What Are You Eating?

The labels on packaged food contain a wealth of nutritional information. In this module, you use these labels and linear programming to help select foods that satisfy nutritional requirements.

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Introduction

Reduced fat. High in fiber. Low calorie. Light. Lite? To satisfy health-conscious consumers, many food manufacturers added health claims to their labels and packaging. Some of these claims were confusing; others were downright misleading.

In order to regulate this situation, the U.S. Congress passed the Nutrition Labeling and Education Act in 1990. By May 1994, nutrition labeling became mandatory for almost all processed foods. Regulations for labeling of meat and poultry became effective later that year.

The Food and Drug Administration (FDA) has found sufficient scientific evidence to support the following health claims:

- higher calcium intake reduces the risk of osteoporosis
- lower sodium use reduces high blood pressure
- high amounts of dietary saturated fat and cholesterol can lead to coronary heart disease
- high dietary fat intake increases the risk of cancer
- a diet high in fiber, fruits, and vegetables helps prevent cancer
- dietary fiber reduces the risk of coronary heart disease.

As a result of this evidence, the FDA recommends that no more than 30% of a day's calories be obtained from fat—with no more than 10% of each day's calories consisting of saturated fat. Carbohydrates should provide approximately 60% of each day's calories, while proteins should account for about 10%.

The FDA's recommendations, along with accurate food labels like the sample shown in Figure **1**, can help people plan diets and make informed nutritional choices. The upper part of the label provides some specific information about the nutritional content of the food, while the lower portion describes Percent Daily Values of fat, cholesterol, sodium, carbohydrates, and dietary fiber for both 2000-calorie and 2500-calorie per day reference diets.

Nutrition Facts

Serving Size 1 cup (228 g)
Servings Per Container 2

Amount Per Serving

Calories 260 Calories from Fat 120

	% Daily Value*
Total Fat 13 g	20%
Saturated Fat 5 g	25%
Cholesterol 30 mg	10%
Sodium 660 mg	28%
Total Carbohydrate 31 g	10%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	

Vitamin A 4%	Vitamin C 2%
Calcium 15%	Iron 4%

* Percent Daily Values are based on a 2,000 calorie diet. Your daily values may be higher or lower depending on your calorie needs:

	Calories:	2,000	2,500
Total Fat	Less than	65 g	80 g
Sat Fat	Less than	20 g	25 g
Cholesterol	Less than	300 mg	300 mg
Sodium	Less than	2,400 mg	2,400 mg
Total Carbohydrate		300 g	375 g
Dietary Fiber		25 g	30 g

Calories per gram:
Fat 9 • Carbohydrate 4 • Protein 4

Figure 1: Sample Food Label

Source: *FDA Consumer Special Issue on Food Labeling*, May 1993.

The information on food labels is based on a reference diet of 2000 calories per day. This 2000-calorie diet is not recommended for all people. Some might need more calories; others might require less. For example, very active people, children under the age of four, and pregnant women have nutritional needs that vary from the 2000-calorie reference diet. On average, a moderately active teenage male needs at least 2800 calories per day, while a moderately active teenage female needs at least 2200 calories per day. A healthy diet involves more than just counting calories; it requires eating a variety of foods in moderation.

Discussion

a. Use the health claims cited by the FDA to describe examples of healthy and unhealthy meals.

b.
1. How did the FDA calculate the number of calories from fat shown on the sample label in Figure **1**?
2. What percentage of the calories in this food come from fat?
3. How is the Percent Daily Value (% DV) of total fat calculated from the information on the label?

c. How might you use the information on a typical food label to make your own nutritional decisions?

d.
1. Consider a person who requires 2500 calories per day and eats one serving of the food whose label is shown in Figure **1**. Describe the Percent Daily Values of total fat, total carbohydrate, and fiber received by this person.
2. How do these Percent Daily Values compare with those for a 2000-calorie diet?

e. One serving of the food on the label in Figure **1** has a mass of 228 g. However, the mass of the fat, cholesterol, sodium, carbohydrates, and protein in that serving totals only about 50 g. If the remainder of the serving is water, then what percentage of this food product is water?

f. One serving of a certain food provides 25% of the Daily Value for both carbohydrates and sodium for a person on a 2000-calorie diet.

1. How many servings of this food would provide 100% of the Daily Value of carbohydrates for a person on a 2000-calorie diet?
2. How many servings would provide 100% of the Daily Value of carbohydrates for a person on a 2500-calorie diet?
3. Use the nutrient information for 2000- and 2500-calorie diets listed on the label in Figure **1** to explain why more than four servings of this food would not be wise for a person who must limit sodium intake.
4. How might a person obtain 100% of their Daily Value for carbohydrates without consuming too much sodium?

Research Project

Develop a chart to keep track of the food you consume each day for a five-day period. Make an entry on your chart for every meal or snack. For each item, record the food type and amount eaten, as well as a nutritional summary. The summary should include calories, total fat, saturated fat, cholesterol, sodium, carbohydrates, and protein. Calculate totals for each day. Write a summary of your diet that includes:

- a comparison with the reference diet listed on food labels
- a consideration of your own caloric needs (this may vary from the reference diet)
- a comparison with the FDA recommendations for percentages of total fat, saturated fat, carbohydrates, and proteins.

Activity 1

The nutrients in foods include vitamins, minerals, fats, protein, and carbohydrates. When planning healthy meals, nutritionists sometimes evaluate each nutrient separately. Many people, for example, need to minimize their intakes of fat and sodium while increasing their consumption of dietary fiber. One method of analyzing the nutrient content of a meal involves **linear programming**.

Mathematics Note

Linear programming can be used to solve problems involving variables that are subject to linear **constraints**. The system of linear inequalities determined by these constraints defines a **feasible region**, a set of points that satisfies the system.

Typically, such problems require the identification of an optimal value for an **objective function**, either a maximum or a minimum. According to the **corner principle**, if the objective function has a minimum or maximum value, it will occur at a corner point (or vertex) of the feasible region.

For example, suppose that you wanted to find the minimum value of the function $f = 4x + 3y$ given the following system of constraints:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \geq 12 \\ 4x + 2y \geq 16 \\ x + y \leq 9 \end{cases}$$

Figure **2** below shows a graph of a feasible region that satisfies these constraints.

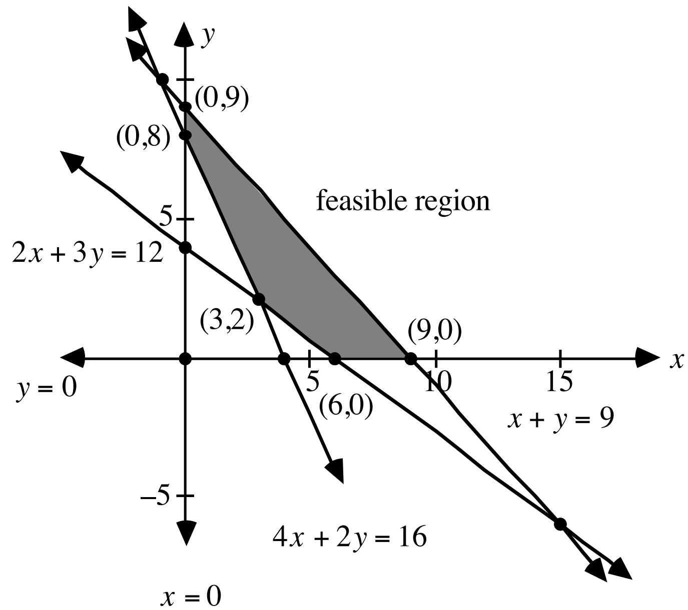

Figure 2: Feasible region defined by five constraints

In this case, the feasible region has five corner points—(0,9), (0,8), (3,2), (6,0), and (9,0)—all of which are contained in the feasible region. Using the corner principle, the minimum value of the objective function $f = 4x + 3y$ must occur at one of these points. By evaluating the function at each corner point, you can determine that the minimum value of 18 occurs at the point with coordinates (3,2).

Discussion 1

a. The five constraints described in the previous mathematics note correspond with the five lines graphed in Figure **2**. How many points of intersection are there for these lines?

b. The intersection of the lines $x = 0$ and $2x + 3y = 12$ in Figure **2** is not a corner point. Explain why this point is not included in the feasible region.

c. Identify the other pairs of lines in Figure **2** whose points of intersection are not corner points.

d.
1. Describe how you could find the y-intercept of the line $4x + 2y = 16$ without using a graph.
2. Describe how you could find the x-intercept without using a graph.

e.
1. Which corner points in Figure **2** are located on a coordinate axis?
2. How could you determine the coordinates of each of these points using algebra?
3. How could you determine the coordinates of the remaining corner point using algebra?

Exploration

Imagine that you are a nutritionist who wants to plan a lunch of tomato soup and rolls. The meal must contain at least 20% of the Daily Value of iron, at least 40% of the Daily Value of calcium, no more than 700 calories, and a minimum amount of fat. The percentages given in Table **1** are based on a 2000-calorie reference diet.

Table 1: Nutrients in one serving of tomato soup and rolls

Food	% DV Iron	% DV Calcium	Calories	Fat
tomato soup	4	17	140	4 g
roll	6	2	120	3 g

a. Using t to represent the number of servings of tomato soup and r to represent the number of servings of rolls, write inequalities for each of the following constraints.

1. The number of servings of tomato soup in the meal must be at least 0.
2. The number of rolls in the meal must be at least 0.
3. The total number of calories from a meal of soup and rolls can be no more than 700.
4. The meal of soup and rolls should provide at least 20% of the Daily Value of iron.
5. The meal of soup and rolls should provide at least 40% of the Daily Value of calcium.

b. Using the inequalities from Part **a**, graph the feasible region for the number of servings of soup and rolls in this meal, with values for t represented on the horizontal axis. Your graph should include the following:

1. the equations that determine the boundaries of the feasible region
2. the coordinates of the vertices of the polygon that defines the feasible region.

c. The two inequalities you determined in Steps **4** and **5** of Part **a** define the constraints placed on iron and calcium consumption. List the coordinates of a point in the first quadrant that:

1. does not satisfy either of these constraints
2. satisfies exactly one of these constraints
3. satisfies both of these constraints.

d.
1. Let f represent the grams of fat consumed in the meal. Write an equation that relates f, t, and r for a meal of tomato soup and rolls.
2. Suppose that the meal must contain 4 g of fat. Substitute $f = 4$ into your equation from Step **1**, and sketch a graph of the result on the same coordinate system as in Part **b**.
3. Repeat Step **2** for $f = 8$, $f = 12$, and $f = 16$.
4. For various values of f, what patterns do you notice in the resulting lines?

e.
1. Determine the approximate number of servings of rolls and soup that will minimize the amount of fat consumed while providing at least 20% of the Daily Value of iron and at least 40% of the Daily Value of calcium.
2. One serving of soup has a volume of 240 mL, while one serving of rolls has a mass of 120 g. Express your solution in Step **1** in terms of milliliters of soup and grams of rolls.
3. Calculate the number of calories in this meal.

Discussion 2

a. What is the significance of each point in a feasible region?

b. Why is it appropriate that the feasible region in the exploration is located in the first quadrant?

c. Describe how you graphed the inequalities and combined their graphs to find the feasible region.

d. The feasible region in the exploration appears to be bounded by three segments. Which constraint determines each of these segments?

e.
1. In Part **d** of the exploration, you wrote an equation that relates grams of fat to servings of soup and rolls. Consider the graph of this equation for $f = 8$ and explain the meaning of several points along the line.
2. The line determined when $f = 8$ does not pass through the feasible region. What does this indicate in terms of the meal of tomato soup and rolls?

f.
1. In Part **e** of the exploration, you were asked to minimize fat consumption while providing at least 20% of the Daily Value of iron and at least 40% of the Daily Value of calcium. Where in the feasible region did this occur?
2. Describe a simple way of finding the minimum or maximum values of an objective function over a feasible region.

g. If you were asked to specify the combination of soup and rolls in terms of whole numbers of servings, how would you report the combination that minimizes the fat content?

Assignment

1.1 A farm family would like to plant up to 100 acres of their land in corn and wheat. They can afford a maximum of $26,500 on crop expenses. Corn costs about $300 per acre to plant, grow, and harvest. Wheat costs about $250 per acre to plant, grow, and harvest.

a. Write a system of inequalities that describes the constraints in this situation.

b. Graph the feasible region.

c. Identify the pairs of equations whose intersections are corner points and find the coordinates of these points.

d. Write an objective function that describes the farm's profit if the profit per acre for corn is $100 and the profit per acre for wheat is $90.

e. Determine the maximum profit the family can make by planting corn and wheat. Justify your response.

1.2 A nutritionist would like a meal of cheese pizza and fruit salad to provide at least 40% of the Daily Value of vitamin C and at least 20% of the Daily Value of calcium, but no more than 500 calories. The table below shows the Percent Daily Values for these nutrients in one serving of each food, based on a 2000-calorie reference diet:

Food	% DV Vitamin C	% DV Calcium	Calories	Fat
cheese pizza	14	18	290	8 g
fruit salad	14	4	124	0 g

a. Using p for the number of servings of pizza and s for the number of servings of salad, express all the constraints as inequalities.

b. Graph the feasible region.

c. The intersection of the constraints $p \geq 0$ and $s \geq 0$ does not define a corner point. Describe the region defined by these constraints and explain how it is related to the feasible region.

d. Write an objective function based on the desire to minimize calories.

e. One serving of pizza has a mass of 210 g, while one serving of fruit salad has a mass of 248 g. Determine how many grams of pizza and fruit salad a person should eat to minimize the intake of calories while providing at least 20% of the Daily Value of calcium and at least 40% of the Daily Value of vitamin C.

f. Find the total calories and total fat for your response to Part **e**.

1.3 **a**. Choose two healthy foods. Use their labels to determine how you can receive at least 100% of the Daily Values of any two nutrients while minimizing your intake of sodium.

b. In order to confirm your response to Part **a**, use both an algebraic method and a graphical method to identify the vertices of the feasible region.

1.4 One average-size plum provides approximately 3% of the Daily Value of vitamin A and 10% of the Daily Value of vitamin C. One large slice (500 g) of watermelon provides approximately 24% of the Daily Value of vitamin A and 80% of the Daily Value of vitamin C.

Explain why it is not possible to obtain at least 80% of the Daily Value of vitamin C, but no more than 10% of the Daily Value of vitamin A, by eating a combination of plums and watermelon.

Note: Vitamin C is water soluble, but vitamin A is not. Unused amounts of vitamin A accumulate in the body.

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1.5 A landscaping firm uses two brands of fertilizer, Green Grass and Grow More. Green Grass fertilizer contains 4 kg of phosphates and 2 kg of nitrates per bag, while Grow More fertilizer contains 6 kg of phosphates and 5 kg of nitrates per bag. One customer's lawn needs a mixture of at least 24 kg of phosphates and at least 16 kg of nitrates.

a. Write a system of inequalities that describes these constraints.

b. Graph the feasible region and identify its corner points.

c. Write the objective function that describes the total cost of fertilizer for this customer if Green Grass costs \$6.99 per bag and Grow More costs \$17.99 per bag.

d. Determine the minimum cost for fertilizer for this customer's lawn. Justify your response.

1.6 Two carpentry classes at school make and sell desks and chairs. One class assembles the furniture, while a second class varnishes each piece. It takes an average of 1 hr for the first class to assemble a chair and 4 hr to assemble a desk. It takes the second class an average of 2 hr to varnish each chair and 1 hr to varnish each desk. Each class has a maximum of 200 student hours to assemble or varnish furniture each week.

a. Write a system of inequalities that describes these constraints.

b. Graph the feasible region and list the coordinates of the corner points.

c. Write the objective function that describes the total profit if the two classes make \$15 per chair and \$25 per desk.

d. Determine the maximum profit the two carpentry classes can make in one week. Justify your response.

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Activity 2

When using linear programming to analyze meals, the feasible region is defined by a system of inequalities. In order to maximize your consumption of fiber or minimize your intake of fat, you may determine an objective function, then apply the corner principle and find a solution. In these situations, finding the coordinates of the vertices of the feasible region is an important task.

You already are familiar with at least two different methods for finding the intersection of two lines: graphing and substitution. In the following activities, you examine a third useful method for solving systems of linear equations.

Exploration

In this exploration, you investigate the use of matrices to represent systems of linear equations.

a. Solve each of the following systems of equations.

1. $\begin{cases} 2s + 3t = 2 \\ -5s + 0.5t = -21 \end{cases}$

2. $\begin{cases} 12x - y = 10 \\ 4x + 2y = 1 \end{cases}$

b. Check your solutions by substitution.

c. Simplify the following matrix equations, then solve them:

1. $\begin{bmatrix} 2 & 3 \\ -5 & 0.5 \end{bmatrix} \bullet \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -21 \end{bmatrix}$

2. $\begin{bmatrix} 12 & -1 \\ 4 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

d. Compare the systems of equations and solutions from Part **a** with the simplified matrix equations and solutions from Part **c**.

e. Based on your observations in Part **d**, express each of the following systems of equations as a matrix equation.

1. $\begin{cases} 5t + 7r = 30 \\ 2t + 8r = 25 \end{cases}$

2. $\begin{cases} x - 3y = -2 \\ 4x + 2y = 7 \end{cases}$

3. $\begin{cases} y = 3 - 2x \\ 3x + y = 4 \end{cases}$

Mathematics Note

Any system of linear equations of the form

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

may be written as a matrix equation:

$$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \end{array}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The matrix **M** is the **coefficient matrix**, since it represents the coefficients of the variables. Similarly, the matrix **X** is the **variable matrix**, since it represents the variables of the system, while the matrix **C** is the **constant matrix**, since it represents the constants of the system.

For example, the system

$$\begin{cases} 2s + t = 10 \\ 3s + 2t = 5 \end{cases}$$

may be written as the following matrix equation:

$$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \end{array}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \bullet \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Discussion

a. Identify the coefficient matrix, the variable matrix, and the constant matrix in each of the matrix equations in Part **c** of the exploration.

b. How many solutions are possible for a system of two linear equations with two variables?

c. How does solving a system of equations help you analyze a linear programming problem?

Assignment

2.1 **a.** Write the following system of equations as a matrix equation and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} -2x + 3y = -19 \\ 5x - 2y = 31 \end{cases}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.2 **a.** Write the following system of equations as a matrix equation and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} 2d = 14 + c \\ -5c + 10d = 70 \end{cases}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.3 **a.** Write the system of equations represented by the matrix equation below:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.4 A nutritionist is planning a meal of pasta and vegetables for a client. The client should receive no more than 40% of the Daily Value of cholesterol and at least 700 calories from the meal. The following table shows some of the nutritional information for single servings of pasta and mixed vegetables.

Food	% DV of Cholesterol	% DV of Sodium	Calories
pasta	35%	25%	450
vegetables	1%	6%	150

a. Write a system of inequalities that represents all of the combinations of pasta and vegetables that meet the client's requirements.

b. Create a graph of the feasible region.

c. Identify the corner point of the feasible region that does not lie on a coordinate axis. Write a system of equations whose solution defines this corner point.

d. Express the system of equations in Part **c** as a matrix equation.

e. One serving of pasta has a mass of 210 g, while one serving of vegetables has a mass of 200 g. Determine the number of grams of pasta and vegetables the client should eat to receive 40% of the Daily Value of cholesterol and 700 calories from the meal.

2.5 The following table lists the Percent Daily Values of several nutrients for one serving of crackers (30 g) and one serving of apples (100 g).

Food	Vitamin A	Iron	Calcium	Fat	Calories
crackers	2	6	3.3	3	120
apples	1.8	0.6	4.7	0.9	56

a. Write a system of inequalities that represents all of the combinations of crackers and apples which provide at least 20% of the Daily Value of vitamin A but no more than 25% of the Daily Value of fat.

b. Create a graph of the feasible region.

c. Identify the corner point of the feasible region that does not lie on a coordinate axis. Write a system of equations whose solution defines this corner point.

d. Express the system of equations in Part **c** as a matrix equation.

e. Determine the numbers of grams of crackers and apples that provide 20% of the Daily Value of vitamin A and 25% of the Daily Value of fat.

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2.6 **a.** Write the system of equations represented by the matrix equation below:

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \end{bmatrix}$$

b. Solve the system of equations given in Part **a** and verify your solution.

2.7 The Jewelry Emporium makes rings and necklaces. Each ring requires 5 g of metal and 1.5 hr of labor, while each necklace requires 20 g of metal and 1 hr of labor. Each week, the staff uses 500 g of metal and works a total of 80 hours.

a. Write both a system of equations and a matrix equation that describe this situation.

b. Determine the number of rings and necklaces that the Jewelry Emporium makes each week.

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Activity 3

In Activity **2**, you expressed systems of equations in matrix form. You also reviewed some methods of solving systems of equations. In this activity, you discover how to solve a matrix equation using matrix operations.

Exploration

This exploration introduces the use of matrices to solve a matrix equation.

a. Represent the following system as a matrix equation in the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ and identify the coefficient matrix, the variable matrix, and the constant matrix.

$$\begin{cases} -8s - 3t = 10 \\ 4s + 6t = 5 \end{cases}$$

Mathematics Note

If $a \bullet b = b \bullet a = 1$, where 1 is the **identity** for multiplication, then a and b are **multiplicative inverses**. The product of a and the identity 1 is a. In other words, $a \bullet 1 = 1 \bullet a = a$. In the real number system, for example, the multiplicative inverse of $3/5$ is $5/3$ since

$$\frac{5}{3} \bullet \frac{3}{5} = \frac{3}{5} \bullet \frac{5}{3} = 1$$

The number 0 has no multiplicative inverse.

The multiplicative inverse of a, where $a \neq 0$, may be written as a^{-1}. The product of a and a^{-1} is always the identity, 1. In other words, $a \bullet a^{-1} = a^{-1} \bullet a = 1$. The multiplicative inverse of $3/5$ can be written as $(3/5)^{-1}$, where $(3/5)^{-1} = 5/3$.

A **multiplicative identity matrix** (**I**) is always a square matrix with entries of 1 along the diagonal that passes from the upper left to the lower right. All the other elements in an identity matrix are 0.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The product of a square matrix **M** and the identity **I** is **M**. In other words, $\mathbf{M} \bullet \mathbf{I} = \mathbf{I} \bullet \mathbf{M} = \mathbf{M}$. If it exists, the multiplicative inverse of the matrix **M** may be written as $\mathbf{M}^{-1}$ and $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$.

Given a matrix **A** of the form shown below, its **determinant** is $ad - bc$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrix **A** has an **inverse matrix** $\mathbf{A}^{-1}$ if and only if $ad - bc \neq 0$. Although only square matrices have inverses, not every square matrix has an inverse.

b. Using appropriate technology, determine the multiplicative inverse of the coefficient matrix you identified in Part **a**.

c.
1. Determine the multiplicative identity **I** for any 2×2 matrix.
2. The product of any matrix **A** and the identity **I** is **A**. Prove that the matrix you found in Step **1** is the 2×2 identity by multiplying it by matrix **A** below:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

d. Solve the matrix equation you wrote in Part **a** by multiplying both sides of the equation by the multiplicative inverse of **M**. **Note:** Each side of the matrix equation must be multiplied on the *left* by $\mathbf{M}^{-1}$.

e. Check your solution in Part **d** using each of the following methods.

1. Substitute your values for *s* and *t* back into the original equations.

2. Graph the original equations and check their point of intersection.

3. Substitute your values for *s* and *t* into the variable matrix **X** and verify that $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$ using matrix multiplication.

Discussion

a. Compare the process for solving the linear equation $m \bullet x = c$ with the process for solving the matrix equation $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$.

b. Given a matrix equation $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$, which of the following equations could represent a first step in solving for the variable matrix **X**? Justify your response.

1. $\mathbf{M}^{-1} \bullet (\mathbf{M} \bullet \mathbf{X}) = \mathbf{M}^{-1} \bullet \mathbf{C}$

2. $\mathbf{M}^{-1} \bullet (\mathbf{M} \bullet \mathbf{X}) = \mathbf{C} \bullet \mathbf{M}^{-1}$

3. $(\mathbf{M} \bullet \mathbf{M}^{-1}) \bullet \mathbf{X} = \mathbf{C} \bullet \mathbf{M}^{-1}$

4. $(\mathbf{M} \bullet \mathbf{X}) \bullet \mathbf{M}^{-1} = \mathbf{C} \bullet \mathbf{M}^{-1}$

c. Describe how each of the following systems can be represented as a matrix equation in the form $\mathbf{M} \bullet \mathbf{X} = \mathbf{C}$.

1. $$\begin{cases} 3y - 2z = 3 \\ 2y + 5z = 21 \end{cases}$$

2. $$\begin{cases} s + 2t - 1v = 7 \\ 5s + t + 5v = 5 \\ s + t - v = 4 \end{cases}$$

d. Suggest a general rule for representing a system of three equations in three unknowns in matrix form.

e. **1.** Describe how to solve each system of equations in Part **c** of the discussion using matrix multiplication and multiplicative inverses.

2. Identify at least two different ways that you could check your solutions.

f. **1.** Describe the $n \times n$ identity for matrix multiplication.

2. What is the product of this identity and an $n \times n$ matrix **A**?

g. **1.** How can you determine whether or not a 2×2 matrix has an inverse?

2. What can you conclude about a system if its coefficient matrix does not have an inverse?

h. **1.** If a system of two equations in two unknowns has no solution, how can you tell this from looking at a graph of the equations?

2. Consider the system of two linear equations in two unknowns shown below.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

This system can be represented by the following matrix equation:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

Using the slopes of the two lines, determine what the relationship between a/d and b/e indicates about the number of solutions to the system.

3. If the coefficient matrix in Step **2** has no inverse, write a true equation expressing the relationship among a, b, d, and e.

i. How can solving a matrix equation help you analyze a linear programming problem?

Assignment

3.1 **a**. Find the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

b. Verify that your response to Part **a** is correct.

c. Write a 2×2 matrix that does not have an inverse. Defend your response.

3.2 Solve each of the following matrix equations and justify your solutions.

a. $\begin{bmatrix} 7 & 5 \\ 4 & 10 \end{bmatrix} \bullet \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$

b. $\begin{bmatrix} 20 & 12 \\ 5 & 3 \end{bmatrix} \bullet \begin{bmatrix} s \\ m \end{bmatrix} = \begin{bmatrix} 25 \\ 10 \end{bmatrix}$

c. $\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \bullet \begin{bmatrix} f \\ r \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \end{bmatrix}$

3.3 A nutritionist is planning a meal of pasta and vegetables for a client. The client should receive at least 300 calories from the meal but no more than 15% of the Daily Value of sodium. The following table shows some of the nutritional information for single servings of pasta and mixed vegetables.

Food	Serving Size	% DV of Cholesterol	% DV of Sodium	Calories
pasta	210 g	35%	25%	450
vegetables	200 g	1%	6%	150

a. Write a system of four inequalities using two variables that describes the constraints in this situation.

b. Graph the feasible region.

c. Each corner of the feasible region is determined by two intersecting lines. Write the equations for the lines related to the 15% limit on the Daily Value of sodium and the minimum of 300 calories.

d. Write a matrix equation for the pair of equations from Part **c**.

e. Solve the matrix equation you wrote in Part **d** and determine whether or not your solution lies in the feasible region.

f. Find the remaining vertices of the feasible region and determine which one minimizes the cholesterol in the meal.

3.4 The following table lists the Percent Daily Values of several nutrients for one serving of crackers (30 g) and one serving of apples (100 g).

Food	Vitamin A	Iron	Calcium	Fat	Calories
crackers	2	6	3.3	3	120
apples	1.8	0.6	4.7	0.9	56

a. Imagine that you would like to design a snack of crackers and apples that provides at least 20% of the Daily Value of calcium and at least 10% of the Daily Values of vitamin A and iron, while providing no more than 15% of the Daily Value of fat. Write six inequalities using two variables that describe these constraints.

b. Graph the feasible region.

c. Determine the coordinates of the vertices of the feasible region.

d. Determine the combination of crackers and apples (in grams) that minimizes the number of calories.

e. Does your solution provide a reasonable snack of crackers and apples? If not, how do you think the snack should be modified?

* * * * *

3.5 As described in Problem **2.7**, the Jewelry Emporium manufactures rings and necklaces. Each week, the staff uses a maximum of 500 g of metal and works a maximum of 80 hours. Each ring requires 5 g of metal and 1.5 hr of labor, while each necklace requires 20 g of metal and 1 hr of labor. The profit on each ring is $90; the profit on each necklace is $40.

a. Write a system of inequalities that describes all the constraints in this situation.

b. Graph the system and shade the feasible region.

c. Determine the vertices of the feasible region. When a vertex does not lie on a coordinate axis, use matrices to find it.

d. Determine the maximum amount of profit that the Jewelry Emporium can earn each week. Justify your response.

3.6 The table below shows some of the nutrients in one piece of fried chicken and one ear of corn.

	Vitamin A	**Potassium**	**Iron**	**Calories**
fried chicken	100 units	0 mg	1.2 mg	122
ear of corn	310 units	151 mg	1.0 mg	70

a. Imagine that you want to plan a meal of chicken and corn that contains a minimum of 1000 units of Vitamin A, 200 mg of potassium, 6 mg of iron, and 600 calories. Write a system of inequalities that describes these constraints.

b. Determine the coordinates of the vertices of the feasible region.

c. Determine the minimum cost of a meal that meets the constraints in Part **a** if one piece of fried chicken costs $0.90 and one ear of corn costs $0.75. Justify your response.

* * * * * * * * * *

Activity 4

Healthy meals typically contain more than two kinds of food. As the number of variables and constraints increases, using linear programming to analyze nutrition can become more complicated. The usual graphical representation of a feasible region is difficult to extend to three variables, and impossible for four or more. Fortunately, the corner principle is valid for any number of variables and constraints. To identify these corner points, however, you must solve systems of equations algebraically.

Exploration 1

Nutritionists recommend a diet that includes fruit. Table **2** lists the Percent Daily Values of some nutrients for several kinds of fruit:

Table 2: Percent Daily Values for Nutrients in Several Fruits

One Serving (100 g)	Vit. A	Vit. C	Iron	Calcium	Fat	Carbo-hydrates	Fiber	Calories
apples	1.8	11.7	0.6	0.3	0.9	4.7	4.0	56
bananas	0.0	23.3	2.2	1.0	1.8	8.5	1.6	97
oranges	4.0	83.3	2.2	4.1	0.3	4.1	2.0	49
pears	0.4	6.7	1.7	0.8	1.1	5.1	5.6	61
peaches	26.6	11.7	2.8	0.9	0.2	3.2	2.4	38
peaches (in syrup)	7.1	4.0	2.0	0.3	0.0	4.8	1.2	56
raisins	0.4	1.5	11.7	8.6	0.3	25.7	3.2	78

Imagine that you must create a recipe for fruit salad using apples, bananas, and oranges. The salad should consist of at least one serving (100 g) of each fruit and provide at least 20% of the Daily Values of vitamin A, iron, and calcium. These constraints are described by the following system of inequalities, where a, b, and r represent servings of apples, bananas, and oranges, respectively:

$$\begin{cases} a \geq 1 \\ b \geq 1 \\ r \geq 1 \\ 1.8a + 0.0b + 4.0r \geq 20 \text{ (vitamin A)} \\ 0.6a + 2.2b + 2.2r \geq 20 \text{ (iron)} \\ 0.3a + 1.0b + 4.1r \geq 20 \text{ (calcium)} \end{cases}$$

a. Determine at least three ordered triples (a, b, r) that are solutions to the equation $1.8a + 0b + 4.0r = 20$.

b. Describe the graph of all the possible solutions in Part **a**.

c. Determine at least three ordered triples (a, b, r) that are solutions to the equation $0.6a + 2.2b + 2.2r = 20$.

d. Describe the graph of all the possible solutions in Part **c**.

Mathematics Note

The solution set of an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all 0, is a plane.

The graph of the equation $x + 2y + z = 10$, for example, is a plane. A portion of this plane is shown in Figure **3**. The triangular region represents the portion of the plane that falls in the first octant.

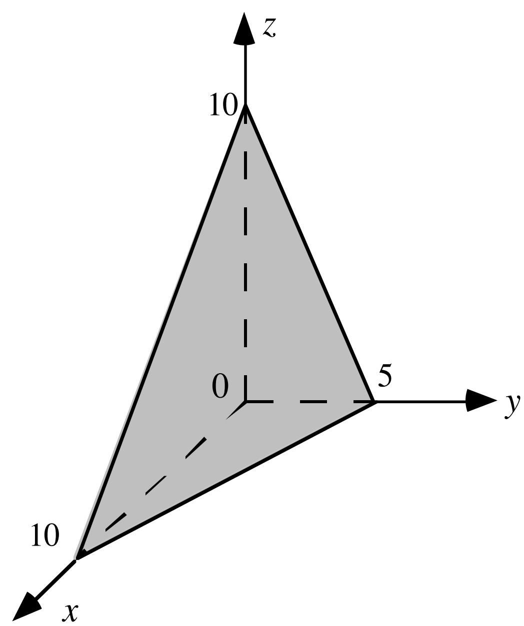

Figure 3: A portion of the graph of the equation $x + 2y + z = 10$

e. Use three index cards to represent three planes in space. Assuming that parallel planes do not intersect, make a model of each of the following situations:

1. three planes that are parallel to each other

2. two of three planes that are parallel to each other

3. three planes that intersect in a line

4. three planes that intersect in a point

5. three planes that are not parallel and have no points common to all of them.

Discussion 1

a. Why do you need three dimensions to graph the equations in Parts **a** and **c** of Exploration **1**?

b. If your models in Part **e** of Exploration **1** represent graphs of constraints, which, if any, includes a potential corner point for a set of feasible solutions? Explain your response.

c. What is the least number of planes that can intersect in a single point?

d. In a two-variable system, the graphs of inequalities are half-planes. Describe the graph of an inequality in a three-variable system.

e. Imagine that you are working with a problem that involves three variables.

1. Determine a system of inequalities for which the feasible region is the entire first octant.
2. Determine a system of inequalities for which the feasible region is a rectangular prism.

Exploration 2

In problems involving two variables, you used the corner principle to find the values of variables that minimized or maximized some objective function. The corner principle works equally well for problems with three or more variables. In this exploration, you solve some selected systems of three equations in three unknowns, without considering all the intersections necessary to analyze the situation completely. To ensure that a particular intersection yields a corner point, you should verify that its coordinates satisfy all the given constraints.

The corner points of the feasible region described by the six inequalities in Exploration **1** are located at the intersections of the following sets of planes:

$$\text{I: } \begin{cases} a = 1 \\ b = 1 \\ 0.6a + 2.2b + 2.2r = 20 \end{cases} \qquad \text{IV: } \begin{cases} a = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.6a + 2.2b + 2.2r = 20 \end{cases}$$

$$\text{II: } \begin{cases} b = 1 \\ r = 1 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases} \qquad \text{V: } \begin{cases} b = 1 \\ 0.6a + 2.2b + 2.2r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

$$\text{III: } \begin{cases} r = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases} \qquad \text{VI: } \begin{cases} 1.8a + 0.0b + 4.0r = 20 \\ 0.6a + 2.2b + 2.2r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

a. Write each of these systems of equations as a matrix equation.

b. Determine the corner point described by each set of planes, either by solving the system of equations using substitution, or by solving the corresponding matrix equation. Check each solution.

c. Write an equation for the total number of calories in a fruit salad containing *a* servings of apples, *b* servings of bananas, and *r* servings of oranges.

d. Using the corner points you found in Part **b** and the equation you wrote in Part **c**, determine a fruit-salad recipe that minimizes calories. Write the recipe in terms of the number of grams of each fruit in the salad.

e. Demonstrate that your solution in Part **d** appears to minimize calories by testing at least three other points in the feasible region.

Discussion 2

a. What do the three equations in the following system represent in terms of the situation described in Explorations **1** and **2**?

$$\begin{cases} r = 1 \\ 1.8a + 0.0b + 4.0r = 20 \\ 0.3a + 1.0b + 4.1r = 20 \end{cases}$$

b. In a two-variable system, an objective function defines a family of lines, some of which pass through the feasible region. Describe the graph of an objective function in a three-variable system.

c. Use geometry to explain why the corner principle works in three-variable systems.

Assignment

4.1 Defend or refute each of the following statements.

a. The solution to a system of equations can be a single ordered pair of numbers.

b. The solution to a system of equations can be a single ordered triple of numbers.

c. A system of parallel lines represents a system of linear equations with at least one solution.

d. A system of parallel planes represents a system of equations with no solutions.

e. If two of three planes are parallel, the system of equations represented by the planes has no solutions.

f. If three planes intersect in such a way that no two of them are parallel, then the system of equations represented by the three planes has at least one solution.

4.2 Consider the following system of three equations with the three variables x, y, and z:

$$\begin{cases} 0x + 0y + z = 1 \\ 0x + 0y + z = 2 \\ 0x + 0y + z = 3 \end{cases}$$

a. Describe the graph of this system.

b. Rewrite the system in matrix form.

c. Describe what happens when you solve this system using matrices.

4.3 Solve each of the following systems of equations and use geometric terms to describe the intersections of planes.

a. $\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{cases}$

b. $\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 5 \\ 9x + 10y + 11z = 12 \end{cases}$

c. $\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 5 \\ 3x + 6y + 9z = 7 \end{cases}$

d. $\begin{cases} 2x - y + 3z = -9 \\ x + 3y - z = 10 \\ 3x + y - z = 8 \end{cases}$

4.4 Imagine that you want to design a snack of bananas, peaches, and raisins that provides at least 10% of the Daily Values for vitamin A, iron, calcium, and fiber. Use the variables b, p, and r to represent the bananas, peaches, and raisins, respectively.

a. Use the information in Table **2** to write the four inequalities that represent these constraints.

b. The intersection of these four inequalities forms a three-dimensional region with four corners. Each of these corners is the intersection of the edges defined by three inequalities. Identify these four corners.

c. Explain why only one of these corners is a possible solution to the problem.

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4.5 In Exploration **2**, you designed a recipe for fruit salad that provides 20% of the Daily Values of vitamin A, iron, and calcium while minimizing calories. Using the corner points determined in Exploration **2**, design another recipe for fruit salad that provides 20% of the Daily Values of the same three nutrients, while maximizing fiber.

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Summary Assessment

Since May 1994, the U.S. Food and Drug Administration (FDA) has required food manufacturers to show the Percent Daily Values of several nutrients on food labels. Find labels for two foods appropriate for a meal. The two labels should report more than 0% of the Daily Values for at least three of the same nutrients. Assuming that you are restricted to a 2000-calorie diet, determine how many servings of each food you must eat to obtain at least 50% of your Daily Values for the three nutrients, while minimizing your consumption of fat. Explain how you determined each amount and describe the nutritional benefits or drawbacks of the meal.

Module Summary

- **Linear programming** can be used to solve problems involving variables that are subject to linear **constraints**. The system of linear inequalities determined by these constraints defines a **feasible region**, a set of points that satisfies the system. Typically, such problems require the identification of an optimal value for an **objective function**, either a maximum or a minimum.
- According to the **corner principle**, if the objective function has a minimum or maximum value, it will occur at a corner point (or vertex) of the feasible region.
- Any system of linear equations of the form

  $$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

  may be written as the matrix equation:

  $$\begin{array}{ccccc} \mathbf{M} & \bullet & \mathbf{X} & = & \mathbf{C} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \bullet & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} e \\ f \end{bmatrix} \end{array}$$

  where matrix **M** is the **coefficient matrix**, matrix **X** is the **variable matrix**, and matrix **C** is the **constant matrix**.
- If $a \bullet b = b \bullet a = 1$, where 1 is the **identity** for multiplication, then a and b are **multiplicative inverses**. The product of a and the identity 1 is a. In other words, $a \bullet 1 = 1 \bullet a = a$. The number 0 has no multiplicative inverse.
- The multiplicative inverse of a, where $a \neq 0$, may be written as a^{-1}. The product of a and a^{-1} is always the identity, 1. In other words, $a \bullet a^{-1} = a^{-1} \bullet a = 1$.
- A **multiplicative identity matrix** (**I**) is always a square matrix with entries of 1 along the diagonal that passes from the upper left to the lower right. All the other elements in an identity matrix are 0.

  $$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- The product of a square matrix **M** and the corresponding identity **I** is **M**. In other words, $\mathbf{M} \bullet \mathbf{I} = \mathbf{I} \bullet \mathbf{M} = \mathbf{M}$.

- If it exists, the multiplicative inverse of the matrix **M** may be written as $\mathbf{M}^{-1}$ and $\mathbf{M} \bullet \mathbf{M}^{-1} = \mathbf{M}^{-1} \bullet \mathbf{M} = \mathbf{I}$, where **I** is the identity for matrix multiplication.
- Given a matrix **A** of the form shown below, its **determinant** is $ad - bc$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  Matrix **A** has an **inverse matrix** $\mathbf{A}^{-1}$ if and only if $ad - bc \neq 0$. Although only square matrices have inverses, not every square matrix has an inverse.
- The solution set of an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all 0, is a plane.

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