Finding Gold

In this module, you investigate some connections among the golden section, the Fibonacci sequence, and Pythagorean triples.

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Introduction

During the period from the sixth century B.C. to the fifth century A.D., Greek scholars raised mathematics to a higher level than ever before in the Middle East or in Europe. One of the legendary figures of Greek mathematics was the philosopher and mystic, Pythagoras (ca. 580–500 B.C.). Historians credit Pythagoras and his followers with many important achievements.

After the fall of the Roman empire, Europe entered a period known as the Dark Ages. During these times, few advances were made in the studies of art, mathematics or science. Not until about 1000 A.D. did widespread interest in mathematical knowledge begin to re–emerge. An Italian merchant named Leonardo of Pisa (ca. 1180–1250), better known as Fibonacci, was an important contributor to this revival.

In this module, you examine some of the mathematical contributions of both Pythagoras and Fibonacci.

Activity 1

Many mathematicians have studied objects and shapes that can be characterized by specific ratios. For example, another Greek mathematician and scientist, Archimedes (ca. 287–212 B.C.), used circles to develop an approximation for π, the ratio of the circumference of a circle to its diameter. In the following exploration, you discover a classical ratio in some rectangles.

Before beginning the exploration however, consider the rectangles in Figure **1**. Which of these rectangles looks "most pleasing" to you? To determine if others share your preference, use the template supplied by your teacher to survey at least 10 other people and record your results.

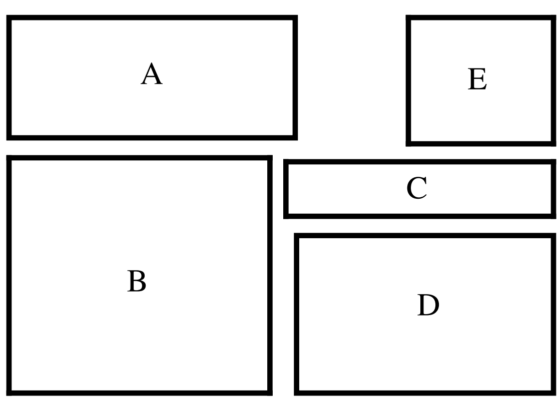


Figure 1: Five rectangles

Exploration 1

a. Draw a rectangle.

b. Measure the rectangle's longer side (l) and its shorter side (s).

c. Calculate the ratio l/s.

d. Use the class data from the rectangle survey to complete Table **1**.

Table 1: Rectangle survey data

Rectangle	Number of People	Percentage of People	Ratio l/s
A			
B			
C			
D			
E			

Discussion 1

a. How does the ratio l/s of the rectangle you drew compare to those of others in the class?

b. **1.** According to the class survey, which rectangle was most popular?

2. What is the ratio l/s of this rectangle?

3. What percentage of those surveyed chose the most popular rectangle?

c. How does the ratio l/s of the rectangle you drew compare to that of the most popular rectangle?

Mathematics Note

The **golden ratio** or **golden section**, often denoted by the Greek letter ϕ (phi), is the irrational number $\left(1+\sqrt{5}\right)/2$ which may be approximated as 1.618.

In a **golden rectangle**, the ratio of the lengths of the longer side to the shorter side is the golden ratio ϕ.

d. What general conclusion can you make about the dimensions of the most popular rectangle?

Exploration 2

In 1876, the German psychologist Gustav Fechner conducted a rectangle survey similar to the one you completed in the introduction. About 75% of those surveyed selected a golden rectangle as "most pleasing." The golden ratio is not limited to visually attractive rectangles, however. In fact, it appears in the dimensions of many objects in both art and nature. Some researchers, for instance, believe that the Greek artist Phidias (ca. 490–430 B.C.) used the golden ratio in his sculptures of the human form.

In the following exploration, you investigate the ratios of some human dimensions.

a. Select a sample of students from your class.

b. For each person in your sample, measure the length of one arm from shoulder to fingertips (*a*) and the width of the back from shoulder to shoulder (*b*). Calculate the ratio a/b.

c. For each person in your sample, measure the length of one arm from shoulder to fingertips (*a*) and the length of that same arm from elbow to fingertips (*e*). Calculate the ratio a/e.

d. Compile the class data for the ratios a/b and a/e and calculate the mean for each ratio.

Discussion 2

a. Considering the data you collected in Exploration **2**, would it seem reasonable for Phidias to use ϕ in a sculpture of a person in your class?

b. Identify some other objects that appear to contain the golden ratio.

Assignment

1.1 Golden rectangles have some interesting properties. For example, they can be constructed recursively from other golden rectangles. Recall that a recursive process uses the result of one procedure as the input for the next repetition of the same procedure.

a. 1. Draw a rectangle in which the ratio of the lengths of the longer side to the shorter side is about 1.6. This rectangle approximates a golden rectangle.

2. Draw a line segment that divides the rectangle into a square and another rectangle.

b. Measure the side of the square.

c. **1.** Calculate the ratio of the longer side to the shorter side of the smaller rectangle.

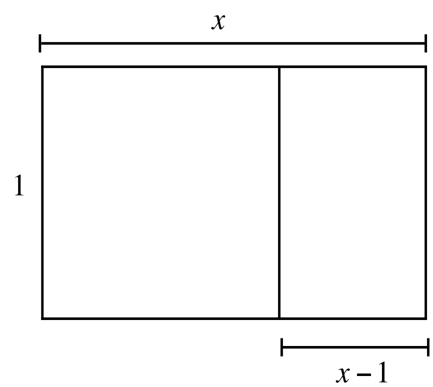
2. Is the smaller rectangle a golden rectangle? Justify your response.

d. Divide the smaller rectangle into a square and another rectangle. Is the new rectangle a golden rectangle? Justify your response.

e. Repeat Parts **c** and **d** three more times.

f. Describe what occurs each time you repeat the procedure in Part **e**.

1.2 The diagram below shows two golden rectangles, the smaller of which was formed by the process described in Problem **1.1**.



a. Use these two rectangles to write a true proportion involving x.

b. Solve this proportion for x.

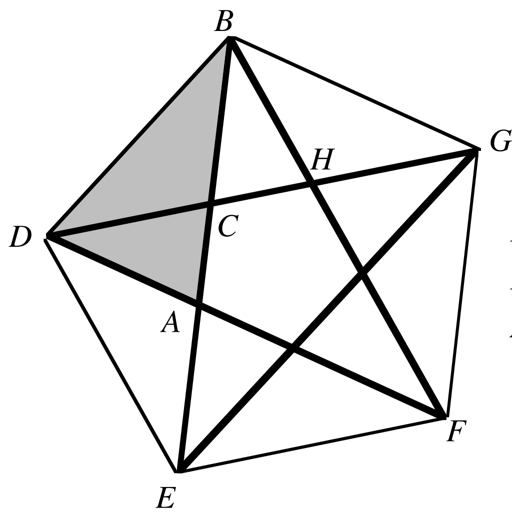
c. Since the two rectangles are golden rectangles, the value of x equals ϕ. Use the value of ϕ to determine the arithmetic relationships that exist between each of the following pairs of values:

1. ϕ and $1/\phi$

2. ϕ and ϕ^2

3. ϕ^2 and $1/\phi$

1.3 The followers of Pythagoras, known as the Pythagoreans, gave special significance to a pentagram, a star formed by connecting each vertex of a regular pentagon with its nonadjacent vertices. The diagram below shows a pentagram inscribed in regular pentagon $BGFED$. (All lengths given are approximate measures.)



$BD \approx 3.565$ cm
$BC \approx 2.203$ cm
$AC \approx 1.362$ cm

a. The ratio BC/AC provides one example of the golden ratio. Find another pair of segments that appear to form the golden ratio.

b. Name each isosceles triangle in the shaded portion of the figure.

c. An isosceles triangle with a leg-to-base ratio of ϕ is a golden triangle. Which triangles in Part **b**, if any, are golden triangles? Justify your response.

d. Determine the approximate measures of the angles in any golden triangle identified in Part **c**. What general conclusion can you make about the angles in a golden triangle?

1.4 Determine whether each statement below is true or false. Provide a counterexample for each false statement.

a. Every isosceles triangle is a golden triangle.

b. Every isosceles triangle with a 36° angle is a golden triangle.

c. Every isosceles triangle with a 36° non-base angle is a golden triangle.

d. Every triangle that contains the golden ratio is a golden triangle.

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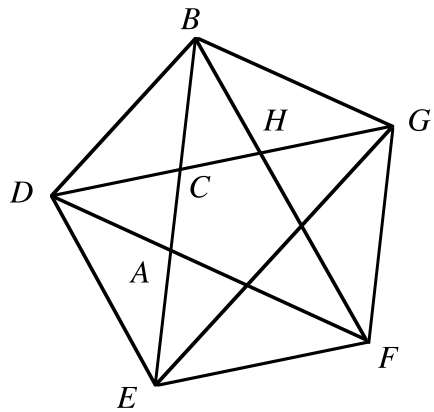
1.5 Use the labeled points in the regular pentagon below to complete Parts **a–c**.

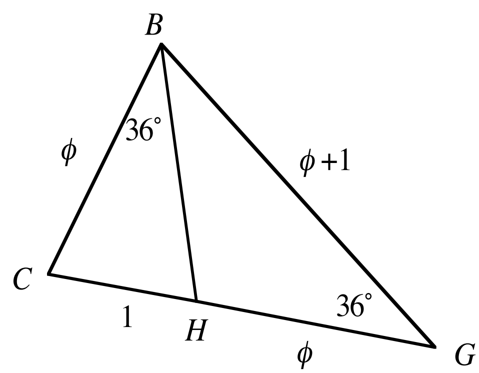


a. Name three pairs of noncongruent, similar triangles.

b. Write a proportion for each pair of triangles in Part **a** that verifies their similarity.

c. Are all golden triangles similar to each other? Explain your response.

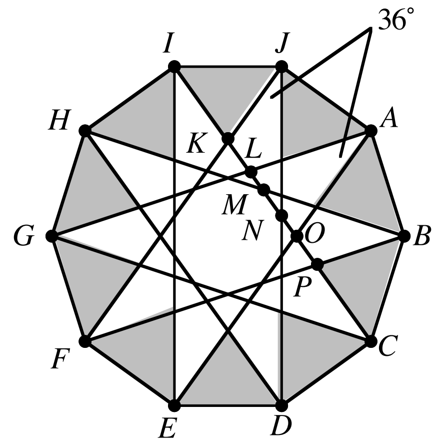
1.6 The diagram below shows a portion of the drawing from Problem **1.5**.



Triangles *CBH* and *CBG* are golden triangles. Use these triangles to verify that

$$\phi = \frac{1 + \sqrt{5}}{2}$$

1.7 The unshaded figure below shows a regular "star" polygon formed by connecting every fourth vertex of a regular decagon. (All lengths given are approximate measures.)



$KI \approx 2.0$ cm
$LI \approx 2.8$ cm
$MI \approx 3.3$ cm
$NI \approx 4.0$ cm
$OI \approx 4.5$ cm
$PI \approx 5.3$ cm
$CI \approx 7.3$ cm

a. Using the given distances, find three pairs of line segments whose ratio of lengths appears to form the golden ratio.

b. Find a golden triangle. Give evidence to support your choice.

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Activity 2

In modern mathematics, Fibonacci is probably best known for the sequence of numbers that bears his name. The first two terms of the **Fibonacci sequence** are both 1. Successive terms are generated by adding the previous two terms. In other words, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, … . Any sequence in which successive terms are formed by adding the previous two terms is referred to as a **Fibonacci-type** sequence.

In this activity, you investigate how Fibonacci-type sequences relate to some of the mathematics of the Pythagoreans.

Exploration 1

a. Generate the first 25 terms of the Fibonacci sequence.

b. Write a recursive definition of the Fibonacci sequence, where F_n is the nth term.

c. **1.** For each pair of consecutive Fibonacci numbers in Part **a**, F_n and F_{n+1}, calculate the following ratios:

$$\frac{F_{n+1}}{F_n} \text{ and } \frac{F_{n+2}}{F_{n+1}}$$

2. For $n = 1, 2, 3, \ldots$, each ratio you calculated in Step **1** also forms a sequence. Recall that the **limit** of a sequence, $k_1, k_2, k_3, \ldots, k_n, \ldots$, is a number L if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of L.

As n increases, what limit do the two sequences of ratios appear to approach?

d. Using any two nonzero natural numbers of your choice, create a Fibonacci-type sequence of your own. Generate the first 25 terms of the sequence.

e. Repeat Part **c** using your sequence from Part **d**.

f. Create a sequence in which each successive term is the sum of the previous three terms. Begin the sequence with any three nonzero natural numbers. Generate the first 25 terms.

g. For each pair of consecutive numbers in your sequence in Part **f**, F_n and F_{n+1}, calculate the following ratio:

$$\frac{F_{n+1}}{F_n}$$

h. Generate the first 10 terms of the sequence whose explicit formula is shown below:

$$S_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

Discussion 1

a. Considering the terms of a Fibonacci-type sequence, describe what happens to the following ratios as n increases:

$$\frac{F_{n+1}}{F_n}, \text{ and } \frac{F_{n+2}}{F_{n+1}}$$

b. What is the relationship between a Fibonacci-type sequence and ϕ?

c. Describe the sequence generated by the explicit formula in Part **h** of Exploration **1**.

Exploration 2

a. Write any four consecutive terms of the Fibonacci sequence.

b. Calculate the product of the first and fourth terms and set this value equal to a.

c. Calculate twice the product of the second and third terms and set this value equal to b.

d. Perhaps the most famous mathematical relationship associated with Pythagoras is the one that bears his name. The Pythagorean theorem states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

For example, in triangle ABC of Figure **2**, $AB^2 = AC^2 + CB^2$, or $5^2 = 3^2 + 4^2$.

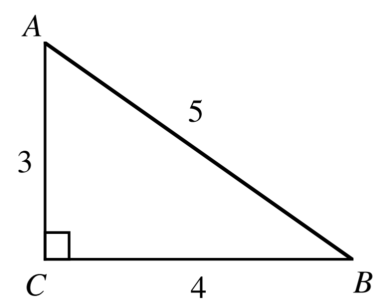


Figure 2: Right triangle *ABC*

Determine the length of the hypotenuse c of the right triangle with legs of lengths a and b as determined in Parts **b** and **c**.

e. Determine if c is a term in the original sequence.

f. Calculate the area of the right triangle with sides a, b, and c.

g. Calculate the product of the four terms in Part **a**. Compare this number with the area of the triangle you calculated in Part **f.**

h.
1. Repeat Parts **b–g** using a different set of four consecutive terms of the Fibonacci sequence.
2. Repeat Parts **b–g** using four nonzero, consecutive terms of a Fibonacci-type sequence.

Discussion 2

a. In Part **h** of Exploration **2**, why were the four consecutive terms of a Fibonacci-type sequence restricted to nonzero numbers?

b. Natural numbers that satisfy the Pythagorean theorem are **Pythagorean triples**. For example, since $3^2 + 4^2 = 5^2$, (3,4,5) is a Pythagorean triple.

Did the values for *a, b*, and *c* in Exploration **2** form a Pythagorean triple when using four successive terms from:

1. the Fibonacci sequence?
2. a Fibonacci-type sequence?

c. Was the length of the hypotenuse found in Part **d** a term of the original sequence when using:

1. the Fibonacci sequence?
2. a Fibonacci-type sequence?

d. What relationship exists between four consecutive terms of any sequence and the area of a right triangle created using the process described in Parts **b–d** of Exploration **2**?

Assignment

2.1

a. Using terms in the Fibonacci sequence, generate three Pythagorean triples.

b. Is the length of each hypotenuse (c) a term in the Fibonacci sequence?

c. For each Pythagorean triple generated in Part **a**, calculate the area of the corresponding right triangle. Describe how you found your solutions.

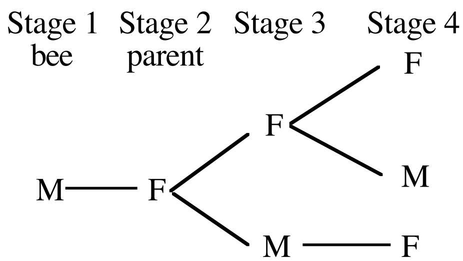
2.2

a. Create three ordered triples (a,b,c) where a is an odd number greater than 1, $b = (a^2 - 1)/2$, and $c = b + 1$. Verify that each of these triples is a Pythagorean triple.

b. Explain why any three numbers a, b, and c that satisfy the constraints given in Part **a** form a Pythagorean triple.

c. Is the value for c in this case always a term in the Fibonacci sequence? Explain your response.

2.3 While female bees have both a female and a male parent, male bees have only a female parent. The tree diagram below shows four ancestral generations of a male bee. Stage 1 represents the bee itself, stage 2 represents the bee's parent, stage 3 represents the bee's grandparents, and stage 4 represents the bee's great-grandparents.



a. Continue the tree diagram for stages 5 and 6.

b. List the first six terms of the sequence $t_1, t_2, \ldots, t_n$, where t_n is the number of bees in stage *n*.

c. **1.** How many bees are there in stage 9 of the tree diagram?

2. How many bees are there in stage 15 of the tree diagram?

d. How could you find the number of bees in stage *n* of the tree diagram?

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2.4 The **Lucas sequence**, named after French mathematician Edouard Lucas, is a Fibonacci-type sequence with 1 and 3 as the first two terms.

a. Choose any four consecutive terms in the Lucas sequence. Using the method described in Parts **b–d** of Exploration **2**, find values for *a*, *b*, and *c*.

b. Is (a,b,c) a Pythagorean triple? Explain your response.

c. Do you think that *c* will always be a term in the Lucas sequence? Explain your response.

2.5 Choose any four consecutive terms in the Fibonacci sequence.

a. Using the method described in Parts **b–d** of Exploration **2**, find the value for a hypotenuse *c*. Which term in the sequence is *c*?

b. Describe the relationship between the term numbers of the four consecutive terms and the term number of the hypotenuse.

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Activity 3

To create convincing arguments, mathematicians use a method based on logical reasoning. This process is referred to as **proof**. Although diagrams and pictures may be used to support arguments, a conclusion based solely on observation may or may not be true.

The Pythagorean theorem is one of the most well-known theorems in mathematics. Although the theorem bears the name of Pythagoras, many mathematicians throughout history have developed unique proofs for it. In fact, over 350 different proofs appear in E. S. Loomis' *The Pythagorean Proposition*. In this activity, you examine several proofs of this famous theorem.

Exploration

Many demonstrations of the Pythagorean theorem show, either by adding or subtracting areas, that the area of a square on the hypotenuse of a right triangle is equal to the combined areas of the squares on the legs. In this exploration, you develop the motivation for a proof first written by H. E. Dudeney in 1917.

a. Draw a right triangle.

b. As shown in Figure **3** below, draw a square on each side of the right triangle.

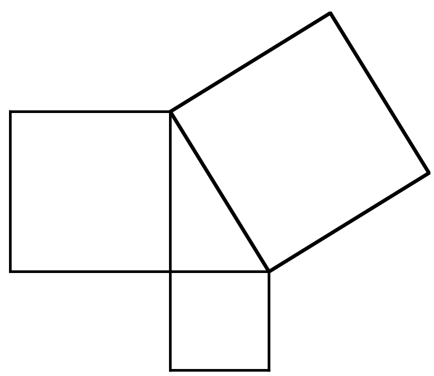


Figure 3: Squares on sides of right triangle

c. Find the center of the square on the longer leg.

d. Through this center, draw lines both parallel and perpendicular to the hypotenuse of the right triangle. These lines divide the square on the longer leg into four congruent quadrilateral regions.

e. Cut out the four quadrilateral regions found in Part **d** and the square on the shorter leg.

f. Arrange the pieces in Part **e** so that they fill the square on the hypotenuse without leaving gaps or overlapping.

g. Repeat Parts **a–f** using a different right triangle.

Discussion

a. How does Figure **3** relate to the Pythagorean formula $a^2 + b^2 = c^2$?

b. What can you conclude about the combined areas of the squares on the legs and the area of the square on the hypotenuse?

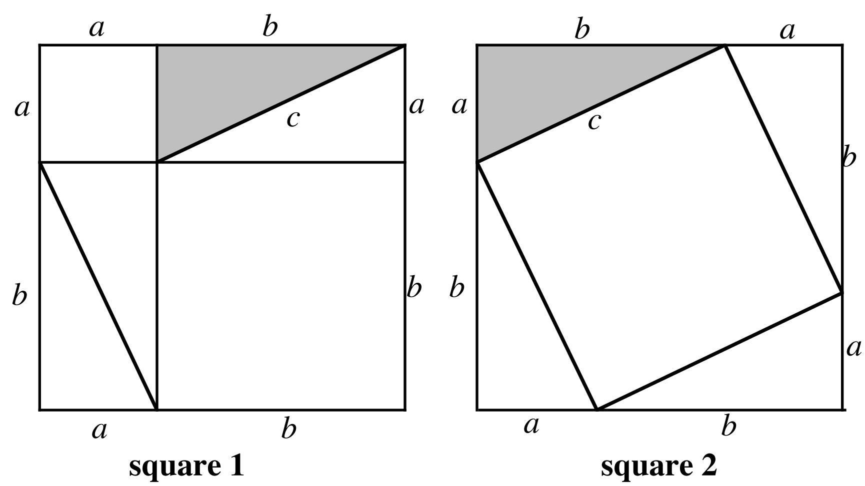
c. Does the procedure you followed in the exploration prove the Pythagorean theorem?

d. H. E. Dudeney's proof of the Pythagorean theorem is described as a "dissection" proof. Why do you think this term is used?

Assignment

3.1 No evidence exists for the original proof of the Pythagorean theorem, but it is generally attributed to Pythagoras himself. According to legend, Pythagoras sacrificed an ox to celebrate the significance of this proof. Hence, its nickname the "ox-killer" proof.

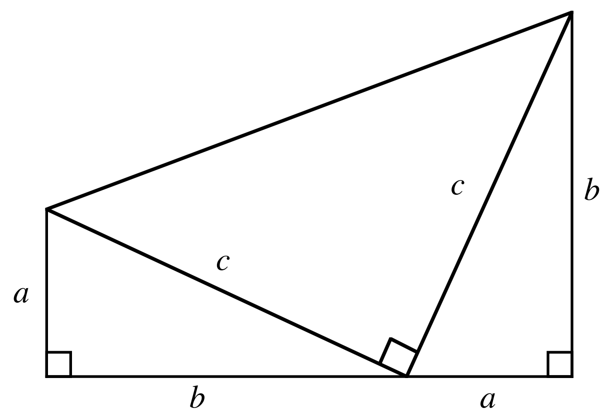
Pythagoras' proof considered two squares, each with side length of $a + b$. As shown below, square 1 is divided into six non-overlapping regions while square 2 is divided into five non-overlapping regions.



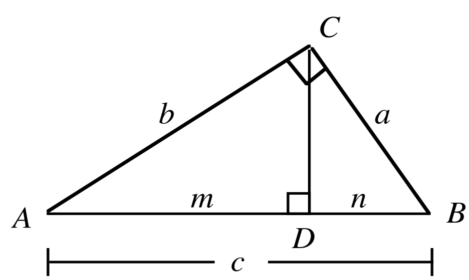
a. What type of quadrilateral is the inner figure in square 2? Justify your response.

b. Use the areas of squares 1 and 2 to prove the Pythagorean theorem. (Hint: Express each square's area as the sum of the areas of its regions.)

3.2 Five years before he became president of the United States, James A. Garfield discovered a creative proof of the Pythagorean theorem. In his proof, he calculated the area of a trapezoid in two ways: by using the area formula for a trapezoid, and by adding the areas of the three right triangles that compose the trapezoid, as shown below. Verify that his method produces the formula $a^2 + b^2 = c^2$.



3.3 In his proof of the Pythagorean theorem, British mathematician John Wallis used similar right triangles, as shown in the diagram below.

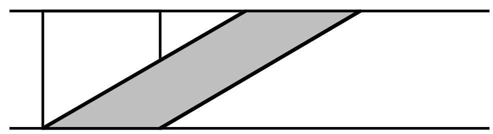


a. Prove that $\Delta ACD \sim \Delta CBD \sim \Delta ABC$.

b. Use measures a, b, m, n, and c to write two true proportions.

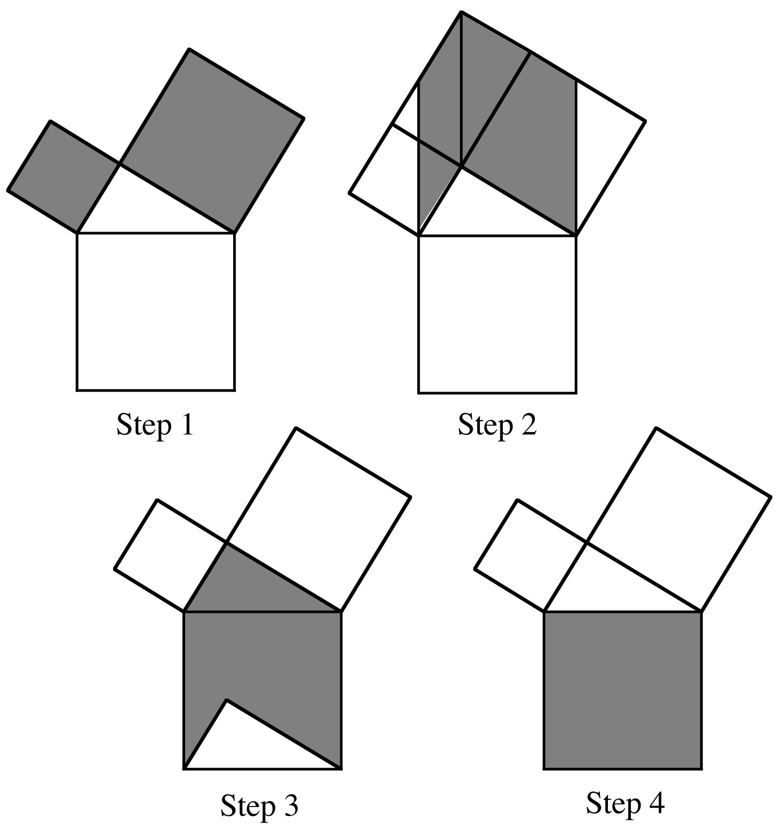
c. Use the proportions in Part **b** to prove the Pythagorean theorem.

3.4 Shearing is a transformation that can be used to create parallelograms of equal area by holding one side of a square fixed and sliding the opposite side along the line containing this side. The diagram below shows one example of this process



In his proof of the Pythagorean theorem, Euclid began with squares on the sides of a right triangle, as in Step **1**. He sheared the squares to obtain the two shaded parallelograms in Step **2**.

The shaded polygon in Step **2** was then transformed to reach Step **3**. Finally, another transformation of a part of the shaded polygon was performed to reach Step **4**.



a. Use one of the shaded squares in Step **1** and its sheared image in Step **2** to demonstrate that the areas are equal.

b. Describe in detail the mathematics that occur at each step and explain how this process demonstrates the Pythagorean theorem.

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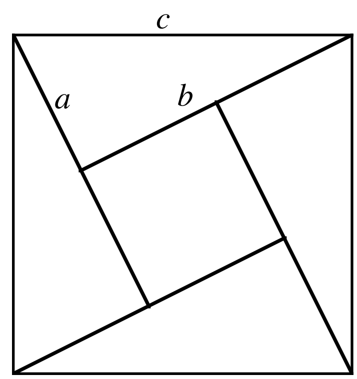
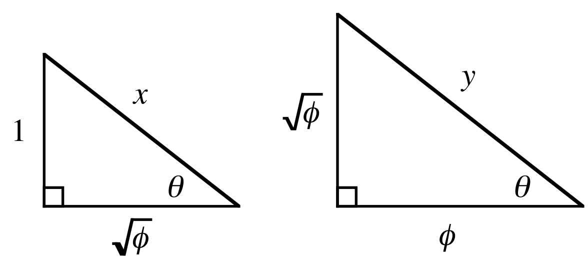
3.5 The Hindu scholar Bhaskara contributed a short, yet elegant, proof of the Pythagorean theorem. For his proof, he drew the following figure with the single word, "Behold!" Calculate the area of the larger square in two different ways, then use the results to prove that $a^2 + b^2 = c^2$.



3.6 Use the following diagram to complete Parts **a–e** below, where ϕ is the golden ratio.



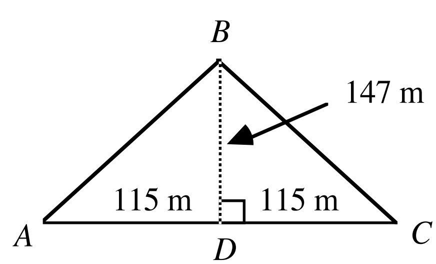
a. Determine the values of x and y.

b. Explain why the two triangles are similar.

c. Explain why the triangles are not congruent, even though five of their parts are the same.

d. Find tan θ and cos θ and compare their values.

e. The following diagram shows some measurements of a cross section of the Great Pyramid of Giza. Find the ratio of AB to AD, then compare triangle ABD to the two triangles given above.



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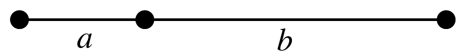
Research Project

In ancient Greece, mathematicians used a compass and straightedge to demonstrate various properties of geometry and find exact measures.

As shown in Figure **4**, a point divides a line segment into the golden ratio when the ratio of the longer segment to the shorter segment is the same as the ratio of the whole line segment to the longer segment.



$$\frac{b}{a} = \frac{a+b}{b}$$

Figure 4: The golden ratio

The ratio of the shorter segment to the longer segment, or a/b, is the reciprocal of the golden ratio. Given that the value of the golden ratio is ϕ, the value of its reciprocal is $1/\phi$ or $\phi - 1$.

Using a compass, a straightedge, and the right triangle in Figure **5** below, construct a segment whose length is equal to the reciprocal of the golden ratio. Describe the process that you used and explain why it works.

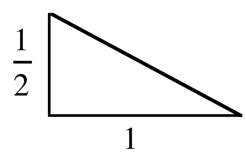


Figure 5: A right triangle

Summary Assessment

1. The following calculator algorithm generates terms of a sequence *S*:

Step 1: Choose a number between 1 and 10 as the first term.
Step 2: Add 1 to the term.
Step 3: Take the reciprocal of the result. This is the next term in *S*.
Step 4: Repeat Steps 2–3 until you have 10 terms of sequence *S*.

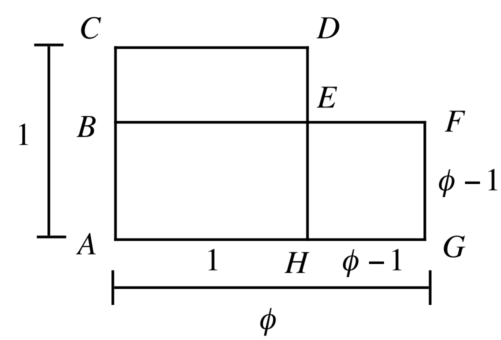
a. Use the algorithm to generate one sequence of 10 terms.

b. Repeat the algorithm using a different value in Step 1.

c. Does the sequence appear to approach a limit? Explain your response.

d. When Step 2 and Step 3 are interchanged in the algorithm, does the sequence appear to approach a limit? Explain your response.

2. Euclid determined that a point divides a line segment into the golden ratio ϕ when the ratio of the longer segment to the shorter segment is the same as the ratio of the whole line segment to the longer segment. Euclid also found a similar relationship in the figure below.



In this diagram, the area of the larger square, *ACDH*, equals the area of the larger rectangle, *ABFG*. Use this fact and the appropriate area formulas to determine the exact value of ϕ.

Module Summary

- The **golden ratio**, or **golden section**, often denoted by the Greek letter ϕ (phi), is the irrational number $\left(1+\sqrt{5}\right)/2$ or approximately 1.618.
- In a **golden rectangle**, the ratio of the measures of the longer side to the shorter side is the golden ratio (ϕ).
- The first two terms of the **Fibonacci sequence** are both 1. Successive terms are generated by adding the previous two terms. In other words, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, … .
- Any sequence in which successive terms are formed by adding the previous two terms is a **Fibonacci-type** sequence.
- The **limit** of a sequence, $k_1,\ k_2,\ k_3,\ \ldots,\ k_n,\ \ldots$, is a number *L* if for any prescribed accuracy, there is a term k_m such that all terms after k_m are within this given accuracy of *L*.
- The **Pythagorean theorem** states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse. In other words, if *a* and *b* are the lengths of the legs of a right triangle and *c* is the length of the hypotenuse, then:

$$a^2 + b^2 = c^2$$

- Natural numbers *x*, *y*, and *z* that satisfy the Pythagorean theorem are **Pythagorean triples** and may be represented by (x,y,z).
- The reciprocal of the golden section, $1/\phi$, is approximately 0.618.

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