

Fair Is Fair



How would you divide a cake between two hungry people? How would you divide a car among three teenage drivers? And what would you do if everyone in the family wanted the old grandfather clock? In this module, you'll learn that a fair division doesn't always have to be equal.

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Introduction

Have you ever argued with someone over who should get the last piece of pie? Or bickered over the right to read the only copy of an exciting book? Whenever two or more parties want the same item, the potential for conflict arises.

Dividing inheritances often causes ill will among previously agreeable family members. The distribution of water rights among resource users can be subject to court battles. The problem of dividing land among nations has resulted in bloody wars. In this module, you investigate some methods of obtaining fair—and peaceful—divisions.

Discussion

- a. In most situations, there is more than one way to divide items among interested individuals. Some ways are fair while others are not. Describe the characteristics of a division you would consider fair.
- b. Give examples of both fair and unfair divisions in each of the following situations. Discuss why each division is fair or unfair.
 1. \$25 among 3 people
 2. 12 equal slices of pizza among 5 people
 3. a sports car valued at \$22,000 and a diamond ring appraised at \$7,000 between 2 people
 4. \$100,000 cash and a motorcycle valued at \$12,000 among 3 people
 5. a 2-L bottle of mineral water between a large adult and a small child marooned on a desert island for a week
- c. In some situations, an item that several people want must be awarded as a single unit. What methods might be used to facilitate a division that is acceptable to all?

Activity 1

In practice, not all divisions are fair. Some people may use their authority to make divisions without concern for fairness. For example, a military officer assigning duties or a construction manager delegating tasks may consider efficiency or safety first, and fairness later. In a bankruptcy court, on the other hand, even a judge who tries to distribute assets fairly among all creditors may not seem completely fair to those involved.

Mathematics Note

A fair division problem exists when individuals must divide a set of items among themselves. A **fair division** occurs when all individuals, by their own assessment, consider the portions they are awarded as fair. “Fairness” depends on each individual’s opinion. This opinion may not agree with the opinion of others involved.

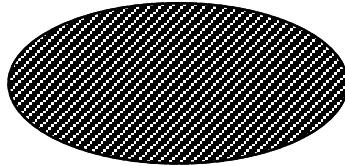
An item is considered **continuous** if it may be awarded in parts in a fair division. For example, a cake is continuous while the plate it is served on is not.

Exploration

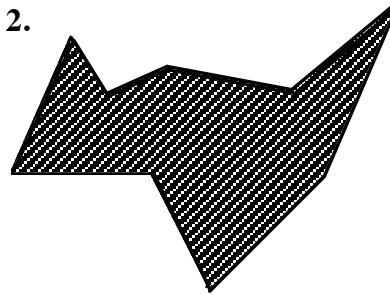
- a. One technique for dividing a continuous object between two persons is the **cut-and-choose method**. In this method, one person cuts the object into two shares. The other person then chooses the desired piece.

Create figures similar to those illustrated below. Use the cut-and-choose method to achieve a fair division in each case.

1.

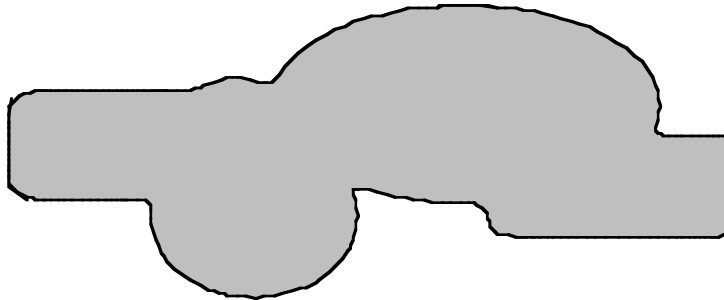


2.



- b. A second technique for dividing a continuous object is the **continuous-knife method**. In this technique, a knife is held above the left edge of the object and moved slowly from left to right. Either person can stop the knife to cut off a piece that represents a fair share. The person who stops the knife receives the portion to the left. The second person receives the remaining portion. (The continuous-knife method may also be performed by moving the knife from top to bottom instead of left to right.)

Create a figure similar to the one illustrated below and use the continuous-knife method to achieve a fair division.

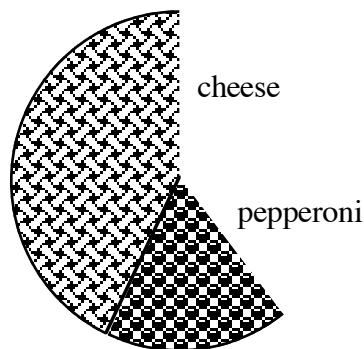


Discussion

- a. Did the cut-and-choose method result in a fair division? Why or why not?
- b. Describe any difficulties that arose when using the cut-and-choose method.
- c. Why would you expect the continuous-knife method to result in a fair division?
- d. What difficulties arose when using the continuous-knife method?
- e. What kinds of objects can be divided by the methods described in the exploration?
- f. What kinds of items should not be divided using either of these methods? Explain your response.

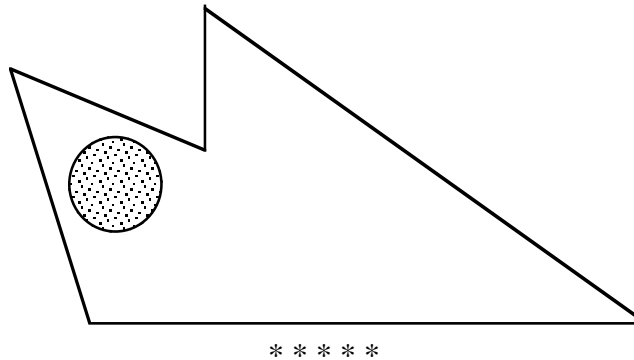
Assignment

- 1.1 John and Leticia have some pizza to share. As shown in the diagram below, part of the pizza is cheese and part is pepperoni.

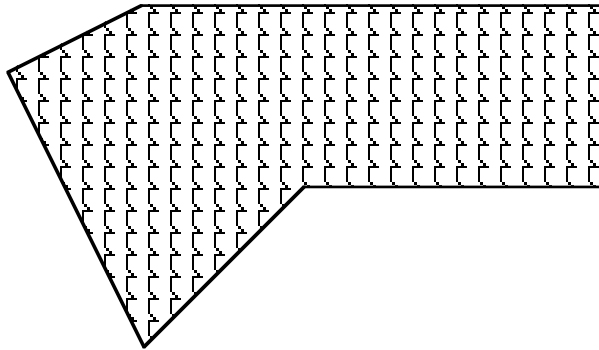


- a. Describe how the two friends could divide the pizza fairly and explain how this method results in a fair division.
 - b. How might the friends change their method if John does not like pepperoni and Leticia does?
- 1.2 In this problem, you use a geometry utility to model the continuous-knife problem for two people.
- a. Create an irregular polygon. Draw a line segment to act as the “knife,” then move the segment across the polygon until it appears to be fairly divided into two portions.
 - b. Determine the areas of the resulting portions of the polygon.
 - c. Would this method result in a fair division? Explain your response.
- 1.3 Xang and Katelyn are willing to share the last submarine sandwich. Describe at least two different methods, other than cut-and-choose or continuous-knife, that might be used to accomplish a fair division. Rank your methods in order of preference and justify your decisions.

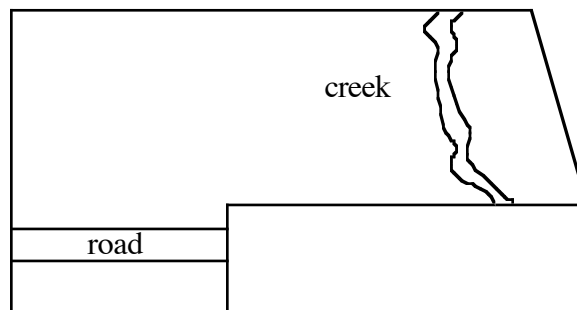
- 1.4** The figure below shows a plot of land with a large circular swimming pool. Describe how two people might divide this plot of land into two fair portions.



- 1.5** To complete a home economics project, Willis and Karissa each need enough cloth to sew a pair of shorts. The diagram below shows the piece of material they must divide between them.

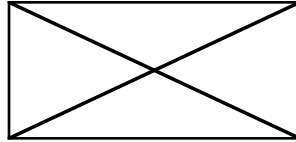


- a. Describe how this piece of material could be divided fairly.
 - b. After the cloth was divided, Willis was able to finish his project, but Karissa ran into trouble. What problems might Karissa have encountered, even though she originally considered the division a fair one?
- 1.6** Mr. and Mrs. Summers would like to give the property shown in the map below to their two grown children.

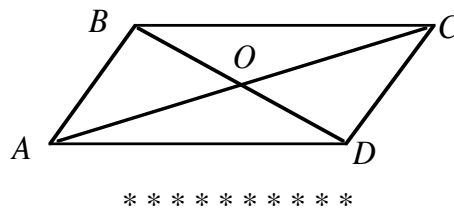


Both children would like to build homes on this land. Describe how the property could be divided fairly.

- 1.7 A figure has **point symmetry** if it can be rotated 180° about a point and each point in the image coincides with a point in the preimage. For example, the rectangle below has point symmetry about the point where its two diagonals intersect.



In the parallelogram below, point O is the point where the diagonals intersect. Explain why any line through point O divides the parallelogram into two regions with equal areas.



Activity 2

Not all fair division problems involve only two parties. When three or more individuals are involved, the situation can become much more complicated.

In order to obtain a fair division among several people, some additional assumptions must be made. First, all persons involved must be capable of determining a fair portion. The value that each individual places on a particular portion may depend on more than just its size. Each person should get a fair share based on his or her own assessment.

In the division of land, for example, one person may consider lake frontage more valuable than forests. In such a case, that person may accept a smaller piece of land with lake frontage as a fair share.

Exploration

One way to divide a continuous object among three or more people is the **reduction method**. Parts **a–f** below describe the steps that must be followed to divide an item among three people using the reduction method. Use these steps to simulate the division of a candy bar.

- a. The individuals determine the order in which each will participate in the division.
- b. The first person “reduces” the item by cutting off a fair share.

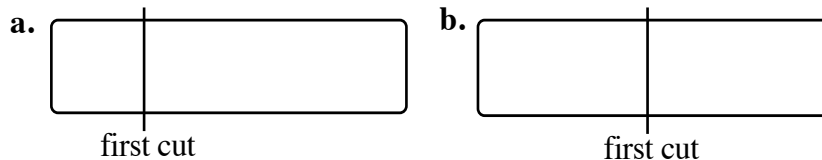
- c. The second person has a choice: either reduce the portion cut by the previous person or leave it intact. If the second person leaves the portion intact, then the third person has the same choice.
- d. The last person to reduce the piece receives that portion and is now out of the process.
- e. The remaining two people divide the rest of the item by repeating the reduction method or by using any other method for fairly dividing an object between two people. Any pieces removed through reduction, but not yet awarded, are included in this division.

Discussion

- a. In the reduction method, when would a person choose to reduce a share?
- b. Who receives the first piece if no one reduces it?
- c. Do you think that the reduction method will result in a fair division of a candy bar? Explain your response.
- d. Would this method result in a fair division for two people? Why or why not?
- e. In any division of a continuous item, the fractional value assigned to each portion may vary from person to person. The sum of these fractional values, however, is always 1. Why is this true?
- f. Could the cut-and-choose method, the continuous-knife method, or the reduction method be used when more than three people are involved in the division of a continuous object? Explain your response.

Assignment

- 2.1 Imagine that you and two friends have decided to share a candy bar using the reduction method. One of your friends makes each of the first cuts shown in Parts **a** and **b** below. In each case, would you reduce the portion on the left or let this cut stand? Explain your responses.



- 2.2
- a. Describe a method other than those previously mentioned in this module of dividing a continuous object fairly among three people.
 - b. Will your method work when five people are involved? If so, explain how. If not, modify your method so that it will work.
- 2.3 Create an irregularly shaped object out of paper. Describe a fair division of this object among three or more people.

- 2.4** Chadd, Rusty, and Kristi own a lawn-mowing service. They have been hired to mow a park that contains about 2 acres of grass. The park includes flat areas, hilly regions, and groves of trees—all of which must be mowed.
- Why would the reduction method fail to provide a fair division in this case? Explain your response.
 - Describe how the park can be divided so that each of the three must mow a fair portion of the grass.
 - Both Chadd and Kristi use push mowers, while Rusty uses a riding mower. How does this fact affect your response to Part **b**?

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- 2.5** Use a geometry utility to model the continuous-knife method of dividing an irregular polygon among four or more people. Describe how the method was altered to accommodate additional people.
- 2.6** Mr. and Mrs. Estrada have purchased a new compact disc (CD) player for their four children: Gisela, Milo, Hank, and Rozella. Since each of the four likes a different type of music, they must find a way to divide the time with the CD player fairly.
- Explain why it might not be fair simply to assign each child 6 hr a day with the CD player.
 - Devise a method that fairly divides access to the CD player during one month.
 - If Rozella and Hank suddenly start listening to the same type of music, should that change the method you devised in Part **b**? Explain your response.

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Activity 3

In the previous activities, you examined the fair division of continuous objects. Some objects, however, are difficult or impractical to divide into parts. For example, how would you divide a car or a boat?

In this activity, you investigate the fair division of **discrete** items. An item is considered discrete if it can only be awarded as an entire unit in a fair division. For example, houses, cars, and boats are all discrete items.

Exploration

- Along with a partner, choose a discrete item that you would both like to own but must divide fairly.
- Develop a process that could be used to decide who will get the item.

- c. Determine how the student who does not get the item should be compensated to make the division fair.
- d. Determine the value of this settlement for each person.

Discussion

- a. How was the person who did not receive the item compensated?
- b. Did your method result in a fair division? Explain why or why not.
- c. Is it necessary for the values of the item and the compensation to be equal in order to have a fair division?
- d. Could your method be adjusted to accommodate more people and more items?

Mathematics Note

Mathematicians have developed some approaches to fair division that include steps you may not have considered. The **bid-and-divide** method, for instance, involves assigning a cash value to an item through a bidding process. The secret bids represent the amounts that individuals would be willing to pay for the item and determine the value to be divided fairly. The item is awarded to the highest bidder. The individual who receives the item then compensates the others for a fair share.

For example, imagine that you and a friend share a calculator. Your friend is moving to another town. Since the calculator is a discrete item, the two of you must determine who will keep it. Using the bid-and-divide method, you each submit a secret bid that reflects the value you place on the calculator. Suppose that you bid \$60 and your friend bids \$70. Since \$70 is the higher bid, the value to be divided fairly is \$70. This is referred to as the **value pool**.

Because there are two of you, your share of the bids is half the value that you placed on the calculator, while your friend's share is half the value that he or she placed on it. Since you bid \$60, your share is $0.5(\$60)$ or \$30. Your friend's share is $0.5(\$70)$ or \$35. The sum of these shares, \$30 and \$35, accounts for \$65 of the \$70 in the value pool. The remainder of the value pool, \$5, is the **value-pool balance**. Each person is awarded half this balance, or \$2.50.

Since your friend's bid is higher than yours, your friend is awarded the calculator. Once the item is awarded, each person's compensation must be calculated. A chart similar to the one illustrated in Figure 1 may help you keep track of the calculations in the bid-and-divide method.

As shown in Figure 1, your **total fair share** is the sum of a share of your bid and an equal share of the value-pool balance. In this example, your total fair share is \$32.50, while your friend's is \$37.50.

Compensation for each individual is determined by calculating the difference between the total fair share and the value of the item received. Your compensation, \$32.50, equals the difference between your total fair share and the value of the item awarded—in this case, \$0. Your friend's compensation is –\$32.50, the difference between his or her total fair share and the value of the calculator.

	You	Friend	Value Pool	
Bids	60.00	70.00	Total of High Bid(s)	70.00
			–Total of Shares of Bid(s)	65.00
Sum of Bid(s)	60.00	70.00	Value-pool Balance	5.00
Share of Bid(s)	30.00	35.00		
Share of Value-pool Balance	2.50	2.50		
Total Fair Share	32.50	37.50		
(Item Awarded)	(none)	(calculator)		
Total Value Awarded	0.00	70.00		
Compensation	+32.50	–32.50		
Final Settlement Value	32.50	37.50		

Figure 1: Fair division using bid-and-divide method

In a fair division of the calculator by this method, your friend keeps the calculator and compensates you \$32.50 for your fair share. The **final settlement value** is the sum of each person's compensation and the value of the item awarded, if any. When a fair division has been properly administered, each person's final settlement value equals that person's total fair share. The sum of the final settlement values equals the total of bids in the value pool. The sum of the compensations equals 0.

- e. In the bid-and-divide method, why do you think the high bid is used to determine the value pool?
- f. In the final settlement, why does each person always receive more than a fair share of that person's original bid?
- g. Why must the sum of the final settlement values equal the value pool?
- h. Why must the sum of the compensations equal 0?

Assignment

- 3.1** Yoshi and Shiho have inherited an ancient Japanese sword. They decide to use the bid-and-divide method to determine who should keep the sword. Yoshi bids \$900. Shiho bids \$1050.

Complete the table below to determine a fair division in this situation.

	Yoshi	Shiho	Value Pool	
Bids	900	1050	Total of High Bid(s)	
			– Total of Shares of Bid(s)	
Sum of Bid(s)			Value-pool balance	
Share of Bid(s)				
+ Equal Share of Value-pool balance				
Total Fair Share				
(Item Awarded)				
Total Value Awarded				
Compensation				
Final Settlement Value				

- 3.2** Milo and Dena have inherited an antique car from their grandfather. To determine a fair division, they decide to use the bid-and-divide method. Milo bids \$24,000 for the car, while Dena bids \$32,000.
- Determine the amount of cash that Dena must pay Milo.
 - What is each person's final settlement value?
- 3.3** Leif and Neva are the co-winners of an essay contest. The first prize is a bicycle. They decide to use the bid-and-divide method to determine who will keep the bike. Leif bids \$120 while Neva bids \$130. Determine the value of the final settlements.
- 3.4** Suppose Neva knows that Leif thinks the bike in Problem **3.3** is worth at least \$150. However, she does not think that it is worth more than \$175. Neva wants the bike. However, even if she doesn't get the bike, she would like to receive as much compensation as possible. What should she bid? Explain your response.

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- 3.5** Alexi and Norjar share first place in an academic challenge. Later, they realize that the real challenge is how to share the first prize—a new computer—and still remain friends.

They decide to use the bid-and-divide method to find a settlement that allocates possession of the computer to one of them, fair compensation to the other, and lasting friendship to both. If Alexi bids \$800 and Norjar bids \$900, determine the values of the final settlement.

- 3.6** By working together, Miranda and Willis have won first place in an essay contest. The prize is an all-expenses paid trip to Europe for one person. Both students contributed equally to the final draft of the essay. Now they must determine a fair method for dividing the prize.

The total dollar value of the trip is \$1995. This amount includes travel (\$1150), housing (\$500), and food (\$345).

- a. Determine a procedure for dividing the prize.
- b. Do you think that your method is fair? Explain your response.

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Activity 4

The division of a family estate can involve several survivors and many items. In this activity, you adapt the bid-and-divide method to handle such a situation.

Exploration

Design a chart similar to the one in Figure 1 to track the division of an estate that involves several items and several survivors. Your chart should be flexible enough for use in a number of different cases.

Discussion

- a. What changes did you make to the chart in Figure 1?
- b. How is the value pool determined in your chart?
- c. How are the shares of the bids in each column determined?
- d. Is it possible to modify your chart so that a spreadsheet can complete all the necessary calculations?
- e. Could your chart be used to determine a fair division for two people who must divide a single object?

Assignment

- 4.1** Dena and Milo must use the bid-and-divide method to distribute an antique car, a coin collection, and the family house. Dena bids \$32,000 for the car, \$6000 for the coins, and \$126,000 for the house. Milo bids \$24,000 for the car, \$5000 for the coins, and \$151,000 for the house.
- Determine a fair division of the three items.
 - Who must pay cash as a compensation? How much will be paid?
 - Does anyone receive more than a fair share?
 - Explain the significance of the value-pool balance.
 - If Dena changes her bid on the coin collection to \$12,000, how will the final settlement values change?
 - If Milo does not want the car, would it be a good strategy for him to bid \$0? Explain your response.
 - What is the best bidding strategy for someone who does not wish to receive an item but wants the final settlement value to be as large as possible?
- 4.2** Imagine that a wealthy neighbor has left you and a friend a computer, a valuable painting, and a single \$40,000 scholarship to attend the college of your choice.
- Use appropriate technology to determine a fair division of the inheritance.
 - Use a chart or spreadsheet to show a fair division in which you receive the computer and the scholarship.
 - Use a chart or spreadsheet to show a fair division in which your friend receives the painting and the scholarship.
- 4.3** The four Hersey children all want the family grandfather clock. Since they are unable to decide who will keep it, they visit a lawyer. The lawyer asks each of the four to submit a secret bid. All four bids are shown in the table below.

Sibling	Jon	Kris	Anne	Dean
Bid	\$950	\$1000	\$1250	\$1300

Determine the value of the final settlement for each sibling.

- 4.4** The Hersey children from Problem **4.3** also inherited a home stereo, a color television, and a sports car. Determine how these three items would be divided fairly if each sibling submitted the bids shown in the following table.

Bids				
Items	Jon	Kris	Anne	Dean
stereo	\$1500	\$1750	\$1000	\$1200
television	\$900	\$600	\$750	\$500
car	\$10,000	\$12,000	\$9500	\$11,500

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- 4.5** In your own home, identify a minimum of four objects of some value which you might want to divide fairly. Ask at least three people (parents, siblings, or friends) to submit bids on these objects, then determine a fair division.

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Summary Assessment

1. Maria, Gisele, and Micah want to divide a continuous object into three parts. They follow the process described below.
 - Maria divides the object into two parts.
 - Gisele divides each of the two parts into three pieces.
 - Micah selects one piece of the object.
 - Maria selects one piece.
 - Gisele selects two pieces.
 - Maria selects one piece.
 - Micah is awarded the remaining piece.
 - a. Explain why this procedure may not result in a fair division.
 - b. Modify this process to ensure that a fair division occurs.
2. Tawnya and Vasu have received an apple strudel as a gift. Vasu would be content to have $\frac{3}{8}$ of the strudel. Tawnya thinks her fair share is $\frac{1}{2}$ of the strudel. Would a procedure that awarded $\frac{1}{2}$ of the strudel to each of them be a fair division? Explain your response.
3. Miguel, Rolf, and Tristan are graduating from college. In their four years as roommates, they have made several purchases together, including a stereo, a calculator, and a compact disc collection. Now that each will be moving to a different town, they must divide these possessions. Devise a method that the roommates can use to achieve a fair division.
4. The three roommates in Problem 3 also own an old car that they used for grocery shopping and trips to the beach. They decide to use the bid-and-divide method to see who will keep the car. Miguel bids \$400, Rolf bids \$600, and Tristan bids \$375.
 - a. Determine the value of the final settlement for each person.
 - b. If Tristan knows the amount of the other two bids and is not interested in keeping the car, how can he maximize the value of his final settlement?

Module Summary

- A fair division problem exists when individuals must divide a set of items among themselves. A **fair division** occurs when all individuals, by their own assessment, consider the portions they are awarded as fair. “Fairness” depends on each individual’s opinion. This opinion may not agree with the opinion of others involved.
- An item is considered **continuous** if it may be awarded in parts in a fair division.
- One technique for dividing a continuous object between two persons is the **cut-and-choose** method. In this method, one person cuts the object in two shares. The other person then chooses the desired piece.
- Another technique for dividing a continuous object is the **continuous-knife** method. In this method, a knife is held above the left edge of the object and moved slowly from left to right. Either person can stop the knife to cut off a piece that represents a fair share. The person who stops the knife receives the portion to the left. The second person receives the remaining portion.
- One approach for dividing a continuous object among three or more people is the **reduction** method.
 1. The first person cuts a fair share of the object.
 2. The second person has a choice: either reduce the portion or leave it intact.
 3. The third person has the same choice: either reduce the portion or leave it intact.
 4. The last person to reduce the piece is awarded that portion and is now out of the process.
 5. The remaining two people divide the rest of the object using this or any method for fairly dividing an object between two people.
- An item is considered **discrete** if it can be awarded as a unit in a fair division.
- The **bid-and-divide** method may be used to divide discrete items. The technique involves assigning a cash value to an item through a bidding process. The bids represent the amounts that individuals would be willing to pay for the item and determine the value to be divided fairly. The item is awarded to the highest bidder. The individual who receives the item then compensates the others for a fair share.

Selected References

Bennett, S., D. DeTemple, M. Dirks, B. Newell, J. M. Robertson, and B. Tyus. "Fair Divisions: Getting Your Fair Share." High School Mathematics and Its Applications (HiMAP) Project. Module 9. Arlington, MA: COMAP, 1987.

Consortium for Mathematics and Its Applications (COMAP). *For All Practical Purposes: Introduction to Contemporary Mathematics*. New York: W. H. Freeman and Co., 1988.

Steinhaus, H. *Mathematical Snapshots*. New York: Oxford University Press, 1960.