What’s Your Orbit?

How in the world do astronomers predict the paths of planets? In this module, you’ll examine some planetary data and model it mathematically.

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Introduction

Italian astronomer Galileo Galilei (1564–1642) was also a mathematician, physicist, musician, painter, and inventor. He built his first telescope in 1609 and is credited with many astronomical discoveries, including the first observations of sunspots, lunar mountains and craters, Saturn’s rings, and four of Jupiter’s moons.

During Galileo’s time, most scientists and philosophers believed that the earth was the center of the universe. When his findings cast doubt on this theory, he was branded a heretic.

Modern astronomers now understand that the earth revolves around the sun—and that our sun is only one star in an immense galaxy of stars. Our galaxy, in turn, is just one galaxy among countless other galaxies in the universe. Furthermore, most scientists agree that the universe itself has been expanding for some time.

Activity 1

Although new innovations like the space shuttle and the Hubble telescope have greatly increased our knowledge of the universe, much remains unknown. In order to make predictions about the characteristics of planets and stars, scientists must rely on their abilities to develop mathematical models of current data. In this module, you focus on three types of models: power equations of the form $y = ax^b$, exponential equations of the form $y = ab^x$, and polynomial equations of degree 1, 2, and 3.

Exploration 1

Before you model data mathematically, you should be aware of the characteristics of the graphs of potential models. In this exploration, you examine graphs of power equations of the form $y = x^b$. With proper restrictions on the domain and the values of b, it is possible to make some generalizations about the graphs of these equations.

a. Choose a non-integer, rational value for b greater than 1.

1. Express the value of b as a fraction m/n in lowest terms.
2. Determine a decimal approximation of b.

b.

1. Using the fractional representation of b in Part **a**, graph the power equation $y = x^b$ over the domain [–10, 10].
2. On a separate coordinate system but using the same domain, graph the power equation $y = x^b$ using the decimal approximation of b.
3. Compare the two graphs created in Steps **1** and **2**.

c. Repeat Parts **a** and **b** for another value of b greater than 1.

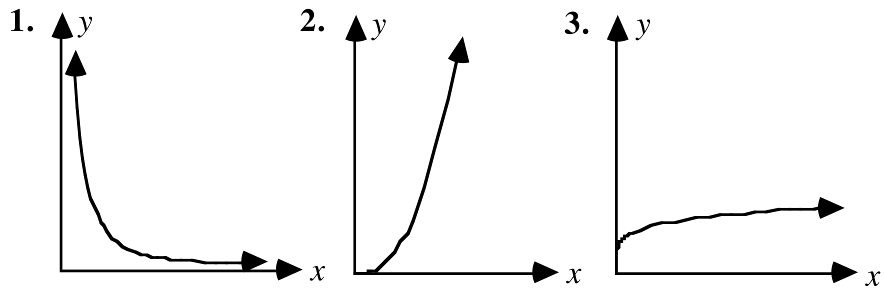
d. Repeat Parts **a** and **b** for two rational values of b between 0 and 1.

e. Repeat Parts **a** and **b** for two rational values of b less than 0.

Discussion 1

a. Judging from your investigations in Exploration **1**, how does the graph of a power equation using a fractional representation of b compare with the graph of the power equation using the corresponding decimal approximation?

b. Describe some possible values of b for each of the following graphs of equations of the form $y = x^b$.



c. In the Level 2 module "Atomic Clocks Are Ticking," you examined the following properties of exponents.

- If d is a real number greater than 0, m and n are positive integers, and m/n is in lowest terms,

$$d^{m/n} = \left(d^{1/n}\right)^m = \left(\sqrt[n]{d}\right)^m = \sqrt[n]{d^m}$$

- If d is a nonzero real number and n is an integer,

$$d^{-n} = \frac{1}{d^n}$$

- If d is a real number greater than 0,

$$\left(d^m\right)^n = d^{m \bullet n} \text{ and } \frac{d^m}{d^n} = d^{m-n}$$

1. How could you express the equation $y = x^{-1/3}$ without using a negative exponent?
2. How could you express the equation $y = x^{-1/3}$ using a radical sign?
3. How could you express the equation $y = x^{0.3}$ using a radical sign?

d. **1.** Why does a graph of $y = x^{1/2}$ only show values in the first quadrant?

2. Why would you expect the graph of $y = x^b$, where b is the decimal approximation of a rational number, to only show values in the first quadrant?

e. Why is the expression x^b, when $b < 0$, undefined for $x = 0$?

f. In the Level 3 module "Graphing the Distance," you examined the effect of various values of *a* on polynomial equations of the form $y = a(x - c)^2 + d$ and $y = a(x - c)^3 + d$. How do you think the value of *a* will affect graphs of equations of the form $y = ax^b$?

g. For what values of *b* are power equations of the form $y = ax^b$ also polynomial equations?

Exploration 2

In this exploration, you investigate the relationship between the volume and circumference of a balloon. One way to measure the volume of a balloon is to count the number of "breaths" it contains. Before beginning this experiment, practice taking several even breaths.

a. Inflate your balloon one breath at a time. After each breath, measure the balloon's circumference. Record a minimum of six data points.

b. Create a scatterplot of your data.

c. One possible model for the data is a power equation. Suggest appropriate values for *a* and *b* in a model of the form $y = ax^b$.

d. Graph your suggested model on the same coordinate system as the scatterplot in Part **b**.

e. Create a spreadsheet with headings like those in Table **1** below.

Table 1: Balloon spreadsheet

No. of Breaths (x)	Circumference (y)	$y = ax^b$

1. Enter your experimental data in the appropriate columns.

2. In the right-hand column, enter a spreadsheet formula that will calculate the approximate circumference given the number of breaths using an equation of the form $y = ax^b$. **Note:** Make sure to use this formula in each cell of the column.

f. Recall that a **residual** is the difference between an observed value and the corresponding value predicted by a model, and that the sum of the squares of the residuals can be used to evaluate how well a model fits a data set.

1. Use the spreadsheet to calculate the sum of the squares of the residuals for your model.
2. Adjust the values of *a* and *b* in your model to identify a power equation that closely approximates the data.
3. Record the corresponding sum of the squares of the residuals.

Discussion 2

a. Describe how you determined an appropriate value for *a* in Part **c** of Exploration **2**.

b.
1. Do you think that anyone in the class found the power equation with the least possible sum of the squares of the residuals for their data set? Explain your response.
2. If the power equation that produces the least possible sum of the squares of residuals could be found, how well do you think it would fit the data?

c. What types of information might help you evaluate the appropriateness of mathematical models?

Assignment

1.1 For each value of *b* below, sketch the graph of a power equation of the form $y = x^b$ over the domain (0, 10].

a. 5/6

b. 7/4

c. −10/3

d. 0.6667

e. 2

1.2 The table below contains information about the planets in our solar system. The mean distance represents the average distance of a planet from the sun during its orbit. The period represents the time, measured in earth days, required for a planet to complete one orbit around the sun.

Planet	Mean Distance from Sun (millions of km)	Period (earth days)
Mercury	57.9	87.97
Venus	108.2	224.70
Earth	149.6	365.26
Mars	227.9	686.98
Jupiter	778.3	4331.87
Saturn	1427.0	10,760.27
Uranus	2869.6	30,684.65
Neptune	4496.6	60,189.55
Pluto	5899.9	90,468.77

a. Create a scatterplot of this data.

b. Determine an equation of the form $y = ax^b$ that models the data.

c. Is the model you selected a good one? Explain your response.

1.3 Imagine that it is the year 2039. Earth's inhabitants have colonized the moon. There are five lunar colonies, each housed in a pressurized hemispherical dome. The table below shows the diameter and volume of each dome.

Colony	Diameter (km)	Volume (km^3)
Alpha	5.0	32.73
Beta	6.4	68.63
Gamma	5.6	45.98
Delta	8.2	144.35
Epsilon	7.6	114.92

a. Considering the relationship between the diameter and volume of a sphere, would a linear, exponential, or power equation provide the best model for this data?

b. Find an equation of the type identified in Part **a** to model the data.

c. Use the equation determined in Part **b** to predict the volume of a hemispherical dome with a diameter of 7.0 km.

1.4 The table below shows the population of Beta colony during its first 20 years.

Year	Population	Year	Population
1	1000	11	6192
2	1200	12	7430
3	1440	13	8916
4	1728	14	10,699
5	2074	15	12,839
6	2488	16	15,407
7	2986	17	18,488
8	3583	18	22,186
9	4300	19	26,623
10	5160	20	31,948

a. Make a scatterplot of this data.

b. What type of equation do you think would provide a good model for the population growth of Beta colony?

c. Find an equation that describes the trend in the data.

d. Graph your equation on the same coordinate system as the scatterplot from Part **a**.

e. The population of Beta colony cannot exceed 1000 people per km^3. Use your model, along with the dimensions of Beta colony's dome from Problem **1.2**, to determine how long it will take the population to reach this limit.

1.5 Find at least five round, flat objects of different sizes.

a. Measure and record their diameters and circumferences.

b. Based on the relationship between the diameter and circumference of a circle, what type of model would you expect to closely approximate the data collected in Part **a**?

c. Create a mathematical model of this data.

d. Use your model to predict the circumference of a disk with a radius of 1 m.

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1.6 A chemistry class at Centerville High School conducted an experiment to determine how much potassium nitrate would dissolve in 100 mL of water at various temperatures. The results of their experiment are shown below.

Temperature (˚C)	Amount of Potassium Nitrate Dissolved (g)
5	27
20	54
40	77
60	94
80	109

a. Determine a power equation that closely models the data.

b. How much potassium nitrate do you think will dissolve in 100 mL of water at 70˚ C? Explain your response.

c. Do you think that it is reasonable to use your model to predict the amount of potassium nitrate that will dissolve in 100 mL of water at 110˚ C? Explain your response.

1.7 The table below shows the atomic radii and melting points for a family of elements known as the alkali metals. Because atoms are so small, their radii are often measured in angstroms, where 1 angstrom equals $1 \bullet 10^{-10}$ m. Melting point refers to the temperature at which a solid turns into a liquid.

Alkali Metals	Atomic Radius (angstroms)	Melting Point (˚C)
Lithium	1.52	180
Sodium	1.86	98
Rubidium	2.44	39
Cesium	2.62	29

a. Create a scatterplot of melting point versus atomic radius.

b. Does there appear to be a positive or a negative association between atomic radius and melting point? Explain your response.

c. Find an equation that closely models this data and explain why you chose this type of model.

d. Potassium is also an alkali metal. Its atomic radius is approximately 2.31 angstroms. Use your model from Part **c** to predict the melting point of potassium.

e. The actual melting point of potassium is approximately 64˚ C. Compare this value with your prediction in Part **c** and suggest a possible explanation for any difference that occurs.

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Activity 2

In the module "Graphing the Distance," you used technology to find linear and quadratic regression equations to model data. In this activity, you examine power and exponential regressions and investigate another method for evaluating the appropriateness of models.

Exploration 1

Table **2** below shows data collected during a balloon experiment like the one described in Activity **1**.

Table 2: Balloon experiment data

No. of Breaths	Circumference (cm)
1	31
2	42
3	53
4	58
5	65
6	70

a. Use technology to find a power regression equation for the data in Table **2**.

b. Determine the sum of the squares of the residuals for this regression equation.

c. Describe how well the graph of the regression equation fits a scatterplot of circumference versus number of breaths.

d. Repeat Parts **a–c** using an exponential regression.

Discussion 1

a. Which regression equation—power or exponential—appears to provide a better model for the data? Explain your response.

b. Would you use this model to predict the circumference of a balloon after 50 breaths? Explain your response.

c. If asked to make a prediction based on a given data set, what steps would you take to find an appropriate model?

d. What other criteria might help you select the most appropriate model from several possibilities?

Exploration 2

In this exploration, you examine another tool for evaluating mathematical models: the **residual plot**. (You may recall the use of residual plots from the Level 2 module, "If the Shoe Fits . . . .")

a. Most of the asteroids in our solar system lie between the orbits of Mars and Jupiter. Table **3** shows the orbital period and mean distance from the sun for nine of these asteroids, listed in order of discovery. Use this data to make a scatterplot of period versus mean distance from the sun.

Table 3: Orbital period of nine asteroids

Name	Mean Distance from Sun (millions of km)	Period (earth years)
Ceres	411.20	4.60
Pallas	411.84	4.61
Juno	396.48	4.36
Vesta	350.88	3.63
Astraea	382.88	4.14
Hebe	360.32	3.78
Iris	354.24	3.68
Flora	327.04	3.27
Metis	354.72	3.69

b. Select three of the following five types of equations: linear, quadratic, cubic, exponential, and power. For each type you select, determine a corresponding regression equation to model the data in Table **3**.

c. Create a graph of each regression equation on the same coordinate system as the scatterplot from Part **a**. To simplify comparison, use the same scales on the axes of all graphs.

Mathematics Note

A **residual plot** is a scatterplot created using the ordered pairs (x-value of the data, residual). If the sum of the squares of the residuals is relatively small, a residual plot in which the points are randomly scattered above and below the x-axis typically indicates that a reasonable model has been selected.

For example, Table **4** shows the x- and y-coordinates of a set of data points, the corresponding y-values predicted by the linear regression model $y = 0.75x + 1.14$, and the value of the residual for each data point.

Table 4: Data points, predicted y-values, and residuals

x-value	y-value	Predicted y-value	Residual
1	2	1.89	0.11
2	2	2.64	–0.64
3	4	3.39	0.61
4	5	4.14	0.86
5	4	4.89	–0.89
6	5	5.64	–0.64
7	7	6.39	0.61

Figure **1** shows the corresponding residual plot. Since no pattern appears to exist, this regression equation may be a reasonable model for the data.

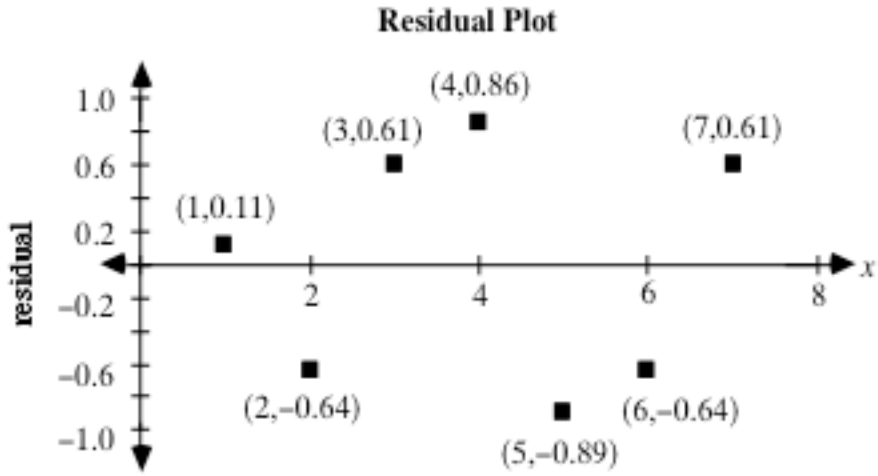

Figure 1: A residual plot

d. Create a residual plot for each model selected in Part **b**.

Discussion 2

a. Of the three regression equations you determined in Exploration **2**, which one appears to provide the best model of the data? Defend your choice.

b. Suppose that a new asteroid is discovered with a mean distance from the sun of $3.50 \bullet 10^8$ km.

1. Describe how you would use your model to predict the asteroid's orbital period.
2. How confident would you feel about the accuracy of this prediction? Explain your response.

c. Another known asteroid orbits the sun at a mean distance of $5.90 \bullet 10^9$ km.

1. Using the model you selected in Part **a** of the discussion, predict the asteroid's orbital period.
2. How confident do you feel about the accuracy of your prediction?

d. German astronomer Johannes Kepler (1571–1630), who lived during the same time as Galileo, also studied the motion of planets. He developed a mathematical description of planetary motion that included three basic laws.

Kepler's third law stated that the ratio of the cube of the mean distance (r) from the sun to the square of the period (p) is a constant for every planet in the solar system. This can be represented algebraically as follows:

$$\frac{r^3}{p^2} = k$$

How does this information affect your choice of an appropriate model for the data in Table **3**?

Assignment

2.1 A meteorite is a chunk of space debris that falls to the earth's surface. The force of impact sometimes forms a crater. The table below shows the depth and diameter of seven meteorite craters on earth.

Name of Crater	Diameter (m)	Depth (m)
Barringer	1240	210
Herault	230	50
Odessa 1	160	40
Odessa 2	21	5
Explosion 1	120	27
Explosion 2	47	14
Explosion 3	32	6

a. Create a scatterplot of crater depth versus diameter.

b. Select a regression equation to model the data. Use the sum of the squares of the residuals and a residual plot to support your choice.

c. Graph your equation on the same coordinate system as the scatterplot in Part **a**.

d. How closely does your model approximate the data? Explain your response.

e. **1.** How confident would you be in predicting the depth of a crater with a diameter of 600 m? Explain your response.

2. How confident would you be in predicting the depth of a crater with a diameter of 5000 m? Explain your response.

f. The Wolf Creek crater has a diameter of approximately 820 m and a depth of approximately 30 m.

1. What depth would your model predict for a crater with a diameter of 820 m?

2. Compare this prediction with the actual depth of the Wolf Creek crater and suggest some possible explanations for any difference that occurs.

2.2 The distances involved when considering objects in the solar system are immense. For example, the mean distance from the sun to the earth is approximately $1.496 \bullet 10^8$ km. Even though light travels at an incredible speed, it still takes several minutes for light from the sun to reach earth.

The following table shows the approximate time required for light from the sun to reach each of the other planets in our solar system, as well as each planet's mean distance from the sun.

Planet	**Mean Distance from Sun (millions of km)**	**Time for Light to Reach Planet (sec)**
Mercury	57.9	195
Venus	108.2	360
Mars	227.9	763
Jupiter	778.3	2601
Saturn	1427.0	4757
Uranus	2869.6	9562
Neptune	4496.6	14,991
Pluto	5899.9	19,670

a. Find a regression equation that fits this data. Defend your selection.

b. Describe how well the equation models the data.

c. Use your model to determine the approximate time required for light from the sun to reach earth.

d. Light travels at a speed of approximately $3 \bullet 10^8$ m/sec. Given this information, did your model provide a reasonable prediction of the time required for light from the sun to reach earth?

2.3 Planets travel along their orbital paths at high speeds. The following table shows the mean distance from the sun and the orbital speed of eight planets in our solar system. Use this data to estimate earth's orbital speed, given that its mean distance from the sun is approximately $1.496 \bullet 10^{8}$ km. Defend your prediction.

Planet	**Mean Distance from Sun (millions of km)**	**Orbital Speed (km/sec)**
Mercury	57.9	47.8
Venus	108.2	35.0
Mars	227.9	24.1
Jupiter	778.3	13.1
Saturn	1427.0	9.6
Uranus	2869.6	6.8
Neptune	4496.6	5.4
Pluto	5899.9	4.7

2.4 The farther a planet is from the sun, the colder its surface temperature is likely to be. The surface temperature of planets, as well as other extreme temperatures, are usually measured in degrees Kelvin. **Note:** The Kelvin temperature scale was invented by Sir William Thomson (1824–1927), also known as Lord Kelvin. The relationship between the Kelvin and Celsius scales is approximately $K = ℃ + 273$.

The table below shows the mean surface temperature for seven planets, along with their mean distances from the sun.

Planet	**Mean Distance from Sun (millions of km)**	**Mean Surface Temperature (Kelvin)**
Mercury	57.9	373
Mars	227.9	250
Jupiter	778.3	123
Saturn	1427.0	93
Uranus	2869.6	63
Neptune	4496.6	53
Pluto	5899.9	43

a. Use the information in the table to determine a model for predicting a planet's surface temperature given its mean distance from the sun. Defend your choice of models.

b. Because of their many similarities, Earth and Venus are often referred to as "sister" planets.

1. Use your model to predict the surface temperature of the two planets, given that Earth's mean distance from the sun is approximately $1.496 \bullet 10^8$ km, while Venus' is approximately $1.082 \bullet 10^8$ km.

2. Do you think that your predictions are reasonable? Explain your response.

c. Earth's actual mean surface temperature is approximately 295 K while Venus' is approximately 753 K. This difference can be attributed to several factors other than the mean distance from the sun, including the composition of each planet's atmosphere.

Given these facts, discuss the dangers of making predictions about complex phenomena using models that only describe the relationship between two quantities.

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2.5 A weak solution of hydrogen peroxide is a common household antiseptic. When heated, hydrogen peroxide decomposes to form water and oxygen gas. The table below shows the change in concentration over time for a heated solution of 1% hydrogen peroxide.

Time (min)	**Percent Concentration**
2	0.90
5	0.78
10	0.60
20	0.37
30	0.22
40	0.13
50	0.08

a. Find a regression equation that models this data. Justify your choice.

b. Use your model to predict the percentage of hydrogen peroxide that remains after 2 hr.

c. How confident are you in the prediction you made in Part **b**?

2.6 The pressure on a fixed amount of gas can be measured using a column of mercury. At a pressure of 500 mm of mercury, a certain amount of gas occupies 10 L. The information in the following table shows how the volume of the gas decreases as the pressure increases.

Volume (L)	**Pressure (mm of mercury)**
10	500
9	556
8	625
7	714
6	833
5	1000
4	1250
3	1667
2	2500

a. Find a regression equation that models this data. Justify your choice.

b. Use your model to predict the pressure on the gas when its volume is 7.33 L.

c. Do you think that your model will provide good predictions for the volumes of gas at very high pressures? Explain your response.

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Research Project

Through taxes and user fees, the U.S. government collects money to pay for national defense, health care, road construction, and numerous other goods and services. These programs often cost more than the government's annual revenue. To make up the difference, the government must borrow money. The total amount owed is known as the national debt.

Conduct some research on the national debt from about 1980 to the present. (Make sure to record the source of the data you collect.) Determine a reasonable model for this data.

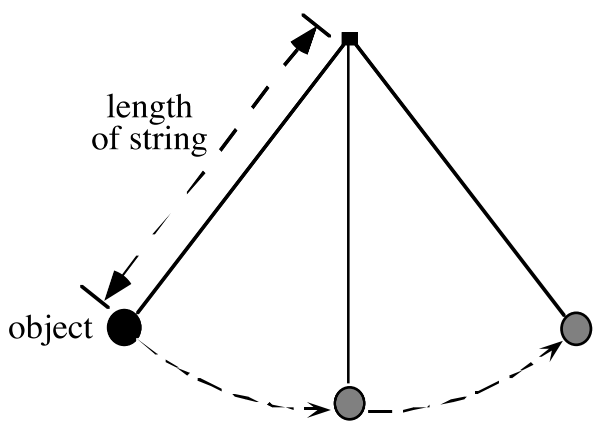
Use your model to predict the size of the national debt in 20 years from now. Describe some of the consequences that may occur if the debt continues to follow the trend described in your model. Finally, discuss why it might be risky to make long-term predictions based on your model.

Summary Assessment

Galileo's inventions and observations helped other astronomers make sense of the motion of the planets and stars. He also made important contributions to the study of other types of motion, including balls rolling on inclined planes, freely falling objects, and swinging pendulums.

While at the chapel of the University of Pisa, Galileo noticed that one of the chandeliers was swinging. Using his own heartbeat as a timer, he measured the time required for the chandelier to complete one swing. Upon returning to his room, he performed a series of experiments that resulted in "the law of the pendulum."

As shown in the diagram below, use a length of string to suspend an object from a fixed location.



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1. A pendulum completes one swing when it returns to the same side as its initial release. The time required for a pendulum to complete one swing is its **period**.

a. Determine a method for measuring one period of your pendulum.

b. Record the length of the string to the nearest 0.01 m and the period of the pendulum to the nearest 0.1 sec.

c. Change the length of the string significantly. Record the new string length and the period for this pendulum.

d. Repeat Part **c** for six different lengths of string.

2. **a.** Find a regression equation that fits the data and explain why you chose this type of model.

b. What does the equation you chose in Part **a** reveal about the motion of pendulums?

c. Describe some of the limitations of your equation for modeling the motion of pendulums.

3. In general, the relationship between the period p of a pendulum and the length l of its string can be described by $p = 2\pi\sqrt{l/g}$, where g is the acceleration due to gravity (about $9.8\ \text{m/sec}^2$). How does the relationship expressed by your model compare to this one?

Module Summary

- If d is a real number greater than 0, m and n are positive integers, and m/n is in lowest terms,

$$d^{m/n} = \left(d^{1/n}\right)^m = \left(\sqrt[n]{d}\right)^m = \sqrt[n]{d^m}$$

- If d is a nonzero real number and n is an integer,

$$d^{-n} = \frac{1}{d^n}$$

- If d is a real number greater than 0,

$$\left(d^m\right)^n = d^{m \bullet n} \text{ and } \frac{d^m}{d^n} = d^{m-n}$$

- A **residual** is the difference between the y-coordinate of a data point and the corresponding y-value predicted by a model.
- A **residual plot** is a scatterplot created using the ordered pairs (x-value of the data, residual). If the sum of the squares of the residuals is relatively small, a residual plot in which the points are randomly scattered above and below the x-axis typically indicates that a reasonable model has been selected.
- When trying to determine a mathematical model for a data set, you may wish to follow these steps:
  1. Create a scatterplot of the data.
  2. Use the shape of the scatterplot and the situation in which the data was collected to identify appropriate types of model equations.
  3. Determine the corresponding regression equations.
  4. Evaluate each potential model using the sum of the squares of the residuals and a residual plot.

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