Marvelous Matrices

How do businesses—large and small—keep track of their sales, inventory, and profits? In this module, you explore some of the ways that matrices are used to store and analyze information.

Kyle Boyce • Pete Stabio

Marvelous Matrices

Introduction

Welcome to the Age of Information. Office workers and company presidents, homeowners and rent payers, consumers and manufacturers—all are bombarded daily by mountains of data. Somehow, some way, you have to make sensible decisions based on that information. Those decisions range from simple day-to-day choices to those with major personal or financial impact.

A typical business manager juggles information like monthly sales, production costs, company inventory, and employee productivity. A **matrix** is one mathematical tool used to organize the information needed to make good decisions. In this module, you look at how a small business grapples with the same problems that confront corporate giants such as General Motors and Motorola.

Activity 1

Family Snack, a branch of the Family Corporation, sells nuts, beef jerky, and jam. These three products are sold separately, as shown in Figure **1**, as well as in assortment packages. Because they cannot be divided into smaller products, these three items are considered **simple components**.

Figure 1: Simple components

Family Snack's assortment packages contain different combinations of the three simple components. The Snack Pack contains 2 boxes of nuts and 6 pieces of jerky, the Gift Pack contains 3 jars of jam and 2 Snack Packs, and the Family Pack contains 3 jars of jam and 3 Gift Packs. These **composite products** are shown in Figure **2**.

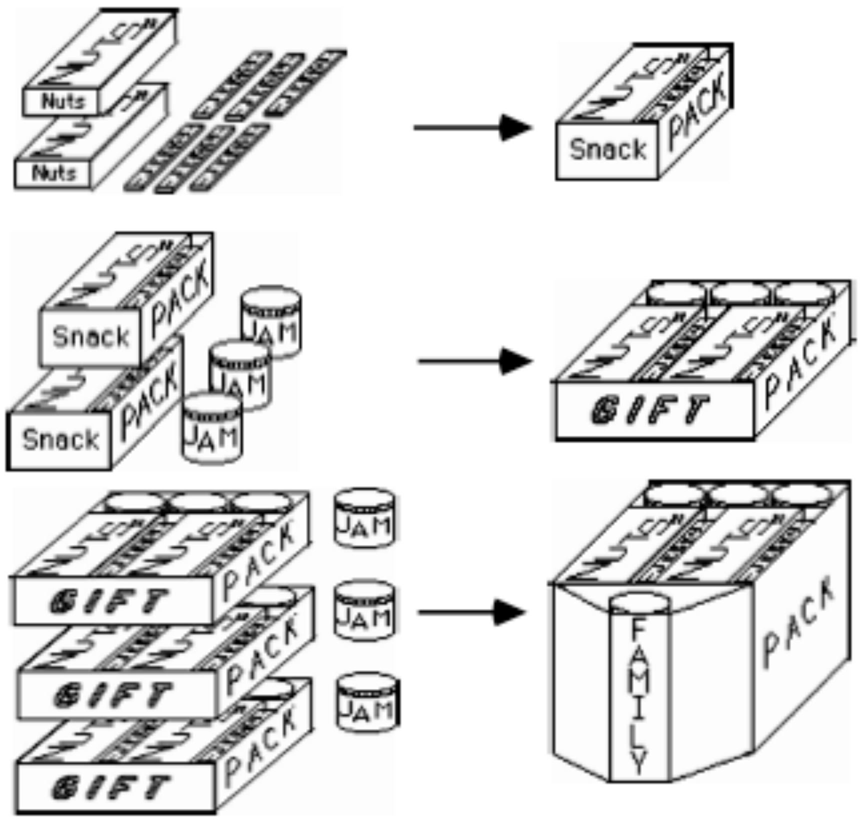


Figure 2: Composite products

Family Snack has received an order for 8 boxes of nuts, 20 pieces of jerky, 25 jars of jam, 10 Snack Packs, 5 Gift Packs, and 40 Family Packs. How many boxes of nuts, pieces of jerky, and jars of jam are needed to fill this order? In the following exploration, you use matrices to help answer this question.

Exploration

In this exploration, you discover how graphs and matrices can help you organize data and make informed decisions.

Business Note

One way to represent a company's product line is with a **requirement graph**. A requirement graph is a tree diagram that shows each product's components at each level of its assembly. A complete requirement graph has simple components at the end of each branch.

For example, consider a company that sells mathematics supplies. This company's simple components are graph paper, rulers, compasses, and protractors. It also sells two composite products: the Euclid Set and the Complete Set. The Euclid Set contains 10 rulers and 5 compasses, while the Complete Set contains 2 Euclid Sets, 10 packs of graph paper, and 10 protractors.

A requirement graph for this product line is shown in Figure **3**.

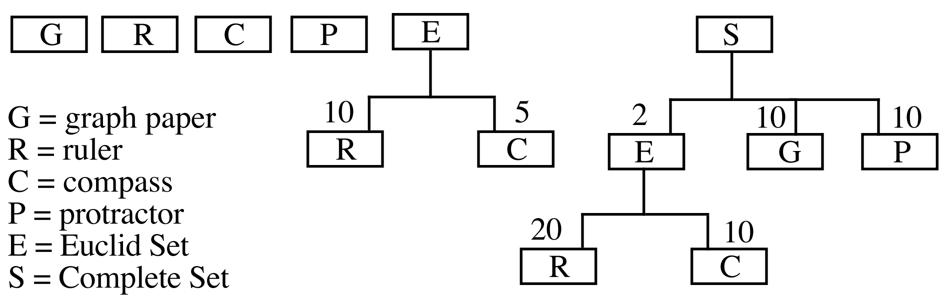


Figure 3: A requirement graph

a. Draw a requirement graph for Family Snack's product line (see Figures **1** and **2**).

b. Use the requirement graph from Part **a** to complete Steps **1–5**.

1. Describe the Snack Pack in terms of its simple components.
2. Describe the Gift Pack in terms of its simple components and composite products.
3. Describe the Gift Pack in terms of its simple components only.
4. Describe the Family Pack in terms of its simple components and composite products.
5. Describe the Family Pack in terms of its simple components only.

Mathematics Note

A **matrix** (plural **matrices** or **matrixes**) is a rectangular arrangement of rows and columns used to organize information. A matrix of i rows and j columns has **dimensions** $i \times j$ (read "i by j"). Matrices are named using bold, uppercase letters or descriptive words. Each item in a matrix is an **element**.

A company can use a requirement graph to create a **total requirement matrix**. In a total requirement matrix, each item sold is represented in a column heading, while each simple component or composite product is represented in a row heading. Each element indicates the number of that simple component or composite product required to produce the corresponding item sold.

For example, the matrix **T** shown below is a total requirement matrix for the product line described by the requirement graph in Figure **3**. In this matrix, G represents graph paper, R represents ruler, C represents compass, P represents protractor, E represents Euclid Set, and S represents Complete Set. In matrix T, the values 10, 20, 10 in the S column represent the total number of simple components required for a Complete Set. This total includes the simple components for the two Euclid sets that are part of a Complete Set.

$$\mathbf{T} = \begin{array}{c} \\ G \\ R \\ C \\ P \\ E \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} G & R & C & P & E & S \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 10 & 20 \\ 0 & 0 & 1 & 0 & 5 & 10 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Since matrix **T** has 6 rows and 6 columns, its dimensions are 6×6. The element in the third row, fifth column indicates that there are 5 compasses in each Euclid Set.

c. Use the requirement graph from Part **a** to create a total requirement matrix **R** for Family Snack's product line.

Discussion

a. What does the element 5 represent in the requirement graph in Figure **3**?

b.
1. In the total requirement matrix you created for Family Snack's product line, why is each element along the diagonal the number 1?
2. What does the element 0 represent in this matrix?

c. Describe the relationship between a requirement graph and its corresponding requirement matrix.

d. As mentioned earlier, Family Snack has received an order for 8 boxes of nuts, 20 pieces of jerky, 25 jars of jam, 10 Snack Packs, 5 Gift Packs, and 40 Family Packs. How can you use a requirement matrix to help determine the number of boxes of nuts, pieces of jerky, and jars of jam needed to fill this order?

Assignment

1.1 Family Snack has received an order for 20 boxes of nuts, 60 pieces of jerky, 48 jars of jam, 24 Snack Packs, 12 Gift Packs, and 2 Family Packs. How many boxes of nuts, pieces of jerky, and jars of jam are needed to fill the entire order?

1.2 Family Snack has decided to introduce three more products—individual jars of honey, the Honey Pack, and the Hive Pack. The Honey Pack contains 3 jars of honey and 2 Snack Packs, while the Hive Pack contains 3 jars of honey and 3 Gift Packs.

a. Create a requirement graph for Family Snack's product line that includes these new products.

b. Create a total requirement matrix for this product line that includes the new products.

c. Determine the number of jars of jam in one Hive Pack.

1.3 The owners of Sammie's Spreadables sell a variety of cracker spreads. The requirement matrix for the company's product line is shown below.

$$\mathbf{S} = \begin{array}{r} \\ \text{shrimp} \\ \text{lobster} \\ \text{spinach} \\ \text{Six-Pack} \\ \text{Ten-Pack} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{shrimp} & \text{lobster} & \text{spinach} & \text{Six-Pack} & \text{Ten-Pack} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a. Create a requirement graph that corresponds with matrix **S**.

b. How can you identify the simple components in matrix **S**?

c. Sammie's Spreadables has received an order for 8 Ten-Packs. Describe how you could use matrix **S** to determine the number of each simple component needed to fill the order.

* * * * *

1.4 The Holiday Hands Company sells packages of cheese and meat products. There are six items in their product line: cheddar cheese logs, cheese-and-bacon logs, summer sausage logs, Snack Packs, Hand-Out Packs, and Pig-Out Packs. The Snack Pack contains two cheddar cheese logs and three summer sausage logs. The Hand-Out Pack contains three cheddar cheese logs and two cheese-and-bacon logs. The Pig-Out Pack contains one Snack Pack and two Hand-Out Packs.

a. Create a requirement graph for Holiday Hands' product line.

b. Create a total requirement matrix for Holiday Hands' product line.

c. Determine the number of summer sausage logs in one Pig-Out Pack.

d. Determine the number of cheddar cheese logs in one Pig-Out Pack.

1.5 Use the requirement matrix shown below to complete Parts **a–c**.

$$\mathbf{T} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 20 \\ 0 & 1 & 0 & 3 & 6 & 18 \\ 0 & 0 & 1 & 2 & 7 & 21 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a. Create a requirement graph that corresponds with this requirement matrix.

b. Identify the simple components in matrix **T** and describe how you recognized them.

c. Describe how you could use the total requirement matrix to determine the amount of each simple component needed to fill an order for 10 pieces of item F.

1.6 In addition to those described in Problem **1.3**, Sammie's Spreadables has added four more items to its product line. These include a pepperoni spread, a jalapeño spread, a Twelve-Pack, and a Combo Pack. The Twelve-Pack contains a Six-Pack, 3 pepperoni spreads, and 3 jalapeño spreads. The Combo Pack contains a Ten-Pack, 4 pepperoni spreads, and 3 jalapeño spreads.

a. Create a total requirement graph for Sammie's expanded product line.

b. Create a total requirement matrix for Sammie's expanded product line.

c. Determine the number of spinach spreads in one Combo Pack.

* * * * * * * * * *

Activity 2

In Activity **1**, you used requirement graphs and requirement matrices to represent Family Snack's product line. In this activity, you use matrices to organize and analyze other types of data.

Exploration

Family Snack employs three salespeople: Keyes, Zhang, and Troy. In September, Keyes sold 8 packages of nuts, 12 packages of jerky, 0 jars of jam, 16 Snack Packs, 28 Gift Packs, and 8 Family Packs.

In the same month, Zhang sold 12 packages of nuts, 4 packages of jerky, 24 jars of jam, 8 Snack Packs, 24 Gift Packs, and 12 Family Packs.

Troy's September sales included 12 packages of nuts, 0 packages of jerky, 12 jars of jam, 12 Snack Packs, 36 Gift Packs, and 4 Family Packs.

a. Use a matrix to organize this information in each of the following ways. Record the dimensions of each matrix.

1. Designate each row by a salesperson's name and each column by a product name.
2. Designate each row by a product name and each column by a salesperson's name.

Mathematics Note

Two matrices are **equal** if they have the same dimensions and if corresponding elements are equal.

For example, consider matrices **A** and **B** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 7 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 7 & 6 \end{bmatrix}$$

These matrices are equal because the dimensions of both are 3×2 and because the element in row 1, column 1 of matrix **A** is the same as the element in row 1, column 1 of matrix **B**; the element in row 1, column 2 of matrix **A** is the same as the element in row 1, column 2 of matrix **B**; the element in row 2, column 1 of matrix **A** is the same as the element in row 2, column 1 of matrix **B**; and so on.

b. Are the matrices created in Part **a** equal? Explain your response.

c. In October, Keyes sold 8 packages of nuts, 4 packages of jerky, 24 jars of jam, 12 Snack Packs, 8 Gift Packs, and 36 Family Packs.

In the same month, Zhang sold 16 packages of nuts, 8 packages of jerky, 28 jars of jam, 16 Snack Packs, 20 Gift Packs, and 32 Family Packs.

Troy's October sales included 8 packages of nuts, 12 packages of jerky, 4 jars of jam, 16 Snack Packs, 8 Gift Packs, and 0 Family Packs.

Select one of the matrices you created in Part **a** for September's sales. Using the dimensions of this matrix, create the corresponding matrix for October's sales.

d.
1. Create a matrix that displays the total number of each product sold by each salesperson during September and October.
2. Create a matrix that shows the change in the sales of each product by each salesperson from September to October.

e. As the holiday season approaches, the company expects November sales to be twice that of the previous month. Create a matrix that shows the predicted sales of each product for each salesperson.

Mathematics Note

Matrix addition can be performed on two matrices with the same dimensions by adding the corresponding elements of each matrix. For example, the addition of two 2×3 matrices is shown below.

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -8 & 9 \\ 10 & 11 & -2 \end{bmatrix} = \begin{bmatrix} (1+7) & (2-8) & (3+9) \\ (-3+10) & (4+11) & (5-2) \end{bmatrix} = \begin{bmatrix} 8 & -6 & 12 \\ 7 & 15 & 3 \end{bmatrix}$$

Matrix subtraction can be performed in a similar manner. For example, the following equation shows the subtraction of one 2×2 from another.

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} (1-0) & (-2-1) \\ (-3-2) & (4-(-5)) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -5 & 9 \end{bmatrix}$$

In **scalar multiplication**, each element of a matrix is multiplied by a constant, or **scalar**. The multiplication of a matrix **M** by a scalar k is denoted by $k \bullet \mathbf{M}$.

For example, consider the matrix **M** below.

$$\mathbf{M} = \begin{bmatrix} 5 & 4 \\ -2 & 8 \\ 0 & -1 \end{bmatrix}$$

Multiplying **M** by the scalar 5 produces the following result.

$$5 \bullet \mathbf{M} = 5 \bullet \begin{bmatrix} 5 & 4 \\ -2 & 8 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5(5) & 5(4) \\ 5(-2) & 5(8) \\ 5(0) & 5(-1) \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ -10 & 40 \\ 0 & -5 \end{bmatrix}$$

f. Use technology to complete Steps **1–3** below. Assume that November sales are represented by the matrix you created in Part **e**.

1. Determine a matrix that displays the total sales of each product by each salesperson during October and November.
2. Determine a matrix that shows the change in the sales of each product by each salesperson between September and November.
3. Determine a matrix that shows total sales of each product by each salesperson for September, October, and November.

g. The mean of two numbers can be found by dividing their sum by 2 or multiplying their sum by $1/2$. Use scalar multiplication to complete Steps **1** and **2** below.

1. Determine a matrix that shows mean sales of each product by each salesperson during September and October.
2. Determine a matrix that shows mean sales of each product by each salesperson during September, October, and November.

Discussion

a. Describe how you determined the mean sales for all three months in Part **g** of the exploration.

b. Do you think that any two matrices **A** and **B** can be added together? Explain your response.

c. When matrices are added, subtracted, or multiplied by a scalar, what are the dimensions of the resulting matrix? Why does this occur?

d. Do you think that matrix addition is commutative? In other words, does $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, where **A** and **B** are matrices? Justify your response.

Assignment

2.1 In Parts **a–e** below, perform the indicated operation, if possible. If the operation is not possible, explain why.

a. $\begin{bmatrix} 5 & 9 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 2 & 8 & -2 \end{bmatrix}$

c. $\pi \bullet \begin{bmatrix} 5 & 7 \\ 11 & -4 \\ 3 & 0 \end{bmatrix}$

d. $4 \bullet \begin{bmatrix} 2 & 6 & -9 \\ 3 & 7 & 11 \end{bmatrix} - \begin{bmatrix} 8 & 15 & 2 \\ -5 & 6 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 4 \\ 1 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.2 To promote the company's products, Family Snack's business manager plans to visit six U.S. cities. In the matrix below, each element represents the distance in kilometers between two of these cities.

	Miami	New York	Chicago	Kansas City	Seattle	Los Angeles
Miami	0	1747	1901	1986	4374	3742
New York	1747	0	1141	1755	3853	3922
Chicago	1901	1141	0	662	2779	2792
Kansas City	1986	1755	662	0	2410	2170
Seattle	4374	3853	2779	2410	0	1534
Los Angeles	3742	3922	2792	2170	1534	0

a. What does the element in row 4, column 5 represent?

b. The element 2779 appears twice in this matrix. Describe both of its positions and identify what each represents.

c. What do the zeros indicate in this matrix?

2.3 Each of Family Snack's four production employees is capable of performing the following daily tasks:

Task 1: Making and packaging jam.
Task 2: Roasting and packaging nuts.
Task 3: Preparing and packaging jerky.
Task 4: Filling and delivering orders.

Since their skill levels differ from task to task, the time necessary for each employee to complete each task varies as described below.

Employee 1 can do task 1 in 8 hr, task 2 in 7 hr, task 3 in 4 hr, and task 4 in 10 hr.

Employee 2 can do task 1 in 10 hr, task 2 in 4 hr, task 3 in 5 hr, and task 4 in 8.5 hr.

Employee 3 can do task 1 in 6 hr, task 2 in 3.5 hr, task 3 in 4 hr, and task 4 in 9 hr.

Employee 4 can do task 1 in 8 hr, task 2 in 6.5 hr, task 3 in 8 hr, and task 4 in 6 hr.

a. Create a matrix to organize this data. Designate each employee by a row heading.

b. Assign one task to each employee so that all the tasks are completed in the least amount of time. (Assume that all tasks can be performed simultaneously.)

c. Is there more than one possible response to Part **b**? If so, describe another solution.

2.4 A Family Corporation sportswear outlet, Family Funwear, sells sweat pants in three colors—blue, red, and yellow—and four sizes: S, M, L, and XL. The number of each item in their current inventory is shown in matrix **I** below.

$$\mathbf{I} = \begin{array}{c} \\ \text{blue} \\ \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \begin{array}{cccc} \text{S} & \text{M} & \text{L} & \text{XL} \end{array} \\ \begin{bmatrix} 11 & 25 & 24 & 8 \\ 12 & 27 & 22 & 10 \\ 9 & 15 & 18 & 12 \end{bmatrix} \end{array}$$

a. What are the dimensions of matrix **I**?

b. What does the element in row 2, column 3 represent?

c. After receiving a new shipment of sweat pants, the store's stock of each item increases by 20. Create a matrix to represent this new inventory.

d. In preparation for an upcoming sale, the manager triples the store's original inventory. Create a matrix to represent this new inventory.

e. Create a matrix that, when added to matrix **I**, results in the matrix for Part **c**.

* * * * *

2.5 Family Bicycles, another division of the Family Corporation, builds and sells three types of bicycles: mountain bikes, touring bikes, and cross-country bikes. Each type is sold in three different frame sizes: 43 cm, 48 cm, and 54 cm.

In the 43-cm frame size, Family Bicycles has 112 mountain bikes, 117 touring bikes, and 111 cross-country bikes in stock. In the 48-cm frame size, they have 190 mountain bikes, 122 touring bikes, and 92 cross-country bikes. In the 54-cm frame size, they have 101 mountain bikes, 216 touring bikes, and 132 cross-country bikes.

- **a.** Create a matrix to represent the company's entire stock of bicycles. Designate each frame size by a row heading.
- **b.** The company's marketing department predicts that sales of each style and size of bicycle will increase by 20% next month.
  1. Describe how to create a matrix that displays the number of each bicycle that must be in stock to meet next month's sales.
  2. If the current inventory represents just enough stock to supply this month's orders, how many of each bicycle should be stocked next month? Explain your response.

2.6 One of the stores that sells Family Bicycles handles only mountain bikes and touring bikes. The inventory matrices $\mathbf{I}_1$ and $\mathbf{I}_{31}$ show the number of bicycles in stock at the beginning of sales on December 1 and at the close of sales on December 31, respectively:

$$\mathbf{I}_1 = \begin{array}{c} \\ \text{mountain} \\ \text{touring} \end{array} \begin{array}{c} \begin{array}{ccc} 43 & 48 & 54 \end{array} \\ \begin{bmatrix} 5 & 17 & 13 \\ 11 & 9 & 18 \end{bmatrix} \end{array} \qquad \mathbf{I}_{31} = \begin{array}{c} \\ \text{mountain} \\ \text{touring} \end{array} \begin{array}{c} \begin{array}{ccc} 43 & 48 & 54 \end{array} \\ \begin{bmatrix} 3 & 11 & 5 \\ 6 & 0 & 10 \end{bmatrix} \end{array}$$

Assuming the shop did not receive any new bicycles during December, complete Parts **a–g** below.

- **a.** How many mountain bikes did the store have in stock on December 1?
- **b.** How many 48-cm bikes were in stock at the start of the month?
- **c.** What was the total number of bicycles in stock on December 1?
- **d.** Write a 2×3 matrix that represents the number of bicycles sold during December. Explain how you determined the elements in this matrix.
- **e.** Which individual model and size sold best during the month?
- **f.** How many 54-cm mountain bikes were sold during December?
- **g.** At the end of the month, the store manager orders more of each model and size for which no more than one-third of the initial inventory remains. Which bicycle(s) should be reordered?

2.7 Family Bicycle offers four different frame colors: white, red, blue, and green. Its current inventory of mountain, touring, and cross-country bicycles is represented in matrix **C** below.

$$\mathbf{C} = \begin{array}{r} \\ \text{white} \\ \text{red} \\ \text{blue} \\ \text{green} \end{array} \begin{array}{c} \begin{array}{ccc} \text{MT} & \text{TR} & \text{CC} \end{array} \\ \begin{bmatrix} 100 & 100 & 67 \\ 121 & 135 & 89 \\ 82 & 125 & 34 \\ 100 & 95 & 145 \end{bmatrix} \end{array}$$

Next week, the company will add the bicycles represented in matrix **N** to its inventory.

$$\mathbf{N} = \begin{array}{r} \\ \text{white} \\ \text{red} \\ \text{blue} \\ \text{green} \end{array} \begin{array}{c} \begin{array}{ccc} \text{MT} & \text{TR} & \text{CC} \end{array} \\ \begin{bmatrix} 6 & 12 & 6 \\ 12 & 0 & 4 \\ 0 & 12 & 6 \\ 16 & 6 & 4 \end{bmatrix} \end{array}$$

a. **1.** Describe how matrix addition can be used to determine a matrix that represents the company's total inventory after next week.

2. Write the matrix that represents the company's total inventory after next week.

b. The matrix **S** represents the inventory the company must have in stock before the summer sales begin. How many more bicycles should the company produce to reach this goal?

$$\mathbf{S} = \begin{array}{r} \\ \text{white} \\ \text{red} \\ \text{blue} \\ \text{green} \end{array} \begin{array}{c} \begin{array}{ccc} \text{MT} & \text{TR} & \text{CC} \end{array} \\ \begin{bmatrix} 116 & 132 & 87 \\ 145 & 147 & 103 \\ 94 & 152 & 54 \\ 130 & 117 & 158 \end{bmatrix} \end{array}$$

2.8 Family Funwear plans to add four different styles of knitted sweaters to its clothing line. To accomplish this goal, the company has purchased four knitting machines and hired four new workers. During some initial trials, the company gathered the following production data:

In one day, worker 1 can produce 3 of style A, 6 of style B, 7 of style C, or 4 of style D.

Worker 2 can produce 4 of style A, 5 of style B, 5 of style C, or 6 of style D.

Worker 3 can produce 6 of style A, 3 of style B, 4 of style C, or 4 of style D.

Worker 4 can produce 5 of style A, 5 of style B, 3 of style C, or 6 of style D.

a. Create a matrix to organize this data. Designate each worker by a row heading.

b. In order to produce the maximum number of sweaters each day, which worker should knit each style? Justify your response.

c. What is the maximum number of sweaters that the four workers can produce in one day?

d. Is there more than one possible response to Part **b**? If so, describe another solution.

* * * * * * * * * *

Activity 3

All businesses record information on profit and loss. One way to organize and analyze this information is with matrices. In this activity, you explore how Family Snack uses matrices to keep track of its profits.

Exploration 1

Family Snack's employees earn a commission on every sale. As part of the company's bookkeeping procedures, the business manager monitors both the weekly sales and the weekly commissions of each salesperson. The matrix **W** below shows one week's sales of cases of Snack Packs, Gift Packs, and Family Packs by Keyes and Zhang.

$$\mathbf{W} = \begin{matrix} & \begin{matrix} S & G & F \end{matrix} \\ \begin{matrix} K \\ Z \end{matrix} & \begin{bmatrix} 4 & 7 & 2 \\ 2 & 6 & 3 \end{bmatrix} \end{matrix}$$

The company sells Snack Packs for \$45.00 per case, Gift Packs for \$70.00 per case, and Family Packs for \$95.00 per case. This information can be represented in a **column matrix**, as shown below. Each element in the matrix $\mathbf{M_P}$ represents the price per case of a particular item.

$$\mathbf{M_P} = \begin{array}{c} \\ S \\ G \\ F \end{array} \begin{array}{c} \text{Price} \\ \begin{bmatrix} 45.00 \\ 70.00 \\ 95.00 \end{bmatrix} \end{array}$$

a. **1.** Determine the value of the sales made by each employee in matrix **W**.

2. Describe how you determined the values in Step **1**.

Mathematics Note

When a matrix **P** with dimensions $k \times m$ is multiplied by a matrix **Q** with dimensions $m \times n$, the dimensions of the **product matrix** $\mathbf{P} \bullet \mathbf{Q}$ are $k \times n$. The number of columns in the first matrix must equal the number of rows in the second matrix.

To obtain the product matrix, row 1 of the first matrix is "multiplied" with column 1 of the second matrix, as shown in Figure **4**. This process is repeated until each row of the first matrix has been multiplied with each column of the second matrix.

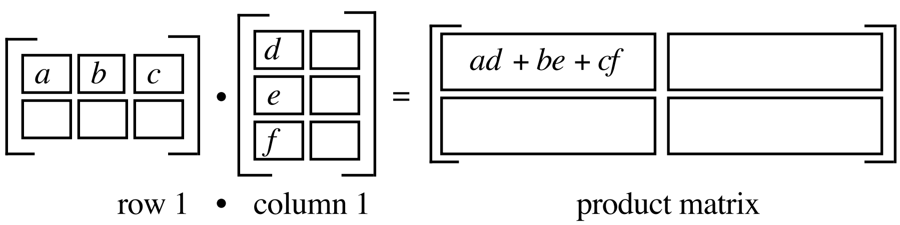


Figure 4: Multiplying row 1 by column 1

When row 1 of the first matrix is multiplied with column 1 of the second matrix, the sum of the products is placed in row 1, column 1 of the product matrix. In general, when row *i* of the first matrix is multiplied with column *j* of a second matrix, the sum of the products is placed in row *i*, column *j* of the product matrix.

For example, the equation below shows a 2×3 matrix **P** multiplied with a 3×2 matrix **Q**. The dimensions of the product matrix $\mathbf{P} \bullet \mathbf{Q}$ are 2×2.

$$\begin{bmatrix} 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \bullet \begin{bmatrix} 2 & 7 \\ 5 & 8 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 2 \bullet 2 + 6 \bullet 5 + 9 \bullet 9 & 2 \bullet 7 + 6 \bullet 8 + 9 \bullet 0 \\ 1 \bullet 2 + 3 \bullet 5 + 4 \bullet 9 & 1 \bullet 7 + 3 \bullet 8 + 4 \bullet 0 \end{bmatrix} = \begin{bmatrix} 115 & 62 \\ 53 & 31 \end{bmatrix}$$

b. The commission earned for selling a case of Snack Packs is \$2.25, for a case of Gift Packs is \$4.20, and for a case of Family Packs \$7.20.

1. Write a column matrix $\mathbf{M_C}$ to represent the commission earned by selling a case of each product.
2. Use matrix multiplication to determine the commission earned by each salesperson for the sales shown in matrix **W**.

c. In Parts **a** and **b**, you used two different column matrices to represent Family Snack's selling prices and commissions. As shown below, these two matrices could also be combined into a single 3×2 matrix.

$$\begin{array}{cc} & \begin{array}{cc}\text{Price} & \text{Com.}\end{array} \\ \mathbf{M} = \begin{array}{c}\text{S}\\ \text{G}\\ \text{F}\end{array} & \begin{bmatrix} 45.00 & 2.25 \\ 70.00 & 4.20 \\ 95.00 & 7.20 \end{bmatrix} \end{array}$$

1. Use the process described in the mathematics note to find the product $\mathbf{W} \bullet \mathbf{M}$.
2. Record the dimensions of the product matrix.
3. Describe the information contained in the product matrix.

d.
1. Use the process described in the mathematics note to find the product $\mathbf{M} \bullet \mathbf{W}$.
2. Record the dimensions of the product matrix.
3. Describe the information contained in the product matrix.

Discussion 1

a. Compare the methods you used to determine the values in Parts **a** and **b** of the exploration. Which appears to be the better method and why?

b. If the product matrix $\mathbf{A} \bullet \mathbf{B}$ exists, what must be true about the number of columns in matrix **A** and the number of rows in matrix **B**?

c. What advantages are there to performing the operation $\mathbf{W} \bullet \mathbf{M}$ instead of the operations $\mathbf{W} \bullet \mathbf{M_P}$ and $\mathbf{W} \bullet \mathbf{M_C}$?

d.
1. Was the information displayed in the product matrix $\mathbf{W} \bullet \mathbf{M}$ meaningful? Explain your response.
2. Was the information displayed in the product matrix $\mathbf{M} \bullet \mathbf{W}$ meaningful? Explain your response.

e. Which employee earned more in commissions for the sales shown in matrix **W**? Justify your response.

Exploration 2

In Exploration **1**, you used matrix multiplication to analyze the sales of two of Family Snack's employees. In this exploration, you use technology to help the company organize and manipulate some additional sales information.

a. Use technology to find the product $\mathbf{W} \bullet \mathbf{M}$. Compare this product with your results in Part **c** of Exploration **1**.

b. Besides Keyes and Zhang, Family Snack employs another salesperson, Troy, as well as a new trainee, Lia. In the week represented in matrix **W**, Troy sold 3 cases of Snack Packs, 9 cases of Gift Packs, and 1 case of Family Packs. Lia sold 6 cases of Snack Packs, 0 Gift Packs, and 0 Family Packs.

Combine this information with the information in matrix **W** to create a new matrix $\mathbf{W_2}$.

c. Determine the product matrix $\mathbf{W_2} \bullet \mathbf{M}$ and record its dimensions. Describe the information contained in this matrix.

d. Evaluate the product matrix $\mathbf{M} \bullet \mathbf{W_2}$ and interpret the results.

e. Matrix **V** below shows the cases of nuts, jerky, and jam sold by each salesperson during the week represented in matrix $\mathbf{W_2}$. Combine this information with that in matrix $\mathbf{W_2}$ to create a new matrix $\mathbf{W_3}$.

$$\mathbf{V} = \begin{array}{c} \\ K \\ Z \\ T \\ L \end{array} \begin{array}{c} \begin{array}{ccc} N & J & M \end{array} \\ \begin{bmatrix} 2 & 3 & 0 \\ 3 & 1 & 6 \\ 3 & 0 & 3 \\ 2 & 4 & 1 \end{bmatrix} \end{array}$$

f. Family Snack sells nuts for \$6.00 per case, jerky for \$10.00 per case, and jam for \$8.00 per case.

The commission earned for selling a case of nuts is \$0.18, for a case of jerky is \$0.30, and for a case of jam is \$0.32.

Combine this information with that in matrix **M** to create a new matrix **S**.

g. Evaluate the product matrix $\mathbf{W_3} \bullet \mathbf{S}$ and interpret the results.

h. Evaluate the product matrix $\mathbf{S} \bullet \mathbf{W_3}$ and interpret the results.

Discussion 2

a. Describe the process of multiplying an $m \times n$ by a $p \times q$ matrix.

b. Why is the matrix multiplication $\mathbf{S} \bullet \mathbf{W}_3$ not possible?

c. When using real numbers, multiplication is commutative. For example, $6 \bullet 14 = 14 \bullet 6$. Matrix multiplication, however, is not commutative. Explain why this is the case.

Assignment

3.1 Consider the five matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 2 & 9 & 1 \\ 0 & 6 & 1 \\ 1 & 2 & 6 \\ 0 & 5 & 1 \\ 8 & 3 & 1 \end{bmatrix}$$

If possible, find each of the following product matrices. When matrix multiplication is possible, show how the elements in the product matrix are formed. If multiplication is not possible, explain why not.

a. $\mathbf{A} \bullet \mathbf{B}$

b. $\mathbf{C} \bullet \mathbf{D}$

c. $\mathbf{E} \bullet \mathbf{B}$

3.2 Consider the following two matrices.

$$\mathbf{X} = \begin{bmatrix} 7 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 2 \end{bmatrix}$$

a. **1.** Find the product matrix $\mathbf{X} \bullet \mathbf{Y}$ and describe its dimensions.

2. How are the dimensions of the product matrix related to the dimensions of **X** and **Y**?

b. **1.** Find the product matrix $\mathbf{Y} \bullet \mathbf{X}$ and describe its dimensions.

2. How are the dimensions of the product matrix related to the dimensions of **X** and **Y**?

c. Describe how your responses to Parts **a** and **b** show that matrix multiplication is not commutative.

3.3 A store has placed an order with Family Snack represented by the matrix **O** shown below. Each element in the matrix indicates the number of a specific item in the order.

$$\mathbf{O} = \begin{array}{c} N \\ J \\ M \\ S \\ G \\ F \end{array}\begin{bmatrix} 20 \\ 60 \\ 48 \\ 24 \\ 12 \\ 2 \end{bmatrix}$$

a. The total requirement matrix for Family Snack's product line is matrix **R** shown below. Use matrix **R** to determine the number of each simple component needed to fill the order.

$$\mathbf{R} = \begin{array}{c} \\ N \\ J \\ M \\ S \\ G \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} N & J & M & S & G & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 0 & 6 & 12 & 36 \\ 0 & 0 & 1 & 0 & 3 & 12 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

b. Another customer has ordered 150 boxes of nuts, 600 pieces of jerky, 48 jars of jam, 60 Snack Packs, and 12 Gift Packs. Determine the number of each simple component needed to fill this order.

3.4 Last year, Family Snack offered an incentive program to its sales staff. Employees received 10 incentive points for each month in which they ranked first in sales, 6 points for each month in which they ranked second, 4 points for third, and 3 for fourth.

a. Create a matrix **S** that displays the incentive points awarded for each ranking.

b. Employee 1 ranked first during 4 months, third during 4 months, and fourth during 4 months. Employee 2 ranked first during 3 months, second during 4 months, and fourth during 5 months. Employee 3 ranked first 2 months, second 5 months, and third 5 months. Employee 4 ranked first, second, third, and fourth for 3 months each.

Create a matrix **R** that displays the number of months at each rank for each of the four employees.

c. The employee with the most incentive points receives an all-expenses-paid vacation. Determine the order in which the four employees finished in the contest, along with their scores.

* * * * *

3.5 Matrix **R** is the total requirement matrix for Family Snack's product line. Family Snack has received orders O_1, O_2, O_3, O_4, and O_5 from five different customers, as shown in matrix **O** below.

$$\mathbf{R} = \begin{array}{c} \\ N \\ J \\ M \\ S \\ G \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} N & J & M & S & G & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 0 & 6 & 12 & 36 \\ 0 & 0 & 1 & 0 & 3 & 12 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{O} = \begin{array}{c} \\ N \\ J \\ M \\ S \\ G \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} O_1 & O_2 & O_3 & O_4 & O_5 \end{array} \\ \begin{bmatrix} 12 & 20 & 60 & 20 & 8 \\ 10 & 18 & 45 & 12 & 2 \\ 16 & 12 & 14 & 22 & 8 \\ 8 & 10 & 18 & 8 & 8 \\ 4 & 4 & 8 & 4 & 2 \\ 2 & 4 & 12 & 8 & 0 \end{bmatrix} \end{array}$$

a. How many boxes of nuts are needed to fill O_3?

b. How many boxes of nuts are needed to fill all five orders?

c. How many pieces of jerky are needed to fill O_2?

d. How many pieces of jerky are needed to fill all five orders?

e. How many jars of jam are needed to fill all five orders?

3.6 A bicycle shop use matrices to track its monthly sales. The number of bikes sold in July are displayed in matrix **J** below.

$$\mathbf{J} = \begin{array}{r} \\ \text{mountain} \\ \text{touring} \end{array} \begin{array}{c} \begin{array}{ccc} 43\text{ cm} & 48\text{ cm} & 54\text{ cm} \end{array} \\ \begin{bmatrix} 2 & 6 & 5 \\ 5 & 9 & 8 \end{bmatrix} \end{array}$$

Matrix **S** shows the selling price in dollars for each model.

$$\mathbf{P} = \begin{array}{r} \\ 43\text{ cm} \\ 48\text{ cm} \\ 54\text{ cm} \end{array} \begin{array}{c} \begin{array}{cc} MT & TR \end{array} \\ \begin{bmatrix} 250 & 280 \\ 270 & 300 \\ 290 & 320 \end{bmatrix} \end{array}$$

a. What matrix operation would allow you to calculate the value of the July sales for each model? Perform this operation. Use the column and row headings for matrices **P** and **J** to determine appropriate headings for the product matrix.

b. Describe what each element in the product matrix represents. If some elements have no clear interpretation, identify them and explain why this occurs.

c. Is it possible to perform the matrix operation in Part **a** if the order of the matrices is reversed? If so, describe the elements that have clear interpretations. If the operation is not possible, explain why not.

d. Are the total July sales for the bicycle shop more or less than $15,000? Explain your response.

3.7 Consider the five matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 7 & 8 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 0 & 5 & -2 \\ 1 & 8 & -4 & 9 \\ 11 & 6 & -5 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 5 & -6 \\ 9 & -4 & 0 \\ 9 & 3 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 7 & -6 & 7 \\ 2 & -1 & 1 & 0 \\ 5 & -8 & 3 & -11 \\ 2 & 9 & 1 & 7 \end{bmatrix}$$

If possible, find each of the following product matrices using a matrix manipulator. When matrix multiplication is possible, identify the dimensions of the product matrix. If multiplication is not possible, explain why not.

a. $\mathbf{A} \bullet \mathbf{B}$ and $\mathbf{B} \bullet \mathbf{A}$

b. $\mathbf{A} \bullet \mathbf{C}$ and $\mathbf{C} \bullet \mathbf{A}$

c. $\mathbf{B} \bullet \mathbf{C}$ and $\mathbf{C} \bullet \mathbf{B}$

d. $\mathbf{C} \bullet \mathbf{D}$ and $\mathbf{D} \bullet \mathbf{C}$

e. $\mathbf{B} \bullet \mathbf{E}$ and $\mathbf{E} \bullet \mathbf{B}$

* * * * * * * * * *

Research Project

In this module, you have explored matrix addition, subtraction, and multiplication. Use the general 2×2 matrices below to further investigate matrix arithmetic.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

a. Is multiplication of 2×2 matrices associative? In other words, does $(\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C} = \mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C})$?

b. Is the distributive property of multiplication over addition valid for 2×2 matrices? In other words, does $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$?

c. The identity element for real-number multiplication is 1. For example, $25 \bullet 1 = 25$. Is there an identity for multiplication of 2×2 matrices?

d. For all real numbers, multiplication by 0 results in a product of 0 (the Zero Product Property). Is there a matrix which, when multiplied by **A**, **B**, or **C**, results in a matrix whose elements are all zeros? If so, is there more than one such matrix?

Summary Assessment

1. Family Snack has decided to add crackers to its product line. The new Snack Pack contains 1 box of nuts, 1 box of crackers, 10 pieces of jerky, and 1 jar of jam; the new Gift Pack contains 2 Snack Packs and 2 jars of jam; and the new Family Pack contains 3 Gift Packs and 4 boxes of crackers.

   a. Construct a requirement graph for Family Snack's new product line.

   b. Use your requirement graph to construct a total requirement matrix for the new product line.

2. An apartment complex offers one-, two-, and three-bedroom apartments. The complex contains 10 furnished and 10 unfurnished one-bedroom apartments; 5 furnished and 10 unfurnished two-bedroom apartments; and 5 furnished and 5 unfurnished three-bedroom apartments.

   A furnished one-bedroom apartment rents for $375 a month; an unfurnished one-bedroom rents for $345. A furnished two-bedroom rents for $400 per month; an unfurnished one rents for $370. A furnished three-bedroom rents for $450 per month; an unfurnished one rents for $420.

   a. Create a 2×3 matrix that displays the apartment inventory.

   b. Create a 3×2 matrix that displays the rent for each type of apartment.

   c. Find the product of the rent matrix from Part **b** and the inventory matrix from Part **a**.

   d. Describe the meaning of each element in the matrix from Part **c**. If some elements in the matrix have no clear interpretation, explain why this occurs.

   e. The managers of the apartment complex wish to raise the rents by 10%. Construct a matrix showing the new rents.

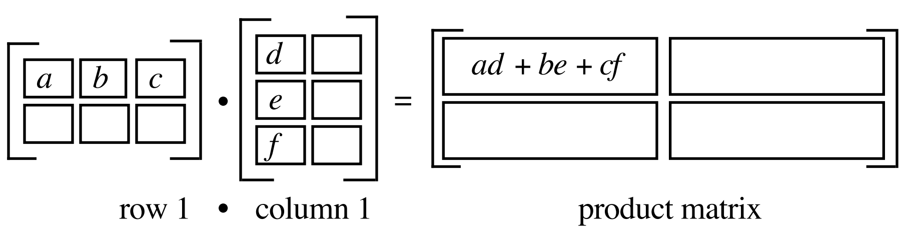
   f. The monthly electric bills average $80 for a one-bedroom apartment, $95 for a two-bedroom apartment, and $105 for a three-bedroom apartment. Show how matrix multiplication can be used to determine the total electric bill for the entire apartment complex.

g. In addition to the electric bills from Part **f**, the apartment managers also must pay bills for other utilities, such as water and sewer. Monthly water bills average \$18 for one-bedroom apartments, \$26 for two-bedroom apartments, and \$42 for three-bedroom apartments. Monthly sewer bills average \$15 for one-bedroom apartments, \$20 for two-bedroom apartments, and \$30 for three-bedroom apartments.

Show how matrix multiplication can be used to find the monthly utility bills for the entire apartment complex. The utility bill should be broken into water, sewer, and electricity.

Module Summary

- One way to represent a company's product line is with a **requirement graph**. A requirement graph is a tree diagram that shows each product's components at each level of its assembly. A complete requirement graph has simple components at the end of each branch.
- A **matrix** (plural **matrices** or **matrixes**) is a rectangular arrangement of rows and columns used to organize information. A matrix of i rows and j columns has **dimensions** $i \times j$ (read "i by j"). Matrices are named using bold, uppercase letters or descriptive words. Each item in a matrix is an **element**.
- A company can use a requirement graph to create a **total requirement matrix**. In a total requirement matrix, each item sold is represented in a column heading, while each simple component or composite product is represented in a row heading. Each element indicates the number of that simple component or composite product required to produce the corresponding item sold.
- Two matrices are **equal** if they have the same dimensions and if corresponding elements are equal.
- **Matrix addition** can be performed on two matrices with the same dimensions by adding the corresponding elements of each matrix. **Matrix subtraction** can be performed in a similar manner.
- **Scalar multiplication** is the multiplication of each element of a matrix by a constant, or **scalar**. The multiplication of a matrix **M** by a scalar k is denoted by $k \bullet \mathbf{M}$.
- When a matrix with dimensions $k \times m$ is multiplied with a matrix with dimensions $m \times q$, the dimensions of the **product matrix** are $k \times q$. To obtain the product matrix, each row of the first matrix is "multiplied" with each column of the second matrix, as shown below. In general, when row i of the first matrix is multiplied with column j of a second matrix, the sum of the products is placed in row i, column j of the product matrix.



Selected References

DeLange, J. *Matrices*. Utrecht, The Netherlands: Research Group on Mathematics Education, 1990.

Harshbarger, R. J., and J. J. Reynolds. *Mathematical Applications for Management, Life, and Social Sciences*. Lexington, MA: D.C. Heath, 1992.

Vazsonyi, A. *Finite Mathematics, Quantitative Analysis for Management*. Santa Barbara, CA: Wiley/Hamilton, 1977.