Skeeters Are Overrunning the World

How does the size of a population change over time? In this module, you use a simple model to shed light on some complicated issues.

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Introduction

At last count, more than 5.4 billion people inhabited the Earth. If each of us laid head to toe, we would make a chain of humanity long enough to wrap around the equator 250 times.

How many people can live on Earth without destroying the environment? How many people can our planet successfully feed? Several organizations studying the ever-increasing human population are concerned with just these questions.

While foretelling the future is never a sure bet, you can gather information about the past and present, find any existing patterns, and use these patterns to make predictions.

Graphs can be useful tools for determining patterns. For example, Figure **1** shows the world population since 1650.

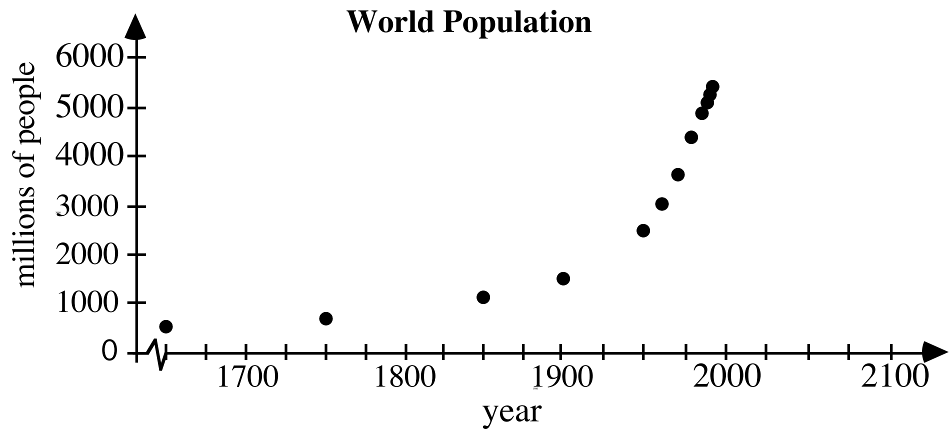

Figure 1: World population from 1650 A.D.

Discussion 1

a. Describe any pattern you see in the world's population since 1650.

b. Based on the pattern you find, predict what you think the world population will be in the year 2075.

c. A prediction is only as good as the information and assumptions on which it is based. The human population has not always grown as rapidly as it has in the past 40 years. Describe any current events that may alter how fast the human population will increase.

Exploration

Statistics like those shown in Figure **1**, along with an appropriate mathematical model, allow researchers to make forecasts about population trends. For example, scientists at the United Nations predict a world population of at least 8.2 billion by the year 2020.

To help make predictions in real-world situations, researchers often use experiments known as **simulations**. The results of the simulations are gathered and analyzed. This data is then compared with known information about the actual population. If the results seem questionable, the simulation may be revised. This modeling process can be summarized by the following five steps:

- creating a model
- translating the model into mathematics
- using the mathematics
- relating the results to the real-world situation
- revising the model.

In the following exploration, you investigate this modeling process using a population of Skeeters.

a. Obtain a large, flat container with a lid, a sack of Skeeters, and several sheets of graph paper.

b. Before beginning the simulation, read Steps **1–7** below and predict how you think the number of Skeeters will change.

1. Place two Skeeters in your container. This is the initial population.
2. After closing the lid, shake the container.
3. Open the lid and count the number of Skeeters with the marked side up.
4. Skeeters reproduce asexually (by themselves). Reproduction is triggered when the marked side of a Skeeter is exposed to light. Add one Skeeter to the container for each mark counted.
5. Record the total number of Skeeters now in the container. This is the end of one "shake."

   The end of each shake represents the end of one time period. The number of Skeeters present at the end of a shake is the total population at that time. (Remember that at shake 0, the number of Skeeters was 2.)
6. Design a method of recording and organizing your data.
7. Repeat Parts **2–5** for 15 shakes.

c. **1.** Create a scatterplot to display the data you recorded. Represent the shake number on the *x*-axis. Select a scale for each axis that will allow you to make predictions for shake numbers through 20.

2. Describe any patterns you see in your data.

d. **1.** Use the pattern described in Part **c** to predict the number of Skeeters after shake 20.

2. How large a box would be necessary to hold this population? Explain how you came to this conclusion.

3. Predict how many shakes it would take for the Skeeter population to reach 1000. Describe how you reached your prediction.

Discussion 2

a. Discuss any similarities or differences you observe between your scatterplot and those of your classmates.

b. How did the number of Skeeters in your population change during the exploration?

c. **1.** Consider your scatterplot as describing the change in the population of Skeeters over time. Use this idea to explain the shape of the graph.

2. How do the graphs obtained in the exploration compare to the linear graphs explored in previous modules?

d. **1.** What other types of living creatures might show the same pattern of population growth as the Skeeters?

2. What limitations might this simulation have in modeling a real-world population?

Activity 1

Within any population, there are differences in appearance and behavior due to genetics and environment. In this activity, you investigate some Skeeter populations with different growth characteristics.

Exploration

In this exploration, each color of Skeeter has its own growth characteristics and initial population. Table **1** shows a list of these characteristics for each color.

Table 1: Skeeter growth characteristics

Color	Growth Characteristics	Initial Population
green	For every green Skeeter with or without a mark showing, add 2 green Skeeters.	1 green
yellow	For every yellow Skeeter with or without a mark showing, add 1 yellow Skeeter.	1 yellow
orange	For every orange Skeeter with a mark showing, add 1 orange Skeeter.	1 orange
red	For every red Skeeter with a mark showing, add 1 red Skeeter.	2 red
purple	For every purple Skeeter with a mark showing, add 1 purple Skeeter.	5 purple

a. Consider the information given in Table **1**.

1. Predict what will happen to the population of green Skeeters for the first 3 shakes.
2. Predict which population will be largest after 10 shakes.

b. Obtain a large, flat container with a lid, a sack of Skeeters of different colors, and a sheet of graph paper. Place the initial population of each color of Skeeters (indicated in Table **1**) in the box.

c. Place the lid on the container and shake it.

d. At the end of each shake, use the growth characteristics from Table **1** to add the appropriate number of Skeeters of each color.

e. Record the total number of Skeeters of each color at the end of each shake. (Record the initial population as the number at shake 0.)

f. Repeat Parts **c–e** for 10 shakes.

g. After 10 shakes, graph the data for each Skeeter population on the same coordinate system, using different colors to indicate the different populations. **Note:** Save your data for the orange, red, and purple populations for Activities **2** and **3**.

Discussion

a. Describe the relationship between the numbers of yellow Skeeters at the end of two consecutive shakes.

b. **1.** Describe the relationship between the number of yellow Skeeters at the end of a shake and the shake number.

2. Restate this relationship as a mathematical equation.

c. Does your equation from Part **b** describe the population of yellow Skeeters after any shake? Explain your response.

d. For which colors of Skeeters is the relationship between shake number and population a function? Recall that a set of ordered pairs (x, y) is a function if every value of x is paired with a value of y and every value of x occurs in only one ordered pair.

e. In the relations you graphed in Part **g** of the exploration, which values represent the domain and which values represent the range?

Mathematics Note

The **growth rate** of a population from one time period to the next is the percent increase or decrease in the population between the two time periods.

For example, Table **2** shows the population of wild horses on an island over three years.

Table 2: A horse population

Year	Total Population
1992	15
1993	18
1994	24

The growth rate of the horse population from 1993 to 1994 is:

$$\frac{24-18}{18} \approx 0.33 = 33\% \text{ per year}$$

f. Is the growth rate constant from shake to shake for each population of Skeeters in the exploration? Explain your response.

Assignment

1.1 **a.** Complete the following table for the yellow Skeeter population.

Shake Number	Total Population	Expanded Notation	Exponential Notation
0	1	$1 \bullet 1$	$1 \bullet 2^0$
1	2	$1 \bullet 2$	$1 \bullet 2^1$
2	4	$1 \bullet 2 \bullet 2$	$1 \bullet 2^2$
⋮	⋮	⋮	⋮
10			

b. Write an equation that relates shake number to the total population after that shake.

c. Is the equation you wrote in Part **b** a function? Explain your response.

d. **1.** Describe how to predict the total population of yellow Skeeters for shake numbers greater than 10.

2. Predict the total population of yellow Skeeters for shake 20.

e. **1.** By what factor is the population of yellow Skeeters increased after each shake?

2. Explain how this factor relates to the equation you wrote in Part **b**.

f. **1.** What is the growth rate for the population of yellow Skeeters?

2. Explain how this growth rate relates to the equation in Part **b**.

1.2 Create a table like the one in Problem **1.1** for your data for the green Skeeter population.

a. Use the table to determine how the shake number relates to the total population after that shake. Write an equation for finding the population based on the shake number.

b. Predict the shake number at which the population of green Skeeters will be close to the population of yellow Skeeters at shake 24.

c. **1.** By what factor is the population of green Skeeters increased after each shake?

2. Explain how this factor relates to the equation you wrote in Part **a**.

d. **1.** What is the growth rate for the green Skeeter population?

2. How does the growth rate relate to the equation in Part **a**?

Mathematics Note

When the values for a variable depend on the outcome of another variable, that variable is **dependent**.

When the values for a variable do not depend on the outcome of another variable, that variable is **independent**.

In a savings account, for example, the amount of time for which money remains in the account determines the amount of interest earned. In this situation, time is the independent variable, while interest earned is the dependent variable.

When drawing a graph, the values for the independent variable are plotted along the horizontal axis. The values for the dependent variable are plotted along the vertical axis.

1.3 **a.** Compare the equations you wrote in Problems **1.1b** and **1.2a**. Describe any similarities or differences you observe.

b. Which quantity represents the dependent variable in these equations, the population or the shake number? Explain your response.

c. Rewrite each equation in terms of a dependent variable (y) and an independent variable (x).

1.4 Consider some Skeeters whose population growth can be modeled by the equation $y = 5^x$, where y represents the total population after a shake and x represents the shake number.

a. Use this equation to predict the population after 10 shakes.

b. How does the population after each shake compare to the population before the shake?

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1.5 In January of 1990, Maridee deposited \$5000 in a savings account. At the end of each year, the interest earned was added to the account. The following table shows the account balance, after interest was added, during the five years from 1990 to 1994.

Year	**Account Balance**
1990	\$5000.00
1991	\$5250.00
1992	\$5512.50
1993	\$5788.13
1994	\$6077.53

a. Create a scatterplot of the information in the table.

b. What is the growth rate in the account per year?

c. Use the growth rate determined in Part **b** to calculate the account balance at the end of 1995.

d. In what year will the account balance have doubled Maridee's original deposit?

1.6 One type of Skeeter produces 3 offspring after every shake, whether the marked side is showing or not.

a. Using an initial population of 1 of these Skeeters, create a table that shows the shake number and total population for the first 6 shakes.

b. Write an equation in the form $y = 1 \bullet b^x$ that models the growth in this population.

c. Determine the total population after 10 shakes.

d. Does this population have a constant growth rate? If so, calculate this growth rate. If not, determine the growth rates between consecutive shakes for the first 6 shakes.

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Activity 2

In Activity **1** you looked at Skeeter populations that doubled or tripled after each shake. What happens if the ratio of consecutive populations is not an integer value?

Discussion 1

a. Recall the orange Skeeter population from Activity **1**. After each shake, only the Skeeters with the marked side showing produced offspring. If you shook a box containing 10 of these Skeeters, how many would you expect to land with the marked side up?

b. How does the probability of a Skeeter landing with the marked side up compare with the probability of a tossed coin landing heads up?

c. What growth rate would you expect to find between consecutive shakes of the orange Skeeter population?

Exploration

a. **1.** Create a spreadsheet with headings like those in Table **3** below.

Table 3: Orange Skeeter population and growth rate

Shake	Expected Population	Actual Population	Actual Growth Rate (from Previous Shake)
0	1	1	
⋮	⋮	⋮	
10			

2. Use the growth rate determined in Part **c** of Discussion **1** to calculate the expected population after each shake. Record this in the appropriate column of the table.

3. Enter the actual data for the orange Skeeter population obtained in Activity **1** in the appropriate column of the spreadsheet.

4. Use the spreadsheet to calculate the actual growth rates between consecutive shakes. Record these values in the appropriate column.

b. On the same set of axes, create scatterplots of the expected data and the actual data for the orange Skeeter population.

c. **1.** Use the growth rate from Part **c** of Discussion **1** to write an equation that describes the orange Skeeter population after x shakes.

2. Sketch a graph of the equation on the same set of axes as the scatterplots from Part **b**.

Discussion 2

a. In Part **b** of the exploration, how does the graph of the actual data compare with the graph of the expected data?

b. How well does the equation from Part **c** of the exploration model the actual data for the orange Skeeter population?

Assignment

2.1 Imagine that you have a container of 20 Skeeters. After shaking the container, you add one Skeeter for every Skeeter with its marked side up.

a. How many Skeeters would you expect to add at the end of the first shake? What would you expect the total population to be after the first shake?

b. How many Skeeters would you expect to add at the end of the second shake? What would you expect the total population to be after the second shake?

c. If you had p Skeeters before a shake, how many would you expect to add after the shake?

d. What is the growth rate for this population?

e. Write an equation that describes the total expected population after a shake if the population before the shake is p.

f. Use your equation from Part **e** to describe the number of Skeeters you would expect to add after the next shake.

g. Describe the formula you would use in a spreadsheet to find the total expected population after any shake.

2.2 When Skeeters are shaken in a container, is the probability of a Skeeter landing marked side up always $1/2$? Explain your response.

2.3 In 1990, the population of Tanzania was approximately 27,000,000 people. The expected growth rate is 3.5% per year.

a. Calculate the expected population in each of the 10 years after 1990.

b. Make a scatterplot of the expected population data from Part **a**.

c. How do you think the expected values for Tanzania's population will compare with the actual values? Explain your response.

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2.4 Chauncy's parents have decided to offer him a weekly allowance. During the first year, he will receive \$10 per week. In each of the following years, they have given him the choice of either a \$7 raise or a 40% increase in his weekly allowance.

If Chauncy plans on living at home for the next 5 years, which proposed increase should he choose? Explain your response.

2.5 In each year between 1990 and 1995, Sue earned a gross salary of \$30,000. In 1990, she paid \$4338 in federal income taxes. In 1991, she paid \$3905 in federal income taxes.

a. Calculate the growth rate in Sue's federal income taxes between 1990 and 1991.

b. Use this rate to predict the taxes Sue can expect to pay in 1995.

c. Describe some possible limitations in using this model to predict Sue's taxes.

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Activity 3

How does the initial population size influence future Skeeter populations? In this activity, you use technology to model the growth of three Skeeter populations from Activity **1**.

Discussion 1

a. Compare your data for the orange, red, and purple Skeeter populations from Activity **1**. Explain any similarities or differences you see.

b. What effect, if any, does the initial population appear to have on the growth of each population?

Exploration

a. Create a spreadsheet with headings like those in Table **4** below. Use initial populations of 1 orange Skeeter, 2 red Skeeters, and 5 purple Skeeters.

Table 4: Three Skeeter populations

Shake No.	Orange	Red	Purple
0	1	2	5
⋮	⋮	⋮	⋮

b. Using an expected growth rate of 0.5 Skeeters per shake, generate a table of values for each population for 20 shakes.

c. On the same set of axes, create a scatterplot of the expected data for each population for the first 5 shakes.

d. On another set of axes, create a scatterplot of the expected data for each population for 20 shakes.

Discussion 2

a. How do the growth rates you observed in the red, purple, and orange Skeeter populations in Activity **1** compare to the expected growth rate of 0.5?

b. Compare the population data you collected in Activity **1** with the expected data generated by the spreadsheet. What similarities or differences do you see?

c.
1. In Part **c** of the exploration, what similarities or differences do you see in the scatterplots for the three populations?
2. Is it possible to determine the initial size of each population by looking at the graph?

d. Describe the mathematical operations used to calculate the population of purple Skeeters in your spreadsheet.

e. If a current population (p) of Skeeters has a growth rate of r, explain why the equation for the total population (T) after the next shake can be expressed as $T = p(1 + r)$.

f.
1. If the population before a given shake is $p(1 + r)$, what expression can be used to describe the total population after that shake?
2. Describe a relationship between initial population (p), growth rate (r), shake number (n), and total population (T) after that shake number.

Assignment

3.1 How closely does the relationship found in Part **f** of Discussion **2** model the data collected in Activity **1** for the orange and red Skeeter populations? Explain your response.

3.2 One equation that can be used to determine the total Skeeter population (T) given an initial population p, a growth rate r, and a shake number n, is

$$T = p(1+r)^n$$

This is one example of an **exponential equation**. The general form of an exponential equation is $y = ab^x$.

a. What would each variable in the general form of an exponential equation represent in terms of a Skeeter population?

b. What effect would a change in the initial population have on the equation?

c. Use appropriate technology to create a graph that shows the growth of a Skeeter population with an initial size of 4 and a growth rate of 150% per shake.

3.3 Explain why a constant growth rate of 0.5 per shake produces little change in population size from one shake to the next when the shake number is small, but results in greater increases in the size of each successive population after many shakes.

Mathematics Note

The exponential equation $y = a \bullet b^x$ can be used to describe a pattern of **exponential growth**. If this equation describes population growth, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the growth rate). The independent variable x represents a time period.

Considering a population of Skeeters, for example, the independent variable x represents the shake number at which the population is counted. The dependent variable y represents the total population. A population of Skeeters with an initial population of 6 and a growth rate of 0.5% per shake can be modeled by the equation $y = 6 \bullet 1.005^x$.

3.4 Two of your classmates have used a spreadsheet to model the growth of a population of Skeeters. The initial population was 7. After shake 8, the total population was 2,734,375. What growth rate did they use? Hint: Substitute the appropriate values into an equation of the form $y = a \bullet b^x$ to find b; then use this value to determine the growth rate.

3.5 **a.** The relation $y = 3 \bullet 2^x$ models the growth in a population of Skeeters. Describe the values of the domain and range in this setting.

b. Is this relation a function? Explain your response.

3.6 **a**. The two equations below represent two different populations of Skeeters. Graph both equations on the same set of axes. When will these populations be approximately the same size?

1. $y = 10 \bullet 1.5^x$

2. $y = 1 \bullet 2^x$

b. Find the size of each population when $x = 0$. What do these values represent?

3.7 The table below shows the population of Skeeters in a container after each of 5 shakes. Write an equation which could be used to describe this data. (Your equation may not describe the data exactly.)

Shake Number	Population Total
0	2
1	7
2	25
3	86
4	300
5	1050

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3.8 The following table shows the growth in a savings account with an initial deposit of $4000.

Year	Account Balance
1990	$4000.00
1991	$4160.00
1992	$4326.40
1993	$4498.50
1994	$4679.40

a. Write an equation in the form $y = a \bullet b^x$ to describe this data.

b. Using your equation, what is the value of *y* when $x = 0$?

c. What is the value of *x* when the initial deposit of $4000 has doubled?

3.9 In 1987, Vincent Van Gogh's painting *Irises* was auctioned for $53.9 million. Assume that the painting's value grew exponentially since 1889. If the painting initially sold for $50, by what percentage did its value increase each year?

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Research Project

Imagine your own population of creatures with a growth rate and initial population different from those used in the explorations.

a. Use a spreadsheet to simulate the growth in this population for at least 10 time periods. Display your simulated data in both a table and a graph.

b. Find an equation that models the growth in the population over time.

c. Write a story about your population. Include a description of the growth rate and explain the consequences this rate will have on the population over time.

Activity 4

In 1991, the world's human population was approximately 5.3 billion. This total was increasing by about 250,000 per day, or 3 people every second. At this rate, over 1 billion people will have been added to the Earth's population by the end of the decade.

As a citizen of Earth and the United States, you have some questions to consider when making future personal and political decisions. What is the growth rate of the world population? How does this rate compare with the growth rate of the U.S. population? Is a growth rate of zero desirable? If so, how can zero population growth be obtained?

Exploration

Table **5** shows the population of the United States at 10-year intervals from 1790 to 1990.

Table 5: U.S. population from 1790 to 1990

Year of Census	Number of People	Year of Census	Number of People	Year of Census	Number of People
1790	3,929,214	**1860**	31,443,321	**1930**	122,775,046
1800	5,308,483	**1870**	39,818,449	**1940**	131,669,275
1810	7,239,881	**1880**	50,155,783	**1950**	150,697,361
1820	9,638,453	**1890**	62,947,714	**1960**	179,323,175
1830	12,866,020	**1900**	75,994,575	**1970**	203,302,031
1840	17,069,453	**1910**	91,972,266	**1980**	226,545,805
1850	23,191,876	**1920**	105,710,620	**1990**	248,709,873

Source: U.S. Bureau of the Census.

a. Identify any patterns you see in the data in Table **5**.

b. In the previous activities, you calculated growth rates for Skeeter populations between consecutive shakes. Use a similar technique to calculate the growth rates for the U.S. population for each 10-year period.

Discussion

a. What historical events may have affected U.S. population growth during the past 200 years?

b. How would you find a representative growth rate for the U.S. population for the 200-year period from 1790 to 1990?

c. If a population with a growth rate of 0 has a birth rate of 0.05 per year, what is the death rate?

d. Consider a population with a growth rate of –0.05 per year and an initial population of 65. What equation could you use to model this population?

e. How does a negative growth rate affect the total population over time?

Assignment

4.1 By describing the patterns found in the U.S. population data and displaying these patterns in graphs or equations, you can make predictions about the future.

- **a.** Predict what the U.S. population would have been in 1990 if the growth rate had remained unchanged since 1790. Compare this number to the actual population in 1990.
- **b.** Describe how you could use either the data in Table **5** or an exponential equation to predict the U.S. population in the year 2040.
- **c.** If the future growth rate remains the same as it was from 1980 to 1990, predict the U.S. population in the year 2040.

4.2

- **a.** Select a growth rate that produces a decreasing population in each successive 10-year period.
- **b.** Use this growth rate and the U.S. population in 1990 to predict the U.S. population in 2040.
- **c.** What conditions might cause a population to decrease?

4.3 From 1920 to 1930, the U.S. population grew by approximately 16%

- **a.** Use this growth rate to estimate the U.S. population in 1940 and 1950.
- **b.** Compare your estimates with the actual values for the U.S. population given in Table **5**.

4.4 The table below shows the growth rates, per year, in the populations of three cities. Use this data to predict the population of each city in the year 2040. What do you notice about the predicted populations for these cities?

City	1992 Population	Annual Growth Rate
New York	14,628,000	0.00%
London	9,168,000	–1.00%
Lagos	8,487,000	4.80%

Source: U.S. Bureau of the Census.

4.5 How does a growth rate of 0 affect total population numbers over time?

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4.6 In 1930, the U.S. national debt totaled $16,185,000,000. By 1940, it had risen to $42,968,000,000.

a. Calculate the percent increase in the national debt over the decade from 1930 to 1940.

b. What would the national debt have been in 1990 if the growth rate from 1930 to 1940 had remained unchanged?

c. Since 1930, there have been periods of increase and decrease in the growth rate of the national debt. The actual debt in 1990 was about $3,233,000,000,000.

1. In 1980, the national debt was $907,700,000,000. Calculate its growth rate from 1980 to 1990.
2. What will the national debt be in 2040 if the growth rate from 1980 to 1990 remains unchanged?

4.7 The enrollment at Eagle Canyon High School has been increasing steadily in the past few years. The present high school building is designed for a maximum of 1000 students. Within the next 5 years, the school board wants to build an addition that will increase the high school's capacity by 200 students.

a. Last year's enrollment at Eagle Canyon High was 947 students. This year's enrollment is 958 students. If this annual growth rate remains unchanged, will the new addition be needed within 5 years? Explain your response.

b. Assume that the growth rate in the student population remains unchanged. Even with the completion of the high school addition, will the school still have enough capacity for the next 20 years? Explain your response.

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Summary Assessment

The table below contains data on the population, growth rate, land area, and population density for four nations. Use this information to complete Problems **1–4**.

Nation	1993 Population	Annual Growth Rate	Land Area (km^2)	Density ($people/km^2$)
Canada	27,700,000	1.0%	8,968,000	3.1
China	1,177,600,000	1.4%	9,327,000	126.3
Hungary	10,324,000	–0.3%	92,000	112.2
India	903,159,000	2.1%	2,972,000	303.9

Source: U.S. Bureau of the Census.

1. For each nation listed in the table, write an exponential equation that describes its population growth.

2. Assuming that the given growth rates remain unchanged, predict the years in which the 1993 populations of Canada and Hungary will have doubled.

3. Although India has a smaller population than China, it is growing at a faster rate.

a. Determine the year in which India's population will surpass that of China.

b. What will India's population density be at that time?

4. If these four nations continue to grow at their current rates until 2093, which nation will have the highest population density? Support your conclusion with figures and graphs.

Module Summary

- A **simulation** is an experiment conducted to investigate real-world situations.
- The **growth rate** of a population from one time period to the next is the percent increase or decrease in the population between the two time periods.
- When the values for a variable do not depend on the outcome of another variable, that variable is **independent**.
- When the values for a variable depend on the outcome of another variable, that variable is **dependent**.
- When drawing a graph, the values for the independent variable are plotted along the horizontal axis. The values for the dependent variable are plotted along the vertical axis.
- The exponential equation $y = a \bullet b^x$ can be used to describe a pattern of **exponential growth**. If this equation describes population growth, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the growth rate). The independent variable x represents a time period.

Selected References

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