Classical Crystals

What do diamonds, quartz, and salt all have in common? In this module, you'll explore the properties of some familiar crystals.

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Introduction

Common table salt, also known as halite, is a mineral composed of sodium and chlorine atoms. Molecules of table salt form crystals that have a cubic shape. (You may want to use a magnifying glass or microscope to get a better look at them.)

Chemists sometimes describe a crystal as a body consisting of a single type of mineral. In a crystal, the forces that bind atoms together cause them to combine in a definite shape. This shape is determined by the number and types of atoms in the mineral. Many minerals other than table salt also form crystals. Some are too small to see with an ordinary microscope, others can be several meters in diameter.

Large crystals are formed by many, many repetitions of small "building blocks" called unit cells. A **unit cell** is the smallest piece of a crystal that possesses the same properties as a larger crystal of the same mineral.

For example, a molecule of table salt consists of one sodium atom and one chlorine atom bonded together. The unit cell of a salt crystal is composed of four molecules of salt—in other words, four sodium atoms and four chlorine atoms.

Figure **1** compares a salt molecule to the unit cell of a salt crystal and to a larger crystal of salt. The larger crystal shows how unit cells join together. One interesting property of the larger crystal is that its shape and symmetry are the same as that of the unit cell.

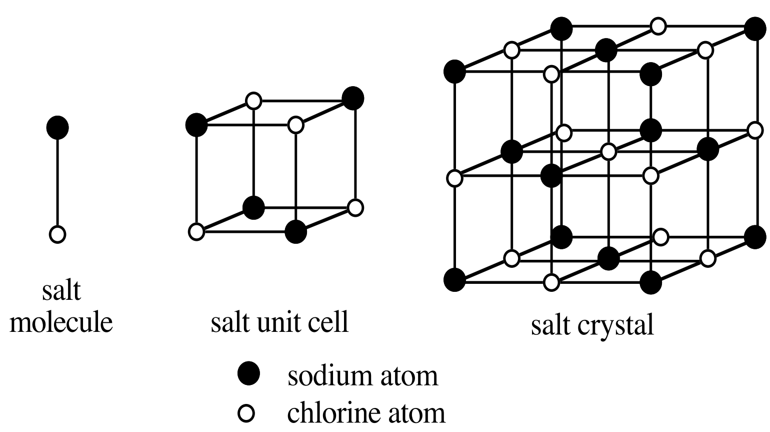


Figure 1: Structure of a salt crystal

The symmetrical properties of crystals have fascinated mathematicians and scientists for centuries. Although crystals may have many different shapes, the more symmetrical a crystal, the more interest it attracts. In fact, the symmetry of certain crystals—like diamonds or emeralds—can even make them more valuable.

Activity 1

A cube is one of the **regular polyhedra**, first described in the sixth century B.C. by a society of Greek mathematicians known as the Pythagoreans. In this activity, you investigate some of the characteristics of regular polyhedra.

Mathematics Note

A **polyhedron** is a simple three-dimensional closed surface made up of faces that are polygons. The plural of polyhedron is **polyhedra** or polyhedrons.

A **regular polyhedron** is a three-dimensional convex solid in which all the faces are congruent regular polygons and the same number of faces meet at each vertex.

The cube in Figure **2** is one example of a regular polyhedron. All of its faces are congruent regular polygons (each a 3-cm square) and the same number of faces (3) meet at each vertex.

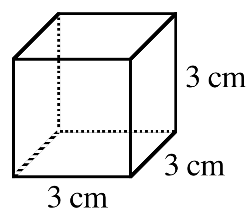


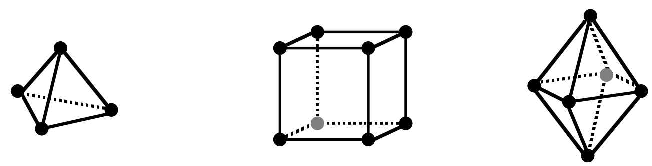
Figure 2: A 3-cm cube

The regular polyhedra were named the **Platonic solids** after the Greek scholar Plato, who lived around 400 B.C.

About 300 B.C., Euclid—another Greek mathematician—discussed the Platonic solids in his book, the *Elements*. Many people attributed mystical powers to these shapes. The Pythagoreans, for example, thought they represented the elements of the universe—fire, earth, air, and water—as well as the universe itself. They were not entirely wrong. Platonic solids do occur in nature. For example, crystals of both table salt and pyrite (an iron-containing mineral) have the outward appearance of a cube. Crystalline gold can take the shape of an octahedron. Diamond crystals occur in many different shapes, including that of tetrahedrons, octahedrons, and dodecahedrons.

Exploration

Figure **3** shows three different Platonic solids. The name of each solid appears below it, along with the names of some minerals that have crystals with that shape. The distances between adjacent vertices are equal.



tetrahedron	**hexahedron**	**octahedron**
tetrahedrite tennantite chalcopyrite	pyrite halite galena flourite	diamond magnetite spinel

Figure 3: Three Platonic solids

a. Build models of the three different Platonic solids shown in Figure **3**. Use toothpicks of equal lengths for the edges, and miniature marshmallows (or some other soft material) for the vertices.

b. Record the characteristics of each solid.

Discussion

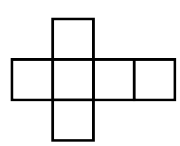
a. What do you think the prefixes *tetra-*, *hexa-*, and *octa-* mean? Explain your responses.

b. Which polygon makes up the faces of the tetrahedron? of the hexahedron? of the octahedron?

c. Describe some of the similarities and differences you observe among the three polyhedra in the exploration.

Assignment

1.1 A net is a two-dimensional pattern that can be folded into a three-dimensional solid. The net for a cube (or Platonic hexahedron) consists of six squares. Many different combinations of six squares will form a cube. Sketch a net for a cube different than the one shown below**.**



1.2 Sketch nets for the octahedron and tetrahedron you built in the exploration.

1.3 **a.** On a sheet of stiff paper, draw a net for a cube with an edge length of 6 cm.

b. Use your net to build the cube. **Note:** Save your cube for use later in the module.

c. Describe the characteristics of a cube that make it a Platonic solid.

1.4 **a.** On a sheet of stiff paper, draw a net for a regular tetrahedron with an edge length of 6 cm.

b. Use your net to build the tetrahedron.

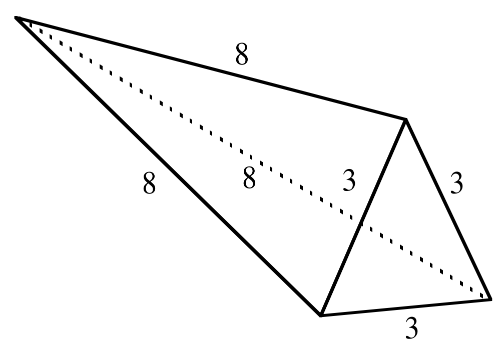
c. Describe the characteristics of a regular tetrahedron that make it a Platonic solid.

1.5 Place your tetrahedron from Problem **1.4** on a mirror.

a. Describe the solid formed by the tetrahedron and its image.

b. Is this solid a Platonic solid? Justify your response. **Note:** Save your tetrahedron for use later in the module.

1.6 The sketch of a tetrahedron below displays the measurements of its edges. Is this tetrahedron a Platonic solid? Explain your response.



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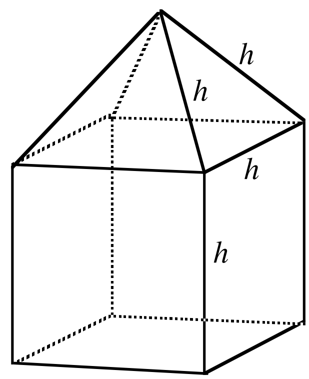
1.7 Using toothpicks and marshmallows (or some other soft material), build a model of a pyramid with a square base. All the toothpicks should be the same length. **Note:** Save your pyramid for use later in the module.

a. Is your pyramid a Platonic solid? Give at least two arguments supporting your answer.

b. Place the square base of the pyramid on a mirror. Does the figure formed by the pyramid and its image represent a Platonic solid? Explain your response.

1.8 Consider the polyhedron that results when a pyramid is constructed on one face of a cube, as shown in the diagram below. The faces of the pyramid are all equilateral triangles with sides of length h. If a pyramid is constructed on each of the other faces of the cube, will the resulting polyhedron be a Platonic solid? Justify your response.



1.9 **a.** Determine the surface area of a cube with an edge length of 10 cm.

b. Describe an efficient way to determine the surface area of any of the Platonic solids.

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Activity 2

The three Platonic solids you built in the previous exploration—the tetrahedron, the hexahedron, and the octahedron—all occur naturally in certain mineral crystals. Do you think that these are the only Platonic solids that occur in nature?

Exploration

In this exploration, you explore the possible existence of other Platonic solids.

a. Cut six congruent equilateral triangles out of stiff paper.

b. Select one triangle. Label an angle X, as shown in Figure **4** below, and record its measure.

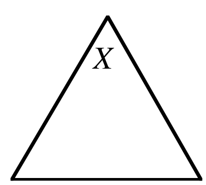

Figure 4: An equilateral triangle

c. Tape two triangles together along one edge, as shown in Figure **5**. Record the sum of the measures of angle *X* and the angle adjacent to it.

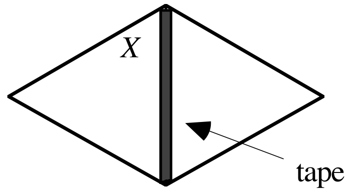

Figure 5: Two triangles taped together

d. Fold the two triangles along the tape. Could these triangles form a vertex of a three-dimensional solid?

e. Tape a third triangle to the others so that all three share a single vertex, as shown in Figure **6**. Record the sum of the measures of the angles at the shared vertex.

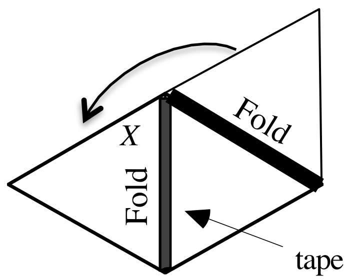

Figure 6: Three triangles taped together

f. Fold the triangles along the tapes. Could these three triangles form a vertex of a three-dimensional solid?

g. Continue taping additional triangles to the others (up to six) so that all share a single vertex. Record the number of triangles and the sum of the measures of the angles at the shared vertex, and note whether or not the combination will form a vertex of a solid.

h. Repeat the procedure described in Parts **a–g** using four congruent squares, four congruent regular pentagons, and four congruent regular hexagons.

Discussion

a. What conditions appear to be necessary to form a vertex of a regular polyhedron?

b. Recall that a tessellation is a repeated pattern that covers an entire plane without gaps or overlaps. Figure **7** shows a portion of a tessellation that uses isosceles triangles to cover the plane.

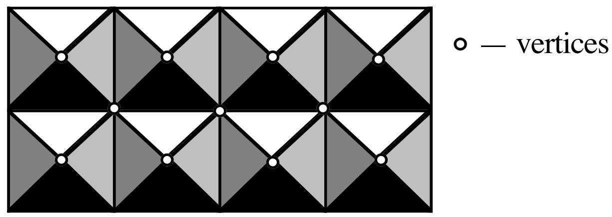

Figure 7: Tessellation using isosceles triangles

1. What is the sum of the measures of the angles at each vertex in this tessellation?
2. Do you think it is possible to create a polyhedron in which the sum of the measures of the angles at each vertex is 360˚? Explain your response.

c. Describe the numbers and types of the regular polygons in the exploration that can be used to form a vertex of a solid.

d. Do you think that any other regular polygons could be used as the faces of a Platonic solid? Explain your response.

e. The problem you investigated in the exploration also interested the Greek scholar Theaetetus (419–369 B.C.). Theaetetus is given credit for determining the exact number of Platonic solids. How many do you think there are? Explain your response.

Assignment

2.1 A regular **dodecahedron** has 12 regular pentagonal faces. Draw a regular pentagon. Using your pentagon as a template, cut 12 congruent pentagons out of stiff paper. Use these pentagons to build a regular dodecahedron.

2.2 Sketch a net that will produce a regular dodecahedron.

2.3 A regular **icosahedron** has 20 regular triangular faces. Fold the net provided by your teacher to produce an icosahedron.

a. Describe the characteristics of an icosahedron.

b. Determine the surface area of your model.

2.4 Crystals of boron take the shape of an icosahedron. The length of one edge of the unit crystal for boron is $8.5 \bullet 10^{-9}$ cm. What is the surface area of a unit crystal of boron?

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2.5 Consider a jewel cut in the shape of a dodecahedron with an edge length of 0.5 cm. What is the surface area of this jewel?

2.6 A typical die is a cube. Each face of the cube represents one of the integers from 1 to 6. Recall that on a fair die, each face has the same chance of turning up. Identify the Platonic solid that should be used to make a fair die in which each face represents one of the integers from 1 to 10. Justify your selection.

2.7 In the exploration in Activity **2**, you discovered that the sum of the measures of the angles at a vertex of a Platonic solid must be less than 360˚. Examine several non-Platonic convex solids, such as containers and packaging boxes. Make a conjecture about the sum of the measures of the angles at a vertex of any convex solid.

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Activity 3

Another distinguishing characteristic of the Platonic solids is their symmetry. In this activity, you investigate the symmetry of some Platonic solids by first examining the symmetries of regular polygons.

Mathematics Note

A **line of symmetry** divides a two-dimensional figure into two congruent parts, each a mirror image of the other.

For example, Figure **8** shows two polygons and their lines of symmetry.

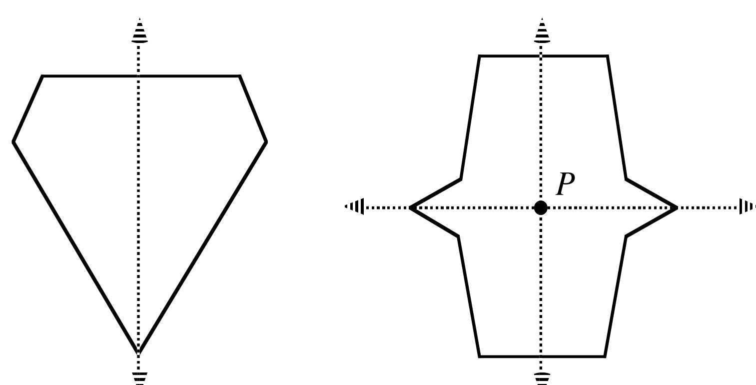


Figure 8: Lines of symmetry for two polygons

Exploration 1

Some regular polygons have several lines of symmetry. The relationships among these lines and the sides of the polygon can be used to determine basic shapes that make up crystals.

a. By placing a mirror perpendicular to the plane containing a polygon and on the polygon's line of symmetry, you should be able to see that each side of the polygon is a reflection of the other side.

Use a mirror to find the lines of symmetry of a regular triangle. Sketch these lines on the triangle.

b. Record any relationships you observe among the lines of symmetry and the sides and angles of the triangle.

c. Two mirrors can be placed on the lines of symmetry in Figure **9** so that the shaded region combines with its reflections to form the entire polygon. The smallest region that can be reflected to form the entire polygon is a **reflecting polygon**.

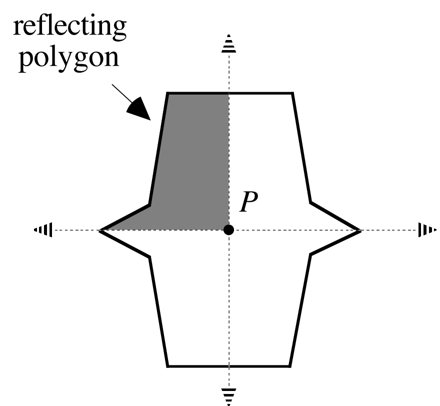


Figure 9: A reflecting polygon

Using the triangle from Part **a**, place two mirrors along two different lines of symmetry to find the reflecting polygon.

d. Measure the angle formed by the mirrors.

e. Sketch the reflecting polygon of the triangle.

f. Repeat Parts **a–e** using a regular quadrilateral, a regular pentagon, and a regular hexagon.

Discussion 1

a. Compare the reflecting polygon for a regular triangle to those for a regular quadrilateral, pentagon, and hexagon.

b. For each regular polygon, how many images of a reflecting polygon are needed to recreate the original?

c. How is the measure of the angle formed by the two mirrors in the exploration related to the number of sides in the regular polygon?

d. Recall that the measure of an interior angle of a regular polygon can be found using the formula below, where m is the measure of each interior angle and n is the number of sides in the polygon:

$$m = \frac{180(n-2)}{n}$$

How does the measure of an interior angle of a regular polygon compare with the measure of the angle between the two mirrors used to find a reflecting polygon?

Exploration 2

In Exploration **1**, you examined lines of symmetry for regular polygons. In this exploration, you investigate **planes of symmetry** for Platonic solids.

Mathematics Note

A **plane of symmetry** divides a three-dimensional object into two congruent three-dimensional objects, each a mirror image of the other. For example, Figure **10** shows a rectangular prism and one of its planes of symmetry.

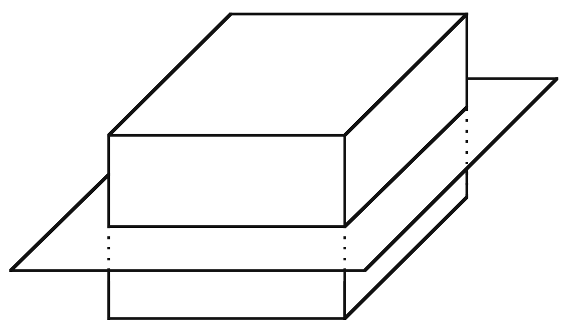


Figure 10: A rectangular solid and a plane of symmetry

Just as lines of symmetry divide polygons into congruent parts, planes of symmetry divide polyhedrons into congruent parts. Crystals shaped like Platonic solids have several planes of symmetry. These planes may be located using mirrors.

To find a plane of symmetry, place a single mirror on one face of the polyhedron. Hold the mirror perpendicular to the face. If the mirror is located in the plane of symmetry, the visible part of the polyhedron and its reflection will form a polyhedron congruent to the original. As shown in Figure **11**, it may be necessary to view the reflection from above the mirror to see the polyhedron formed.

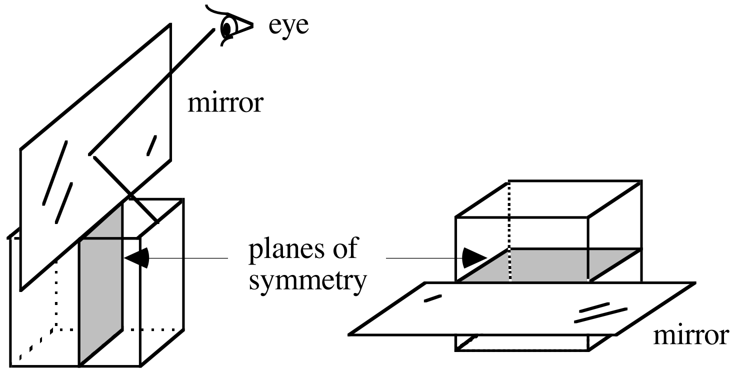


Figure 11: Using mirrors to find planes of symmetry

a. Obtain the cube you made in Problem **1.3**. Use a single mirror to find as many planes of symmetry as you can for the cube.

b. Record the position of each plane of symmetry.

c. Repeat Parts **a** and **b** using the regular tetrahedron you built in Problem **1.4**.

Discussion 2

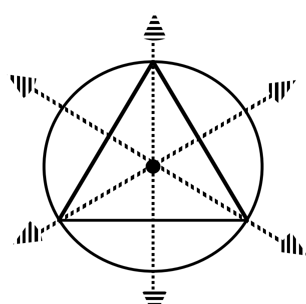
a. How many different planes of symmetry are there for a cube? Where is each one located?

b. How many different planes of symmetry are there for a regular tetrahedron? Where is each one located?

c. How many planes of symmetry do you think there are for a regular octahedron? Where would these planes of symmetry be located?

Assignment

3.1 The point at which a regular polygon's lines of symmetry intersect is the center of the circle that circumscribes the polygon. For example, the diagram below shows a regular triangle, its lines of symmetry, and the circle that circumscribes it.



Describe how you could use a circle to construct a regular polygon with *n* sides, as well as all of the polygon's lines of symmetry.

3.2 **a.** Make a table with headings like those in the table below. Complete the table for three different regular polygons.

No. of Sides	**Measure of an Interior Angle (*I*)**	**Measure of Angle Formed by Two Adjacent Lines of Symmetry (*S*)**	**Ratio $I:S$**

b. Using the patterns you observe in the table, describe the relationship between the measure of an interior angle of a regular polygon and the measure of an angle formed by the intersection of two adjacent lines of symmetry.

c. Complete the table for a regular polygon with n sides.

3.3 **a.** Describe how to use two mirrors to locate a reflecting polygon for any regular polygon.

b. Test your method using a regular polygon with more than six sides.

3.4 Determine the number of different planes of symmetry for each of the Platonic solids you have constructed in this module. **Note:** Make sure to count each plane only once.

3.5 Do you observe any relationship between the symmetry of the face of a Platonic solid and the symmetry of the solid itself? Explain your response.

3.6 Describe a relationship among the sides, vertices, and lines of symmetry for regular polygons with:

a. an odd number of sides

b. an even number of sides.

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3.7 Consider the relationships you described for lines of symmetry of regular polygons in Exploration **1** of Activity **3**. How could you use these relationships to help locate lines of symmetry using a ruler rather than a mirror?

3.8 Consider a pyramid with a square base, such as the one you constructed in Problem **1.7**. Describe the planes of symmetry for such a pyramid.

3.9 **a.** Describe the locations of the planes of symmetry for a sphere.

b. How many planes of symmetry are there for a sphere? Explain your response.

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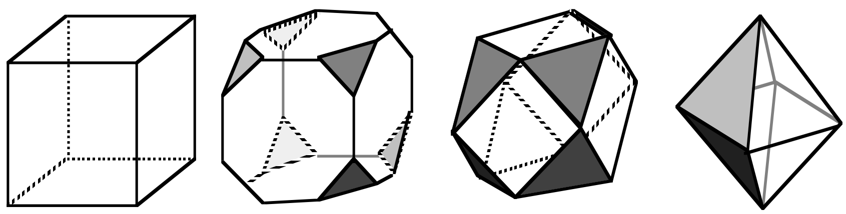
Activity 4

The mineral galena is composed of lead and sulfur. Also known as lead ore, it has been our most important source of lead for centuries. In the past, lead was used extensively in paints, gasoline, metal alloys, batteries, and glass. Because of its toxic effects on humans and the environment, however, the use of lead is now much less widespread.

As shown in Figure **12**, crystals of galena occur in several different shapes: cubes, truncated cubes, cuboctahedrons, and octahedrons.



cube truncated cube cuboctahedron octahedron

Figure 12: Different shapes of galena crystals

A **truncated cube** can be thought of as a cube with its "corners" cut off. A **cuboctahedron** is a special kind of truncated cube. Its corners have been "cut" so that the remaining faces are squares and equilateral triangles. The length of the sides of the squares equals the length of the sides of the triangles.

A cuboctahedron also can be visualized as an octahedron with each of its vertices cut off in the same way. Each square face is joined to four triangular faces, and each triangular face is joined to three square faces. Both the truncated cube and the cuboctahedron are classified as **Archimedean solids**, after the Greek mathematician Archimedes (287–212 B.C.).

Mathematics Note

Archimedean solids, or **semiregular polyhedra**, are solids whose faces consist of two or three different types of congruent regular polygons. Each vertex is formed by the intersection of the same numbers and types of these polygonal faces.

For example, the faces of a truncated cube are equilateral triangles and regular octagons. Two octagons and one triangle meet at each vertex.

Exploration

a. Determine the number of squares and equilateral triangles that make up the faces of a cuboctahedron in Figure **12**.

b. Build a model of a cuboctahedron.

c. Draw a net for a cuboctahedron.

Discussion

a. How many squares and how many equilateral triangles are required to create the faces of a cuboctahedron?

b. How do you think the name *cuboctahedron* was derived?

c. How many faces intersect to form a vertex of a cuboctahedron? Is this number the same at every vertex?

d. What is the sum of the measures of the interior angles at each vertex?

Assignment

4.1 How do the planes of symmetry of a cuboctahedron compare with the planes of symmetry of a cube and an octahedron?

4.2

a. Create a table that shows the number of vertices, faces, and edges for each Platonic solid.

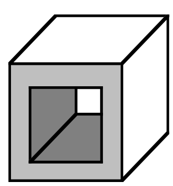
b. Using the table from Part **a**, find the sum of the number of faces and the number of vertices for each polyhedron.

c. How does the sum of the numbers of faces and vertices compare to the number of edges?

d. Write a formula that describes the relationship among the number of faces (F), vertices (V), and edges (E) of the polyhedron. This relationship is known as **Euler's formula**.

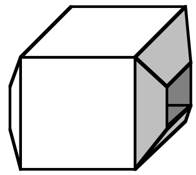
4.3 The relationship you discovered in Problem **4.2**, Euler's formula, is true for any polyhedron. In Parts **a** and **b** below, you determine if this relationship also applies to solids with holes in them.

a. Consider the solid shown in the diagram below, a cube with a square hole through its middle.



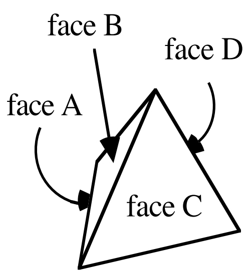
1. Does Euler's formula also apply to this solid? Why do you think this is so?
2. Is this solid a polyhedron? Explain your response.

b. Like the solid in Part **a**, the following solid also has a hole through it.

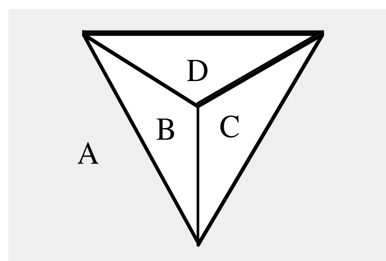


1. Explain why Euler's formula is not true for this solid.
2. Does this prove that Euler's formula does not apply to all solids with holes? Justify your response.

4.4 In a **planar map** of a polyhedron, there is a one-to-one correspondence between the faces of the polyhedron and the regions of the planar map. In addition to this correspondence, the regions which represent faces that share an edge also share a boundary. For example, the diagram below shows a planar map of a tetrahedron.



tetrahedron



planar map of tetrahedron

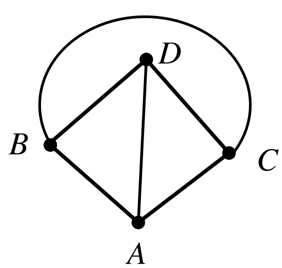
a. Does Euler's formula apply to the planar map of a tetrahedron? Explain your response.

b. Create a planar map of a cube.

c. Does Euler's formula also apply to the planar map of a cube?

4.5 In a **graph** of a planar map, there is a one-to-one correspondence between the regions of the map and the vertices of the graph. In addition to this correspondence, the vertices which represent regions that share a boundary are joined by an edge.

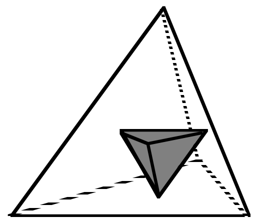
If it is possible to draw such a graph so that the edges intersect only at vertices, then the graph is a **planar graph**. For example, the diagram below shows a planar graph of the map of a tetrahedron from Problem **4.4**.



Draw a planar graph for the map of a cube from Problem **4.4b**.

4.6 A **dual** of a polyhedron is the solid formed by the segments joining the center of each face of the polyhedron to the center of each adjacent face. A dual is therefore **inscribed** in the original solid.

The diagram below shows a tetrahedron with its inscribed dual. The dual of a tetrahedron is also a tetrahedron.



a. What is the dual of a regular octahedron?

b. A cuboctahedron also has a dual. How many vertices, faces, and edges does this dual have?

4.7 As noted in the introduction to this activity, a cuboctahedron can also be thought of as an octahedron with its vertices cut off. Two squares and two equilateral triangles meet at each vertex of the cuboctahedron. The lengths of the sides of both polygons are equal. What is the height of the corners cut from the regular octahedron to form the regular cuboctahedron? Explain your response.

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4.8 The surface of a soccer ball appears to be made up of regular pentagons and regular hexagons. Each pentagon is surrounded by five hexagons. If each face were actually a polygon, would the pattern used on a soccer ball result in an Archimedean solid? Explain your response.

4.9 A cuboctahedron has 14 faces. Could this solid be used to make a fair die in which each face represents an integer from 1 to 14? Explain your response.

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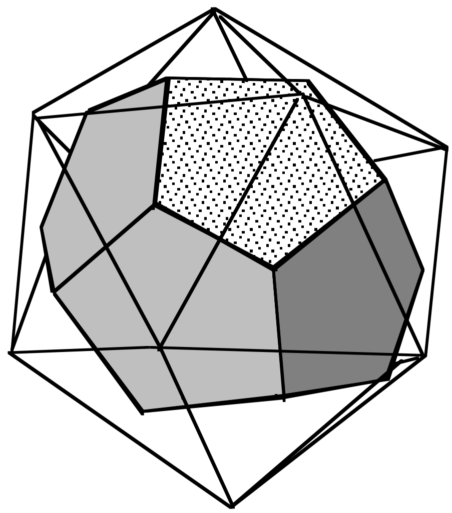
Research Project

Write a report that describes each Archimedean solid and explains its relationship to one of the Platonic solids. When appropriate, include models or drawings of these solids as part of your report.

Summary Assessment

It is possible to position a Platonic solid inside another Platonic solid. These special pairs of "solids within solids" are referred to as **nested solids**. Every Platonic solid can be "nested" in every other Platonic solid—including itself. The figure below shows a dodecahedron nested in an icosahedron.



In a pair of nested solids, the vertices of the inner solid must be points of the outer solid. In some—such as the pair shown in the diagram above—the vertices of the inner polyhedron are the centers of the faces of the outer solid. The inner solid may even share the same vertices as the outer solid.

Build a model or make a sketch of a pair of nested polyhedra (other than a solid within itself or a solid and its dual). Write a description of the nested solids, explaining where the vertices of the inner solid lie. Compare the lengths of the edges of both solids and explain how you determined these lengths.

Module Summary

- A **unit cell** is the smallest piece of a crystal that possesses the same properties as a larger crystal of the same mineral.
- A **line of symmetry** divides a two-dimensional figure into two congruent parts, each a mirror image of the other.
- A **polyhedron** is a simple three-dimensional closed surface made up of faces that are polygons. The plural of polyhedron is **polyhedra** or polyhedrons.
- A **regular polyhedron** is a three-dimensional convex solid in which all the faces are congruent regular polygons and the same number of faces meet at each vertex. These solids are referred to as the **Platonic solids**.
- A regular **tetrahedron** has four faces and four vertices. Three regular triangular faces intersect at each vertex.
- A **cube** or regular **hexahedron** has six faces and eight vertices. Three square faces intersect at each vertex.
- A regular **octahedron** has eight faces and six vertices. Four regular triangular faces intersect at each vertex.
- A regular **icosahedron** has 20 faces and 12 vertices. Five regular triangular faces intersect at each vertex.
- A regular **dodecahedron** has 12 faces and 20 vertices. Three regular pentagonal faces intersect at each vertex.
- A **plane of symmetry** divides a three-dimensional object into two congruent three-dimensional objects, each a mirror image of the other.
- A **dual** of a polyhedron is the solid formed by the segments joining the center of each face of the polyhedron to the center of each adjacent face.
- Each vertex of an **inscribed** polyhedron is a point of the outer solid.
- **Archimedean solids**, or **semiregular polyhedra**, are solids whose faces consist of two or three different types of congruent regular polygons. Each vertex is formed by the intersection of the same numbers and types of these polygonal faces.

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