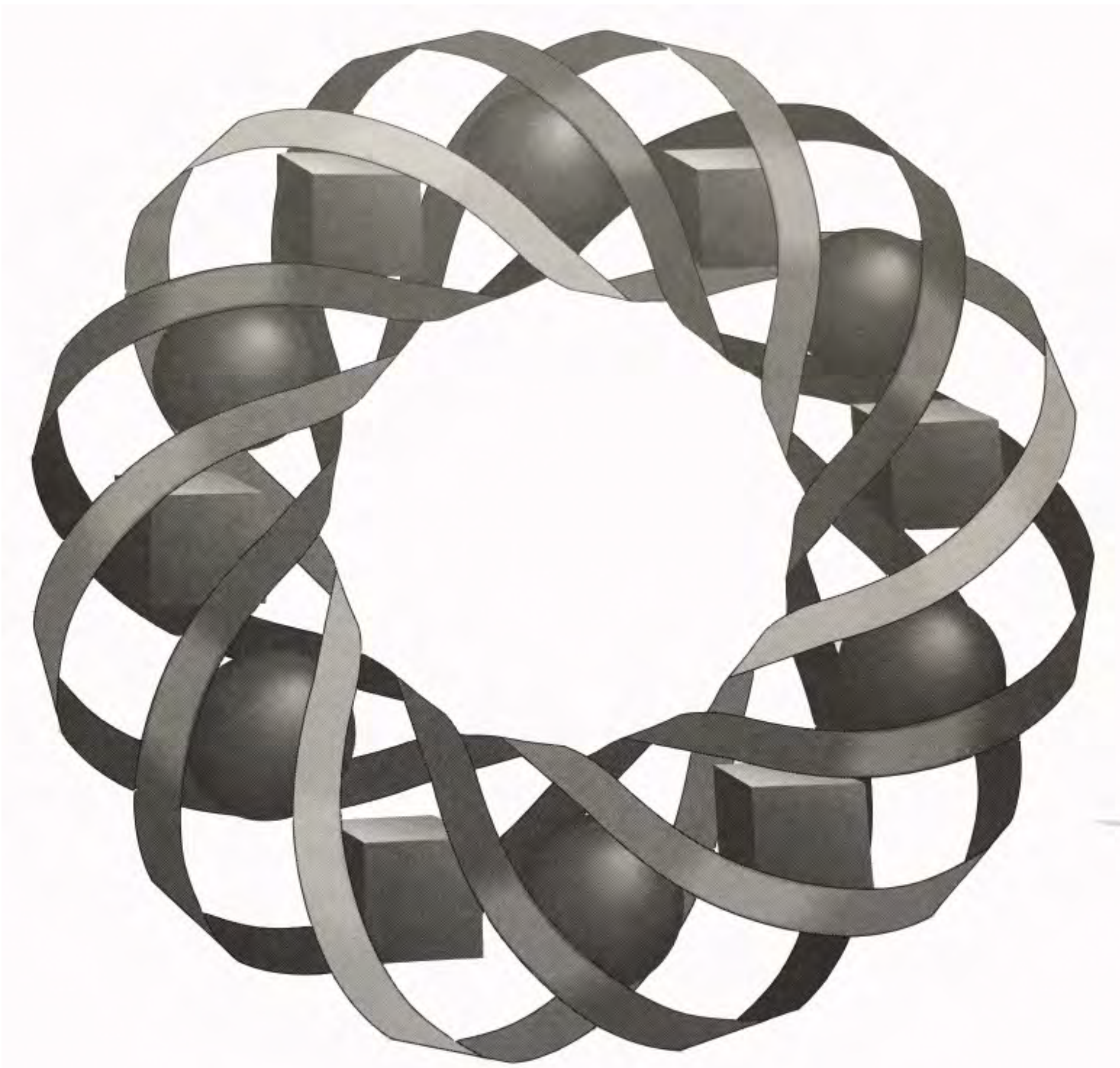
A New Angle on an Old Pyramid



What mathematical skills did the ancient Egyptians use to build the pyramids? In this module, you explore some of the theories regarding the construction of these architectural wonders—and see how modern mathematical knowledge could have made part of the job much simpler.

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Introduction

Tutankhamen and Khufu were kings of ancient Egypt. Khufu ruled around 2500 B.C., while Tutankhamen (King Tut) governed about 1200 years later. Although King Tut's artifacts are among the most popular items displayed in museums today, it was King Khufu who built the Great Pyramid at Giza, one of the "seven wonders" of the ancient world.

Still standing after 4500 years, the Great Pyramid is an amazing architectural monument. Designed as Khufu's burial tomb, the structure is a regular square pyramid which originally stood 147 m high. Each side of its square base measured 230 m, as shown in Figure **1**.

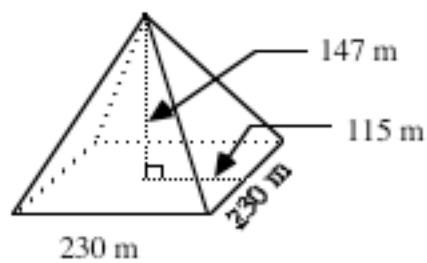


Figure 1: Dimensions of the Great Pyramid

The Great Pyramid consists of about 2.3 million stone blocks, each with a mass of approximately 1000 kg. As many as 100,000 laborers may have worked for 20 summers to complete this engineering marvel. Two basic kinds of stones were used: rectangular blocks and casing blocks. The casing blocks gave the faces of the pyramid a smooth, continuous slope.

The base of the Great Pyramid is nearly a perfect square. To build a structure of this size to such exacting specifications, the Egyptian engineers and surveyors must have applied some mathematics. Due to the lack of written records, however, historians are unsure about what mathematics the builders actually knew. In this module, you investigate some mathematical ideas that historians believe may have been used to build the pyramid.

Activity 1

Before beginning construction, modern engineers use precise measuring tools to stake out the corners of a building's foundation. How did the Egyptians locate the corners of the Great Pyramid without the benefit of 20th-century surveying tools? In this activity, you review some of the difficulties involved in performing this task.

Exploration

a. On a large sheet of paper, construct one corner of a square by following Steps **1–3** below. When marking points and line segments on the paper, make them dark enough to show through another sheet of paper.

1. Mark a point *C* near the lower left-hand corner of the paper, 5 cm from each edge.
2. Draw a line segment from point *C* to the lower right-hand corner of the paper. As shown in Figure **2**, place point *D* on the segment 30 cm from point *C*.

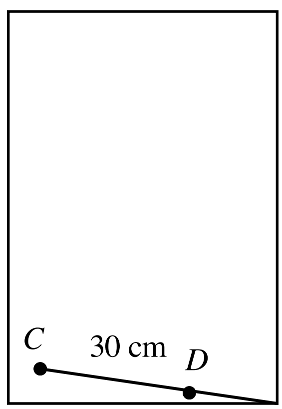


Figure 2: Segment from point *C* to point *D*

3. Use a protractor and a meterstick to draw a line segment that passes through point *C*, is perpendicular to $\overline{CD}$, and has length of 1 m. Label the point at the end of the segment as *P*.

b. Compare the corner you created with those of others in the class by completing the following steps.

1. Select one sheet of paper from Part **a** on which to record the class results.
2. Place the selected sheet on top of another sheet and align the two drawings of $\overline{CD}$.

3. Mark the location of point P on the lower sheet on the top sheet.
4. Repeat Steps **2** and **3** until the information for the entire class has been collected on the selected sheet.

c.
1. On the common sheet of paper, identify the two locations of P that are farthest apart. Measure the distance between these points to the nearest millimeter.
2. Label these two points X and Y.
3. Express the length of $\overline{XY}$ in centimeters.
4. Draw ΔCXY. The common sheet should now resemble the diagram in Figure **3**.

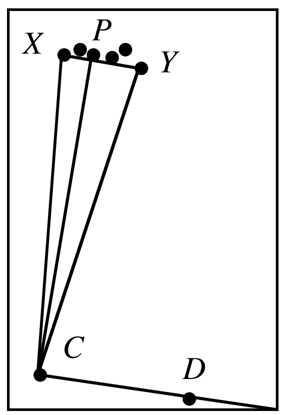


Figure 3: ΔCXY on the common sheet of paper

d. Imagine R is a point on $\overrightarrow{CX}$ 100 m from C and S is a point on $\overrightarrow{CY}$ 100 m from C. Predict the distance between R and S.

Discussion

a. Why was there a difference between the corner you drew and those drawn by others in your class?

b. What type of triangle is ΔCXY? What are its special features?

c. Two figures are **similar** if they have the same shape and corresponding sides are proportional. The symbol used to indicate similarity is $\sim$. If ΔABC is similar to ΔDEF, explain why the notation $\Delta ABC \sim \Delta EDF$ is an incorrect way of expressing this similarity.

Mathematics Note

In plane geometry, if the angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar. This is referred to as the **Angle-Angle-Angle (AAA) Property**.

The symbol used to indicate congruence is $\cong$. For example, consider triangles ABC and DEF in Figure **4**. As shown by the markings, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. Since their corresponding angles are congruent, $\Delta ABC \sim \Delta DEF$.

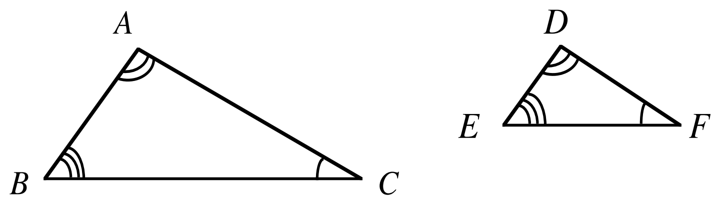


Figure 4: Two similar triangles

Since these triangles are similar, the measures of their corresponding sides are proportional, as shown below.

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

d. In the exploration, which triangle is similar to ΔCXY? Explain your response.

e. How could you use the pair of similar triangles identified in Part **d** of the discussion to determine the distance between R and S?

Assignment

1.1 Suppose that King Khufu ordered two different surveyors to stake the corners of the Great Pyramid. At 1 m from the first corner, the distance between the two surveyors' lines equaled the distance you measured in Part **c** of the exploration. How far apart will their two lines be after 230 m, the side length of the base of the Great Pyramid? Explain how you determined your response.

1.2 As mentioned in the introduction, the Great Pyramid is a regular square pyramid. This means that the base is square, the altitude passes through the center of the base, and the four lateral faces are congruent isosceles triangles. To investigate some of the properties of these lateral faces, complete Parts **a–e** below.

a. Cut an isosceles triangle from the template supplied by your teacher. Fold the triangle along its line of symmetry.

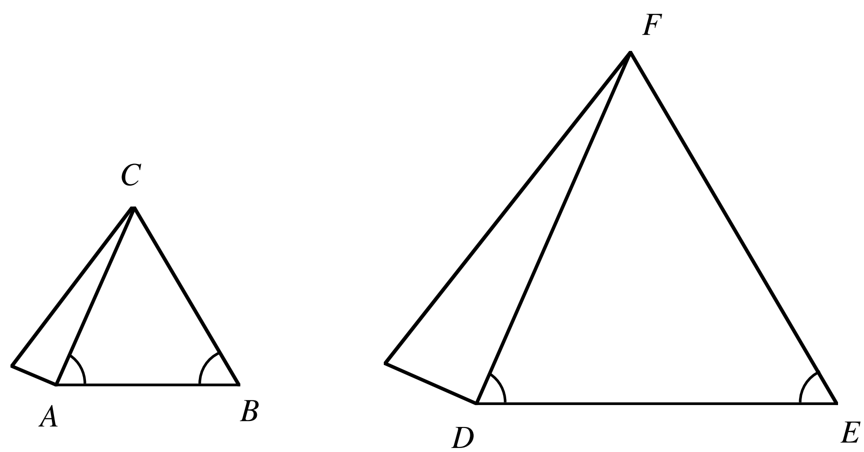
b. Use a protractor and a ruler to determine the relationship between the line of symmetry and the base of the triangle.

c. What is the relationship between the triangle's altitude and its line of symmetry?

d. When the isosceles triangle is folded along its line of symmetry, what type of triangle is formed?

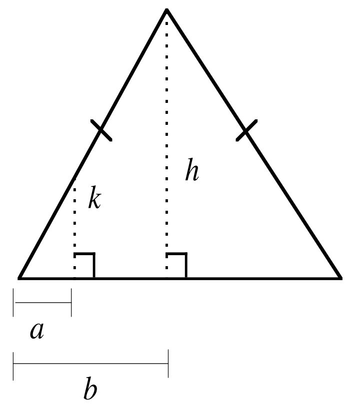
e. What is the relationship between the line of symmetry for the isosceles triangle and its vertex angle?

1.3 Two pyramids are similar when their bases are similar and their faces are similar. The diagram below shows two regular square pyramids. As shown by the corresponding markings, $\angle CAB$ and $\angle FDE$ are congruent. Are these pyramids similar? Explain your response.



1.4 What is the least amount of information necessary to determine if two triangles are similar? Explain your response.

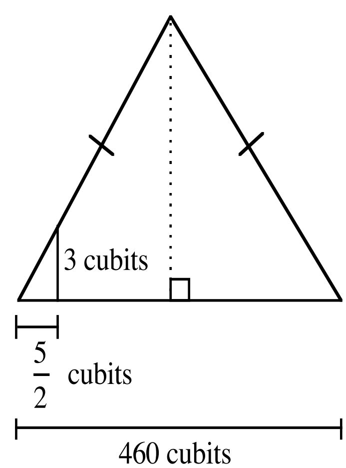
1.5 **a.** The diagram below shows a cross section of a regular square pyramid where h is the height of the pyramid. The ancient Egyptians were aware that the ratios h/k and b/a are equal. Explain why this relationship is true.



b. Describe a simple rule for determining when two right triangles are similar.

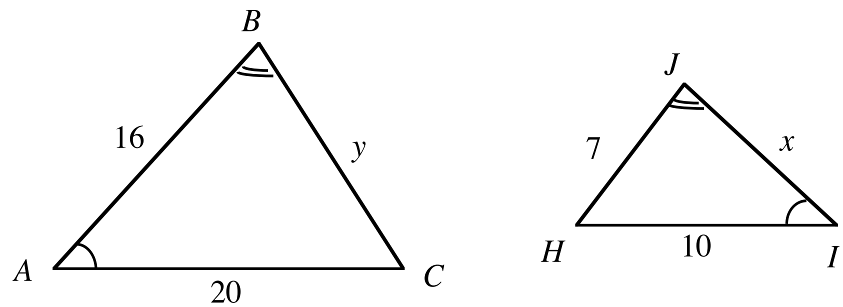
1.6 One source of information on ancient Egyptian mathematics is the Rhind Papyrus, a scroll about 5.5 m long and 33 cm wide. Written about 1650 B.C. by a scribe named Ahmes, the Rhind Papyrus is a practical handbook of mathematical problems.

As shown in the diagram below (not drawn to scale), one of these problems asks for the height of a square pyramid given certain measurements.

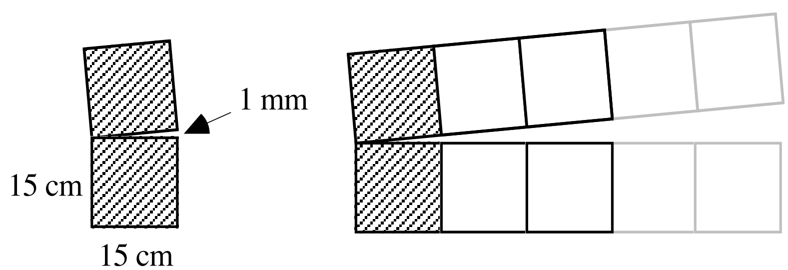


Find the height of the pyramid in cubits. **Note:** In ancient Egypt, the *cubit* was a unit of measurement based on the distance from the elbow to the tip of the fingers in an average person's forearm.

1.7 Use proportions to determine the values of x and y in the following triangles. (Angles which are marked in the same manner are congruent.)



1.8 When placing the giant stone blocks on some levels of the Great Pyramid, workers had to align the blocks edge to edge and corner to corner. The task was similar to installing a tile floor—except on a much larger scale. In the diagram below, two square tiles have been placed with one pair of corners touching. The other pair of corners, however, has a gap of 1 mm. If the next tiles are aligned exactly with these two (as shown in the diagram), then how wide will the gap be after 3 m of tile have been installed? Explain your answer using similar triangles.



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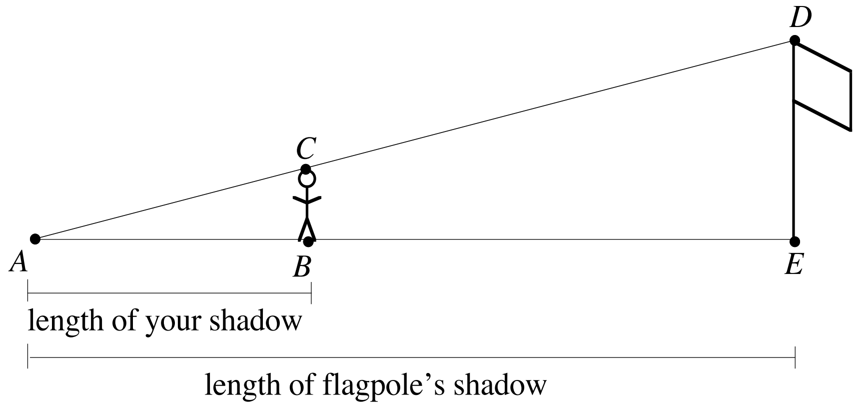
1.9 Your math class has been asked to determine the height of the school flagpole. Since you have no tools to actually measure the flagpole, the class devised a method which involves similar triangles. The diagram below shows a sketch of this situation.

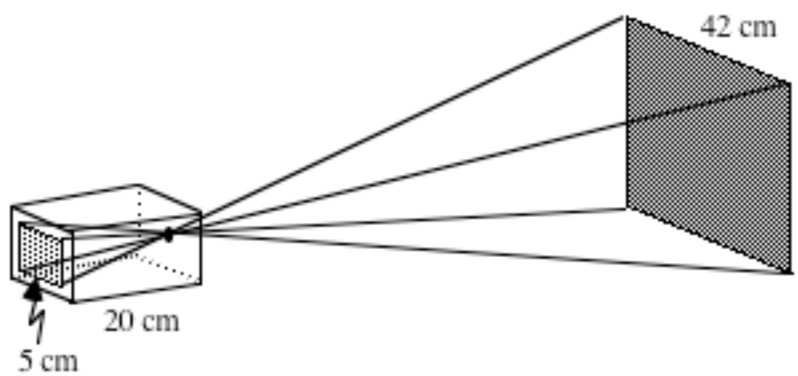


a. Explain why triangles *ABC* and *AED* are similar.

b. Describe how to use the properties of similar triangles to find the height of the flagpole.

1.10 A pinhole camera consists of a sealed box with a pinhole opening at one end and a piece of film at the opposite end. When light enters the box through the pinhole, an inverted image is projected on the film.

Consider a pinhole camera 20 cm long, as shown in the diagram below, containing a square piece of film, 5 cm on each side. The object to be photographed occupies a square 42 cm on each side.



If the photographer wants the image to exactly fill the film, how far away should the camera be placed? Explain your reasoning. (Hint: The lines in the diagram indicate the corresponding vertices of the object and the image. The two pyramids formed are similar.)

1.11 If two triangles are similar, are their corresponding altitudes proportional? Use diagrams to help support your response.

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Activity 2

Since the ancient Egyptians did not measure angles in degrees, they must have used other methods than those in Activity **1** to make their corners "right." One theory suggests that the Egyptians used a rope subdivided by knots into 12 congruent segments. To make a corner, the rope was formed into a triangle with sides of 3, 4, and 5 units. As shown in Figure **5**, the special characteristics of this kind of triangle might have allowed the Egyptians to form a right angle. In this activity, you examine such triangles and develop a method for deciding when a corner is "right."

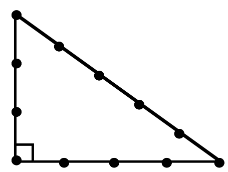


Figure 5: A 3-4-5 triangle

Mathematics Note

The **Pythagorean theorem** states that, in a right triangle, the square of the length of the longest side (the hypotenuse) equals the sum of the squares of the lengths of the other sides (the legs).

In Figure **6**, for example, $\angle C$ is a right angle, $\overline{AB}$ is the hypotenuse, and $\overline{BC}$ and $\overline{AC}$ are the legs. Since ΔABC is a right triangle, $a^2 + b^2 = c^2$, where *a* and *b* are the lengths of the legs and *c* is the length of the hypotenuse.

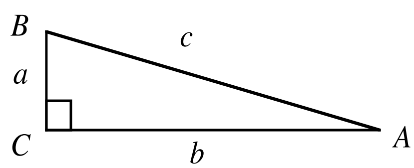


Figure 6: A right triangle

Notice that the vertices of the triangle in Figure **6** are labeled with capital letters, while the sides opposite these vertices are identified with the corresponding lowercase letters. This is a conventional way to label the parts of triangles.

For example, consider ΔRST in Figure **7**. In this triangle, $s = 13 \text{ m}$, $t = 5 \text{ m}$, and $r = 12 \text{ m}$. Because ΔRST is a right triangle, $s^2 = t^2 + r^2$ or $13^2 = 5^2 + 12^2$.

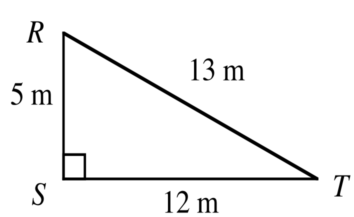


Figure 7: Right triangle *RST*

Exploration

Although the Pythagorean theorem is evident in the Egyptian rope triangle, it is the **converse** of this theorem that would have been more useful to the Egyptians.

Mathematics Note

The **converse** of a statement in the form "If A, then B" is the statement "If B, then A." The converse of a true if-then statement may or may not be true.

For example, consider the following if-then statement:

"If the country is Egypt, then the Nile River passes through the country."

The converse of this statement is:

"If the Nile River passes through a country, then the country is Egypt."

Since the Nile flows through other countries besides Egypt, the first if-then statement in this example is true, while its converse is not.

a.
1. Write the converse of the Pythagorean theorem.
2. Do you think that the converse is true?

b. Use a geometry utility to test the truth of the converse of the Pythagorean theorem by completing Steps **1–5**.

1. Construct ΔABC and two lines perpendicular to the base $\overline{AB}$, as shown in Figure **8**.

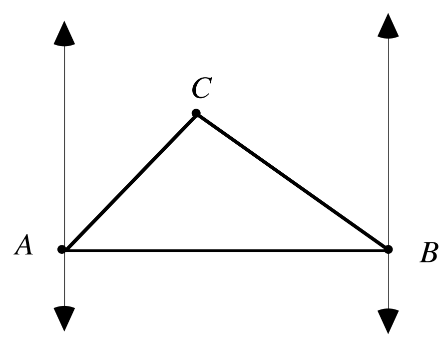


Figure 8: Construction of ΔABC

2. Create a table with headings like those in Table **1** below.

Table 1: Triangle data

$m\angle C$	AB	AC	BC	$(AB)^2 - [(AC)^2 + (BC)^2]$

3. Record the appropriate measurements for ΔABC in Table **1**.
4. Move point C to another location between the two lines (but without touching either line). Record the appropriate measurements in Table **1**.
5. Repeat Step **4** for several other locations of C. Include some positions that make $m\angle C$ acute (less than 90˚), some that make it a right angle (exactly 90˚), and some that make it obtuse (greater than 90˚ but less than 180˚).

c.

1. When the value of $(AB)^2 - \left[(AC)^2 + (BC)^2\right]$ is positive, is $\angle C$ acute, right, or obtuse?
2. When the value of $(AB)^2 - \left[(AC)^2 + (BC)^2\right]$ is negative, is $\angle C$ acute, right, or obtuse?
3. When the value of $(AB)^2 - \left[(AC)^2 + (BC)^2\right]$ is 0, is $\angle C$ acute, right, or obtuse?

Discussion

a. How is the expression $(AB)^2 - \left[(AC)^2 + (BC)^2\right]$ related to the equation for the Pythagorean theorem, $a^2 + b^2 = c^2$?

b. In an obtuse triangle, two angles are acute and one is obtuse. The longest side is opposite the obtuse angle. Based on your data in Table **1**, describe how the sides of an obtuse triangle are related.

c. In an acute triangle, all three angles are acute. Based on your data in Table **1**, describe how the sides of an acute triangle are related.

d. Does the converse of the Pythagorean theorem appear to be true? Explain your response.

e. How could you use the converse of the Pythagorean theorem to determine if a corner is "square"?

Assignment

2.1 The scholars of Babylon recorded much of their mathematics on clay tablets. Many of these tablets still exist. One remarkable tablet, designated "Plimpton 322," dates from sometime around 1900 B.C. This fragment of a once larger tablet contains several columns of numbers.

The table below shows a portion of the contents of Plimpton 322, as well as some numbers (in parentheses) that historians believe were in the original tablet. Each row of three numbers in the table is a *triple*.

Side 1	Side 2	Side 3
(120)	119	169
(72)	65	97
(60)	45	75
(360)	319	481

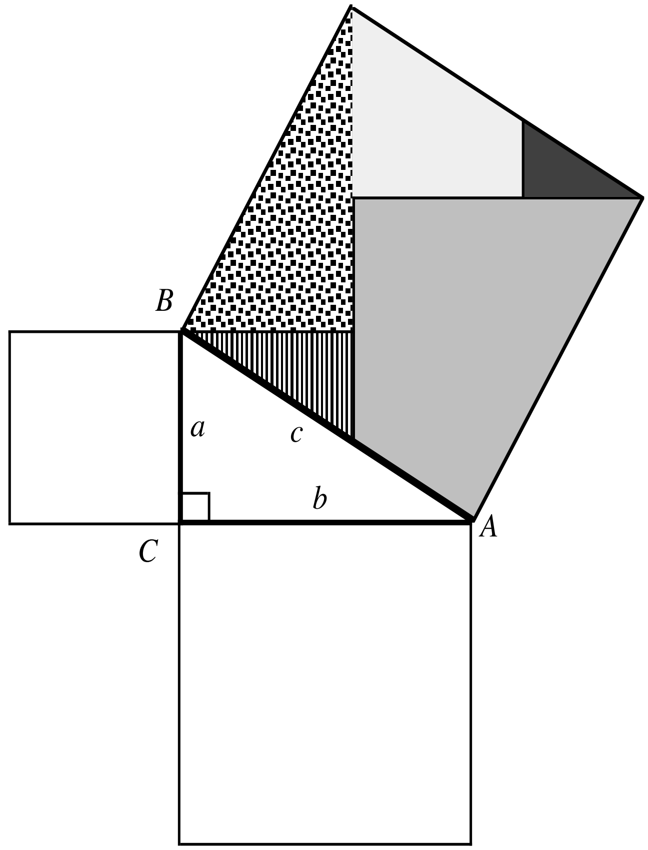
a. Use the converse of the Pythagorean theorem to determine what type of triangle—acute, obtuse, or right—is formed when each triple in the table represents the lengths of the sides of a triangle.

b. If you double each number in a triple from Plimpton 322, does the resulting triple still show the same relationship you described in Part **a**? Explain your response.

2.2 To create a surveying tool similar to the one the Egyptians might have used, mark 12 congruent segments on a string. Considering the length of each segment as 1 unit, form a square whose sides are exactly 3 units long. (Do not use protractors or other tools to help set your corners.) Verify that your result is a square.

2.3 Although the early Babylonians were aware of the relationship represented in the Pythagorean theorem, they had not yet developed the algebraic notation we use today. Instead, they used a sort of "geometric algebra."

The following diagram shows a right triangle *ABC* with squares constructed on each side. Five shaded pieces subdivide the largest square. Obtain a template from your teacher and cut out the shaded pieces from the largest square. Arrange the pieces so that they exactly fill the two smaller squares. Explain how your arrangement geometrically demonstrates the algebraic expression $a^2 + b^2 = c^2$.



2.4 Given three segments, it is not always possible to use them to form a triangle.

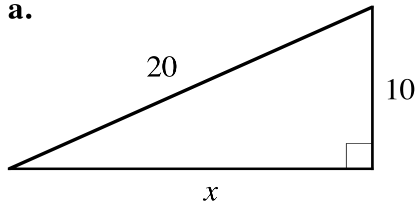
a. Identify at least three different sets of three segments which do not form triangles.

b. Explain why each set of segments does not form a triangle.

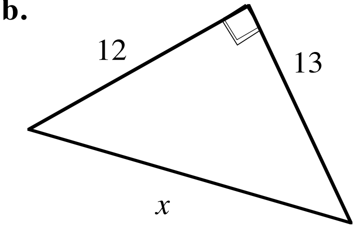
c. Describe the relationship that exists among the lengths of three segments when they cannot be used to form a triangle.

2.5 Determine the unknown length x in each of Parts **a–e** below. The pyramids in Parts **c–e** are regular square pyramids.

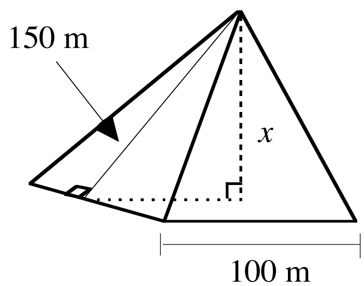
a.



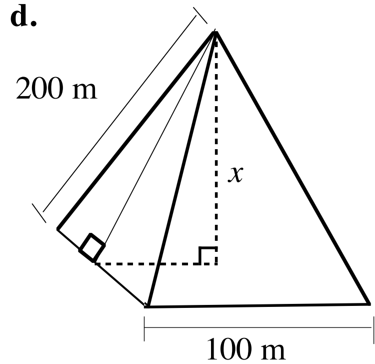
b.



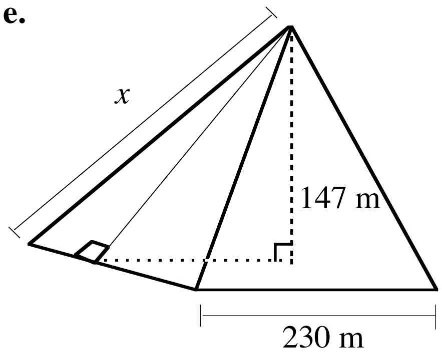
c.



d.



e.



2.6 Without constructing the triangles, determine if the three lengths in each of the following would form an acute triangle, an obtuse triangle, a right triangle, or no triangle.

a. 5, 12, 13

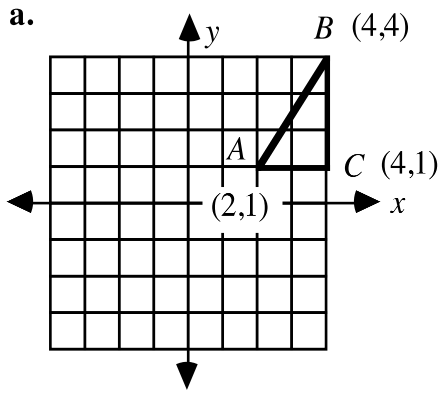
b. 6, 8, 12

c. 7, 10, 19

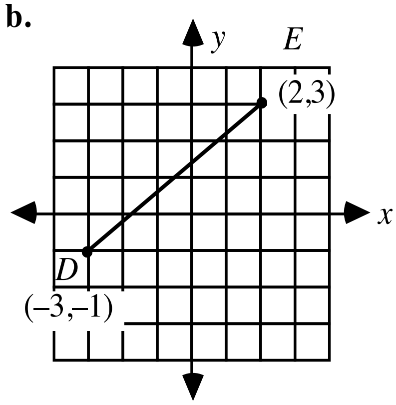
d. 13, 8, 15

2.7 Use the properties of right triangles to find the distance between each pair of points in Parts **a–d**. (In Part **a**, an appropriate triangle has been drawn for you. In Part **d**, use an expression to represent the distance.)

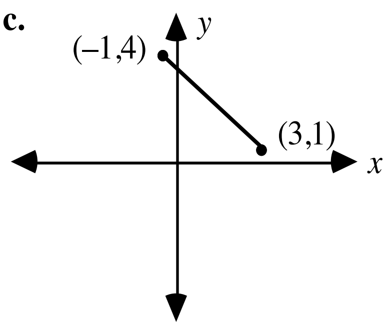
a.



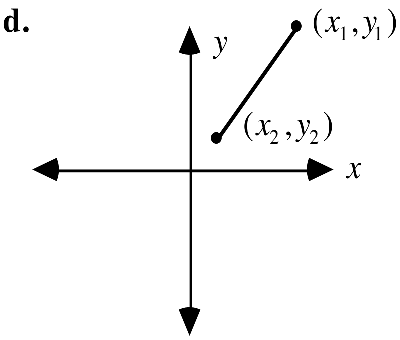
b.



c.

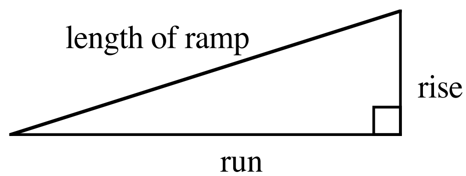


d.

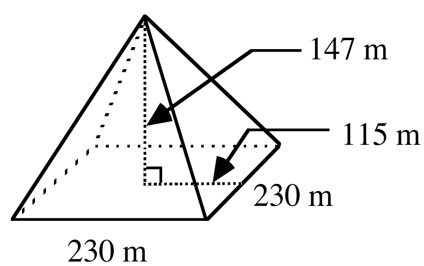


2.8 Some of the granite blocks used in the Great Pyramid are 8.2 m long and 1.2 m thick. Each block has a mass of 49 metric tons, and some of them are set 61 m above the base of the pyramid.

Egyptologists believe that the builders used inclined ramps to raise the stone blocks to these levels. As shown in the diagram below, some researchers have suggested that the incline ratio (slope or rise:run) of these ramps was 1:3.



The dimensions of the Great Pyramid are shown in the diagram below. If the capstone (the stone on the top) is 1 m tall, how long was the ramp required to raise it?



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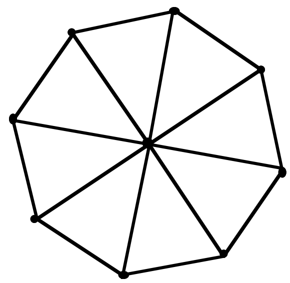
2.9 Leah and Harlan are building a concrete foundation for their new garage. The foundation is a square 8 m on each side. After marking the boundaries, they decide to measure the diagonals to confirm that the corners are right angles. If the corners are right angles, how long should the diagonals be?

2.10 After finishing their garage, Leah and Harlan decide to build a gazebo in their backyard. (A gazebo is a small, roofed structure with open sides.) As shown in the following diagram, the floor of their gazebo is a regular octagon. The floor supports lie along the perimeter and the diagonals of the octagon.



a. The diagonals of the octagon form eight congruent triangles. What type of triangles are these?

b. An altitude drawn from the center of the octagon to the base of each triangle is 2 m long. Each diagonal is 5 m long. Determine the total length of the wood required for the floor supports.

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Activity 3

Judging from the information recorded on Plimpton 322, the ancient Babylonians (and the ancient Egyptians too, perhaps) were aware of some uses for the ratios of the sides of right triangles. These ratios are important in many modern applications, from architecture to navigation. The study of these ratios and their properties is the focus of **trigonometry**. In this activity, you explore how the ancient Egyptians might have used some basic trigonometric ideas in the construction of the pyramids.

Exploration

The Great Pyramid of Giza has 203 distinct levels or steps, like those shown in Figure **9**. These steps are not all the same height.

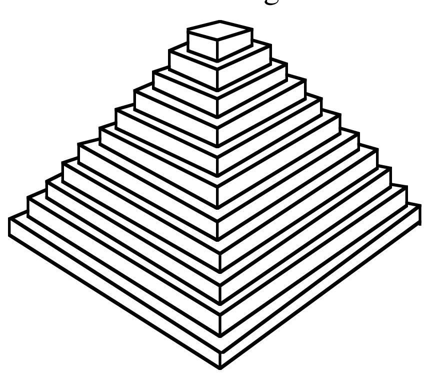


Figure 9: Pyramid steps

After the steps were built, casing blocks were used to give the pyramid's walls their smooth outer surface. To create a smooth slope, the outer faces of all the casing blocks—regardless of their height—had to be slanted at the same angle to their bases. Figure **10** shows a portion of a pyramid wall with three casing blocks.

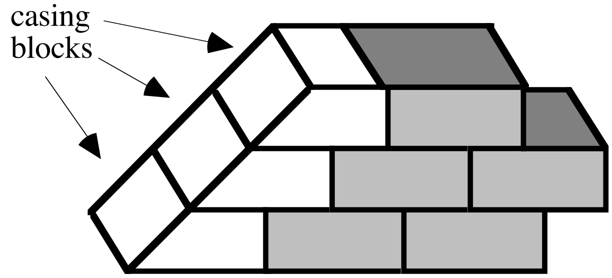


Figure 10: A portion of a pyramid wall with casing blocks

An angle formed by two intersecting planes is a **dihedral angle**. The measure of a dihedral angle is the measure of the angle whose sides are the two rays formed by the intersections of the faces and a plane perpendicular to the edge. As shown in Figure **11**, the measure of the dihedral angle of the Great Pyramid is 52˚.

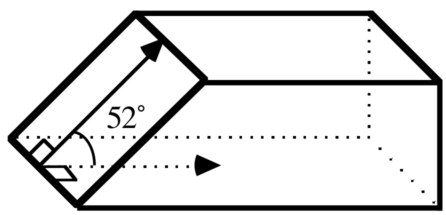


Figure 11: The dihedral angle of the Great Pyramid's casing blocks

Figure **12** shows cross-sectional drawings of the casing blocks for the bottom and top steps of the Great Pyramid. The bottom step is 141 cm thick, while the top step is 56 cm thick. The heights of the remaining 201 steps lie somewhere between these two values.

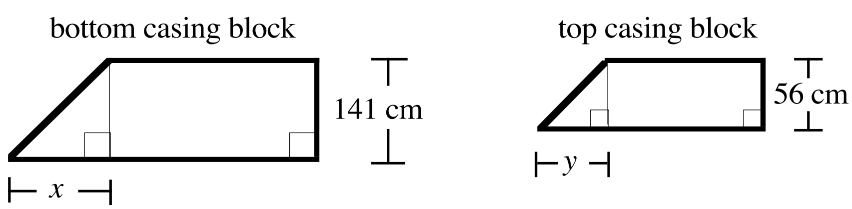


Figure 12: Cross-sections of two casing blocks

To maintain a constant dihedral angle measure of 52˚, the stonecutters had to be very precise. However, the Egyptians did not measure blocks using angles or degrees. Instead, they used a relationship called the *seqt*, the ratio of the horizontal "run" of a slope to its vertical "rise." The seqt of the bottom casing block in Figure **12**, for example, is $x/141$. In this exploration, you experiment with a ratio from trigonometry related to the Egyptian seqt.

a. 1. On a geometry utility, construct a horizontal line. Construct and label two points A and C on the line.

2. Construct a line perpendicular to $\overleftrightarrow{AC}$ through C. Label a point B on the perpendicular.

3. Construct $\overline{AB}$. Your construction should now resemble the one shown in Figure **13**.

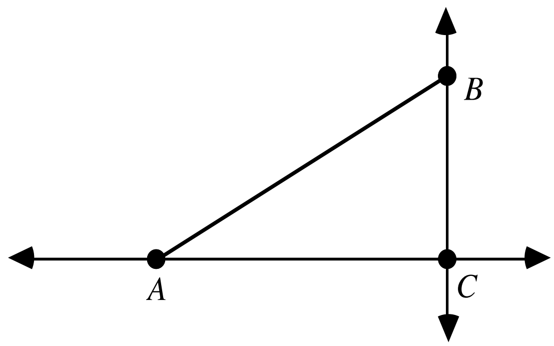


Figure 13: Construction of right triangle

4. Hide the lines and construct $\overline{AC}$ and $\overline{BC}$, as shown in Figure **14**. **Note**: Save this construction for use in Activity **4**.

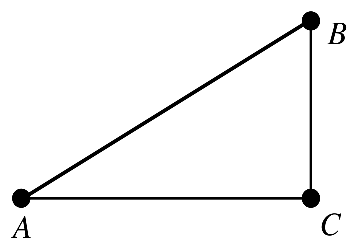


Figure 14: Right triangle

5. Drag point B until $m\angle A = 52^\circ$ (the measure of the dihedral angle of the Great Pyramid).

b. Measure the legs of right triangle ABC and calculate the ratio below.

$$\frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$

Note: For convenience, this ratio will be referred to in this module as the ratio "opposite/adjacent."

c. Drag point C along the horizontal line to form other right triangles.

1. As the lengths of the legs change, describe any patterns you observe in the ratios of the lengths.

2. Explain why these right triangles are similar.

d. Create a table with headings like those in Table **2**. Drag point *B* to change the measure of $\angle A$ and record the ratio of the length of the leg opposite $\angle A$ to the length of the leg adjacent to $\angle A$.

Table 2: Ratios of length of opposite leg to adjacent leg

$m\angle A$	opposite/adjacent
52°	
15°	
30°	
45°	
60°	
75°	

Mathematics Note

In any right triangle, the **tangent** of the measure of an acute angle is the ratio of the length of the leg opposite the angle to the length of the leg adjacent to the angle.

For example, Figure **15** shows a right triangle *ABC*.

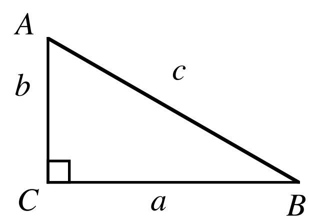


Figure 15: A right triangle *ABC*

In this case, the tangent of $\angle A$ is the ratio of *a* to *b*. This can be written as follows:

$$\tan \angle A = \frac{a}{b}$$

The tangent of $\angle B$ is the ratio of *b* to *a*. This can be written as $\tan \angle B = b/a$.

The tangent ratio is useful for determining unknown lengths in triangles. Consider the triangle in Figure **16**, for example.

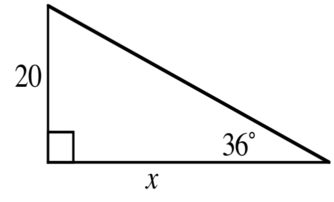


Figure 16: Triangle with the length of one leg unknown

Since $\tan 36^\circ$ equals the ratio of the length of the opposite leg to the length of the adjacent leg, the value of x can be found by solving the following equation.

$$\tan 36^\circ = \frac{20}{x}$$
$$x(\tan 36^\circ) = 20$$
$$x = \frac{20}{\tan 36^\circ} \approx 27.53$$

e. Add a column to Table **2** and label it "$\tan \angle A$." Using technology, find and record the tangent of each angle measure in the left-hand column. Compare your results with the ratios in the column labeled "opposite/adjacent."

f. Many calculators feature a key labeled "$\tan^{-1}$." This represents the inverse tangent command. Use technology to determine and record the inverse tangent of each value you determined in Part **e**. Describe any patterns you observe.

Discussion

a. In Part **c** of the exploration, what pattern did you observe in the ratio opposite/adjacent? Explain why this pattern occurs.

b. Compare the values you recorded in Table **2** with others in your class. Describe any similarities or differences you observe.

c. What happens to the tangent as the measure of an acute angle changes?

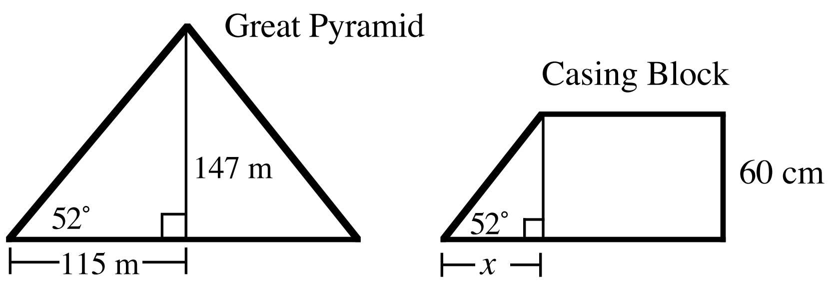
d. How is the tangent related to the Egyptian seqt?

e.
1. Given that the tangent of an angle is approximately 1.73, how could you use the values in Table **2** to determine the measure of the angle?
2. Describe how this process is related to the $\tan^{-1}$ key on a calculator.

f. Why would you expect $\tan 45^\circ$ to equal 1?

Assignment

3.1 The figure below shows a cross-sectional drawing of the Great Pyramid and a cross-sectional drawing of a casing block.



a. Explain why the right triangle shown in the casing block is similar to the right triangle shown in the Great Pyramid.

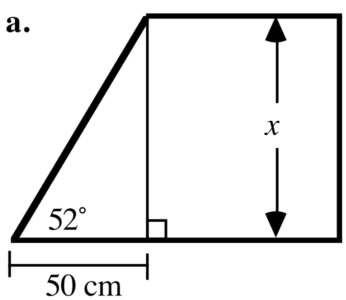
b. Determine the value of x in the casing block using each of the following methods:

1. proportions and similar triangles

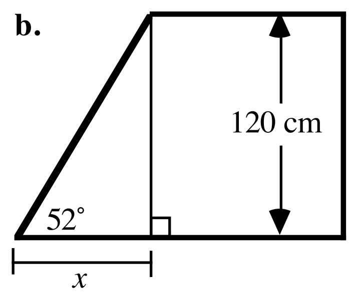
2. the tangent ratio.

3.2 The following diagrams show three casing blocks from the Great Pyramid. Determine the unknown length in each case.

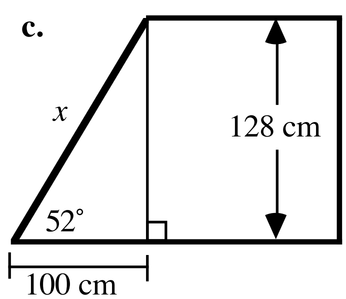
a.



b.



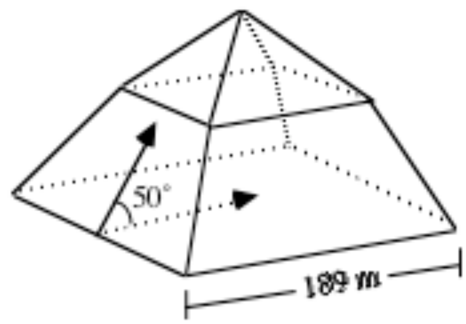
c.



3.3 Consider a pyramid whose base has the same dimensions as the Great Pyramid's. If its casing blocks have dihedral angle measures of 71° instead of 52°, how tall is the pyramid? (Hint: Use the given information to sketch an appropriate triangle.)

3.4 Describe what a calculator displays for tan 90° and explain why this occurs.

3.5 King Khufu's father, Snefru, built a pyramid at Dashur known as the Bent Pyramid. As shown in the diagram below, the base of the Bent Pyramid is a square 189 m on each side. The 50° dihedral angle measure at its base was abandoned after construction reached a height of 73.5 m. A smaller angle measure, about 37°, was used to finish the pyramid.

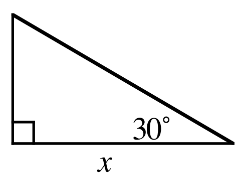


a. How tall would the Bent Pyramid have been if completed using the 50° dihedral angle measure?

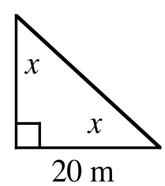
b. How tall is the actual Bent Pyramid?

3.6 Find the unknown lengths and angle measures in each of the following triangles.

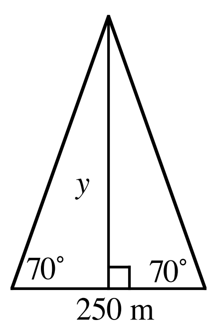
a.



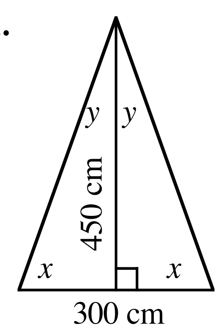
b.



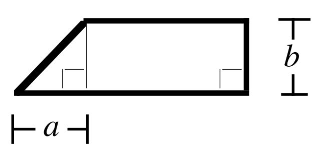
c.



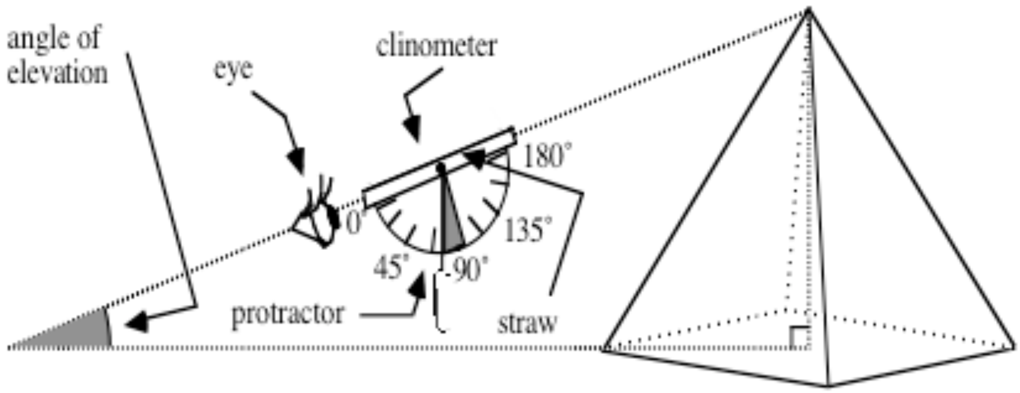
d.



3.7 The ancient Egyptians used the ratio a/b to measure the slant of the casing block below. Express this ratio in terms of the tangent of the measure of the block's dihedral angle.



3.8 A clinometer measures angle of elevation. The simple clinometer shown in the diagram below consists of a drinking straw, a weighted string, and a protractor. Using this tool, the angle of elevation (the shaded angle in the diagram) is the difference between the angle indicated by the weighted string and 90˚. Use this diagram to describe how a clinometer—along with some knowledge of trigonometry—could be used to calculate the height of a pyramid.



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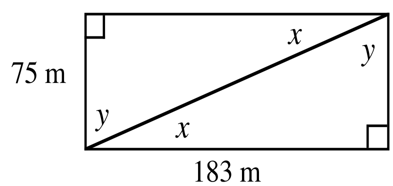
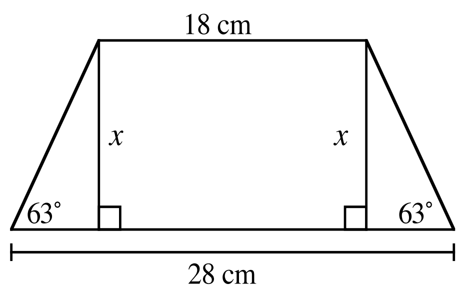
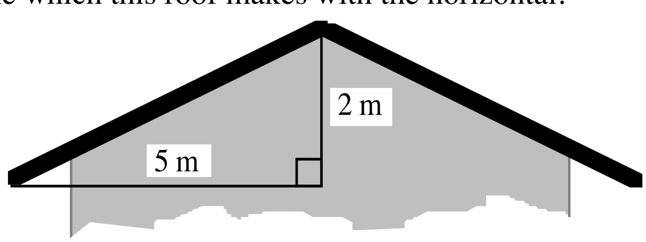
3.9 **a.** Find the unknown angle measures in the figure below.



b. Find the unknown lengths in the following figure.



3.10 When describing the roof of a house, the ratio of the vertical distance to the horizontal distance is often referred to as *pitch*. The figure below shows a roof with a pitch of $2/5$. Determine the measure of the angle which this roof makes with the horizontal.



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Research Project

Construct a clinometer like the one described in Problem **3.8**. Use your clinometer to estimate the height of your school, a flagpole, or some other tall object. Describe the methods you used to make each estimate. If possible, find the actual height of the object and compare this value to your estimate. Discuss some possible explanations for any differences you observe.

Activity 4

The word *trigonometry* is derived from the Greek words for "three-angle measurement." Right-triangle trigonometry involves the relationships among the sides and angles of right triangles. The tangent ratio is only one of these relationships.

As noted in Activity **3**, the Egyptians used the seqt. In Plimpton 322, the Babylonians worked with yet another such ratio. In this activity, you explore some other trigonometric ratios and examine their usefulness as problem-solving tools.

Exploration

a. Consider ΔABC in Figure **17**. In Activity **3**, you found that the tangent ratio for a given measure of $\angle A$ remains the same for all right triangles that contain that angle. List all the other ratios that you think will remain constant for $\angle A$.

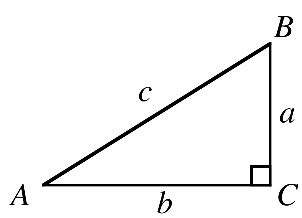


Figure 17: Right triangle *ABC*

b. Using your construction of a right triangle from Activity **3**, measure $\angle A$ and the sides of ΔABC. Calculate all the ratios that exist among the lengths of the sides.

c. Drag point C to create other, similar right triangles. Verify that the ratios you identified in Part **a** remain constant.

d. Create a table with headings like those in Table **3** below. Use your geometry utility to complete the table.

Table 3: Ratios of lengths in right triangles

$m\angle A$	**opposite/hypotenuse**	**adjacent/hypotenuse**
5°		
15°		
30°		
45°		
60°		
75°		

Mathematics Note

In any right triangle, the **sine** of the measure of an acute angle is the ratio of the length of the leg opposite that angle to the length of the hypotenuse.

For example, Figure **18** shows a right triangle *ABC*.

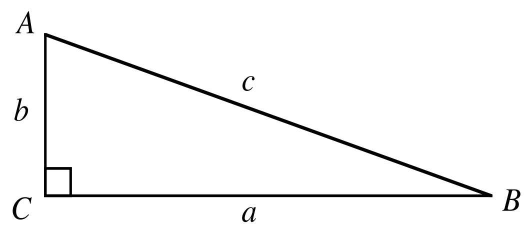


Figure 18: A right triangle *ABC*

In this right triangle, the sine of $\angle A$ is the ratio of a to c. This can be written as:

$$\sin\angle A = \frac{a}{c}$$

In any right triangle, the **cosine** of the measure of an acute angle is the ratio of the length of the leg adjacent to that angle to the length of the hypotenuse. In the right triangle in Figure **18**, the cosine of $\angle A$ is the ratio of b to c. This can be written as:

$$\cos\angle A = \frac{b}{c}$$

For example, consider right triangle *DEF* in Figure **19**. In this case, $\sin\angle E = 8/10 = 0.8$ and $\cos\angle E = 6/10 = 0.6$.

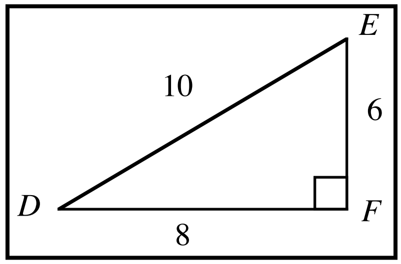


Figure 19: Right triangle *DEF*

e. Add two columns to Table **3** using the headings "$\sin\angle A$" and "$\cos\angle A$." Use technology to determine the sine and cosine of each angle measure in the left-hand column. Compare your results with the ratios in the columns labeled "opposite/hypotenuse" and "adjacent/hypotenuse."

f. Use your geometry utility to determine the minimum and maximum values of $\sin\angle A$ and $\cos\angle A$ in right triangle *ABC*.

g. Many calculators feature keys labeled "$\sin^{-1}$" and "$\cos^{-1}$." These represent the inverse sine and inverse cosine commands, respectively. Use the values you determined in Part **e** to experiment with these commands. Describe any patterns you observe.

Discussion

a. Describe how the two ratios in Table **3** are related to $\sin\angle A$ and $\cos\angle A$.

b. What trends did you observe in the values of sine and cosine as the measure of $\angle A$ increased?

c. What is the measure of $\angle A$ when $\sin\angle A$ and $\cos\angle A$ are equal?

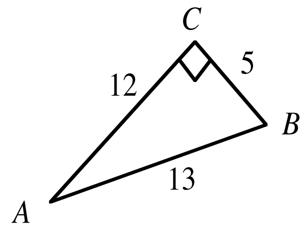
d. **1.** Describe the information you would need to find the measure of an angle using the inverse sine command.

2. Describe the information you would need to find the measure of an angle using the inverse cosine command.

e. In a right triangle *ABC* where $\angle C$ is the right angle, $\sin\angle A$ equals $\cos\angle B$. Explain why this occurs.

Assignment

4.1 Use the right triangle below to answer Parts **a–d**.



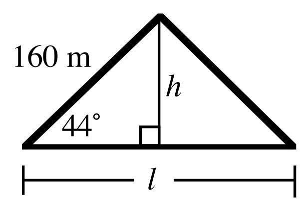
a. For $\angle B$, which trigonometric ratio is represented by $5/13$?

b. For $\angle A$, which trigonometric ratio is represented by $5/13$?

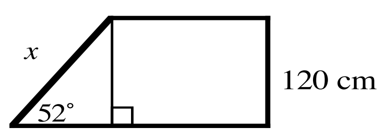
c. For $\angle B$, which trigonometric ratio is represented by $12/5$?

d. For $\angle A$, which trigonometric ratio is represented by $12/13$?

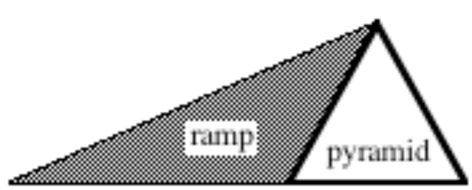
4.2 **a.** The figure below shows a cross-section of a pyramid. Use trigonometric ratios to determine the height *h* and the length *l*.



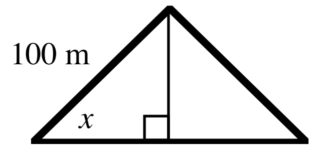
b. The figure below shows a cross-section of a casing block. Determine the unknown length *x*.



c. To move stones to the top of a pyramid, the Egyptians may have built a ramp of sand. If the pyramid in the diagram below is 100 m tall and the incline of the ramp is 25°, determine the length of the ramp.



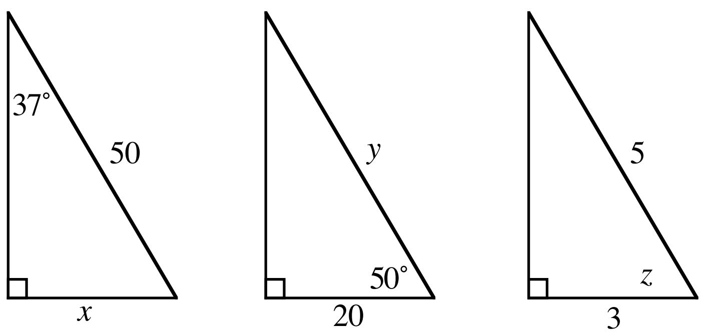
4.3 Use appropriate technology and the inverse cosine command to find the unknown angle measure x in the figure below.



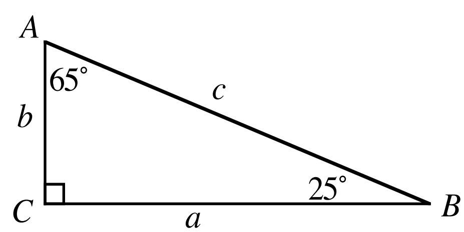
4.4 Use the definitions of the sine, cosine, and tangent ratios to prove that the following equation is true.

$$\tan \angle A = \frac{\sin \angle A}{\cos \angle A}$$

4.5 Determine the unknown length or angle measure in each of the three right triangles below.

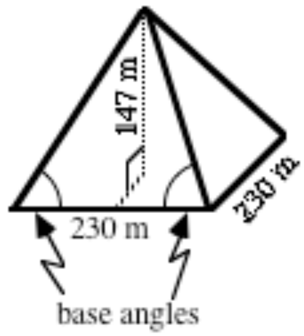


4.6 **a.** Using ΔABC below, explain why $\sin 65° = \cos 25°$ and $\tan 65° = 1/\tan 25°$.



b. Generalize your response to Part **a** for all pairs of acute angles whose measures add up to 90°. Use technology to verify this conjecture.

4.7 The diagram below shows the base angles on a face of the Great Pyramid. Determine the measure of these angles.



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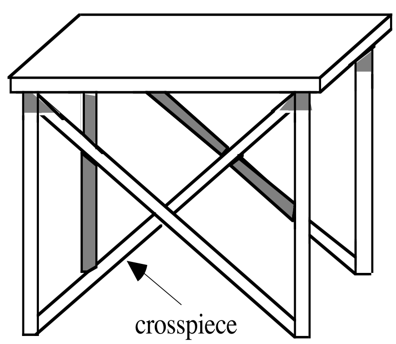
4.8 On a highway, a uniform grade of 4% means that there is a rise of 4 m for every 100 m of horizontal distance.

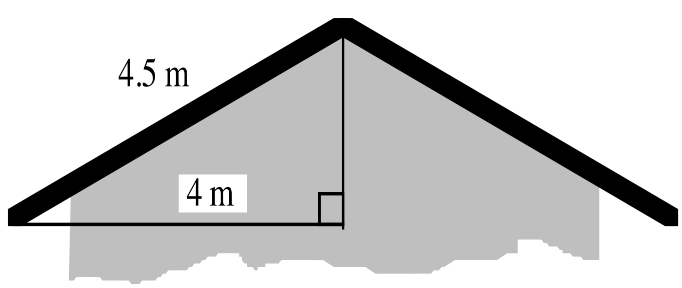
a. If you are driving up a road with a 4% grade, what angle measure does the path of your car make with the horizontal?

b. What is your change in elevation after traveling 32 m on this road?

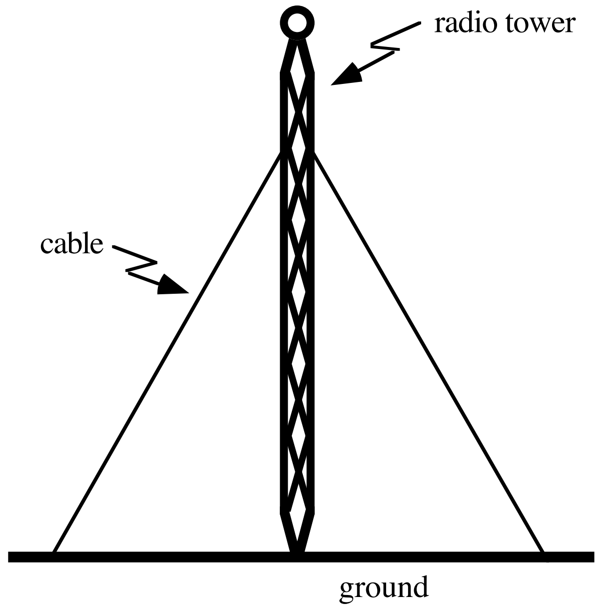
4.9 During construction of the Great Pyramid, workers used scaffolding to raise themselves to an appropriate height. The scaffold in the diagram below lifted stone masons to a height 4 m above the previous level. The measure of the angle between each crosspiece and vertical support is 60˚. What is the length of the crosspiece? Describe how you determined your response.



4.10 Determine the measure of the angle which the roof in the following diagram makes with the horizontal.



4.11 A radio tower is anchored to the ground by four cables, two of which are shown in the figure below. Each cable is bolted to the tower 20 m above the ground. The angles formed by the cables with the ground measure 60˚.



a. How many meters of cable have been used to anchor the tower?

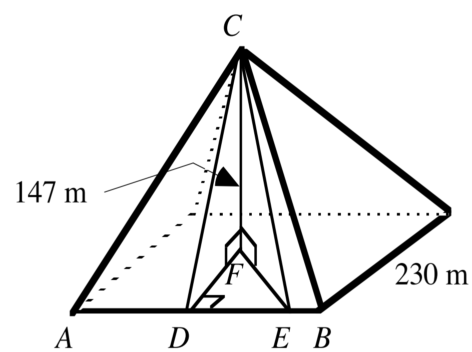
b. How far is it from the bottom of the tower to the anchor point of each cable?

* * * * * * * * * *

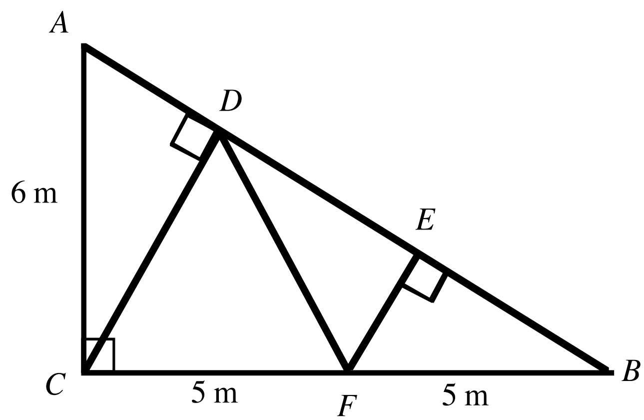
Summary Assessment

1. The annual flooding of the Nile River was an important part of life for the ancient Egyptians. Imagine that you must determine the width of the Nile at its crest near the Great Pyramid. You know the distance from the edge of the pyramid to the near bank is 1 km, and that you are 3.5 km from the opposite bank. From where you are standing, the angle of elevation to the pyramid's peak is approximately 1.5˚. Use the figure below to help calculate the width of the Nile.

2. In the diagram of the Great Pyramid below, $\angle CDF$ and $\angle CEF$ are angles with one side in the plane of the pyramid's base and the other in the plane of one of its faces. The dihedral angle *CDF* measures 52˚. The length of $\overline{DE}$ is 77 m. Triangles *CFD*, *CFE*, and *FDE* are all right triangles. Use the Pythagorean theorem and trigonometric ratios to show that the measure of the dihedral angle is different from $m\angle CEF$.

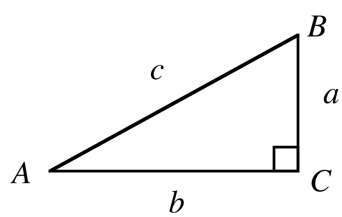


3. A truss is a combination of beams used to support the roof of a building. The diagram below shows some dimensions for a truss for a house. Use the measures indicated in the diagram to determine the lengths of segments *EF*, *DF*, and *CD*.



Module Summary

- Two figures are **similar** if they have the same shape and corresponding sides are proportional.
- In plane geometry, if the angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar. This is referred to as the **Angle-Angle-Angle (AAA) Property**.
- The symbol used to indicate congruence is $\cong$.
- The **Pythagorean theorem** states that, in a right triangle, the square of the length of the longest side (the hypotenuse) equals the sum of the squares of the lengths of the other sides (the legs).
- The **converse** of a statement in the form "If A, then B" is the statement "If B, then A." The converse of a true if-then statement may or may not be true.
- An angle formed by two intersecting planes is a **dihedral angle**. The measure of a dihedral angle is the measure of the angle whose sides are the two rays formed by the intersections of the faces and a plane perpendicular to the edge.
- In any right triangle, the **tangent** of the measure of an acute angle is the ratio of the length of the leg opposite the angle to the length of the leg adjacent to the angle. For right triangle *ABC* in the figure below, $\tan \angle A = a/b$.



- In any right triangle, the **sine** of the measure of an acute angle is the ratio of the length of the leg opposite that angle to the length of the hypotenuse. For right triangle *ABC* in the figure above, $\sin \angle A = a/c$.
- In any right triangle, the **cosine** of the measure of an acute angle is the ratio of the length of the leg adjacent to that angle to the length of the hypotenuse. For right triangle *ABC* in the figure above, $\cos \angle A = b/c$.
- The sine, cosine, and tangent are **trigonometric** ratios.

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