I’m Not So Sure Anymore

Are you betting on the lottery to change your life? The odds of hitting that big jackpot are not as good as you might think. In this module, you’ll explore the mathematical probability of waking up a winner.

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Introduction

Sarah looked at the clock. The drawing was five minutes away. She clutched the week's lottery ticket in both hands, trying to convince herself that it was a winner. She knew that the probability of winning the jackpot was extremely low. Still, she had hope. After all, she'd bought her ticket at the Yummy Mart, where three other winning tickets had been sold, and she'd chosen lucky numbers, numbers that had shown up in many previous drawings.

As the numbers appeared on the television screen, Sarah moved to the edge of her chair. Her first number matched. So did the second. Her heart raced. The next two numbers also matched. As the last number was being drawn, the lights flickered and the screen went blank. A power outage!

Was the power restored in time for Sarah to see the fifth number? Did she win? And what were the rules of the game she was playing? Although you can't answer all of these questions, you may be able to determine the probability that Sarah is holding the winning ticket.

Exploration

In this exploration, you investigate a simple lottery game in which the prizes are small. In the Apple Lottery, players win red, green, or yellow apples. A sample ticket is shown in Figure **1**.

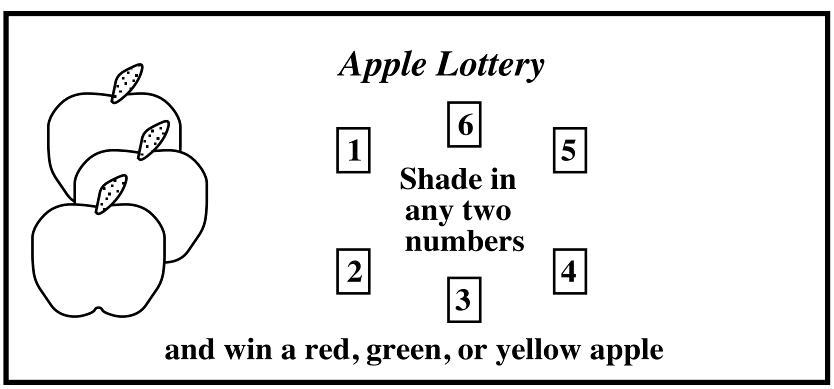


Figure 1: A ticket for the Apple Lottery

All lottery games have rules, often printed on the back of the ticket. Figure **2** shows the back of an Apple Lottery ticket.

How to Play

- Shade two different numbers from 1 to 6 on the front of the ticket.

How to Win

Apple Lottery officials randomly draw two different numbers from 1 to 6.

- Win a yellow apple by matching 2 numbers.
- Win a green apple by matching 1 number.
- Win a red apple by matching 0 numbers.

Figure 2: The back of an Apple Lottery ticket

a. When numbers are chosen at random, there is no way to predict which numbers will be chosen. Suggest a method that Apple Lottery officials might use to draw two different numbers from 1 to 6 at random.

b. In the Apple Lottery, there are three different **events**—winning a red apple, winning a green apple, and winning a yellow apple.

Predict the number of each type of apple you would win if you played the Apple Lottery 1000 times. Record your predictions.

c. Pick two numbers from 1 to 6 that you would shade on an Apple Lottery ticket. Record these numbers.

d. Recall that a simulation is a model of a real-world occurrence. The results of a simulation are often used to make predictions.

Use the following steps to simulate the Apple Lottery officials drawing two numbers.

1. Place 6 objects in a container. Each object should be marked with a different number from 1 to 6.
2. Shake the container. Without looking, draw two of the numbered objects.

e. Compare the numbers drawn in Part **d** with the two numbers you picked in Part **c**. Record the type of apple won.

f. Repeat Parts **d** and **e** 24 more times.

Mathematics Note

One way of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event can be calculated using the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

For example, suppose that you counted 60 heads in 100 trials of a coin toss. The experimental probability of obtaining a head on any one toss is:

$$\frac{60}{100} = \frac{3}{5}$$

g. Using your results from Part **f**, determine the experimental probability of each of the following events:

1. winning a red apple
2. winning a yellow apple
3. winning a green apple
4. winning an apple of any color.

h. Combine your results from Part **f** with those of the rest of the class. Use the combined results to determine the experimental probability of each of the following events:

1. winning a red apple
2. winning a yellow apple
3. winning a green apple
4. winning an apple of any color.

i.

1. Predict how many apples of each color you would win in 1000 games of the Apple Lottery.
2. Compare your response to the prediction made in Part **b** of the exploration.

Discussion

a. Compare the experimental probabilities found in Parts **g** and **h** of the exploration. Which do you believe give better estimates of the true chances of winning? Explain your response.

b. Explain why the experimental probability of winning an apple in the Apple Lottery is 1.

c.
1. What other methods could you use to generate two random numbers for the Apple Lottery?
2. Do these methods guarantee that two different numbers will be generated?
3. What problem may occur when using a simulation that can generate identical numbers?
4. How could you modify the simulation to ensure that the two numbers are different?

d. What advantages might there be in using technology to simulate the Apple Lottery?

Activity 1

When using experimental probability to make predictions, a large number of trials provides a better estimate of the true likelihood of an event. In the following exploration, you use technology to help you simulate the results of many games of the Apple Lottery.

Exploration

a. Select two numbers from 1 to 6 for a new Apple Lottery ticket. Record these numbers.

b. When Apple Lottery officials draw two numbers, these may or may not match your numbers. The first column in Table **1** lists all the possible pairs of numbers in the Apple Lottery. This is the **sample space** for the lottery.

Make a copy of Table **1**. For each possible pair of numbers, record the number of matching digits and the color of the apple you would win with your ticket.

Table 1: Apple Lottery sample space

Pair of Numbers	Number of Matching Digits	Color of Apple Won
1, 2		
1, 3		
1, 4		
1, 5		
1, 6		
2, 3		
2, 4		
2, 5		
2, 6		
3, 4		
3, 5		
3, 6		
4, 5		
4, 6		
5, 6		

c. Use technology to simulate the Apple Lottery by completing the following steps.

1. Randomly generate the first number.
2. Randomly generate the second number.
3. If the second number is the same as the first, generate another number. Repeat until you obtain a number different from the first.

d. Use Table **1** to determine which apple you won.

e. Repeat Parts **c** and **d** 99 more times, recording the number of times you won each color of apple.

f. Use the results of your 100 trials to determine the experimental probability of each of the following events:

1. winning a red apple
2. winning a yellow apple
3. winning a green apple

Mathematics Note

The set of all possible outcomes for an experiment is the **sample space**.

An **event** is a subset of the sample space.

If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

For example, the sample space for tossing two fair coins can be represented as $\{\text{HH, TH, HT, TT}\}$, where H stands for head and T stands for tail. The event of getting one tail when tossing two coins consists of 2 outcomes: TH and HT. Because the total number of outcomes in the sample space is 4, the theoretical probability of getting one tail is:

$$\frac{2}{4} = \frac{1}{2}$$

g. Use the information in Table **1** to determine the theoretical probability of each of the following events:

1. winning a red apple
2. winning a yellow apple
3. winning a green apple
4. winning an apple of any color

Note: Save your results for use in Problem **3.1**.

Discussion

a. Compare the theoretical probabilities of the events in the Apple Lottery with the experimental probabilities you determined in Part **f** of the exploration.

b. Using the combined class results for Parts **c–e** of the exploration, determine the experimental probability of each of the following events:

1. winning a red apple
2. winning a yellow apple
3. winning a green apple

c. Compare the experimental probabilities determined using the class results with their corresponding theoretical probabilities.

d. How does the pair of numbers you selected for your Apple Lottery ticket affect the theoretical probability of winning each type of apple?

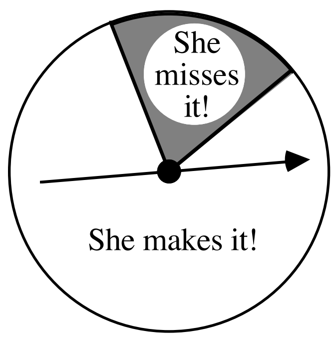
Assignment

1.1 Explain why picking a new ticket in the Apple Lottery does not change the theoretical probability of winning a particular type of apple.

1.2 Judging from the theoretical probabilities, how many apples of each color do you think you would win after playing the Apple Lottery 1000 times?

1.3 Time has expired at the divisional basketball championship. The game is tied and Charrette is at the foul line. During the season, she made 4 out of every 5 of her free throws.

One way to simulate this situation is to use a spinner, as shown in the diagram below.



a. What does one spin represent in this simulation?

b. Use a pencil and a paper clip to simulate the arrow in the spinner. On a copy of the diagram above, spin the paper clip to determine whether Charrette makes or misses the free throw. If the paper clip lands on a boundary segment, spin again.

c. Record the results of 30 trials. Find the experimental probability of Charrette making the free throw.

d. Compare the experimental and theoretical probabilities of Charrette making the free throw.

1.4 Describe the sample space for each of the following situations.

a. At the end of the school year, a student receives a letter grade for a science class.

b. Two ordinary dice are rolled and the numbers added. For example, if one die shows a 6 and the other shows a 2, the result is 8.

1.5 Camie has asked Alicia to play a game of cards. This game involves two piles of four cards each. The first pile contains the ace, king, queen, and jack of diamonds. The second pile contains the ace, king, queen, and jack of spades. The object of the game is to select a card from the first pile, then match it with a card from the second pile.

a. Determine the sample space for this game.

b. What is the theoretical probability of obtaining a winning combination of cards?

c. Does this game seem fair? Explain your response.

d. If all eight cards were shuffled together in one pile, would the sample space and theoretical probabilities remain the same? Explain your response.

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1.6 A football coach wants to know the probability of winning the coin toss in the next three games if the team captain calls heads each time.

a. One player decides to use a simulation to predict the probability of this event. The table below shows the results of 250 trials of tossing three coins.

Three Heads	Two Heads	One Head	No Heads
29	109	91	21

Use these results to determine the experimental probability of getting three heads.

b. Another player tells the coach that he can determine the probability of getting three heads using the following sample space:

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Use this sample space to determine the theoretical probability of getting three heads.

c. Do the results of the simulation agree with the theoretical probability? Explain your response.

1.7 In another version of the Apple Lottery, players choose two different numbers from 1 to 5. After lottery officials randomly select two different numbers from 1 to 5, prizes are awarded as in the original Apple Lottery.

a. Determine the sample space for this version of the Apple Lottery.

b. What is the theoretical probability of winning a red apple? a yellow apple? a green apple?

c. Devise a way to simulate playing this game 100 times.

1. Describe your simulation.
2. Run the simulation 100 times and record the results.
3. Using this data, what is the experimental probability of winning a red apple? a yellow apple? a green apple?

d. Do the results of the simulation agree with the theoretical probabilities you calculated in Part **b**? Explain your response.

e. In which version of the Apple Lottery are you more likely to win a yellow apple? Explain your response.

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Research Project

A new game called the Match Lottery has the same rules as the Apple Lottery, except that the second number drawn need not be different from the first. In the Match Lottery, for example, players may select the numbers 3, 3.

Write a report on the Match Lottery that includes the following:

a. a description of the sample space

b. an explanation of how the difference in rules affects its simulation

c. the results of at least 100 trials in which the lottery ticket chosen has two different numbers

d. the results of at least 100 trials in which the lottery ticket chosen has two identical numbers

e. the experimental probabilities of winning each prize

f. the theoretical probabilities of winning each prize

g. an explanation of whether or not the results of the simulations support the theoretical probabilities

h. an explanation of whether or not any tickets have a better chance of winning a yellow apple than other tickets.

Activity 2

In order to calculate the probability of winning a lottery, you must determine the number of outcomes in the sample space. While the sample space for the Apple Lottery is relatively small, other lotteries may have more than a million possible outcomes.

Exploration

By counting the number of outcomes in small sample spaces, you may observe some patterns that can help you determine the size of large sample spaces.

In one simple lottery, for example, players select 1 number from the set $\{1, 2, 3\}$. The sample space for this lottery contains 3 singles:

1	2	3

If players must select 2 different numbers from the set $\{1, 2, 3\}$, the sample space contains 3 pairs:

1, 2	1, 3	2, 3

If players must select 3 different numbers from this set, the sample space contains 1 triple:

1, 2, 3

The numbers of outcomes in these sample spaces are recorded in the third row of Table **2** below. (Because this lottery only has three available numbers, there are no quadruples or quintuples.)

Table 2: Size of sample space for different lotteries

Numbers for the Lottery	No. of Singles	No. of Pairs	No. of Triples	No. of Quadruples	No. of Quintuples
$\{1\}$	1	0	0	0	0
$\{1, 2\}$	2	1	0	0	0
$\{1, 2, 3\}$	3	3	1	0	0
$\{1, 2, 3, 4\}$					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{1, 2, 3, \ldots, 50\}$	50				

a. Create a spreadsheet with headings like those in Table **2**.

b. Consider a lottery in which players pick from the set $\{1, 2, 3, 4\}$. As you complete Steps **1–4**, record your responses in the spreadsheet.

1. If players select 1 number from the set, how many singles are there in the sample space?
2. If players select 2 different numbers from the set, how many pairs are there in the sample space?
3. If players select 3 different numbers from the set, how many triples are there in the sample space?
4. If players select 4 different numbers from the set, how many quadruples are there in the sample space?

c. Continue to determine the size of the sample spaces for other sets of lottery numbers. As you work, look for patterns that will allow you to quickly fill in all the cells of the spreadsheet. **Note:** Save your completed table for use later in this module.

Discussion

a. What patterns do you observe in the spreadsheet?

b. How did you complete the spreadsheet?

c. What does the number in each cell represent?

d. How would you use the spreadsheet to determine the theoretical probability of winning a lottery in which 3 different numbers are picked from the set $\{1, 2, 3, \ldots, 20\}$?

Assignment

2.1 Using your spreadsheet, determine the number of quadruples you can select from 40 available numbers.

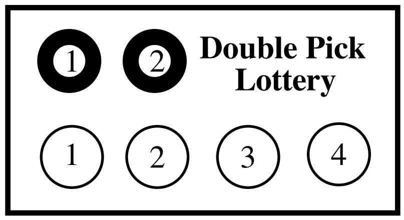
2.2 In one state lottery, players choose 4 different numbers from a set of 24. To win the jackpot, a player must match all 4 numbers.

a. Use your spreadsheet to determine the probability of winning the jackpot with one ticket.

b. What is the probability of winning the jackpot with 10 different tickets for the same drawing?

2.3 In the Double Pick Lottery, players pick a number from 1 to 4 from a white panel and a number from 1 to 2 on a black panel. Lottery officials randomly draw one ball from a container of four white balls and one ball from a container of two black balls. To win a prize, players must match the numbers on both the white and black balls.

A ticket for the Double Pick Lottery is shown below.



a. List the sample space for this game and determine the number of possible outcomes.

b. Describe how to determine the size of the sample space using the spreadsheet created in the exploration.

2.4 Another popular lottery game involves picking five numbers from 1 to 35 on a white panel and one number from 1 to 35 on a black panel. Lottery officials draw five balls from a container of 35 white balls and one ball from a container of 35 black balls. To win the lottery, players must match all five white balls and the black ball.

a. Determine the size of the sample space when selecting five numbers from a set of 35.

b. Determine the size of the sample space for selecting one number from a set of 35.

c. Using the method you described in Problem **2.3b**, determine the size of the sample space for this game.

d. Determine the theoretical probability of matching the five white balls and one black ball.

2.5 Determine the number of different groups of 5 that there are in your math class.

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2.6 In Lottery A, officials randomly draw six numbers from the set $\{1, 2, 3, ..., 20\}$. In Lottery B, officials randomly draw eight numbers from the set $\{1, 2, 3, ..., 18\}$.

Josh thinks that he is more likely to match all the numbers in Lottery B than in Lottery A because the set of available numbers is smaller. Do you agree with Josh? Explain your response.

2.7 A drawer contains an assortment of 5 different pairs of gloves (a total of 10 single gloves). If you randomly select two gloves from the drawer, what are your chances of getting a matching pair? Explain your response.

2.8 In one state lottery, players pick five numbers from the set $\{1, 2, 3, ..., 45\}$ and one number from the set $\{1, 2, 3, ..., 45\}$.

a. Determine the size of the sample space for randomly selecting five numbers from the set $\{1, 2, 3, ..., 45\}$.

b. Determine the size of the sample space for randomly selecting one number from the set $\{1, 2, 3, ..., 45\}$.

c. Determine the size of the sample space for making both selections in Parts **a** and **b**.

d. Determine the theoretical probability that a player will pick the same six numbers as the lottery officials on a single ticket.

2.9 In another lottery game, officials randomly select four numbers from the set $\{1, 2, 3, ..., 12\}$ and a fifth number from a different set. The probability that a player will pick the same five numbers as lottery officials on a single ticket is $1/2970$. How many numbers are there in the second set?

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Activity 3

In many lotteries, the cost of playing is relatively small, while the potential winnings could be very large. Typically, the probability of winning a large prize with any one ticket is low. Will playing the game many times increase your chances of winning? In the following activity, you learn about expected lotteries values.

Exploration

The Apple Lottery has decided to change its prizes. In the new version of the game, players that match neither of the two numbers win nothing, players that match exactly one of the numbers receive \$1.00, and players that match both of the numbers receive \$3.00.

The cost of a New Apple Lottery ticket is \$1.00. In the following exploration, you examine how much a player might expect to win at this game.

a. Create a spreadsheet with headings like those in Table **3** below.

Table 3: **Experimental results for New Apple Lottery**

Event	Prize	No. of Wins	Total Winnings
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum	20	

b. Play the New Apple Lottery 20 times. Determine the number of times you won each prize and enter your results in the appropriate column of the spreadsheet.

c. Determine the total winnings for each row in the spreadsheet.

d. Find the sum of the winnings for all three events.

e. Calculate the mean amount won per game.

f. Determine the experimental probability of winning each prize in the New Apple Lottery. Record these probabilities in a spreadsheet with headings like those in Table **4** below.

Table 4: **Experimental probabilities for New Apple Lottery**

Event	Prize	Experimental Probability	Expected Winnings
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum		

g.
1. Multiply the value of each prize by its experimental probability and enter the product in the expected winnings column of Table **4**.
2. Find the sum of the experimental probabilities and the sum of the expected winnings for the three events.
3. Compare the sum of the expected winnings to the mean amount won per game calculated in Part **e**.

Discussion

a. Compare the mean amount you won per game with others in the class.

b. Why does the sum of the expected winnings determined in Part **g** equal the mean amount won per game?

c. On average, how much do you think you would win or lose by playing the New Apple Lottery?

Mathematics Note

The mean value of an experiment is the **expected value**. Expected value can be calculated by adding the products of the value of each event and its corresponding theoretical probability.

For example, consider a game in which players predict heads or tails, then flip a coin. If the prediction matches the result of the coin toss, the player wins \$1.00. If the prediction does not match, the player wins \$0.00. The products of the value of each event and its corresponding theoretical probability are shown in Figure **3**.

Event	Value	Theoretical Probability	Product
match	\$1.00	1/2	\$0.50
no match	\$0.00	1/2	\$0.00
	Sum	1	\$0.50

Figure 3: Expected value of a coin game

Since the sum of the products is \$0.50, the expected value of the game is \$0.50.

A **fair game** is one in which the expected value equals the cost of playing. For example, if you paid \$0.50 to play the coin game described above, the game would be mathematically fair.

d. If the coin game described in the mathematics note cost \$1.00 to play, how might the prizes be changed to make it a fair game?

e. Judging from your experimental results, do you believe that the New Apple Lottery is a fair game? Explain your response.

f. Why do you think most lotteries are not fair games?

Assignment

3.1 In order to determine if the New Apple Lottery is a fair game, it must be analyzed using expected value.

a. Create and complete a spreadsheet with headings like those in the table below.

Event	Value	Theoretical Probability	Product
two matches	\$3.00		
one match	\$1.00		
no matches	\$0.00		
	Sum		

b. Explain why the New Apple Lottery is not a fair game.

c. Change the values of the prizes to make the lottery a fair game.

d. Are there other prize values that make this a fair game? Explain your response. Hint: Use the spreadsheet to help you examine possible prize values.

3.2 In one state lottery, players select five numbers from 1 through 37. Tickets cost $1.00 each. The table below shows the values of the prizes in this lottery.

No. of Matches	Prize	Probability of Winning with One Ticket
5	$20,000.00	$\frac{1}{435,897}$
4	$200.00	$\frac{160}{435,897}$
3	$5.00	$\frac{4960}{435,897}$

a. If a player buys one ticket, what is the expected value of the game? Describe how you determined your response.

b. Can players make the lottery a fair game by buying more than one ticket? Explain your response.

3.3 In a carnival game, players pay 10 cents for one roll of a 20-sided die. Each side of the die shows a different number from 1 to 20. The number rolled is the value of the prize in cents.

a. Find the theoretical probability of rolling each of the following:

1. 20

2. 15

b. Use expected value to determine if this is a fair game.

3.4 A basketball player has an 80% chance of making a free throw. Determine the number of shots you would expect the player to make in:

a. 30 attempts

b. 500 attempts

c. n attempts.

3.5 Imagine that you have purchased one ticket for a benefit raffle. A total of 750 tickets have been sold at $2 each. From these tickets, one winner will be chosen at random. The prize is worth $300.

a. Determine the probability that you will win the $300 prize.

b. What is the expected value of this raffle for one ticket?

c. Is the raffle a fair game? Explain your response.

3.6 A typical roulette wheel has 38 compartments, each of which has an equal chance of being selected during one spin of the wheel.

a. If a player wins \$30 for selecting the right compartment, what is the expected value of the game?

b. In order for this to be a fair game, how much should it cost to play? Explain your response.

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3.7 As part of its annual fund drive, the local hospital sells 400 raffle tickets for \$5.00 each. From these tickets, one winner will be chosen at random to receive a prize of \$100.00. If you buy one ticket, what is the expected value for the raffle? Explain your response.

3.8 In the Red and Blue Lottery, players choose four numbers from 1 to 20 on a red panel and one number from 1 to 20 on a blue panel. Tickets cost \$1.00 each. The table below shows the probability of winning the two smaller prizes in the game.

Number of Matches	Prize	Probability of Winning with One Ticket
four red, one blue		$\frac{1}{96,900}$
three red, one blue	\$200.00	$\frac{16}{24,225}$
two red, one blue	\$5.00	$\frac{12}{1615}$

a. Describe how to determine the probability of matching four numbers on the red panel and one number on the blue panel.

b. If the expected value for one play is approximately \$0.70, what is the value of the prize for matching four red numbers and one blue number?

c. If the lottery commission sells 100,000 tickets for one game, how much money can it expect to make?

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Summary Assessment

Your state legislature has decided to start a new lottery. Earnings from the game will fund the construction of an amusement park. In order to attract an innovative and appealing design, the governor has agreed to pay 10% of all lottery profits to the creator of the new game.

Design a new lottery for your state. Your proposal to the state gaming commission should include:

- a description of how to play the lottery and the cost to play (you may wish to include a sketch of a sample lottery ticket)
- a list of prizes and a description of how each prize is won
- the theoretical probability of winning each prize
- the expected value of the lottery for one ticket and for 1 million tickets
- the amount you expect to earn for one ticket and for 1 million tickets
- a summary of the experimental results obtained from a simulation of your lottery.

Show all calculations and explain how your experimental results support the theoretical probabilities and expected values.

Module Summary

- One way of predicting the likelihood of an event is to perform many trials under controlled conditions. The results of these trials provide the **experimental** (or **empirical**) **probability** of the event occurring. The experimental probability of an event can be calculated using the following ratio:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}$$

- The set of all possible outcomes for an experiment is the **sample space**.
- An **event** is a subset of the sample space.
- If each outcome in a sample space has the same chance of occurring, then the **theoretical probability** of an event can be calculated using the following ratio:

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

- The mean value of an experiment is the **expected value**. Expected value can be calculated by adding the products of the value of each event and its corresponding theoretical probability.
- A **fair game** is one in which the expected value equals the cost of playing.

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