Crazy Cartoons

How are cartoon characters made? How do they change shape? How can you make an inanimate object appear to move? A cartoonist—or a math student—can help answer these questions.

Monty Brekke • John Carter • Janet Kuchenbrod

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Introduction

Think about your favorite animated cartoon characters. You've probably seen them stretched, flipped, smashed, and racing away in a blur of speed. In order to create these sequences, cartoonists once drew hundreds of individual frames.

In each frame, the size, shape, or position of the character was slightly altered. When the frames were transferred to film and run through a projector, the character appeared to move.

Although many of today's animated cartoons are created with computers, the process still follows much of the same logic. In this module, you develop the geometry skills that can help you create your own cartoon.

Discussion

a. Figure **1** below shows several different images of a cartoon bug. Describe how the bug in frame A could be changed to create the image in each of the other frames.

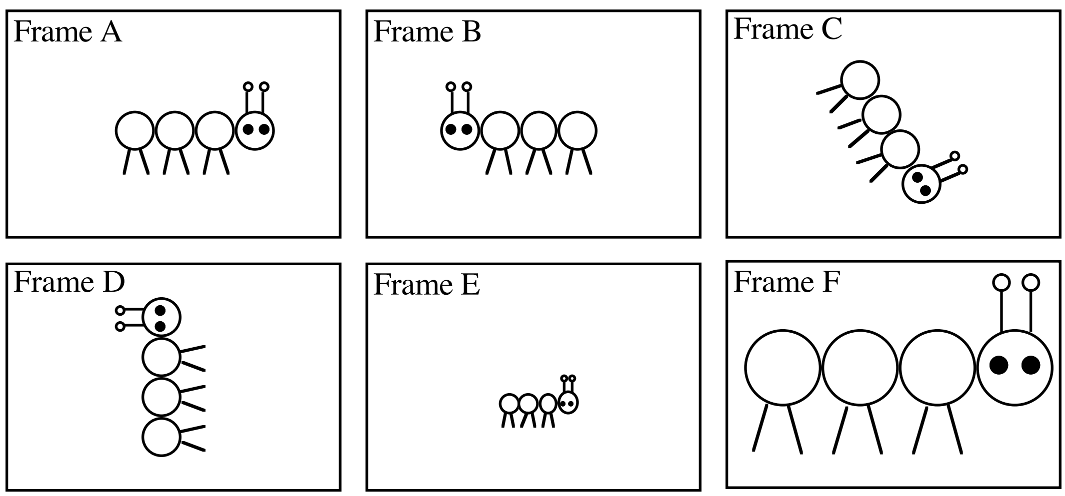


Figure 1: Transformations of a bug

b. Describe how the bug in frame A could be changed to the bug in frame B through two or more modifications.

Activity 1

Each change of the bug in Figure **1** is an example of a **transformation**. In this activity, you investigate a type of transformation that alters the size of a figure proportionally.

Mathematics Note

A **one-to-one correspondence** is a function between two sets: the domain and the range. It pairs each element in the domain with exactly one element in the range, and each element in the range with exactly one element in the domain.

For example, Figure **2** shows a one-to-one correspondence between the fingers of the left hand and the fingers of the right hand.

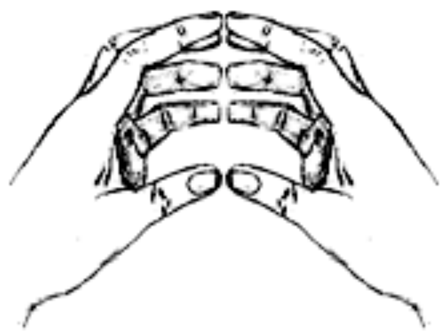


Figure 2: A one-to-one correspondence between two hands

A **transformation** in a plane is a one-to-one correspondence of the plane onto itself. For example, Figure **3** shows a reflection in the *y*-axis. In this transformation, point $A(2,5)$ is paired with $A'(-2,5)$. This type of pairing occurs for every point in the plane.

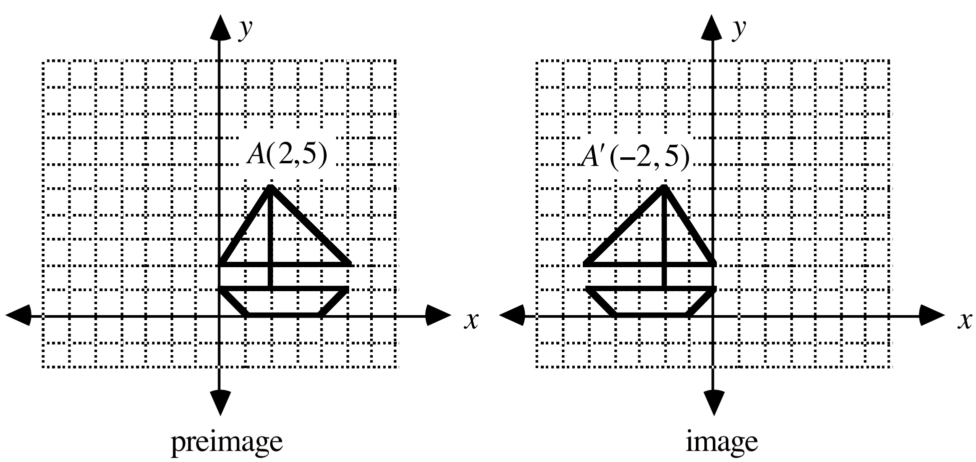


Figure 3: A transformation of a sailboat

Exploration

Before drawing a cartoon of your own, some experience with simple figures may be helpful. In this exploration, you use graph paper to construct a **point-perspective drawing**. Artists use point-perspective drawings to represent three-dimensional objects in two dimensions. For example, Figure **4** shows two rectangular prisms drawn using this method, with point P as the **point of perspective**.

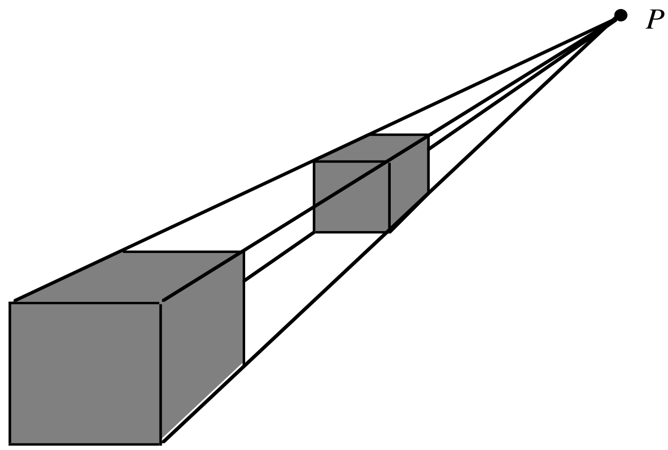


Figure 4: Point-perspective drawing of two rectangular prisms

a. On a sheet of graph paper, plot the ordered pairs $A(1,1)$, $B(2,1)$, and $C(1,3)$. Draw triangle ABC.

b. On the same sheet of graph paper, plot the following ordered pairs: $A'(3,3)$, $B'(6,3)$, and $C'(3,9)$. Draw triangle $A'B'C'$.

c. Use your drawing, the Pythagorean theorem, and the inverse trigonometric functions to complete Table **1** below.

Table 1: Measures of corresponding parts of ΔABC and $\Delta A'B'C'$

Preimage		Image	
AB		$A'B'$	
BC		$B'C'$	
CA		$C'A'$	
m$\angle ABC$		m$\angle A'B'C'$	
m$\angle BCA$		m$\angle B'C'A'$	
m$\angle CAB$		m$\angle C'A'B'$	

d. Determine each of the following ratios:

1. $A'B'/AB$
2. $B'C'/BC$
3. $C'A'/CA$

e. Draw $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. Extend these segments until they intersect at a common point, the point of perspective. Label this point P and identify its coordinates.

Mathematics Note

As shown in Figure **5**, the Pythagorean theorem can be used to derive a formula for the distance between any two points, (x_1,y_1) and (x_2,y_2), on a coordinate plane.

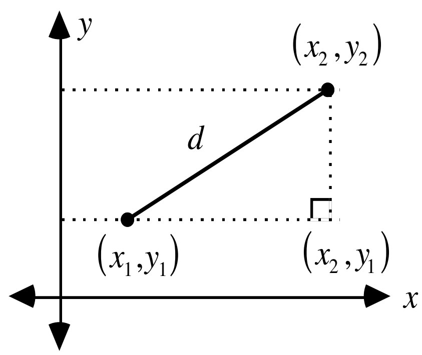


Figure 5: Distance between two points on a coordinate plane

According to the **distance formula**, the distance d between (x_1,y_1) and (x_2,y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, consider the points (–2,5) and (–4,–6). Using the distance formula, the distance between these two points can be found as follows:

$$\begin{aligned} d &= \sqrt{(-4-(-2))^2 + (-6-5)^2} \\ &= \sqrt{(-2)^2 + (-11)^2} \\ &= \sqrt{125} \approx 11.18 \end{aligned}$$

f. Use the distance formula to find the lengths of each of the following segments: $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PA'}$, $\overline{PB'}$, and $\overline{PC'}$.

g. Determine each of the following ratios:

1. PA'/PA

2. PB'/PB

3. PC'/PC

h. Compare the ratios you determined in Part **g** with those found in Part **d**.

Discussion

a. How is the Pythagorean theorem used to derive the distance formula?

b. What did you observe about the ratios of corresponding lengths in Parts **d** and **g** of the exploration?

Mathematics Note

A **dilation** is a transformation that pairs a point P, the **center**, with itself and any other point X with a point X' on ray PX so that $PX'/PX = r$, where r is the **scale factor**.

A dilation with its center at the origin is a transformation such that every point Q with coordinates (x,y) has an image Q' with coordinates (rx,ry), where r is the scale factor, and $r \neq 0$. The scale factor also may be written as a ratio of the corresponding dimensions of the image to the preimage.

For example, Figure **6** shows ΔDEF with vertices at the points with coordinates (1,1), (2,1) and (1,2) and its dilation with center at the origin O and scale factor 2. The vertices of the image, $\Delta D'E'F'$, are the points with coordinates (2,2), (4,2) and (2,4), respectively.

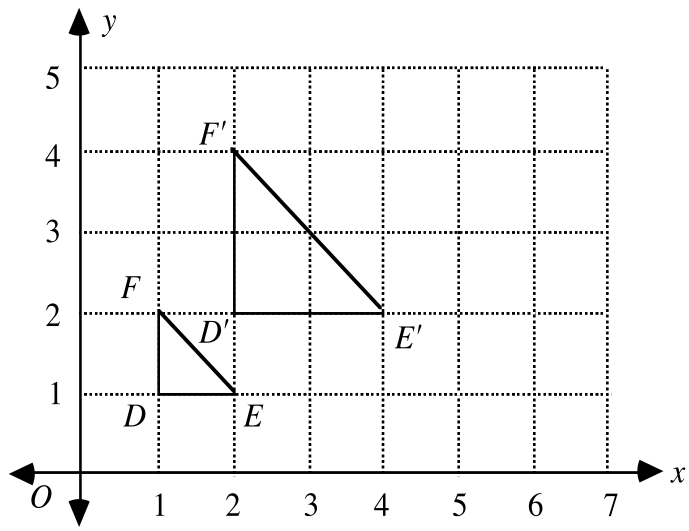


Figure 6: Preimage and image of a triangle under a dilation

c. The transformation of ΔABC to $\Delta A'B'C'$ is a dilation. Describe the similarities between a dilation and a point-perspective drawing.

d. In Part **c** of the exploration, what did you observe about the measures of corresponding angles of ΔABC and $\Delta A'B'C'$?

e. In a dilation, what geometric properties appear to remain the same in both the image and the preimage?

f.
1. Describe the geometric relationship between ΔABC and $\Delta A'B'C'$.
2. Identify any other triangles on your graph that share this same relationship.

g. **1.** How does the scale factor for the dilation compare to the ratios of corresponding lengths for ΔABC and $\Delta A'B'C'$?

2. How does the scale factor compare to the ratio of the perimeters of the two triangles?

3. How does the scale factor compare to the ratio of the areas of the two triangles?

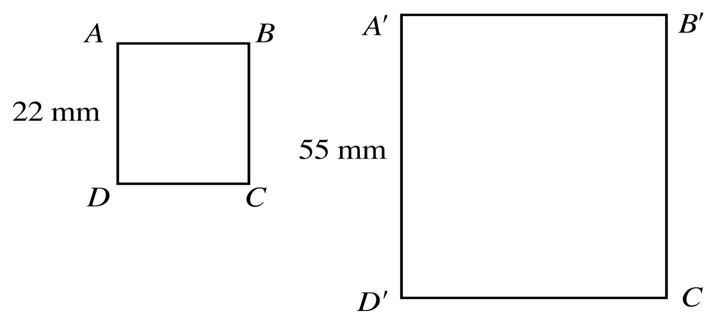
h. If you moved the center of the dilation, do you think that the relationships identified in Parts **f** and **g** would still be true?

i. How could you change the scale factor to produce an image that is smaller than the preimage?

j. In general, what is the ratio of the distance from the center of dilation to a point on the preimage to the distance from the center of dilation to the corresponding point on the image?

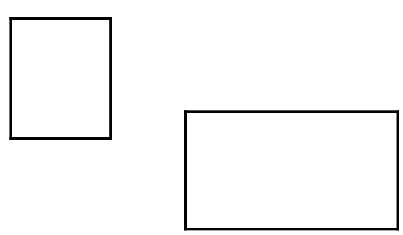
Assignment

1.1 **a.** The two figures in the following diagram are similar. Describe how to find the center of the dilation that transforms $ABCD$ to $A'B'C'D'$.



b. Find the scale factor of the dilation.

1.2 Is it possible to find a point of perspective in the drawing below, where either figure can be the preimage and the other its image? Explain your response.



1.3 **a.** Draw the triangle formed by the points $A(5,10)$, $B(-5,5)$, and $C(10,-15)$.

b. Find the image of the triangle under a dilation with center at the origin O and a scale factor of $2/5$.

c. Label the vertices of the image and identify their coordinates.

d. Determine the scale factor of the dilation of ΔABC for each of the following:

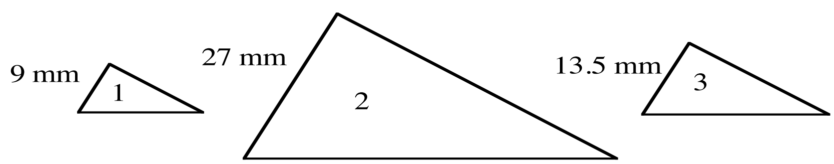
1. $A'(10,20)$, $B'(-10,10)$, and $C'(20,-30)$
2. $A'(-1,-2)$, $B'(1,-1)$, and $C'(-2,3)$
3. $A'(12.5,25)$, $B'(-12.5,12.5)$, and $C'(25,-37.5)$

1.4 **a.** On the same sheet of graph paper, draw the images of the triangle from Problem **1.3a** in a series of four dilations with center at the origin O and scale factors of $5/4$, $3/2$, $7/4$, and 2.

b. Suppose that each image was taped to a separate file card and the cards arranged in the order given in Part **a**. If you held the cards on one side and flipped through them—as if flipping the edge of a telephone book—what would appear to happen to the original triangle?

c. For any pair of images in Part **a**, describe how to find a dilation that transforms one image to the other.

1.5 The similar triangles below represent different images of a cartoon character's eye.



a. Determine the scale factor of a dilation that transforms triangle 1 to triangle 2.

b. Determine the scale factor of a dilation that transforms triangle 2 to triangle 3.

c. Determine the scale factor of a dilation that transforms triangle 1 to triangle 3.

d. Considering the arrangement of the triangles in the diagram above, could the dilations in Parts **a** and **b** have the same center? Explain your response.

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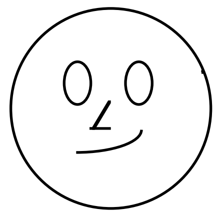
1.6 The coordinates of the vertices of quadrilateral *ABCD* are $A(0,0)$, $B(0,3)$, $C(5,3)$, and $D(5,0)$. The coordinates of the vertices of its image are $A'(0,0)$, $B'(0,6)$, $C'(10,6)$, and $D'(10,0)$. Find the center of dilation and the scale factor for this transformation.

1.7 Sketch a dilation of the face shown below using any point on a coordinate grid as the point of perspective. On your sketch, label the point of perspective, one point in the preimage, and the corresponding point in the image. Describe how to determine the scale factor of the dilation.



1.8 The coordinates of the vertices of quadrilateral *ABCD* are $A(1.9,1.2)$, $B(4.9,1.2)$, $C(4.9,3.2)$ and $D(1.9,3.2)$. The coordinates of the vertices of its image are $A'(5.3,6.7)$, $B'(11.3,6.7)$, $C'(11.3,10.7)$ and $D'(5.3,10.7)$.

a. Graph the two quadrilaterals on the same coordinate plane.

b. Show that the quadrilaterals are similar.

c. What is the scale factor of the dilation that transforms *ABCD* to $A'B'C'D'$?

d. **1.** On your graph from Part **a**, draw the lines of perspective.

2. Estimate the coordinates of the point of perspective.

3. Determine the exact coordinates of the point of perspective. (Hint: Find the equations of two lines of perspective.)

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Activity 2

Welcome to Skip's world! Although Skip is relatively easy to draw, his face still requires 19 ordered pairs to define. In this activity, you use matrices to store the ordered pairs that define Skip and his images.

Exploration

a. Figure **7** shows a picture of Skip. Graph Skip on a coordinate grid with his chin on the x-axis and the highest point on his cap on the y-axis.

Label your axes so that the coordinates for Skip's cap can be represented by the following points: (9,6), (0,6), (0,8), (7,7), and (7,6).

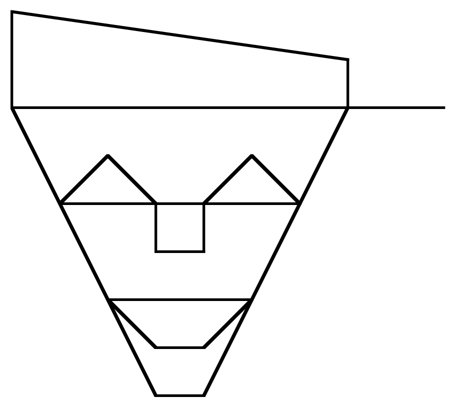


Figure 7: Skip, a cartoon character

Mathematics Note

A point P with coordinates (x,y) can be represented in a matrix as shown below.

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Using matrices to represent several points in this way can be helpful in many applications. For example, one of Skip's eyes is a triangle with vertices at (1,4), (2,5), and (3,4). These points can be mathematically described by the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \end{bmatrix}$$

b. List the coordinates for Skip's mouth in a matrix **M**.

c. Using matrix **M**, draw Skip's mouth on another sheet of graph paper. Draw a dilation of Skip's mouth with center at the origin and a scale factor of 3.

d. Write a matrix $\mathbf{M}'$ for the ordered pairs that define the image of Skip's mouth from Part **c**.

e. Dilations, along with some other types of transformation which you will examine later, can be accomplished using matrix multiplication. For example, Skip's mouth can be dilated by multiplying **M** on the left by a 2×2 matrix. Determine the dilation matrix **D** necessary to transform **M** to $\mathbf{M}'$, where **D** is of the following form:

$$\mathbf{D} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(Hint: Find the values for a, b, c, and d so that $\mathbf{D} \bullet \mathbf{M} = \mathbf{M}'$.)

f. List the coordinates for Skip's entire face in a matrix **S**.

Note: When using some forms of technology to draw closed figures, you should enter the coordinates of each vertex in the order in which you would like them to be connected.

To determine this order, trace your graph of Skip without lifting your pencil from the paper or retracing any segments. Mark both your starting and stopping points. You may have to list both these points in the matrix—even if they have the same coordinates.

Enter the coordinates of your starting point in the first column, the coordinates of the second point in the second column, those of the third point in the third column, and so on. List the coordinates of the stopping point in the last column of the matrix.

g. Multiply **D**, the dilation matrix found in Part **e**, by **S**. Compare the elements in **S** to those in $\mathbf{S}'$.

Discussion

a. Describe how a dilation of Skip's mouth by a scale factor of 3, with center at the origin, relates to scalar multiplication.

b. Why must matrix **S** be multiplied on the left by the 2×2 dilation matrix **D**?

c.
1. By what 2×2 matrix could you multiply matrix **M** to dilate Skip's mouth by a scale factor of 3, with center at the origin?
2. By what 2×2 matrix could you multiply matrix **M** to dilate Skip's mouth by a scale factor of $1/4$, with center at the origin?
3. By what 2×2 matrix could you multiply matrix **M** to dilate Skip's mouth by a scale factor of 1, with center at the origin?

d. In the module "Marvelous Matrices," you learned that the 2×2 identity matrix for matrix multiplication is:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

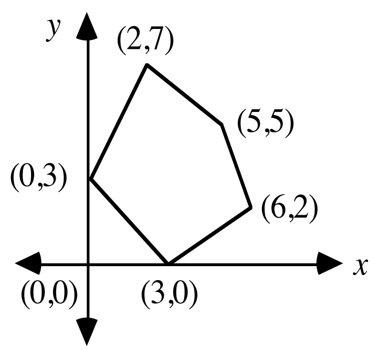
Describe how the product of a scalar and the 2×2 identity matrix relates to a matrix that produces a dilation with center at the origin.

e. Suppose Skip is dilated by a scale factor of n, with center at the origin. Describe how to determine the matrix that represents Skip's image.

f. In a dilation, what is the geometric relationship between an image and its preimage?

Assignment

2.1 **a.** Write a matrix to represent the vertices of the polygon below so that connecting the points in order results in a closed figure.



b. **1.** Write a matrix expression that produces a dilation of the polygon in Part **a** by a scale factor of 3.5, with center at the origin.

2. Write a different matrix expression from the one in Part **b1** that produces the same dilation of the polygon.

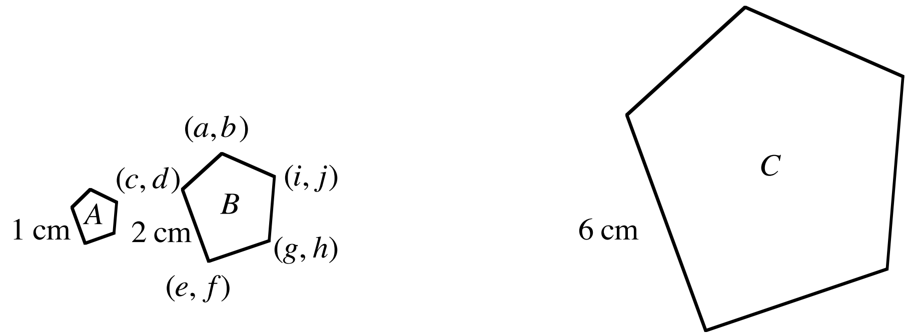
c. Use technology to find the coordinates of the image polygon. Write the results in a matrix.

2.2 **a.** Write a matrix expression to show the dilation of Skip by a scale factor of 5, with center at the origin.

b. Write a matrix expression to show the dilation of Skip's cap by a scale factor of $3/4$, with center at the origin.

c. Use technology to find the coordinates of the image of Skip's mouth under a dilation with center at the origin and a scale factor of $3/4$. Write the coordinates of the image in a matrix.

2.3 The diagram below shows three similar polygons.



a. Write a matrix for polygon B.

b. If polygon B is the preimage, what scale factors are needed to create polygons A and C, respectively?

c. Consider a dilation, with center at the origin, in which polygon B is the preimage and polygon A is the image. Write a matrix equation to describe this transformation.

d. Consider a dilation, with center at the origin, in which polygon A is the preimage and polygon C is the image. Write a matrix equation to describe this relationship.

2.4 Use technology to determine the image matrix that represents a dilation of Skip by a scale factor of 3, with center at the origin. Graph both the preimage and the image.

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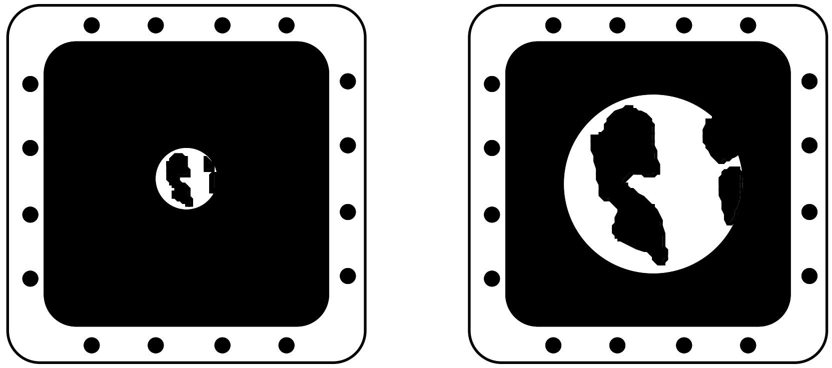
2.5 The following matrix represents a company's daily production of sweatshirts, in two colors and four sizes.

$$\begin{array}{c} \\ \text{blue} \\ \text{red} \end{array} \begin{array}{c} \begin{array}{cccc} \text{small} & \text{medium} & \text{large} & \text{x-large} \end{array} \\ \begin{bmatrix} 30 & 70 & 30 & 40 \\ 50 & 80 & 20 & 10 \end{bmatrix} \end{array}$$

The company plans to quadruple its daily production. Write a matrix to represent this information.

2.6 The following figure shows two views from the window of a spacecraft traveling at a constant velocity. As the spacecraft approaches earth, the planet's image appears to dilate.

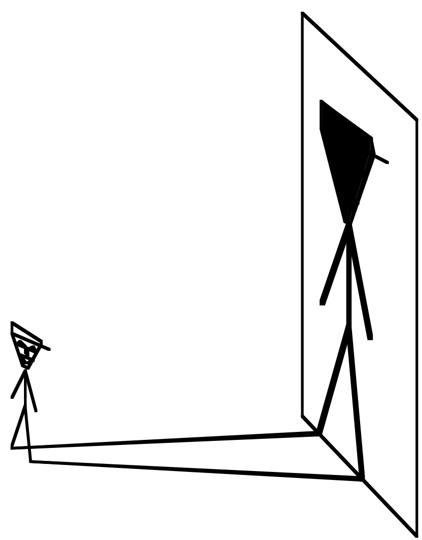


6:00 P.M. Tuesday 1:00 A.M. Wednesday

a. Use a ruler to determine the difference in scale between the two images of earth.

b. If the spacecraft continues to approach earth at the same velocity, what will the diameter of the earth's image be after another 7 hr?

2.7 Skip's friend Chip used a flashlight to produce a giant shadow of Skip, as shown in the illustration below.



a. Use a copy of this illustration to determine where Chip held the flashlight.

b. If Chip wants Skip's shadow to be five times as tall as Skip, describe where he should hold the flashlight.

c. If Chip keeps moving the flashlight toward Skip, what will appear to happen to Skip's shadow? Explain your response.

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Activity 3

Imagine Skip going down a slide. When he reaches the bottom, only his position in the frame has changed. This type of transformation is a **translation**. As shown in Figure **8**, Skip's change in position can be represented by a **translation vector**. The length of the translation vector represents the distance Skip has moved; the arrow indicates the direction of travel. Any translation vector may be represented as the sum of its horizontal and vertical components.

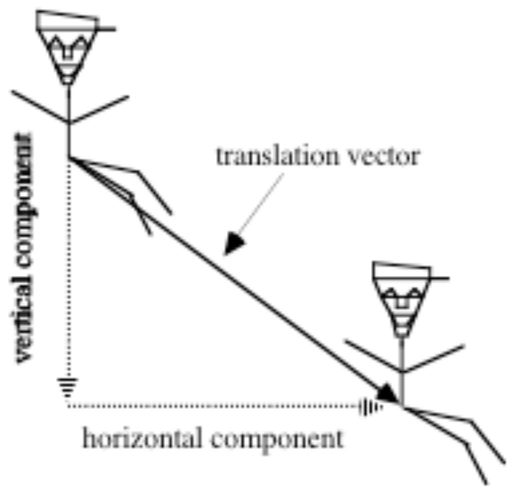


Figure 8: Skip on the slide

Exploration

a. **1.** Trace a picture of Skip from Figure **8**. Align your tracing with Skip's position at the top of the slide, then move Skip along the translation vector. Record your observations.

2. Beginning again at the top of the slide, move Skip along the vertical component of the translation vector shown in Figure **8**, then along the horizontal component. Record your observations.

b. Matrix **C** below represents Skip's cap at the top of the slide, while matrix **C′** represents his cap at the bottom of the slide.

$$\mathbf{C} = \begin{bmatrix} 9 & 0 & 0 & 7 & 7 \\ 6 & 6 & 8 & 7 & 6 \end{bmatrix} \qquad \mathbf{C}' = \begin{bmatrix} 13 & 4 & 4 & 11 & 11 \\ 3 & 3 & 5 & 4 & 3 \end{bmatrix}$$

Graph the positions of Skip's cap as defined by these matrices.

c. Determine the changes that have occurred in the coordinates of each point of Skip's cap.

d. A matrix equation of the following form, where **T** is a translation matrix, can be used to represent the movement of Skip's cap from the top of the slide to the bottom: $\mathbf{C} + \mathbf{T} = \mathbf{C}'$.

Determine the matrix **T** that makes this equation true.

e. Use technology to verify your equation from Part **d**.

Discussion

a. In Part **a** of the exploration, how did Skip's image at the bottom of the slide after moving him along the vertical and horizontal components compare with the image after moving him along the translation vector?

b. What is the geometric relationship between the image at the bottom of the slide and its preimage?

c. Describe the change in position of Skip's cap from the top of the slide to the bottom.

Mathematics Note

A **translation** is a transformation that pairs every point $P(x,y)$ with an image point $P'(x+h, y+k)$. A translation of this type moves every point h units horizontally and k units vertically.

A translation can be described using matrix addition. In this case, the number of columns in the translation matrix must match the number of points used to defined the figure to be translated. A translation matrix is represented in the following form:

$$\mathbf{T}_{P,P'} = \begin{bmatrix} h & h & h & \\ k & k & k & \cdots \end{bmatrix}$$

For example, consider the triangle whose vertices are represented in matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 5 \\ 3 & 8 & -1 \end{bmatrix}$$

To find the vertices of the image of this triangle under a translation –7 units horizontally and 3 units vertically, you can perform the following matrix addition:

$$\mathbf{A} + \mathbf{T} = \mathbf{A}'$$

$$\begin{bmatrix} 2 & 7 & 5 \\ 3 & 8 & -1 \end{bmatrix} + \begin{bmatrix} -7 & -7 & -7 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -2 \\ 6 & 11 & 2 \end{bmatrix}$$

d. What translation matrix would you use to translate Skip's cap 4 units to the left and 5 units down?

e. The lengths of the vertical and horizontal components represent vertical and horizontal distances. How could you use these distances to find the length of the translation vector?

f. How could you find the measure of the angle formed by the translation vector and its vertical component?

g. Describe how to find the distance from any point in Skip's preimage to the corresponding point in the image.

h. Figure **9** shows the translation of point $P(x,y)$ to $P'(x+h, y+k)$.

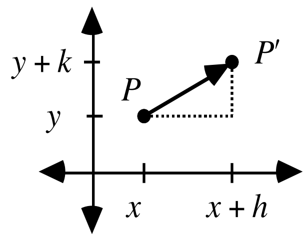


Figure 9: A translation of point *P*

1. What is the length of the translation vector?
2. How could you determine the direction of the translation vector?

Assignment

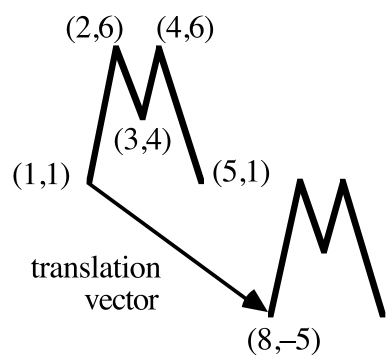
3.1 Use your results from Part **d** of the exploration to describe the length of the translation vector in Figure **8**, along with the measure of the angle it forms with its vertical component.

3.2 Consider a polygon with vertices at (1,1), (5,1), (4,4), and (0,4).

a. Write a matrix equation that describes a translation of this polygon 4 units to the right and 2 units down.

b. Show this translation on a sheet of graph paper.

3.3 The following diagram shows the translation of a letter M.



a. Describe the matrix that accomplishes the translation.

b. Write a matrix equation to summarize the translation.

c. What is the measure of the angle formed by the translation vector and its vertical component?

d. What is the length of the translation vector?

3.4 **a.** Use technology to draw the preimage and image of the letter M in Problem **3.3**.

b. Suppose that you drew five more images of the letter M, all evenly spaced between the preimage and image in Part **a**. If you taped each image to a separate card and flipped the cards like the pages of a book, what would you see?

c. Describe how to find a translation matrix that transforms each image in Part **b** to the next, given only the coordinates of the original preimage and final image.

3.5 **a.** Describe a translation matrix that could be used to slide Skip's face to a region below and to the left of its original position on a grid.

b. Describe the matrix that contains the coordinates of the image points.

3.6 Consider the hexagon described by matrix **P** below.

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & -1 & -2 & -1 & 2 \\ 1 & 7 & 7 & 1 & -4 & -4 \end{bmatrix}$$

a. **1.** Translate this figure 3 units horizontally and –8 units vertically, then dilate it by a scale factor of 2, with center at the origin. Write a separate matrix equation to describe each transformation.

2. Create a graph of the preimage and the final image.

3. If possible, write a single matrix equation that describes the combination of transformations. If a single matrix equation is not possible, explain why not.

b. Reverse the order of the transformations, then repeat Part **a**.

c. Compare your results from Parts **a** and **b**. Did you think they would be the same? Were they? Why or why not?

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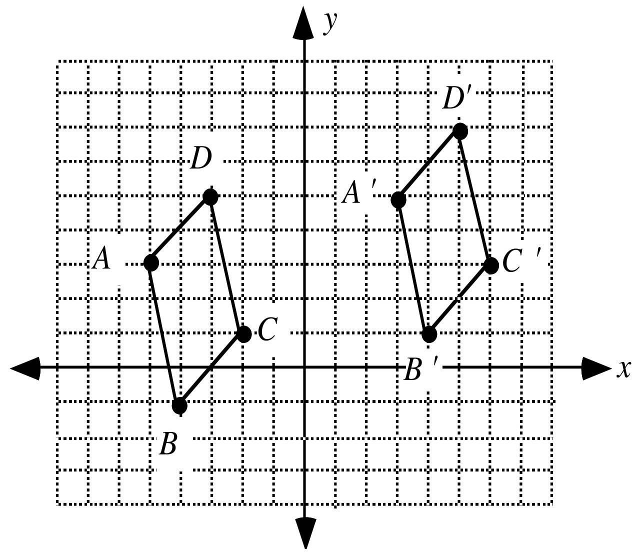
3.7 **a.** Describe the translation shown in the diagram below.



b. Write a matrix that can be used to translate *ABCD* to $A'B'C'D'$.

c. Determine the distance between each point in the preimage and its corresponding point in the image.

3.8 Consider ΔABC with vertices $A(2,4)$, $B(-4,6)$ and $C(1,-2)$. A translation using the following matrix results in the image $\Delta A'B'C'$.

$$\begin{bmatrix} 5 & 5 & 5 \\ -4 & -4 & -4 \end{bmatrix}$$

A translation of $\Delta A'B'C'$ by the matrix below produces the image $\Delta A''B''C''$.

$$\begin{bmatrix} -7 & -7 & -7 \\ 2 & 2 & 2 \end{bmatrix}$$

a. Find the coordinates of $\Delta A''B''C''$.

b. **1.** Describe the translation of ΔABC to $\Delta A''B''C''$.

2. Write a translation matrix that describes this transformation.

c. Determine the distance between each point in ΔABC and its corresponding point in $\Delta A''B''C''$.

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Activity 4

Skip's friend Chip is washing dishes when the phone rings. While talking on the phone Chip's head is tilted as shown in Figure **10**. The **rotation** of his head represents another type of transformation. In this activity, you explore some special cases of rotations.

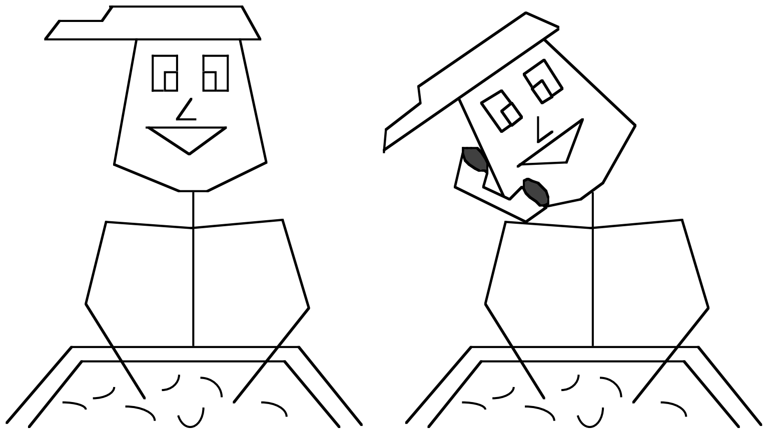


Figure 10: Chip washing dishes and talking on the phone

Exploration

a. Chip's mouth can be represented by matrix **P** below.

$$\mathbf{P} = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \end{bmatrix}$$

1. Plot Chip's mouth on a coordinate grid.
2. On a sheet of tracing paper, trace Chip's mouth, the x- and y-axes, and the origin O.

b. Position the tracing over the original so that it aligns perfectly. Place the tip of your pencil on the origin. While holding this point with the pencil gently turn the tracing so that the x-axis rotates 180° counterclockwise. Write the matrix $\mathbf{P}'$ for the image of Chip's mouth.

c. Make a conjecture about the relationship between the matrix of the preimage and the matrix of its image under a counterclockwise rotation about the origin of 180°.

d. Find a 2×2 matrix **R** so that $\mathbf{R} \bullet \mathbf{P}$ produces a counterclockwise rotation about the origin of 180°.

e. Repeat Parts **b–d** using a counterclockwise rotation about the origin of each of the following measures:

1. 360°
2. 90°.

Discussion

a. Describe the image that results from a rotation of 0° about the origin.

b. What 2×2 matrix, when multiplied on the left of a preimage matrix, would produce a rotation of 0° about the origin?

Mathematics Note

A **rotation** is a transformation that pairs one point C, the center, with itself and every other point P with a point P' that lies on a circle with center C such that $m\angle PCP'$ is the magnitude of the rotation.

Counterclockwise rotations about a point are denoted by positive degree measures. Clockwise rotations are represented by negative degree measures.

In Figure **11**, for example, $\Delta P'Q'R'$ is the image of ΔPQR under a 90° rotation about point C.

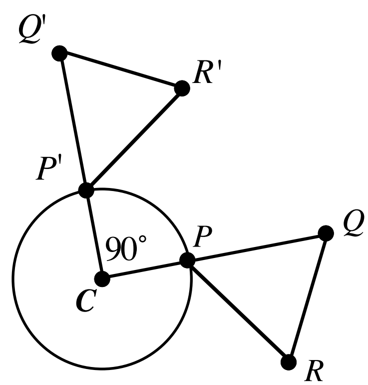


Figure 11: Triangle rotated 90° about point *C*

c. Name at least two rotations that yield the same result as a counterclockwise rotation about the origin of 360°.

d. What clockwise rotation would produce the same result as a counterclockwise rotation about the origin of 90°? How can you generalize the relationship between equivalent clockwise and counterclockwise rotations?

e. Do you think that two counterclockwise rotations of 90° with the same center are equivalent to one counterclockwise rotation of 180°? Explain your response.

f. In a rotation, what is the geometric relationship between an image and its preimage?

Assignment

4.1 Skip's mouth can be described by matrix **M** below.

$$\mathbf{M} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

a. Use matrix multiplication to determine the coordinates of the image that results from each of the following:

1. a 360° rotation about the origin

2. a 180° rotation about the origin

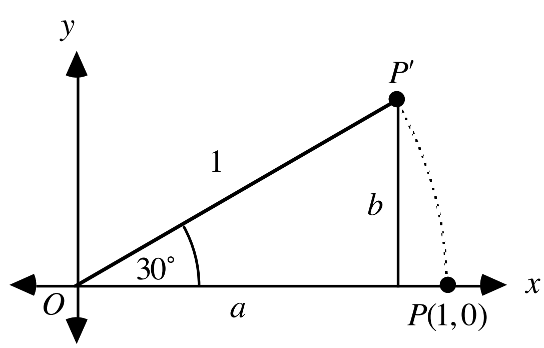
3. a 90° rotation about the origin.

b. Graph the image of Skip's mouth produced by each rotation in Part **a**.

4.2 **a.** What matrix can be used to describe the image of Skip's mouth after a 270° rotation about the origin?

b. Find a 2×2 matrix **R** so that $\mathbf{R} \bullet \mathbf{M}$ produces a counterclockwise rotation about the origin of 270°.

4.3 As shown in the diagram below, the image of a point $P(1,0)$ under a 30° rotation with center at the origin is P' .



a. **1.** Use the cosine ratio to determine the distance a.

2. Use the sine ratio to determine the distance b.

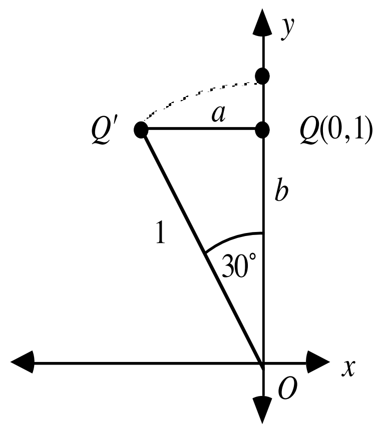
3. Write the coordinates of point P' in a matrix.

b. Determine the coordinates of P' under a 30° rotation, with center at the origin, for each of the following:

1. $P(2,0)$

2. $P(c,0)$.

4.4 As shown in the following diagram, the image of a point $Q(0,1)$ under a 30° rotation with center at the origin is Q'.



a. **1.** Use the sine ratio to determine the distance a.

2. Use the cosine ratio to determine the distance b.

3. Write the coordinates of point Q' in a matrix.

b. Determine the coordinates of Q' under a 30° rotation, with center at the origin, for each of the following:

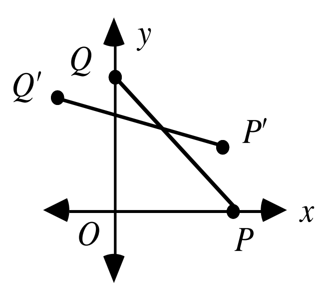
1. $Q(0,2)$

2. $Q(0,d)$.

4.5 Matrix **S** below represents the line segment connecting $P(1,0)$ and $Q(0,1)$.

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a. If $\overline{PQ}$ is rotated 30° about the origin, as shown in the following graph, what matrix could be used to represent $\overline{P'Q'}$?



b. If $\overline{PQ}$ is rotated 20° about the origin, what matrix could be used to represent $\overline{P'Q'}$?

c. If $\overline{PQ}$ is rotated n° about the origin, where $0 \le n < 90$, what matrix could be used to represent $\overline{P'Q'}$?

4.6 **a.** The following matrix **S** represents the line segment connecting $P(2,0)$ and $Q(0,2)$.

$$\mathbf{S} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

If $\overline{PQ}$ is rotated 30° about the origin, what matrix could be used to represent $\overline{P'Q'}$?

b. The matrix **S** below represents the line segment connecting $P(c,0)$ and $Q(0,c)$.

$$\mathbf{S} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

1. If $\overline{PQ}$ is rotated 30° about the origin, what matrix could be used to represent $\overline{P'Q'}$??

2. If $\overline{PQ}$ is rotated n° about the origin, where $0 \leq n < 90$, what matrix could be used to represent $\overline{P'Q'}$?

4.7 Use matrix multiplication to perform a rotation of Skip's cap. Make a sketch of the preimage and image and list the appropriate matrix equation.

4.8 A triangle can be represented by matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 2 \end{bmatrix}$$

Determine a matrix $\mathbf{A}'$ that represents the image of this triangle under each of the following transformations:

a. a 180° rotation about the origin

b. a 90° rotation about the origin.

Mathematics Note

An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.

For example, Figure **12** shows an equilateral triangle *ABC* and its center *O*. This object has rotational symmetry about *O*. When ΔABC is rotated 120° about *O*, each point in the image coincides with a point in the preimage.

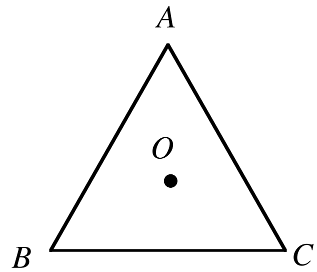


Figure 12: An equilateral triangle

4.9 When ΔABC in Figure **12** is rotated 120° about *O*, each point in the image coincides with a point in the preimage. Identify at least two other degree measures for which this is true.

4.10 Determine which cards in an ordinary deck of playing cards have rotational symmetry.

4.11 Does Skip's face have rotational symmetry? Explain your response.

4.12

a. Draw a polygon that has rotational symmetry when rotated 90° about its center.

b. What regular polygon has rotational symmetry when rotated 45° about its center?

c. Determine the smallest positive degree measure which shows rotational symmetry for each of the following:

1. a regular hexagon
2. a regular pentagon
3. a regular decagon
4. a regular polygon with *n* sides.

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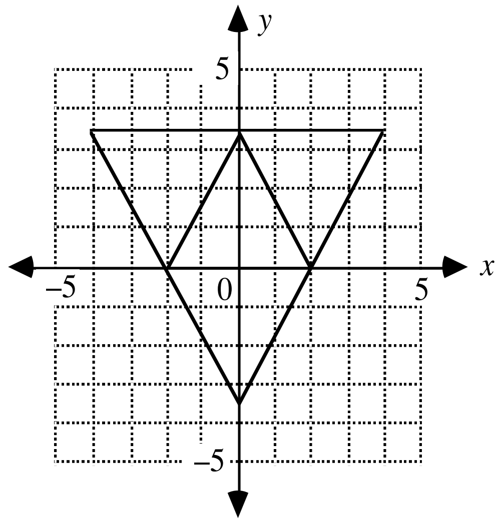
4.13 The following graph shows a net for a triangular pyramid.



This net can be represented by matrix **S** below.

$$S = \begin{bmatrix} -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \\ 0 & 3.4 & 3.4 & 0 & -3.4 & 3.4 & 3.4 & 0 \end{bmatrix}$$

a. Determine a matrix $\mathbf{S}'$ that represents the image of this net under each of the following rotations with center at the origin:

1. 180˚

2. 270˚

3. 360˚

4. 0˚

5. 90˚.

b. Identify the degree measures, where $0 < n < 360$, for which the net shows rotational symmetry about its center.

4.14 Consider a figure contained entirely in the first quadrant. What angle of rotation, with center at the origin, is required to produce an image of this figure that is contained entirely in the second quadrant?

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Activity 5

In previous activities, you have seen transformations of Skip and Chip using dilations, translations, and rotations. Another common type of transformation is a **reflection**, as shown in Figure **13**.

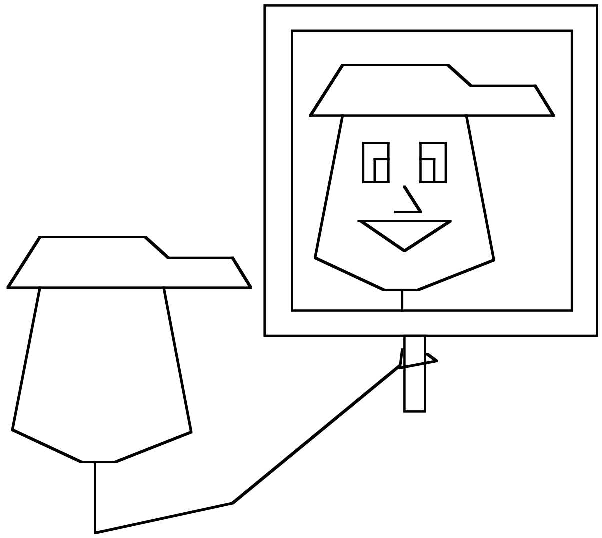


Figure 13: Chip looking in the mirror

Exploration

a. Matrix **P** below describes the vertices of a triangle.

$$\mathbf{P} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

1. Graph this triangle on a coordinate plane.
2. Consider the x-axis as a mirror. Graph the reflection of the triangle in the x-axis.
3. Write the coordinates of the image in matrix form.
4. Draw line segments connecting the corresponding vertices of the preimage and the image.
5. Describe the relationship between the line of reflection and each segment you drew in Step **4**.

b. Matrix **H** below describes the vertices of a quadrilateral.

$$\mathbf{H} = \begin{bmatrix} -1 & 2 & 6 & 8 \\ 0 & 3 & 4 & -2 \end{bmatrix}$$

1. Graph this quadrilateral on a coordinate plane.
2. Graph the reflection of the quadrilateral in the x-axis.
3. Write the coordinates of the image in matrix form.

c. Repeat Parts **a** and **b** using the y-axis as a mirror.

d.

1. On a new coordinate system, graph the line $y = x$.
2. Reflect the triangle described by matrix **P** in this line and write the coordinates of the image in matrix form.
3. Repeat Steps **1** and **2** using the quadrilateral defined by matrix **H**.

e. Reflect the polygons described by matrices **P** and **H** in the line $y = -x$
.

Mathematics Note

A **reflection** in a line is a transformation that pairs each point on the line with itself and each point in the preimage with a corresponding point in the image so that the line of reflection is the perpendicular bisector of the segment connecting the point in the preimage with its image.

For example, Figure **14** shows the reflection of a polygon in line m.

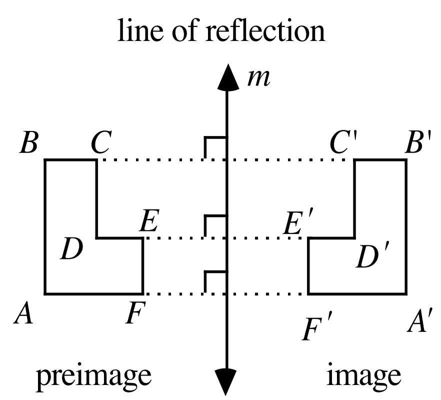


Figure 14: Reflection of a polygon in a line

A reflection in line m is denoted by r_m. For example, a reflection in the x-axis may be written as r_x.

f. When a matrix of ordered pairs is multiplied on the left by one of the four matrices below, the product is one of the reflections explored in Parts **a–e**. Identify the line of reflection associated with each matrix below.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Discussion

a.
1. What patterns did you observe in the matrices when reflecting an object in the x-axis?
2. What patterns did you observe when reflecting in the y-axis?

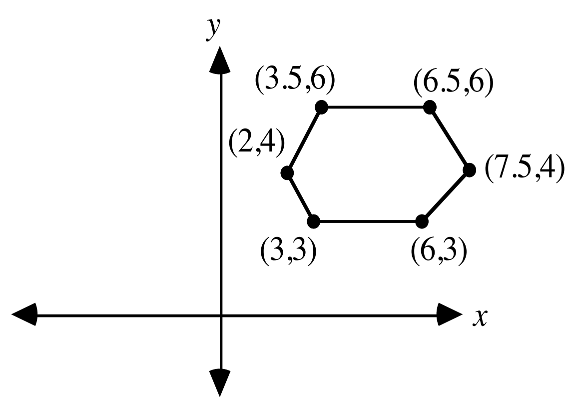
b. Using your results from Parts **d** and **e** of the exploration, write a general statement that describes a reflection in each of the following lines:
1. $y = x$
2. $y = -x$.

c. What similarities or differences are there in the matrix operations for the four types of transformations you have explored so far?

d. How would you summarize the geometric relationships between the preimage and the image in these four types of transformations?

Assignment

5.1 **a.** The following graph shows the coordinates of the vertices of a hexagon. Write a matrix expression that describes the reflection of this hexagon in the y-axis.



b. List the coordinates of the image in a matrix.

c. Graph the preimage and the image of the hexagon in Part **a**. Reflect each of these figures in the x-axis.

d. Select any one of the four figures from Part **c**. Using this figure as the preimage, can each of the other figures be generated by a single reflection in a line? Explain your response.

5.2 Use matrices to produce a reflection of Skip in the line $y = x$ or the line $y = -x$, followed by a reflection in the x-axis or the y-axis. Show the resulting image on a sheet of graph paper.

5.3 When more than one transformation is performed on a figure, the result is a **composite transformation**.

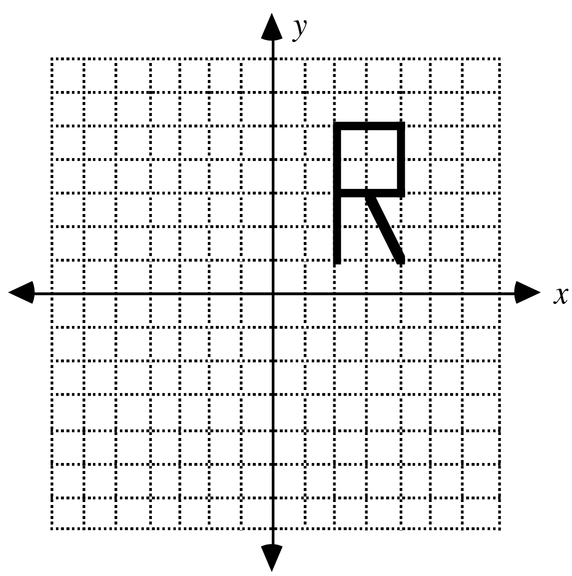
a. Reflect the quadrilateral described by matrix **M** below in the x-axis.

$$\mathbf{M} = \begin{bmatrix} -1 & 2 & 8 & 2 \\ 5 & 8 & 5 & 2 \end{bmatrix}$$

b. Dilate the image from Part **a** by a scale factor of $1/2$, with center at the origin. Show your results graphically and identify the coordinates of the vertices.

c. Write a matrix representation of the composite transformation.

5.4 The figure below shows a letter R graphed on a coordinate plane.



a. Reflect the letter R in the x-axis, then reflect the resulting image in the y-axis.

b. Rotate the letter R 180° about the origin.

c. Compare the images you obtained in Parts **a** and **b**. Use matrix multiplication to confirm your observations.

Mathematics Note

An object has a **line of symmetry** if, when reflected in that line, each point in the image coincides with a point in the preimage.

The equilateral triangle in Figure **15**, for example, has three lines of symmetry. When ΔABC is reflected in line l, m, or n, each point in $\Delta A'B'C'$ coincides with a point in ΔABC.

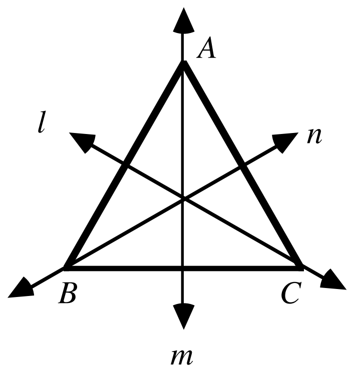
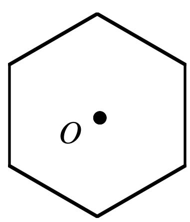


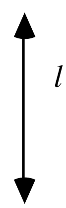
Figure 15: Lines of symmetry of an equilateral triangle

5.5 Identify the line(s) of symmetry for each of the following.

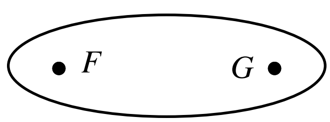
a.



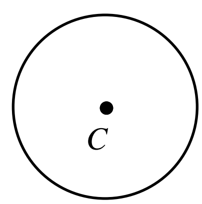
b.



c.



d.



5.6 Does Skip's face have a line of symmetry? Explain your response.

* * * * *

5.7 If the quadrilateral described by matrix **D** is reflected in a line, the resulting image is described by matrix **Z**.

$$\mathbf{D} = \begin{bmatrix} -3 & -6 & -4 & -2 \\ 1 & 2 & 4 & 3 \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} -1 & -2 & -4 & -3 \\ 3 & 6 & 4 & 2 \end{bmatrix}$$

a. What is the equation of the line of reflection?

b. Determine a matrix that may have been used to perform the reflection.

c. Verify your response to Part **b** using a matrix equation.

5.8 Parts **a–c** below each describe two composite transformations. For each part, use matrix operations to determine if the two composite transformations produce the same final image.

a. **1.** a dilation with center at the origin and a scale factor of 4, followed by a translation 2 units to the left and 5 units up

2. a translation 2 units to the left and 5 units up, followed by a dilation with center at the origin and a scale factor of 4

b. **1.** a reflection in the line $y = x$, followed by a dilation with center at the origin and a scale factor of 7

2. a dilation with center at the origin and a scale factor of 7, followed by a reflection in the line $y = x$

c. **1.** a reflection in the y-axis, followed by a reflection in the line $y = -x$

2. a 90° rotation about the origin

5.9 Make a sketch of an object that has at least three different lines of symmetry.

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Research Project

Create your own animated cartoon using the transformations examined in this module. (Your collection of Skip's transformations should help you consider the possibilities.) Your cartoon should be at least 20 frames long. So that others may view your cartoon, tape each image onto a 3×5 card. Include a written summary with a mathematical description of the transformations from each frame to the next. You may also create a cartoon on a programmable calculator. See your teacher for details.

Summary Assessment

1. When composite transformations are repeated over and over again, some interesting patterns may develop. **Note:** Read Parts **a–c** below before beginning work on this problem.

a. Graph the triangle described by matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 0 & 16 & 16 \\ 0 & 0 & 16 \end{bmatrix}$$

b. Perform each of the following transformations in the order given below. Graph the image that results from this composite transformation on the same grid as the triangle in Part **a**.

1. a translation of –2 units horizontally and 6 units vertically

2. a rotation of 90° about the origin

3. a dilation with center at the origin and a scale factor of $3/4$

c. Using the image that resulted from the composite transformation in Part **b** as the preimage, repeat Steps **1–3** above. Graph the resulting image on the same grid as the triangle in Part **a**.

Continue this process until you have graphed five triangles. Describe the resulting figure.

2. Create your own picture using repeated composite transformations. Use a geometry utility to help explore some possibilities. Summarize the transformations using matrix equations.

Module Summary

- A **one-to-one correspondence** is a function between two sets: the domain and the range. It pairs each element in the domain with exactly one element in the range, and each element in the range with exactly one element in the domain.
- A **transformation** in a plane is a one-to-one correspondence of the plane onto itself.
- The Pythagorean theorem can be used to derive a formula for the distance between any two points, (x_1, y_1) and (x_2, y_2), on a coordinate plane. According to the **distance formula**, the distance d between (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- A **dilation** is a transformation that pairs a point P, the **center**, with itself and any other point X with a point X' on ray PX so that $PX'/PX = r$, where r is the **scale factor**.
- A dilation with its center at the origin is a transformation such that every point Q with coordinates (x, y) has an image Q' with coordinates (rx, ry), where r is the scale factor, and $r \neq 0$. The scale factor also may be written as a ratio of the corresponding dimensions of the image to the preimage.
- A point P with coordinates (x, y) can be represented in a matrix as shown below.

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- An object's change in position can be represented by a **translation vector**. The length of the translation vector represents the distance the object has moved; the arrow indicates the direction of travel. Any translation vector may be represented as the sum of its horizontal and vertical components.
- A **translation** is a transformation that pairs every point $P(x, y)$ with an image point $P'(x + h, y + k)$. A translation of this type moves every point h units horizontally and k units vertically.
- A translation can be described using matrix addition. In this case, the number of columns in the translation matrix must match the number of points used to defined the figure to be translated. A translation matrix is represented in the following form:

$$\mathbf{T}_{P,P'} = \begin{bmatrix} h & h & h & \\ k & k & k & \cdots \end{bmatrix}$$

- A **rotation** is a transformation that pairs one point C, the center, with itself and every other point P with a point P' that lies on a circle with center C such that $m\angle PCP'$ is the magnitude of the rotation.

  Counterclockwise rotations about a point are denoted by positive degree measures. Clockwise rotations are represented by negative degree measures.

- An object has **rotational symmetry** about a point if, when rotated through an angle about that point, each point in the image coincides with a point in the preimage.
- A **reflection** in a line is a transformation that pairs each point on the line with itself and each point in the preimage with a corresponding point in the image so that the line of reflection is the perpendicular bisector of the segment connecting the point in the preimage with its image.
- When more than one transformation is performed on a figure, the result is a **composite transformation**.
- An object has a **line of symmetry** if, when reflected in that line, each point in the image coincides with a point in the preimage.

Selected References

Barasch, M. *Theories of Art: From Plato to Winkelman*. New York: New York University Press, 1985.

Hulse, C. *The Rule of Art: Literature and Painting in the Renaissance*. Chicago: University of Chicago Press, 1990.

Papy, G. *Modern Mathematics*. London: Collier-Macmillan, 1968.