One Dish and Two Cones

Shapes based on conic sections are commonplace in our technological society. In this module, you learn to recognize conic shapes and investigate some of their properties.

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Introduction

In today's fast-paced world of telecommunications, satellite dishes have become common sights. They sprout from backyards and office buildings, roam the streets on mobile transmission trucks, and bring a world of information to schools and communities. How does the shape of a satellite dish affect how the dish works? You can answer this question by investigating **conic sections**.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a right circular cone. Depending on the slope of the plane, the intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**, as shown in Figure **1** below.

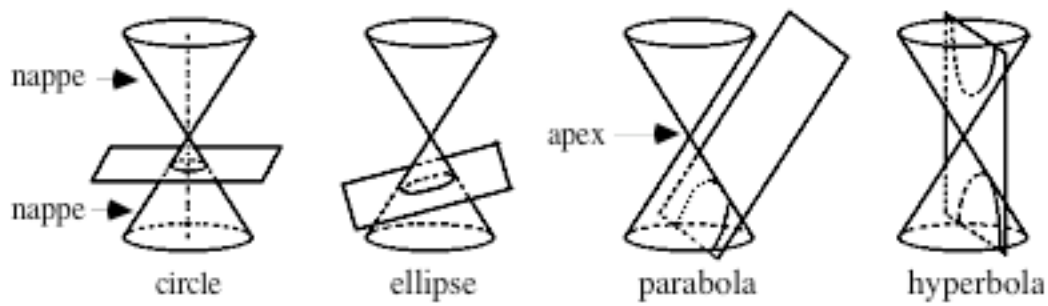


Figure 1: Conic sections

Mathematicians have written about these shapes for well over 2000 years. In fact, the Greek geometer Appollonius of Perga wrote eight books on conic sections in the third century B.C. Since that time, mathematicians have continued to investigate and explore the special properties of these curves.

Exploration

A flashlight produces a partial cone of light. As shown in Figure **2**, the light defines a shape when it strikes a flat surface.

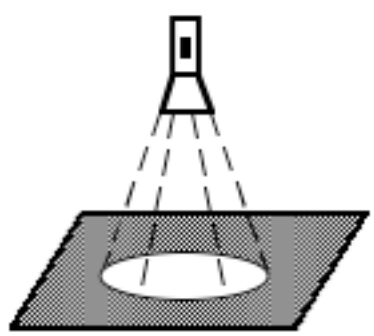


Figure 2: Flashlight, flat surface, and cone of light

a. Use a flashlight to create a partial cone of light. Holding the flashlight still, position a flat surface so that the light forms a circle.

b. Record the relationship between the cone of light and the position of the flat surface.

c. Repeat Parts **a** and **b** to form an ellipse, a parabola, and one-half of a hyperbola.

Discussion

a. Describe how to create each of the following conic sections using a flashlight and a flat surface:

1. a circle
2. an ellipse
3. a parabola
4. one-half of a hyperbola.

b. When the intersection of a plane and a cone contains the cone's apex, geometric figures other than a circle, an ellipse, a parabola, or a hyperbola are formed. These other intersections are **degenerate conic sections**.

1. Describe the geometric shapes of the three degenerate conic sections.
2. For each one, describe the location of the plane relative to the cone.

Activity 1

Conic sections can be found in many real-world situations and are used in many practical applications. The orbits of the planets in our solar system, for example, are elliptical. The shape of a parabola appears in automobile headlights, telescopes lens, microwave transmitters, and satellite dishes. In this activity, you continue your exploration of conics through paper folding.

Exploration 1

a. Cut out the circle template provided by your teacher.

b. Mark a point P inside the circle but not at the center.

c. Select a point on the circle and label it P_1. Label the remaining points P_2 through P_{20}.

d. As shown in Figure **3**, fold and crease the paper so that P_1 coincides with P.

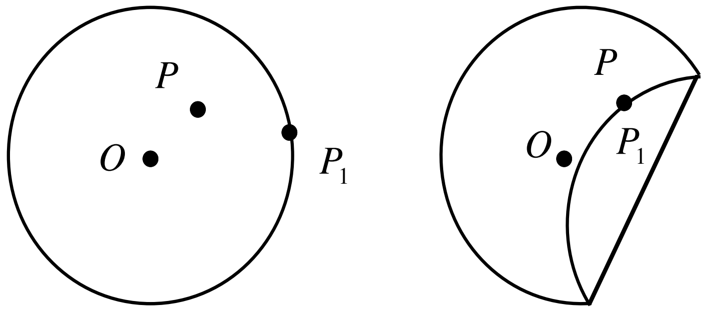


Figure 3: Folding paper so that P_1 coincides with P

e. Unfold the paper. As shown in Figure **4**, use a straightedge to mark a point where the crease intersects $\overline{OP_1}$, a radius of circle O. Label this point X_1.

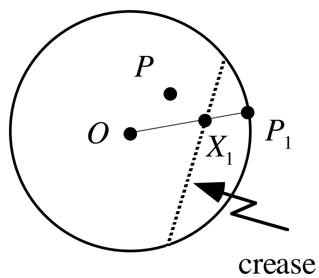


Figure 4: Marking point X_1

f. Repeat Parts **d–e** for each labeled point on the circle. This creates the points X_2, X_3, X_4, …, X_{20}.

g. Draw a smooth curve by connecting X_1 through X_{20}. **Note:** Save your construction for reference later in the module.

Discussion 1

a. What conic section results from the process described in Exploration **1**?

b. Would you expect this conic to be exactly the same for everyone in the class? Explain your response.

c. What happens to the shape of the conic when P is moved closer to the center of the circle?

d. In Part **c** of Exploration **1**, you formed a crease by folding the paper so that P_1 on the circle coincides with P. Describe how the line represented by the crease is related geometrically to $\overline{PP_1}$.

Mathematics Note

Each conic separates a plane into three regions: the conic itself, an "interior," and an "exterior." In Figure **5** below, the interior of each conic is shaded. Note that the boundaries are not included in the interior, and that a hyperbola has two branches.

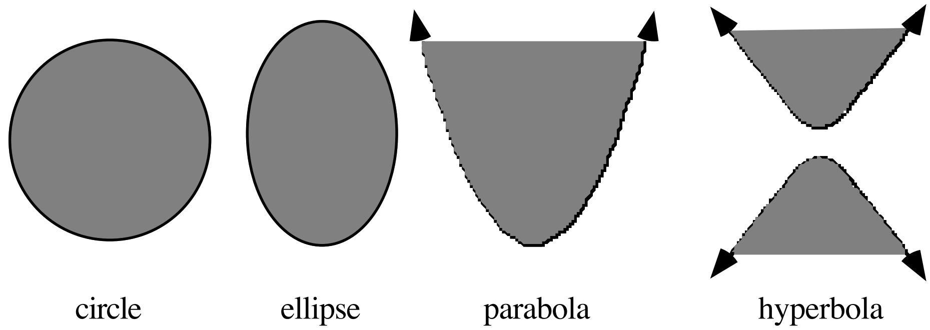


Figure 5: Interiors of a circle, ellipse, parabola, and hyperbola

A **tangent line** to a conic is a line in the plane of the conic that intersects the curve at exactly one point and contains no points in the interior. The point at which the conic and the tangent line intersect is the **point of tangency**.

For example, Figure **6** shows a line tangent to a circle. A radius drawn to the point of tangency on a circle is perpendicular to the tangent line.

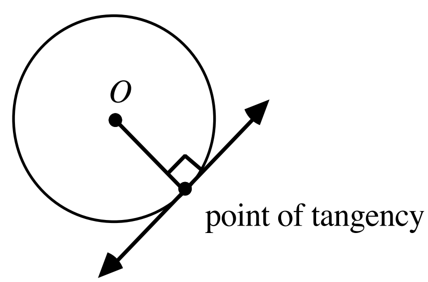


Figure 6: A tangent line to circle *O*

e. How are the lines represented by creases in the paper related to the conic section?

f. What do the points $X_1, X_2, X_3, \ldots, X_{20}$ represent in relation to the lines described in Part **e** of Discussion **1**?

Exploration 2

In this exploration, you use a geometry utility to create three of the four conics. This method of constructing conics is directly related to the paper-folding you did in Exploration **1**.

a. Construct a large circle and label its center O. Construct a point P_1 on the circle so that it moves freely around the circle without changing the circle's size.

b. Construct a point P in the interior of the circle, but not at the center.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct $\overleftrightarrow{OP_1}$.

f. Mark the intersection of the perpendicular bisector from Part **d** and $\overleftrightarrow{OP_1}$. Label this point X. Your construction should now resemble the one shown in Figure **7**.

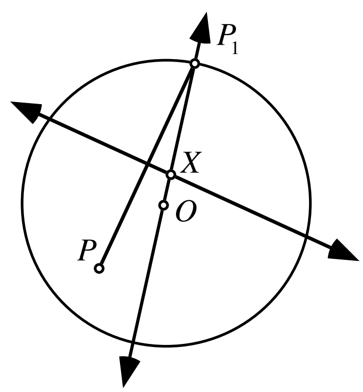


Figure 7: Beginning conic construction

g. Trace the path of X as P_1 is moved around the circle. Note which conic is formed by the path.

h. Move P to a different location inside the circle and repeat Part **g**. Experiment with several other locations of P, including the center of the circle as well as points on and outside the circle. **Note:** Save this construction for use in Activity **2**.

Discussion 2

a. Which step in Exploration **2** corresponds with forming a crease in the paper in Exploration **1**? Explain your response.

b. Which conic is formed when P is at each of the following locations?

1. inside the circle but not at the center

2. at the center of the circle

3. outside the circle

4. on the circle

c. In Exploration **2**, where are O and P located in relation to the interiors of each conic?

Assignment

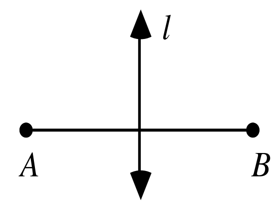
1.1 Complete Parts **a–c** below for each conic generated in Exploration **2**.

a. Describe or draw any axes of symmetry.

b. The **center** of each of these conics is its point of rotational symmetry. Describe the locations of the center in relation to the lines of symmetry.

c. Describe where *O* and *P* are located in relation to the center of each conic and its axes of symmetry.

1.2 In the diagram below, line *l* is the perpendicular bisector of $\overline{AB}$.



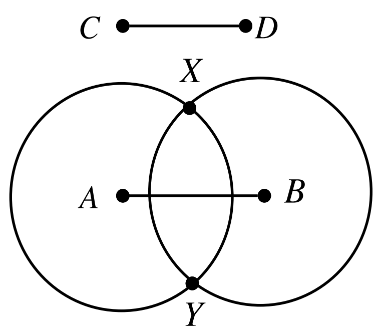
a. Draw a copy the diagram above. Select a point on line *l* and label it *C*.

b. Draw $\overline{AC}$ and $\overline{BC}$.

c. What conclusions can you make about the measures of $\overline{AC}$ and $\overline{BC}$? Explain your response.

d. What conclusions can you make about the measures of $\angle CAB$ and $\angle CBA$? Explain your response.

1.3 **a.** Use a geometry utility to create the diagram below. Read Steps **1–5** before beginning the construction.



1. Construct a line segment and label the endpoints *A* and *B*.
2. Construct another line segment and label the endpoints *C* and *D*.
3. Construct two different circles centered at *A* and *B* with radii *CD*.
4. Label the two intersections of the circles *X* and *Y*.
5. Trace the path of points *X* and *Y* as the length of $\overline{CD}$ changes.

b. The line created by tracing *X* and *Y* has certain properties in relationship to $\overline{AB}$. Describe these properties.

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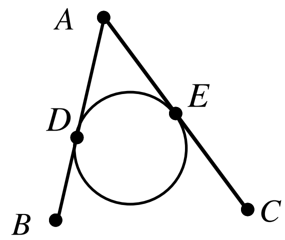
1.4 In the following diagram, $\overline{AB}$ and $\overline{AC}$ are tangent to the circle at points D and E, respectively.



a. Describe how you could use the tangents to locate the circle's center.

b. Describe the relationship between $\overline{AD}$ and $\overline{AE}$.

1.5 Consider a circle with center at point O. A line intersects the circle at points P and Q.

a. Can $\overline{QP}$ lie on a tangent to the circle? Explain your response.

b. Can $\overline{OP}$ be perpendicular to $\overline{QP}$? Explain your response.

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Activity 2

One way to define the conics is to consider each shape in geometric terms. In the following exploration, you use a geometry utility to investigate the relationship between the distances from some specific points not on the conics to points on the conics themselves.

Exploration

a. Using your construction from Exploration **2** of Activity **1,** begin with point P in the interior of the circle but not the center, as shown in Figure **8** below. Measure the radius $\overline{OP_1}$.

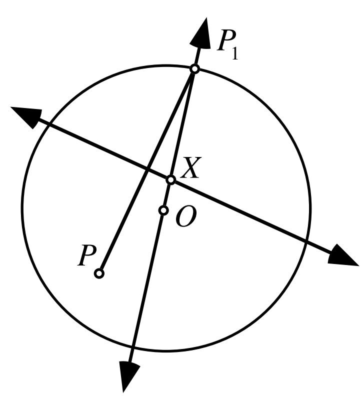


Figure 8: Beginning conic construction

b. Measure the distances from O to X and from P to X.

1. Find the sum of OX and PX.
2. Find the absolute value of the difference of OX and PX.

c. Retrace the path of X as P_1 is moved around the circle. Note the conic formed and describe what happens to the sum of OX and PX and the absolute value of the difference of OX and PX.

d. Repeat Parts **b** and **c** when P is located outside the circle.

e. Change the size of the circle, then repeat Parts **a–d**. Note any changes you observe in the sum of OX and PX and the absolute value of the difference of OX and PX as the conics are generated.

Discussion

a. As P_1 is moved around the circle with P located inside the circle, which value remains constant: the sum of OX and PX or the absolute value of the difference of OX and PX? How could you use this constant value to describe the conic generated?

b. When P is located at the center of circle in the exploration, the conic generated by the path of X is a circle. In this situation, what would be true about the sum of OX and PX and the absolute value of the difference of OX and PX?

c. Why do you think a circle can be thought of as a special case of an ellipse?

d. As P_1 is moved around the circle with P located outside the circle, which value remains constant: the sum of OX and PX or the absolute value of the difference? How could you use this constant value to describe the conic generated?

Mathematics Note

An **ellipse** is a set of all points in a plane such that the sum of the distances from any point in the set to two fixed points is a constant. Each of these fixed points is a **focus** (plural **foci**). Figure **9** shows one example of an ellipse. Points F_1 and F_2 are the foci, and $F_1X + F_2X$ is a constant (AB) for any point on the ellipse.

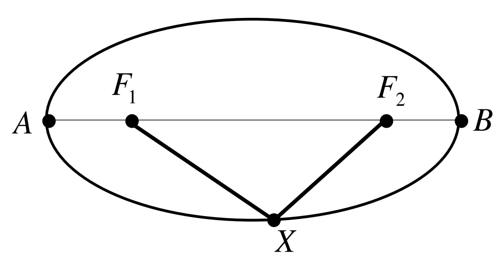


Figure 9: An ellipse

A **circle** is a set of all points in a plane equidistant from a given fixed point, the center. A circle is a special case of an ellipse, where both foci are located at the center of the circle.

A **hyperbola** is a set of all points in a plane such that the absolute value of the difference in the distances from any point in the set to two fixed points (the foci) is a constant. Figure **10** shows one example of a hyperbola. Points F_1 and F_2 are the foci, and $|F_1X - F_2X|$ is a constant (AB) for any point on the hyperbola.

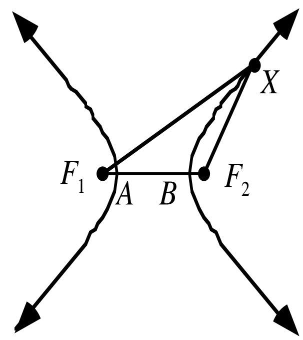


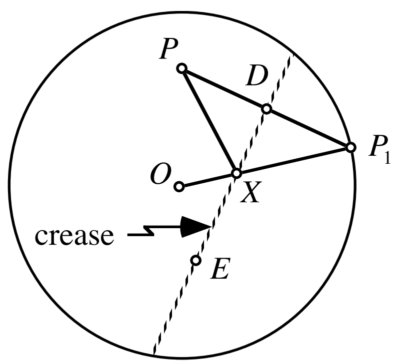
Figure 10: A hyperbola

e. Which points in the exploration correspond to the foci of an ellipse?

f. Which points correspond to the foci of a hyperbola?

Assignment

2.1 In Activity **1**, you used paper folding to model the construction of an ellipse, as shown in the diagram below. Suppose that the crease line in this diagram represents a mirror. As you may recall from the Level 1 module "Reflect on This," when a light ray strikes a flat mirror, the outgoing angle is congruent to the incoming angle.



Describe the path of a light ray passing through O and striking the mirror at X. Justify your response.

2.2 In the diagram given in Problem **2.1**, X is a point on the perpendicular bisector of $\overline{PP_1}$.

a. What is the relationship between the lengths of $\overline{PX}$ and $\overline{P_1X}$?

b. How would this relationship change if X were moved to another location along the crease? State your conclusion as a generalization about all points on the perpendicular bisector of a segment.

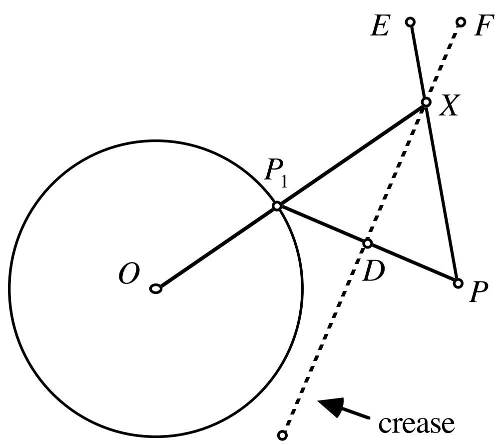
2.3 **a.** Paper folding can be used to demonstrate many geometric relationships in the conics. Use paper folding to model the construction of a hyperbola by completing Steps **1–5** below.

1. Draw a circle on a sheet of paper and label its center O.

2. Mark a point P outside the circle.

3. Select one of the points on the circle and label it P_1.

4. Fold and crease the paper so that P_1 coincides with P as shown in the following diagram.



5. Unfold the paper. As shown in the diagram, mark a point X where the crease intersects $\overleftrightarrow{OP_1}$ and a point D where the crease intersects $\overline{PP_1}$.

b. Write an argument showing that the crease is the perpendicular bisector of $\overline{PP_1}$. (Your argument must demonstrate that the crease bisects $\overline{PP_1}$ and is perpendicular to $\overline{PP_1}$.)

c. Write an argument showing that $\angle PXD$ is congruent to $\angle EXF$.

d. Suppose that the crease line represents a mirror. Describe the path of a light ray passing through E and striking the mirror at X. Justify your response.

2.4 Like the rest of the planets in our solar system, the earth's orbit is approximately an ellipse with the sun located at one focus. When the earth is located at its closest point to the sun, or at its farthest point from the sun, it lies along a line containing the foci of the elliptical path. At the closest point in its orbit, the earth is about $1.47 \bullet 10^8$ km from the sun. At the farthest point, the earth is about $1.52 \bullet 10^8$ km from the sun.

a. Determine the length of the segment that contains the foci of the ellipse and whose endpoints are on the ellipse.

b. What is the distance between the sun and the other focus?

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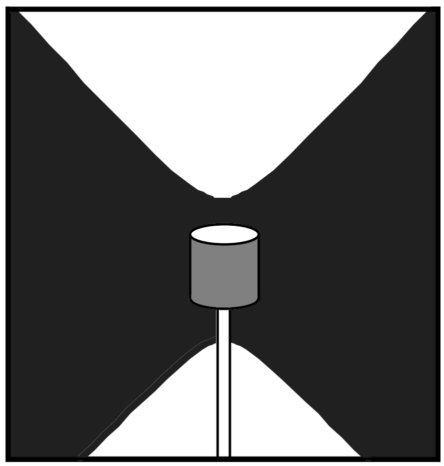
2.5 The shadow created by the lamp in the figure below forms a conic section. Identify this conic and justify your response.



2.6 According to Boyle's law, the volume (V) of a fixed amount of gas at constant temperature varies inversely with the pressure (P) on it. This relationship can be expressed as $PV = c$, where c is a constant.

a. A certain amount of oxygen gas occupies a volume of 500 cm^3 at a pressure of 0.978 atmospheres (atm). Find the constant c for this amount of oxygen.

b. Use the constant c from Part **a** to graph the equation $PV = c$. Represent volume on the vertical axis and pressure on the horizontal axis. Indicate an appropriate domain and range for this setting.

c. The graph in Part **b** is a conic section. Identify this conic and justify your response.

d. Determine the volume of the oxygen when the pressure is 1.5 atm.

e. Determine the volume of oxygen when the pressure is 0.75 atm.

f. If you double the pressure, what is the effect on the volume?

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Activity 3

As you observed in the previous activity, circles, ellipses, and hyperbolas can all be generated in a similar manner. In this activity, you examine the geometric properties of a parabola using a different approach.

Exploration

a. Using a geometry utility, construct a long, horizontal segment. Construct a point P_1 on the segment.

b. Construct a point *P* not on the segment.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct a line perpendicular to the horizontal segment through P_1.

f. Construct a point at the intersection of the line from Part **e** and the perpendicular bisector of $\overline{PP_1}$. Label this intersection *X*. Your construction should now resemble the one shown in Figure **11**.

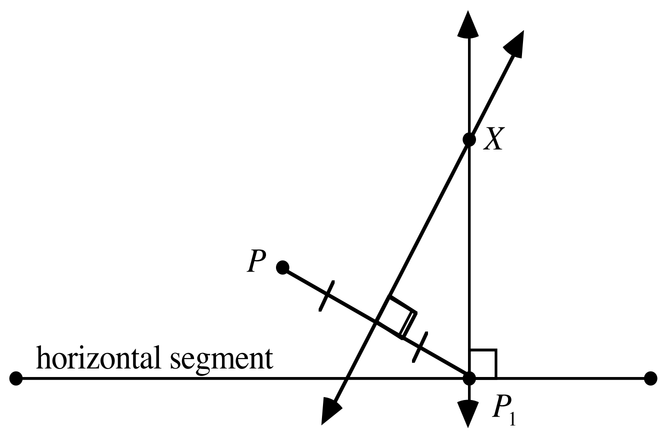


Figure 11: Completed construction

g. Trace the path of *X* as P_1 is moved along the horizontal segment from one endpoint to the other. Note the conic formed. As the path is traced, collect the coordinates of point *X* at 10 locations.

h. Measure the distance from *P* to *X* and from *X* to P_1. Observe how these distances compare as P_1 moves along the horizontal segment.

i. **1.** Create a scatterplot of the data collected in Part **g**.

2. Recall that the graph of a quadratic function is a parabola. Determine a function that models the data.

j. Relocate point P and repeat Parts **g** and **i**. Observe how the location of P affects the shape of the conic formed. **Note:** Save your work for use in the assignment and in the Exploration in Activity **4**.

Discussion

a. Describe the differences between the construction that results in a parabola and the one that results in an ellipse.

b. How is the perpendicular bisector of $\overline{PP_1}$ related to the parabola?

c. Describe the relationship among the perpendicular bisector of $\overline{PP_1}$, point X, and the parabola.

d. In the exploration, $\overline{PX}$ and $\overline{P_1X}$ are congruent. Why is this true?

Mathematics Note

A **parabola** is the set of all points in a plane equidistant from a line and a point not on the line. The line is the **directrix** of the parabola. The point is the **focus** of the parabola.

In Figure **12**, for example, the directrix of the parabola is line l and the focus is point F. The distances from any point P on the parabola to F and l are equal.

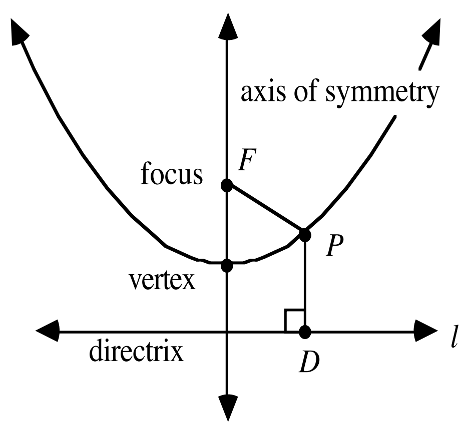


Figure 12: A parabola

e. Describe any symmetries you observe in the parabola in Figure **12**.

f. The **vertex** of a parabola occurs at the point where the axis of symmetry intersects the parabola. How could you use the coordinates of the vertex, the equation of the axis of symmetry, and the distance from the focus to the directrix to locate the focus and directrix?

Assignment

3.1 Use your construction from the exploration to complete Parts **a** and **b**.

a. Describe what happens to the parabola when the focus is moved closer to the directrix.

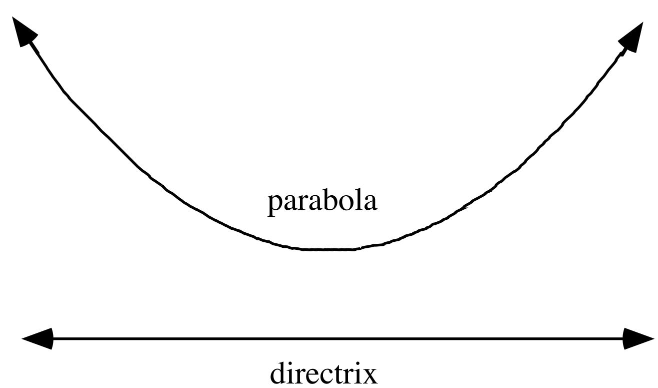
b. Describe what happens to the parabola when the focus is moved away from the directrix.

3.2 **a.** Adapt the paper-folding process from Activity **1** to construct a parabola with a sheet of paper. Describe the new procedure.

b. Describe the axis of symmetry of the parabola you created in Part **a**.

c. Where are the focus and the directrix located in relation to the axis of symmetry?

3.3 The diagram below shows a parabola and its directrix. Describe how you could locate the focus of this parabola.



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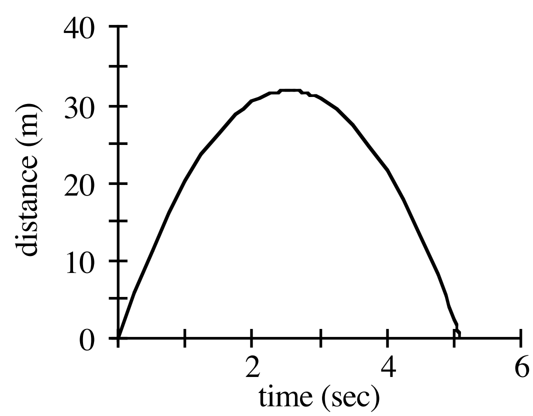
3.4 When a ball is thrown into the air, the graph of its distance above the ground versus time is a parabola. The graph below shows the distance-time graph of a ball thrown straight upward with an initial velocity of 25 m/sec. The ball returns to the ground after approximately 5.1 sec.



a. What is the equation of this parabola's axis of symmetry? Justify your response.

b. Estimate the coordinates of the parabola's vertex. What does this point represent in terms of the motion of the ball?

c. Compare the location of the focus and the directrix for this parabola with those of the parabola in Problem **3.3**.

d. Write an equation that describes the parabola's distance above the ground as a function of time.

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Activity 4

Many of the practical applications of conics are a result of their reflective properties. When light reflects off a flat mirror, as shown in Figure **13**, the outgoing angle is congruent to the incoming angle.

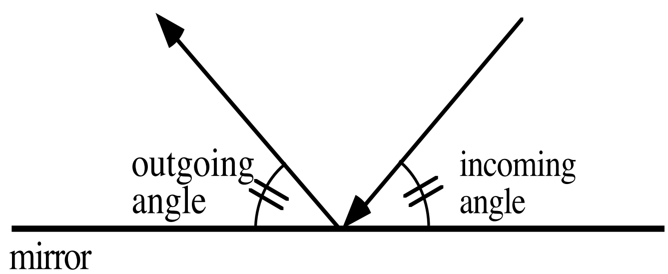


Figure 13: Light reflecting off a flat mirror

But how does light reflect off curved surfaces? In this activity, you examine the reflective characteristics of curved surfaces whose shapes are based on conic sections.

Exploration

In this exploration, you examine a method for determining the angle of reflection for curved surfaces.

a. Using a geometry utility, construct a circle with center at point O.

b. Mark a point X on the circle. Construct and measure $\overline{OX}$.

c. A tangent line to a circle is perpendicular to the radius at the point of tangency. Construct the tangent line to the circle at X.

d. Use the geometry utility to simulate a magnified view of the circle and tangent line near X.

e. Observe how the circle and the tangent line compare near the point of tangency. Record your observations.

f. Again magnify the view of the circle and tangent line near X. Record your observations.

Discussion

a. Imagine that both the tangent line and the circle are mirrors and that a ray of light strikes them very near the point of tangency, as shown in Figure **14**. Describe how you think the light will reflect off each surface.

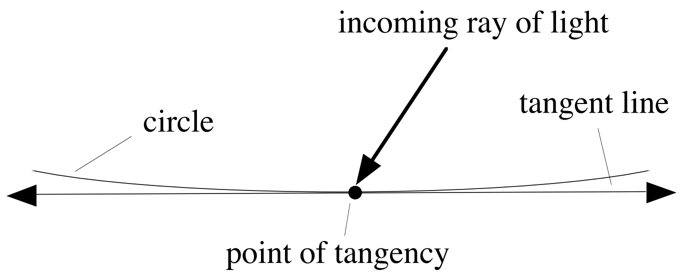


Figure 14: Light striking two mirrors

Mathematics Note

As illustrated in Figure **15**, a portion of a curve near a point of tangency has the characteristics of its tangent line. This property is called **local linearity**. The reflective properties of a point on a curve are the same as the reflective properties of a line tangent to the curve at that point.

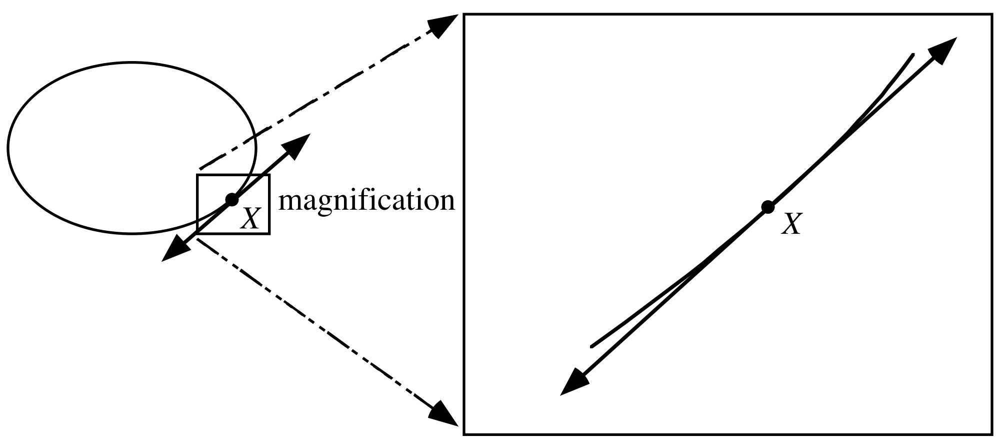


Figure 15: Magnification of a curve and tangent line

b. **1.** Imagine that the parabola in Figure **16** is a cross section of a curved mirror and that a ray of light passing through A strikes the parabola at X. Describe where you think the ray of light will be reflected.

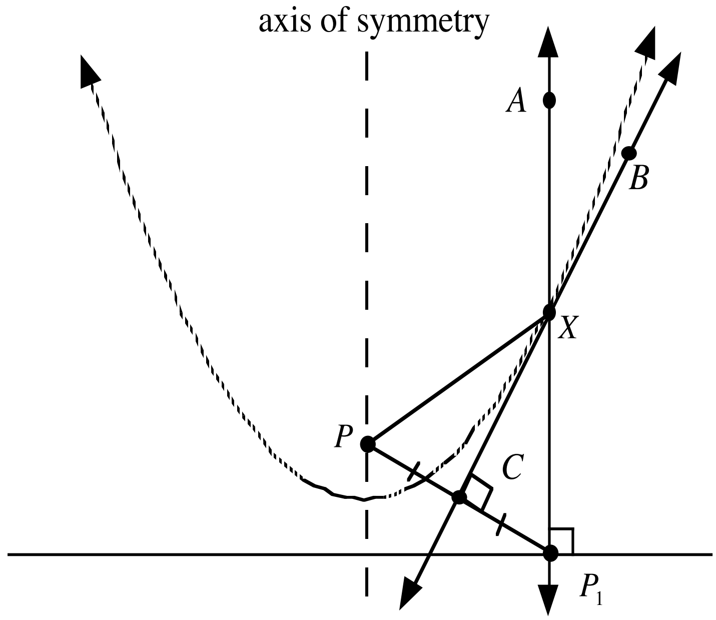


Figure 16: Cross section of a parabolic mirror

2. Suppose that a light bulb is positioned at point P in Figure **16**. What do you think will happen to the light rays emanating from the bulb?

c. In Activity **1**, you used a circle with center at point O to construct an ellipse, as shown in Figure **17**.

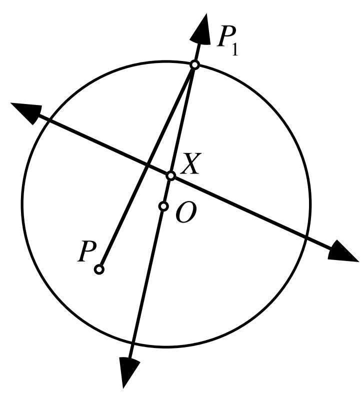


Figure 17: Construction for an ellipse

Suppose that this circle were extremely large.

1. Up close, what would a small piece of this circle look like?

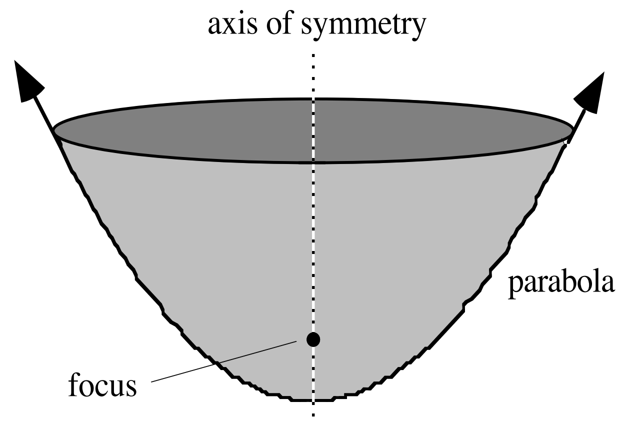
2. How would $\overline{OP_1}$ appear to be related to the circle?

3. What would a tangent line to the circle at P_1 look like?

4. As P_1, moves around the circle, what conic would appear to be formed by the path of point X?

Assignment

4.1 **Conicoids** are three-dimensional objects produced by rotating a conic around the axis of symmetry that contains the focus or foci. As shown in the figure below, a **paraboloid** is created by rotating a parabola around its axis of symmetry. Many satellite dishes are paraboloids.



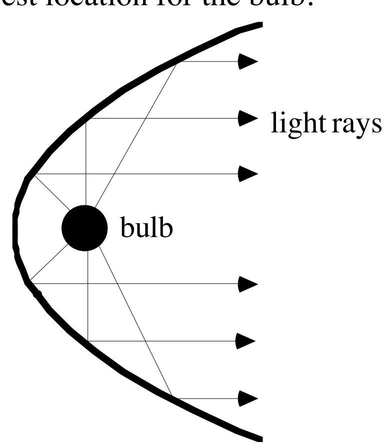
a. What do you think happens to the location of the focus as the parabola is rotated?

b. Satellite dishes are paraboloids. Based on your knowledge of the reflective properties of a parabola, explain how satellite dishes collect and concentrate television signals. Use a diagram in your explanation.

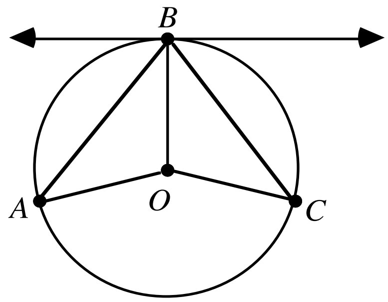
c. If a receiver collects signals sent to the satellite dish, where would you place it in the dish? Explain your response.

d. Why is the direction in which a satellite dish faces important to signal reception?

4.2 Paraboloids are also used in automobile headlights. As shown in the diagram below, a paraboloid around the bulb directs light rays parallel to the axis of symmetry. Use the reflective properties of a parabola to describe the best location for the bulb.



4.3 Consider a mirror whose cross section is a circle, as shown in the diagram below. When a light source at point *A* is directed at point *B*, the ray of light is reflected to point *C*.

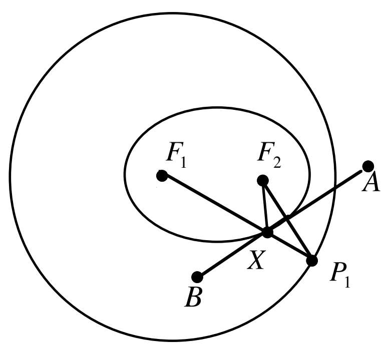


a. Use a geometry utility to investigate the relationship between $m\angle ABC$ and $m\angle AOC$.

b. With the light source at *A*, what must the measure of the incoming angle be so that *A*, *B*, and *C* form the vertices of an equilateral triangle?

c. With the light source at *A*, what must the measure of the incoming angle be to form other regular polygons?

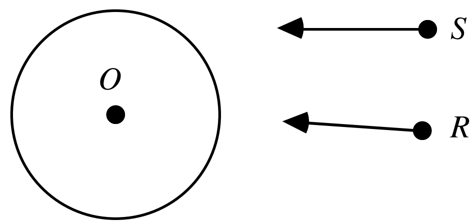
4.4 The following diagram shows the segments used in Activity **2** to construct point *X* on an ellipse. Use this diagram to describe the reflective properties of an ellipse for light emanating from one of the foci, F_1 or F_2.



4.5 In a paragraph, describe some applications for which the reflective properties of an ellipse might be put to use.

* * * * *

4.6 A sphere is a conicoid produced by rotating a circle about a diameter. The figure below shows a cross section of a sphere with a mirrored exterior and two sources of light. The light source at *R* is pointed toward the center *O*. The light source at *S* is not.



a. Describe how each ray of light will reflect off the mirror, including sketches of the incoming and outgoing rays.

b. Imagine that the inside of the sphere also has a mirror. Describe how light emanating from point *O* would reflect inside the sphere, including a diagram to support your response.

4.7 A regular dinner spoon has some interesting reflective properties. When you look at the concave side of the spoon, for example, your reflection is inverted. When you look at the convex side, however, your reflection is not inverted. Use a diagram to help explain why this occurs.

* * * * * * * * * *

Research Project

In Activity **2**, you used a geometry utility to construct a hyperbola. Use your construction to investigate the reflective properties of a hyperbola. Write a report explaining your findings and describe how these reflective properties might be used in practical applications. Include diagrams where appropriate.

Summary Assessment

1. Conic graph paper is printed with two sets of evenly spaced concentric circles and can be useful for sketching hyperbolas and ellipses. The centers of the circles are used to represent the foci.

Using the definition of a hyperbola and a sheet of conic graph paper, sketch a hyperbola such that the difference in the distances between two fixed points and any point on the hyperbola is:

a. 8

b. 4

2. Using the definition of an ellipse and a sheet of conic graph paper, sketch an ellipse such that the sum of the distances between two fixed points and any point on the ellipse is:

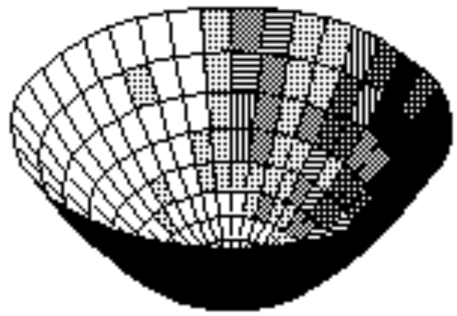
a. 12

b. 16

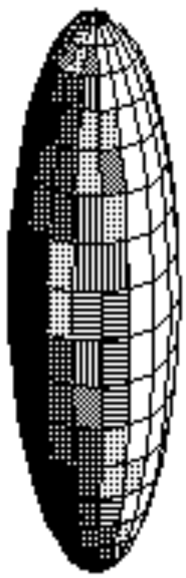
3. Suppose that you were the lighting director of a theater. Describe the shape of the lamp you might use to illuminate an actor's face on the stage. Explain why you chose this particular shape.

4. Write a geometric definition of each conicoid in the diagram below and describe its reflective properties.

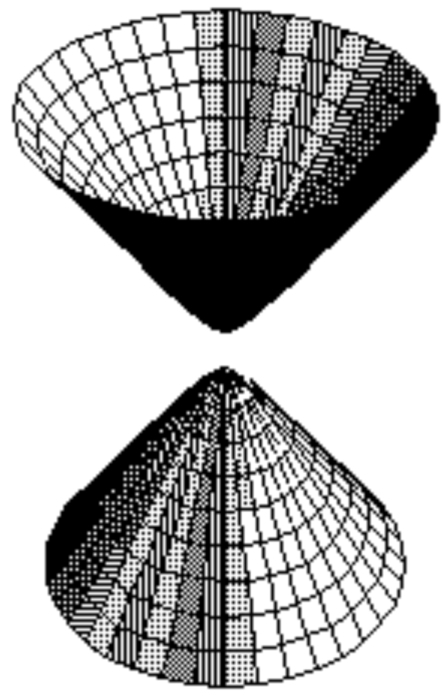
a. Sphere



b. Paraboloid



c. Ellipsoid



d. Hyperboloid

Module Summary

- A **conic section** can be formed by the intersection of a plane with a cone. Depending on the slope of the plane, the intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.
- A **tangent** line to a conic is a line in the plane of the conic that intersects the curve at exactly one point and contains no points in the interior. This intersection is the **point of tangency**.
- The **interior** of a conic is the region or regions of a plane containing the focus (or foci) of the conic.
- A radius drawn to the point of tangency on a circle is perpendicular to the tangent line at that point.
- An **ellipse** is a set of all points in a plane such that the sum of the distances from any point in the set to two fixed points is a constant. Each of these fixed points is a **focus** (plural **foci**).
- A **circle** is a set of all points in a plane equidistant from a given fixed point, the center. A circle is a special case of an ellipse, where both foci are located at the center of the circle.
- A **hyperbola** is a set of all points in a plane such that the absolute value of the difference in the distances from any point in the set to two fixed points (the foci) is a constant.
- A **parabola** is the set of all points in a plane equidistant from a line, the **directrix**, and a point not on the line, the focus.
- A portion of a curve near the point of tangency has the characteristics of its tangent line. This property is called **local linearity**. The reflective properties of a point on a curve are the same as the reflective properties of a line tangent to the curve at that point.
- **Conicoids** are produced by rotating the corresponding conic around the axis of symmetry that contains the focus or foci.

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