The side of the rubber band you stretched shows the path that light rays follow from Q to the mirror to P.

g. **1.** Remove your graph paper from the fiberboard block and unfold it. Draw $\overline{PC}$ and $\overline{CQ}$. These segments show the path of light as it travels from the pushpin at Q to your eye at P by reflecting off the mirror.

2. Measure $\angle QCB$ and $\angle PCA$. Record these angle measures. **Note:** Save your graph paper for use in the assignment and in Activity **3**.

Science Note

Light rays reflecting off a flat surface form rays and angles as shown in Figure **8**.

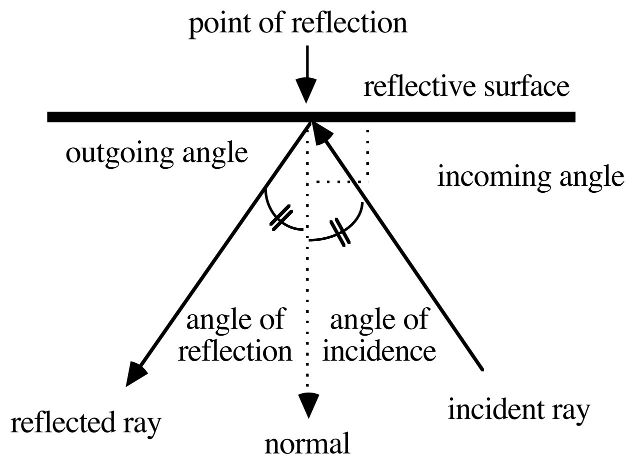

Figure 8: A reflected light ray

The **incident ray** models the path that a light ray follows from the object to the reflective surface, while the **reflected ray** models the path of the light ray away from the reflective surface.

The **incoming angle** is the angle between the reflective surface and the incident (incoming) ray. The **outgoing angle** is the angle between the surface and the reflected (outgoing) ray.

The **normal** of the reflecting surface is the perpendicular line to the surface at the **point of reflection**. The **angle of incidence** is the angle formed by the normal and the incident ray, while the **angle of reflection** is the angle formed by the normal and the reflected ray.

Discussion

a. Using terms from the Science Note, describe the path traveled by a light ray that passes through Q and strikes the mirror before reaching your eye.

b. Make a conjecture about the relationship between the measures of the incoming angle and the outgoing angle.

c. Compare the angle measures you recorded with others in your group. Do any of these measurements contradict your conjecture? If so, do you still believe that your conjecture is correct? Explain your response.

d. Describe several ways in which an error in measurement could occur in the exploration. How might these errors affect your conclusions?

e. Suggest another method for confirming your conjecture in Part **c**.

Mathematics Note

If the sum of the measures of two angles is 90°, then the two angles are **complementary**. In Figure **9**, for example, $\angle BAC$ and $\angle HIG$ are complementary angles because $30° + 60° = 90°$.

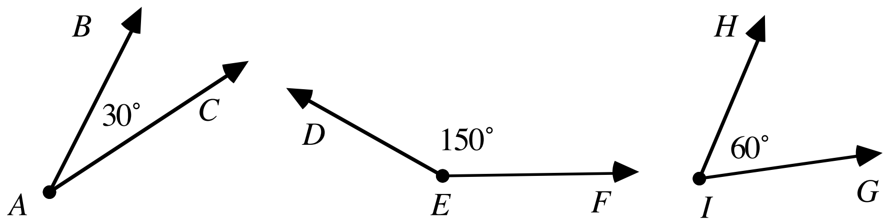

Figure 9: Three angles

If the sum of the measures of two angles is 180°, then the two angles are **supplementary**. In Figure **9**, $\angle BAC$ and $\angle DEF$ are supplementary angles because $30° + 150° = 180°$.

f. Identify the pairs of angles in Figure **8** that are complementary.

Assignment

2.1 Use your graph from the exploration to complete Parts **a–c** below.

a. **1.** Draw and label a normal on the graph at the point where the light rays were reflected.

2. Measure and label the angle of incidence and the angle of reflection.

b. Use the graph to make a conjecture about the relationship between each of the following pairs of angles:

1. the angle of incidence and the angle of reflection

2. the incoming angle and the angle of incidence

3. the outgoing angle and the angle of reflection.

c. Write an argument to convince a classmate that your conjectures are correct.

2.2 **a.** Use a protractor to draw a light ray bouncing off a mirror with an angle of incidence of 55°.

b. Draw and label the normal to the surface at the point of reflection.

c. Label all angles and give their measures.

2.3 **a.** Prepare a labeled diagram that shows what happens when a ray of sunlight reflects off the surface of a watch.

b. Write a summary of the mathematical ideas represented in the diagram.

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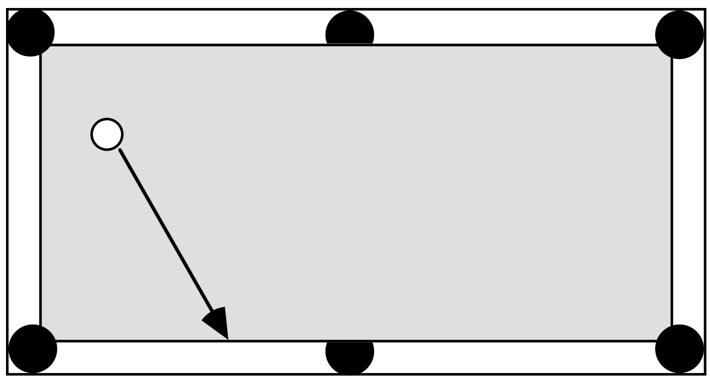
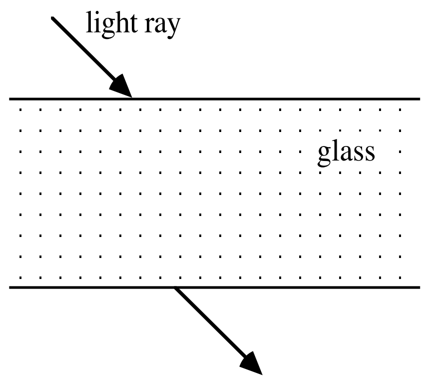
2.4 Using mathematical terms, describe what happens when a driver adjusts the side mirror on a car.

2.5 Reflections occur not only in mirrors, but also in other situations. In the game of pool, players use a cue stick to strike a white ball (the cue ball). When a player hits the cue ball correctly, it strikes one of the other balls and knocks it into one of six holes, or pockets, on the table. In a bank shot, a ball bounces off a side rail before falling into a pocket. The path of a ball bouncing off a side rail is like the path of light reflecting off a mirror.



Suppose a pool ball is hit along the path shown in the diagram. Will the ball fall into the upper right corner pocket after a single bounce off the side rail? Explain your response.

2.6 Light rays passing through glass are bent towards the normal. This bending of light rays is called **refraction.** For example, a stick in water appears bent because of refraction. The light ray enters and exits the glass as shown below.



a. Copy the diagram above and draw the path of light from the entry point to the exit point.

b. Draw and label the normals at the entry and exit points.

c. In your drawing from Parts **a** and **b**, identify and label all pairs of angles that appear to be congruent.

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Research Project

1. Describe how to determine the length of the smallest mirror that would allow every member of your family to see a complete reflection from head to toe. Include a diagram with your report.

2. Write a report about periscopes that answers the following questions.

a. What is a periscope?

b. In what types of situations are periscopes useful?

c. How does a periscope work? (Include a diagram in your report.)

d. How can you make a simple periscope using a tube and two mirrors? (Include a model with your report.)

Activity 3

When you look in a mirror, your reflection sometimes appears to be behind the mirror's surface. This illusion, often called a **virtual image**, is created by light rays reflecting off the mirror. In this activity, you explore the relationship between the position of an object and the apparent position of its reflection in a mirror.

Exploration 1

a. Unfold your sheet of graph paper from the exploration in Activity **2** and lay it flat. Position a tinted plastic reflector along the *x*-axis with points P and Q on your side of the reflector.

Mathematics Note

The reflection of an object is its **image**. The object itself is the **preimage**. If point A is the preimage, then the image of a point A can be represented as A' (read "A prime").

b.

1. Find the reflection of point Q in the reflector.

2. Place your pencil behind the reflector and mark the point on the graph paper where the reflection of Q appears to be. This point is the image of Q. Label it Q'.

3. Remove the reflector and draw $\overline{PQ'}$. As shown in Figure **10**, points P, C, and Q' should be collinear.

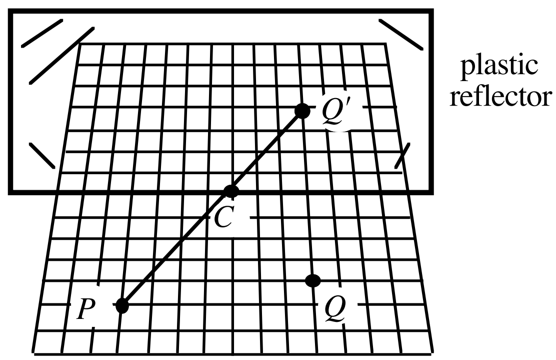


Figure 10: Finding Q'

c. **1.** Use the plastic reflector to mark the image of point P on your graph paper. Label this point P'.

2. Remove the reflector and draw $\overline{QP'}$.

d. The **distance from a point to a line** is the distance along a path perpendicular to the line. Measure the distance from the x-axis to each of the following points:

1. Q and Q'

2. P and P'.

e. Make a conjecture about the relationship between the distance from an object to the mirror line and the distance from its image to the mirror line.

Discussion 1

a. In Activity **2**, you pushed pins through two layers of folded graph paper at points P and Q. How do the locations of the pinholes in the second layer compare with the locations of P' and Q'?

b. How is the distance from Q to the x-axis related to the distance from Q' to the x-axis?

c. **1.** Where do $\overline{PQ'}$ and $\overline{QP'}$ intersect?

2. What does this tell you about the paths traveled by reflected light?

d. Does the class data support the conjecture you made in Part **e** of Exploration **1**? Explain your response.

Exploration 2

In this exploration, you use technology to model the process described in Exploration **1**. Most geometry utilities have reflection tools for finding the images of objects. To use this feature, you must first define a mirror line and the points to be reflected.

a.

1. Construct a mirror line segment, $\overline{AB}$.

2. Construct two points, P and Q, on the same side of $\overline{AB}$.

3. Reflect points P and Q in $\overline{AB}$ and label the images P' and Q', respectively.

4. Construct $\overline{PQ'}$ and $\overline{QP'}$.

5. Construct a point C at the intersection of $\overline{AB}$ and $\overline{PQ'}$.

6. Measure $\angle PCA$ and $\angle QCB$.

7. Construct $\overline{PP'}$ and $\overline{QQ'}$.

8. Label the intersection of $\overline{AB}$ and $\overline{PP'}$ as point D and the intersection of $\overline{AB}$ and $\overline{QQ'}$ as point E. Your construction should now resemble the one in Figure **11** below.

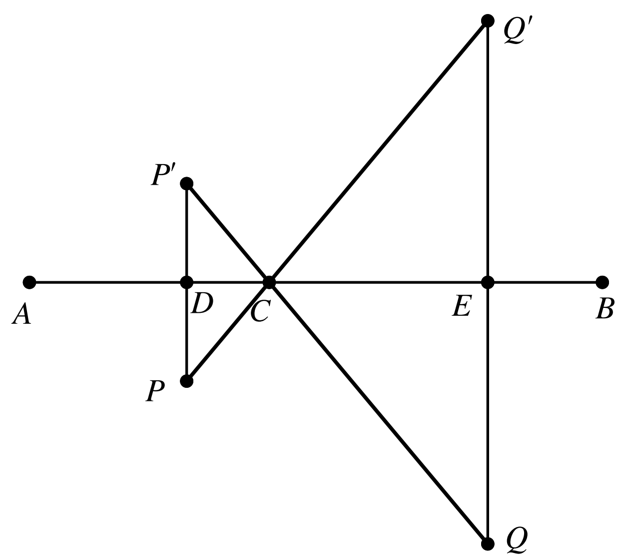


Figure 11: Reflection modeled on a geometry utility

9. Measure $\angle PDA$ and $\angle QEB$.

10. Measure the distances from $\overline{AB}$ to points P, P', Q, and Q'.

11. Measure $\overline{EQ}$ and $\overline{EQ'}$.

b. Use your measurements in Part **a** to make a conjecture about the relationship between $\overline{AB}$ and $\overline{QQ'}$.

c. Select and move various points on your construction. Note any relationships you observe among the measurements of angles and segments as you move the points.

d. Complete Steps **1–3** below using your construction from Part **a**.

1. Construct a point S anywhere on $\overline{AB}$.
2. Measure the total distance traveled when moving from Q to S and then from S to P.
3. By moving S along $\overline{AB}$, find the point where the total distance traveled in Step **2** is shortest.

Mathematics Note

The **perpendicular bisector** of a segment is the line that is perpendicular ($\perp$) to the segment and divides the segment into two congruent parts.

In Figure **12**, for example, line m is the perpendicular bisector of $\overline{CC'}$ because line m and $\overline{CC'}$ are perpendicular and $\overline{CM} \cong \overline{C'M}$.

A **reflection** in a line is a pairing of points in a plane so that the **line of reflection** (or **mirror line**) is the perpendicular bisector of every segment connecting a point in the preimage to its corresponding point in the image. Every point on the line of reflection is its own image.

In Figure **12**, parallelogram $C'D'E'F'$ is the image of parallelogram $CDEF$. Line m is the line of reflection since it is the perpendicular bisector of each segment joining a point in the preimage to its corresponding point in the image.

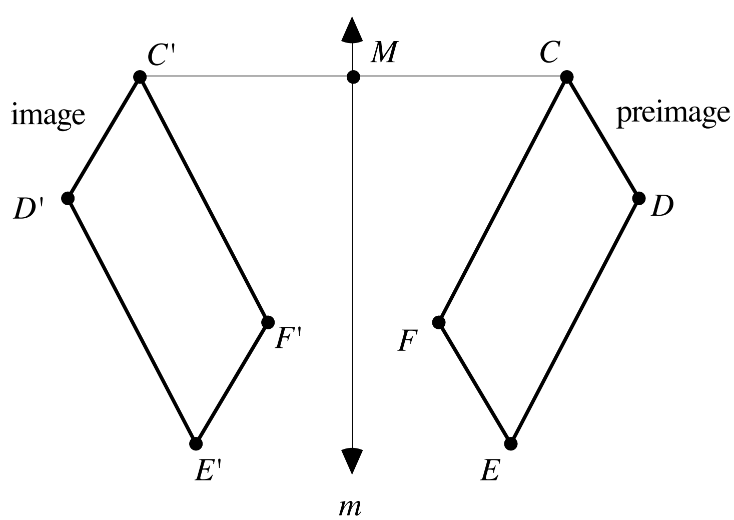


Figure 12: Reflection of parallelogram *CDEF* in line *m*

Discussion 2

a.

1. Make a conjecture about the relationship between $\overline{AB}$ and $\overline{PP'}$ in Figure **11**.
2. Do your measurements in Exploration **2** support this conjecture?

b. What conjecture can you make about the path which results in the shortest total distance in Part **c** of Exploration **2** and the point of reflection?

c. Imagine that $\overline{AB}$ in Figure **11** is on the x-axis.

1. What does the measure of $\angle QEB$ tell you about the x-coordinates of points Q and Q'?
2. What do the lengths of $\overline{EQ}$ and $\overline{EQ'}$ tell you about the y-coordinates of Q and Q'?
3. Which segments would you measure to compare the y-coordinates of points P and P'?

d.

1. Describe the relationship between the coordinates of point Q and its image Q'.
2. Describe the relationship between the coordinates of point P and its image P'.
3. Are these relationships also true for each preimage and image in Exploration **1**?

e. What generalization might you make about the relationship between the coordinates of a point in the preimage and the coordinates of its image under a reflection in each of the following lines:

1. the x-axis?
2. the y-axis?

f. Many mathematical ideas are based on observed patterns. When conjectures based on patterns are stated in a general form and proved, they become accepted mathematical **theorems**.

1. In Activity **2**, you made a conjecture about the relationship between the measures of the incoming and outgoing angles. Do your measurements in Exploration **2** support this conjecture?
2. If so, could this conjecture now become a theorem?

Assignment

3.1 **a.** Plot three points on a coordinate grid so that one point has all positive coordinates, one point has all negative coordinates, and one point has one positive and one negative coordinate. Label these points A, B, and C.

b. Plot the reflections of A, B and C in the y-axis. Label the image points A', B', and C', respectively.

c. List the coordinates of each point and its image.

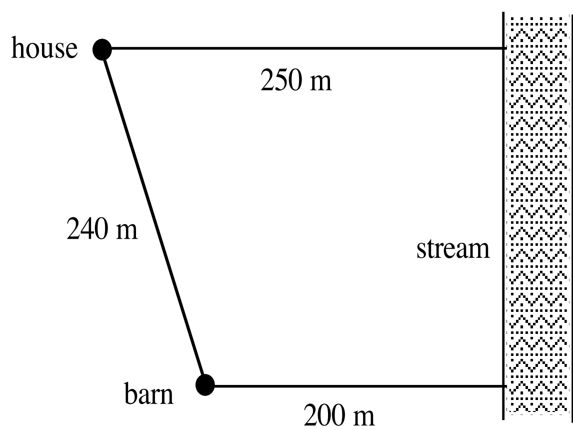
d. Consider a point D with coordinates (x,y). If D is reflected in the y-axis, what are the coordinates of D'?

3.2 Use your graph paper from the Exploration in Activity **2** to complete Parts **a** and **b** below.

a. Describe the shortest path from Q' to P.

b. How does the length of this path compare to the length of the path that light travels from Q to P? Explain your response.

3.3 A family wants to build the shortest possible trail from their house to the stream and then to the barn. The distances between these locations are shown in the diagram below.



a. Make a scale drawing of this situation.

b. Use your drawing to determine where the trail should reach the stream to make the shortest trail from the house to the stream to the barn.

c. Explain how you used the properties of reflections to determine your solution.

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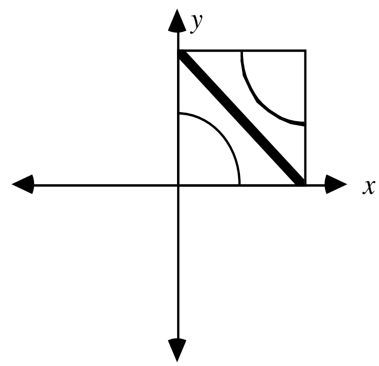
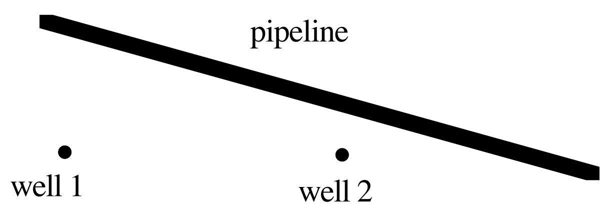
3.4 The diagram below shows a square tile with its lower left-hand corner at the origin of an xy-coordinate system. Sketch a copy of this diagram on a sheet of graph paper.



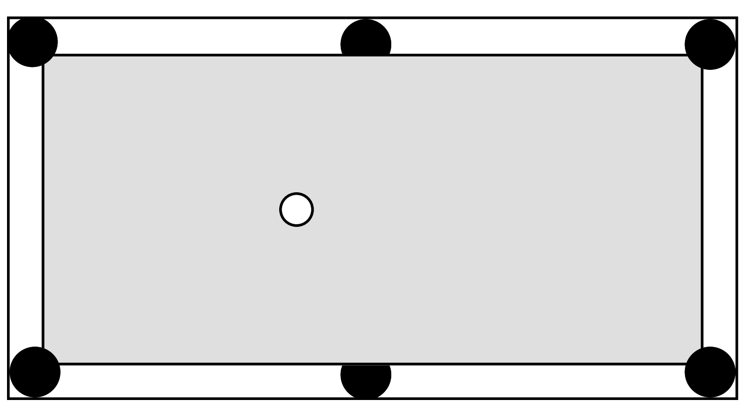
a. Sketch the image of the tile reflected in the x-axis.

b. Sketch the image of your response to Part **a** reflected in the y-axis.

3.5 The diagram below shows a section of an oil pipeline along with two nearby wells. To transport oil from the wells to the pipeline, the company plans to build a new pumping station somewhere on the line. If the company wants to minimize the amount of pipe needed to connect the two wells to the pipeline, where should it locate the pumping station?



3.6 The diagram below shows the top of a pool table along with a cue ball.



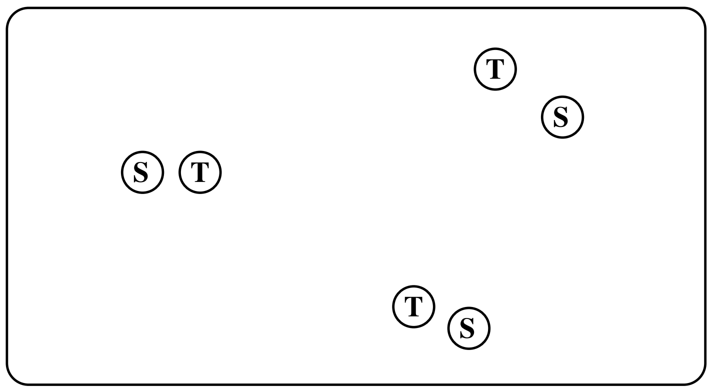
a. Trace a path that banks the cue ball off exactly one side and places it in a pocket.

b. Describe how you determined the path you chose in Part **a**.

c. Find another path to the same pocket, using a different side of the table to bank the ball.

3.7 In a hockey game between the Saints and the Tornadoes, six players are positioned as shown in the diagram below. Each circle that contains a letter S represents a member of the Saints; each circle that contains a T represents a member of the Tornadoes.

Draw all the paths that show a successful pass from the Saint on the left to one of the other two Saints. Each pass must reflect off one of the boards (sides). Explain why your pass is possible.



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Activity 4

In the previous activities, you examined how light reflects off a single flat mirror. The kaleidoscope you built in the introduction, however, used two mirrors. In this situation, light reflects off one mirror, then the other. In the following exploration, you investigate the virtual images produced by light reflecting off two mirrors.

Exploration 1

a. On a sheet of graph paper, draw two perpendicular line segments and a point Q, as shown in Figure **13** below.

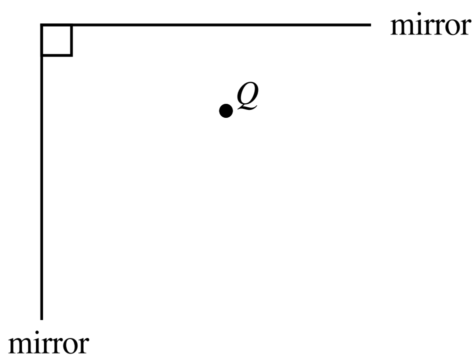


Figure 13: Two perpendicular mirrors

b. If hinged mirrors like those used in Activity **1** were placed along the line segments, how many images of point Q would you see in the mirrors? Record your prediction.

c. Use two hinged mirrors to test your prediction in Part **b**. Record how many images of point Q you actually observe.

d. To discover why this number of images occurs, use your diagram from Part **a** to complete Steps **1–6** below.

1. Reflect Q in one of the line segments and label the image Q_1'.

2. Reflect Q in the other line segment and label this image Q_2'.

3. Place the hinged mirrors on the segments again. Observe that two of the virtual images of Q in the mirrors correspond to the two images of Q found in Steps **1** and **2**.

4. Remove the hinged mirrors. Reflect Q_1' in the other line segment and label its image Q'' (read "Q double-prime").

5. Repeat Step **4** using Q_2' to locate Q_2''. What do you observe about the positions of Q'' and Q_2''?

6. Place the hinged mirrors back on the line segments and observe the virtual image that corresponds to Q''. **Note:** Save your diagram for use in Exploration **2**.

Discussion 1

a. How did the number of images you observed in Part **c** of Exploration **1** compare with your prediction in Part **b**?

b. Explain why the two mirrors in Exploration **1** produce three virtual images.

c. In Exploration **1**, you labeled the image of a single reflection Q', and the image of a double reflection Q''. How would you label the image of a triple reflection?

d. How do your observations in Part **d** of Exploration **1** help explain the patterns you saw in the kaleidoscope?

Exploration 2

In this exploration, you use the properties of single reflections to investigate the path of light in a double reflection.

a. Add a point E to your diagram from Exploration **1**, as shown in Figure **14** below. In this diagram, point E represents the location of your eye.

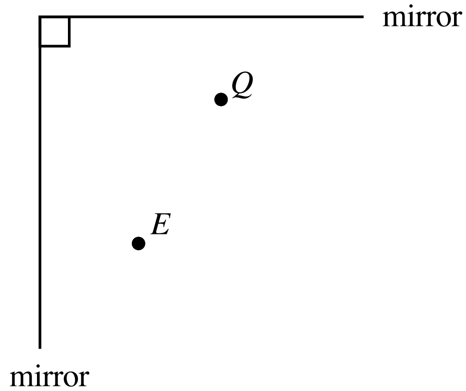


Figure 14: Perpendicular mirrors with points *E* and *Q*

b. To find the path light travels in a double reflection from Q to E, complete Steps **1–4** below.

1. Use a straightedge to connect E and Q''.
2. Label the point where $\overline{EQ''}$ intersects the mirror line segment as C_2.
3. Use a straightedge to connect C_2 and Q_1'.
4. Label the point where $\overline{C_2Q_1'}$ intersects the other mirror line segment as C_1.

c. Draw the path from Q to C_1, then to C_2, and finally to E. **Note:** Save your diagram for use in the assignment.

Discussion 2

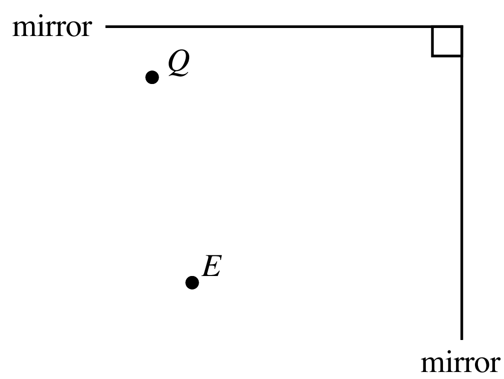
a. What does the path drawn in Part **c** of Exploration **2** represent?

b. What do the points C_1 and C_2 represent?

c. Using light rays, explain why the virtual image that corresponds to Q'' appears where it does in the mirror.

Assignment

4.1 The diagram below shows two perpendicular mirrors and the points Q and E. On a copy of this diagram, draw the path light travels in a double reflection from Q to E.



4.2 Use the diagram of two perpendicular mirrors you created in Exploration **2** to complete Parts **a–e** below.

a. Label the intersection of the two mirror line segments as point Z (0,0) and graph two more points: A (0,–4) and B (4,0).

b. Use a protractor to measure $\angle ZC_2C_1$ to the nearest degree. Record the measurement on your graph paper and label the angle as either outgoing or incoming.

c. Explain how the measurements of the remaining angles can be determined by knowing only the measure of $\angle ZC_2C_1$.

d. Label the remaining three angles as incoming or outgoing and record their measures.

e. **1.** Calculate the sum of the measures of the two incoming and two outgoing angles.

2. Use the sum of the four angles to make a conjecture about the path of a light ray in a double reflection or the path of a pool ball in a double bank shot.

4.3 To model a pair of mirrors, draw two perpendicular segments on a sheet of graph paper as in Exploration **2**. Label the intersection of the two segments point Z (0,0) and graph two more points: A (0,–4) and B (4,0). Find the path that light would travel from point O (4,–1) to point E (3,–7) if it reflected off both mirrors. Label all points, angles, and segments that correspond to those found in Exploration **2**. Identify the incoming and outgoing angles and give their measures.

4.4 **a.** Use a protractor to draw two line segments representing mirrors hinged at 70˚ on a sheet of blank paper.

b. Label two points between the mirrors: point O, representing an object, and point E, representing the perspective of an eye.

c. Find the locations of the images of O. Label these points O' and O'' .

d. Sketch the path of light from O to E in a double reflection.

e. Measure one of the incoming or outgoing angles and record this measurement on your drawing.

f. Determine the measures of the remaining three angles and record these measures on your drawing. Label all angles either as incoming or outgoing.

g. Calculate the sum of the measures of all four angles. Does this sum support the conjecture you made in Problem **4.2**?

4.5 A full-size pool table measures approximately 126 cm ∞ 255 cm.

a. Make a scale drawing of a pool table, including the six pockets.

b. Mark the location of a ball somewhere on your scale drawing, then draw the path of a shot that requires the ball to strike at least two side rails before reaching a pocket.

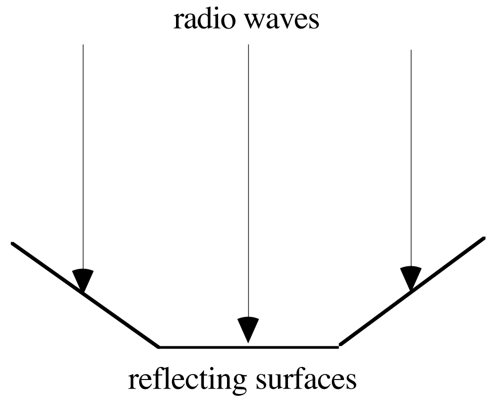
c. Label the appropriate points (with measurements in centimeters), so that a pool player could:

1. locate the starting position of the ball on an actual pool table
2. locate the point where the ball must hit the first side rail in order to make the shot.

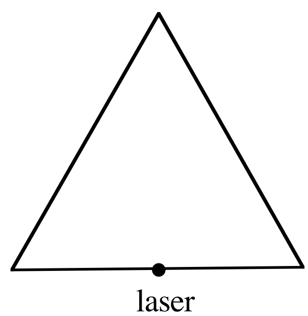
d. If possible, test your calculations on a real pool table, and write an account of your test.

* * * * *

4.6 Some large radio telescopes use several reflecting surfaces positioned in special ways. The diagram below shows three such surfaces in a radio telescope built in Arecibo, Puerto Rico, in 1963. On a copy of this diagram, draw the path of each reflected radio wave and explain why these paths occur.



4.7 The diagram below shows three hinged mirrors positioned in the shape of an equilateral triangular prism. A hole in the center of one mirror allows a laser beam to enter the prism.



a. Draw a path for the laser beam that allows it to reflect off each of the other two mirrors exactly once and pass back through its starting point.

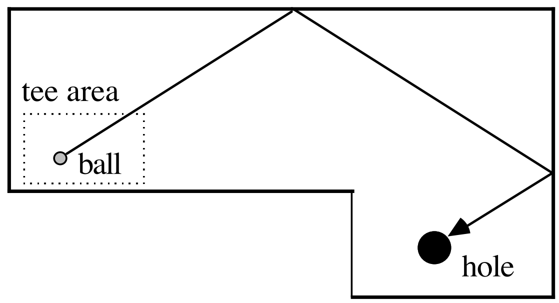
b. Determine the measures of every incoming and outgoing angle and the measures of every angle of incidence and angle of reflection.

c. Identify the shapes formed by the lines of reflection. How are these shapes related to the original triangle?

* * * * * * * * * *

Summary Assessment

1. Most miniature golf courses have holes requiring a player to hit the ball off at least one wall to score a hole-in-one. An example is shown in the diagram below.



Design and draw some miniature golf holes according to the following three rules.

- The drawing must be to scale, with all dimensions indicated.
- Only line segments may be used for walls.
- A tee area must be provided.

As you design holes, sketch some possible paths for a hole-in-one. In your final drawing for each hole, however, do not reveal your winning strategy. Design one of each of the following types of holes.

a. A hole that looks simple, but where a hole-in-one is probably impossible.

b. A hole that looks difficult, but has a simple path for the ball.

c. A hole that has many possible paths.

d. A hole that requires a player to bank the ball off exactly three walls to get a hole-in-one.

2. To test your designs, trade drawings with a classmate. Try to sketch the path of a hole-in-one in each of your classmate's drawings.

3. After your designs have been tested, present one of them to the rest of the class. Use mathematical ideas and the language of this module to explain your design.

Module Summary

- A **polygon** is a union of coplanar segments intersecting only at endpoints. At most two segments intersect at any one endpoint and each segment intersects exactly two other segments. Each segment in a polygon is a **side**; each endpoint is a **vertex** (plural **vertices**).
- A polygon is **equiangular** if its interior angles are congruent. A polygon is **equilateral** if its sides are congruent. A polygon is **regular** if its sides are congruent and its interior angles are congruent.
- An angle formed by two rays drawn from the center of a circle is a **central angle**.
- If a regular polygon is inscribed in a circle (all vertices of the polygon lie on the circle), then the central angle formed by rays drawn from the center of the circle to two consecutive vertices of the polygon divide the polygon into congruent isosceles triangles. The measure of a central angle of a regular polygon with n sides is:

$$\frac{360°}{n}$$

- The measure of each **interior angle** of a regular polygon can be determined using the following formula, where n represents the number of sides:

$$180^\circ - \frac{360^\circ}{n}$$

- The sum of the measures of the interior angles of a regular polygon can be determined using the following formula, where n represents the number of sides:

$$\left(180^\circ - \frac{360^\circ}{n}\right)(n) = 180^\circ n - 360^\circ$$

- The **incident ray** models the path that a light ray follows from the object to the reflective surface, while the **reflected ray** models the path of the light ray away from the reflective surface.
- The **incoming angle** is the angle between the reflective surface and the incident (incoming) ray. The **outgoing angle** is the angle between the surface and the reflected (outgoing) ray.
- The **normal** of the reflecting surface is the perpendicular line to the surface at the **point of reflection**. The **angle of incidence** is the angle formed by the normal and the incident ray, while the **angle of reflection** is the angle formed by the normal and the reflected ray.

- If the sum of the measures of two angles is 90°, then the two angles are **complementary**.
- If the sum of the measures of two angles is 180°, then the two angles are **supplementary**.
- The incoming angle and the angle of incidence are **complementary**, in other words, the sum of their measures is 90°. The same is true for the outgoing angle and the angle of reflection.
- The **distance from a point to a line** is the distance along a path perpendicular to the line.
- The reflection of an object is its **image**. The object itself is the **preimage**. If point *A* is the preimage, then the image of a point *A* can be represented as A' (read "*A* prime").
- The **perpendicular bisector** of a segment is the line that is perpendicular ($\perp$) to the segment and divides the segment into two congruent parts.
- A **reflection** in a line is a pairing of points in a plane so that the **line of reflection** is the perpendicular bisector of each segment connecting a preimage point to its corresponding image point. Every point on the line of reflection is its own image.
- When conjectures based on patterns are generalized and shown to be true for all cases, they become accepted mathematical **theorems.**

Selected References

Iowa Academy of Science. *Physics Resources and Instructional Strategies for Motivating Students*. Cedar Falls, IA: University of Iowa, 1985.

Murphy, J., and R. Smoot. *Physics Principles and Problems*. Columbus, OH: Charles E. Merrill Publishing Co., 1977.