# Our Town

In this module, you use one community's bridges and traffic patterns to explore the applications of graph theory to city planning.

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# Our Town

## Introduction

The old city of Königsberg (now Kaliningrad, Russia) was situated along the banks of the Pregol River. As shown in Figure **1**, two islands in the river were connected to each other and to the rest of the city by a system of seven bridges.

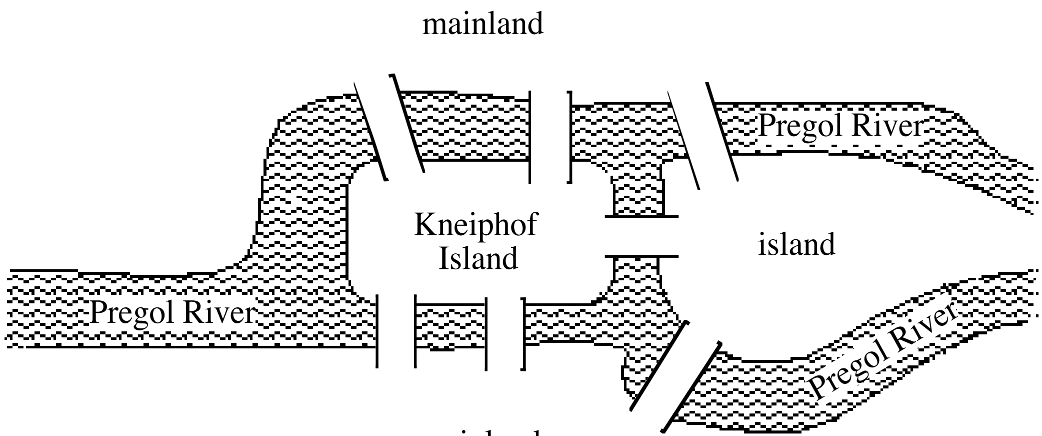



**Figure 1: The bridges of Königsberg**

The bridges of Königsberg inspired a famous mathematics problem: Is it possible to plan a route through the city that crosses each of the seven bridges exactly once before returning to the starting point?

To solve this problem, Swiss mathematician Leonhard Euler (1707–1783) created a simplified map or graph of Königsberg by representing each area of the city as a point and each bridge as an arc. Today, Euler is considered the founder of **graph theory**.

In this module, you use graph theory to investigate some planning problems in the imaginary community of Our Town.

## *Activity 1*

Welcome to Our Town! The town council has called a meeting to discuss the construction of a new bridge. As a newcomer, you will shake hands with many of your fellow citizens for the first time today. Exactly how many handshakes depends on the number of people who attend the meeting. In this activity, you use some of the basic concepts of graph theory to model this situation.

## Exploration

Since the new bridge has become a hot topic in the past week, the town meeting is likely to involve some spirited debate. Hoping to begin the discussion on a friendly basis, the council has organized a cooperative activity. Each person in attendance has been assigned to a small group. Group members will shake hands with each other and tell a little bit about themselves.

**a.** Determine the total number of handshakes for a group of two people.

**b.** Suppose a third person joins the group in Part **a**. Determine the total number of handshakes that will occur if each member shakes hands with every other member of the group.

**c.** Determine the total number of handshakes that will occur in a group containing each of the following numbers of people:

**1**. 4 people

**2**. 5 people

**3**. 6 people

**4**. 7 people

**5**. 10 people.

### Mathematics Note

A **graph** is a non-empty set of **vertices** and the **edges** that connect them. A **loop** is an edge that connects a vertex to itself. A pair of vertices may be connected by more than one edge.

A **simple graph** is a graph without loops in which any pair of vertices has at most one connecting edge.

For example, the two graphs in Figure **2** consist of the same set of vertices. Since the graph on the right has no loops and each pair of vertices is connected by no more than one edge, it is a simple graph.

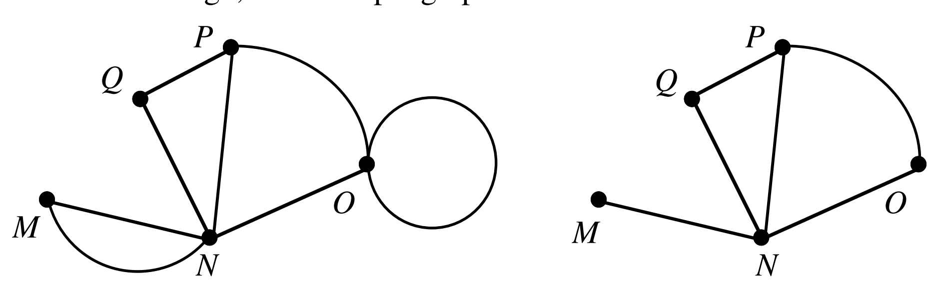



**Figure 2: Two graphs with five vertices each**

**d.** Draw graphs to model the situations in three of the examples in Part **c**. Use vertices to represent people and edges to represent handshakes.

**e.** Develop a method you could use to find the number of handshakes that would occur in a group of 100 people.

**f**. Write a formula for the number of handshakes that would occur in a group of *n* people.

## Discussion

**a.** In your graphs in Part **d** of the exploration, how are the numbers of vertices and edges related to the numbers of people and handshakes?

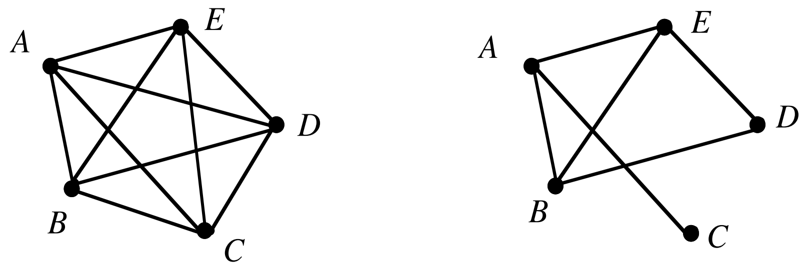
**b.** How can you determine when a graph represents all possible handshakes?

**c.** Compare your graphs with those of others in the class. Describe two examples of graphs that look different, but still model the same situation.

**d.** How do your graphs support the formula you wrote in Part **f** of the exploration?

## Mathematics Note

A **complete graph** is a simple graph in which every distinct pair of vertices is connected by exactly one edge. For example, Figure **3** shows two graphs with the same set of vertices. (Note that the intersection of two edges is not necessarily a vertex.) The graph on the left is complete, while the graph on the right is not.




**a.** complete **b.** not complete

**Figure 3: Two graphs with the same set of vertices**

The **degree** of a vertex is the number of edges that meet at that vertex. If the number of edges that meet at a vertex is even, then the vertex has an **even degree**. If the number of edges that meet at a vertex is odd, then it has an **odd degree**.

For example, the degree of each vertex in the graph in Figure **3a** is 4. The degree of vertex *A* in the graph in Figure **3b** is 3. In Figure **2**, the degree of vertex *O* is 4. (Without the loop, the degree of vertex *O* would be 2.)

**e.** Are the graphs you created in the exploration complete graphs? Justify your response.

## Assignment

**1.1** **a.** To determine the relationships among the number of vertices, the degree of each vertex, and the number of edges in a complete graph, complete the following table.

| No. of Vertices | Sketch of Complete Graph | Degree of Each Vertex | Total No. of Edges | No. of Odd Vertices | No. of Even Vertices |
|---|---|---|---|---|---|
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |

**b.** Determine the degree of each vertex in a complete graph with $n$ vertices.

**c.** Determine the number of edges in a complete graph with $n$ vertices.

**1.2** In a complete graph, what is the relationship between the number of edges and the sum of the degrees of all vertices? Justify your response.

**1.3** Can a complete graph contain exactly one vertex with an odd degree? Justify your response.

**1.4** **a.** Draw five vertices and label them with the names of people you know. Draw an edge between any pair of vertices that represents two people who know each other.

**b.** Is your graph a complete graph? Explain your response.

**c.** Find the degree of each vertex and state whether it is odd or even.

### Mathematics Note

A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices can occur more than once in a path.

For example, Figure **4** shows a graph with five vertices: $A$, $B$, $C$, $D$, and $E$. The sequence of vertices $E$–$B$–$A$–$D$ is a path.

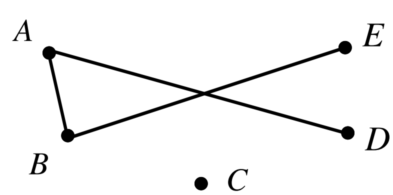



**Figure 4: A graph with five vertices and three edges**

A graph is **connected** if a path exists from each vertex to every other vertex in the graph. Figure **5** shows an example of a connected graph.

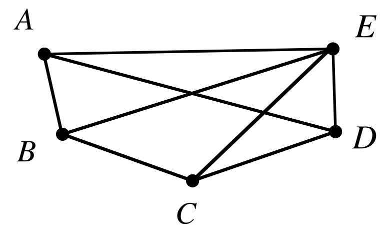



**Figure 5: A graph with five vertices and seven edges**

The graph in Figure **4** is not connected, because there is no path to vertex *C* from any other vertex.

A **closed path** is a path that begins and ends at the same vertex.

A **circuit** is a closed path in which no intermediate vertex is repeated. A circuit may or may not contain all the vertices of a graph.

In Figure **5**, for example, closed path *D–C–B–A–D* is a circuit. Closed path *A–E–B–C–E–D–A*, however, is not a circuit.

A **Hamiltonian circuit** is a circuit in which every vertex in a graph is visited exactly once. In Figure **5**, for example, the path *A–B–C–D–E–A* is a Hamiltonian circuit.

**1.5** **a.** Obtain a map of your school, town, or state. Select five places on the map that you have visited. Draw a vertex to represent each of these locations. Indicate any route you have traveled between locations by drawing an edge between the appropriate pair of vertices.

**b.** Is your graph a complete graph? Justify your response.

**c.** Does your graph contain any circuits? If so, describe one. If not, explain why not.

**d.** Does your graph contain a Hamiltonian circuit? If not, add the edge(s) needed to create one.

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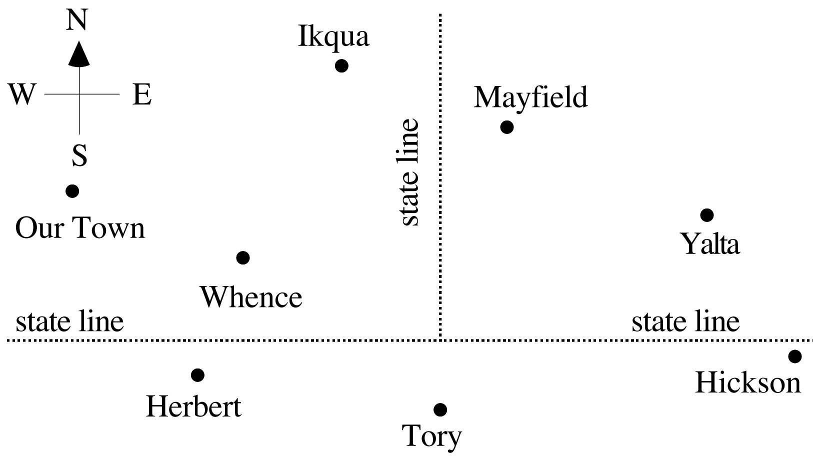
**1.6** A map of the region around Our Town is shown below:




The airlines that serve the region offer the following connections:

- Our Town and Ikqua
- Ikqua and Mayfield
- Yalta and Whence
- Yalta and Tory
- Tory and Hickson
- Our Town and Yalta
- Mayfield and Yalta
- Whence and Herbert
- Hickson and Herbert

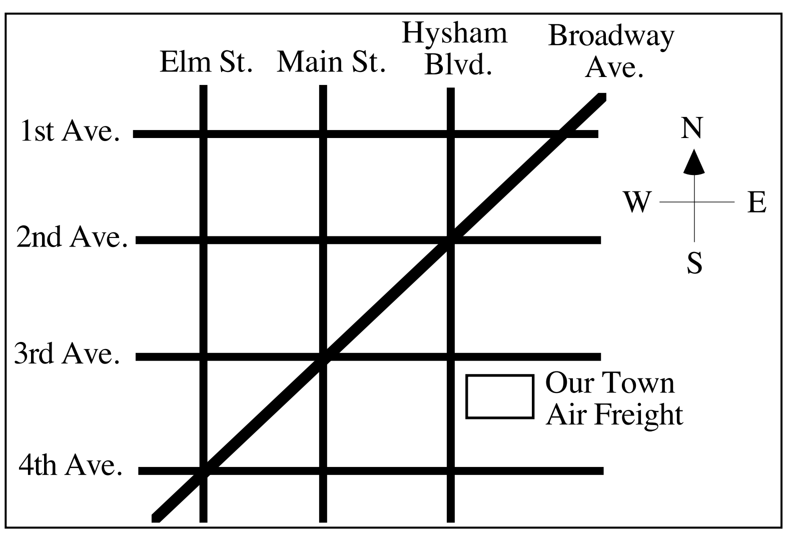
**a.** Draw a graph that models these flight connections.

**b.** Is your graph a complete graph? Explain your response.

**c.** Imagine that you must fly on a business trip that includes Our Town, Mayfield, Yalta, Whence, and Ikqua. Is it possible for you to follow a path that forms a Hamiltonian circuit? Explain your response.

**d.** If a Hamiltonian circuit does exist, should your business trip follow that path? Why or why not?

**1.7** The Our Town Air Freight Company offers pick-up and delivery service in the neighborhood shown below.



The freight truck begins its route at the company's office at the corner of Third and Hysham. The driver's scheduled stops are outlined in the following table. In the left-hand column, the letter P represents a pick-up, while the letter D represents a delivery.

| Pick-up or Delivery | Time | Corner |
|---|---|---|
| P | 8:15 A.M. | Third & Elm |
| D | after 2:00 P.M. | Broadway & Hysham |
| P | 9:00 A.M. | Main & Fourth |
| D | anytime | Broadway & Hysham |
| P | 11:00 A.M. | Third & Hysham |
| P | 11:45 A.M. | Second & Main |
| D | anytime | Third & Main |
| D | anytime | First & Elm |
| D | before 10:00 A.M. | Main & Broadway |
| D | anytime | Fourth & Hysham |

**a.** Draw a graph that models the driver's stops.

**b.** Is the graph a complete graph? Explain your response.

**c.** Is the graph connected? Explain your response.

**d.** Plan an efficient path for the driver to follow. Use terms from graph theory to explain why your route is the best one.

**e.** Is the route you planned in Part **d** a Hamiltonian circuit? Explain your response.

**1.8** Identify each of the following statements as true or false and explain how you determined your response.

**a.** Every polygon forms a complete graph.

**b.** Every polygon forms a connected graph.

**c.** Every polygon contains a Hamiltonian circuit.

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## Research Project

Hamiltonian circuits are named after Sir William Rowan Hamilton (1805–1865). Find out more about Hamilton's life and his contributions to graph theory.

## *Activity 2*

Like the old city of Königsberg, Our Town was built on the banks of a river and contains two islands. As shown in Figure **6**, however, Our Town is connected to its two islands by only five bridges.

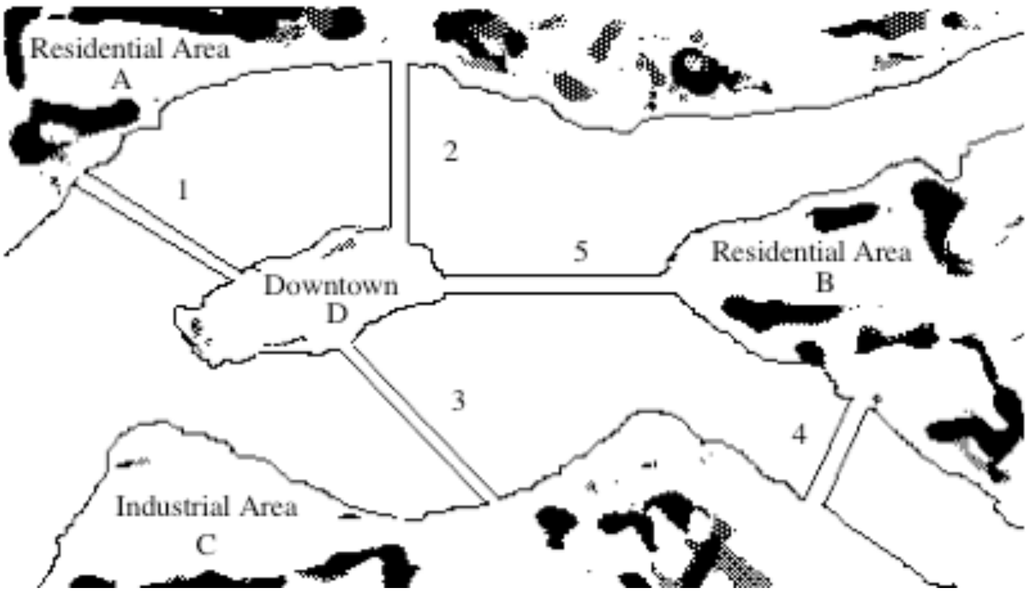



**Figure 6: The bridges of Our Town**

### Exploration 1

The students of Our Town have challenged each other to select a starting point, then follow a path that crosses each bridge exactly once before returning to that point.

**a.** Draw a graph to represent the map of Our Town in Figure **6**.

**b**. Select an area of the town as your starting point. Beginning from this point, is there a closed path that crosses each bridge exactly once? If so, describe this path. If no such path exists, explain why not.

**c.** To accommodate Our Town's growing population, city planners have decided to build two more bridges, bringing the total to seven. These bridges (labeled 6 and 7) are shown in Figure **7**.

Draw a graph to represent a map of the town that includes all seven bridges, then repeat Part **b**.

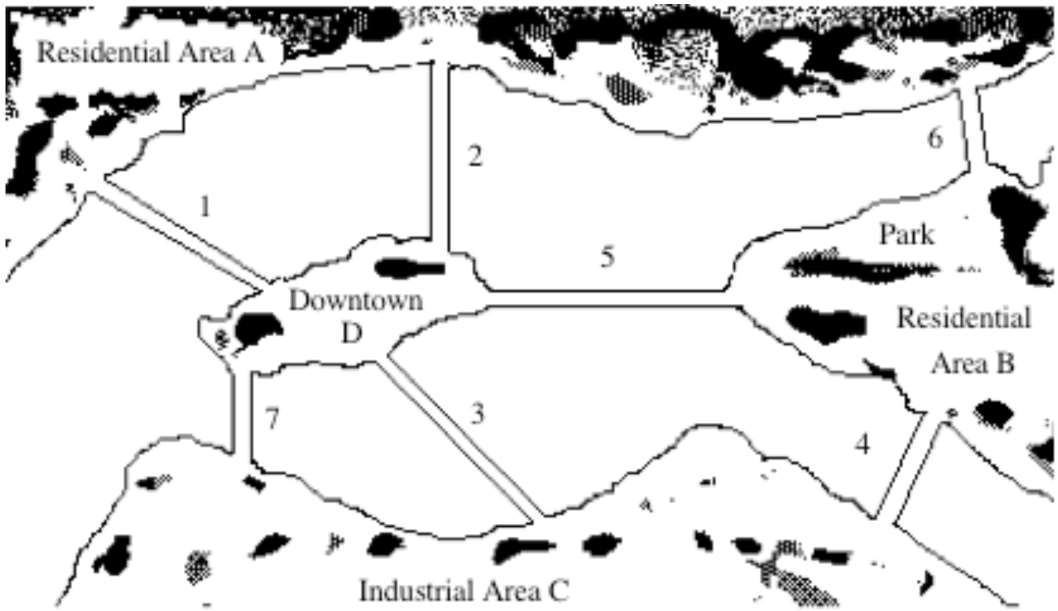



**Figure 7: Our Town with seven bridges**

## Discussion 1

**a.** Are the graphs you created in Exploration **1** simple graphs? Explain your response.

**b.** How did the number of bridges affect your ability to find a closed path that crosses each bridge exactly once?

**c.**
**1.** Is either of the graphs you created in Exploration **1** a complete graph? Explain your response.

**2.** Is either graph a connected graph? Explain your response.

## Mathematics Note

A connected graph is **traversable** if it is possible to traverse every edge of the graph exactly once.

For example, Figure **8** shows a graph with two edges between vertices *B* and *D*. Since path *A–B–D–B–C–D–A* traverses every edge exactly once, this graph is traversable.

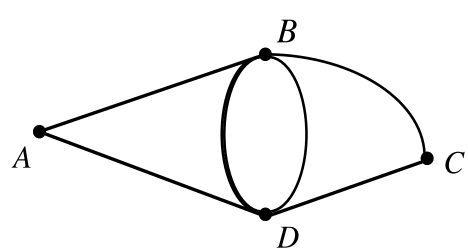



**Figure 8: A traversable graph**

**d.** Are all graphs that have a Hamiltonian circuit traversable?

**e.** Do all traversable graphs have a Hamiltonian circuit?

## Exploration 2

While studying the problem of the bridges of Königsberg, Euler discovered a relationship between the degrees of the vertices of a graph and its traversability. In this exploration, you investigate some of the reasoning behind his discovery.

Figure **9** shows eight different graphs. Use a copy of these graphs supplied by your teacher to complete Parts **a–d**.

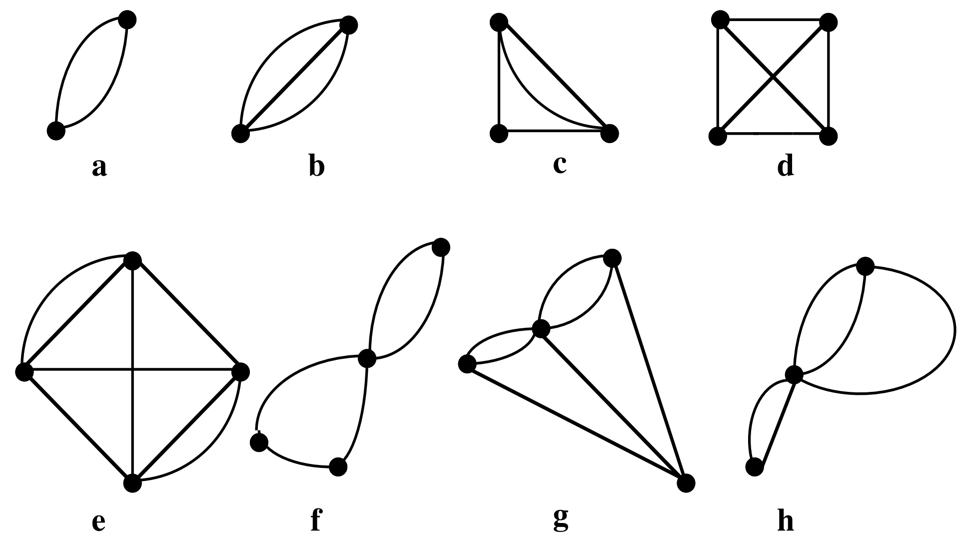



**Figure 9: Eight graphs**

**a.** **1**. Specify the degree of each vertex in the graphs in Figure **9**.

**2**. Determine if there is a path that traverses each graph.

**b.** Use your responses to Part **a** to complete Table **1** below.

**Table 1: Degrees of vertices and traversability of graphs**

| Graph | No. of Odd Vertices | No. of Even Vertices | Traversable? (Yes or No) |
|---|---|---|---|
| a | | | |
| b | | | |
| c | | | |
| d | | | |
| e | | | |
| f | | | |
| g | | | |
| h | | | |

**c.** Use Table **1** to find a rule that describes when a graph is traversable.

## Discussion 2

**a.** What relationship did you find between the traversability of a graph and the degrees of its vertices?

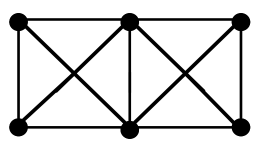
**b.** Using the relationship identified in Part **a** of Discussion **2**, determine if the graph of each of the following is traversable:

1. the system of five bridges in Our Town
2. the system of seven bridges in Our Town.

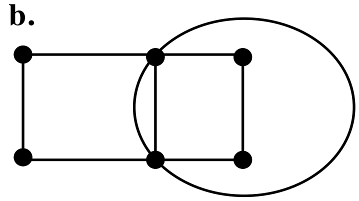
## Assignment

**2.1** Determine whether or not each of the following graphs is traversable. Justify your responses.

**a.**



**b.**



**2.2** Determine if a complete graph with each of the following numbers of vertices is traversable. Explain your responses.

**a.** 5

**b.** 6

**c.** 101

**d.** *n*

**2.3** Draw a traversable graph with five vertices that is not a complete graph.

**2.4** Because of its seven beautiful bridges, Our Town has become a popular tourist destination. To show visitors some of the community's other attractions, the city council has asked you to design a tour that begins and ends downtown and visits all four areas of the city.

**a.** Using the map of Our Town in Figure **7**, design a tour and model it with a graph.

**b.** Is your graph in Part **a** traversable? Justify your response.

**c.** Is the tour you designed in Part **a** a Hamiltonian circuit? Justify your response.

**2.5** **a.** Design a tour that begins downtown and visits all seven bridges in Our Town. Model the tour with a graph.

**b.** Is your graph in Part **a** traversable? Justify your response.

**c.** Is the tour you designed in Part **a** a Hamiltonian circuit? Justify your response.

**2.6** Select four locations in your town or county. Use these locations as the vertices of a graph. Connect the vertices with edges representing major travel routes. Use this graph to show an efficient snowplowing, mail delivery, or garbage removal route. Explain why your route is an efficient one.

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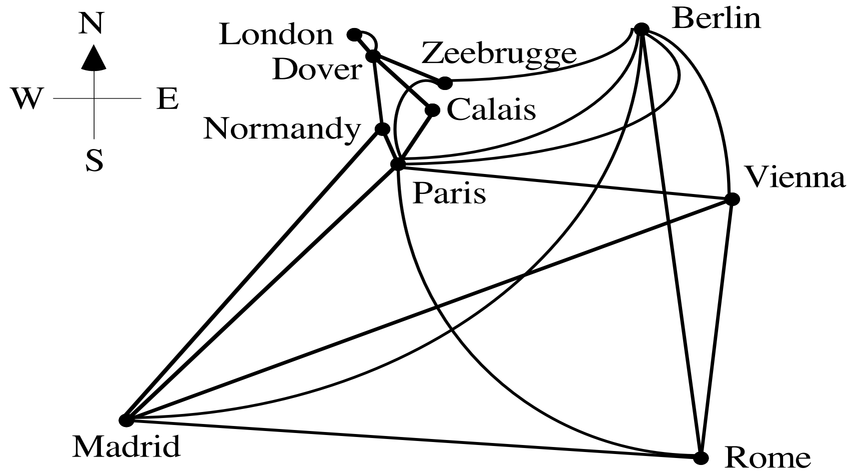
**2.7** The map below shows the six cities included in a European tour sponsored by Our Town Travel.


**a.** Each of the six cities is connected to every other one by a direct flight. Draw a graph to model these connections.

**b.** Use your graph from Part **a** to design a tour that is a Hamiltonian circuit.

**2.8** Malcolm lives in London and would like to visit the same cities offered in the tour in Problem **2.7**. His travel agent suggests that he take a bus from London to Dover, then take a ferry across the English Channel—either to Normandy, Calais, or Zeebrugge. Once on the European continent, he can then visit the other six cities by train. The available connections are shown in the following graph.




Malcolm would like to experience each connection on the graph, without repeating any one of them, if possible. Is it possible to create a tour that begins and ends in London and traverses each connection exactly once? If so, describe such a tour. If not, explain why not.

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## Research Project

Select a city built along one or more rivers, such as Pittsburgh, St. Louis, or New York. Find a map of the city and draw a graph that represents the city's system of bridges. Demonstrate the traversability of the bridge system to the class and discuss the system in terms of Hamiltonian circuits. Describe how the system changes if some of the bridges allow only one-way traffic.

## *Activity 3*

To speed the flow of traffic in Our Town, planners have restricted movement on some of the city's bridges. As shown in Figure **10**, bridges 1, 2, 3, and 7 allow only one-way traffic. Bridges 4, 5, and 6 allow two-way traffic. When travel on any bridge is blocked, access to different parts of the city can be severely affected. In this activity, you use graphs to determine routes that avoid possible blockages.

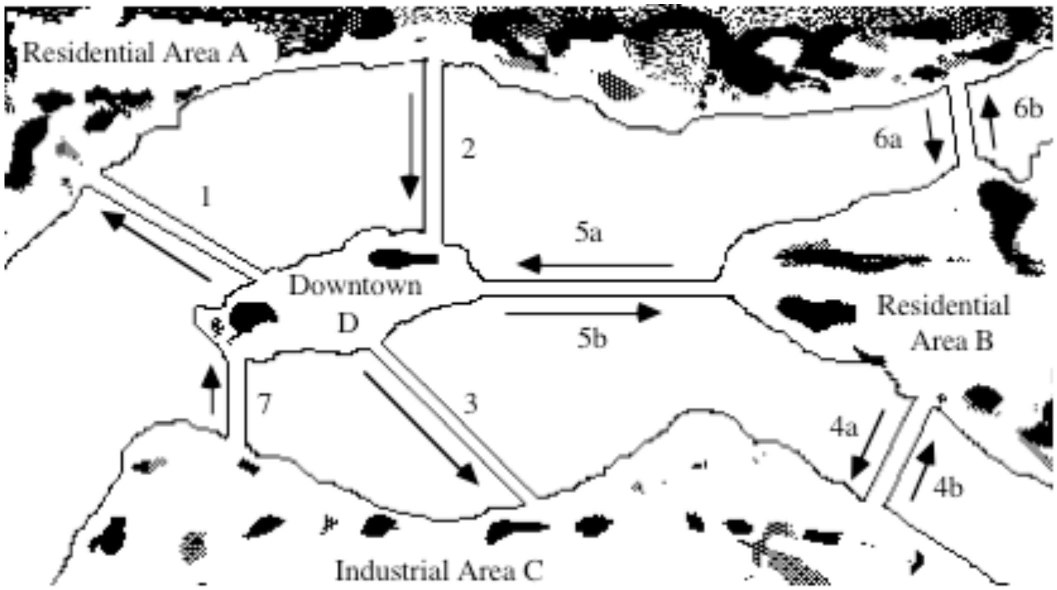



**Figure 10: One-way and two-way bridges of Our Town**

## Exploration

The high school in Our Town is located on the downtown island. On your way home from school one afternoon, you discover that the bridge connecting your neighborhood to the downtown area is closed due to an accident. What alternate route would you use to get home as quickly as possible? In this exploration, you use matrices to examine the possibilities.

## Mathematics Note

A directed graph or **digraph** is a graph in which each edge indicates a single direction.

Figure **11** shows one example of a digraph. Notice that vertices *A* and *C* are connected by paths in both directions.

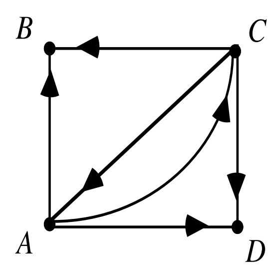



**Figure 11: A digraph**

**a.** Using a copy of the map in Figure **10** supplied by your teacher, create a digraph of the bridges of Our Town showing the possible directions of traffic. Let each vertex represent one of the four areas of Our Town.

**b.** A one-bridge route requires only one bridge crossing to get from one area to another. Table **2** represents a partially completed matrix of one-bridge routes from each area of Our Town to every other area of the town. For example, the element in column B, row A indicates that from area A to area B there is 1 one-bridge route. Use your digraph to complete the matrix.

**Table 2: Matrix of one-bridge routes (all bridges open)**

| | | **to** | | | |
|---|---|---|---|---|---|
| | **Area** | **A** | **B** | **C** | **D** |
| | **A** | 0 | 1 | 0 | 1 |
| **from** | **B** | | 0 | | |
| | **C** | | | 0 | |
| | **D** | | | | 0 |

**c.** Suppose that bridge 5 is closed for repairs. In this case, it would be impossible to drive directly from area D to area B. Using bridge numbers, describe all alternate routes from area D to area B that cross:

1. two bridges
2. three bridges.

**d.**
1. Delete edge 5b from your digraph in Part **a** to create a digraph representing the traffic flow when bridge 5b is closed.
2. Create a matrix like the one in Part **b** to represent the one-bridge routes between areas of Our Town after bridge 5b has closed.

**e.**
1. Use your digraph from Part **d** to count the number of two-bridge routes between each area of Our Town. Use these values to create a matrix representing the two-bridge routes. **Note**: A two-bridge route requires exactly two bridge crossings. The two crossings can be over the same bridge.
2. In your two-bridge matrix, identify the element that represents the number of two-bridge routes from D to B. How does this number compare to your response to Part **c1**?

**f.**
1. Square the matrix you created in Part **d2**.
2. Compare the information in the squared matrix with the values in your matrix from Part **e1**.

**g.**
1. Cube the matrix created in Part **d2**.
2. Identify the element in row D, column B of the cubed matrix. How does this number compare to your response to Part **c2**?

## Discussion

**a**. If the matrix of one-bridge routes in Table **2** were squared, what information would be obtained?

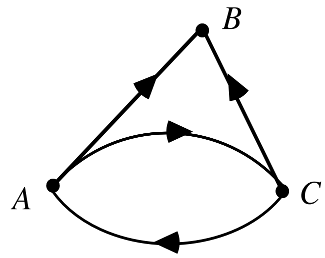
**b**. If the matrix of one-bridge routes were cubed, what information would be obtained?

**c.** If the matrix of one-bridge routes were raised to the *n*th power, what information would be obtained?

**d.** In Part **g** of the exploration, you cubed the matrix of one-bridge routes after bridge 5b had closed. The elements in the resulting matrix represent the number of three-bridge routes when bridge 5b is closed. Are each of these routes also paths?

## Assignment

**3.1** **a.** Create a matrix of one-edge routes for the digraph below.



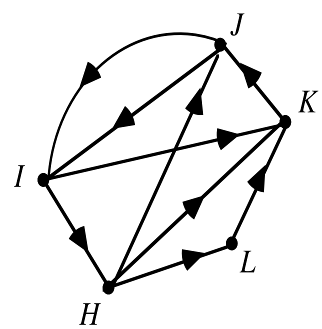

**b.** Create a matrix of two-edge routes for this digraph.

**c.** List all the possible two-edge routes from *C* to *B*.

**d.** Create a matrix of three-edge routes for the digraph.

**e.** List all the possible three-edge routes from *C* to *B*.

**3.2** **a.** Create a matrix of one-edge routes for the digraph below.




**b.** Create a matrix of two-edge routes for this digraph.

**c.** List all the possible two-edge routes from *J* to *J*.

**d.** Create a matrix of three-edge routes for this digraph.

**e.** List all the possible three-edge routes from *J* to *J*.

**3.3** Our Town is planning to build a new hospital. Before selecting a site, city planners must analyze the traffic flow to each potential location. Since traffic moves more slowly over bridges, they would like to maximize the number of one-bridge routes to the hospital. However, since individual bridges are often blocked or congested, they would also like to maximize the number of two-bridge routes.

**a.** Imagine that you are a city planner in Our Town. Using a copy of the map in Figure **10** supplied by your teacher, recommend an area to build the new hospital.

**b.** Prepare a presentation to the town council that supports your recommendation.

**3.4** As suggested in the exploration, bridge 5b, the oldest bridge in Our Town, is often closed for repairs. With this in mind, the city planner must consider how hospital access would be affected if bridge 5b were closed.

**a.** Using a copy of the map in Figure **10** supplied by your teacher, recommend a site for the new hospital for which access will be least affected when bridge 5b is closed.

**b.** Prepare a presentation to the town council that supports your recommendation.

**3.5** Draw a digraph that corresponds with the matrix of one-edge routes below.

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} H & I & J & K \end{array} \\ \text{from} \begin{array}{c} H \\ I \\ J \\ K \end{array} & \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

* * * * *

**3.6** Use the matrix and digraph from Problem **3.5** to complete Parts **a–d**.

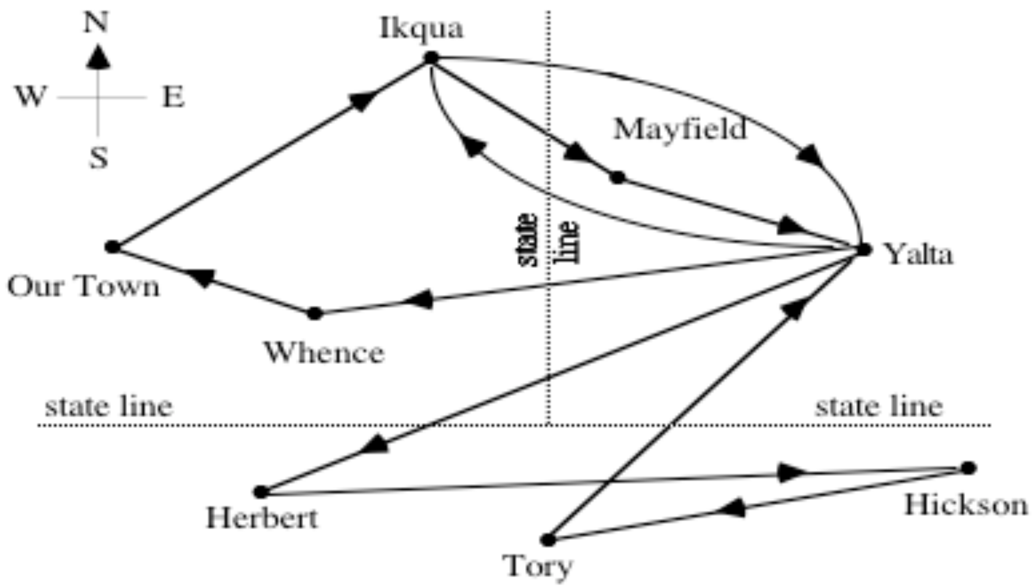
**a.** Describe all the two-edge routes from $I$ to $J$.

**b.** Describe all the two-edge routes from $J$ to $H$.

**c.** Determine the number of three-edge routes that exist between each pair of vertices.

**d.** Describe all the three-edge routes from $J$ to $I$.

**3.7** As illustrated in the digraph below, Friendly Skies Airlines offers commuter service to and from the cities near Our Town.




**a**. Create a matrix of the one-edge routes in this digraph.

**b**. Which city has the most direct flights to the other cities? Describe how your matrix supports your response.

**c.** A company new to the region around Our Town must decide where to locate its corporate headquarters. During a typical week, each member of its sales staff travels to a different city each day from Monday to Wednesday, then returns home on Thursday night. On Friday, all company employees must report to the home office.

1. Select the best location for the corporate headquarters and explain how you made your choice.
2. Using the location you recommended in Step **1**, create a schedule that shows the possible weekly itineraries for the members of the company's sales staff. The schedule should include the day, the town from which the person is leaving, and the town to which the person is traveling.

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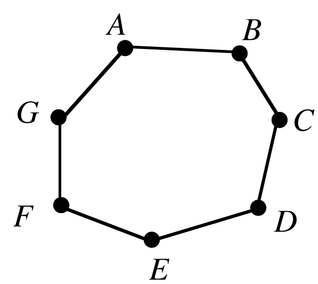
## *Summary Assessment*

**1.** Imagine that you are the president of a computer company in Our Town. A law firm has hired your company to install a computer network for their seven attorneys. Using the network, each attorney must be able to communicate with every other member of the firm.

**a.** Draw a graph showing the necessary connections between the seven lawyers.

**b.** Why does the graph of the connections in Part **a** need to be a complete graph?

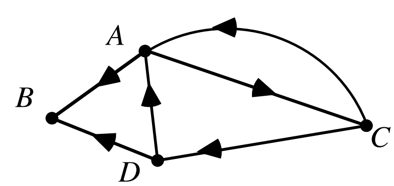
**c.** The actual wiring of a computer network may differ from the communication paths. The graph below shows a possible wiring diagram for a seven-computer network.




Does this graph contain a Hamiltonian circuit? Explain your response.

**d.** Is the graph in Part **c** traversable? Explain your response.

**2.** **a**. Create a matrix of the one-edge routes between each pair of vertices in the digraph below.



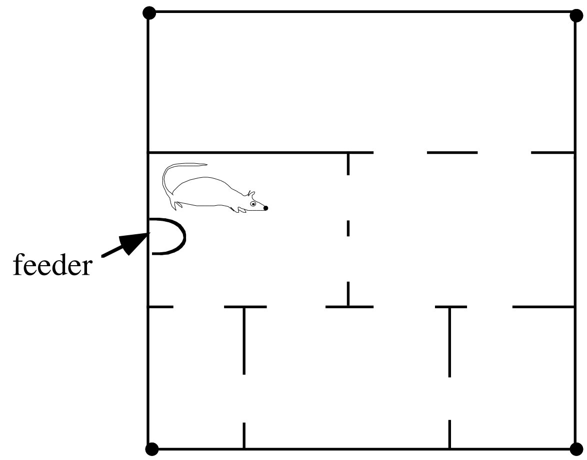

**b**. How would you determine the number of two-edge routes that exist between each pair of vertices?

**c**. How would you determine the number of three-edge routes that exist between each pair of vertices?

**d**. Beginning from vertex $A$, is there a closed path that traverses every edge exactly once? If not, add one or more directed edges so that the graph does contain such a path.

**e**. Does the graph contain a Hamiltonian circuit that begins at vertex *A*? If not, add one or more directed edges so that the graph does contain such a circuit.

**3.** To study the intelligence of mice, a biologist has designed the maze shown in the diagram below.




**a.** In one experiment, a mouse starts at the feeder. It is rewarded only if it passes through every doorway exactly once before returning to the feeder. What type of path must the mouse traverse to earn a reward?

**b.** Is it possible for the mouse to earn a reward in the experiment described in Part **a**? Explain your response.

**c.** In another experiment, the mouse starts at the feeder and is rewarded only if it passes through every room exactly once before returning to the feeder. What type of path must the mouse traverse to earn a reward?

**d.** Is it possible for the mouse to earn a reward in the experiment described in Part **c**? Explain your response.

**e.** By adding or subtracting doors, create a maze in which it is possible for the mouse to earn a reward in the experiment in Part **c** but not possible in the experiment in Part **a**.

## *Module Summary*

- A **graph** is a non-empty set of **vertices** and the **edges** that connect them. A **loop** is an edge that connects a vertex to itself. A pair of vertices may be connected by more than one edge.
- A **simple graph** is a graph without loops in which any pair of vertices has at most one connecting edge.
- A **complete graph** is a simple graph in which every distinct pair of vertices is connected by exactly one edge.
- The **degree** of a vertex is the number of edges that meet at that vertex. If the number of edges that meet at a vertex is even, then the vertex has an **even degree**. If the number of edges that meet at a vertex is odd, then it has an **odd degree**.
- A **path** is a sequence of vertices connected by edges in which no edge is repeated. Vertices can occur more than once in a path.
- A graph is **connected** if paths connect each vertex to every other vertex in the graph.
- A **closed path** is a path that starts and stops at the same vertex.
- A **circuit** is a closed path in which no intermediate vertex is repeated.
- A **Hamiltonian circuit** is a circuit in which every vertex in a graph is visited exactly once.
- A connected graph is **traversable** if it is possible to traverse every edge of the graph exactly once.
- A directed graph or **digraph** is a graph in which each edge indicates a single direction.

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