Digging into 3-D

How do you record the position of a sunken treasure? How do you plot the site of a buried dinosaur bone? How can you describe locations in the world around, above, and below you? The answers to these questions lie in another dimension.

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Introduction

On September 12, 1857, about 300 km off the coast of South Carolina, disaster struck the *S.S. Central America*. During a hurricane, the ship sank with 578 passengers and three tons of gold aboard. Although the loss of human life can never be given a dollar value, the lost gold was worth nearly $40 million at the time. In today's market, the gold's value would be in the hundreds of millions of dollars.

The Columbus America Discovery Group was formed in 1985 for the purpose of finding sunken treasure. For three summers, the group used side-scan sonar to search for the wreck of the *Central America*. The ship was eventually located at a depth of 2500 m. Using a custom-designed, remote-operated vehicle, the *Nemo*, the group recovered 1.4 tons of gold and other artifacts.

Three-dimensional mappings of the underwater site played a vital role in recovering the treasure. In this module, you examine the basics of three-dimensional graphing.

Activity 1

To describe the locations of artifacts on the sunken ship, the computer used by the Columbus America Discovery Group required three values: latitude, longitude, and depth. In some situations, however, latitude and longitude may not provide the most useful method of describing a location. In this activity, you use distances in three dimensions to describe the location of an object in your classroom.

Mathematics Note

A **three-dimensional coordinate system** in which three planes subdivide space into eight **octants** is shown in Figure **1**. Each octant is bounded by the *xy*-plane, the *yz*-plane, and the *xz*-plane. The octant which includes positive *x*-, *y*-, and *z*-values is referred to as the **first octant**.

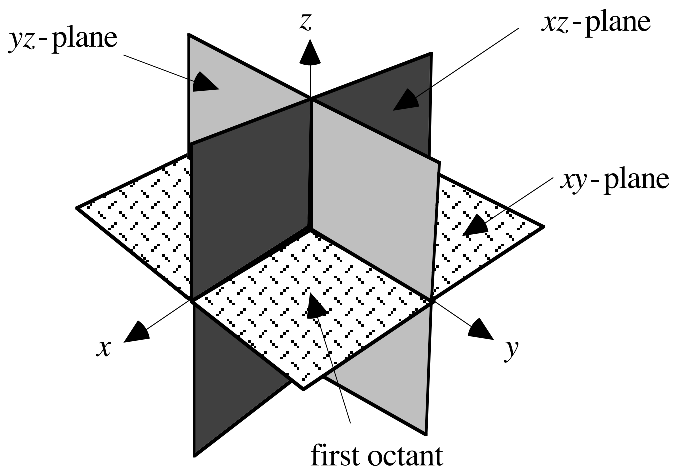


Figure 1: Octants in a three-dimensional coordinate system

Exploration

a. Use the templates provided by your teacher to build a model of a three-dimensional coordinate system.

1. Tape or glue the templates to four pieces of cardboard and cut slots as shown in Figure **2**.

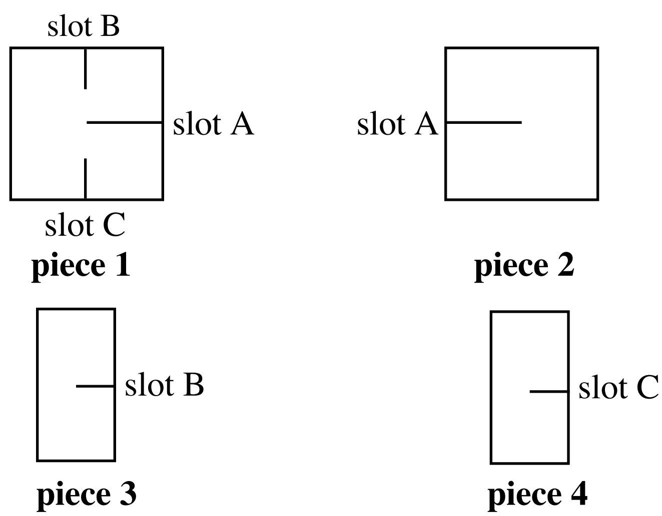
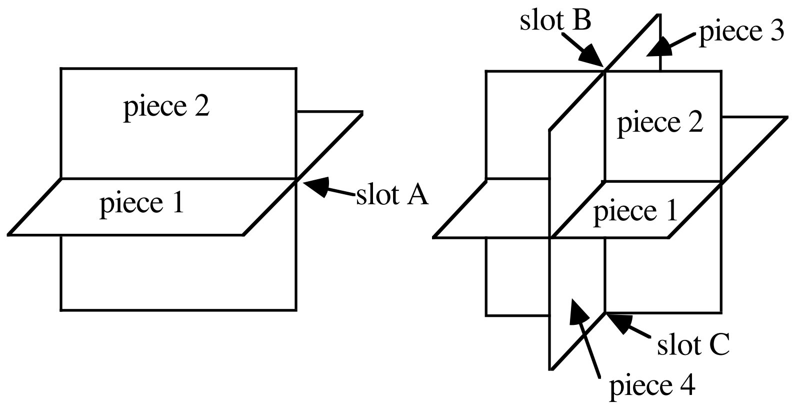


Figure 2: Cutouts for three-dimensional coordinate system

2. As shown in Figure **3a**, slide pieces 1 and 2 together along slot A. Add pieces 3 and 4 by sliding them into the corresponding slots. Your model should now resemble the diagram in Figure **3b**.



a. **b.**

Figure 3: Assembly of three-dimensional coordinate system

b. Select a corner of your classroom (or some other room in the shape of a rectangular prism). Let the intersection of one wall and the floor represent the *x*-axis in a three-dimensional coordinate system. Let the intersection of the other wall and the floor represent the *y*-axis.

The intersection of the two walls represents the *z*-axis. The intersection of the three axes (represented by the intersection of the two walls and the floor) is the origin.

1. Let 1 m be the unit of measure on each axis in your coordinate system. Use tape to mark the units for each axis.
2. Choose one part of some object in the room and designate it as the location of a point. Identify the location of the point in terms of the *x*-axis, the *y*-axis, and the *z*-axis.

c. The location of a point on the model coordinate system from Part **a** can be represented using a piece of graph paper.

For example, consider the location of the point shown in Figure **4**. This point is 4 m from one wall, 5 m from the adjacent wall, and 3 m from the floor.

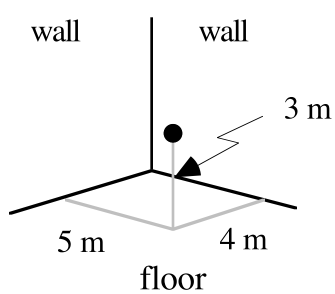


Figure 4: Location of a point in a room

This point's location can be modeled by folding a piece of graph paper as shown in Figure **5**.

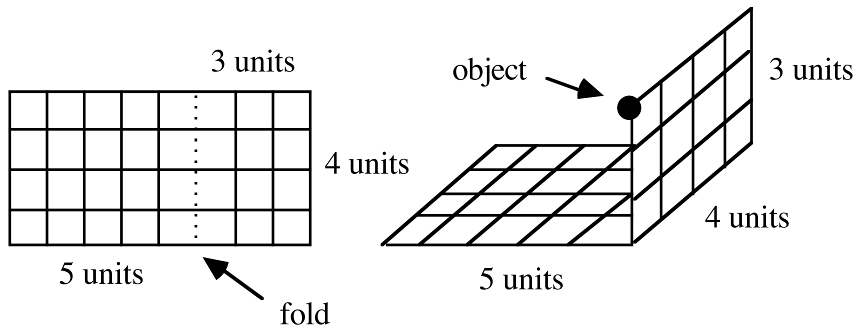


Figure 5: Graph-paper model

1. Use the process described above to model the location of your point with a folded piece of graph paper.
2. To represent this location on your three-dimensional coordinate system from Part **a**, place the appropriate corner of the folded paper at the origin.

d. Ask a classmate to identify the object you selected in Part **b** using the location of your point on the model coordinate system.

Mathematics Note

Three-dimensional coordinate systems are typically represented by graphs like the ones shown in Figure **6**. The *x*- and *y*-axes determine the *xy*-plane. The *z*-axis is used to help depict points above, below, or on the *xy*-plane. The *x*- and *z*-axes determine the *xz*-plane, while the *y*- and *z*-axes determine the *yz*-plane.

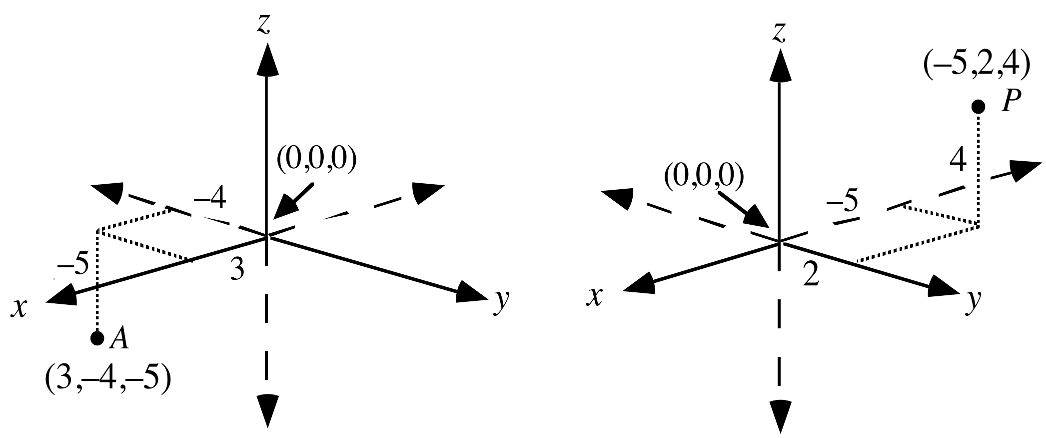


Figure 6: Three-dimensional graphs

The coordinates of a point in this type of three-dimensional coordinate system form an **ordered triple**, (x,y,z). For example, the coordinates of point A in Figure **6** are (3,–4,–5). In other words, the x-coordinate of point A is 3, the y-coordinate is –4, and the z-coordinate is –5, which indicates that point A is located 5 units below the xy-plane. Similarly, the coordinates of point P are (–5,2,4), which means that its x-coordinate is –5, its y-coordinate is 2, and it is located 4 units above the xy-plane.

The **origin** is the point where the three axes intersect. The origin has coordinates (0,0,0) because at that point $x = 0$, $y = 0$, and $z = 0$.

Discussion

a. Describe how you determined the coordinates of the point you selected in Part **b** of the exploration.

b. What do all the points on the floor of the room have in common?

c. How would you model the position of points located below the floor of your classroom?

d.
1. In what octant are the coordinates for every point in your classroom?
2. Are the coordinates for a point located in an adjacent room also in this octant? Explain your response.

e. Use your model of a three-dimensional coordinate system to describe the coordinates of points in each of the eight octants.

f. Compare an octant in space to a quadrant in a plane.

g. As long as the axes are labeled as in Figure **6**, does it make any difference in what order the x-, y- and z-coordinates are used to plot a point? Explain your response.

h. How could you describe the precise location of a large object in your classroom?

Assignment

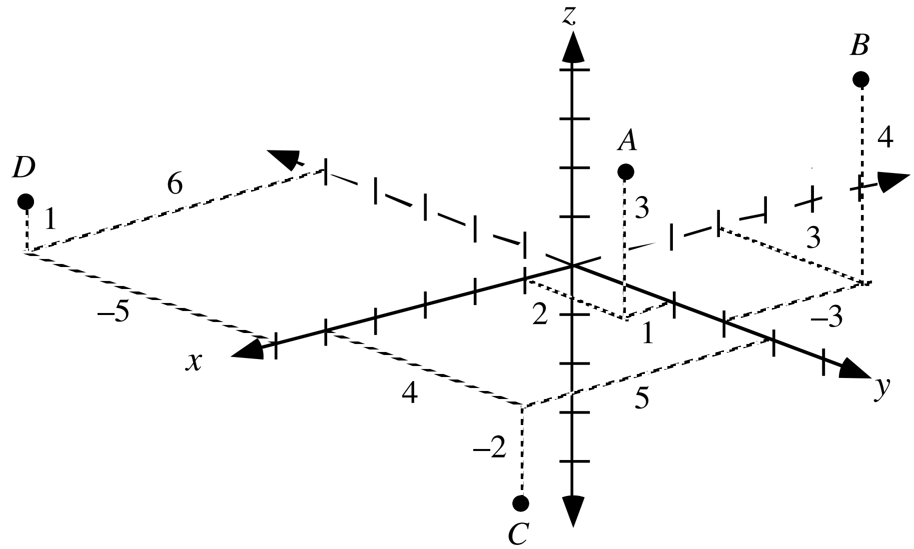
1.1

a. Is the z-coordinate of a point above the xy-plane positive or negative?

b. What is the x-coordinate of a point on the yz-plane?

c. If the y-coordinate of a point is positive, what is its relationship to the xz-plane?

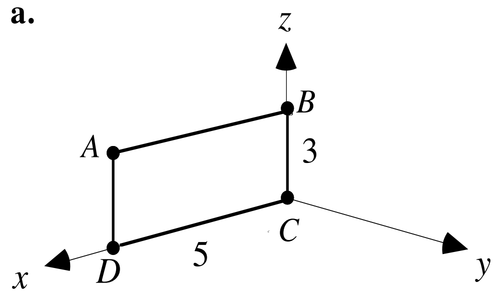
1.2 Identify the coordinates of points A, B, C and D in the graph below.



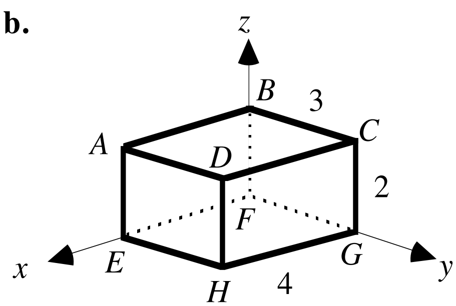
1.3 Describe the location of a point with coordinates (–5,–4,–6) on a three-dimensional coordinate system.

1.4 Identify the coordinates of each labeled vertex in Parts **a–c**.

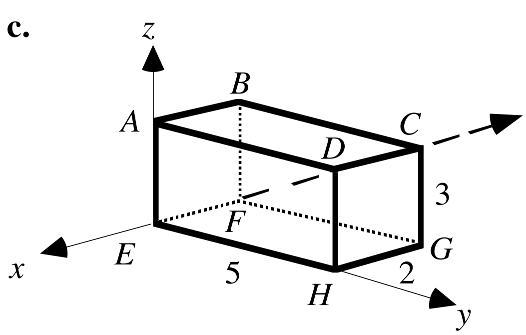
a.



b.



c.



1.5 Use a coordinate system like the one shown in Problem **1.2** to sketch the graph of each point below.

a. *A* (4,5,3)

b. *B* (2,0,5)

c. *C* (0,5,0)

d. *D* (2,3,–2)

e. *E* (4,1,–5)

1.6 Complete Parts **a–c** to make a sketch of a three-dimensional figure.

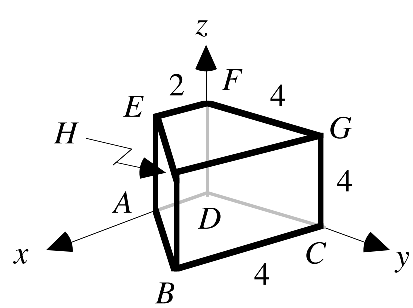
a. Plot each of the following ordered triples and label each as a vertex:

A (2,2,0), *B* (6,2,0), *C* (6,6,0), *D* (2,6,0), *E* (4,4,5)

b. Use segments to connect each vertex with every other vertex. Use solid segments to represent the edges that can be seen from your point of view. Use dotted segments to depict any unseen edges.

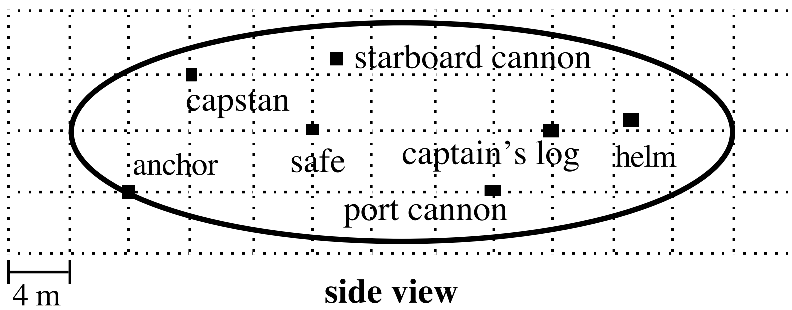
c. What kind of three-dimensional figure did you draw?

1.7 List the coordinates of the vertices of the following figure.

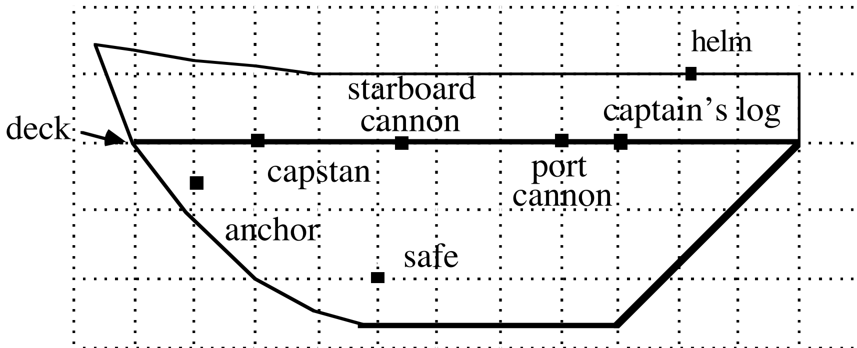


1.8 A team of archaeologists has discovered the wreck of the Spanish galleon *Isabella*. The diagrams below show both top and side views of the *Isabella*, along with the locations of several objects the team wants to bring to the surface.

top view



side view



a. To help the scientists create a map of each artifact's location, complete Steps **1–3** below.

1. Select and label a point on the deck as the origin of a three-dimensional coordinate system with its *x*- and *y*-axes on the deck.

2. Select a scale for the axes.

3. Identify the coordinates of each artifact as an ordered triple.

b. Could you have selected a different point as the origin of your coordinate system? Why might one point be a better choice for the origin than another?

* * * * *

1.9 Describe the set of all points in space that are:

a. 3 cm from a point

b. 3 cm from a segment

c. equidistant from two parallel lines

d. 3 cm from a plane.

1.10 Consider a room shaped like a rectangular prism. Describe the set of all points in the room that are:

a. 180 cm from a point at the center of the floor

b. equidistant from the four corners of the ceiling.

1.11 While examining a wreck, a team of divers collected information about each artifact on the ship. As described in Parts **a–c** below, they recorded this data as ordered triples in which each unit represents a distance of 1 m. Make a sketch of each artifact on a three-dimensional coordinate system.

a. a mast with its endpoints at (1,3,0) and (0,6,8)

b. a storage box with its vertices at (5,2,0), (7,2,0), (5,3,0), (7,3,0), (5,2,1), (7,2,1), (5,3,1), and (7,3,1)

c. a cannonball whose center is located at (8,8,1)

* * * * * * * * * *

Research Project

On a typical two-dimensional map, differences in elevation can be hard to visualize. For this reason, hikers and others often use topographic maps, which show elevation on contour lines. Although this feature makes it easier to identify the elevation of a specific location, forming a mental picture of the terrain can still be difficult.

When a clear representation of the landscape is needed, mapmakers create relief maps. Relief maps show the features of the terrain in three dimensions. In this research project, you make a relief map of a region of your choice. (The area you select should have identifiable differences in elevations.)

a. Create a two-dimensional map of your area on graph paper. For example, Figure **7**, shows a map of Montana on graph paper.

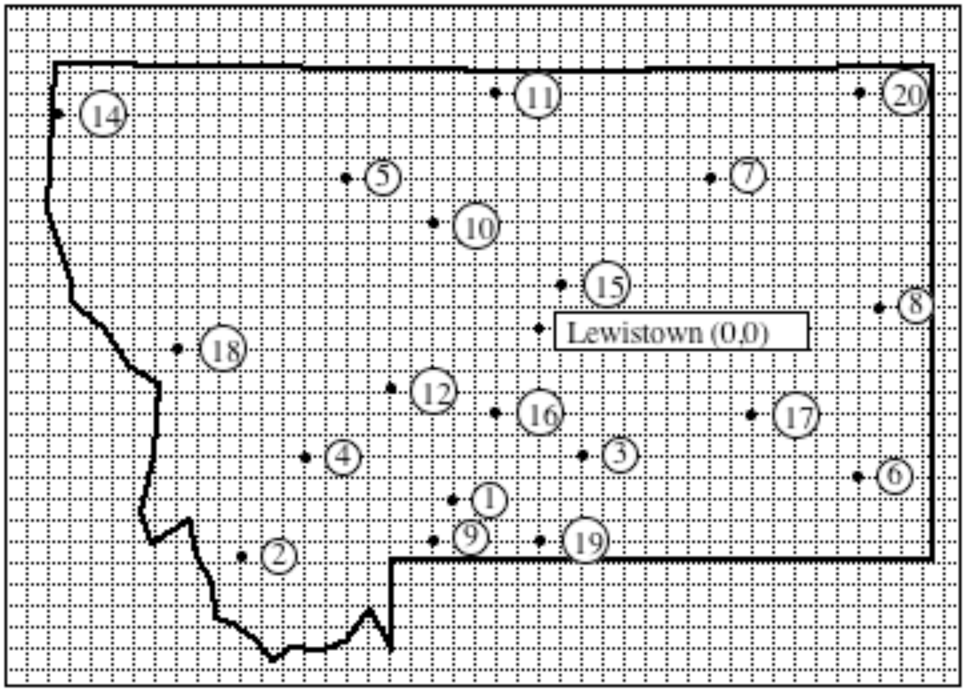


Figure 7: Map of Montana on graph paper

b. Identify several locations of interest on your map and number them as illustrated in Figure **7**. In this example, the numbered locations represent important historical sites.

c. Determine the elevation of each site selected in Part **b**. In Figure **7**, for example, the elevation of site 12 (Helena, the state capital) is 1267 m above sea level. Record these values in a table.

d. Select a location to represent the origin.

e. Tape the map onto a piece of cardboard. To select a scale for your three-dimensional map, complete the following steps.

1. Place the map and cardboard on a table. To determine the thickness of the cardboard, push a toothpick into the cardboard until it hits the table. Mark the toothpick where it intersects the top of the cardboard.
2. Measure the distance from this mark to the top of the toothpick in millimeters.
3. Use the elevations you recorded in Part **c** to identify the highest site on the map.
4. Use the distance from Step **2** and the elevation from Step **3** to create a scale that relates the elevation of a site in meters to the length of the toothpick in millimeters. Round the scale to the nearest meter.

5. Push a toothpick into the map at the location of the highest site. Using your scale, the elevation of this site is represented by the length of the toothpick above the cardboard.

f. Gather enough toothpicks to identify all the sites on the map.

1. Mark each toothpick with the thickness of the cardboard.
2. Use proportions to calculate the number of millimeters above the mark needed to represent the elevation of each site. (How would you show a site with an elevation below sea level?)
3. Cut each toothpick to the appropriate length and push it into the map at the corresponding site.

g. Write a paragraph identifying one of the sites by its location on your map and by its latitude, longitude, and elevation. In a second paragraph, explain the historical significance of that site.

Activity 2

In this activity, you investigate another kind of three-dimensional map: a **surface plot**. Surface plots are similar to the underwater maps used by the Columbus America Discovery Group. Their computer systems used sonar to obtain ordered triples. In the following exploration, you use a mechanical process to produce surface plots.

Mathematics Note

A **surface plot** is a three-dimensional graph used to display the surface of an object. Figure **8** shows one example of a surface plot.

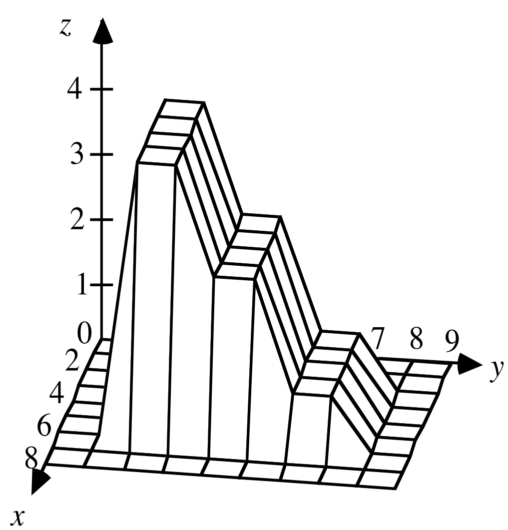


Figure 8: Surface plot

Exploration

Obtain a box with a lid, some stiff wire cut into equal lengths, and a sheet of centimeter grid paper. The box simulates the cargo hold of a sunken ship. Inside the ship is an artifact of unknown size and shape. Before you can recover the artifact, you must first determine its size and shape.

a. **1.** Cover the lid of the box with centimeter graph paper.

2. Place the origin of a three-dimensional coordinate system in one corner of the graph paper. Draw the x- and y-axes as shown in Figure **9**.

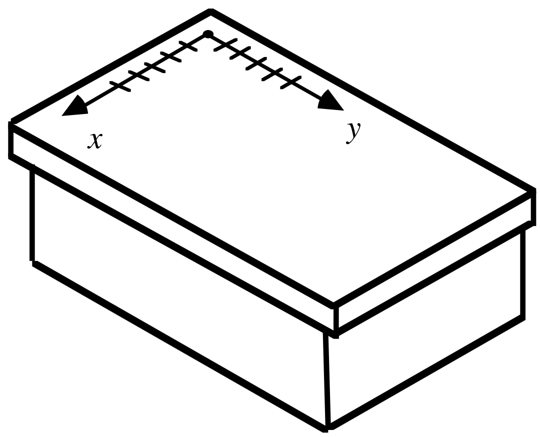


Figure 9: Placement of coordinate system

b. **1.** At each point where the grid lines intersect (the **lattice points**), press a piece of wire through the lid until it strikes either the artifact or the bottom of the box. Leave each wire in the box.

2. Measure the distance from the lid to the top of each wire in millimeters. Record your measurements in a data table or **matrix** like the one in Figure **10**. Each entry in the table represents an ordered triple that describes the location of the top of a wire in a three-dimensional coordinate system.

	0	1	2	3	…	$y-1$	y
0							
1							
2							
3							
⋮							
$x-1$							
x							

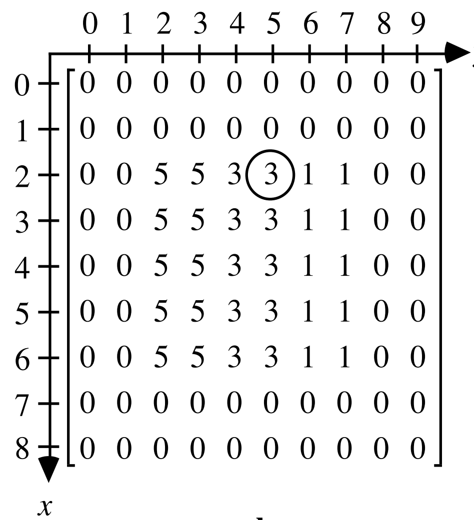
Figure 10: Data table

Mathematics Note

A **matrix** is a rectangular arrangement of data used for storing information in an organized fashion. Matrices can be used to create surface plots. For example, the matrix in Figure **11a** was used to create the surface plot in Figure **8**. The entries in this matrix represent z-values in the ordered triples (x, y, z).

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

a.



b.

Figure 11: Matrix used to create a surface plot

The manner in which data is entered into a matrix may vary for different types of technology. In this module, the x- and y-coordinates are related to the matrix as illustrated in Figure **11b**. For example, the circled entry in this matrix represents the ordered triple (2,5,3).

c. Use a symbolic manipulator to make a surface plot of the artifact in your box. Print a copy of the surface plot.

d. On the basis of the surface plot, what do you think is in the box?

e. "Recover" your artifact by removing the lid of the box.

Discussion

a. How does the object described by your surface plot compare with the actual artifact?

b. What is the relationship between the tops of the wires and the surface of the artifact?

c. What is the relationship between the numbers in the data table and the surface plot?

Assignment

2.1 For each of the following data tables, sketch a surface plot on a three-dimensional coordinate system.

a.

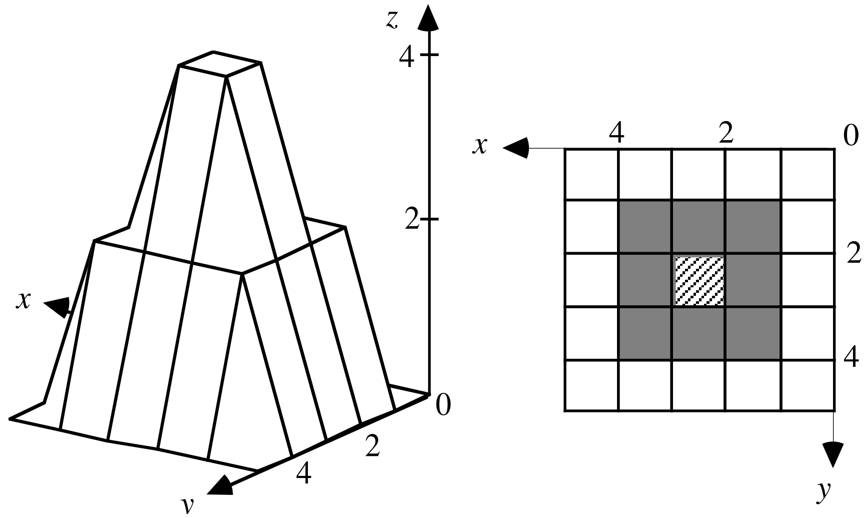
$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&5&5&7&7&5&5&0&0\\ 0&0&5&5&7&7&5&5&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

b.

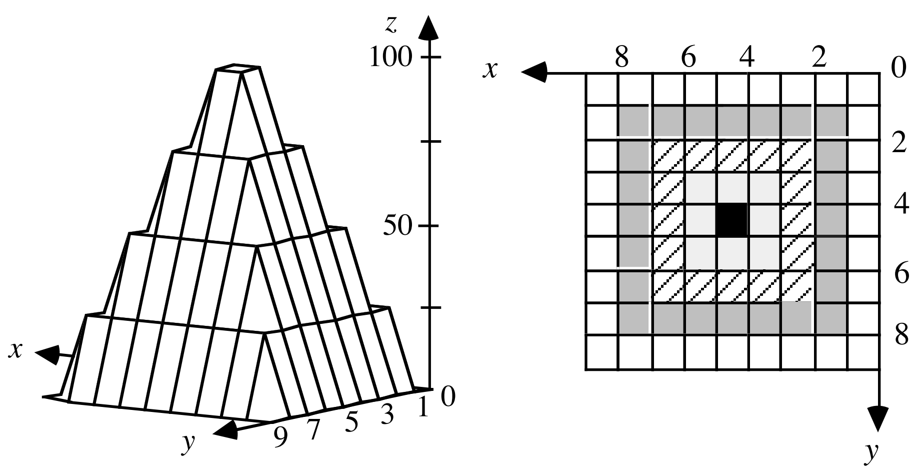
$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&5&5&5&5&0&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&5&5&0&0&5&5&0&0\\ 0&0&5&5&0&0&5&5&0&0\\ 0&0&5&5&5&5&5&5&0&0\\ 0&0&0&5&5&5&5&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

2.2 Complete a matrix for each of the following surface plots. In each case, the figure on the left shows the object as viewed from a perspective slightly to one side of it. The figure on the right shows the object as viewed from a perspective directly above it. The different shadings represent different elevations.

a.



b.



2.3 Create a surface plot that resembles a river flowing through a section of a deep canyon.

2.4 How would the data table in Problem **2.1a** be affected in each of the following situations?

a. The surface plot is moved 2 units upward.

b. The height of the surface plot is tripled.

c. The surface plot is reflected in the *xy*-plane.

2.5 Describe the surface plot that occurs when all the entries in a data table are equal.

* * * * *

2.6 **a.** On a three-dimensional coordinate system, sketch a surface plot of the object represented by the following matrix.

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

b. Determine the angle that the surface plot makes with the *xy*-plane.

2.7 In a one-dimensional coordinate system, the graph of the equation $x = 3$ is a point. In a two-dimensional coordinate system, the graph of this equation is a line. Describe the graph of each of the following in a three-dimensional coordinate system.

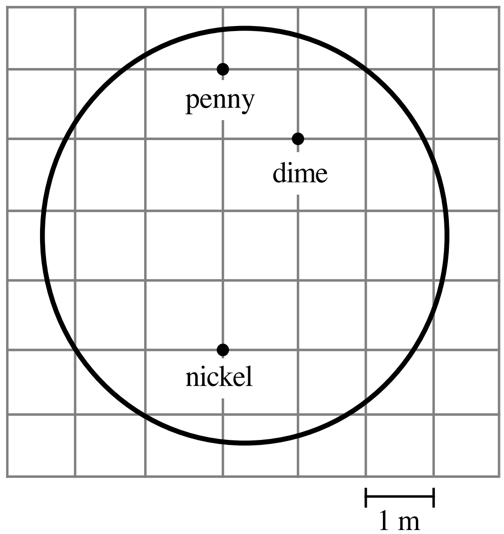
a. $x = 3$

b. $y = 3$

c. $z = 3$

Summary Assessment

1. At the local wishing well, you throw in three coins—a dime, a nickel, and a penny. They land in the well as shown in the following diagram.



a. Label a point in the diagram as the origin and record the coordinates of each coin.

b. The depth of the water in the well is 0.5 m. If the surface of the water is the *xy*-plane, identify the location of each coin as an ordered triple.

2. Imagine that you have a front-row seat at AquaLand's dolphin show. The star performer at AquaLand is Daphne. The pool is a rectangular prism 80 m long, 30 m wide, and 10 m deep.

Your seat is at water level at the center of the long side of the pool. The announcer stands by the corner of the pool to your left, on the opposite side from your seat. The trainer stands on a platform whose base is at the center of the side of the pool on your right. The platform extends 1 m from the edge of the pool and 2 m above the water. At the center of the bottom of the pool is a drain.

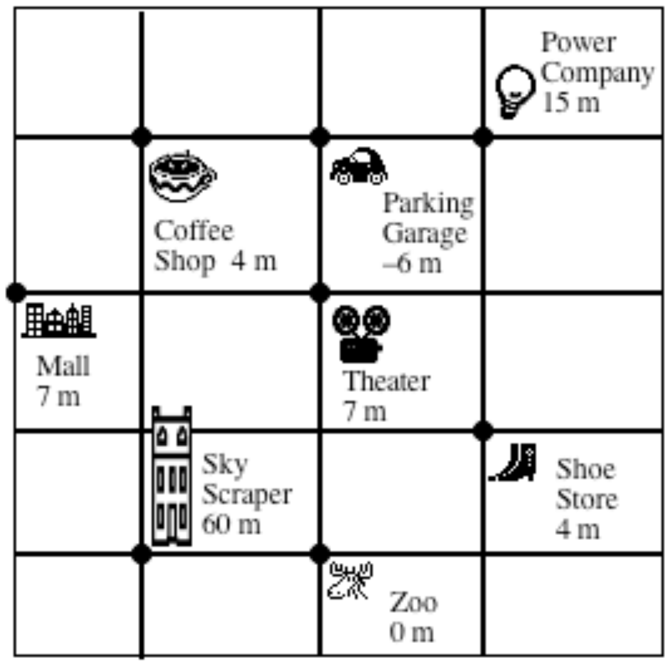
a. Make a sketch of the pool on a three-dimensional coordinate system. Let the surface of the pool be located in the *xy*-plane, with the narrator at the origin.

b. Identify the coordinates of the narrator, the trainer, and the drain.

c. The starting position for Daphne's routine is the point (25,50,–6). Use X to represent this point on your drawing.

d. On the surface of the water, directly above Daphne's starting position, is a plastic hoop. What are the coordinates of the hoop's position?

3. The figure below shows the heights of eight structures in a city. One of these, the parking garage, is located underground.



a. Use coordinates to identify the height and location of each structure.

b. Organize this information in a matrix.

4. An architect's new home contains a rectangular room 8 m long and 6 m wide. The ceiling is 4 m high.

a. Make a sketch of the room on a three-dimensional coordinate system.

b. The electrician wants to install a light at the center of the ceiling. What are the coordinates of the point where the light should be placed?

c. The door to the room is located in one of the 8-m walls. The door is 1 m wide and its hinges are 1 m from the adjacent 6-m wall. The door opens to the inside, toward the 6-m wall. The carpenter wants to install a doorstop to keep the door from hitting the wall. Where should the carpenter place the doorstop? Describe this location as an ordered triple.

d. The thermostat control is located 1.5 m above the floor on one wall of the room, directly opposite the door. Describe the thermostat's location as an ordered triple.

Module Summary

- In one type of **three-dimensional coordinate system**, three planes subdivide space into eight **octants**. Each octant is bounded by the *xy*-plane, the *yz*-plane, and the *xz*-plane. The octant which includes positive *x*-, *y*-, and *z*-values is referred to as the **first octant**.
- The coordinates of a point in this type of three-dimensional coordinate system form an **ordered triple**.
- The **origin** in this three-dimensional coordinate system is the point where the three axes intersect. The origin has coordinates (0,0,0).
- A **lattice point** is a point whose coordinates are integer values.
- A **surface plot** is a three-dimensional graph used to display the surface of an object.
- A **matrix** is a rectangular arrangement of data used for storing information in an organized fashion.

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Seanor, D. "The Case with the Midas Touch." *ABA Journal* (May 1990): 50–55.