Atomic Clocks Are Ticking

You've found the remains of an old cooking fire. How do you tell if that lump of charcoal is 10 years old, or 10,000? In this module, you explore one method that archaeologists use to tell time.

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Introduction

Andrea shone her flashlight at the walls of a cavern. One wall was covered with drawings of animals and people, just like in a museum. Jared guessed that the pictures could be hundreds—or even thousands—of years old. Excited about their discovery, the two friends decided to tell Professor Cordova about the cave.

Three years after Professor Cordova's first visit, there have been many changes. A sign warns visitors not to touch the ancient drawings. The floor at the cave's mouth has been carefully excavated to reveal layers of charcoal and ash. Professor Cordova explains to visitors that people have used this cave for thousands of years. Samples of the charcoal from old cooking fires are being used to determine the approximate age of the site.

One visitor asks how anyone can possibly determine the age of a lump of charcoal. Professor Cordova then describes the **carbon dating** process. Anything that has once been alive contains carbon atoms, including a charred piece of wood. A very small fraction of those atoms are radioactive. They emit tiny particles that scientists can measure. As time passes, the number of radioactive atoms decreases in a predictable way, acting as an atomic clock. Because scientists know the original level of radioactive carbon in living things, they can read the clock by determining the amount that remains.

Activity 1

The process by which the number of radioactive atoms decreases is **radioactive decay**. In this activity, you learn more about the pattern of decay described by Professor Cordova.

Exploration

In this exploration, you use a model to simulate the process of radioactive decay. You need a container with a lid, 32 chips, and graph paper or a spreadsheet for recording data.

Read Parts **a–g** before beginning the exploration. The chips represent radioactive atoms. Predict how the number of chips will change during the simulation. Record your prediction, then begin.

a. Put a mark on one side of all 32 chips and place them in the container.

b. Close the lid, then shake the container.

c. Open the container and remove each chip with a mark showing.

d. Count the number of chips remaining.

e. Record both the number of chips remaining and the shake number. (After shake 0, there are 32 chips.)

f. Repeat Parts **b–e** until no chips remain in the container.

g. Graph the data. Plot the shake number along the x-axis and the number of chips remaining along the y-axis. **Note:** Save your data for use in the assignment.

Discussion

a. Describe any patterns you observed in your data or graph.

b. What similarities or differences are there among the patterns found by the class?

c. What fraction of the previous number of chips would you expect to remain after each shake?

d. What happens to any number representing a population when it is multiplied by a factor between 0 and 1?

Mathematics Note

An equation of the form $y = a \bullet b^x$ is an exponential function. The function can be used to describe a pattern of **exponential growth** or **exponential decay**.

When this equation describes the growth or decay in a population, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the rate of growth or decay). The independent variable x represents number of time periods, while the dependent variable y represents the total population after x time periods. When used to model growth or decay, the expontial function will be of the form $y = a(1 \pm r)^t$.

In exponential growth, r is positive and represents the **growth rate**. In exponential decay, r is negative and represents the **decay rate**.

For example, the value of business equipment often depreciates exponentially. If the value of a machine originally worth $1000 decreases by 19% each year, the equation that models its depreciation is $y = 1000(1 - 0.19)^x$ or $y = 1000(0.81)^x$. In this case, x represents the number of years since the machine was purchased and y represents the value of the machine after x years. The graph in Figure **1** shows the depreciation of this machine over a 10-year period.

Figure 1: Depreciation of equipment over 10 years

e. Given the example outlined in the **Mathematics Note**, how would you interpret the value of y in each of the following equations?

1. $y = 1000(0.81)^{1/2}$

2. $y = 1000(0.81)^{1/3}$

f. Describe how you could use a calculator to find the value of each of the following expressions.

1. $(0.81)^{1/2}$

2. $(0.81)^{1/3}$

g. What is the mathematical meaning of $(0.81)^{1/2}$?

h. From your response to Part **g**, what do you think is the mathematical meaning of $(0.81)^{1/3}$? Verify your response.

Mathematics Note

The roots of a number can be represented using exponents of the form $1/n$, where n is a natural number. If the nth root of a exists, it can be represented as follows:

$$\sqrt[n]{a} = a^{1/n}$$

For example, $\sqrt[4]{64} = 64^{1/4}$, $\sqrt[5]{32} = 32^{1/5}$, and $\sqrt{16} = 16^{1/2}$.

i. In the equation $y = a \bullet b^x$, $b = 1 + r$.

1. What are the possible values of b when the equation models exponential growth?
2. What are the possible values of b when the equation models exponential decay?

Assignment

1.1 Evaluate each of the following expressions:

a. $125^{1/3}$

b. $81^{1/4}$

c. $343^{1/3}$

d. $\sqrt[10]{1024}$

1.2 Use the data you collected in the exploration to complete Parts **a–e**.

a. Find the percent decrease in the population after each shake. For example, if there are 17 chips remaining after shake 1, the percent decrease can be found using the following ratio:

$$\frac{32-17}{32} = \frac{15}{32} \approx 0.47 = 47\%$$

b. **1.** Determine the mean of the percents of decrease for all shakes.

2. Considering the theoretical probability of a chip landing with the marked side up, what would you expect this mean to be? Explain your response.

c. Describe the relationship between the rate of decay and the mean of the percents of decrease.

d. Use the mean from Part **b** to determine the value of b, where $b = 1 + r$.

e. Write an equation of the form $y = a \bullet b^x$ that models the exponential decay of the population of chips.

1.3 Complete the following table.

	Initial Population	Rate of Growth or Decay	Model Equation of the Form $y = a \bullet b^x$
a.	500	8%	
b.			$y = 20{,}000 \bullet (0.89)^x$
c.	100	–32%	

1.4 A 1-g sample of carbon contains $5.02 \bullet 10^{22}$ atoms, of which 1 out of every 1 trillion ($1 \bullet 10^{12}$) is radioactive. How many radioactive atoms are there in this sample?

1.5 Suppose that you began the exploration in Activity **1** with $4 \bullet 10^{14}$ chips. Predict the number of chips remaining after the third shake. Explain your reasoning.

1.6 An object contains $3.8 \bullet 10^{13}$ radioactive atoms. The annual rate of decay is 0.5. Estimate the number of radioactive atoms the object contained in the previous year. Describe how you found your solution.

1.7 A sample contains $1 \bullet 10^{12}$ radioactive atoms which decay at the rate of 50% every decade. How many decades will it take for the number of radioactive atoms to be less than 1 billion? Explain your reasoning.

1.8 Each of the equations below models a pattern of either exponential decay or exponential growth. For each equation, complete the following steps.

- Identify whether the equation models growth or decay.
- Determine the value of r.
- Use appropriate technology to graph the function and describe the basic shape of the graph.

a. $y = 350 \bullet (1.055)^x$

b. $y = 100 \bullet (0.50)^x$

c. $y = 5000 \bullet (0.75)^x$

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1.9 The balance of an interest-bearing account also can be modeled by an exponential equation. For example, consider an initial deposit of $1000 at an annual interest rate of 8%. The interest earned each year is deposited in the account at the end of the year.

a. Assuming that no withdrawals or deposits are made, write an equation that models the account balance after x years.

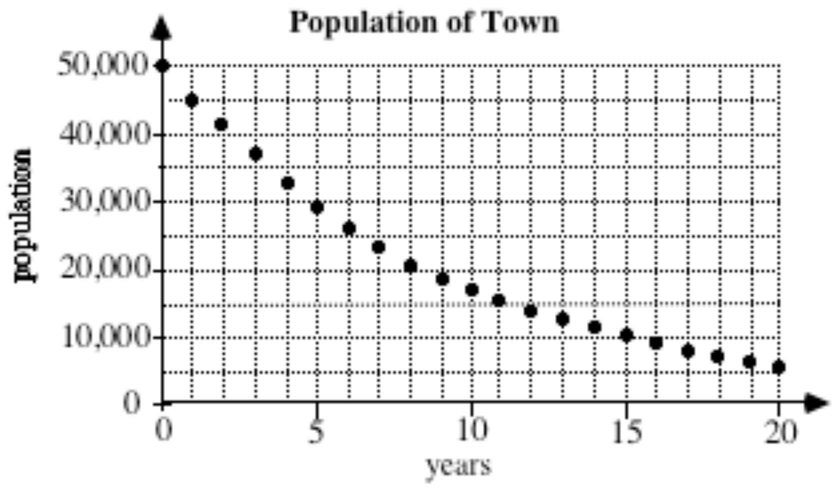
b. What is the account balance at the end of 5 years?

c. **1.** Create a graph that shows the increase in the account balance over the next 20 years.

2. Using the graph, estimate how many years it will take for the initial deposit to double.

d. If you deposited $1000 in an account of this type, what would the account balance be when you are 65 years old?

1.10 The following graph shows the change in a town's population over a 20-year period.



a. What was the town's initial population?

b. About how many years did it take for the population to decrease by half?

c. Judging from the graph, what was the population at year 15?

d. Using the method described in Problem **1.2**, determine the mean of the percents of decrease in the population over the 20-year period.

e. **1.** Find an equation of the form $y = a \bullet b^x$ that models this data, where y represents population and x represents years.

2. Use this equation to verify your response to Part **c**.

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Activity 2

In living plants and animals, only about 1 in every $1 \bullet 10^{12}$ carbon atoms is radioactive. When a plant or animal dies and new carbon is no longer absorbed or metabolized, these radioactive carbon atoms decay. When determining the age of an artifact using carbon dating, scientists can estimate the number of radioactive carbons remaining. This number, along with carbon's known rate of decay, allows scientists to determine the age of the sample.

Given carbon's rate of decay and the number of radioactive atoms present at a particular time, it is also possible to calculate the number of radioactive atoms at any time in the past. Such calculations may require the use of negative exponents.

In this activity, you examine other ways to represent the model of exponential decay from the exploration in Activity **1**.

Exploration 1

a. Create a spreadsheet with the headings shown in Table **1** below. Complete the table for values of x from 4 to –4, where x is an integer.

Table 1: Spreadsheet for Exploration 1

x	2^x	$(1/2)^x$	$1/2^x$	2^{-x}
4	16	0.0625		
3	8			
2				
$\vdots$				
–2				
–3				
–4				

b. Describe any patterns you observe in the spreadsheet.

c. **1.** Create a scatterplot of the values of 2^{-x} versus the values of x.

2. On the same coordinate system from Step **1**, create a scatterplot of the values for 2^x versus the values for x.

Discussion 1

a. Based on your results in Table **1**, what is the value of 2^0?

b. Describe the relationship among the following expressions: $(1/2)^x$, $1/2^x$, and 2^{-x}.

c. What is the relationship between 2^x and each of the other expressions in Table **1**?

d. In the Level 1 module "Are You Just a Small Giant," you learned that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

where n is a non-negative integer and $b \neq 0$. From your spreadsheet, it appears that

$$\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$$

Are these two relationships in conflict? Explain your response.

e. Describe some other ways to represent the quantity 2^{-3}.

f. How do the graphs from Part **c** of the exploration compare?

Mathematics Note

If a is a nonzero real number, then $a^0 = 1$.

If a is a nonzero real number and n is an integer, then

$$a^{-n} = \frac{1}{a^n}$$

For example, $2^{-3} = 1/2^3$ and $3^4 = 1/3^{-4}$. This relationship is also true when $a \neq 0$ and n is a real number.

g. Describe how to express each of the following using a negative exponent:

1. 2^3

2. $a \bullet b^2$

h. Describe how to express each of the following without using a negative exponent:

1. 2^{-3}

2. $a \bullet b^{-2}$

Exploration 2

In this exploration, you examine some properties of integer exponents that can help you determine rates of decay.

a. Create a spreadsheet with headings like those in Table **2** below. The spreadsheet should be created so that the values you select for b, x, and y in the first three cells of each row are used to complete the rest of the entries in that row.

Table 2: Spreadsheet for Exploration 2

b	x	y	b^{x+y}	b^{x-y}	$b^{x \bullet y}$	$(b^x)^y$	$b^x \bullet b^y$	b^x/b^y

b. Select values for b, x, and y and enter them in the appropriate columns. Note the values generated in the remaining columns. Describe any relationships you observe.

c. Change any combination of the values of b, x, and y. Determine if your observations from Part **b** hold true for these new values.

Discussion 2

a. Compare your observations with those of others in the class. Do the generalizations you made seem to hold true for all examples?

Mathematics Note

When two exponential expressions containing the same base are multiplied, their product is the base raised to a power equal to the sum of the exponents. For natural numbers *m* and *n*,

$$\begin{aligned} a^m \bullet a^n &= \overbrace{a \bullet a \bullet \cdots \bullet a}^{m \text{ terms}} \bullet \overbrace{a \bullet a \bullet \cdots \bullet a}^{n \text{ terms}} \\ &= \overbrace{a \bullet a \bullet \cdots \bullet a}^{m+n \text{ terms}} = a^{m+n} \end{aligned}$$

This result also is true for any real numbers *m* and *n* when $a > 0$.

For example, $3^{-2} \bullet 3^4 = 3^{-2+4} = 3^2 = 9$.

When an exponential expression is raised to a power, the result equals the base in the original expression raised to the product of the powers. For natural numbers *m* and *n*,

$$\left(a^m\right)^n = \overbrace{a^m \bullet a^m \bullet \cdots \bullet a^m}^{n \text{ terms}} = a^{m \bullet n}$$

This result is also true for any real numbers *m* and *n* when $a > 0$.

For example, $(3^{-2})^3 = 3^{-2 \bullet 3} = 3^{-6} = 1/729$.

When two exponential expressions containing the same base are divided, their quotient is the base raised to a power equal to the exponent of the dividend minus the exponent of the divisor. For natural numbers *m* and *n*,

$$\frac{a^m}{a^n} = a^m \bullet \frac{1}{a^n} = a^m \bullet a^{-n} = a^{m+(-n)} = a^{m-n}$$

This result also is true for any real numbers *m* and *n* when $a > 0$.

For example, $3^7/3^4 = 3^{7-4} = 3^3 = 27$.

b. Explain how you could use the relationships described in the mathematics note above to find the value of *x* in each of the following:

1. $4^{-5} \bullet 4^x = 4^{10}$

2. $\left(4^{-5}\right)^x = 4^{10}$

c. What value of *x* satisfies the equation $\left(5^3\right)^x = 5^1$? Describe how you identified this value.

d. What number do you think is represented by each of the following expressions?

1. $16^{0.25}$
2. $16^{3/4}$

Mathematics Note

When $a \geq 0$, a rational exponent in the form m/n can be represented as follows, where m/n is in lowest terms:

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

For example, $8^{0.4} = 8^{4/10} = 8^{2/5} = \left(8^{1/5}\right)^2 = \left(\sqrt[5]{8}\right)^2 \approx 2.30$.

e. The number of radioactive atoms in a sample at a given time can be modeled by the equation $y = 64 \bullet 0.67^{4.3}$. What does 4.3 represent in this situation?

f. Describe another way to represent the expression $b^{4.3}$.

g. Because $2/6 = 1/3$, one might incorrectly believe that $(-8)^{2/6} = (-8)^{1/3}$.

1. Express $(-8)^{2/6}$ and $(-8)^{1/3}$ in as many different forms as you can. Evaluate each expression and compare the results.
2. According to the mathematics note above, the relationship below is true only when the exponent m/n is in lowest terms.

$$a^{m/n} = \sqrt[n]{a^m}$$

Using $(-8)^{2/6}$ and $(-8)^{1/3}$ as examples, explain why you think this condition is necessary.

Assignment

2.1 The number of radioactive atoms in a sample decays at a rate of 50% per year and can be modeled by an equation of the form $y = a \bullet b^x$.

a. Determine the proportion of the original number of radioactive atoms that remain for each of the following values of x:

1. 2
2. 0
3. –2

b. In this situation, what does $x = -2$ represent?

2.2 The equation $y = 32 \bullet 2^x$ models the number of bacteria present in a population after x minutes.

a. In this situation, what does the expression $32 \bullet 2^4$ represent?

b. Given this same population of bacteria, what does the expression $32 \bullet 2^{-4}$ represent?

c. Write the expression in Part **b** without using a negative exponent.

2.3 **a.** In Activity **1**, you modeled the number of chips remaining after x shakes using the equation $y = 32 \bullet (1/2)^x$. Write an equivalent equation using a negative exponent.

b. Use the equation $y = 32 \bullet (1/2)^x$ to predict the shake number at which each of the following populations of chips would remain:

1. 8 chips

2. 2 chips

3. 64 chips

4. 128 chips

5. 32 chips.

2.4 The two relationships below were described in Part **d** of Discussion **1** and in the previous **Mathematics Note** on page 278, respectively:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \text{and} \quad a^{-m} = \frac{1}{a^m}$$

Use these relationships to show that the following relationship is true, where $a \neq 0$ and $b \neq 0$.

$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$$

2.5 **a.** Rewrite the expression $(1/3)^3$ using an integer as the base.

b. Rewrite the expression $(1/a)^b$ using a as the base, where $a \neq 0$.

c. Rewrite the expression $(3/5)^2$ without using parentheses.

d. Rewrite $(3/5)^2$ using a negative exponent.

e. Rewrite $(4/3)^{-2}$ using a positive exponent.

2.6 A fragment of animal bone originally contained $5.12 \bullet 10^{14}$ radioactive carbon atoms. Now it contains only $3.2 \bullet 10^{13}$ radioactive carbons.

a. Given a rate of decay of 50% for each interval of time, represent this situation in an equation of the form $y = a \bullet b^x$.

b. Determine the age of the artifact.

2.7 **a.** How does a change in the initial population affect the graph of the equation $y = a \bullet b^x$?

b. How does a change in the rate of decay affect the graph of the equation $y = a \bullet b^x$?

2.8 The equation $y = 4000 \bullet (17/20)^x$ models the decay of radioactive atoms in an object. In a paragraph, describe the situation represented by this equation.

2.9 To solve the equation $64 = x^3$ for x, both sides of the equation can be raised to the exponent $1/3$, as follows:

$$64 = x^3$$
$$(64)^{1/3} = (x^3)^{1/3}$$
$$64^{1/3} = x$$

a. Explain why $(x^3)^{1/3}$ can be replaced with x.

b. Find the value of x in the equation $64^{1/3} = x$.

2.10 Solve each of the following equations for x and describe how you reached your solutions.

a. $x^6 = 729$

b. $x^{0.125} = 2$

c. $10^5 \bullet x = 10^{16}$

2.11 Each of the following equations is a model of exponential growth or decay in the form $y = a \bullet b^x$. For each one, determine the value of b, explain whether it models growth or decay, and identify the rate.

a. $1024 = 64 \bullet b^4$

b. $13 = 10 \bullet b^3$

c. $1.6 \bullet 10^{13} = 5.0 \bullet 10^{14}(b^5)$

d. $107 = 1000 \bullet b^{10}$

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2.12 The table below shows some data collected while a container of boiling water cooled to room temperature.

Number of 2-min Intervals Elapsed	Degrees above Room Temperature (°C)
1	74
2	70
3	67
4	65
5	63
6	60
7	58
8	56

a. Create a scatterplot of the data.

b. Find the mean of the percents of decrease in the temperature for these 2-min intervals.

c. Using the mean from Part **b** as a rate, write an equation of the form $y = a \bullet b^x$ that models the data.

d. When the water was first poured, its temperature was 100° C. Use this fact and your model to estimate the temperature of the room.

e. Should your equation from Part **c** be used to make predictions about the temperature of the water several minutes before the time frame given in the table? Explain your response.

2.13 Identify the equation in column B that is equivalent to each equation in column A.

Column A	Column B
$y = 5 \bullet (1 - 0.5)^3$	$y = 5 \bullet (1/2)^{-3}$
$y = 5 \bullet (2)^3$	$y = 5 \bullet (0.5)^{-1/3}$
$y = 5 \bullet (2)^{1/3}$	$y = 5 \bullet (2)^{-3}$

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Activity 3

All living organisms contain carbon atoms. A relatively small proportion of these carbon atoms are unstable. Over time, these unstable atoms, known as carbon-14 atoms, decay. As they decay, they emit energy. Scientists can estimate the age of a sample by measuring the energy emitted from these radioactive atoms.

The time required for one-half of the carbon-14 atoms in a given sample to decay is the **half-life**. Carbon-14 has a half-life of 5730 years. Other unstable atoms have half-lives that range from fractions of seconds to billions of years. For example, the half-life of carbon-11 is about 21 min, while the half-life of uranium-238 is 4.5 billion years. These unstable atoms rarely have half-lives that are exactly 1 unit of time. In the following exploration, you modify models of exponential decay to include half-lives that are not whole numbers.

Exploration

Although a typical artifact may contain trillions of radioactive atoms, in this exploration, you again consider a sample that has only 32 unstable atoms.

a. Create a table with headings like those in Table **3**. Round 0 represents the time when the number of radioactive atoms in the sample is 32.

Table 3: Radioactive atom simulation

Round	Number of Radioactive Atoms
0	32
1	
2	
⋮	
9	
10	

b. Suppose the rate of decay is $1/6$. This means each radioactive atom has 1 chance in 6 of decaying.

1. To model a rate of decay of $1/6$, randomly generate an integer from 1 to 6 for each radioactive atom remaining in the sample. The last entry in the second column of Table **3** represents the number of radioactive atoms remaining in the sample.

   For example: In Round 1, generate a list of 32 random integers since 32 is the last entry in the second column of Table **3**.

2. Count the number of times that the integer 1 appears in the list. Reduce the quantity of radioactive atoms in the sample by this number.

3. Record the remaining number of radioactive atoms in the second column of Table **3**. This ends the round. The number recorded represents the number of radioactive atoms that remain in the sample.

c. Repeat Part **b** for 10 rounds, or until no radioactive atoms remain.

d. Create a scatterplot of the data. Represent the round number on the *x*-axis and the number of radioactive atoms on the *y*-axis.

e. **1.** What equation of the form $y = a \bullet b^x$ would you expect to model this simulation?

2. Graph this equation on the coordinate system from Part **d**.

f. **1.** Use the data to estimate the half-life, in number of rounds, of these radioactive atoms.

2. Use your equation from Part **e** to determine the half-life of the atoms.

Discussion

a. What fraction of the number of atoms did you expect to remain after each round? Explain your response.

b. How does the half-life of the atoms in this simulation compare with the half-life of the atoms in the simulation in Activity **1**?

c. **1.** Describe how you determined a model equation of the form $y = a \bullet b^x$ in Part **e** of the exploration.

2. How does this equation compare with the one that modeled the simulation in Activity **1**?

Assignment

3.1 A sample originally contained $1.05 \bullet 10^{11}$ radioactive atoms, which have been decaying at a rate of 0.00003 atoms/year.

a. Write an equation of the form $y = a \bullet b^x$ to model this situation.

b. The number of radioactive atoms currently in the sample is $2.3 \bullet 10^{10}$. Use your model to estimate the age of the sample.

3.2 **a.** Find a value for *x* when:

1. $3^x = 9$

2. $3^x = 27$

b. If the equation $3^x = 16$ has a solution, between what two whole numbers must *x* lie?

c. Use a guess-and-check method to find *x* to the nearest hundredth.

d. Find *x* to the nearest hundredth when $2.5^x = 20$.

3.3 In the exploration in this activity, you conducted a simulation with 32 radioactive atoms and a rate of decay of $1/6$. This situation can be modeled by the exponential equation $y = 32 \bullet (5/6)^x$.

The half-life is the time required for the number of radioactive atoms to decrease to 16. Substituting this value into the model equation yields $16 = 32 \bullet (5/6)^x$. Use this equation to estimate the half-life.

3.4 **a.** Each of the following equations models the radioactive decay of a substance over time, where x represents years. Determine the half-life for each substance and describe the methods you used.

1. $y = 4000(0.61)^x$

2. $y = 5.3 \bullet 10^7 (0.61)^x$

3. $y = 0.61^x$

b. Considering your responses to Part **a**, how does the initial number of radioactive atoms appear to affect the half-lives of substances with the same rate of decay?

3.5 Kim is a doctor in the Department of Nuclear Medicine at the county hospital. She uses radioactive tracers to detect disease or injury in her patients. When comparing levels of radioactivity in the tracers, Kim designates the initial level as 100%. Later measurements are then recorded as percentages of that level. Using this method, radioactive decay can be modeled by equations of the form $y = 100 \bullet b^x$. When $y = 50$, the value of x equals the half-life.

a. In one case, Kim has monitored the radioactivity of a tracer at the same time each day for two consecutive days. The second day's radioactivity is 80% of the first day's. Write an equation of the form $y = 100 \bullet b^x$ to model this situation, where x represents time in days.

b. Create a graph of the model for the next 10 days.

c. Estimate the half-life of the tracer.

d. Kim received the radioactive tracer 2 days after it was manufactured. If the rate of decay is constant, what was the radioactivity of the tracer when it was first produced? Explain your response.

3.6 The comparative equation $50 = 100 \bullet b^x$ can be used to calculate the value of b for a given half-life. Determine the value of b and identify the rate of decay for each of the following half-lives:

a. 12.5 hours

b. 57.3 centuries

3.7 Imagine that you are examining a sample that contains radioactive atoms with a half-life of 5.0 years.

a. The radioactive decay of these atoms can be modeled by the equation $y = 100 \bullet b^x$ where x represents time in years. What is the value of b in this equation?

b. What is the radioactive material's rate of decay?

c. If the level of radioactivity remaining is 93% of the original level, how old is the sample you are examining? Explain your response.

3.8 Adrian's foot has bothered him since Wednesday. The X-rays taken on Friday looked normal, but his doctor suspects a possible stress fracture. On Monday, the doctor will inject a radioactive tracer to help detect the site of the fracture. The tracer has a half-life of 6.0 hr.

a. What percentage of the tracer's radioactivity remains after each hour?

b. Write an equation of the form $y = 100 \bullet b^x$, where x represents hours, to model the decay of this tracer.

c. After the tracer has been in Adrian's body for 45 min, the doctor will scan Adrian's foot for signs of radioactivity. What percentage of the tracer is left in his body at this time?

d. How many hours will pass before only 10% of the tracer remains?

e. The radioactive tracer was manufactured 3 hr before it was injected into Adrian. How much more radioactive was the tracer at the time of its manufacture?

3.9 As mentioned in the introduction to Activity **3**, the half-life of carbon-14 is 5730 years. Use an equation of the form $y = a \bullet b^x$ to determine the rate of decay for carbon-14 per century. Hint: If the initial amount of carbon-14 is 100%, the amount remaining after 57.3 centuries is 50%.

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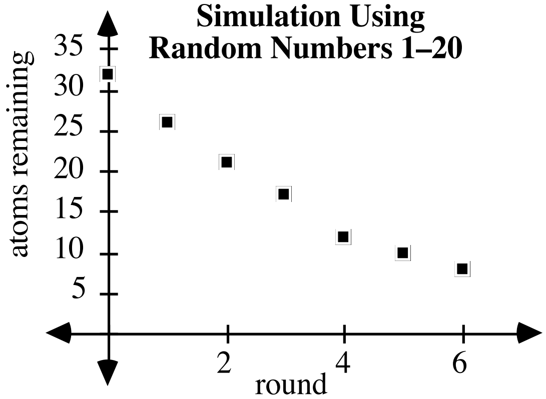
3.10 Use technology to find a value of x so that $256^x = 4$. Express this value as a decimal and as a fraction.

3.11 One of your classmates has conducted a simulation like the one described in Activity **3**. Instead of randomly generating integers from 1 to 6 however, she generated integers from 1 to 20, inclusive, and removed an atom for each multiple of 5 that appeared in the list.

a. What fraction of the atoms would you expect to remove after each round? Explain your response.

b. What fraction of the atoms would you expect to remain in the container after each round?

c. A scatterplot of the data from your classmate's simulation is shown below. Use it to estimate the half-life of the substance.



d. Find an equation of the form $y = a \bullet b^x$ that models the data.

3.12 The Trojan Nuclear Power Plant near Portland, Oregon, is shutting down. One of the waste products of its operation is radioactive strontium-90. Strontium-90 is extremely hazardous to people and other living things. Its half-life is 28.0 years. As an environmental engineer, you must design leak-proof containers to hold strontium-90 until less than 1% of its radioactivity remains.

a. What percentage of a sample of strontium-90's initial radioactivity remains after 1 year?

b. How many years will it take for a sample of strontium-90 to have less than 1% of its present radioactivity?

c. Most of the plant's strontium-90 was buried in temporary storage containers 5 years ago. How much more radioactive was this material at the time of its burial?

d. To inform the public of the radioactivity present in the new, leak-proof containers, your firm has decided to attach a warning label to each one. Design a label that includes a graph showing the percentage of radioactivity remaining as a function of the year.

3.13 In 1972, a candy bar cost $0.25. In 1996, the same candy bar sold for $0.65.

a. Assuming that the cost of the candy bar grew exponentially, determine the annual rate of growth.

b. Write an equation that could be used to model the cost of the candy bar in a given year.

c. Use your equation to estimate the year in which the candy bar cost only $0.10.

d. Predict the year in which the candy bar will cost $1.00.

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Summary Assessment

Professor Cordova found many layers of charcoal in the fire pit near the mouth of Andrea and Jared's cave. Each layer contained a different percentage of the carbon-14 activity of living trees. Her data is shown in the table below. Layer 1 was taken from the top of the pit; layer 11 was taken from the bottom.

Layer	Percentage of Carbon-14 Radioactivity
1	91.6
2	83.3
3	81.6
4	78.5
5	75.0
6	71.2
7	65.4
8	47.4
9	42.9
10	39.9
11	32.1

1. Professor Cordova believes humans have occupied this site for no more than 12,000 years. Given that the half-life of carbon-14 is about 57.3 centuries, the rate of decay is approximately 0.012. Find the approximate age of each layer of charcoal.

2. Near the cave, Professor Cordova found what appears to be a mammoth bone. Mammoths became extinct about 11,500 years ago. What percentage of carbon-14 radioactivity would you expect to find in this artifact? Explain your reasoning.

3. As trees grow, new carbon-14 is absorbed only by the outermost ring of the trunk. The amount of carbon-14 radioactivity in the inner rings decays as if the wood were actually dead. Because of this phenomenon, the age of trees can be determined by carbon dating.

- **a.** To verify her carbon-dating procedure, Professor Cordova decides to test a newly fallen tree near the cave. The outermost ring of the tree has 112% of the carbon-14 radioactivity of the innermost ring. Use this information to estimate the age of the tree.
- **b.** The age of a tree also can be found by counting the growth rings in the trunk, one ring for each year of life. The rings of the tree indicate that it was about 940 years old. How does your estimate in Part **a** compare with this value?

Module Summary

- The process by which the number of radioactive atoms decreases is **radioactive decay**.
- An equation of the form $y = a \bullet b^x$ is an exponential function. The function can be used to describe a pattern of **exponential growth** or **exponential decay**.

  When this equation describes the growth or decay in a population, a represents the size of the initial population. The value of b is the sum of two percentages: 100 (representing the initial population) and r (representing the rate of growth or decay). The independent variable x represents number of time periods, while the dependent variable y represents the total population after x time periods. When used to model growth or decay, the expontial function will be of the form $y = a(1 \pm r)^t$.
- In exponential growth, r is positive and represents the **growth rate**. In exponential decay, r is negative and represents the **decay rate**.
- The roots of a number can be represented using exponents of the form $1/n$, where n is a natural number. If the nth root of a exists, it can be represented as follows:
  $$\sqrt[n]{a} = a^{1/n}$$
- If a is a nonzero real number, then $a^0 = 1$.
- If a is a nonzero real number and n is an integer, then
  $$a^{-n} = \frac{1}{a^n}$$
- When two exponential expressions containing the same base are multiplied, their product is the base raised to a power equal to the sum of the exponents. For natural numbers m and n,
  $$a^m \bullet a^n = \overbrace{a \bullet a \bullet \cdots \bullet a}^{m \text{ terms}} \bullet \overbrace{a \bullet a \bullet \cdots \bullet a}^{n \text{ terms}}$$
  $$= \overbrace{a \bullet a \bullet \cdots \bullet a}^{m+n \text{ terms}} = a^{m+n}$$

  This result also is true for any real numbers m and n when $a > 0$.
- When an exponential expression is raised to a power, the result equals the base in the original expression raised to the product of the powers. For natural numbers m and n,
  $$\left(a^m\right)^n = \overbrace{a^m \bullet a^m \bullet \cdots \bullet a^m}^{n \text{ terms}} = a^{m \bullet n}$$

  This result is also true for any real numbers m and n when $a > 0$.

- When two exponential expressions containing the same base are divided, their quotient is the base raised to a power equal to the exponent of the dividend minus the exponent of the divisor. For natural numbers *m* and *n*,

$$\frac{a^m}{a^n} = a^m \bullet \frac{1}{a^n} = a^m \bullet a^{-n} = a^{m+(-n)} = a^{m-n}$$

  This result also is true for any real numbers *m* and *n* when $a > 0$.

- When $a \geq 0$, a rational exponent in the form m/n can be represented as follows, where m/n is in lowest terms:

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Selected References

Libby, W. F. *Radiocarbon Dating*. Chicago, IL: University of Chicago Press, 1952.

Taylor, R. E. *Radiocarbon Dating: An Archaeological Perspective*. Orlando, FL: Academic Press, 1987.