Algorithmic Thinking

What do baking a cake, programming a VCR, and adding a pair of two-digit numbers have in common? In this module, you explore how algorithms affect everything from cooking to computers.

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Introduction

Computers come in many shapes and in many forms, from high-tech factory robots to children's toys. In the 21st century, computers will have new applications and new abilities beyond the comprehension of today's users.

The only sure thing in this rapidly changing environment is the need for clever and competent programmers. Before a computer can perform any task, it must be programmed. Skilled programmers dedicate many working hours to crafting clear directions in a coded language that machines can understand. Each set of directions is written in the form of an **algorithm**, a step-by-step process for completing a task.

In this module, you will plan and write algorithms, represent them graphically using flowcharts, and explore some of their applications. During this process, you will also examine some other aspects of mathematics that have been important since ancient times.

Historical Note

The word *algorithm* is derived from the name of Mohammed ibn-Musa al-Khwarizmi, an Arabian scholar and mathematician who died around 850 A.D. When his influential book on the use of Hindu numerals (based on the work of an earlier Indian mathematician, Brahmagupta) appeared in Latin translation, many readers mistakenly attributed the number system to an Arabic source. (As a result, the numerals we use are commonly referred to as Hindu-Arabic numerals.)

The notation described in the translation came to be known as that of al-Khwarizmi, or *algorismi*. This latter word later evolved to *algorism* and, eventually, to *algorithm*.

Activity 1

The ability to write clear, concise directions is an important skill in industries ranging from food products to fiber optics. Consumers often form their most lasting impressions of a product not through the endless hours of advertising, but through those first 15 minutes with an instruction manual. In this activity, you take the initial steps in developing your algorithmic thinking skills by investigating some familiar types of algorithms.

Exploration

a. Draw a simple picture that includes at least three geometric shapes. Do not allow anyone else to see the picture.

b. Write a set of instructions that a person could use to duplicate the picture.

c. Trade instructions with a partner.

d. Perform the instructions as written and record how long it takes you to complete them.

e. Compare your original picture with your partner's drawing. Make any revisions necessary to your set of instructions.

f. Exchange instructions with another partner and repeat Part **d**. Note any difference in the time required to recreate the drawing.

Discussion

a. How did the picture your partner created using your instructions compare with the original picture?

b. Which picture took the least amount of time to reproduce?

c. Which set of instructions resulted in the most accurate reproduction?

d. Which set of instructions were the most efficient? Why? Explain your definition of *efficient*.

e. Is it possible for more than one set of instructions to generate the same picture?

f. How could your instructions have been made more clear?

g. How would you change your instructions if the reader was an elementary student?

Mathematics Note

An **algorithm** is a step-by-step process for completing a task. It typically ends in a finite number of steps. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.

For example, the following algorithm could be used by bus drivers who must cross railroad tracks.

1. Stop between 50 and 100 feet from the railroad tracks.
2. Open the bus door.
3. Look and listen for a train.
4. If there is no train approaching, close the door and proceed.
5. If there is a train approaching, close the door and wait for the train to pass.
6. After the train passes, return to step 2.

Assignment

1.1 Consider the two sets of instructions below.

Washing Clothes	**Preparing a Bath**
Put clothes in washing machine.	Turn on the water in the tub.
Add soap.	Make sure the temperature is warm, but not too hot.
Turn on the machine.	Lay out clean towels.

Complete Parts **a–c** for each set of instructions.

a. Write a short story about someone who followed the instructions explicitly.

b. Explain whether or not the instructions are successful algorithms.

c. Rewrite the set of instructions for washing clothes so that they are appropriate for elementary students.

1.2 Most recipes provide a list of ingredients and a set of directions for preparing a particular dish or meal. The following recipe is from a collection gathered by the Indian Studies Program in Harlem, Montana.

> **Raised Fry Bread**
>
> 1 to $1\frac{1}{4}$ cup warm milk
>
> 1 package dry yeast
>
> 2 tablespoons soft shortening
>
> 2 teaspoons salt
>
> 2 tablespoons sugar
>
> 3 cups sifted flour
>
> Dissolve yeast in milk. Add shortening, salt, sugar, and one-fourth of the flour. Beat until smooth, scraping sides and bottom of the bowl frequently. Add remaining flour and blend until smooth. Let rise in warm place about 30 minutes. Punch down and knead. Let rise again. Break into doughnut-size pieces and shape flat, cutting a hole in the center of the dough. Deep fry in hot grease, turning the bread until both sides are golden brown. Drain on a paper towel and serve while warm.

a. Most recipes are designed so that any two people who follow it will obtain comparable results. Do you think this recipe will always produce the same results? Explain your response.

b. In an algorithm, a **decision point** is a place where a judgment must be made. Are there decision points in the fry-bread recipe that influence later instructions or results? If so, identify them and describe some possible results of these decisions.

1.3 A **prime number** is a natural number greater than 1 with exactly two divisors. The ancient Greeks apparently knew about prime numbers since the time of the Pythagoreans—and specifically since the time of Philolaus (died ca. 390 B.C.).

a. List the first 20 prime numbers.

b. Write a set of instructions for determining if a natural number is prime.

1.4 **a.** Consider the purchase of an item which costs less than $1.00 with a dollar bill. Beginning with the two steps described below, create an algorithm that will determine the appropriate number and type of coins to be given in change. The total number of coins should be kept to a minimum and the types of coins should include pennies, nickels, dimes, and quarters.

Step 1: Subtract the cost of the item from 100 cents.

Step 2: Divide the difference from step 1 by 25.

b. Use an example to show that the algorithm works.

c. Describe what basic knowledge might be required in order to understand the algorithm in Part **a**.

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1.5 Some steps in an algorithm may include algebraic notation. For example, the following algorithm converts temperature in degrees Fahrenheit to temperature in degrees Celsius.

- Let F represent the temperature in degrees Fahrenheit and C the temperature in degrees Celsius.
- Substitute the value of F into the following equation:

$$C = \frac{5}{9}(F - 32)$$

- Solve the equation for C.

a. Use this algorithm to determine the temperature in degrees Celsius that corresponds with a temperature of 99° F.

b. Rewrite the algorithm so that it converts temperature in degrees Celsius to temperature in degrees Fahrenheit.

1.6 A small programmable robot can travel a specific path if given the correct algorithm. For example, the algorithm "Triangle(x)" shown below instructs the robot to trace an equilateral triangle whose sides are x meters long.

- Forward(x)
- Right(120°)
- Forward(x)
- Right(120°)
- Forward(x)
- Right(120°)

a. What path would the robot trace if given the algorithm "Triangle(10)"?

b. Write an algorithm that will allow the robot to trace a square.

c. Write an algorithm that allows the robot to trace a path of your choice. Have another person complete the algorithm by drawing the path the robot will follow.

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Activity 2

In Activity **1**, you examined sets of instructions that form algorithms. Algorithms can also be represented graphically as **flowcharts**.

Mathematics Note

A **flowchart** can be used to represent the steps of an algorithm. As seen in Figure **1**, it may consist of input and output boxes, data processing boxes, and decision boxes. Ovals are used for "start" and "stop" commands.

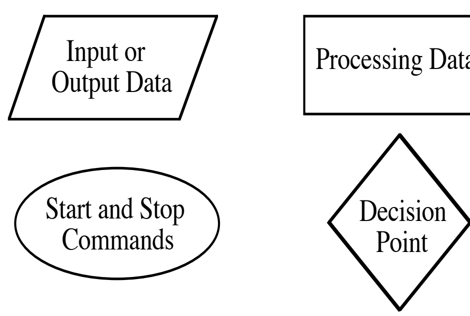


Figure 1: Flowchart symbols

To illustrate a step-by-step sequence of instructions, flowchart symbols are usually connected by arrows. The flowchart in Figure **2**, for example, shows an algorithm for finding two matching socks in a drawer that contains several pairs of unmatched socks.

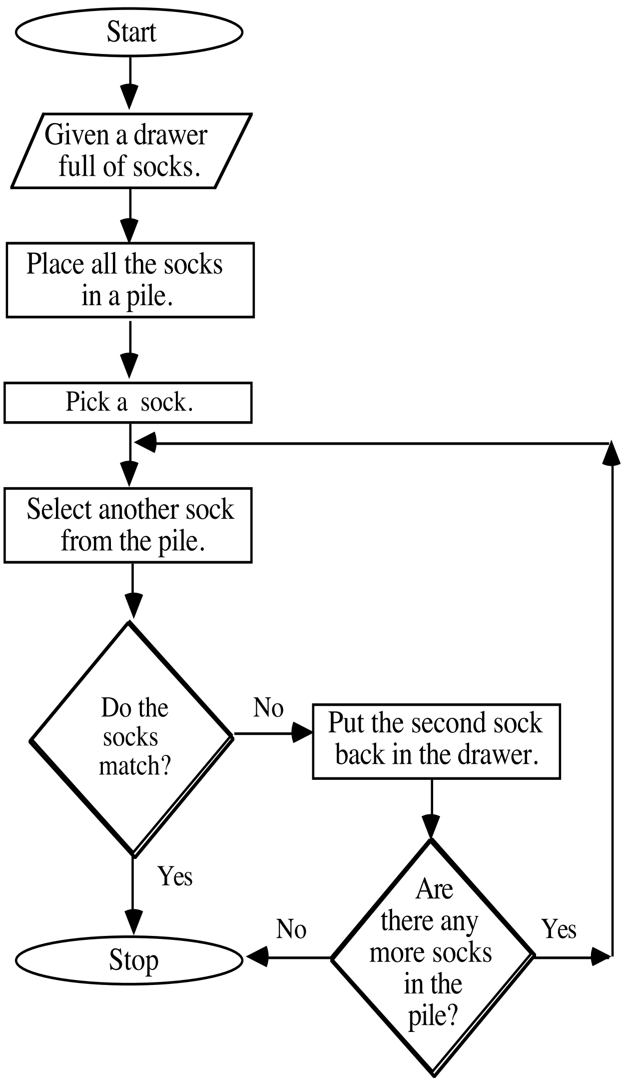


Figure 2: Flowchart for finding a pair of matching socks

Exploration 1

A flowchart must follow the same logical process as the algorithm it represents. In this exploration, you organize the parts of a flowchart into a logical order.

a. Obtain a copy of the flowchart template for subtracting two natural numbers from your teacher. Cut out each of the command boxes.

b. Organize the command boxes so that a person could use the algorithm to subtract two natural numbers. Draw arrows to show the appropriate directions and include "yes" and "no" answers for decision boxes.

c. Exchange your flowchart with another student. Follow the commands as indicated to verify that the flowchart is properly organized.

Discussion 1

a. Compare your flowchart with those of others in the class. Are all the flowcharts that correctly model the subtraction algorithm identical?

b. Describe the method you used to organize the flowchart.

c. What characteristics make one flowchart easier to read than another?

Exploration 2

In the following exploration, you create a flowchart for addition.

a. Select a pair of two-digit numbers, such as 84 and 39. Working with pencil and paper, find the sum of these numbers using the algorithm you learned in elementary school.

b. Write a set of instructions that someone else could use to add these two numbers.

c. Draw a flowchart to illustrate the algorithm you developed in Part **b**.

d. Exchange flowcharts with a partner. Follow your partner's flowchart exactly as written to find the sum of a pair of two-digit numbers. When you have finished, return the flowchart and sum to the original writer.

e. Revise your flowchart so that it may be used to add any pair of two-digit numbers.

Discussion 2

a. How does your flowchart compare with those of others in the class?

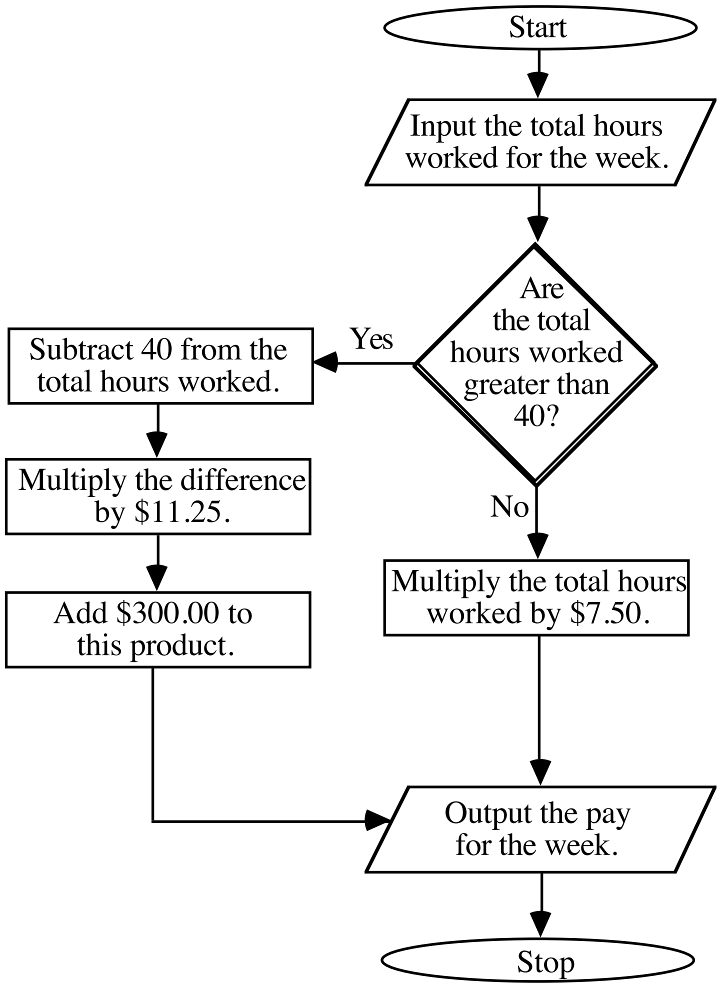
b. **1.** Which flowchart from Exploration **2** seems to be the best?

2. What criteria did you use to make this decision?

c. Describe how your flowchart could be adapted for finding the sum of a pair of three-digit numbers.

Assignment

2.1 To calculate an employee's weekly wages, Company A uses the number of hours worked during the week, along with the algorithm shown in the flowchart below:



a. Use the flowchart to calculate the weekly pay for an employee who works each of the following numbers of hours:

1. 46 hours

2. 36 hours

b. Company B calculates its employee's wages on a daily rate. The company pays $7.50 per hour for 8 or fewer hours, and an additional $3.75 per hour for any time over 8 hours.

Create a flowchart that could be used to determine an employee's daily wage for Company B.

c. For which company would you prefer to work? To support your response, use an example of the wages paid by each company for a specific number of hours worked.

2.2 Design a flowchart to determine if any natural number is prime.

2.3 In one version of the game of Nim, two players take turns adding either 1 or 2 to a sum until a total of 21 is reached. The rules for this game are listed below:

- The first player starts with either the number 1 or the number 2.
- The second player adds either 1 or 2 to the first player's total.
- Players continue to take turns adding 1 or 2 to the previous player's total until one player reaches exactly 21. That player wins the game.

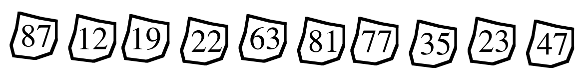
a. Play the game several times.

b. Describe an algorithm for winning at this game.

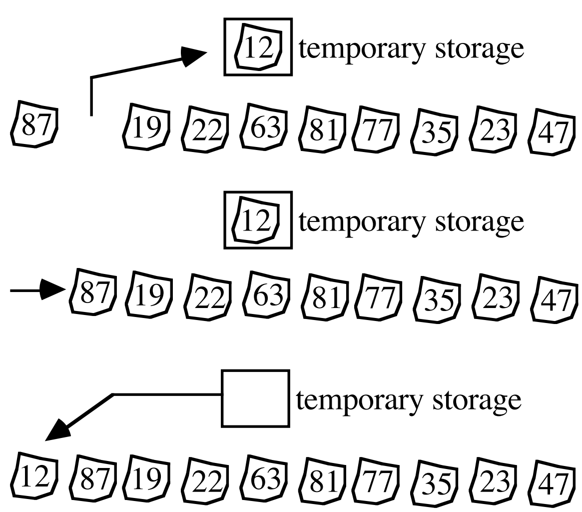
c. Do you think a computer could be programmed to win at this game? Explain your response.

2.4 Sorting algorithms are used to place items such as addresses, numbers, and names in order. Computer scientists have developed many algorithms that can sort quickly.

For example, suppose that you want to sort the 10 numbers in the following set from least to greatest: {87, 12, 19, 22, 63, 81, 77, 35, 23, 47}. Write each number on a slip of paper and arrange them as shown in the following diagram:



The **insertion sort algorithm** starts by removing the second number from the list, leaving an empty position. This number is placed in a temporary storage location and compared with the first number. If the first number is greater, it is moved to the right, into the empty position. The number in temporary storage is then inserted back into the list wherever the open position is located, as shown in the diagram below.



At this point in the sort, one comparison has been made. Next, the third number is moved into the temporary storage location and compared with the second number. If the second number is greater, it is moved to the right. If the first number is greater, it is also moved to the right. The number in storage is then moved into the empty position. Two more comparisons have been made at this point in the sort, for a total of three comparisons.

a. Complete the insert sort algorithm for all 10 numbers, keeping track of the number of comparisons made at each step in the sort. How many comparisons were needed to completely sort the list?

b. Draw a flowchart to model the insertion sort algorithm.

c. Two other sorting algorithms used by computer programmers are the bubble sort and the merge sort. The bubble sort algorithm requires 45 comparisons to sort this same list of numbers, while the merge sort algorithm takes 24 comparisons. Which algorithm appears to be the most efficient: the insertion, bubble, or merge sort?

2.5 Draw a flowchart for finding the sum of any pair of three-digit numbers.

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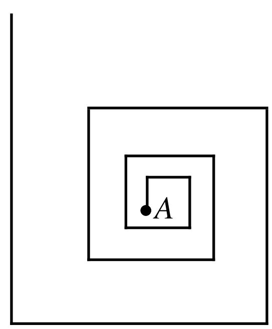
2.6 In Problem **1.6**, you wrote an algorithm that instructed a programmable robot to trace a path. Create a flowchart that will allow this robot to trace a path similar to the one shown in the diagram below, beginning at point *A*.



2.7 The *Elements* of the Greek mathematician Euclid (ca. 300 B.C.) consist of 13 books on topics from elementary geometry to number theory. Book VII contains what has come to be known as the "Euclidean algorithm"—a scheme for finding the greatest common divisor of two natural numbers.

In the Euclidean algorithm, the smaller number s is subtracted from the greater number g. If the difference d is greater than s, then s is subtracted from d, creating a new d. This process is repeated (subtracting s from d) until d is less than s.

If d is a divisor of the original two numbers, then the process is complete and the greatest divisor is found. If d is not a divisor, then g is replaced with s, s is replaced with d, and the entire process is repeated until a divisor is found.

For example, the Euclidean algorithm can be used to find the greatest common divisor of 15 and 6 as shown below.

$$\begin{array}{r} 15 \\ -6 \\ \hline 9 \end{array} \qquad \begin{array}{r} 9 \\ -6 \\ \hline 3 \end{array}$$

Since 3 is a divisor of 15 and 6, it is the greatest common divisor of the two numbers.

Similarly, the Euclidean algorithm can be used to find the greatest common divisor of 24 and 9 as shown below:

$$\begin{array}{r} 24 \\ -9 \\ \hline 15 \end{array} \qquad \begin{array}{r} 15 \\ -9 \\ \hline 6 \end{array}$$

The difference found after the second subtraction is less than 9. However, since 6 is not a divisor of 24 and 9, the algorithm is continued using 6 and 9.

$$\begin{array}{r} 9 \\ -6 \\ \hline 3 \end{array}$$

Since 3 is a divisor of 24 and 9, it is the greatest common divisor of the two numbers.

a. Use the Euclidean algorithm to find the greatest common divisor of each of the following pairs of numbers:

1. 54 and 24

2. 51 and 9

3. 12 and 5

b. Create a flowchart to model the Euclidean algorithm.

c. Do you think the Euclidean algorithm could be modified to find the greatest common divisor of any two integers? Explain your response.

2.8 **a.** The process of division can be defined as "repeated subtraction." Rewrite the Euclidean algorithm described in Problem **2.7** so that it uses division, rather than repeated subtraction, to find the greatest common divisor.

b. Create a flowchart to model this algorithm.

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Research Project

Working with a partner, develop a generalized version of the game of Nim described in Problem **2.3**. For example, you might consider a game in which players add the digits 1 or 3 to reach a total of 36. In your generalized version of the game, let T be the target number and the digits 1 and n the numbers to be added. Your report should include precise directions for playing the game, rules of play, and a description of any algorithms for winning the game.

Activity 3

The algorithms you explored in previous activities used concepts from number theory. In the Level 1 module "Going in Circuits," you investigated some algorithms derived from graph theory. In this activity, you examine an algorithm that involves geometry.

Exploration

You have investigated several methods of writing algorithms: step-by-step instructions, flowcharts, and formulas. In this exploration, the algorithms consist of a set of information and a procedure to perform on the given information.

a. Use the algorithm in Table **1** to create a geometric shape.

Table 1: Algorithm for creating a geometric shape

Given:	A triangle.
Procedure:	1. Locate the midpoints of each side. 2. Connect the midpoints consecutively to create a new inscribed polygon. 3. Repeat the procedure on the new inscribed polygon.

Mathematics Note

A **recursive definition** consists of some specific information, a procedure that is performed on the given information, and continuing steps that generate new information using the same procedure on previously generated information. The process of using a recursive definition is known as **recursion**.

For example, the following formula uses recursion to describe the arithmetic sequence: 1, 8, 15, 22, 29, … .

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 7,\ n > 1 \end{cases}$$

In this case, the first term, $a_1 = 1$, represents the given information. The process of repeatedly adding 7 to the previous term to obtain a new term is the procedure.

b. Write an algorithm that uses recursion to create a sequence of inscribed squares like the one shown in Figure **3**. Each square is a dilation by a scale factor of 0.8 of the next larger square. The center of dilation is the center of the largest square.

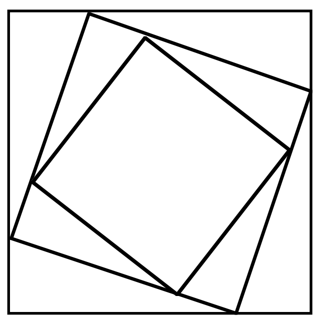


Figure 3: Inscribed squares

c. Use your algorithm to continue the pattern until your drawing includes eight inscribed squares.

Discussion

a. Which step in the formula for an arithmetic sequence given in the mathematics note creates recursion?

b. Which step in the algorithm in Table **1** creates recursion?

c. Since the algorithm described in Table **1** has no terminating step, the recursion is indefinite. If the algorithm continues indefinitely, what is the limit of the perimeters of the inscribed polygons? Explain your response.

d.
1. How could you change the algorithm in Table **1** so that it creates exactly 10 inscribed triangles?
2. How could you change the algorithm to create a geometric figure with inscribed quadrilaterals?

Assignment

3.1 The following algorithm uses recursion to create a geometric sequence.

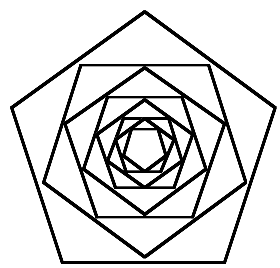
$$\begin{cases} g_1 = 3 \\ g_n = 4g_{n-1}, n > 1 \end{cases}$$

a. Use the algorithm to write the first five terms of the sequence.

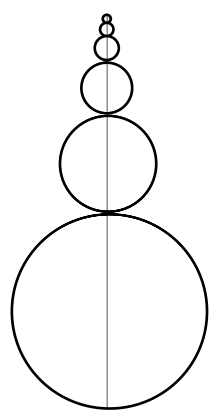
b. What part of the algorithm defines the given information?

c. What part of the algorithm defines the procedure?

3.2 Design an algorithm that could be used to recreate the diagram below. Each pentagon is a dilation by a scale factor of 0.8 of the next larger pentagon. The center of dilation is the center of the largest pentagon.



3.3 Design an algorithm that could be used to recreate the diagram below.



3.4 Write an algorithm that creates a geometric sequence.

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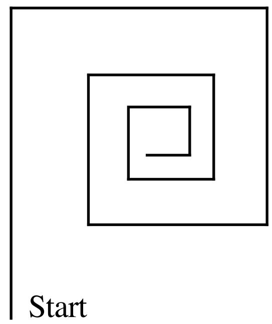
3.5 Consider the geometric figure shown in the following diagram.



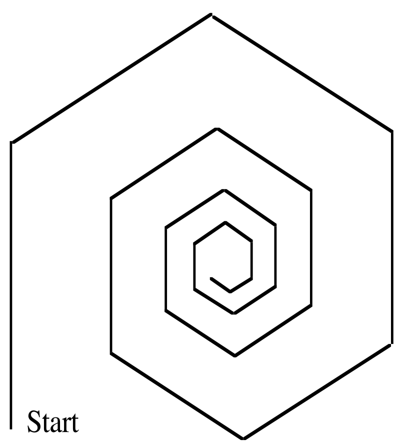
Using the algorithm described below, "Squiral(x)," a programmable robot can trace a similar figure.

- Squiral(x)
  - Forward(x)
  - Right(90˚)
- Squiral($0.8x$)

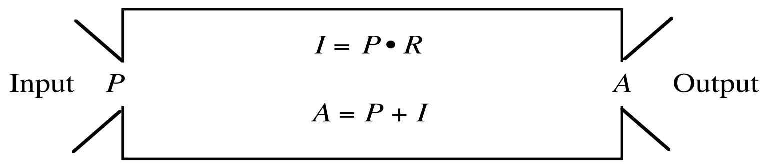
a. If x is 5 m, how long is the first side of the figure? How long is the second side?

b. According to the algorithm, how many times will the robot go forward and turn?

c. Write a recursive algorithm that will allow the robot to trace the following figure.



3.6 Interest is the amount earned on invested money. A bank offers an investment account that pays interest at an annual rate of 8%. To demonstrate the potential earnings of this account, a bank officer uses the "interest machine" shown below. In this case, I represents interest, P represents the initial investment, R represents the annual interest rate, and A represents the account balance after 1 year. The output at the end of each year becomes the input for the following year.



a. Given an initial investment of $500 in this account, determine the account balance at the end of each of the next three years.

b. Is the algorithm used by the "interest machine" recursive? Explain your response.

c. **1.** If no withdrawals are made from the account, what will it be worth at the end of the 20th year?

2. What is the total amount of interest earned during this period?

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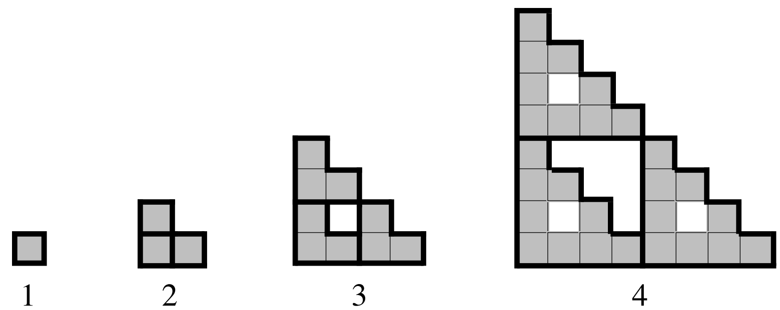
Summary Assessment

1. **a.** The diagram below shows a sequence of figures. Term 2 can be created with three copies of term 1, term 3 can be created with three copies of term 2, and so on.



Write an algorithm that allows you to create each figure using three copies of the previous figure.

b. Create a flowchart to illustrate the algorithm.

c. Is the algorithm recursive? Explain your response.

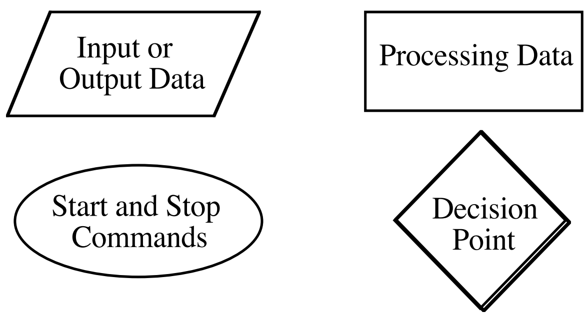
d. Use your algorithm to create the next figure in the sequence.

2. **a.** Create your own sequence of geometric figures.

b. Write an algorithm that another person could use to recreate the same sequence.

Module Summary

- An **algorithm** is a step-by-step process for completing a task. It typically ends in a finite number of steps. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.
- A **decision point** in an algorithm is a place where a judgment must be made.
- A **prime number** is a natural number greater than 1 with exactly two divisors.
- A **flowchart** can be used to represent the steps of an algorithm. As seen in the diagram below, flowchart symbols may consist of input and output boxes, data processing boxes, and decision boxes. Ovals are used for "start" and "stop" commands. To illustrate a step-by-step sequence of instructions, these symbols are usually connected by arrows.



- A **recursive definition** consists of some specific information, a procedure that is performed on the given information, and continuing steps that generate new information using the same procedure on previously generated information. The process of using a recursive definition is known as **recursion**.

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