# Making Concessions

If there's going to be a prom this spring, the class needs to raise some cash. What to sell? How much to buy? How much to charge? Unless the class makes good choices, the big spring dance could be doomed.

*Bonnie Eichenberger • Paul Swenson*

# Making Concessions

## Introduction

Even before the class treasurer had made her report, everyone knew that cash was in short supply. Without more money, the spring dance would have to be canceled. The class discussed several fund-raising suggestions before deciding to sell concessions at the remaining home sporting events.

Since other school organizations already controlled the candy sales, the class would have to sell something else. Before they could start this project, however, they needed to answer some key questions. What should they sell? How much should they sell it for? How much could they afford to buy? And how could they make the most profit?

Anyone who opens a store or restaurant must consider these same questions. In the following activities, you examine some methods which could help the class make sound business decisions.

## *Activity 1*

To increase their chances for success, businesses gather as much information as possible before making important decisions. Before entering a new market or introducing a new product, businesses often conduct surveys or **feasibility studies**. These studies may identify both potential customers and appropriate prices.

## Exploration

When the class conducted a feasibility study, they found that students at their school were most interested in buying pizza, canned soda pop, and nachos. The class has asked you to make some recommendations on the remaining questions.

They have about $100 to spend on pizzas, pop, and nachos, and want to earn as much money as possible. Each pizza contains eight slices. Soda pop comes in six-packs. Nachos must be purchased in kits consisting of four bags of tortilla chips and one container of cheese. Each kit makes 16 servings.

- **a.** Estimate the cost to the class of each pizza, six-pack, and nacho kit.
- **b.** Describe how the class should sell each item—by weight, volume, piece, or some other method.
- **c.** Determine how much they should charge for each item sold.
- **d.** Keeping in mind the $100 limit, how many of each item—pizza, pop, and nachos—should the class purchase for resale?

**e.** Record your decisions from Parts **a–d** in a table with headings like those in Table **1** below. Use this information to estimate the potential profit for the class.

**Table 1: Estimated profit for concessions**

| Item | How Sold | Selling Price | Cost per Item | Quantity Purchased | Amount Spent | Potential Profit |
|---|---|---|---|---|---|---|
| pizza | | | | | | |
| pop | | | | | | |
| nachos | | | | | | |
| | | | | **Total** | | |

**f.** Summarize your recommendations in a report. **Note:** Save your report for use in the assignment.

## Discussion

**a.** What difficulties did you encounter in reaching your decisions?

**b.** What additional information would have been helpful?

**c.** What other questions might be important to ask in a feasibility study?

**d.** What assumptions did you make when estimating potential profit?

**e.** Compare your recommendations with those of others in the class.

1. For each item, what is the least quantity recommended for purchase?
2. For each item, what is the greatest quantity recommended for purchase?
3. What is the range of estimated profit?

**f.** What would be the least possible profit in this situation? Explain your response.

## Assignment

**1.1** **a.** Assuming it is possible for the class to sell none of the items they purchase, what is the range of their estimated profit?

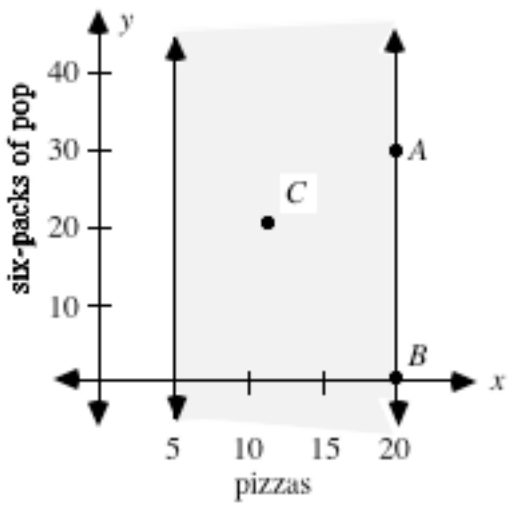
**b.** Represent the range of estimated profit on a number line.

**c.** Write an inequality to represent the range for estimated profit.

**d.** In your response to Part **c**, did you use the symbols $<$ and $>$ or the symbols $\leq$ and $\geq$? Explain how you made your choice.

**1.2** Suppose the class recommended that 5–20 pizzas should be purchased for resale. The illustration below shows a graphical representation of this information on an *xy*-coordinate system.

The recommended number of pizzas is represented on the *x*-axis. The recommended number of six-packs—which has not yet been determined—is represented on the *y*-axis.




**a.** Interpret the meaning of the coordinates for each of the following:

**1.** point *A*

**2.** point *B*.

**b.** If point *C* represents any point in the shaded region, describe its possible range of coordinates.

**c.** Is there any portion of the shaded region in which the coordinates of the points are not relevant to this setting? Explain your response.

**d.** Suppose the class recommended that 10–40 six-packs of pop be purchased. On a copy of the coordinate system shown above, represent the range of six-packs that might be ordered.

**e.** Graph the intersection of the region that describes the recommended range of 5–20 pizzas with the region that describes the recommended range of six-packs.

**1.3** In Part **e** of the discussion, your class described its recommended range for each item.

**a.** On a coordinate system like the one in Problem **1.2**, graph the recommended ranges for pizzas and six-packs. Represent the number of pizzas on the *x*-axis and the number of six-packs on the *y*-axis.

**b.** Describe what the graph represents in terms of items to be purchased.

**1.4** Repeat the process described in Problem **1.3** for each of the following.

**a.** Graph the recommended ranges for six-packs and nacho kits. Represent the number of six-packs on the $x$-axis and the number of nacho kits on the $y$-axis.

**b.** Graph the recommended ranges for pizzas and nacho kits. Represent the number of pizzas on the $x$-axis and the number of nacho kits on the $y$-axis.

## Mathematics Note

A **feasible region**, or set of **feasible solutions**, consists of all the points that satisfy the limitations, or linear **constraints** of a problem. The vertices of the feasible region are **corner points**.

For example, Figure **1** shows the feasible region for the constraints $x \geq 1$, $x \leq 4$, $y \geq 0$, and $y < 3$.

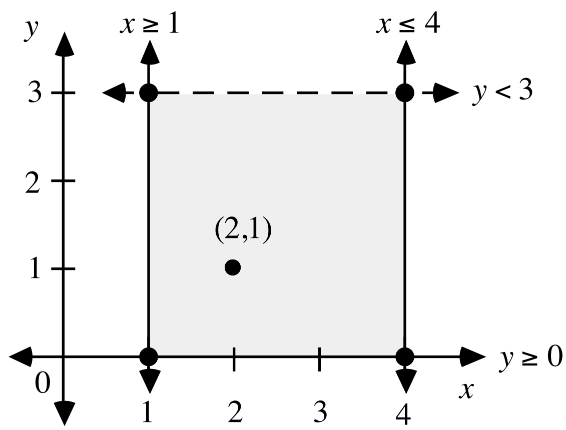



**Figure 1: A feasible region**

The $x$-coordinates of points in the shaded region are greater than or equal to 1 and less than or equal to 4. The $y$-coordinates of these points are greater than or equal to 0 and less than 3. The corner point with coordinates (4,0) is in the feasible region. The corner point with coordinates (4,3) is not in the feasible region.

**1.5** Use your graph from Problem **1.2e** to complete Parts **a–c** below.

**a.** Write inequalities to describe the intersection of the two graphs (the feasible region).

**b.** In your inequalities from Part **a**, did you use the symbols < and > or the symbols $\leq$ and $\geq$? Explain your choice.

**c.** Do all the points in the feasible region indicate reasonable values considering the problem setting? Explain your response.

**1.6** The class has decided to start concession sales using only pizza and pop.

**a.** Estimate a reasonable cost for each pizza and six-pack. You may wish to consult local merchants on prices.

**b. 1.** Write inequalities to represent the number of pizzas that can be purchased for \$100 or less, where $x$ represents number of pizzas. (Your inequalities should indicate that the class cannot purchase a negative number of pizzas.)

**2.** Write inequalities to represent the number of six-packs that can be purchased for \$100 or less, where $y$ represents number of six-packs. (Your inequalities should indicate that the class cannot purchase a negative number of six-packs.)

**3.** Graph these inequalities on a two-dimensional coordinate system where $y$ represents number of six-packs and $x$ represents number of pizzas.

**c. 1.** In Part **b**, you assumed that the entire \$100 could be spent on either pizza or six-packs. Write an inequality in which \$100 is the limit for the total cost of both items.

**2.** Graph this constraint on the same coordinate system as in Part **b**. You may need to solve the inequality for $y$ before graphing.

**d.** Label the feasible region on your graph and explain what it represents in this situation.

**e. 1.** Determine the coordinates of the corner points of the feasible region and describe the method you used to identify them.

**2.** Describe at least one other method you could use to determine the coordinates of the corner points.

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**1.7** A student newspaper plans to conduct a survey on dating. The editors would like to poll at least 30 boys and 30 girls, but no more than a total of 120 students. To reflect their ratio in the school population, the survey group should consist of at least 5 girls for every 7 boys.

**a.** Write an inequality to represent each of the following:

**1.** the number of boys to be surveyed

**2.** the number of girls to be surveyed

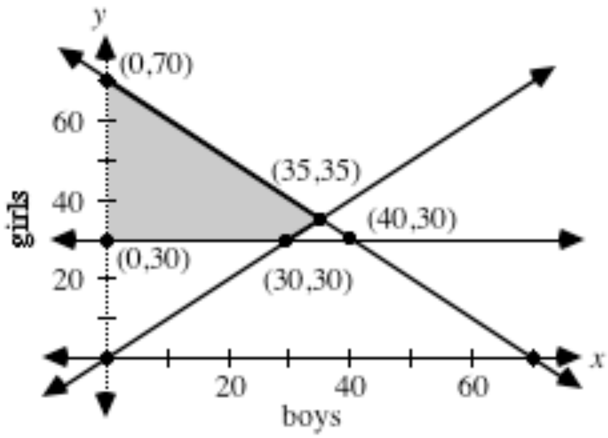
**3.** the total number of students to be surveyed

**4.** the ratio of the number of girls to be surveyed to the number of boys to be surveyed.

**b.** Graph the inequalities from Part **a** on a single coordinate system. Identify the feasible region.

**c.** Determine the coordinates of the corner points and identify those that are included in the feasible region.

**1.8** The feasible region shown in the graph below is defined by four constraints.




**a.** Write each constraint as an inequality.

**b.** Identify the corner points that are included in the feasible region.

**c.** What might these constraints represent in terms of the number and gender of students in a newspaper survey?

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## *Activity 2*

Business analysts often use feasible regions to examine all the possible solutions to a problem—and eliminate those that do not satisfy one or more constraints. These constraints might include limitations on raw materials, on production costs, or on manufacturing output. The relationships among these constraints affect a company's potential profit.

One method for analyzing potential profit is **linear programming**. To use this method, the constraints on a situation must all be linear. After determining a feasible region, an **objective function**—in this case, an equation representing profit—can be evaluated for points in this region. Using this process, it is possible to identify the conditions which will result in the maximum potential profit.

Linear programming also can be used to identify the conditions which will result in minimum values, such as production costs or manufacturing time. In this activity, you use an objective function to determine the maximum profit the class can make from their concession stand.

## Exploration 1

Suppose that the class makes a profit of \$3.00 on each pizza and \$2.00 on each six-pack. If $x$ represents number of pizzas, $y$ represents number of six-packs, and $P$ represents profit, the objective function in this situation is $P = 3x + 2y$.

**a.** Substitute four different values for $P$ in the objective function $P = 3x + 2y$. Use technology to graph these four equations.

**b.** **1.** What similarities or differences do you observe among the graphs?

**2.** How is the value of $P$ related to the $y$-intercept of $P = 3x + 2y$?

## Discussion 1

**a.** What similarities did you observe in the graphs of the four equations?

**b.** Why do you think these similarities occur?

**c.** As the values of $x$ increase in any given profit equation, does the profit increase or decrease? Explain your response.

**d.** In relation to the other graphs, where is the graph that indicates the greatest profit? Why do you think this occurs?

**e.** In relation to the other graphs, where is the graph that indicates the least profit?

## Exploration 2

One method for determining the maximum profit involves substituting the coordinates of a point in the feasible region in the objective function, then calculating the resulting profit. This process is continued for each point until the maximum profit is found. Since it requires checking every point in the feasible region, this method can require an unreasonable amount of time. In this exploration, you develop an alternative method for finding maximum profit.

**a.** Suppose that the class can buy pizzas for \$5.00 each and soda pop for \$2.00 per six-pack. Given that they can spend no more than \$100.00, the constraints on this situation are $0 \le x \le 20$, $0 \le y \le 50$, and $5x + 2y \le 100$, where $x$ represents number of pizzas and $y$ represents number of six-packs.

Use technology to graph the feasible region described by these constraints. **Note:** The technology you choose may not be able to graph the boundary of a constraint that is not a function. In this case, it may be possible to "draw" a line to represent that boundary.

**b.**

1. Select a point in the feasible region. Use the objective equation $P = 3x + 2y$ to calculate the profit at that point.
2. Substitute the profit calculated in Step **1** for $P$ in $P = 3x + 2y$. Graph this equation on the coordinate system from Part **a**.

**c.**

1. As you observed in Exploration **1**, the graphs of $P = 3x + 2y$ for all values of $P$ are parallel.

   Use this fact to help you find the location, with respect to the feasible region, of the line that indicates the maximum profit.
2. Determine the maximum profit in this situation.

**d.** Compare the value you found for the maximum profit with those of others in the class.

## Discussion 2

**a.**

1. What is the minimum number of boundary lines required to determine a vertex of a feasible region?
2. What is the minimum number of equations required to identify the coordinates of a vertex?

**b.** Depending on the technology you used, it may not have been possible to graph all of the boundaries of the feasible region. Describe any problems you encountered when graphing the feasible region and explain why they occurred.

**c.** Where did the line that indicates maximum profit intersect the feasible region?

**d.** In the module "Under the Big Top but Above the Floor," you used the substitution method to solve systems of equations. Describe how you could use this method to find the coordinates of the corner points of a feasible region.

**e.** How could you determine the maximum profit without substituting the coordinates of every point in the feasible region in the objective equation?

**f.** Suppose that the graph of the objective function was parallel to the boundary that contained the point where the maximum profit occurred. How would this affect your ability to determine the maximum value?

**g.** Given the objective function and the coordinates of the point at which the maximum profit occurs, how could you determine the maximum profit?

## Mathematics Note

When the corner points are part of the feasible region, the **corner principle** provides a method of finding the maximum or minimum values of an objective function. According to this principle, the maximum and minimum values of an objective function occur at the corner points of the feasible region.

For example, Figure **2** shows a feasible region with five corner points. According to the corner principle, the maximum value of an objective function is located at one of the corner points.

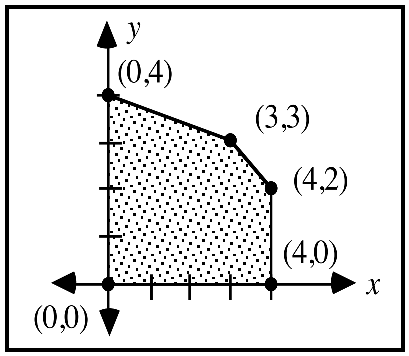



**Figure 2: Feasible region with five corner points**

Given the objective function $P = 2x + y$, the maximum value can be found by substituting the coordinates of each of the five corner points in this equation, then comparing the resulting values for $P$.

As shown in Figure **3** below, the maximum value of 10 occurs at the corner point with coordinates (4,2).

| **Coordinates ($x,y$)** | **Objective Function ($P = 2x + y$)** |
|---|---|
| (0,0) | $2(0) + 0 = 0$ |
| (0,4) | $2(0) + 4 = 4$ |
| (3,3) | $2(3) + 3 = 9$ |
| (4,2) | $2(4) + 2 = 10$ |
| (4,0) | $2(4) + 0 = 8$ |

**Figure 3: Substituting the coordinates of corner points in the objective function**

The minimum value also can be found using the corner principle. For the objective function $P = 2x + y$, the minimum value of 0 occurs at the corner point with coordinates (0,0).

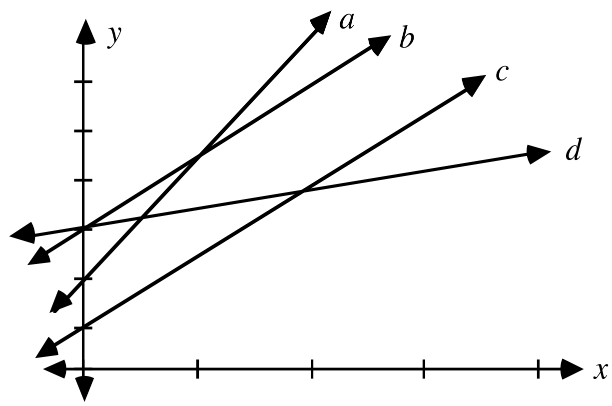
**h.** What if, by applying the corner principle, you obtained a solution which indicated that the class should buy $5/8$ of a six-pack of pop. Is this solution reasonable? Why or why not?

**i.** Suppose that the objective function for the feasible region in Figure **2** is $P = x + 3y$. In this case, the maximum value of $P$ is 12.

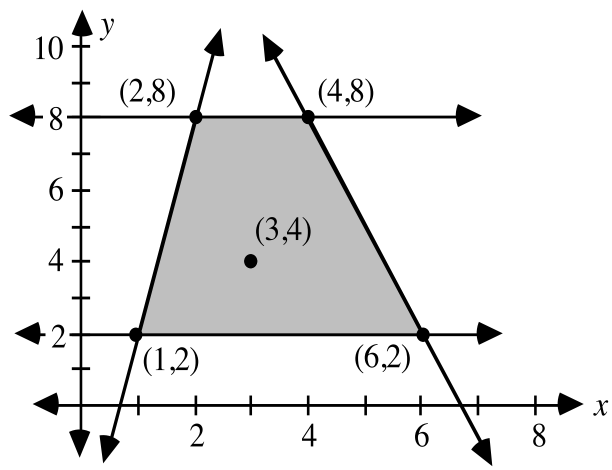
The graph of $12 = x + 3y$ contains the segment joining the vertices (0,4) and (3,3). What does this indicate about where the maximum value occurs?

## Assignment

**2.1** Which of the lines in the graph below, if any, appear to represent the same profit function for different profit values? Explain your response.




**2.2** The figure below shows a feasible region with four corner points.




**a.** Use the information in the graph to represent each of the following with a sketch and with an equation.

1. the maximum value of the objective function $P = 3x + 2y$
2. the maximum value of the objective function $P = 15x + y$
3. the minimum value of the objective function $P = 3x + 2y$
4. the minimum value of the objective function $P = 15x + y$

**b.** Describe how the slope of the objective function affects your use of the corner principle.

**2.3** According to the results of their feasibility study, the class should purchase no more than 16 eight-slice pizzas. The study also indicated that the class should buy no more than 22 six-packs of pop. Each pizza costs $4.96, while each six-pack costs $1.84. The class has $100.00 to spend on these items.

**a.** Write inequalities to represent the constraints in this situation.

**b.** Graph the feasible region defined by these constraints.

**c.** The class plans to charge $2.00 for a slice of pizza and $1.00 for a can of pop. How much profit will the class earn by selling one pizza? How much profit will they earn by selling one six-pack?

**d.** Use your responses to Part **c** to write an objective function for profit from selling pizzas and pop.

**e.** Find the maximum profit the class can earn if they sell all the pizza and pop they purchase. Graph the equation that represents the maximum profit on the coordinate system from Part **b**.

**f.** To realize this maximum profit, how many pizzas and six-packs should the class buy? Does this solution seem reasonable? Explain your response.

## Mathematics Note

The **substitution method** can be used to solve a system of linear equations. For example, Figure **4** shows a feasible region with five corner points.

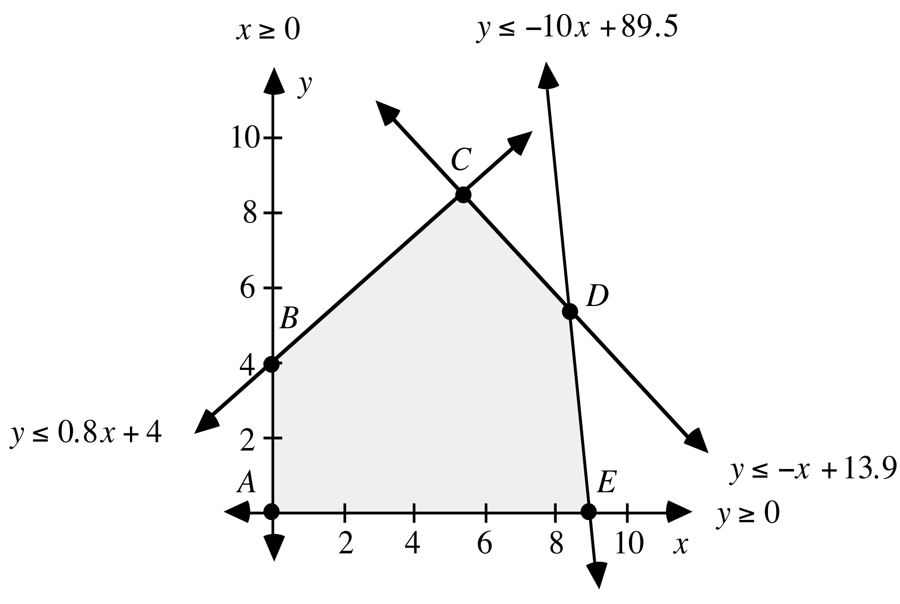



**Figure 4: A feasible region with five corner points**

In this case, the substitution method can be used to identify the coordinates of each corner point. For instance, point $C$ is the intersection of the boundaries defined by the system of equations $y = -x + 13.9$ and $y = 0.8x + 4$. To find the coordinates of $C$, the system can be solved as shown in the steps below.

| | |
|---|---|
| Substitute the value of $y$ in one equation into the other equation. | $0.8x + 4 = -x + 13.9$ |
| Solve for $x$. | $1.8x = 9.9$<br>$x = 5.5$ |
| Substitute this value for $x$ in one of the original equations. | $y = 0.8x + 4$<br>$= 0.8(5.5) + 4$<br>$= 8.4$ |

Since the solution to the system is (5.5,8.4), these are the coordinates of point $C$. To verify this solution, these coordinates should be substituted into each of the original equations.

**2.4** **a.** Verify the coordinates for point $C$ found in the mathematics note.

**b.** Determine the coordinates of the corner points $A, B, D$, and $E$ in Figure **4**. Verify your solutions by substituting these coordinates into the appropriate equations.

**2.5** **a.** Graph the feasible region described by the following constraints.

$$\begin{cases} 3x + y \leq 15 \\ x + 2y \leq 20 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

**b.** Determine the coordinates of the corner points of the feasible region.

**c.** Given the constraints in Part **a**, find the minimum value of each of the following objective functions:

1. $P = 2y - x$
2. $P = 3y + 2x$

**2.6** **a.** Repeat Problem **2.3** if each pizza costs the class \$6.00 and each six-pack costs \$2.00.

**b.** How would you try to find the maximum profit if the corner principle resulted in a solution that requires the purchase of fractional amounts of pizza or six-packs?

**c.** What do you think a businessperson might do if faced with the situation in Part **b**?

**2.7** Imagine that your class decides to sell nachos and canned soft drinks as a fund-raising project. You have \$100.00 available to purchase these items. Determine the maximum potential profit for the class. Describe any assumptions you make in analyzing this situation.

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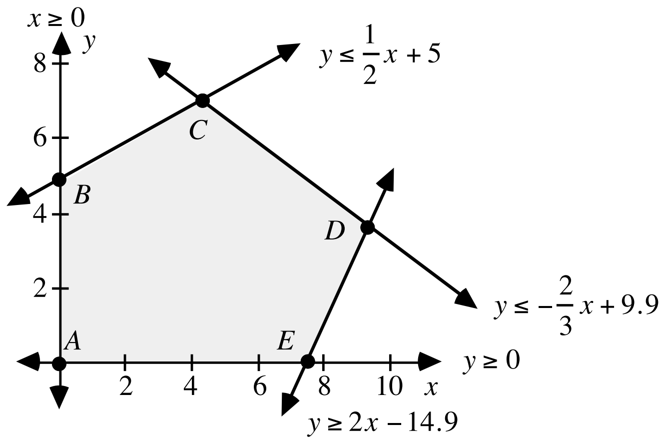
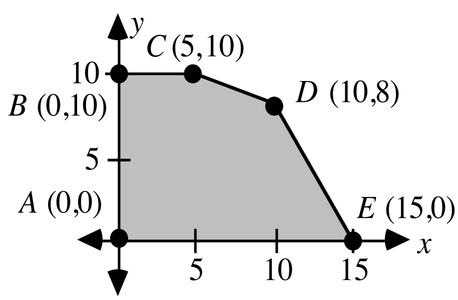
**2.8** The feasible region shaded in the following graph is defined by five constraints. Determine the coordinates of the corner points of this region.




**2.9** The following graph shows a feasible region with five corner points.




**a.** **1.** Which point in the feasible region results in the maximum value of the objective function $P = 5x + 5y$?

**2.** Which point in the feasible region results in the minimum value of the objective function $P = 5x + 5y$?

**b.** **1.** Determine the slope of the line that passes through points $D$ and $E$.

**2.** Determine the value of the objective function $P = 8x + 5y$ at $D$ and $E$.

**3.** Write $P = 8x + 5y$ in slope-intercept form and identify the slope.

**4.** Describe how your responses to Steps **1** and **2** are related.

**5.** A graph of the objective function $P = 8x + 5y$ for three values of $P$ would consist of three lines. Describe how these lines would be related to $\overline{DE}$.

**c.** Find an example of an objective function that has its maximum value at each of the following:

1. both point $D$ and point $C$
2. point $C$ only
3. point $E$ only.

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## Research Project

Plan a fund-raiser for your mathematics class that involves selling two items at a dance, game, or other school-related activity. Design a survey to determine what two items to sell, how much to charge for each item, and how much of each item to purchase. Administer the survey. Assume that you can interest enough investors in your plan to raise $50.00. Use this constraint and the results from the survey to determine the constraints for the fund-raiser.

Use linear programming to determine the maximum profit that each investor could expect to earn. Prepare a report for the class to persuade each student to become an investor. You may wish to include the survey questionnaire, survey results, initial costs, expected profit, graphs, and sample advertising posters.

# *Activity 3*

In Activity **2**, you used the corner principle to determine the maximum or minimum value of an objective function. Identifying the coordinates of corner points often involves solving a system of equations, either by substitution or by some other method. In this activity, you use matrices to solve systems of linear equations.

## Exploration

Using matrices to solve a system of equations involves two steps: writing a matrix equation to represent the system, then solving the resulting matrix equation.

**a.** Consider the following system of equations:

$$\begin{cases} -8x + 10y = 40 \\ 10x + 10y = 139 \end{cases}$$

The **matrix equation** for this system is shown below. Simplify this equation by completing the matrix multiplication on the left-hand side of the equal sign.

$$\begin{bmatrix} -8 & 10 \\ 10 & 10 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 139 \end{bmatrix}$$

**b.** Compare the simplified matrix equation with the original system of equations. Describe any similarities you observe.

## Mathematics Note

Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

may be written as the matrix equation:

$$\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, matrix **X** is the **variable matrix**, since it represents the variables of the system, while matrix **C** is the **constant matrix**, since it represents the constants of the system.

For example, the system

$$\begin{cases} 3x + 4y = 22 \\ -5x + 2y = 1 \end{cases}$$

may be written as the following matrix equation:

$$\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} 3 & 4 \\ -5 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 1 \end{bmatrix}$$

The **multiplicative inverse** of matrix **M** is written $\mathbf{M}^{-1}$ and has the property that $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$, where **I** is the identity matrix for matrix multiplication. The identity matrix for $2 \times 2$ matrices is shown below.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For example, consider the following two matrices:

$$\mathbf{M} = \begin{bmatrix} -5 & 8 \\ -2 & 3 \end{bmatrix} \qquad \mathbf{M}^{-1} = \begin{bmatrix} 3 & -8 \\ 2 & -5 \end{bmatrix}$$

Since $\mathbf{M}^{-1}$ is the multiplicative inverse of **M**, their product is the identity matrix **I**:

$$\begin{bmatrix} 3 & -8 \\ 2 & -5 \end{bmatrix} \bullet \begin{bmatrix} -5 & 8 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**c.** Use technology to verify that the value of $\mathbf{M}^{-1}$ given in the mathematics note is correct.

**d.** Use technology to verify that $\mathbf{I} \bullet \mathbf{A} = \mathbf{A} \bullet \mathbf{I} = \mathbf{A}$ for two different $2 \times 2$ matrices **A**.

**e.** A linear equation of the form $a \bullet x = b$ can be solved for $x$ by multiplying both sides of the equation by the multiplicative inverse of $a$.

Similarly, a matrix equation of the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ can be solved for matrix **X** by multiplying both sides of the equation by the multiplicative inverse of **M**, as shown below, and using the associative property for matrix multiplication:

$$\mathbf{M}^{-1} \bullet (\mathbf{M} \bullet \mathbf{X}) = \mathbf{M}^{-1} \bullet \mathbf{C}$$

Note that each side of the equation must be multiplied on the left by $\mathbf{M}^{-1}$ and recall that $\mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{I}$.

1. Write the following system of linear equations as a matrix equation of the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$:

$$\begin{cases} 5x + 2y = 22 \\ 2x + y = 9 \end{cases}$$

2. Use technology to find the inverse of the coefficient matrix **M**.
3. Solve the matrix equation for the variable matrix **X** by multiplying each side by $\mathbf{M}^{-1}$. Write the resulting equation in the form below:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$$

4. Verify by substitution that $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$.
5. Write the solution of the system as an ordered pair.
6. Verify your solution by substituting these coordinates in the original equations.
7. Check the solution by graphing the two equations on the same coordinate system.

**f.** When a system of equations has no solution, it is said to be **inconsistent**.

1. Verify that the system shown below is inconsistent by graphing the two equations on the same coordinate system.

$$\begin{cases} x + 2y = 4 \\ 2x + 4y = 13 \end{cases}$$

2. When an attempt is made to solve an inconsistent system using matrices, one step of the process cannot be completed. Repeat Part **e** using the system of equations above until you discover where the process fails.

## Discussion

**a.** In Part **c** of the exploration, you used technology to verify that the value of $\mathbf{M}^{-1}$ given in the mathematics note was correct. Describe two different ways that you can use technology to make this verification.

**b.** If a system is inconsistent, what can you conclude about the coefficient matrix $\mathbf{M}$?

**c.** Describe the coefficient matrix for the following system of equations:

$$\begin{cases} 3x = 7 \\ 5x + 2y = 11 \end{cases}$$

**d.** When multiplying both sides of a matrix equation $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ by the multiplicative inverse of $\mathbf{M}$, each side of the equation must be multiplied on the left by $\mathbf{M}^{-1}$. Why is this necessary?

**e.** Matrices also can be used to solve systems of equations involving more than two equations and two unknowns. Consider the following system of three equations and three unknowns:

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 15 \\ x + 3y - 2z = -7 \end{cases}$$

Describe how you could rewrite this system as a matrix equation including a coefficient matrix, a variable matrix, and a constant matrix.

## Assignment

**3.1** **a.** Find the multiplicative inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$$

**b.** Verify that your response to Part **a** is correct.

**c.** Create a $2 \times 2$ matrix that does not have an inverse. Defend your response.

**3.2** Solve each of the following matrix equations and verify your solutions.

**a.** $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

**b.** $\begin{bmatrix} 5 & 2 \\ 4 & -2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \end{bmatrix}$

**c.** $\begin{bmatrix} 1 & 1 \\ 5 & -8 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

**3.3** Use matrices to solve the following system of equations:

$$\begin{cases} 80x + 33y = 466 \\ -20x + 73y = 46 \end{cases}$$

**3.4** **a.** Graph the feasible region described by the following constraints.

$$\begin{cases} 3x + 8y \le 48 \\ 3x + 2y \le 24 \\ -x + y \le 0 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

**b.** Determine the coordinates of the corner points of the feasible region.

**c.** Given these constraints, find the maximum value of the objective function $P = 2y + 3x$.

**3.5** Your class has $100.00 to buy pizza and canned soda pop for a concession booth. Each pizza costs $5.00 and each six-pack costs $2.00. The constraints on your purchases are listed below:

- no more than $100.00 can be spent on pizza and pop
- at least $10.00 must be spent on pop
- no more than $80.00 can be spent on pizza
- the amount spent on pizza cannot exceed 4 times that spent on pop.

**a.** Express these constraints as inequalities.

**b.** Graph the feasible region described by these constraints.

**c.** The class expects a profit of $11.00 on each pizza and $2.00 on each six-pack. Determine the maximum potential profit in this situation.

* * * * *

**3.6** The manager of a municipal water system must decide how to provide the town with at least $20 \bullet 10^6$ L per day. Some of this the water can be drawn from a nearby reservoir. The rest must be purchased from a company that pumps water from a large underground aquifer.

In order to maintain the level of the reservoir, no more than $12 \bullet 10^6$ L per day can be drawn from this source. The town's contract with the water company states that the city must purchase at least $14 \bullet 10^6$ L per day, but no more than $20 \bullet 10^6$ L per day.

**a.** Express these constraints as inequalities.

**b.** Graph the feasible region described by these constraints.

**c.** Water from the reservoir costs the town $80.00 for $1 \bullet 10^6$ L, while water from the water company costs $100.00 for $1 \bullet 10^6$ L.

1. Determine the town's smallest possible daily water bill.
2. Determine the town's largest possible daily water bill.

* * * * * * * * * *

## *Activity 4*

In Activity **1**, you explored the potential sale of three items at a concession stand: pizza, pop, and nachos. Do you think it is possible to use linear programming and the corner principle to determine the maximum profit in this situation?

### Discussion 1

**a.** When considering the sale of two items, the feasible solutions form a region on a coordinate plane. The boundaries of the region are formed by lines whose equations are determined by constraints.

Suppose that the class wanted to decide which two of the three items—pizza, pop, or nachos—to sell. How many coordinate planes would be needed to graph the feasible regions for all the possible pairs?

**b.** How many variables are necessary to model a situation in which three items are sold?

**c.** Is it possible to graph the constraints for all three items on a single coordinate plane? Explain your response.

**d.** Is it possible to represent the constraints on all three items on another type of coordinate system? If not, why not? If so, what geometric figure do you think could be used to represent the boundaries?

### Exploration

**a.** Cut the top, front, and one side from a cardboard box.

**b.** Use the remaining inside corner to create a three-dimensional coordinate system. As shown in Figure **5**, label the two axes that lie in the bottom of the box *x* and *y*, respectively. Label the vertical axis *z*. Divide each axis into intervals of at least 1 cm and label the intervals accordingly.

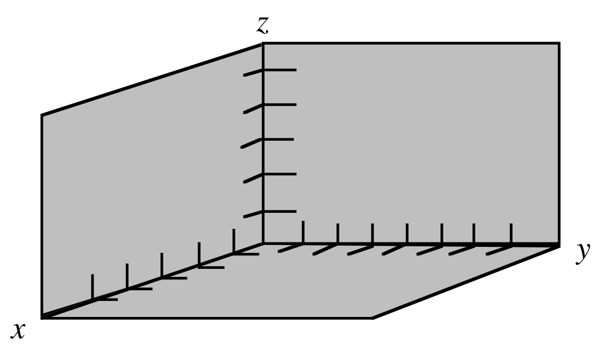



**Figure 5: A three-dimensional coordinate system**

**c.** The model shown in Figure **5** represents the first octant of a three-dimensional coordinate system. On the planes of the first octant, shade and label the regions defined by the following sets of constraints.

$$\text{Region A}: 0 \le x \le 6,\ 0 \le y \le 4,\ z = 0$$

$$\text{Region B}: 0 \le y \le 4,\ 0 \le z \le 5,\ x = 0$$

$$\text{Region C}: 0 \le x \le 6,\ 0 \le z \le 5,\ y = 0$$

**d.** Cut shapes that approximate these regions from leftover cardboard.

**e.**

1. Position the cutout of region A on your coordinate system to represent the constraints $0 \le x \le 6$, $0 \le y \le 4$, and $z = 1$. Note its position.
2. Move the same cutout along the $z$-axis to the position that represents the constraints $0 \le x \le 6$, $0 \le y \le 4$, and $z = 2$. Compare this position with the position in Step **1**.
3. Repeat the process described in Step **2**, increasing the value of $z$ by 1 unit each time, until the cutout is positioned at the edge of the shaded region on the $yz$-plane.

**f.** Tape the cutouts of regions A, B, and C in the first octant so that they form the boundaries of a shape that includes the set of feasible solutions described by $0 \le x \le 6$, $0 \le y \le 4$, and $0 \le z \le 5$.

**g.** Identify the corner points of the shape formed in Part **f**.

## Discussion 2

**a.** Describe the set of points obtained as you moved the cutout of region A in Part **e** of the exploration.

**b.**

1. What is the least number of planes that can intersect in exactly one point? Explain your response.
2. Imagine that the constraints of a problem indicate that four planes define the set of feasible solutions. Why is 4 the maximum number of corner points in this situation?
3. What is the maximum number of corner points if five planes define the feasible solutions? Explain your response.

**c.**

1. What shape is formed in Part **f** of the exploration? How many corner points does this shape have?
2. Would you expect this shape to be the same for every set of feasible solutions involving six planes? Why or why not?

**d.**

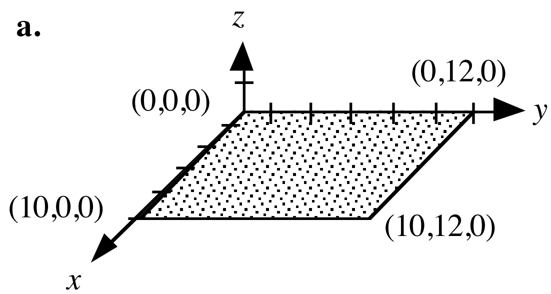
1. Where would you expect the maximum value of an objective function to occur in terms of the set of feasible solutions from Part **f**?
2. How could you verify the location of the maximum value?

**e.** Do you think it is possible to graphically represent a situation in which you consider more than three items for sale? Explain your response.

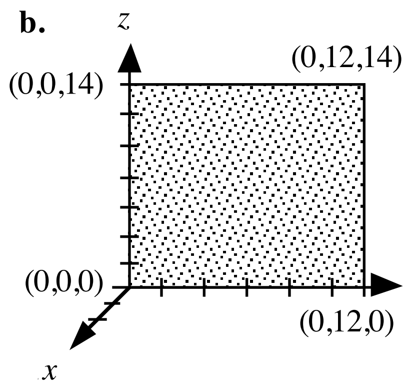
## Assignment

**4.1** In the three graphs below, $x$ represents the number of pizzas, $y$ the number of six-packs, and $z$ the number of nacho kits. For each graph, identify the items whose constraints are represented, and write inequalities to describe the shaded region.

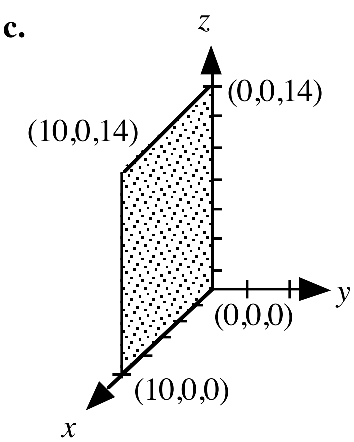
**a.**




**b.**




**c.**




**4.2** Describe what the inequalities in Problem **4.1** mean in terms of the number of pizzas, six-packs, and nacho kits the class should purchase.

**4.3** In Problem **4.1a**, the boundaries of the feasible region are line segments. The equations of the lines that form these boundaries are $x = 10$, $x = 0$, $y = 12$, and $y = 0$. You can use these equations to determine the intersection of this plane with other planes.

**a.** Write the equations that represent the boundary lines for the region in Problem **4.1b**.

**b.** Write the equations that represent the boundary lines for the region in Problem **4.1c**.

**c.** Describe how you could find the points of intersection of the boundary lines given only their equations.

**4.4** Sketch the shape that defines the feasible solutions when the constraints on all three items in Problem **4.1** are considered at once. List the coordinates of the corner points of this shape.

**4.5** Use the corner points from Problem **4.4** to find the maximum value for the objective function $P = x + y + z$. Describe the method you used to find the maximum value.

**4.6** Use matrices to find the intersection of the three planes represented by each of the systems of equations below. Identify any system that is inconsistent.

**a.** $3x - 2y + 3z = 0,\ -x + 4y + z = 8,\ x - 2y + z = 4$

**b.** $2x + y + 2z = 4,\ 3x - 6y + 2z = 12,\ 4x + 2y + 4z = 12$

**c.** $x + y = 3,\ 2x + z = 9,\ 3x - 4y + z = 20$

**4.7** Suppose that the class decides to sell pizza, pop, and nachos. Each pizza costs the class \$5.00 and each six-pack costs \$2.00. Nacho kits contain four bags of tortilla chips and one container of cheese. Each bag of tortilla chips costs \$2.00; each container of cheese also costs \$2.00.

There are eight slices in each pizza and the class plans to sell them for \$2.00 a slice. A can of pop sells for \$1.00. Each nacho kit contains enough chips and cheese to make 16 orders of nachos. One order of nachos sells for \$1.50.

**a.** Determine the profit on each pizza, six-pack, and nacho kit sold.

**b.** Let $x$ represent the number of pizzas, $y$ the number of six-packs, and $z$ the number of nacho kits purchased by the class. Write an objective function to calculate the maximum profit the class can expect to earn.

**4.8** **a.** The class in Problem **4.7** plans to spend no more than \$50.00 on any one of the three items to be sold: pizza, pop, and nachos. Write inequalities to describe these constraints.

**b.** How many planes enclose the feasible solutions for these constraints?

**c.** How many points of intersection must be examined to determine the maximum profit? Explain your response.

**4.9** Using the information determined in Problems **4.7** and **4.8**, find the maximum profit the class can expect to earn.

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**4.10** Using the cost of pizza, pop, and nacho kits from a local store and a \$50 spending limit for each item, find the maximum profit that your class could earn selling concessions. Record all assumptions you make in analyzing this situation.

**4.11** Use matrices to determine the solution to the system of equations below.

$$\begin{cases} x + y + z \le 15 \\ 2x + y + 2z \le 26 \\ 5x + 2y + 3z \le 43 \end{cases}$$

* * * * * * * * * *

## *Summary Assessment*

**1.** As shift manager of a fast-food restaurant, you must make sure that enough meals are prepared to serve customers quickly—but not so many that food is wasted.

During a typical lunch-time rush, you sell no more than 100 single burgers and no more than 90 double burgers. A single burger requires 100 g of hamburger, while a double burger requires 200 g. To help avoid spoilage, you plan to use no more than 20 kg of hamburger during the rush.

Before being served to a customer, each single burger requires 6 seconds in a microwave oven, while each double burgers requires 8 seconds. During the lunch-time rush, there are no more than 16 minutes of available time in the microwave.

The restaurant earns a profit of $0.70 on each single burger and $0.90 on each double burger. Assuming that customers purchase every burger prepared, determine how many of each type should be made in order to maximize profits.

**2.** As class president, you are analyzing the best way for your class to earn money with its concession stand. The class officers have decided to sell hot dogs, potato chips, and soda pop. The table below shows the cost and selling price for each of these items.

| **Item** | hot dog | bag of chips | can of pop |
|---|---|---|---|
| **Cost** | $0.15 | $0.20 | $0.35 |
| **Selling Price** | $1.00 | $1.00 | $1.00 |

The class has $108.00 in its fund-raising account. The total cost for hot dogs, chips, and pop cannot exceed this amount. Judging from last year's sales, the class will sell at least twice as many hot dogs as bags of chips, and at least 3 times as many cans of pop as the combined numbers of chips and hot dogs. Determine the profit at the corner point defined by these three constraints.

## *Module Summary*

- A **feasible region**, or set of **feasible solutions**, consists of all the points that satisfy the limitations, or linear **constraints** of a problem. The vertices of the feasible region are **corner points**.
- **Linear programming** is a method used to maximize or minimize an objective (such as profit), subject to linear constraints. The function that describes the objective is called the **objective function**.
- When the corner points are part of the feasible region, the **corner principle** provides a method of finding the maximum or minimum values of an objective function. According to this principle, the maximum and minimum values of an objective function occur at the corner points of the feasible region.
- The **substitution method** for solving a system of linear equations can be used to solve a system of equations involving any number of variables.
- Any system of linear equations of the form

  $$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

  may be written as the matrix equation:

  $$\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$$

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

  The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, matrix **X** is the **variable matrix**, since it represents the variables of the system, while matrix **C** is the **constant matrix**, since it represents the constants of the system.
- The **multiplicative inverse** of matrix **M** is written $\mathbf{M}^{-1}$ and has the property that $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$, where **I** is the identity matrix for matrix multiplication.

## Selected References

Tan, S. T. *Applied Finite Mathematics*. Boston, MA: PWS Publishing, 1994.