From Rock Bands to Recursion

You've endured hours on the highway, rows of seats in an empty amphitheater, lagging ticket sales, crowded soft drink concessions, bouncing beach balls, and—finally—the notes of an electric guitar. Is this a pattern worth repeating?

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Introduction

Since the time you first learned to count (1, 2, 3), skip-count (2, 4, 6, 8), and stack blocks, you've been investigating patterns of numbers. For example, if you continue to stack blocks as shown in Figure **1**, the number of blocks in each row of a stack forms a pattern. The total number of blocks used to build each stack forms another pattern.

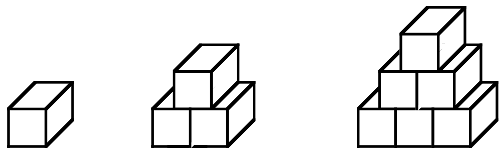

Figure 1: Stacks of blocks

In this module, you go on tour with the rock band Stellar Attraction. Along the way, you use technology and mathematical formulas to examine patterns in the sales of CDs and concert tickets. You also analyze the sequence of notes played by the lead guitarist.

Activity 1

In this activity, you investigate the basic number patterns formed by the number of performances during Stellar Attraction's first world tour.

Exploration

Before starting their world tour, the band's concert career was relatively short—only 6 performances. The upcoming schedule, however, contains 2 performances per week. Table **1** summarizes the band's career number of performances over the next 10-week period.

Table 1: Performances by Stellar Attraction

No. of Weeks on Tour	Total No. of Performances
1	8
2	10
3	12
4	14
⋮	⋮
10	26

a. Complete Table **1** for each of the next 10 weeks.

b. Record any patterns you observe in the table.

c. Use the patterns you discovered to answer the following questions.

1. After how many weeks will the band have given 24 performances?
2. How many performances will they have given after 12 weeks?
3. After 16 weeks, Stellar Attraction will have performed 38 times. What will be the total number of performances after 18 weeks?

Mathematics Note

A **sequence** is an ordered list. Each item in the list is a **term** of the sequence.

The terms of a sequence may be represented by symbols, such as $p_1, p_2, p_3, \ldots, p_n$. These symbols are **subscripted variables**, and the natural numbers $(1, 2, 3, \ldots, n)$ are the **subscripts**. The symbol p_1 (read "*p* sub one") represents the first term of the sequence, the symbol p_2 (read "*p* sub two") represents the second term of the sequence, and so on. The symbol p_n (read "*p* sub *n*"), represents the **general**, or ***n*th**, **term** of a sequence.

For example, consider the following ordered list of numbers 16, 18, 20, 22, 24, … . In this sequence, $p_1 = 16$, $p_2 = 18$, $p_3 = 20$, and so on.

d. 1. Create a spreadsheet with headings like those shown in Table **2** below. The entries in the right-hand column represent the total number of performances at the end of each week. They also represent the terms of a sequence.

Table 2: Performances spreadsheet

No. of Weeks (n)	Total No. of Performances (p_n)
1	8
2	10
3	12
4	14
⋮	⋮
50	

2. Use the spreadsheet to determine the total number of performances after each week for the next 50 weeks.

Mathematics Note

A **recursive formula** is a rule for finding any term in a sequence by using the preceding term(s). The process of using a recursive formula is known as **recursion.**

A sequence in which every term after the first is found by adding a constant value to the preceding term is an **arithmetic sequence**.

The recursive formula for calculating any term in an arithmetic sequence is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, \ n > 1 \end{cases}$$

where a_1 is the first term, a_n is the nth term, a_{n-1} is the term preceding a_n, and d is the **common difference** between any two consecutive terms, $a_n - a_{n-1}$.

For example, consider the sequence in which the first term (a_1) is 27 and the common difference (d) between any two consecutive terms is 5. The recursive formula for this sequence is:

$$\begin{cases} a_1 = 27 \\ a_n = a_{n-1} + 5, \ n > 1 \end{cases}$$

Using this formula, the first four terms of the sequence can be found as follows:

$$\begin{aligned} a_1 &= 27 \\ a_2 &= a_{2-1} + d = a_1 + d = 27 + 5 = 32 \\ a_3 &= a_{3-1} + d = a_2 + 5 = 32 + 5 = 37 \\ a_4 &= a_{4-1} + 5 = a_3 + d = 37 + 5 = 42 \end{aligned}$$

e. Find a recursive formula for the sequence in the right-hand column of Table **2**.

Discussion

a. How did you use the patterns you described in Part **b** of the exploration to complete Table **2**?

b. Use the sequence in the right-hand column of Table **2** to answer the following questions.

1. What is the value of n for p_{20}?
2. What is the value of p_{20}?
3. If $p_n = 20$, what is the value of n?
4. When $n = 8$, what is the value of p_{n-1}, the term before p_n?

c. How can you use subscript notation to express the fact that the 30th term of a sequence is 66?

d. Explain why the sequence found in each column of Table **2** is an arithmetic sequence.

Assignment

1.1 For their next performance, Stellar Attraction must drive from St. Louis, Missouri, to Portland, Oregon: a distance of 3290 km. They can average 100 km per hour during the trip.

a. Create a spreadsheet with headings like those in the table below. Let h represent the number of hours driven and k_h represent the number of kilometers remaining after each hour.

Hours Driven (h)	**Kilometers Remaining (k_h)**
1	3190
2	3090
3	2990
⋮	⋮

b. During which hour does the band arrive in Portland?

c. What is the value of k_5?

d. How many kilometers remain after the band has driven for 13 hr? Express your answer using subscript notation.

e. For how many hours has the band been driving when they are 1390 km from Portland? Express your answer in subscript notation.

f. What is the value of k_{h-1} when $k_h = 2290$?

g. Write a recursive formula that describes the pattern for k_h.

1.2 After arriving in Portland, the band finds that concert tickets have been selling well. The ticket agency sold 790 tickets on the first day. During the next 20 days, they sold an average of 213 tickets per day.

a. Let n represent the number of days that tickets have been on sale and t_n represent the total number of tickets sold after n days. Express the first 5 terms of this sequence using subscript notation.

b. Using the sequence from Part **a**, on what day did total sales exceed 2200 tickets? Express your answer using subscript notation.

c. What is the value of t_{n+1} when $t_n = 2494$?

d. Write a recursive formula that describes the pattern of ticket sales.

e. To complete Parts **a-d**, you assumed that the pattern of ticket sales was an arithmetic sequence where $t_1 = 790$ and $t_{21} = 790 + 20(213)$. Do you think that this is a reasonable assumption?

1.3 In some arithmetic sequences, the numbers increase with each successive term. In others, the numbers decrease with each successive term. What can you say about the common differences used to form these sequences?

1.4 Consider the following recursive formula:

$$\begin{cases} t_1 = 9.0 \\ t_n = t_{n-1} + 0.5, \quad n > 1 \end{cases}$$

a. What is the value of the first term of this sequence?

b. What are the next four terms of this sequence?

1.5 Find a recursive formula for the arithmetic sequence 3, 7, 11, 15, … .

1.6 **a.** Create your own arithmetic sequence.

b. What is the common difference for your sequence?

c. Write a recursive formula for your sequence.

1.7 In a paragraph, compare the two sequences described by the formulas shown below.

$$\begin{cases} t_1 = 5 \\ t_n = t_{n-1} + 3, \quad n > 1 \end{cases} \text{ and } \begin{cases} r_1 = -4 \\ r_n = r_{n-1} + 3, \quad n > 1 \end{cases}$$

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1.8 The *National Geographic* magazine is published 12 times per year. Each edition of the magazine is approximately 0.8 cm thick.

a. Imagine that your parents started collecting *National Geographic* in the year you were born. How much shelf space would they need to display their collection at the end of this year?

b. Write a recursive formula that describes the amount of shelf space needed at the end of a year. Let the first term of the sequence be the width of the shelf when you were in kindergarten.

c. Use your recursive formula to predict how much shelf space will be required when you are 60 years old.

1.9 **a.** Melinda has \$51.00 and spends \$3.00 per week. Write a recursive formula to describe the amount of money m_n Melinda has at the beginning of week n.

b. Kris has \$11.00 and saves an additional \$2.00 per week. Write a recursive formula to describe the amount of money k_n Kris has at the beginning of week n.

c. When will Melinda and Kris have the same amount of money?

1.10 Consider the following recursive formula:

$$\begin{cases} t_1 = 1 \\ t_2 = 1 \\ t_n = t_{n-2} + t_{n-1} \text{ for } n > 2 \end{cases}$$

a. Find t_3.

b. Generate the first 10 terms of the sequence.

c. Create a scatterplot of the first 10 terms of the sequence versus the term number.

d. Is the sequence an arithmetic sequence? Justify your response.

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Activity 2

The band's record company tracks the number of compact discs (CDs) that Stellar Attraction sells each week. During the week of June 10–16, the band sold 9050 copies. Sales projections indicate that the band can expect weekly sales to increase by an average of 2353 copies each week for the next year. These projections are shown in Table **3** below.

Table 3: Weekly CD sales

Week	Week No.	CD Sales
June 10–16	1	9050
June 17–23	2	11,403
June 24–30	3	13,756
July 1–7	4	16,109
⋮	⋮	⋮
December 23–29	29	
⋮	⋮	⋮
June 2–8	52	

If the projections are accurate, how many CDs will the band sell during the last week of December? When will the total number of copies sold exceed 1 million? Since the predicted values for weekly sales represent an arithmetic sequence, you could use a recursive formula to answer these questions. However, using a recursive formula can be time consuming. In this activity, you develop another type of formula that will allow you to respond more quickly.

Exploration

The two left-hand columns in Table **4** show the term number and terms for an arithmetic sequence. The two right-hand columns show expanded (and equivalent) forms of the term values. In this exploration, you use the patterns in this table to develop another type of formula for the sequence.

Table 4: Patterns within arithmetic sequences

Term Number (n)	**Term (p_n)**	**Recursive Form of p_n**	**Another Form of p_n**
1	27	27	27 + (0 • 2)
2	29	27 + 2	27 + (1 • 2)
3	31	27 + 2 + 2	27 + (2 • 2)
⋮			
7			

a. Complete Table **4** for the first seven terms of the sequence. Record any patterns you discover.

b. Write an expression that represents the number of 2s added to the first term to form the nth term.

c. Use the pattern in the far right-hand column to write a formula for p_n in terms of n.

d. Create a scatterplot of the data in the two left-hand columns of the table. Use the horizontal axis for the term number (n) and the vertical axis for the term value (p_n).

e. Find an equation that models the scatterplot and graph it on the coordinate system from Part **d**.

Compare this equation to the one you wrote in Part **c**.

Discussion

a. **1.** In the sequence shown in Table **4**, how many 2s would you have to add to find p_{21}?

2. How many 2s would have to be added to 27 to find p_n?

b. **1.** Describe the formula you wrote for p_n using n as a variable.

2. Use this formula to determine the values of p_{47} and p_{100}.

Mathematics Note

An **explicit formula** for calculating any specific term in an arithmetic sequence is:

$$a_n = a_1 + d(n-1)$$

where a_n is the nth term, a_1 is the first term, and d is the common difference between any two consecutive terms, $a_n - a_{n-1}$.

For example, consider the arithmetic sequence 8, 14, 20, 26, … . In this sequence, the first term is 8 and the common difference is 6. The explicit formula for this sequence is $a_n = 8 + 6(n-1)$. This formula can be used to find the 20th term of the sequence as follows:

$$\begin{aligned} a_{20} &= 8 + 6(20-1) \\ &= 8 + 6(19) \\ &= 122 \end{aligned}$$

The sum of the terms of an arithmetic sequence is an **arithmetic series**.

For example, consider the arithmetic sequence 8, 14, 20, 26. The corresponding arithmetic series is 8 + 14 + 20 + 26, or 68.

c. How does the graph of a linear equation differ from the graph of an arithmetic sequence?

d. Do you think that the scatterplot of an arithmetic sequence can always be modeled by a linear equation in slope-intercept form, $y = mx + b$? Why or why not?

e. Using the distributive property, the explicit formula for an arithmetic sequence, $a_n = a_1 + d(n-1)$, can be written as $a_n = a_1 + dn - d$. Using the associative property, it can also be written as $a_n = dn + (a_1 - d)$.

Assuming that the equation $y = mx + b$ models the same arithmetic sequence, describe the relationship between each of the following:

1. the nth term a_n and y
2. the term number n and x
3. the difference d and the slope m
4. the first term a_1 and the y-intercept b

f. In what types of situations would an explicit formula be easier to use than a recursive formula?

g. Using the relationships described in Part **e** of the discussion, show that the explicit formula for an arithmetic sequence, $a_n = a_1 + d(n-1)$, is equivalent to the linear equation of the form $y = mx + b$ that models the same sequence.

Assignment

2.1 As shown in Table **3**, weekly CD sales for Stellar Attraction are expected to increase by an average of 2353 copies per week for the next 52 weeks after June 10–16.

a. If this projection is accurate, how many CDs will be sold during the week of December 23–29?

b. When will the total number of CDs sold exceed 1 million?

2.2 Stellar Attraction is playing at the Jan-San Amphitheater. The amphitheater has 120 seats in the front row, 136 seats in the second row, and 152 seats in the third row. This pattern continues from row to row. The last row has 584 seats.

a. Write an explicit formula to determine the number of seats in any row of the theater.

b. How many seats are in the 16th row?

c. How many rows are in the theater?

d. How many total seats are in the theater?

2.3 Imagine that you work as an usher at the Jan-San Amphitheater. Your starting wage is \$4.25 per hour. Periodically, you will receive a raise of \$0.10 per hour. Write an explicit formula to calculate your hourly wage after n raises.

2.4 **a.** Before the concert begins, the Jan-San Amphitheater contains 85 employees. The doors open to the public at 6:00 P.M. Between 6:00 P.M. and 6:15 P.M., an average of 9 people per second enter the theater. Let p_n represent the number of people in the theater after n seconds.

1. List the first 5 terms of the sequence.
2. Identify the common difference d.
3. Write an explicit formula of the form $a_n = a_1 + d(n-1)$.
4. Determine the number of people in the amphitheater at 6:15 P.M.

b. When the concession stands open, they have a supply of 8500 L of soft drinks. Soft drink sales average 110 L per minute throughout the evening. Let l_n represent the number of liters remaining after n minutes.

1. List the first 5 terms of the sequence.
2. Identify the common difference d.
3. Write an explicit formula of the form $a_n = a_1 + d(n-1)$.
4. Determine how long the supply of soft drinks will last.

2.5 A cashier starts the evening with \$50.00 in the drawer. During the next 45 minutes, an average of \$2.15 per minute is added to the drawer. Let c_n represent the number of dollars in the drawer at the end of n minutes.

a. Write a recursive formula for this sequence of the form

$$\begin{cases} c_1 = \text{first term} \\ c_n = c_{n-1} + d, \quad n > 1 \end{cases}$$

Explain what each term in the formula represents.

b. Write an explicit formula for this sequence of the form

$$c_n = c_1 + d(n-1)$$

Explain what each term in the formula represents.

c. Write an equation of the form $y = mx + b$ that models this sequence. Explain what each term in the equation represents.

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2.6 Brianna chairs the membership committee of a club with 33 members. To meet club goals, she plans to recruit 5 new members each month.

a. If Brianna meets her membership goals, how many members will the club have after the next 12 months?

b. Each club member receives a monthly newsletter. If each newsletter requires one stamp to mail, how many stamps will be needed during the next year?

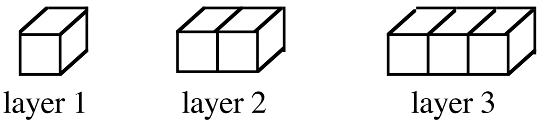
2.7 **a.** Create a scatterplot of the data in the following table.

n	t_n
1	7
2	9
3	11
4	13
5	15

b. Write an equation for the line that fits the data. Graph this equation on your coordinate system from Part **a**. What does this graph tell you about an arithmetic sequence?

c. Write an explicit formula to determine t_n. How does this compare with the equation you wrote in Part **b**?

2.8 Raul works in a supermarket. He is stacking blocks of cheese for a dairy display. The following diagram shows the first three layers, top to bottom, in his display.



a. Make a sketch of the next three layers in the display.

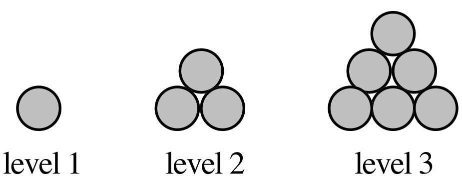
b. Let l_n represent the number of blocks in layer n. Find the first six terms of this sequence.

c. Is the sequence you wrote in Part **b** an arithmetic sequence? Justify your response.

d. Write a formula to describe this sequence.

e. Raul plans to build a display with seven layers. How many blocks of cheese will there be in the stack?

2.9 Another supermarket employee is creating a pyramid of oranges for a produce display. The following diagram shows the first three levels, top to bottom, in this display.



a. Make a sketch of the next three levels in the display.

b. Let l_n represent the number of oranges in level n. Find the first six terms of this sequence.

c. Is the sequence you wrote in Part **b** an arithmetic sequence? Justify your response.

d. Write both recursive and explicit formulas to describe this sequence.

e. How many oranges are there in a 10-level display?

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Activity 3

In Activities **1** and **2**, you examined how arithmetic sequences can be modeled by linear equations. However, just as there are many different types of equations, there are many different types of numerical sequences. In this activity, you explore a type of sequence that can be used to model a projected increase in sales.

Exploration

Before starting their next concert tour, Stellar Attraction releases a second CD. During the first week, 500 copies are sold. The band's manager predicts sales will double each week that the band is on tour.

a. The band's concert tour will last 12 weeks. Table **5** shows the manager's predictions for both weekly sales and total sales during this period. Complete the table.

Table 5: Projected CD sales

Week	Weekly Sales	Total Sales
1	500	500
2	1000	1500
3	2000	3500
⋮	⋮	⋮
12		

b. Record any patterns you observe in the table.

Mathematics Note

A **geometric sequence** is a sequence in which every term after the first is found by multiplying the preceding term by a constant value.

The recursive formula for calculating any term in a geometric sequence is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = g_{n-1}(r), \ n > 1 \end{cases}$$

where g_1 is the first term, g_n is the nth term, g_{n-1} is the term preceding g_n, and r is the **common ratio** between any two consecutive terms, g_n / g_{n-1} .

For example, consider the sequence in which the first term (g_1) is 4 and the common ratio (r) between any two consecutive terms is 5. The recursive formula for this sequence is:

$$\begin{cases} g_1 = 4 \\ g_n = g_{n-1}(5), \ n > 1 \end{cases}$$

Using this formula, the first four terms of the sequence can be found as follows:

$$g_1 = 4$$
$$g_2 = g_{2-1}(r) = g_1(r) = 4(5) = 20$$
$$g_3 = g_{3-1}(r) = g_2(r) = 20(5) = 100$$
$$g_4 = g_{4-1}(r) = g_3(r) = 100(5) = 500$$

The sum of the terms of a geometric sequence is a **geometric series**.

For example, consider the geometric sequence 3, 12, 48, 192. The corresponding geometric series is 3 + 12 + 48 + 192, or 255.

c. In Table **5**, the numbers in the “Weekly Sales” column form a geometric sequence.

1. Find the common ratio for this sequence.
2. Write a recursive formula for this sequence.
3. Determine the corresponding geometric series.

Discussion

a. Describe the patterns you observed in the numbers in Table **5**.

b. Use these patterns to answer the following questions:

1. After 12 weeks, what are the predicted total sales?
2. What are the predicted weekly sales for week 10?
3. During what week are total sales predicted to exceed 1 million?

c. Why do the numbers in the “Weekly Sales” column form a geometric sequence?

d. Do the numbers in the “Total Sales” column also form a geometric sequence? Why or why not?

e. If the band extends their tour for 12 more weeks, should they expect sales to continue the predicted pattern? Explain your response.

Assignment

3.1 Determine a recursive formula for each of the following geometric sequences:

a. 4, 28, 196, 1372, …

b. 9, 18.9, 39.69, …

c. 144, 36, 9, $9/4$, …

3.2 **a**. Create a scatterplot of the first six terms of each geometric sequence in Problem **3.1**. Let x represent the term number and y represent the value of the term.

b. Compare the graphs of the three sequences.

3.3 As soon as the concert tour ends, Stellar Attraction's manager predicts that CD sales will begin to decline. Over the next 8 weeks, sales should fall by 75% each week.

a. Use this information to extend Table **5** for weeks 13–20.

b. The numbers in the "Weekly Sales" column for weeks 12–20 form a geometric sequence. Write a recursive formula for this sequence. (The first term is the weekly sales for week 12.)

c. Does your formula work after week 18? Explain your response.

d. What are the predicted weekly sales for week 20?

e. What are the predicted total sales for week 20?

3.4 Following the concert at the Jan-San Amphitheater, an enterprising group of students offers to clean the arena at the following rate: \$0.01 for the first barrel of garbage, \$0.02 for the second barrel, and \$0.04 for the third barrel, with this doubling pattern continuing for each additional barrel.

a. Create a table with the following column headings. Complete this table for 20 barrels of garbage.

Barrel Number	Charge per Barrel	Total Charge

b. If the students want to earn at least \$500, how many barrels of garbage must they collect?

c. The manager of the amphitheater has budgeted \$1000 to clean up after the concert. How many barrels of garbage must the students collect to exceed this budget?

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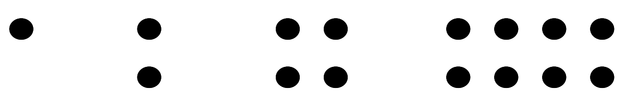
3.5 Consider the pattern of dots shown in the following diagram.



a. Draw the next picture in this pattern.

b. Represent this pattern as a sequence.

c. Is this sequence a geometric sequence? Explain your response.

d. Write a recursive formula for the sequence.

e. Graph the sequence as a scatterplot. Represent the term number on the x-axis and the value of the term on the y-axis.

3.6 As part of a holiday sales promotion, a clothing store plans to reduce the price of its $20.00 shirts by 10% each week.

a. Will the sale prices from week to week represent a geometric sequence? Explain your response.

b. The store originally paid $12.00 for each shirt. In what week will the store begin to lose money on the sale items?

c. The store manager decides to stop the sale in the week before shirt prices fall below $12.00. If you buy one shirt during each week of the sale, how much money will you spend?

3.7 Shahid and Yasmir borrowed money to buy their house. Their monthly payment includes the cost of the loan, insurance, and property taxes. During the first year of the loan, the monthly payment is $350.00. In each year following the first, the monthly payment rises by 2%.

a. Write a recursive formula that describes the monthly payment in any year.

b. Determine when the monthly payment will be more than $500.00.

c. Shahid and Yasmir must pay a total of $150,238.41 to pay off their loan. During what year will this occur?

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Activity 4

As mentioned in Activity **3**, Stellar Attraction's manager predicted that the weekly sales of their CD would double each week. The predicted sales for the next 10 weeks form a geometric sequence. The two left-hand columns in Table **6** show the term number and terms for this sequence. The other three columns show expanded (and equivalent) forms of the term values.

Table 6: Patterns within geometric sequences

Term Number (n)	Term (t_n)	Recursive Form of t_n	Expanded Recursive Form of t_n	Explicit Form of t_n
1	500	500	500	500
2	1000	500 • 2	500 • 2	$500 \cdot 2^1$
3	2000	1000 • 2	500 • 2 • 2	$500 \cdot 2^2$
4	4000	2000 • 2	500 • 2 • 2 • 2	$500 \cdot 2^3$
⋮	⋮	⋮	⋮	⋮

In this activity, you use the patterns in this table to investigate explicit formulas for geometric sequences.

Exploration

a. Develop an explicit formula for the sequence of terms, t_n, in Table **6**.

b. Create a three-column spreadsheet that duplicates the following three columns from Table **6**.

Term Number (*n*)	**Recursive Form of t_n**	**Explicit Form of t_n**

c.
1. Extend the spreadsheet to at least $n = 10$.
2. Compare the values in the two right-hand columns in your spreadsheet.

d. Add a fourth column to the spreadsheet that determines the ratio of each term to its preceding term, t_n / t_{n-1} where $n > 1$.

e. Create a scatterplot of the sequence. Let x represent the term number and y represent the value of each term.

f. Find an equation that models the scatterplot and graph it on your coordinate system from Part **e**.

Compare this equation to the formula you wrote in Part **a**.

Discussion

a.
1. Using the explicit form of t_n described in Table **6**, what power of 2 is multiplied by 500 to determine t_{19}?
2. How is the power of 2 in each row related to the value of n in that row?

b.
1. Describe a formula based on the value of n that could be used to calculate the value of t_n .
2. Use this formula to calculate t_7 .

c. What do you observe about the ratio found in Part **d** of the exploration?

d. How is this ratio related to the formula described in Part **b** of the discussion?

Mathematics Note

An **explicit formula** for calculating any specific term in a geometric sequence is:

$$g_n = g_1 r^{n-1}$$

where g_n is the nth term, g_1 is the first term, and r is the common ratio between any two consecutive terms, g_n / g_{n-1} .

For example, consider the geometric sequence 6, 24, 96, 348, 1536, … . For this sequence, the first term is 6 and the common ratio 4. The explicit formula for this sequence is: $g_n = 6(4)^{n-1}$. Using this formula, the seventh term of the sequence can be found as follows:

$$g_7 = 6(4)^{7-1} = 6(4)^6 = 6(4096) = 24,576$$

e. Do you think that the scatterplot of a geometric sequence can always be modeled by an exponential equation? Why or why not?

f. How does the graph of an exponential equation differ from the graph of a geometric sequence?

g. Using the laws of exponents, the equation $y = ab^x$ can be rewritten as $y = a \bullet b^1 \bullet b^{x-1}$, which is equivalent to $y = (ab)b^{x-1}$. Assuming that this equation models the same geometric sequence as the explicit formula $g_n = g_1 r^{(n-1)}$, describe the relationship between each of the following:

1. the nth term g_n and y
2. n and x
3. the common ratio r and b
4. the first term g_1 and a

h. When is an explicit formula for a geometric sequence easier to use than the recursive formula?

i. Using the relationships described in Part **g** of the discussion, show that the explicit formula for a geometric sequence, $g_n = g_1 r^{(n-1)}$, is equivalent to the exponential equation of the form $y = ab^x$ that models the same sequence.

Assignment

4.1 Identify each of the following sequences as geometric or not geometric. For each geometric sequence, write an explicit formula and find the 10th term. Explain why each of the remaining sequences is not geometric.

a. $0.5, 2.5, 12.5, 62.5, \ldots$

b. $15, 150, 300, 900, \ldots$

c. $4, -12, 36, -108, \ldots$

d. $1000, 250, 62.5, 15.625, \ldots$

4.2 Write the geometric series for each sequence below and find the corresponding sum.

a. $1, 3, 9, 27, 81$

b. $2, 5, 12.5, \ldots, 78.125$

c. $0.5, 1, 2, \ldots, 64$

d. $125, 25, 5, \ldots, 0.04$

4.3 At one of Stellar Attraction's concerts, an exuberant fan hits a beach ball into the air. The ball falls onto the stage from a height of 24 m. The height of each bounce is approximately one-third the height of the preceding bounce. Predict the number of bounces the ball will take before coming to rest. Justify your response.

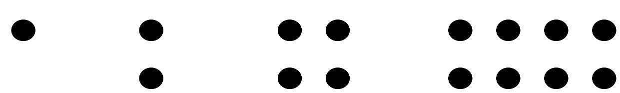
4.4 Describe the possible range of values for the common ratio in each of the following:

a. an increasing geometric sequence

b. a decreasing geometric sequence.

* * * * *

4.5 Consider the pattern of dots shown in the following diagram.



a. If this pattern is expressed as a geometric sequence, what are the values for t_1 and r?

b. Write an explicit formula that describes this pattern.

c. Use your formula to determine the value of t_{14}.

Science Note

Sound is produced by vibrating objects. The frequency of a sound wave equals the frequency of the vibrating object.

The international unit of frequency is the **hertz (Hz)**, which represents one cycle per second. For example, a sound with a frequency of 220 Hz is produced by an object vibrating at 220 cycles per second.

4.6 When a guitar string vibrates, the guitar produces sound. The rate at which the string vibrates determines the note. A string vibrating at 220 Hz produces the note A immediately below middle C. The note A immediately above middle C vibrates at 440 Hz. The frequencies of consecutive A notes form a geometric sequence.

Note	Frequency
A	
A	
A	
A	440 Hz
middle C	
A	220 Hz
A	
A	
A	
A	

a. Complete the table for all A notes above and below middle C.

b. Write a recursive formula that describes this sequence.

c. The average young person can hear sounds with frequencies from 20 Hz to 20,000 Hz. Which of the notes in the table in Part **a** could be heard by concert fans? Defend your response.

4.7 Annaborg has been offered two summer jobs, each for 12 weeks. The job at Plouvier's Pottery pays $5.00 per hour, with a 10% increase in the hourly wage every two weeks. The job at Brocklebank's Bakery pays $6.50 per hour, with a $0.10 increase in the hourly wage every two weeks.

a. Which type of sequence, arithmetic or geometric, best describes the wages offered by Plouvier's Pottery? Justify your response.

b. Which type of sequence, arithmetic or geometric, best describes the wages offered by Brocklebank's Bakery? Justify your response.

c. Write an explicit formula that describes the wages for each job.

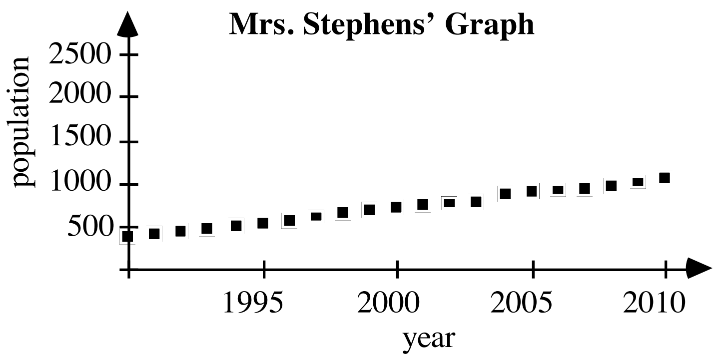
d. If Annaborg wants to make as much money as possible, which job should she take? Justify your response.

* * * * * * * * * *

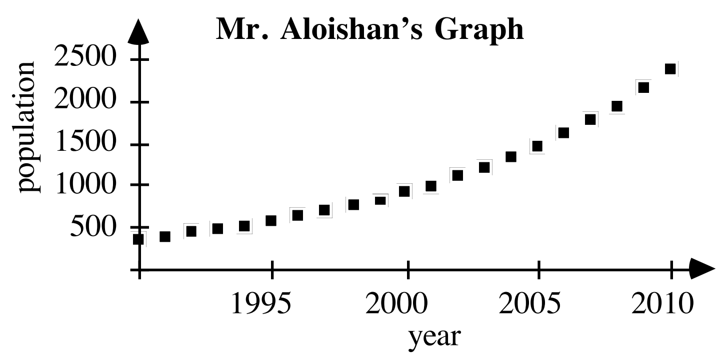
Summary Assessment

1. In 1990, the village of Bone Gap had a population of 350 people. By the end of 1991, the population had grown to 385. Many long-time residents grew concerned about the future of their community. At a town meeting, two different predictions were made.

Mrs. Stephens presented the following graph and argued that residents should not be concerned about growth.



Mr. Aloishan, however, presented the graph below and argued that life in Bone Gap would change dramatically.



Based on these two graphs and the initial data, explain how the two residents reached their conclusions. Include formulas for each graph.

2. Imagine that the following two plans were proposed at the time that the U.S. national debt reached $4 trillion.

- Plan A: Balance the budget to eliminate additional accumulation of debt. Reduce the debt by $1.00 per second until the debt is canceled.
- Plan B: Balance the budget to eliminate additional accumulation of debt. Reduce the debt by $1.00 in the first year, $2.00 in the second year, $4.00 in the third year, and continue to double the reduction until the debt is canceled.

Use the ideas and tools in this module to analyze the two plans. Which one do you think is better? Defend your choice.

Module Summary

- A **sequence** is an ordered list. Each item in the list is a **term** of the sequence.
- The terms of a sequence may be represented by symbols, such as $p_1,\ p_2,\ p_3,\ \ldots,\ p_n$. These symbols are **subscripted variables**, and the natural numbers $(1,\ 2,\ 3,\ \ldots,\ n)$ are the **subscripts**. The symbol p_1 (read "*p* sub one") represents the first term of the sequence, the symbol p_2 (read "*p* sub two") represents the second term of the sequence, and so on. The symbol p_n (read "*p* sub *n*"), represents the **general**, or ***n*th**, **term** of a sequence.
- A **recursive formula** is a rule for finding any term in a sequence by using the preceding term(s). The process of using a recursive formula is known as **recursion.**
- A sequence in which every term after the first is found by adding a constant value to the preceding term is an **arithmetic sequence**.
- The recursive formula for calculating any term in an arithmetic sequence is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d,\ \ n > 1 \end{cases}$$

  where a_1 is the first term, a_n is the *n*th term, a_{n-1} is the term preceding a_n, and *d* is the **common difference** between any two consecutive terms, $a_n - a_{n-1}$.
- An **explicit formula** for calculating any specific term in an arithmetic sequence is:

$$a_n = a_1 + d(n-1)$$

  where a_n is the *n*th term, a_1 is the first term, and *d* is the common difference between any two consecutive terms, $a_n - a_{n-1}$.
- The sum of the terms of an arithmetic sequence is an **arithmetic series**.
- A **geometric sequence** is a sequence in which every term after the first is found by multiplying the preceding term by a constant value.
- The recursive formula for calculating any term in a geometric sequence is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = g_{n-1}(r),\ \ n > 1 \end{cases}$$

  where g_1 is the first term, g_n is the *n*th term, g_{n-1} is the term preceding g_n, and *r* is the **common ratio** between any two consecutive terms, g_n / g_{n-1}.

- The sum of the terms of a geometric sequence is a **geometric series**.
- An **explicit formula** for calculating any specific term in a geometric sequence is:

$$g_n = g_1 r^{n-1}$$

where g_n is the nth term, g_1 is the first term, and r is the common ratio between any two consecutive terms, g_n / g_{n-1}.

Selected References

Stundber, J. *The Science of Musical Sounds*. Sand Diego, CA: Academic Press, 1991.

Taylor, C.A. *The Physics of Musical Sounds*. New York: American Elsevier Publishing Co., 1965.