A New Look at Boxing

What do a box of cereal and a take-out pizza have in common? What are templates, tilings, and tessellations? And how can you catch an apothem without a net? This module answers all of these questions—and more.

Bill Chalgren • Darlene Pugh • Teri Willard • Lisa Wood

A New Look at Boxing

Introduction

Gloria watches the clock: tick, tick, tick. Time always seems to drag when she's waiting for lunch. Finally, the bell rings. It's pizza time!

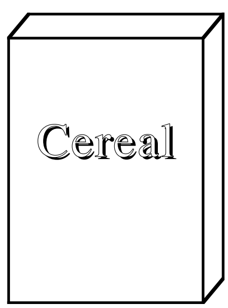
Friends, pop, breadsticks, and pizzas make Little Cheesers the most popular lunch-time hangout at Gloria's school. There's only one problem: time flies when she's having fun, and lunch is only 30 minutes long. As she heads for the door, Gloria asks for a box. She still has one slice of pizza left. On the way back to school, Gloria finishes her meal. Without a second thought, she tosses the box into the trash.

The cashier used a full-size box for Gloria's leftovers. A box big enough to hold a whole pizza may not be the best container for a single slice. If the box had been the right shape and size, less material would have been wasted. In this module, you will look at some different properties of boxes and explore how much material it takes to make them.

Mathematics Note

A **prism** is a three-dimensional figure determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.

For example, Figure **1a** shows a cereal box that, like many pizza boxes, is a prism with rectangular faces. Figure **1b** shows a box for a candy bar in the shape of a triangular prism.



a.

b.

Figure 1: Two boxes

Discussion

a. Explain why a cereal box is a prism.

b. Which faces of the prism in Figure **1b** are the bases?

c. Describe several examples of other objects that are prisms.

d. Is a tennis-ball container a prism? Explain your response.

Activity 1

The cardboard for the cereal box in Figure **1** was cut from a pattern. What does this pattern look like? In this activity, you examine a box and the pattern used to create it.

Exploration

Use a box similar to the cereal box in Figure **1** to complete the following steps.

a. Estimate the total area, in square centimeters, of all the sides of the box.

b. On a sheet of grid paper, make a scale drawing of the pattern you think the manufacturer used to create the box. Use dotted line segments to indicate folds. Cut out your paper pattern and fold it into a box.

c. Without tearing the cardboard, take the box apart and lay it flat. The box probably was constructed using glued tabs.

A pattern with tabs is a **template.** A pattern without tabs is a **net.** Figure **2** shows both a net and a template for a cube.

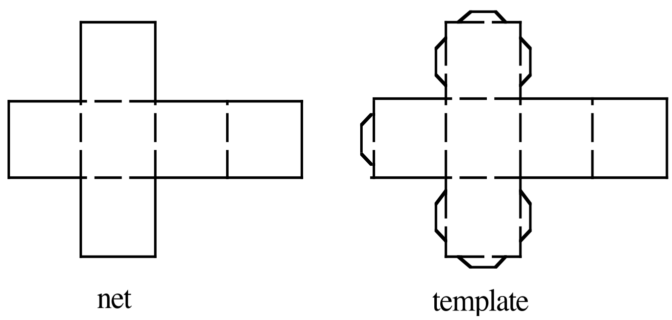


Figure 2: **Cube net and template**

d. Find the area of the net for the box. Compare this area to the estimate you made in Part **a**.

e. Imagine that the template for the box was cut from a rectangle of cardboard. Record the dimensions of the smallest rectangle that will enclose the template.

Discussion

a. Does your folded box from Part **b** of the exploration resemble the original box?

b. How does your paper pattern from Part **b** of the exploration compare to the template for the box?

c. Without making any calculations, compare the area of the net to the area of the template.

d. Why might a box manufacturer be interested in the smallest rectangle that will enclose a template?

e. How do the areas of the net and the template compare to the actual surface area of the box?

Mathematics Note

The **surface area** of a prism is the sum of the areas of its bases and lateral faces.

For example, the surface area of the cube in Figure **3** below is $9+9+9+9+9+9=54 \text{ cm}^2$. The area of the net in Figure **3** is also 54 cm^2.

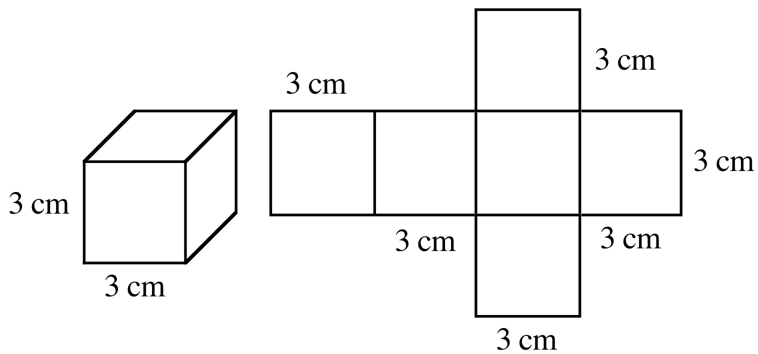


Figure 3: A cube and its net

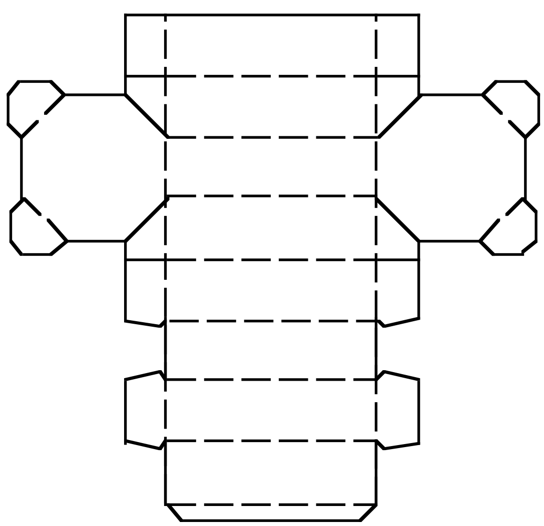
f. Are all containers prisms? Use examples to support your answer.

Assignment

1.1 The template shown below can be used to create an octagonal prism.

a. Draw the corresponding net for the prism.

b. Make a sketch of the three-dimensional figure.



1.2 In Part **e** of the exploration, you recorded the dimensions of the smallest rectangle that would enclose the template for the box.

a. Find the area of this rectangle in square centimeters $\left(\text{cm}^2\right)$.

b. Find the area of the template.

c. When the template is cut from the rectangle in Part **a**, what percentage of the cardboard is wasted?

d. Imagine that a box manufacturer makes 500,000 boxes by cutting templates from cardboard rectangles. If cardboard costs 14 cents per square meter, what is the cost of the wasted cardboard? Describe any assumptions you make in solving this problem.

1.3 **a.** Make a sketch of a container that is not a prism but has faces that are polygons.

b. Draw a possible template for this container.

c. Explain why this container is not a prism.

1.4 A sugar company would like to sell packages of 100 sugar cubes. Each cube is approximately 1 cm on a side.

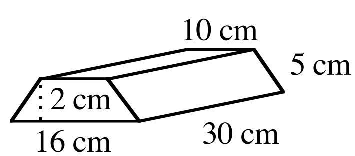
a. Draw a net for a container which could be used to package 100 sugar cubes.

b. Explain how you could position several copies of your net so that the containers could be produced efficiently.

* * * * *

1.5 You are in charge of repainting the red background on all the stop signs in a large city. Before you can order paint, you must determine the area of a stop sign. Draw a regular octagon and describe how to find its area.

1.6 Gold bullion is often molded into blocks. As shown in the diagram below, the cross section of one type of block is an isosceles trapezoid. The trapezoid has a height of 2 cm and bases of 10 and 16 cm.



a. Draw a net that could be used to create a model of the block.

b. Use the net to find the surface area of the block.

* * * * * * * * * *

Activity 2

When trying to minimize waste in box construction, you must consider more than just the shape that encloses the template. How those shapes fit together is also important. For example, a box template may be enclosed by a rectangle. To improve efficiency, a manufacturer might place these rectangles edge to edge on a large sheet of cardboard and cut many templates at once.

Figure **4** shows one way in which smaller rectangles can be arranged on a larger rectangle of cardboard. To cover an even larger sheet without gaps or overlaps, more of the smaller rectangles could be added.

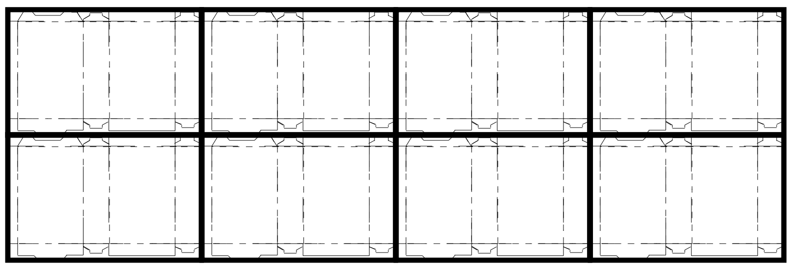
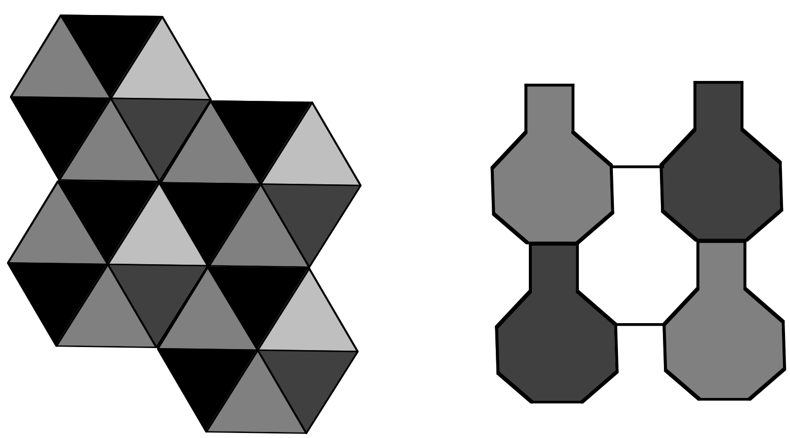


Figure 4: Cereal box templates

> **Mathematics Note**
>
> When a shape is repeated to form a pattern that covers an entire plane without gaps or overlaps, it **tessellates** or **tiles** the plane. The pattern that covers the plane is a **tessellation** or **tiling**.
>
> For example, Figure **5** shows two tessellations.
>
> 
>
> **Figure 5: Two tessellations**

Exploration

Installing floor tiles is similar to covering a plane with regular polygons. Since squares fit together easily without gaps or overlaps, many floor tiles are square. However, it is also possible to tile a floor with other polygons.

a. Cut out templates of all the regular polygons, other than squares, with up to 12 sides. Determine which of these regular polygons can tile a plane. Use drawings to record your results.

b. On a sheet of paper, extend the sides of an equilateral triangle to form three equal exterior angles. In Figure **6** below, $\angle 1$, $\angle 2$, and $\angle 3$ are the exterior angles and $\angle 4$, $\angle 5$, and $\angle 6$ are the interior angles.

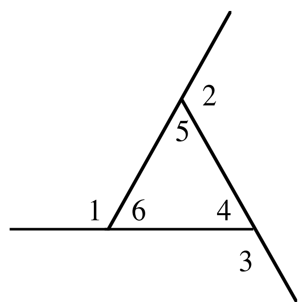


Figure 6: An equilateral triangle with exterior angles

1. Without measuring, determine the sum of the measures of the exterior angles. Hint: Start at the vertex of $\angle 1$ as shown in Figure **6** and visualize walking around the polygon (triangle) until you return to the starting point facing in the same direction. How many degrees did you turn during the walk?
2. Record the measure of a single exterior angle in a spreadsheet with headings like those in Table **1** below.
3. Use your response to Step **2** to determine the measure of an interior angle of a regular polygon and record the result in Table **1**.
4. Determine the number of these regular polygons that would "fit" at one vertex and record the result in Table **1**.

Table 1: Measures of angles of regular polygons

No. of Sides in Polygon	Measure of Exterior Angle (x)	Measure of Interior Angle (m)	No. of Polygons that "Fit" at One Vertex ($360°/m$)
3			
4			
⋮			

c. Repeat Part **b** for all regular polygons with up to 12 sides.

Discussion

a. How can you use Table **1** to determine which regular polygons tessellate a plane?

b. Identify the regular polygons that tessellate a plane. Explain your answer.

c. Will the method of finding the sum of the measures of the exterior angles described in Part **b** of the exploration work for any regular polygon? Explain your response.

d. What is the sum of the measures of the exterior angles of any regular polygon?

e. What is the measure of an exterior angle of a regular polygon with n sides?

f. What is the measure of an interior angle of a regular polygon with n sides?

g. If you knew the number of degrees in an interior angle of a regular polygon, how could you determine the number of sides in the polygon?

Assignment

2.1 Consider a regular polygon with 24 sides.

a. Find the measure of an interior angle of the polygon.

b. Find the sum of the measures of the interior angles of the polygon.

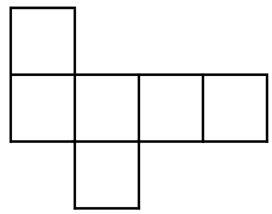
2.2 Find the measure of an interior angle of a regular polygon with 102 sides.

2.3 Is there a regular polygon with more than six sides that will tile a plane? Explain your response.

2.4 Describe one of your tilings from Part **a** of the exploration so that someone unfamiliar with regular polygons and tessellations could reconstruct the pattern.

2.5 A box manufacturer wishes to cut templates from a roll of cardboard 400 cm wide and 6000 m long. The dimensions of the smallest rectangle that will enclose the template are 22.4 cm by 35.5 cm. How many templates can be cut from one roll?

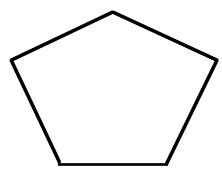
2.6 **a.** The net shown below is just one of several possible nets of a cube. Draw three different cube nets.



b. On a sheet of grid paper, find a cube net that tessellates a plane. Use at least six copies of the net to show that the pattern may be extended in all directions.

* * * * *

2.7 Although a regular pentagon will not tile the plane, there are pentagons that do form tessellations. The pentagon in the following diagram is equilateral but not equiangular.



a. Trace this pentagon and determine whether it will tessellate the plane.

b. Design a pentagon different from the one above that tessellates the plane.

2.8 Do you think that all quadrilaterals will tessellate a plane? Explain your response.

2.9 **a.** Find the measures of an interior angle and an exterior angle of a 16-sided regular polygon.

b. Calculate the sum of the interior angles and the sum of the exterior angles of a 16-sided regular polygon.

c. If the measure of an interior angle of a regular polygon is 170˚, how many sides does the polygon contain?

* * * * * * * * * * *

Research Project

Regular polygons can be arranged in many interesting patterns.

a. Find at least six ways in which a combination of regular polygons will tessellate a plane. In each tiling, use at least two different regular polygons.

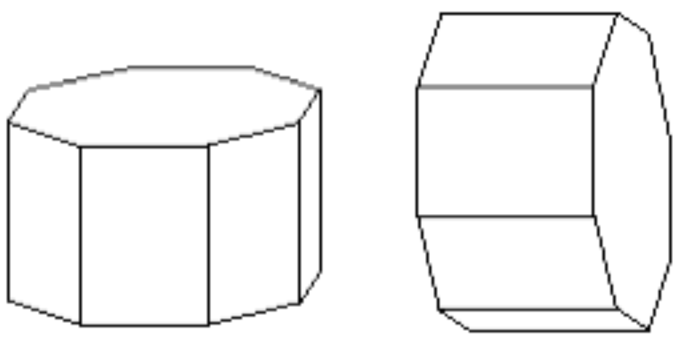
b. Make a careful drawing of each of your patterns.

c. Describe each pattern in a few sentences.

d. Find at least two examples of tilings that use polygons other than squares and rectangles in your home or community.

Activity 3

Not all prisms have rectangular bases. Some have bases that are triangular. Others, such as the octagonal prisms shown in Figure **7**, have bases that are regular polygons.



In this activity, you use your knowledge of the area of triangles and squares to develop a method for finding the area of regular polygons with five or more sides.

Exploration

a. Use a geometry utility to construct a regular pentagon by completing the following steps.

1. Construct a circle. Place five points on the circle. Use segments to connect the center of the circle to each of the five points. Each angle formed by two adjacent radii is a **central angle**.
2. Drag the points on the circle until the measures of all the central angles are equal.
3. Connect the points on the circle to form a regular pentagon.

b. Create a table with headings like those in Table **2** below.

Table 2: Triangles in regular polygons

Polygon	No. of Triangles	Apothem (*a*)	Length of Side (*s*)	Area of Polygon
pentagon	5			
heptagon				
decagon				
n-gon				

c. Measure the perpendicular distance from the center of the polygon to one side. This distance is the length of the **apothem**. In Figure **7**, for example, $\overline{AG}$ is the apothem. Record this measure in the appropriate column in Table **2**.

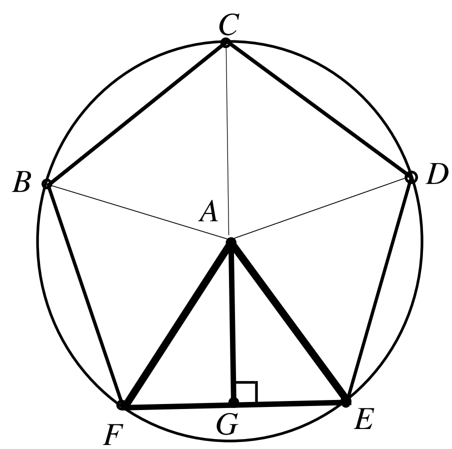


Figure 7: Constructing a regular pentagon

d. Measure the length of one side of the polygon. Record this length in the appropriate column of Table **2**.

e. **1.** Your construction of a polygon contains congruent triangles. Create a formula using the length of the apothem to find the area of one of these triangles.

2. Use the area of one congruent triangle to find the total area of the polygon. Record this area in the appropriate column of Table **2**.

f. Use the geometry utility to find the area of the polygon. Compare this value to the one you determined in Part **e**.

g. Repeat Parts **a–f** for a regular heptagon and a regular decagon.

h. Create a formula for finding the area of a regular *n*-gon. Enter it in the appropriate cell of Table **2**.

Discussion

a. As the number of sides of a polygon increases, what happens to the shape of the polygon?

b. How does the measure of the central angle of a polygon affect the shape of the polygon?

c. How do the areas of the polygons found using your formula compare to the areas of the same polygons found using the geometry utility?

d. The area of a regular polygon with n sides, apothem a, and side length s can be described by the following equation:

$$\text{Area} = \left(\frac{1}{2}as\right)n$$

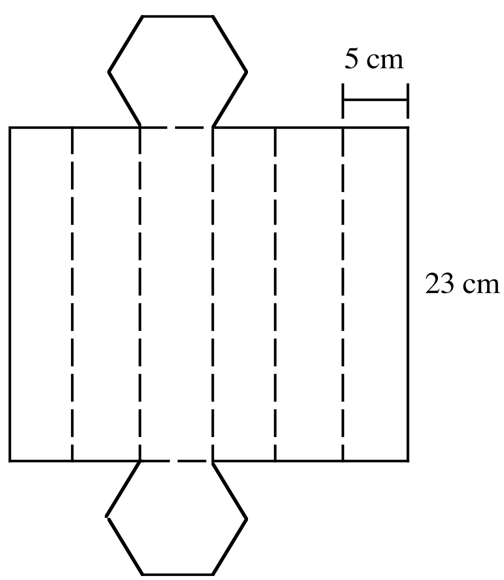
Is this equation equivalent to the formula you developed in Part **h** of the exploration? Explain your response.

Assignment

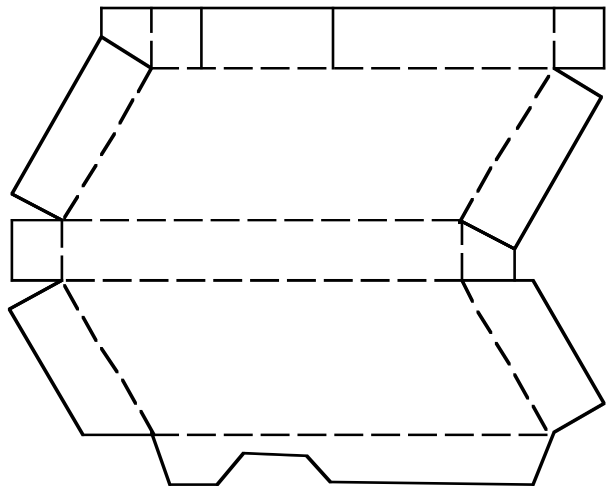
3.1 The floor of Greg's new hot tub is shaped like a regular hexagon. He wants to install a tile floor. What measurements should he make?

3.2 Write a formula for finding the area of a regular polygon in terms of the apothem and the perimeter.

3.3 Use the net below to find the surface area of the corresponding hexagonal box.



3.4 A product engineer has created the template below for a cardboard box.



To hold down production costs, you must design a shape that encloses the template and minimizes waste. Prepare a presentation for the product engineer that includes:

a. the area of the template and the area of the shape

b. a sketch of the tessellation of the shape

c. the percentage of cardboard wasted when the template is cut from the shape.

* * * * *

3.5 Find a box that is not a rectangular prism. Carefully unfold the box and lay its template flat.

a. Draw the smallest shape that encloses the template and also tessellates a plane.

b. Sketch the tessellation of the shape.

c. Find the area of the shape from Part **a**.

d. Calculate the percentage of cardboard wasted when the template is cut from the shape.

3.6 A soccer ball can be modeled with 12 regular pentagons and 20 regular hexagons, each with a side length of approximately 4.5 cm. Find the approximate surface area of a soccer ball. Hint: You may need to draw a sample hexagon and pentagon and measure their apothems.

* * * * * * * * * *

Summary Assessment

You have been asked to design and build an efficient box for one slice of pizza. The specifications and costs are listed below.

- A whole pizza has a diameter of 30 cm. Each pizza is cut into 8 equal slices.
- Cardboard costs 22 cents per square meter.
- The template must assemble into a closed container.
- The shape enclosing the template should both tessellate the plane and minimize waste.

Prepare a report describing your container, including the following:

1. a template

2. the area of the template

3. the shape that encloses the template and tessellates a plane

4. a sketch of the tessellation

5. the cost to make one template

6. the percentage of cardboard wasted in making one template

7. an attractive advertising logo sketched on a template

8. a model of the container.

Module Summary

- A **prism** is a solid determined by two congruent polygons in parallel planes whose corresponding vertices are connected by segments. The two congruent and parallel faces are the prism's **bases**. The parallelograms formed by joining the corresponding vertices of the bases are the prism's **lateral faces**.
- A **template** is a two-dimensional pattern with tabs that can be folded to make a three-dimensional solid.
- A **net** is a two-dimensional pattern without tabs that can be folded to make a three-dimensional solid.
- The **surface area** of a prism is the sum of the areas of its bases and lateral faces.
- When a shape is repeated to form a pattern that covers an entire plane without gaps or overlaps, it **tessellates** or **tiles** the plane. The pattern that covers the plane is a **tessellation** or **tiling**.
- The measure of an exterior angle of a regular polygon with n sides is $360^\circ/n$.
- The measure of an interior angle of a regular polygon with n sides is:

$$180^\circ - \frac{360^\circ}{n}$$

- The **apothem** is the segment whose measure is the perpendicular distance from the center of a regular polygon to one of its sides.
- The area of a regular polygon with n sides, apothem a, and side length s can be described by the following equation:

$$\text{Area} = \left(\frac{1}{2}as\right)n$$

Selected References

Britton, J., and D. Seymour. *Introduction to Tessellations*. Palo Alto, CA: Dale Seymour Publications, 1989.

Consortium for Mathematics and Its Applications (COMAP). *For All Practical Purposes*. New York: W. H. Freeman and Co., 1991.

Grünbaum, B., and C. G. Shephard. *Tilings and Patterns*. New York: W. H. Freeman and Co., 1987.