Taste Test

What's your favorite soft drink? Do you drink it because of its taste, or for some other reasons? In this module, you explore the bewildering number of choices involved in marketing a new soft drink.

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Taste Test

Introduction

Soft-drink manufacturers and other food companies often conduct taste tests to try out new product lines. Marketers know that placement of products on tables and the order in which products are presented may influence consumers' responses. Depending on the number of possibilities, however, it may not be practical to consider all the different ways to arrange products.

Activity 1

Imagine that you are the director of marketing at a soft-drink company. As part of a new advertising campaign, you decide to conduct a taste test using four of your company's products. The four soft drinks you've selected are 6-Down, Dr. Salt, Valley Fog, and Branch Tea. How should you organize the taste test?

Exploration

During the taste test, samples of each soft drink will be served at specially designed stations to a panel of judges. The serving stations will be placed on tables in a large television studio.

Each table must have at least one serving station, but can accommodate no more than three. To heighten visual impact, the tablecloths should be one of the five colors in the company's logo and brand labels: red, white, blue, black, or green.

a. List all the different ways you could design a table according to the above specifications. For example, one possible design places two serving stations on a table with a white tablecloth.

b. Organize your list to verify that no possible design has been overlooked.

c. Recall that a **tree diagram** can be used to show all the possible outcomes of an experiment. Create a tree diagram that shows all the possible table designs for the taste test.

d. Compare the number of possible designs indicated in your tree diagram with the number you determined using the list from Part **b**.

e. Suppose that two different types of tables are available: rectangular or circular. Create a tree diagram that shows how these new choices affect the number of possible designs.

f. Determine a method for finding the number of possible designs in Part **e** without making a list or a tree diagram.

Discussion

a. Describe the method you devised in Part **f** of the exploration.

b. How is this method reflected in the organization of the tree diagram you produced in Part **e**?

Mathematics Note

The **fundamental counting principle** states that if one selection can be made in h ways, and for each of these ways a second selection can be made in k ways, then the number of different ways the two selections can be made is $h \bullet k$.

For example, the number of outfits that can be made with 2 different pairs of pants and 3 different shirts is $2 \bullet 3 = 6$.

c. Describe how the fundamental counting principle can be used to determine the number of possible table designs considering the list of choices below.

- You must choose a table from n possible shapes.
- You must choose a tablecloth from m possible colors.
- You must select a specific number of serving stations per table, from at least 1 to no more than s.
- You must decide whether to test 2, 3, or 4 different soft drinks at each station. (Assume that each station must offer the same number of soft drinks.)

Assignment

1.1 Imagine that you are planning to conduct a taste test at your school. You have decided to use 7 rooms, with 3 tables in each room. Each table has 4 serving stations and each station has 3 unlabeled soft-drink dispensers.

a. Determine a method for identifying each soft-drink dispenser with a distinct label.

b. How many different labels are needed to identify all the dispensers?

1.2 Consider a taste test that involves 20 different soft drinks. In how many different ways can the judges select first-place and second-place winners?

1.3 A sporting goods store stocks ice chests in 3 different sizes: small, large, and jumbo. Each size comes in 5 different colors: red, blue, green, purple, and gray. Each ice chest also can be purchased with or without an electric cooling unit.

a. Create a tree diagram that shows all the possible ice chests.

b. How many different ice chests does the store stock?

1.4 A restaurant offers three different sizes of pizza with thin or thick crust and 5 choices of toppings. How many different kinds of single-topping pizzas can the restaurant make?

1.5 In the United States, each radio station is identified by a set of 4 call letters. These call letters traditionally begin with either the letter K or the letter W. Assuming that the remaining three letters can be selected from any letter of the English alphabet, how many different sets of call letters can be created before a new naming system must be devised?

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1.6 Guida has 3 pairs of dress pants: red, blue, and green. She also has 2 vests—red and blue—along with 3 silk shirts: red, green, and white.

a. Make a tree diagram that shows all the possible outfits that include pants, a vest, and a shirt.

b. How many different outfits can Guida create?

c. If Guida picks an outfit at random, what is the probability that the outfit will be all red?

1.7 A local ice cream parlor features 7 different flavors of ice cream, 5 different toppings, and 3 different types of cones.

a. How many different single-flavor ice cream cones with one topping are possible?

b. How many different single-flavor desserts are possible if a customer may choose to have ice cream in a dish instead of a cone and may choose not to have a topping?

1.8 Professional sports stadiums typically seat thousands of people. To allow reserved seating, each seat must be identified by a unique label—using numbers, letters, or both. Suggest an identification system for an arena with 18,000 seats.

1.9 Imagine that you are chief of security at an art museum. The security code for the electronic lock on the museum's main door contains four digits. Each digit may range from 1 through 9. Because of a recent security violation, you have been asked to assign a new code to the door.

a. From how many different codes can you choose?

b. Suppose that a burglar has enough time to try 10 different security codes before an alarm sounds. What is the probability that the burglar will discover the correct code?

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Activity 2

As the director of marketing at a soft-drink company, you know that a product's packaging can be as important to sales as the product itself. To help boost the popularity of a new beverage, you decide to explore some new color schemes on the can's label.

Exploration

The company's design staff has determined that the most appealing colors to potential consumers are red, blue, white, and silver. Your plan is to produce some model cans using these colors, then conduct a survey to determine which mix of colors is most popular.

a. Imagine that the can design will have four elements: background, company logo, soft-drink name, and highlighting.

1. The first color selected will be used for the label's background. The second color selected will be used for the company logo. This color must be different from the background color.

   Determine how many cans must be made to show all possible arrangements of first and second colors.

2. The third color selected will be used for the name of the soft drink. This color must be different from both the background color and the logo color. Determine how many cans are necessary to show every possible arrangement of first, second, and third colors.

3. The fourth color selected will be used for highlighting. This color must be different than the previous three colors. Determine how many cans are necessary to show every possible arrangement of first, second, third, and fourth colors.

4. Justify your response to Step **3** by creating a tree diagram that shows all the possible arrangements of the four colors among the four design elements.

b. How would the number of cans necessary to show all the possible arrangements of colors among the four design elements change if you added a fifth color—black—to the original set? Hint: To answer this question, repeat Part **a** using a set of five colors.

c. Suppose that the marketing team decides to incorporate a fifth design element in the can.

1. Determine the number of cans necessary to show all the possible arrangements of five colors among five design elements.
2. Express your response to Step **1** using factorial notation.

d. Suppose that the marketing team had identified a set of n different colors. If no color can be used more than once in a design, how many cans would be necessary to show all the possible arrangements of colors for each of the following:

1. a design with 1 element
2. a design with 2 elements
3. a design with 3 elements
4. a design with r elements, where $r < n$
5. a design with n elements (express your response using factorial notation).

Mathematics Note

A **permutation** is an ordered arrangement of items from a set. For example, two permutations of three colors that could be used on the cans described in the exploration are shown in Figure **1**.

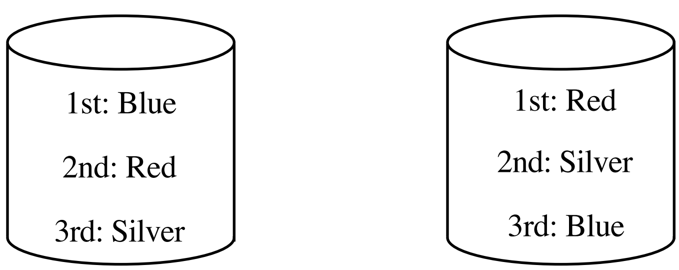

Figure 1: Two possible cans

The total number of permutations of r items chosen from a set of n items can be denoted by $P(n,r)$ or ${}_nP_r$.

For example, the number of ways to select the first, second, and third colors from a set of 5 colors can be denoted by $P(5,3)$, or "the number of permutations of 5 items taken 3 at a time." This number of permutations may be found using $5 \bullet 4 \bullet 3 = 60$.

e. Use the notation $P(n,r)$ to express the number of possible designs for each case considered in Part **d**.

Discussion

a. Describe how the fundamental counting principle could be used to determine the number of possible designs in each case considered in Part **d** of the exploration.

b. Use your response to Part **a** above to suggest a formula for calculating $P(n,r)$.

c. Consider a can with n design elements in which no color can be used more than once. Use factorial notation to describe the number of designs possible given a set of n colors.

d. Consider a can with $(n-r)$ design elements in which no color can be used more than once. Use factorial notation to describe the number of designs possible given a set of $(n-r)$ colors.

e. What does the product of your responses to Parts **b** and **d** of the discussion represent?

f. Use your response to Part **e** to determine an equation for calculating $P(n,r)$.

Mathematics Note

The number of permutations of n items taken r at a time can be calculated using the following formulas:

$$P(n,r) = \overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r \text{ terms}} \text{ or } P(n,r) = \frac{n!}{(n-r)!}$$

For example, the number of possible rankings of 12 soft drinks into first, second, and third places can be found as follows:

$$P(12,3) = \overbrace{12 \bullet 11 \bullet 10}^{3 \text{ terms}} = 1320 \text{ or } P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 1320$$

g. How do the formulas described in the mathematics note compare with the ones you suggested in Parts **b** and **f** of the discussion?

h. Use an example to explain why both formulas in the mathematics note yield the same result.

i. The permutation of n objects arranged n at a time, $P(n,n)$, is $n!$ Use this fact, along with one of the formulas for $P(n,r)$, to suggest a definition for $0!$

Assignment

2.1 Find the value of each of the following:

a. $P(8,3)$

b. $P(8,4)$

c. $P(8,7)$

d. $P(8,8)$

2.2 Use factorials to write an expression for each of the following:

a. $P(12,4)$

b. $P(8,3)$

c. $P(8,7)$

2.3 Suppose that you decide to use 5 tables in the soft-drink taste test described in Activity **1**. To boost publicity for this promotion, you plan to station a television celebrity at each table.

How many different arrangements can you create if you may choose from each of the following?

a. a set of 5 celebrities

b. a set of 8 celebrities

2.4 You and 7 other members of your marketing staff plan to have a group picture taken at the taste test. In how many ways can the group be lined up in a row for the photograph?

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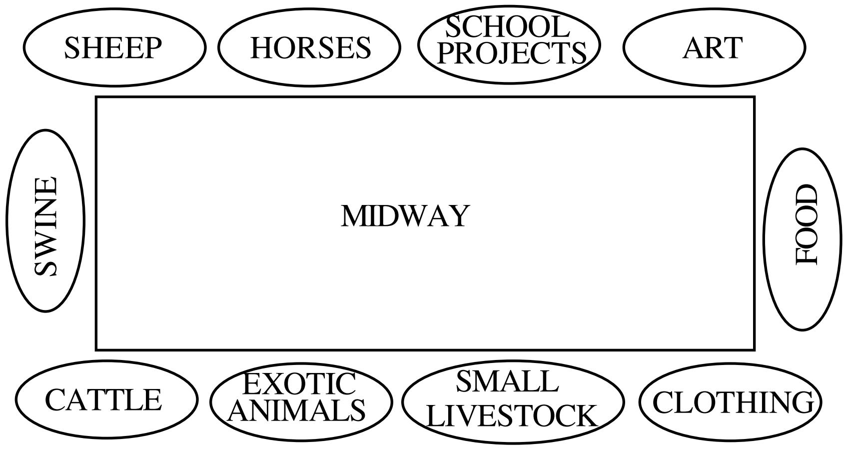
2.5 Imagine that you coach a children's baseball team. While selecting the batting order for the next game, you decide to avoid any appearance of favoritism by randomly drawing the names of 9 players out of a hat. Players will bat in the order in which their names were drawn.

a. If there are 12 players on the team, how many different batting orders are possible?

b. When you announce the batting order, the father of the player who is batting ninth accuses you of favoritism. He claims that if the selection process were actually random, then the chances of his child batting ninth are almost 0. How many different batting orders are possible in which his child is picked ninth?

c. You explain to the father that, since names were drawn at random, every child had the same chance of being picked ninth. Use your responses to Parts **a** and **b** to determine the probability that any particular child would be picked 9th in a random drawing.

2.6 As shown in the diagram below, there are 10 exhibit areas at a county fair. To determine which areas are most popular, you survey fairgoers leaving the exhibit grounds and ask them to rank each location from 1st to 10th.



a. How many different rankings are possible for 1st and 2nd place?

b. How many different rankings are possible for 1st through *r*th place?

c. Suppose you ask people to rank only the exhibit areas they actually visited. How many different responses are possible then?

2.7 A local radio station is holding a random drawing for three prizes—one first prize, one second prize, and one third prize.

a. There are 550 entries in the contest and each player can win only one prize. How many different arrangements are possible for the first-, second-, and third-prize winners?

b. Imagine that your name is included in the 550 entries.

1. What is the probability that you will win the first prize?
2. What is the probability that you will win one of the three prizes?

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Activity 3

As marketing director, you must determine how to conduct a taste test for four different soft drinks. Should you test two soft drinks at each serving station? Or three? Or all four? How many different stations would have to be used to test all possible combinations of the soft drinks?

Exploration

In this exploration, you develop a method for calculating the number of different combinations which can be tested at each serving station.

a. You have four soft drinks to test: 6-Down, Dr. Salt, Valley Fog, and Branch Tea. Imagine that each serving station has two dispensers, labeled 1 and 2.

1. Record each possible permutation for testing 2 of 4 soft drinks at a given station on a separate slip of paper.
2. Rearrange the paper slips into groups so that each permutation in the group contains the same two soft drinks.
3. Record both the number of different groups and the number of permutations in each group.
4. Express the number of permutations in each group as a factorial.
5. What is the product of the number of different groups and the number of permutations in each group?

> ### Mathematics Note
>
> A **combination** is a collection of items from a set in which order is not important.
>
> For example, the members of a committee with no designated chairperson is a combination. The committee composed of Nicholas and Alexandra is exactly the same as the committee consisting of Alexandra and Nicholas.

b.

1. Suppose that the soft-drink dispensers at each serving station are unlabeled. List all of the possible combinations for testing 2 of 4 soft drinks at a given station.
2. Compare the number of combinations in your list to the number of groups you recorded in Part **a**.
3. Describe the relationship among the number of combinations, the total number of permutations you listed in Part **a**, and the number of permutations in each group in Part **a**.

c. Suppose that each serving station has three drink dispensers, labeled 1, 2, and 3. Repeat Parts **a** and **b** assuming that 3 of the 4 soft drinks are tested at each station.

d. Suppose that each serving station has four drink dispensers, labeled 1, 2, 3, and 4. Repeat Parts **a** and **b** assuming that all 4 soft drinks are tested at each station.

e. Determine a general equation that relates the number of possible combinations of r items selected from a set of n items and the number of possible permutations of r items from a set of n items.

Discussion

a. What might a combination describe in terms of sets?

b. Explain how to determine the number of possible combinations of r items selected from a set of n items.

Mathematics Note

The number of different combinations, or subsets, of r items selected from a set of n items can be denoted by $C(n,r)$, ${}_nC_r$, or

$$\binom{n}{r}$$

These symbols are often read as "the combinations of n things taken r at a time" or "n choose r."

Using the fundamental counting principle, the relationship between $C(n,r)$ and $P(n,r)$ can be expressed as follows:

$$C(n,r) \bullet P(r,r) = P(n,r)$$

This equation can be used to write the following formulas:

$$C(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

For example, the number of different combinations of 3 people chosen from a group of 20 can be denoted by $C(20,3)$, ${}_{20}C_3$, or

$$\binom{20}{3}$$

Using one of the formulas given above,

$$C(20,3) = \frac{20!}{(20-3)!\,3!} = \frac{20 \bullet 19 \bullet 18}{3!} = 6840$$

c. Which of the formulas given in the mathematics note corresponds best with the explanation you gave in Part **b** of the discussion?

d. How is the quantity $r!$ in the formulas related to the number of permutations in each group found in Part **a** of the exploration?

e. Express the formulas given for combinations in terms of the notation used for permutations.

f. The value of $C(n,n)$ is 1. Use the formulas given in the mathematics note to explain this result.

Assignment

3.1 **a.** Consider a race involving six runners. In how many different ways can the runners place first, second, and third?

b. The top three runners in the race will qualify for the next round of competition. How many different subsets of runners can finish in the top three?

c. Given a subset of three runners who qualify for the next round, in how many different ways can they place first, second, and third?

d. Write an equation that describes the relationship among your responses to Parts **a**, **b**, and **c**.

3.2 Is the following equation true or false? Justify your response.

$$\frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}$$

3.3 Determine whether each of the following statements is true or false. Justify your responses.

a. $C(7,2) = C(7,5)$

b. $C(10,6) = C(10,4)$

c. $C(n,r) = C(n,n-r)$

3.4 Imagine that you decide to conduct a taste test involving 5 different soft drinks. Assuming that you test only 2 soft drinks at each station, how many different stations would you need to match each soft drink against every other one?

3.5 Suppose that you decide to test 3 of the 5 soft drinks in Problem **3.4** at each station.

a. How many different stations would you need to conduct all the possible three-way competitions simultaneously?

b. Dr. Salt is one of the 5 soft drinks in the taste test. How many different stations will have Dr. Salt as 1 of their 3 soft drinks?

3.6 Of the first 25 people who volunteered for the taste test, 10 are male and 15 are female. You plan to randomly select 5 of them to go from station to station and judge the drinks.

a. In how many different ways can the 5 judges be selected?

b. In how many different ways could the 5 judges turn out to be all males? all females?

c. What is the probability that the 5 judges will turn out to be all males? all females?

3.7 Concerned that the random selection process described in Problem **3.6** might produce a group of 5 judges who are all male or all female, you decide at the last minute to choose 3 of the females and 2 of the males from the 25 volunteers.

a. In how many different ways can the 2 males be selected? In how many different ways can the 3 females be selected?

b. Based on your responses to Part **a**, how many different ways are there to select the panel of 3 females and 2 males? (Hint: Use the fundamental counting principle.)

c. Compare your responses to Problem **3.6a** and **3.7b**. How do you explain the difference?

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3.8 Imagine that you own 200 compact discs, but only have room to store 75 of them. How many different subsets of 75 compact discs can you select?

3.9 Sam's Chinese Restaurant has 12 different main entrees. You and 5 friends are planning a "Taste of China" evening with videos, dinner, and music. You agree to order 6 different entrees and share them at the meal. From how many different groups of 6 entrees can you and your friends select?

3.10 Fifteen points are evenly spaced on a circle. Imagine that you want to create a design by connecting each point to each of the other points.

a. How many different line segments must you draw to complete the design?

b. How many different triangles with each vertex on one of the circle's 15 points does your design contain?

3.11 One of the booths at a fair offers the chance to win your choice of 14 different prizes—if you can toss a softball into the mouth of a milk can. Your friend Jack wins twice. From how many sets of 2 different prizes can Jack choose?

3.12 At basketball camp, the players are divided randomly into 14 teams of 5 players each.

a. In how many different ways can the first team of 5 be selected?

b. In how many different ways can the second team of 5 be selected?

c. The coaches promise that each team will play every other team once before the week is over. How many games will this require?

3.13 Consider the set of numbers $\{1, 2, 3, 4, 5\}$.

a. How many subsets of 3 numbers can be chosen from this set?

b. How many of the possible subsets of 3 numbers include the number 5?

c. How many of the possible subsets of 3 numbers do not include the number 5?

d. How is the sum of your responses to Parts **b** and **c** related to your response to Part **a**?

e. Generalize the relationship you described in Part **d** for subsets of *r* elements chosen from a set of *n* elements.

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Research Project

Imagine that you have been asked to design the schedule for an annual cookie baking competition. There are 13 entrants. Each entrant must bake 36 chocolate chip cookies.

During each round of the competition, the 6 judges each sample 2 different cookies. Each entrants' cookies must be compared with every other entrants' cookies and no entrant may have more than 1 cookie sampled during any round.

a. Design a schedule for the competition and describe it in a report. Your report should address each of the following questions.

1. How many rounds must be held during the competition?

2. How many cookies must each judge sample?

3. Have the entrants baked enough cookies for the competition?

4. How will the winning entrant be determined?

b. Draw a diagram to help others interpret the schedule. Include an explanation of how your diagram works.

Summary Assessment

When selecting the members of its student council, Cooperative High School prides itself on giving an equal voice to each class. The positions are filled by a random drawing of the names of all interested students who pass the minimum school-attendance requirement. Once the council has been selected, another random drawing from among the council members determines the positions of president, vice-president, secretary, and treasurer.

Because representation is proportional to enrollment, this year's freshman and sophomore classes each receive 4 representatives, while the junior and senior classes each receive 3. The chart below shows the number of students, by class, whose names were included in the random drawing.

Class	**Freshman**	**Sophomore**	**Junior**	**Senior**
Female	14	10	9	8
Male	6	7	9	7
Total	20	17	18	15

1. Complete Parts **a–e** below for each class.

- **a.** How many different selections of class representatives are possible for each class?
- **b.** How many possible selections of class representatives consist entirely of females for each class?
- **c.** How many possible selections of class representatives consist entirely of males for each class?
- **d.** What is the probability that all the representatives for a given class will be female?
- **e.** What is the probability that all the representatives for a given class will be male?

2. Use the fundamental counting principle and your responses to Problem **1** to answer each of the following:

- **a.** How many different school councils are possible, before the selection of officers?
- **b.** How many of the possible school councils are all female?
- **c.** How many of the possible school councils are all male?
- **d.** What is the probability that the school council will be all female?
- **e.** What is the probability that the school council will be all male?

3. This year's selection of president, vice president, secretary, and treasurer is causing some concern among the juniors and seniors. As one senior noted, "There's a good chance that all four offices will go to freshmen and sophomores!"

a. In how many different ways can the four offices be filled once the representatives to the council have been selected?

b. In how many different ways can the four offices be filled using only freshmen and sophomores?

c. What is the probability that all four offices will go to freshmen and sophomores?

d. Are the concerns of the senior class justified?

Module Summary

- The **fundamental counting principle** states that if one selection can be made in h ways, and for each of these ways a second selection can be made in k ways, then the number of different ways the two selections can be made is $h \bullet k$.
- A **permutation** is an ordered arrangement of items from a set. The total number of permutations of r items chosen from a set of n items can be denoted by $P(n,r)$ or ${}_nP_r$.
- The number of permutations of n items taken r at a time can be calculated using the following formulas:

$$P(n,r) = \overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r \text{ terms}} \text{ or } P(n,r) = \frac{n!}{(n-r)!}$$

- A **combination** is a collection of items from a set in which order is not important. The number of different combinations, or subsets, of r items selected from a set of n items can be denoted by $C(n,r)$, ${}_nC_r$, or

$$\binom{n}{r}$$

  These symbols are often read as "the combinations of n things taken r at a time" or "n choose r."
- Using the fundamental counting principle, the relationship between $C(n,r)$ and $P(n,r)$ can be expressed as follows:

$$C(n,r) \bullet P(r,r) = P(n,r)$$

  This equation can be used to write the following formulas:

$$C(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$C(n,r) = \frac{n!}{(n-r)!\, r!}$$

Selected References

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