Take It to the Limit

What do hungry chickens, chain letters, and the perimeters of nested triangles have in common? This module introduces you to infinite sequences—and their limits.

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Take It to the Limit

Introduction

Many advertisements use the power of mathematics to attract consumers. An ad for the latest weight-control plan, for example, might promise to help dieters "lose 10 pounds a week." An ad for a new car might guarantee "payments of only $199 a month for 24 months." And an ad for chicken feed might lure poultry farmers by crowing "the mass of your birds will increase 10% a day."

The numbers implied by these ads can be represented as different types of **sequences**, or ordered lists. Whether you find their claims believable or not depends on how you interpret them.

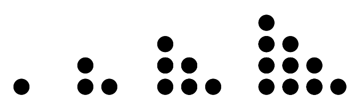
Discussion

a. Determine a rule or pattern that allows you to identify a next possible term or object for each of the following:

1. 7, 13, 19, 25

2. 162, 54, 18, 6

3.



4. –7, 49, –343, 2401

5. $[2, 3], [2, 2.5], [2, 2.\overline{3}], [2, 2.25]$

6. 4, 5, 9, 14, 23

Mathematics Note

Rules and patterns often can be described using formulas.

A **recursive formula** for a sequence identifies the initial term (or terms), then defines all other terms using the previous term(s).

An **explicit formula** for a sequence defines any term based on its term number.

For example, consider the sequence 2, 5, 8, 11, 14, ... A recursive formula for this sequence is shown below:

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 3, \ \ n > 1 \end{cases}$$

The same sequence can be described by the explicit formula $a_n = 2 + 3(n - 1)$.

b. If possible, determine a recursive formula for each sequence in Part **a**.

c. If possible, determine an explicit formula for each sequence in Part **a**.

Mathematics Note

In an **arithmetic sequence**, every term after the first is formed by adding a constant value, the **common difference**, to the preceding term.

For example, the sequence 2, 7, 12, 17, 22 is a finite arithmetic sequence with a common difference of 5.

In a **geometric sequence**, every term after the first is formed by multiplying the preceding term by a constant value, the **common ratio**.

For example, the sequence 2, 10, 50, 250, 1250 is a finite geometric sequence with a common ratio of 5.

d. **1.** Which of the ordered lists in Part **a** can be represented by an arithmetic sequence?

2. Which can be represented by a geometric sequence?

e. Describe how to find the common difference for an arithmetic sequence.

f. Describe how to determine the common ratio for a geometric sequence.

Activity 1

In this activity, you use formulas for arithmetic and geometric sequences to create and interpret sequential patterns.

Mathematics Note

The recursive formula for an arithmetic sequence, where d is the common difference, n is the term number, and a_n is the nth term, is:

$$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d,\ \ n > 1 \end{cases}$$

The explicit formula for such a sequence is $a_n = a_1 + d(n-1)$.

The recursive formula for a geometric sequence, where r is the common ratio ($r \neq 0$), n is the term number, and g_n is the nth term, is:

$$\begin{cases} g_1 = \text{first term} \\ g_n = rg_{n-1},\ \ n > 1 \end{cases}$$

The explicit formula for such a sequence is $g_n = g_1 r^{n-1}$.

Exploration

In this exploration, you use your knowledge of sequences to play a game. Read Parts **a–e** before beginning play.

a. To play this game, each person creates a sequence based on a specific rule or pattern. The first five terms of the sequence are written near the top of a sheet of paper. The paper is then passed to a second person.

b. After examining the first five terms, the second person identifies the sequence as *arithmetic*, *geometric*, or *neither*. The second person writes this identification on the sheet of paper below the sequence.

Using complete sentences, the second person then writes a rule for the sequence, folds the paper so that only the rule shows, and passes it to a third person.

c. Using only the rule written by the second person, the third person writes both a recursive formula and an explicit formula (if possible) for the sequence.

The third person then folds the paper so that only the two formulas are visible, and passes it to a fourth person.

d. Using only the formulas written by the third person, the fourth person writes the first five terms of the sequence on the paper, then passes it back to the sequence's original creator.

e. The creator unfolds the paper and checks to see that the new sequence matches the original five terms. If the two sequences do not match, any discrepancies are identified and recorded.

Discussion

a. Did the sequence found in Part **d** of the exploration match the original five terms of your sequence? If not, explain why the two patterns differed.

b. Describe how to determine whether a sequence is arithmetic, geometric, or neither.

c. Consider a sequence whose first three terms are 1, 2, 3. Describe a rule which allows the next three terms to be:

1. 4, 5, 6

2. 5, 8, 13

3. 6, 11, 20

d. Given the first three terms of an arithmetic sequence, must the next three terms follow the same pattern? Explain your response.

Assignment

1.1 Write the first five terms of the sequence defined by each of the following explicit formulas:

a. $k_n = 9 - 3(n - 1)$

b. $k_n = -5(1.3)^{n-1}$

1.2 Determine the number of terms in each of the following sequences. Hint: Find an explicit formula for the sequence, then solve the formula for the term number *n*.

a. 11, 13, 15, …, 99

b. 20, 16, 12.8, …, 6.5536

c. 7, 10.5, 14, …, 357

d. 3, 6, 12, …, 3072

1.3 Complete Steps **1–3** for each sequence in Parts **a–e** below.

1. Identify the next two terms in the sequence.
2. Write an explicit or recursive formula, if one exists, that describes each pattern. If such a formula does not exist, write a description of the pattern you used to identify the next two terms.
3. Explain whether the sequence is arithmetic, geometric, or neither.

a. 1, 3, 7, 15, 31, …

b. –50, 10, –2, 2/5, …

c. 1, 4, 7, 10, …

d. $2\frac{1}{2}$, $2\frac{1}{3}$, $2\frac{1}{4}$, $2\frac{1}{5}$, …

e. 6, –6, 6, –6, 6, –6, …

1.4 **a.** Write the first five terms of a sequence that is both arithmetic and geometric. Explain why your sequence fits in both categories.

b. Write an explicit formula for your sequence when it is considered an arithmetic sequence.

c. Write an explicit formula for your sequence when it is considered a geometric sequence.

1.5 **a.** If the first term of an arithmetic sequence is 13 and the 50th term is 93, what is the common difference? Describe the process you used to solve this problem.

b. If the first term of a geometric sequence is 12 and the fifth term is 192, what is the common ratio? Describe the process you used to solve this problem.

c. If the first term of an arithmetic sequence is 11, the second term is 14, and the last term is 62, what is the common difference? How many terms are there in this sequence?

d. If the first term of a geometric sequence is 4, the second term is 20, and the last term is 7,812,500, what is the common ratio? How many terms are there in this sequence?

1.6 Obesity in pets can cause joint problems and is related to arthritis and other diseases. In some cases, veterinarians may recommend specially formulated pet foods to help control this condition. One such product claims that it can "help your pet lose 0.9 kg a month." Imagine that a veterinarian has prescribed this diet for a dog with a mass of 54 kg. What would you expect the mass of the dog to be at the end of 12 months? Explain your reasoning.

1.7 In its ad for chicken feed, a supplier promises a 10% daily gain in mass. A newly hatched chick has an approximate mass of 36 g. When the chicken reaches 8 weeks of age, it is considered a marketable broiler.

a. Consider the chicken's mass as a geometric sequence. If the ad's claim is true, what will the chicken's mass be after 8 weeks?

b. Do you think your answer to Part **a** is reasonable? Explain your response.

c. After 56 days on the advertised feed, a broiler chicken has a mass of 1487 g. What percentage daily gain in mass should the advertiser claim? Explain your response.

1.8 As the term number *n* becomes increasingly large, predict what will happen to a geometric sequence with a common ratio:

a. between –1 and 0

b. between 0 and 1

c. less than –1

d. greater than 1.

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1.9 Cut out a square of paper that measures 16 cm on each side.

a. **1.** Fold the paper in half.

2. Record the fold number, the number of layers created by the fold, and the area of the top surface.

b. Repeat Part **a** as many times as you can.

c. Write both explicit and recursive formulas for the area of the top surface after n folds.

d. Write both explicit and recursive formulas for the number of layers after n folds.

e. What value does the area of the top surface appear to approach as the number of folds increases?

f. What value does the number of layers appear to approach as the number of folds increases?

1.10 In Problem **1.9**, you folded the paper in half at each stage. After seven or eight folds, you probably noticed that continued folding became nearly impossible. Instead of folding the paper at each stage, suppose you cut it in half and stacked the cut sheets.

a. How big would the original sheet of paper have to be in order for the area of the top surface of the stack to be 1 cm^2 after 50 cuts?

b. After 50 cuts, how tall would the stack of paper be? Give your estimate in kilometers and describe how you arrived at your response.

1.11 Describe the pattern for each of the following sequences. If possible, identify the sequence as arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference. If it is geometric, find the common ratio.

a. $1, 1, 2, 3, 5, 8, \ldots, k_n$

b. $\begin{cases} k_1 = 2 \\ k_n = k_{n-1}(1.05), \; n > 1 \end{cases}$

c. $k_n = 2000 - (1000)(n - 1)$

d. $\begin{cases} k_1 = 0 \\ k_n = k_{n-1} + (n - 1), \; n > 1 \end{cases}$

e. $3, 3/2, 1, 3/4, 3/5, \ldots, k_n$

1.12 For each sequence in Problem **1.11**, describe what happens to the terms as n becomes increasingly large.

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Activity 2

Carl Friedrich Gauss (1777–1855) was one of the most productive and influential mathematicians of the 19th century. He made many important mathematical discoveries while still a student at the University of Göttingen.

Even as a child, Gauss showed a remarkable skill with numbers—in particular with the set of natural numbers $\{1, 2, 3, 4, \ldots\}$. According to mathematical lore, one day his teacher asked the class to add all the natural numbers from 1 to 100. Students were instructed to place their slates on the table when finished. To the surprise of the teacher, young Gauss placed his slate on the table after only a few moments.

Mathematics Note

The indicated sum of the terms of a finite sequence is a **finite series**. A finite series of n terms, denoted S_n, may be written in expanded form as follows:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

where a_n represents the nth term of the sequence.

The sum of the terms of an arithmetic sequence is an **arithmetic series**. For example, the finite arithmetic sequence shown below has seven terms:

$$2, 4, 6, 8, 10, 12, 14$$

The sum of those terms, in expanded form, is the following arithmetic series:

$$S_7 = 2 + 4 + 6 + 8 + 10 + 12 + 14 = 56$$

To find the sum of the first 100 natural numbers, Gauss used a method involving a finite series. For example, the sum of the first 100 numbers can be written as the arithmetic series S_{100}:

$$S_{100} = 1 + 2 + \cdots + 99 + 100$$

This series can also be written in reverse order, as shown below.

$$S_{100} = 100 + 99 + \cdots + 2 + 1$$

These two series can then be added as follows:

$$\begin{array}{rcrcrcccrcr} S_{100} & = & 1 & + & 2 & + & \cdots & + & 99 & + & 100 \\ S_{100} & = & 100 & + & 99 & + & \cdots & + & 2 & + & 1 \\ \hline 2S_{100} & = & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

Since the resulting equation contains 100 terms of 101, the sum of the two equations can be written as:

$$2S_{100} = 100(101)$$

Solving this equation for S_{100}, the sum of the first 100 natural numbers can be found as follows:

$$S_{100} = \frac{100(101)}{2} = 5050$$

Exploration

a.

1. Use a method like Gauss's to find the sum of the first 1000 natural numbers.

2. Generalize this method to find the sum of the first n natural numbers.

b. Use a similar method to find the sum of the first 50 even natural numbers.

c. Consider an arithmetic sequence with a first term of 2 and a common difference of 4. Find the sum of the first 75 terms of this sequence. Hint: Determine the 75th term of the sequence before trying to find the sum.

d. Consider an arithmetic sequence of n terms with a first term a_1 and a common difference d.

1. Using a_1 and d, write expressions for the second, third, and fourth terms of the sequence.

2. Using a_1 and d, write expressions for the nth term of the sequence and the three terms preceding the nth term.

3. Find the sum of the first n terms of this arithmetic sequence.

Mathematics Note

The sum of the terms of a finite arithmetic sequence with n terms and a common difference d can be found using the following formula:

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the sequence and a_n is the nth term.

For example, consider the sequence 7, 11, 15, …, 59. The number of terms in this arithmetic sequence can be found using the general form of the explicit formula, $a_n = a_1 + (n-1)d$. In this case, $a_1 = 7$, $a_n = 59$, and $d = 4$. By substituting as shown below, the equation can be solved for n, the number of terms.

$$\begin{aligned} 59 &= 7 + (n-1)4 \\ 59 &= 7 + 4n - 4 \\ 56 &= 4n \\ 14 &= n \end{aligned}$$

Since $n = 14$, the sum of those terms, S_{14}, can be found as follows:

$$S_{14} = \frac{14}{2}(7 + 59) = 462$$

Discussion

a. Verify that the following two expressions are equivalent:

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

b. Consider the sequences whose terms you added in Parts **a** and **b** of the exploration. Are these sequences arithmetic, geometric, or neither?

c.
1. Multiplying the first 100 natural numbers by 2 produces the first 100 even natural numbers. Describe how to find the sum of this set of numbers.

2. Multiplying the first 100 natural numbers by k produces the first 100 multiples of k. Describe how to find the sum of this set of numbers.

d. Suppose that the following arithmetic series continued indefinitely:

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

What do you think the sum of the series would be?

Assignment

2.1 Write a formula to find the sum of the first n even natural numbers.

2.2 Find the sum of the first 2000 natural numbers.

2.3
a. Generalize your response to Part **c** of the discussion to find a formula for the sum of the first n multiples of k.

b. Use your formula from Part **a** to find the sum of the first 600 multiples of 3.

2.4 Consider the arithmetic sequence 3, 7, 11, ..., 451.

a. What is the first term of the sequence?

b. What is the common difference?

c. How many terms are there in the sequence?

d. What is the sum of the terms of the sequence?

2.5 Can Gauss's method be used to find the sum of a geometric sequence? Explain your response.

2.6 To attract new customers, a car dealership displays the following advertisement.

New Car for Only $8,888.00 **Starting at $106.26 a Month!*** ***36-month lease**		
Refundable Security Deposit	$	150.00
Down Payment	$	1,600.00
First Month's Payment	$	106.26
Cash Due at Lease Inception	$	1,856.26

a. Do the monthly payments described in this ad form an arithmetic sequence? Explain your response.

b. Does the total amount paid over the term of the lease form an arithmetic series? Explain your response.

c. According to this ad, how much money will the lessee pay over the entire 36-month period?

d. Explain any discrepancies you observe between the amount determined in Part **c** and the advertised price of $8,888.00.

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2.7 Consider a weekly newspaper with 15,000 subscribers. During the next year, the paper plans to increase its number of subscribers by 50 every week. If this plan succeeds, what will be the total number of newspapers delivered for the entire year?

2.8 Consider a sequence described by the explicit formula $a_n = -7 + 5n$, for $n \geq 1$.

a. Write the first three terms of the sequence.

b. Find a_{150}.

c. Find S_{150}.

2.9 Consider an arithmetic sequence with a first term of 5 and a 41st term of 120.

a. Find the sum of the first 41 terms of this sequence.

b. Identify the common difference of the sequence.

c. Write the second and third terms of the sequence.

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Activity 3

Have you ever received an e-mail from a friend warning you about a computer virus? Such warnings usually urge you to send the warning to all the names in your e-mail address book.

What happens if the warning is a hoax? What are the consequences for all the servers that route the e-mail messages? Knowledge about geometric series helps answer such questions.

Exploration

You just received an e-mail warning your about Red Ant, particularly destructive computer virus that destroys all data stored on a computer. Concerned that this virus could cause major damage in your computer and those of your friends, you immediately e-mail a copy of the warning to two friends.

a. Your two friends each e-mail a copy of the warnings to three of their friends. What is the total number of e-mail warnings your two friends send?

b. Suppose each of the people to whom your friends send e-mail warnings in Part **a** also send e-mail warnings to three of their friends. How many total e-mail warnings will they send?

c. Assuming every recipient continues to e-mail a copy of the warning to three of their friends, continue this pattern for three more stages. Record the total number of e-mail warning sent at each stage.

d. Write the geometric sequence formed by the number of e-mail warnings sent at each stage.

Mathematics Note

The indicated sum of the terms of a geometric sequence is a **geometric series**. For example, the finite geometric sequence below has five terms.

$$2, 6, 18, 54, 162$$

The sum of those terms, in expanded form, is the following finite geometric series:

$$S_5 = 2 + 6 + 18 + 54 + 162 = 242$$

e. Write the expanded form of the geometric series that represents the total number of e-mail warnings sent. List at least three terms at the beginning of the series and three terms at the end of the series. Use an ellipsis to indicate the middle terms.

Set the series equal to S_{200} and call this "equation 1."

f. If a geometric series has many terms, determining its sum by simple addition can be time consuming. Use the following steps to determine the total number of e-mail warnings sent.

1. Multiply both sides of equation 1 by the common ratio of the sequence. Call the result "equation 2."
2. Subtract equation 1 from equation 2.
3. Solve the resulting equation for S_{200}.

g. A geometric series with n terms, where g_1 is the first term and r is the common ratio, also can be represented in the following expanded form:

$$S_n = g_1 r^0 + g_1 r^1 + g_1 r^2 + \cdots + g_1 r^{n-3} + g_1 r^{n-2} + g_1 r^{n-1}$$

Use the process described in Part **f** to simplify this representation of the series.

Discussion

a. Assuming every recipient continues to e-mail warnings to three of their friends, how many total e-mail warnings would you be sent? Is this a reasonable expectation? Why or why not?

b. Suppose the virus warning is a hoax and people do e-mail warnings to everyone in their address list. What are some possible consequences?

Mathematics Note

The sum of the terms of a finite geometric sequence with n terms and a common ratio r can be found using the following formula:

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

where g_1 is the first term of the sequence and $r \neq 1$.

For example, consider the geometric sequence 2, 6, 18, 54, 162. In this case, $n = 5$, $r = 3$, and $g_1 = 2$. Using the formula given above, the sum of the sequence can be found as follows:

$$S_5 = \frac{2(3)^5 - 2}{3 - 1} = 242$$

c. How does the formula given in the mathematics note above compare to the one you found in Part **g** of the exploration?

d. Why must r not equal 1 in the formula given in the mathematics note?

e. **1.** Write a geometric series S_n where $r = 1$.

2. Find a formula for the geometric series S_n when $r = 1$.

f. Describe how to demonstrate that the formula for a geometric series

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

is equivalent to each of the following expressions. Hint: $g_1 r^{n-1} = g_n$.

1. $S_n = \dfrac{g_1\left(r^n - 1\right)}{r - 1}$

2. $S_n = \dfrac{g_n r - g_1}{r - 1}$

g. Given the first three terms and the last term of a geometric series, how could you determine the sum?

Assignment

3.1 **a.** Determine the sum of the terms of a geometric sequence in which $g_1 = 3.5$, $r = 0.6$, and $n = 17$.

b. Determine the sum of the terms of a geometric sequence in which $g_1 = 10$, $r = 10$, and $n = 10$.

3.2 Find each of the sums below.

a. $20 + 16 + 12.8 + \cdots + 6.5536$

b. $3 + 6 + 12 + \cdots + 3072$

3.3 Identify each of the following series as arithmetic, geometric, or neither and find the sum.

a. $11 + 13 + 15 + 17 + 19 + 21 + 23 + \cdots + 99$

b. $2 + (-3) + (-8) + \cdots + (-28)$

c. $2\frac{1}{2} + 2\frac{1}{3} + 2\frac{1}{4} + \cdots + 2\frac{1}{7}$

d. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32}$

3.4 The Greek philosopher, Zeno of Elea, used paradoxes to demonstrate the difficulties of dividing time and space into an infinite number of parts. The following situation is similar to one of Zeno's paradoxes.

Imagine a race between the legendary Greek hero, Achilles, and a tortoise. Since Achilles can run 10 times faster than the tortoise, he gives the tortoise a 10-m head start. During the time it takes Achilles to run the first 10 m, the tortoise travels a distance of 1 m. Because of its head start, the tortoise still leads, but now by only 1 m.

During the time it takes Achilles to make up that 1 m, the tortoise travels 0.1 m. While Achilles runs that 0.1 m, the tortoise travels 0.01 m. As Achilles runs 0.01 m, the tortoise moves another 0.001 m. Imagine that this pattern continues for each successive interval of time. **Note:** As the race continues, the intervals of time used to analyze the situation become smaller and smaller.

a. If the tortoise has traveled a distance of 1.11111 m, how many time intervals have elapsed since the start of the race?

b. Find the distance covered by Achilles when the tortoise has traveled a distance of 1.11111 m. Describe the process you used.

c. Compare the distance traveled by Achilles in Part **a** with the total distance traveled by the tortoise, including its 10-m head start. Did Achilles catch the tortoise? Explain your response.

d. If the race continues in this fashion, do you think Achilles will ever catch the tortoise? Explain your response.

3.5 In one ancient legend, the king of Persia offered a reward to the inventor of the game of chess. The inventor suggested using the chessboard itself to determine the size of the reward: for the first square on the board, the reward would be 1 kernel of wheat. For the second square, the reward would be twice that, or 2 kernels. The inventor would receive twice as many again for the third square (4 kernels), and so on, until all 64 squares on the board had been filled.

The king agreed to this plan and sent a servant to bring the appropriate amount of grain.

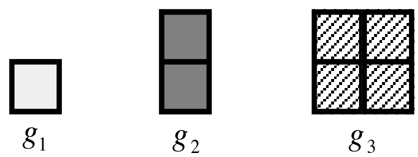
a. How many kernels of wheat were in the reward?

b. When kernels of wheat are spread one layer thick, $7.5 \bullet 10^{10}$ kernels will cover 1 km^2. If the reward were spread one layer thick, how many square kilometers would it cover?

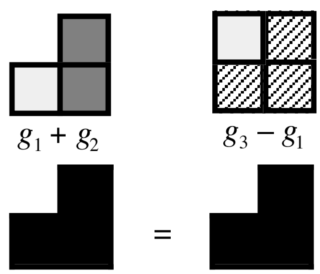
c. The area of the United States is approximately $1.0 \bullet 10^7$ km^2. If the reward were spread over the entire country, how many layers thick would it be?

3.6 Imagine that the inventor of chess asked the king to triple the amount of grain on each successive square of the chessboard. How many wheat kernels would there be in this reward? How does this amount compare to the reward in Problem **3.5**?

3.7 **a.** Consider the sequence described by the explicit formula $g_n = 2^{n-1}$. On a sheet of graph paper, shade and label a rectangular region for each of the first six terms. The area of each shaded rectangle should correspond to the value of the term. A sample graph of the first three terms is shown below:



b. The figure below shows the sum of the first two terms of the sequence. Since $g_1 + g_2$ equals $g_3 - g_1$, the sum can be expressed in two different, but equivalent, ways.



Demonstrate a similar property for each of the following:

1. the sum of the first three terms

2. the sum of the first four terms

3. the sum of the first five terms.

c. Generalize the pattern for the sum of the first n terms of the sequence.

d. Show how the formula you obtained in Part **c** is related to the general formula for a finite geometric series.

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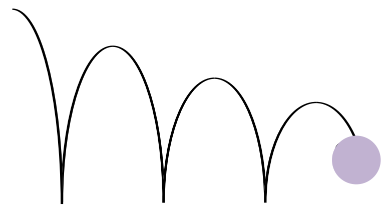
3.8 Consider a ball bouncing as shown in the figure below.



On the first bounce, the ball reaches a height of 6.00 m. On the second bounce, the ball reaches a height of 4.80 m. On the third bounce, the ball reaches a height of 3.84 m.

a. Assuming that this pattern continues, determine a geometric sequence which models the height of the ball on each bounce.

b. Write the expanded form of the geometric series for the first *n* terms of the sequence.

c. Determine the value of the geometric series for each of the following:

1. the first 10 terms

2. the first 20 terms

3. the first 30 terms

4. the first 40 terms.

d. **1.** What value does the geometric series appear to approach?

2. What does this value represent?

3.9 A tree in a tropical rain forest grows 9 m in one year, 6 m in the next year, and 4 m in the following year.

a. Assuming that this pattern continues, write a geometric series that models the tree's growth for 5 years.

b. What is the common ratio?

c. Given that the tree's initial height was 4 m, use a geometric series to determine the height of the tree after:

1. 10 years

2. 20 years

2. 30 years

d. What value does the height of the tree appear to approach?

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Activity 4

Paradoxes are statements that seem contrary to common sense, yet somehow may be true. Zeno of Elea used the race between Achilles and the tortoise, for example, to support his argument that time could not be split into an infinite number of parts.

Because it is difficult to describe an infinite number, problems involving infinity have intrigued mathematicians for centuries. In this activity, you begin an informal investigation of some problems involving infinite processes.

Exploration

a.

1. Using a geometry utility, create a triangle large enough to fill the screen. Label the vertices A, B, and C.

2. Locate the midpoint of each side of the triangle and label them D, E, and F, respectively. Connect these points to form ΔDEF.

3. Connect the midpoints of the sides of ΔDEF to form ΔGHJ.

4. Repeat this process to form ΔKLM. Your drawing should now resemble the one shown in Figure **2**.

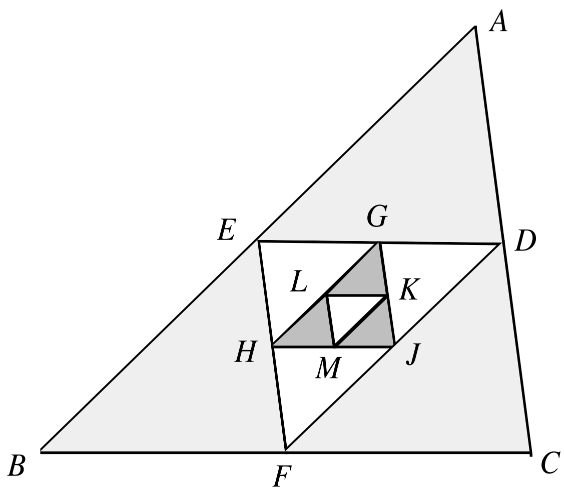


Figure 2: Triangles formed by connecting midpoints of sides

b. Create a spreadsheet with headings like those in Table **1** and complete the first four rows.

Table 1: Data for sequence of triangles

Term No.	**Name of Triangle**	**Area**	**Perimeter**	$\dfrac{\textbf{Area of }\Delta}{\textbf{Area of Previous }\Delta}$	$\dfrac{\textbf{Perimeter of }\Delta}{\textbf{Perimeter of Previous }\Delta}$
1	ΔABC			none	none
2	ΔDEF				
3	ΔGHJ				
4	ΔKLM				
$\vdots$					
15					

c.
1. Use the ratios of areas developed in Part **b** to extend the spreadsheet for 15 triangles.
2. Create a scatterplot of area versus term number.

d.
1. Use the ratios of perimeters developed in Part **b** to extend the spreadsheet for 15 triangles.
2. Create a scatterplot of perimeter versus term number.

e.
1. Use the data for the areas of the triangles to find each of the following sums: S_1, S_2, S_3, …, S_{10}.

   For example, if the area of ΔABC is 100 cm^2, the area of ΔDEF is 25 cm^2, and the area of ΔGHI is 6.5 cm^2, then $S_1 = 100$, $S_2 = 100 + 25 = 125$, and $S_3 = 100 + 25 + 6.5 = 131.5$.
2. Create a scatterplot of the sum versus the number of terms**.**

f. Find the sums S_{20}, S_{30}, S_{50}, and S_{100} using the formula below:

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

Discussion

a.
1. Considering the four triangles from Part **a** of the exploration, what do you notice about the areas of consecutive triangles?
2. What do you notice about the perimeters of consecutive triangles?

b.
1. If your spreadsheet columns containing areas and perimeters could be extended indefinitely, what value would the areas appear to approach?
2. What value would the perimeters appear to approach?

Mathematics Note

In an **infinite sequence**, every term in the sequence has a successor. Infinite sequences do not have a finite number of terms, but continue indefinitely.

The **limit** of a sequence, $k_1,\ k_2,\ k_3,\ \ldots,\ k_n,\ \ldots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L.

For example, consider the infinite sequence below.

$$\frac{0}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ \frac{n-1}{n},\ \ldots$$

Figure **3** shows a graph of the first 32 terms of this sequence.

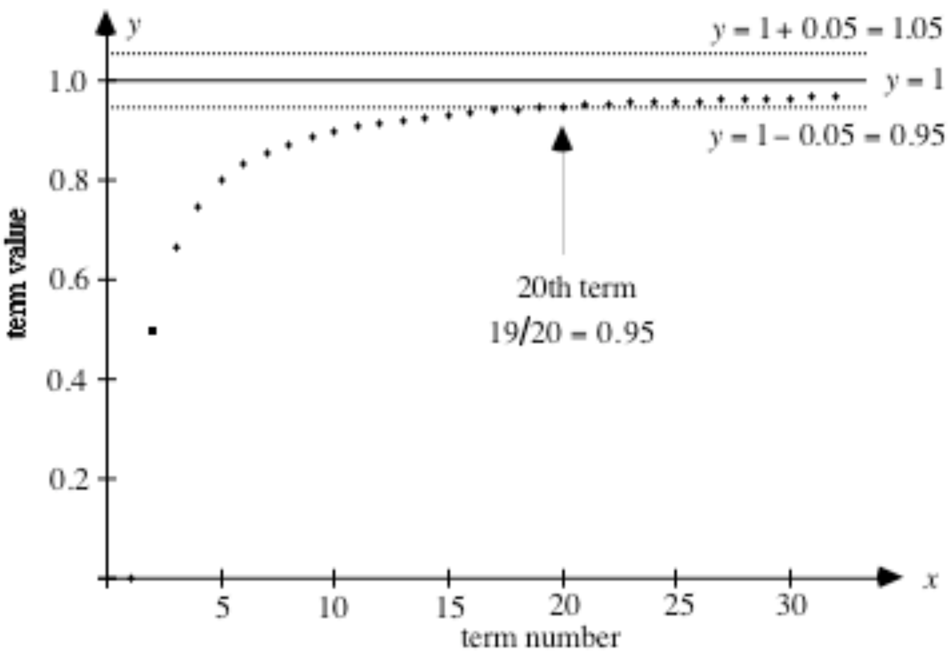


Figure 3: Graph of a sequence whose terms approach a limit

The limit of this sequence appears to be 1 because, for any prescribed level of accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of 1. For example, given a level of accuracy of 0.05, all of the terms of the sequence after k_{20} are within 0.05 of 1.

c. What is meant by "any prescribed accuracy" as mentioned in the mathematics note?

d. How do your responses to Part **b** of the discussion compare with the trends you observe in the scatterplots of the perimeter and area data?

e. **1.** How many triangles would you have to draw for the value of the perimeter to fall within 0.01 units of the limit?

2. How many triangles would you have to draw for the value of the area to fall within 0.01 square units of the limit?

f. In Part **f** of the exploration, you used the following formula to find sums for larger values of n, where $r = 1/2$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of $(1/2)^n$ as n gets very large.

2. Describe what happens to the value of the formula below as n gets very large.

$$S_n = \frac{g_1 \bullet \left((1/2)^n - 1\right)}{(1/2) - 1}$$

g. Consider the following formula for a geometric series where $-1 < r < 1$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of r^n as n increases indefinitely.

2. Describe what happens to the formula for S_n as n increases indefinitely.

h. Consider the following formula for a geometric series where $r \leq -1$ or $r > 1$.

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

1. Describe what happens to the value of r^n as n increases indefinitely.

2. Describe what happens to the formula for S_n as n increases indefinitely.

Mathematics Note

The sum of the terms of an infinite geometric sequence in which $-1 < r < 1$ can be found using the following formula:

$$S = \frac{g_1(-1)}{r - 1} = \frac{g_1}{1 - r}$$

For example, consider the infinite geometric series below:

$$27 + 9 + 3 + 1 + \cdots$$

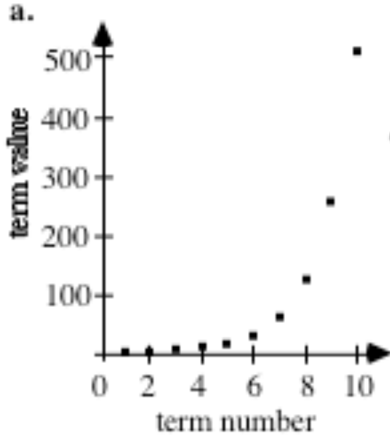
In this case, $g_1 = 27$ and $r = 1/3$. The sum of the terms can be found as follows:

$$S = \frac{27}{1 - (1/3)} = \frac{27}{2/3} = 40.5$$

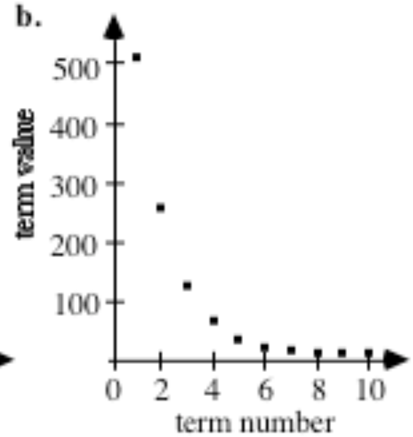
Assignment

4.1. Each graph in Parts **a–c** below shows a scatterplot of the first 10 terms of a sequence. As the number of terms increases indefinitely, which of the sequences appears to have a limit? Explain your responses.

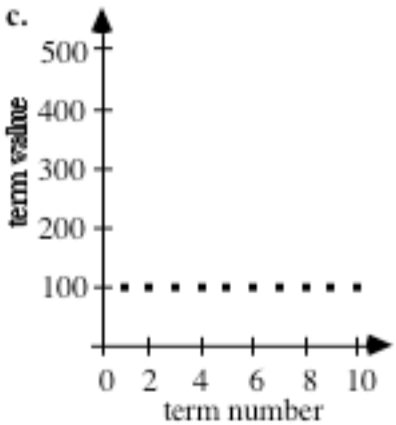
a.



b.



c.



4.2 Using a spreadsheet, enter the sequence 1, 2, 3, 4, ..., 50 in column A. In column B, enter the geometric sequence with a first term of 1 and a common ratio of 10.

a. Suppose that the values in column A represent the term numbers of a sequence while the values in column B represent the term values. As the term number increases indefinitely, do the values in column B appear to approach a limit?

b. Determine the ratio of each term in column B to the preceding term. Enter these values in column C.

c. Suppose that the values in column A represent the term numbers of a sequence while the values in column C represent the term values. As the term number increases indefinitely, do the values in column C appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.01 of the limit.

d. Determine the reciprocal of each term in column B and enter these values in column D.

e. Suppose that the values in column A represent the term numbers of a sequence while the values in column D represent the term values. As the term number increases indefinitely, do the values in column D appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.005 of the limit.

f. Determine the ratio of each term in column D to the preceding term. Enter these values in column E.

g. Suppose that the values in column A represent the term numbers of a sequence while the values in column E represent the term values. As the term number increases indefinitely, do the values in column E appear to approach a limit?

Verify your response by finding a term such that all terms after it are within 0.001 of the limit.

h. **1.** Create a scatterplot of the term value versus the term number for the first six terms of the sequence in column B.

2. Create a scatterplot of the term value versus the term number for the first six terms of the sequence in column D.

3. Describe the differences you observe in the two scatterplots.

4.3 Repeat Problem **4.2** using a geometric sequence with a first term of –1 and a common ratio of 10 in column B.

4.4 Use a geometry utility to construct a square. Connect the midpoints of the sides to form a quadrilateral. Repeat the process two more times.

a. Justify that your original figure is a square.

b. What type of quadrilateral is each of the other three figures in your drawing? Explain your response.

c. Find the areas and perimeters of the four quadrilaterals.

d. **1.** Calculate the ratio of the area of each quadrilateral to the area of the next larger one. What pattern do you observe in these ratios?

2. In the exploration in this activity, the ratio of the areas of consecutive triangles was 1:4. How does this ratio compare with the ratio of the areas of consecutive squares?

e. The ratio of the perimeters of consecutive triangles in the exploration was 1:2. How does this ratio compare with the ratio of the perimeters of consecutive squares?

f. **1.** If you continued to construct quadrilaterals as described above, what would happen to the sequence of areas?

2. What would happen to the sequence of perimeters?

4.5 Determine the sum of the terms of the infinite geometric sequence below:

$$98, 19.6, 3.92, \ldots$$

4.6 Consider the intervals described by the following expression, where n is the term number:

$$\left[3,\ 3+\frac{1}{n}\right]$$

a. Use this expression to write the first four terms of the sequence.

b. As n increases indefinitely, does the sequence appear to approach a limit? Explain your response.

* * * * *

4.7 Consider the following infinite geometric sequence:

$$\frac{6}{10}, \frac{6}{100}, \frac{6}{1000}, \frac{6}{10,000}, \ldots$$

a. What is the common ratio?

b. What does the limit of the sequence appear to be?

c. After how many terms of the sequence are all successive terms within 0.001 of the limit?

d. Find the sums $S_1,\ S_2,\ S_3,\ \ldots,\ S_6$.and write them as a sequence. Hint: $S_1 = 6/10$ and

$$S_2 = \frac{6}{10} + \frac{6}{100} = \frac{66}{100}$$

e. If the sequence of sums continues indefinitely, what limit does it appear to approach?

4.8 While recovering from surgery, a patient takes a 250-mg pain relief tablet every 12 hours for 2 weeks. At the end of 12 hours, 10% of the medication remains in the body.

a. Write a sequence that describes the amount of medication in the body right after taking the first tablet, the second tablet, the third tablet, and the fourth tablet.

b. How much medication is there in the body on the morning of the second day (after the patient takes the third tablet)?

c. What appears to be the limit of the amount of medication in the body?

d. On what day does the amount of medication become close to this limit?

4.9 Express $5/11$ as an infinite series.

4.10 **a.** Write the first four terms of the infinite geometric sequence in which $g_1 = 0.27$ and $r = 0.01$.

b. What is the sum of the infinite geometric series formed by the sequence in Part **a**?

* * * * * * * * * *

Research Project

The idea of figures inside figures has intrigued many artists and mathematicians. To explore this idea in more detail, choose either Part **a** or **b** below.

a. Waclaw Sierpinski, a Polish mathematician, experimented with triangles like those described in the exploration in Activity **4**. Starting with a large triangle, he created a new triangle by connecting the midpoints of the sides of the original triangle. Sierpinski then removed the interior triangle and created new triangles using the midpoints of the sides of the three remaining triangles. He repeated this process over and over, removing the interior triangle each time, and examining the areas and the perimeters of all the remaining triangles.

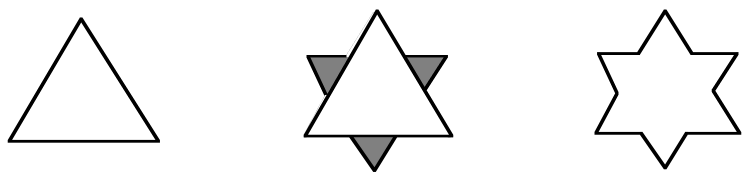
Explore Sierpinski's triangle for yourself using a geometry utility, a graphing calculator, or graph paper.

b. Create a rectangle that is not a square. Construct consecutive quadrilaterals within the rectangle by connecting the midpoints of the sides. Explore the patterns created by the ratios of consecutive areas and consecutive perimeters of the quadrilaterals.

Summary Assessment

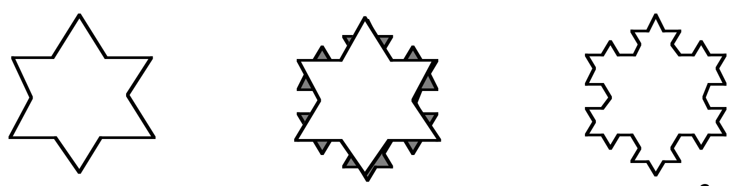
In 1906, Swedish mathematician Helge von Koch introduced a geometric figure that became known as the Koch curve. The Koch curve was later manipulated to create the Koch snowflake. The snowflake can be created with the following steps.

- Begin with an equilateral triangle. This is stage 1.
- Divide each edge of the figure into three equal segments.
- Use the middle segment of each edge to create an equilateral triangle, as shown in the diagram below. The resulting figure is stage 2.



stage 1 stage 2

- Continue dividing each edge of the figure into three equal segments and using the middle segment of each edge to create an equilateral triangle, as shown in the diagram below. The resulting figure is stage 3.



stage 2 stage 3

- To complete another stage, this process is repeated. When the process continues for an infinite number of stages, the figure becomes the Koch snowflake.

1. Assuming that the area of the original equilateral triangle is 1 unit^2, determine the area of the snowflake at each of the following stages:

a. stage 1

b. stage 2

c. stage 15

2. If the process described above is continued, what does the area of the snowflake appear to approach? Explain your response.

Module Summary

- A **sequence** is an ordered list. Each item in the list is a **term** of the sequence. A **finite sequence** has a specific number of terms.
- A **recursive formula** for a sequence identifies the initial term (or terms), then defines all other terms using the previous term(s).
- An **explicit formula** for a sequence defines any term based on its term number.
- In an **arithmetic sequence**, every term after the first is formed by adding a constant value, the **common difference**, to the preceding term. The recursive formula for an arithmetic sequence, where d is the common difference, n is the term number, and a_n is the nth term, is:

  $$\begin{cases} a_1 = \text{first term} \\ a_n = a_{n-1} + d, \ \ n > 1 \end{cases}$$

  The explicit formula for such a sequence is $a_n = a_1 + d(n-1)$.
- In a **geometric sequence**, every term after the first is formed by multiplying the preceding term by a constant value, the **common ratio**. The recursive formula for a geometric sequence, where r is the common ratio ($r \neq 0$), n is the term number, and g_n is the nth term, is:

  $$\begin{cases} g_1 = \text{first term} \\ g_n = rg_{n-1}, \ \ n > 1 \end{cases}$$

  The explicit formula for such a sequence is $g_n = g_1 r^{n-1}$.
- The indicated sum of the terms of a finite sequence is a **finite series**. A finite series of n terms, denoted S_n, may be written in expanded form as follows:

  $$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

  where a_n represents the nth term of the sequence.
- The sum of the terms of an arithmetic sequence is an **arithmetic series**.
- The sum of the terms of a finite arithmetic sequence with n terms and a common difference d can be found using the following formula:

  $$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] = \frac{n}{2}\left(a_1 + a_n\right)$$

  where a_1 is the first term of the sequence and a_n is the nth term.

- The indicated sum of the terms of a geometric sequence is a **geometric series**.
- The sum of the terms of a finite geometric sequence with n terms and a common ratio r can be found using the following formula:

$$S_n = \frac{g_1 r^n - g_1}{r - 1}$$

  where g_1 is the first term of the sequence and $r \neq 1$.
- In an **infinite sequence**, every term in the sequence has a successor. Infinite sequences do not have a finite number of terms, but continue indefinitely.
- The **limit** of a sequence, $k_1,\ k_2,\ k_3,\ \ldots,\ k_n,\ \ldots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L.
- The sum of the terms of an infinite geometric sequence in which $-1 < r < 1$ can be found using the following formula:

$$S = \frac{g_1(-1)}{r - 1} = \frac{g_1}{1 - r}$$

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