Going in Circuits

After leaving for work in the morning, a bus driver must make six stops before returning home. There are more than 500 ways to complete this route. Which ones are best? In this module, you explore some ways to find an efficient route without testing all the possibilities.

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Introduction

The map in Figure **1** shows the floor plan of William R. Hamilton High School.

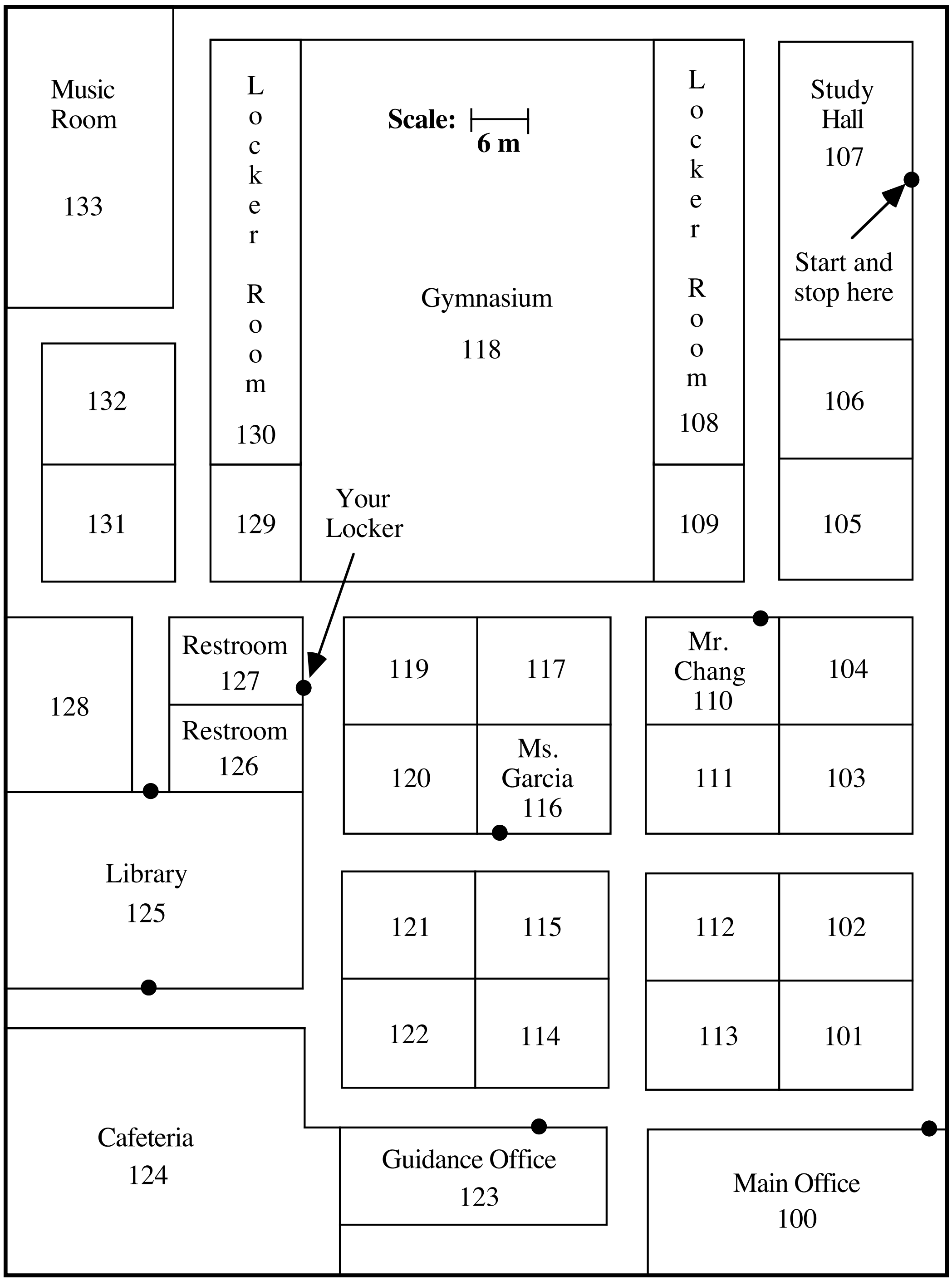


Figure 1: Floor plan of Hamilton High School

As a new student at Hamilton High, you have a lot to do. In order to complete your first day on schedule, you must visit six different rooms during a 55-min study hall. The following list describes each room, the task involved, and the time required to complete the task.

- Main Office: fill out a new student information card (7 min)
- Guidance Office: make an appointment to talk to a counselor (1 min)
- Room 116: question Ms. Garcia about a science experiment (10 min)
- Room 110: talk to Mr. Chang about taking French (10 min)
- Library: find a book for an English assignment (15 min)
- Locker: pick up a notebook (1 min).

You may visit the rooms in any order. However, your route must satisfy the following requirements.

- You must start and stop in Study Hall, Room 107.
- You must visit all the rooms and accomplish all the tasks.
- You must return to Study Hall before the end of the class period (55 min).

How can you find an efficient route to all these rooms and back to Study Hall? A situation such as this one, in which an appropriate solution makes good use of time or other resources, involves **optimization**.

Activity 1

Before beginning to solve an optimization problem, it can help to organize the available information in a table or graph.

By representing each destination as a point, one possible route can be described using the diagram shown in Figure **2**. This route leaves Study Hall, visits the locker first, then eventually returns to Study Hall. Although the diagram is not drawn to scale, it does show the sequence in which the rooms are visited.

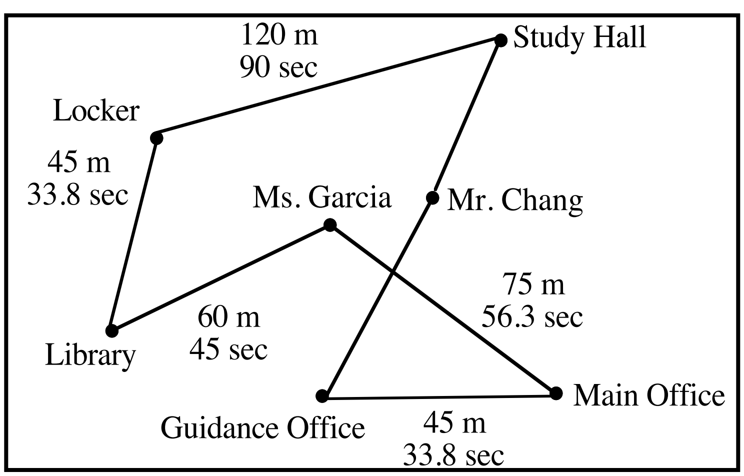


Figure 2: Possible route for completing study hall tasks

Exploration

Figure **2** also shows the distances between some of the stops and the time required to walk those distances. The distances were determined by measuring the hallways on the map in Figure **1**, then using the map's scale to convert these distances to meters. The corresponding walking times were calculated using a rate of 4.8 km/hr.

a.

1. Use Figure **1** to find the distances along the hallways from the Guidance Office to Mr. Chang's room and from Mr. Chang's room to the Study Hall.
2. Using a walking speed of 4.8 km/hr, calculate the walking times for these trips.
3. Determine the walking time for the entire route.
4. Calculate the total time required to walk the route and complete all the tasks.

Mathematics Note

A **graph** is a set of points or **vertices** (plural of **vertex**) and the edges connecting them. For example, Figure **3** shows a graph with five vertices: A, B, C, D, and E. This graph has seven edges. The edge connecting vertices A and D does not intersect the edge connecting vertices B and E.

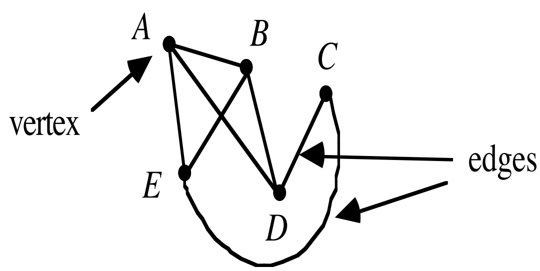


Figure 3: A graph with five vertices and seven edges

In a **weighted graph**, each edge is assigned a numerical value. For example, the numerical value on each edge in Figure **4a** represents the flight time between the corresponding pair of cities. In Figure **4b**, the numbers represent the airfares between two cities.

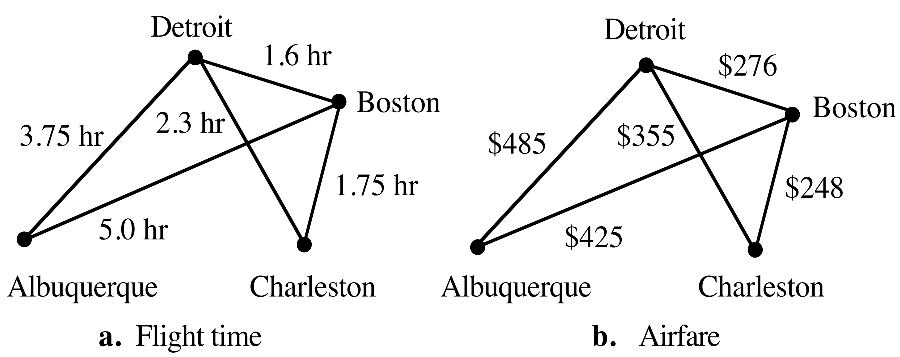


Figure 4: Two weighted graphs

b. Find an acceptable route through Hamilton High School that takes less time than the one shown in Figure **2**.

1. Display the route as a graph.
2. Weight the edges of the graph with the corresponding walking times.
3. Find the total time required to complete the route.

Discussion

a. Which route in the exploration do you think is the better one? Explain your response.

b. What characteristics do you think describe the best possible route?

c. How could you make sure that you found the best possible route?

Mathematics Note

A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices, however, can occur more than once in a path.

In Figure **5**, for example, *A–B–D–C–E* represents a path in which no vertex is repeated, while *B–E–A–B–D* represents a path in which one vertex (*B*) is repeated.

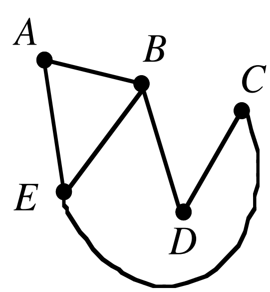


Figure 5: A graph with five vertices

A **circuit** is a path that starts and ends with the same vertex in which no intermediate vertex is repeated. For example, *E–B–D–C–E* and *A–B–E–A* in Figure **5** are circuits.

A **Hamiltonian circuit** is a circuit in which every vertex in the graph is visited exactly once. For example, *A–B–D–C–E–A* in Figure **5** is a Hamiltonian circuit.

d. What do the vertices of the graphs in the exploration represent?

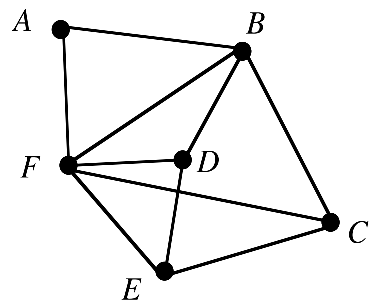
e. What do the edges of the graphs in the exploration represent?

f. Explain why your graph from Part **b** of the exploration is a Hamiltonian circuit.

g. Why is *A–B–D–C–E–B–A* in Figure **5** not a Hamiltonian circuit?

Assignment

1.1 The diagram below shows a graph with six vertices: A, B, C, D, E, and F.



Use this graph to describe an example, if possible, of each of the following:

a. a path

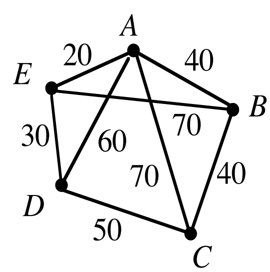
b. a circuit that is not Hamiltonian circuit

c. a path that is not a circuit

d. a circuit that is not a path

e. a Hamiltonian circuit.

1.2 In the weighted graph below, the vertices represent cities and the numbers on the edges represent the time in minutes required to drive from one city to the next.



a. Find the path from A to D that uses the minimum amount of time.

b. Identify one Hamiltonian circuit in the graph and determine the total time required for this route.

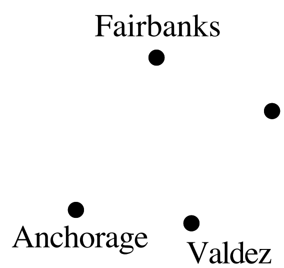
c. Identify another Hamiltonian circuit in the graph that results in a different total time.

d. Which of the circuits from Parts **b** and **c** makes more efficient use of time?

1.3 The following chart shows the distances in kilometers between some Alaskan cities.

	Valdez	**Tok**	**Fairbanks**
Anchorage	478	520	581
Fairbanks	584	333	
Tok	409		

a. The vertices in the diagram below represent the relative locations of the cities. On a copy of this diagram, draw all the possible edges between these vertices.



b. Weight your graph from Part **a** using the distances between cities.

c. What is the shortest path on the graph?

d. Name one circuit in the graph that is not a Hamiltonian circuit.

e. Identify two Hamiltonian circuits that result in different total distances.

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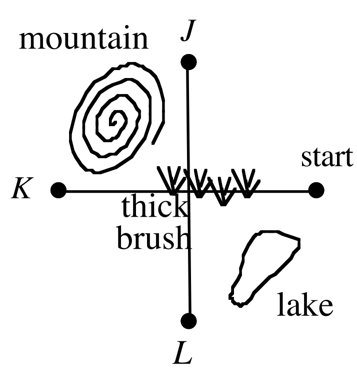
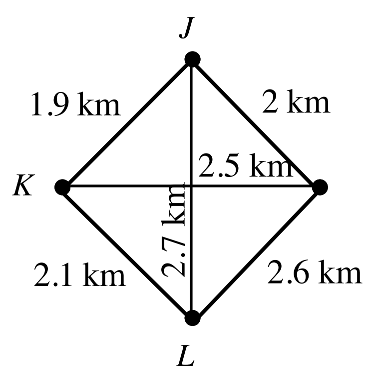
1.4 A scoutmaster has designed an orienteering competition for six scouts. Each scout will be assigned a different course. All six courses, however, start at the same point, visit three locations, then return to the starting point. The scouts must use maps and compasses to complete the assigned course in as little time as possible.

The diagram on the left shows the relative positions of the three locations and the starting point. It also shows the presence of some obstacles on the course, including a mountain, thick brush, and a lake. The weighted graph on the right shows the distances between the locations.





a. Each of the six possible routes is a Hamiltonian circuit. Identify the six routes.

b. Over easy terrain, the scouts can travel at an average speed of 9.6 km/hr. Over rugged terrain, such as mountains or thick brush, the average speed is 7 km/hr.

If you were one of the six scouts participating in the competition, which route would you want to be assigned? Explain your response.

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Activity 2

In the previous activity, you described two different routes through Hamilton High School. To determine the best route in this situation, you might be tempted to consider all the possible Hamiltonian circuits. However, this would mean taking the time to examine over 500 circuits.

In this activity, you investigate a method for quickly determining the number of possible solutions to a problem.

Exploration

Imagine that you are the manager of the most popular band in the nation. The members of the band are in Miami, Florida, and must fly to Los Angeles, California, to sign a recording contract.

There are two flights leaving Miami—one to Chicago, Illinois, and one to Kansas City, Missouri. From both Chicago and Kansas City, there are three flights going west—one to Denver, Colorado; one to Salt Lake City, Utah; and one to Las Vegas, Nevada. There are direct flights to Los Angeles from Denver, Salt Lake City, and Las Vegas.

a. A **directed graph** or **digraph** is a graph in which a direction is indicated on each edge. Draw a directed graph that shows the relative positions of the cities named above and the flights between them.

b. Draw a tree diagram to illustrate all possible routes from Miami to Los Angeles.

c. How many different routes are there from Miami to Los Angeles?

Discussion

a. Describe how you determined the number of different routes from Miami to Los Angeles.

b. Would your method work well if several more cities were involved?

c. The **fundamental counting principle** provides a method for determining the total number of ways in which a task can be performed. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways that the two events can occur is $m \bullet n$.

Describe how the fundamental counting principle can be used to determine the number of different routes from Miami to Los Angeles.

Assignment

2.1 As the manager of a band, you are planning a five-city tour to promote the band's new album. The tour will begin and end in Miami, with stops in Seattle, Washington; Kansas City, Missouri; Chicago, Illinois; and New York City. The band will visit each city only once and the order of the cities is not important.

a. Determine the number of possible routes that the band can take on the tour.

b. Draw a tree diagram to verify your answer from Part **a**.

c. Describe the process you used to create the tree diagram in Part **b**.

Mathematics Note

An **algorithm** is a step-by-step process used to accomplish a task.

In a **brute force algorithm**, every potential solution to a problem is examined. For example, suppose that you wanted to determine the number of possible routes from your home to school. To solve this problem with a brute force algorithm, you could use a tree diagram to list every route.

2.2 On the first day of a new semester, your principal greets you with a list of the seven classes in your school day.

a. In how many different ways could these seven classes be arranged?

b. Why is a brute force algorithm not appropriate in this situation?

Mathematics Note

Factorial notation is often used to simplify the representation of the product of the positive integers from 1 to n. If n is a positive integer, then ***n* factorial** (denoted by ***n*!**) can be expressed as

$$n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 3 \bullet 2 \bullet 1$$

Zero factorial, or **0!**, is defined as 1.

For example, the number of possible arrangements of the letters A, B, and C can be found using the fundamental counting principle as follows: $3 \bullet 2 \bullet 1 = 6$. Using factorial notation, this can be represented as 3!

2.3 As part of the publicity for their tour, the band will appear on the cover of *Singer's Circuit* magazine. Before choosing a photo, the magazine editors want to look at samples with the four band members arranged in different orders.

a. If the band members stand side by side, how many different arrangements are possible?

b. Use factorial notation to represent the number of possible arrangements.

2.4 During its concerts, the band plays 10 songs in a set before taking a break.

a. In how many different orders can 10 songs be played?

b. If the band wants to play its current hit song first, how does this affect the number of possible orders for a 10-song set?

c. How would your answer to Part **b** change if the band chose to play its hit song fourth in the set?

d. If the band wants to play its hit song first and its shortest song second, how does this affect the number of possible orders for a 10-song set?

e. In how many different orders can n songs be played?

2.5 The band is planning several tours during the summer concert season. All of these tours begin and end in the band's hometown.

a. Use the fundamental counting principle to determine the number of possible tours involving each of the following numbers of cities:

1. 2 cities (the band's hometown plus one other city)
2. 3 cities (the band's hometown plus two other cities)
3. 4 cities (the band's hometown plus three other cities)
4. 10 cities (the band's hometown plus nine other cities)

b. Use the pattern you observe in your responses to Part **a** to develop a formula for calculating the number of tours possible for n cities if all tours begin and end in the band's hometown.

2.6 **a.** Determine the number of possible routes through Hamilton High School that begin and end in Study Hall and visit all six destinations described in Activity **1**.

b. Explain why it is not feasible to list all the possible routes.

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2.7 As captain of the softball team, Corrine must determine the order in which the team's nine players will bat. Use the fundamental counting principle to determine the number of different batting orders that Corrine could choose.

2.8 **a.** Find the value of 7!

b. Find the value of $9!/7!$

c. Explain why it is possible to write 200! as 200(199!).

d. One way to find the value of $100!/95!$ is to calculate 100! and divide it by 95!

Describe another way to calculate the value of $100!/95!$

e. Simplify the following expression:

$$\frac{n!}{(n-1)!}$$

2.9 On his first day at Hamilton High School, Marcus forgets the combination to his locker. In an attempt to jog his memory, he tries a few different combinations.

a. The combination to the lock consists of a sequence of 3 different numbers. Each of these numbers can be chosen from 40 different numbers. How many different combinations are possible?

b. If it takes 15 seconds to try each combination, how long would it take Marcus to try them all?

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Activity 3

In Activity **2**, you identified the number of possible routes for your band's upcoming tour. As manager, however, you must also consider the time and expense associated with the tour. Selecting the shortest route possible, for example, might save both time and money.

Exploration

The stops on your concert tour include Miami, Seattle, Chicago, New York, and Kansas City. Table **1** below shows the distances in kilometers between these cities.

Table 1: Distances between cities (in kilometers)

	Seattle	Kansas City	Chicago	New York
Miami	4400	1997	1912	1757
New York	3875	1765	1147	
Chicago	2795	667		
Kansas City	2424			

a. Without using a brute force algorithm, plan a tour that starts in Miami, ends in Miami, visits each of the other cities exactly once, and travels as few kilometers as you think possible.

Mathematics Note

In a **greedy algorithm**, the choice made at each step is the best of all remaining choices. One greedy algorithm is the **nearest neighbor algorithm**. The steps for the nearest neighbor algorithm are described below.

- Starting with any vertex, draw an edge to its nearest vertex. In Figure **6**, for example, vertex *A* is connected to its nearest vertex, *B*.

Figure 6: Connecting the first vertex to the nearest vertex

- Continue this process from the second vertex, drawing an edge to the next nearest vertex not yet visited, and so on, until all vertices have been visited. For example, Figure **7** shows edges drawn from *B* to *C* and from *C* to *D*.

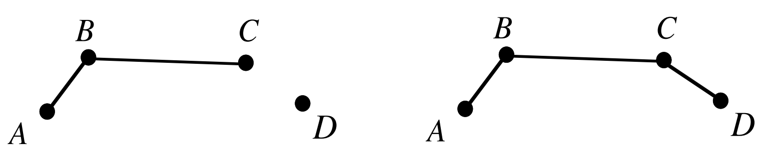


Figure 7: Continuing the nearest neighbor algorithm

- To complete a Hamiltonian circuit, return to the original vertex. For example, Figure **8** shows an edge drawn from *D* back to *A*.

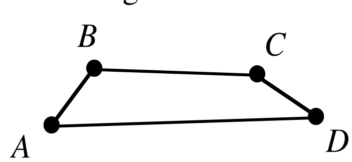


Figure 8: Completing a Hamiltonian circuit

b. Use the nearest neighbor algorithm to draw weighted Hamiltonian circuits that start and stop in each of the five cities in Table **1**.

c.
1. Compare the graphs of the five circuits.
2. Determine which weighted graph represents the tour with the shortest total distance.

d. Compare the shortest tour found using the nearest neighbor algorithm to the tour you identified in Part **a**.

Discussion

a.
1. What advantages are there to using the nearest neighbor algorithm to find a short tour?
2. What advantages are there to using a brute force algorithm to find a short tour?

b. Describe a situation in which the nearest neighbor algorithm would provide an appropriate method for solving a problem.

c. Describe a situation in which a brute force algorithm would provide an appropriate method for solving a problem.

d. In Part **b** of the exploration, you drew Hamiltonian circuits that started and stopped in each of five cities. Which of these circuits could you use to plan a tour that starts and stops in Miami? Explain your response.

Assignment

3.1 The band would like to add Los Angeles, California, to the tour in the exploration. The following table shows the distances in kilometers between Los Angeles and the other five cities.

	Los Angeles
Miami	3764
New York	3945
Chicago	2808
Kansas City	2182
Seattle	1543

a. Use the nearest neighbor algorithm to draw weighted Hamiltonian circuits that start and stop in each city on the tour.

b. Determine which circuit results in the shortest total distance.

c. Using the nearest neighbor algorithm, how many different routes did you have to check?

d. If you used a brute force algorithm to select the shortest tour, how many routes would you have to check?

3.2 **a.** About how many seconds did it take you to generate and calculate the total distance of one circuit in Problem **3.1**?

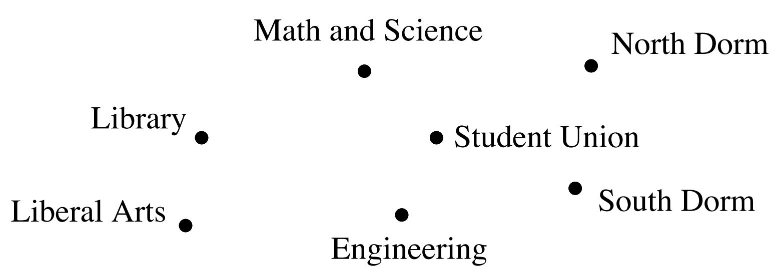
b. Using your response to Problem **3.1d**, about how long would it take you to check all the possible tours?

3.3 One of the band members suggests that this summer's tour begin with a send-off concert in your town, visit 24 other cities, then end with a homecoming concert in your town.

a. Determine the number of possible 25-city tours that begin and end in the same town.

b. Using the time you estimated in Problem **3.2**, determine the total time needed to check all the possible tour routes.

3.4 Katherine works in the admissions office at Applegate University. As part of her duties, she leads prospective students on a walking tour of the campus. The diagram below shows the relative positions of the campus buildings.



a. If she visits each of the seven buildings exactly once, in how many different ways can Katherine organize the walking tour?

b. Since new students live in South Dorm, Katherine likes to begin and end the tour there. How many ways are there to organize this tour?

c. The table below shows the distances in meters between buildings on campus. Use these distances and the nearest neighbor algorithm to graph a possible route for the walking tour you described in Part **b**.

	Student Union	South Dorm	North Dorm	Math/ Science	Library	Liberal Arts
Engineering	20	40	55	40	50	45
Liberal Arts	55	90	100	60	20	
Library	50	85	90	40		
Math/Science	25	50	50			
North Dorm	40	30				
South Dorm	35					

Mathematics Note

The nearest neighbor algorithm is just one of many greedy algorithms that can be used to solve optimization problems. In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each stage, regardless of starting and stopping points.

Individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that is not Hamiltonian, then the next best action is taken. When a Hamiltonian circuit is found, the algorithm is complete.

For example, Figure **9** shows the steps used to draw a Hamiltonian circuit with the cheapest link algorithm.

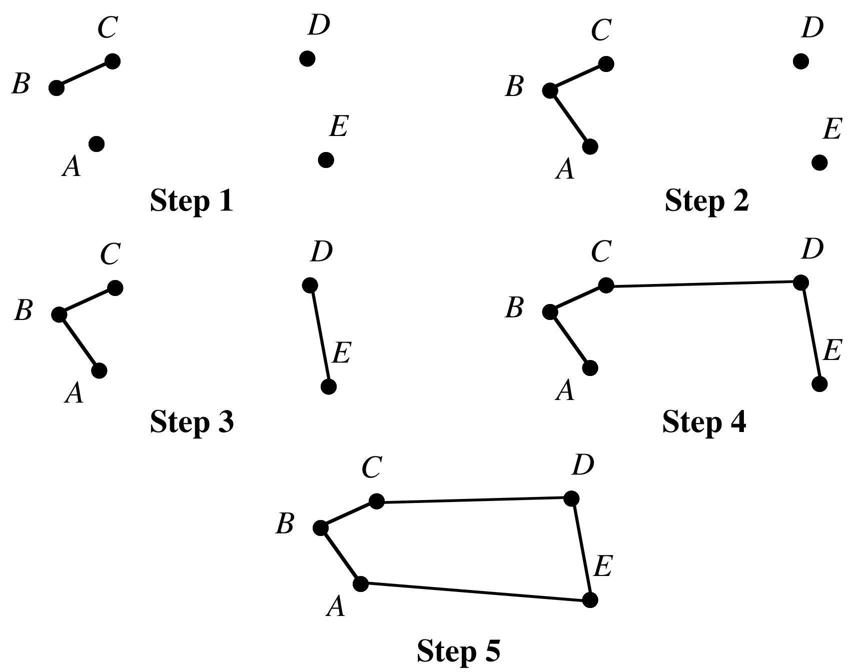


Figure 9: Steps in cheapest link algorithm

In Step **3**, notice that *D* and *E* are connected even though these points are farther apart than *C* and *A*. This occurs because drawing an edge between *A* and *C* would complete a circuit that is not Hamiltonian.

3.5 **a.** Use the cheapest link algorithm to determine a possible route for the walking tour in Problem **3.4b**.

b. How does this tour compare with the one you identified in Problem **3.4c**?

3.6 **a.** Use the cheapest link algorithm to find a tour of the six cities in Problem **3.1**.

b. Calculate the distance covered by the tour.

c. Compare this distance with the distance found using the nearest neighbor algorithm.

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3.7 As the result of a stellar performance on the state math exam, your class has been awarded a trip to four interesting places in your state. Your class may choose both the places and the route, but the trip must start and end in your town.

a. Choose four destinations and create a table that shows the distances from each one to the others, including your town.

b. Determine the number of possible routes for your trip.

c. Use the nearest neighbor algorithm to plan your route. Determine the total distance traveled.

d. Use the cheapest link algorithm to plan your route. Determine the total distance traveled.

e. Is the shorter of the two routes necessarily the shortest possible route? Explain your response.

f. Use a brute force algorithm to support your response to Part **e**.

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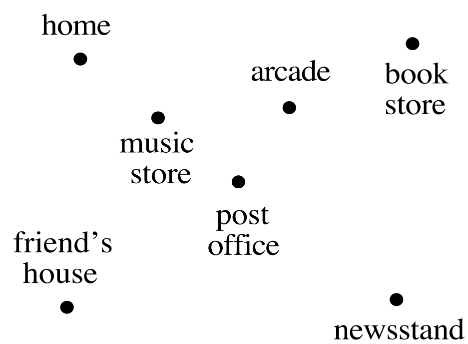
Summary Assessment

At 4:30 P.M., Jack and Jill leave their house to run some errands for their parents. They must stop by a friend's house, the music store, the book store, the newsstand, and the post office. As a reward, their parents have given them enough money to play video games at the arcade. The relative positions of these destinations are shown in the diagram below.



Each errand should take 5 min to complete and Jack and Jill must be home by 6:00 P.M. for dinner. The following table shows the walking time in minutes between the locations.

	Friend's House	Newsstand	Book Store	Post Office	Arcade	Music Store
Home	5.5	9.5	6.5	5.5	4.0	2.5
Music Store	3.0	6.5	6.0	2.5	2.5	
Arcade	5.0	5.5	3.5	2.0		
Post Office	3.0	4.0	5.5			
Book Store	8.5	5.5				
Newsstand	3.5					

1. Select a route that allows Jack and Jill to spend more time at the arcade. Draw a weighted graph to represent this route and indicate the order in which the edges were chosen.

2. The games at the arcade last about 5 min each. Using the route you selected in Problem **1**, how many games can Jack and Jill play?

3. Identify the algorithm you used to select a route. If you devised an algorithm of your own, describe its steps so that any student in the class could repeat your results.

4. If you had to find the quickest possible route that visited all these destinations, how many circuits would you have to check?

Module Summary

- In an **optimization** problem, the goal is to make the best use of time or other resources.
- A **graph** is a set of **vertices** (plural of **vertex**) and **edges**. Each edge connects two vertices.
- In a **weighted graph**, each edge is assigned a numerical value.
- A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices, however, can occur more than once in a path.
- A **circuit** is a path that starts and ends with the same vertex in which no intermediate vertex is repeated.
- A **Hamiltonian circuit** is a circuit in which every vertex in the graph is visited exactly once.
- A **directed graph** or **digraph** is a graph in which a direction is indicated on each edge.
- The **fundamental counting principle** provides a method for determining the total number of ways in which a task can be performed. If an event that can occur in m ways is followed by an event that can occur in n ways, then the total number of ways that the two events can occur is $m \bullet n$.
- An **algorithm** is a step-by-step process used to accomplish a task.
- In a **brute force algorithm**, every potential solution to a problem is examined.
- **Factorial notation** is often used to simplify the representation of the product of the positive integers from 1 to n. If n is a positive integer, then ***n* factorial** (denoted by ***n*!**) can be expressed as

$$n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 3 \bullet 2 \bullet 1$$

- **Zero factorial**, or **0!**, is defined as 1.
- In a **greedy algorithm**, the choice made at each step is the best of all remaining choices.
- To use the **nearest neighbor algorithm**, start with any vertex and draw an edge to its nearest vertex. Continue this process from the second vertex to the next nearest vertex not yet visited, and so on, until all vertices have been visited. To complete a Hamiltonian circuit, return to the original vertex.
- In the **cheapest link algorithm**, the cheapest (or shortest) action is taken at each stage, regardless of starting and stopping points. Individual, disconnected edges may occur at various stages. If the cheapest remaining action completes a circuit that is not Hamiltonian, then the next best action is taken. When a Hamiltonian circuit is formed, the algorithm is complete.

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