Colorful Scheduling

Have you outgrown coloring? Mathematicians haven't. Mathematicians use coloring theory to solve problems in scheduling, organization, and cartography.

Masha Albrecht • Darlene Pugh

Colorful Scheduling

Introduction

As a cartographer, you wish to color a map of the western United States. Any states that share a common border cannot be the same color. States that touch only at one corner, however, can be the same color.

Exploration

Figure **1** shows a map of the western part of the continental United States. Color the map as suggested in the introduction, using as few colors as possible.

Figure 1: Map of the western United States

Discussion

a. How many different colors did you need to color the map in Figure **1**?

b. Using the number of colors identified in Part **a**, do you think you could color a map of the entire United States in the same fashion?

c. What is the least number of colors you would need to color a map of all the world's countries?

Activity 1

Mathematicians have puzzled over map-coloring problems since the 19th century. In this activity, you create several maps and determine the minimum number of colors needed to color each one, provided that regions which share a border cannot be the same color.

Exploration

a. On a blank sheet of paper, draw at least seven straight, nonparallel lines. Extend each line entirely across the paper.

b. Imagine that each region enclosed by intersecting lines (or the edge of the paper) is a country on a map. Given that countries which share a border cannot be the same color, color the map using the least possible number of colors. Record this number.

Mathematics Note

The **chromatic number** of a map is the least number of colors required to color the map. For example, the chromatic number of the map in Figure **2** is 3.

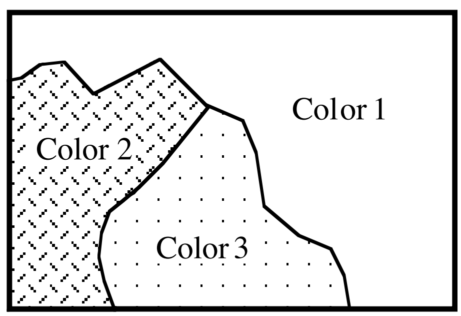


Figure 2: Map with chromatic number 3

Note: In this module, it is assumed that regions which share a border cannot be the same color and that each region is contained in one continuous border.

c. **1.** Design a map of four regions that has a chromatic number of 3.

2. Design another map of four regions that has a chromatic number of 4.

d. Predict the chromatic number you think would be sufficient to color any map drawn on a flat surface.

Discussion

a. **1.** How does the chromatic number of your straight-line map compare with those of others in the class?

2. How do you think the chromatic number would change if you drew a map using 6 straight lines? 15 straight lines? *n* straight lines?

b. Cartographers have long believed that four colors are enough to color any map of the earth. Do you think that a chromatic number of 4 is large enough for any map drawn on paper? Explain your response.

Assignment

1.1 Why does a checkerboard have a chromatic number of 2?

1.2 Using templates provided by your teacher, determine the chromatic number for maps of each of the following continents:

a. South America

b. Australia

1.3 **a.** If a map has a chromatic number of 2, what is the minimum number of regions it can have?

b. If a map has a chromatic number of 2, what is the maximum number of regions it can have? Draw an example to illustrate your answer.

1.4 **a.** If a map has a chromatic number of 4, what is the minimum number of regions it can have? Draw an example to illustrate your answer.

b. If a map has a chromatic number of 4, what is the maximum number of regions it can have?

* * * * *

1.5 Would you expect all regions on a map to have simple closed curves as boundaries? Explain your response.

1.6 **a.** Determine the maximum number of colors required to color a map consisting of 14 regions formed by intersecting straight lines. Justify your response.

b. Determine the minimum number of colors required to color the map described in Part **a**. Justify your response.

1.7 Tie a length of string into a simple loop. Drop the loop onto a flat surface so that it forms several regions. If the separate regions formed by the string represent countries on a map, what is the map's chromatic number? Sketch and color one example to support your response.

* * * * * * * * * *

Activity 2

Besides map making, **coloring theory** can be applied to other real-world situations. For example, the New York City Department of Sanitation uses coloring theory to schedule routes for garbage trucks, while computer scientists use it to analyze printed circuit boards.

Mathematics Note

A **graph** is a non-empty set of points or **vertices** V, along with a set of **edges** E. Each edge is a one- or two-element subset of V.

For example, the graph in Figure **3** is the set of points $\{M, N, O, P, Q\}$ and the set of edges $\{\{M,N\}, \{M,P\}, \{M,Q\}, \{N,O\}, \{N,P\}, \{N,Q\}, \{O\}, \{O,P\}, \{P,Q\}\}$. Notice that the edge represented by $\{O\}$ is a **loop**.

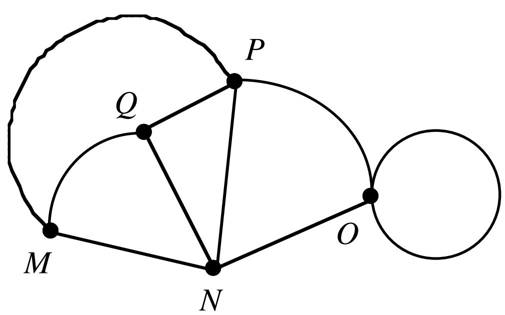


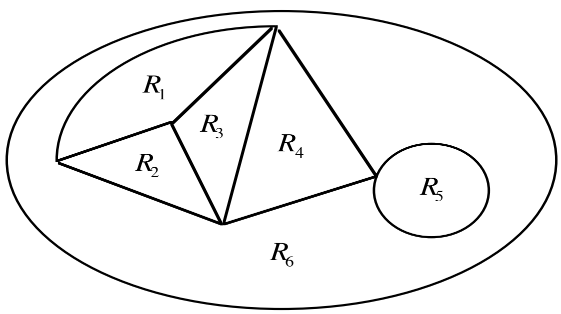
Figure 3: A graph with five vertices and nine edges

The **degree** of a vertex is the number of edges that meet at that vertex. In Figure **3**, the degree of vertex M is 3, the degree of vertex N is 4, the degree of vertex O is 4, the degree of vertex P is 4, and the degree of vertex Q is 3. Note that the loop contributes 2 to the degree of vertex O.

Exploration

Graphs may be used to represent many different types of information. For example, graphs can model computer networks, transportation routes, or molecular structures. In such graphs, the vertices might represent individual computers, cities, or atoms, respectively, while the edges represent their connections.

A map can also be represented by a graph. In this case, vertices correspond to regions, while edges correspond to shared borders. For example, Figure **4a** shows a map of six regions, R_1, R_2, R_3, R_4, R_5, and R_6. In Figure **4b**, these regions are represented by the vertices. The edges of a graph join vertices that represent regions with a common border.



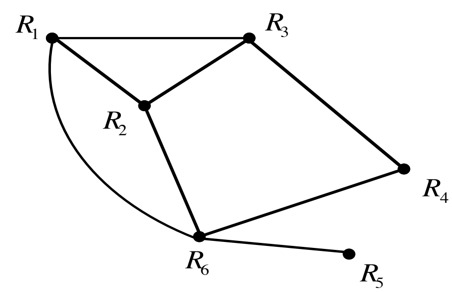


Figure 4: A map with its corresponding graph

On a map, no two regions that share a border, such as R_2 and R_3, can have the same color. Since the edges on the corresponding graph represent shared borders, no two vertices connected by an edge can have the same color. As a result, the chromatic number of the map in Figure **4a** is the same as the chromatic number of the graph in Figure **4b**. In this exploration, you use graphs to investigate a map's chromatic number.

a.
1. Obtain a map of South America from your teacher. Draw a single dot inside each country's borders.
2. If two countries share a border, connect the two dots that represent them with a curve or a segment. The result is a graph of South America.

b.
1. Compare your graph with those of others in the class.
2. Devise a method for determining when two graphs are equivalent.

c. Use your graph to determine the chromatic number of the map of South America.

Discussion

a. In your graph of South America, which vertex has the greatest degree? Which vertex has the least degree?

b. Is your graph equivalent to those of others in the class? Explain your response.

c. When using graphs to depict maps and their colors, would you ever need to use a loop as shown in Figure **3**? Explain your response.

Assignment

2.1 Obtain a map of the counties, parishes, or other subdivisions of your state or local area.

a. Make a graph of the map.

b. Label each vertex with its corresponding degree.

c. Determine the chromatic number of the map.

2.2 Imagine that you are the music director at your school. You must schedule half hour practice sessions for five musical ensembles. Only one student plays each instrument. Because some students play in more than one group, some groups cannot practice at the same time. The table below shows the instruments in each ensemble.

Duo	Trio	Quartet	Quintet	Jazz Ensemble
violin	tuba	violin	flute	clarinet
piano	trumpet	cello	clarinet	saxophone
	piano	viola	oboe	drums
		bassoon	piano	guitar
			bassoon	

A graph can help solve your scheduling problem. Complete Parts **a–d** below to determine how many different practice times are necessary.

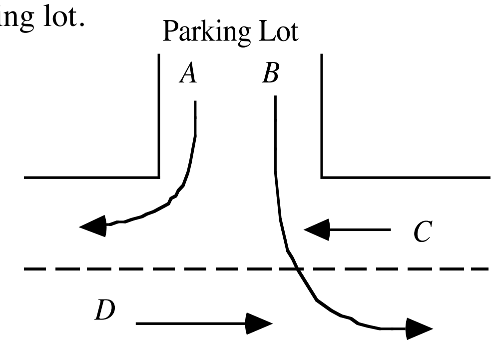
a. Draw a graph in which each vertex represents one of the ensembles. If any two ensembles contain the same instrument, connect the vertices that represent them with an edge. This indicates that the two groups should not be scheduled to rehearse at the same time.

b. Determine the chromatic number of the graph created in Part **a**.

c. The chromatic number of your graph is the minimum number of practice times necessary in the music schedule. Explain why this is true. Hint: Before showing that a number n is a minimum value, you must first demonstrate that it satisfies the constraints of the problem. You must then show that $n - 1$ does not satisfy these constraints.

d. Devise a schedule of possible rehearsal times for the five groups.

2.3. The diagram below shows the intersection of a two-lane, two-way street with the exit from a school parking lot. The arrows in the diagram indicate the flow of traffic through the intersection. Cars leaving the parking lot can turn either to the left or to the right. Cars on the through street cannot enter the parking lot.



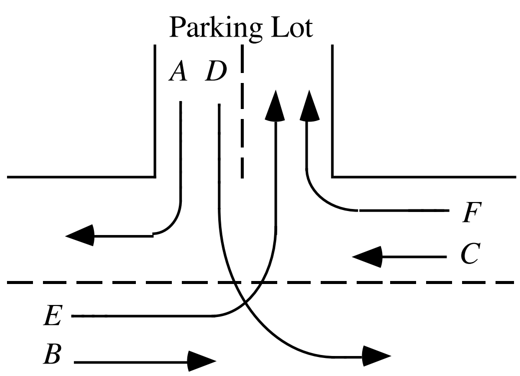
The city plans to install a new traffic light at this intersection. To determine how many settings the light requires, complete Parts **a–c**.

a. Draw a graph in which each vertex represents a direction of traffic. If two cars can collide while following the flow of traffic, connect the vertices that represent these directions with an edge.

b. To prevent collisions, how many different settings should the new traffic light have? Hint: Determine the chromatic number of the graph.

c. Determine which directions of traffic can move at the same time without resulting in collisions. Describe how you identified these directions.

2.4 Due to an increase in school enrollment, city planners have changed the exit from the parking lot in Problem **2.3** to both an entrance and an exit. The city has also added lanes for left and right turns to the through street, thus creating six possible directions of traffic.

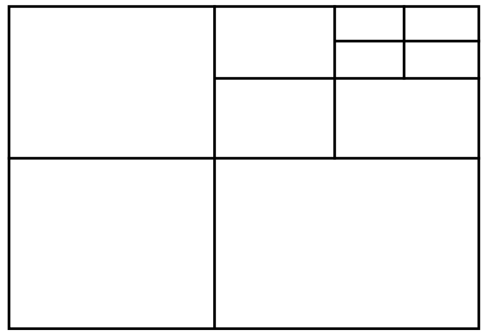


a. Use the directions in Problem **2.3a** to create a graph of this situation.

b. How many traffic-light settings should be used at this intersection? Justify your response.

* * * * *

2.5 Color the following map using only three colors, given that regions which share a border cannot be the same color.



2.6 Use graphs to show that in a party of six people, three of the people either know each other or are complete strangers.

* * * * * * * * * *

Activity 3

Topology is a branch of mathematics that includes the study of the properties of shapes. In this activity, you investigate some connections between topology and coloring theory.

Mathematics Note

If one graph, surface, or shape can be stretched, shrunk, or distorted into another graph, surface, or shape without cutting, tearing, folding, or overlapping, then the two are **topologically equivalent**. When graphs are topologically equivalent, there is a one-to-one correspondence between the vertices so that corresponding edges connect corresponding vertices.

For example, the four graphs in Figure **5** are all topologically equivalent because corresponding edges connect corresponding vertices. **Note:** Edges that appear to overlap or intersect, as in Figure **5d**, do not indicate a vertex unless marked by a dot.)

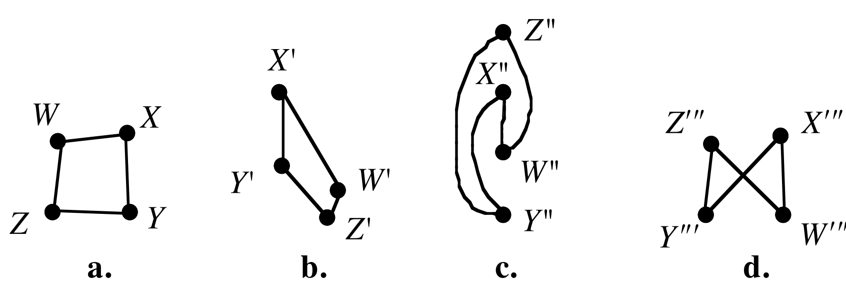
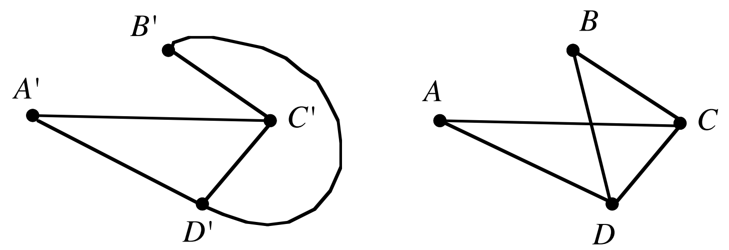


Figure 5: Four topologically equivalent graphs

A graph is **planar** if it is topologically equivalent to a graph drawn on a plane with no overlapping edges.

For example, the graph in Figure **6a** is planar because it is drawn in a plane with no pair of edges crossing or overlapping. The graph in Figure **6b** also is planar because it is topologically equivalent to the one in Figure **6a**.



a. **b.**

Figure 6: Two planar graphs

Discussion 1

a. Is the graph in Figure **5a** topologically equivalent to the one in Figure **6b**? Explain your response.

b. What changes could you make in the graph in Figure **5c** so that the resulting graph is topologically equivalent to the one in Figure **6a**?

Exploration

In this exploration, you distort graphs by stretching and shrinking them.

a.
1. Using a geometry utility, construct a graph with exactly five vertices and eight edges.
2. Determine whether or not your graph is planar.
3. Make a copy of the graph. Place the copy near the original graph.
4. Distort the copy by stretching or shrinking edges and moving vertices, while retaining five vertices and eight edges.
5. Determine whether or not the original graph and the distorted copy are topologically equivalent.

b.
1. Construct a set of four topologically equivalent graphs each of which has exactly 7 vertices and 10 edges.
2. Distort the graphs so that none of them look alike. Delete or add an edge in one or two of the graphs (leaving at least two of the graphs topologically equivalent).
3. Challenge another member of your class to identify the graphs that are topologically equivalent.

Discussion 2

a. Compare your strategy for identifying topologically equivalent graphs with those of others in the class.

b. Why are graphs no longer topologically equivalent when edges are added or deleted?

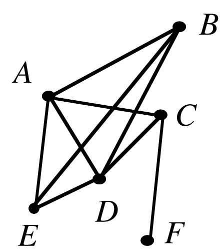
c. What characteristics might make it hard to determine whether or not two graphs are topologically equivalent?

d. Explain why graphs of the same map are always topologically equivalent.

e. Is the graph of any flat map a planar graph? Explain your response.

Assignment

3.1 Draw a graph that is topologically equivalent to the graph below, but has no overlapping edges (except at the vertices).



3.2 Consider two graphs that have the following characteristics in common: each has six vertices—three with a degree of 3, two with a degree of 2, and one with a degree of 1. Would two such graphs always be topologically equivalent? Use an example to support your response.

3.3 **a.** Consider maps of South America, Australia, and the continental United States. Are the graphs of these maps planar? Explain your response.

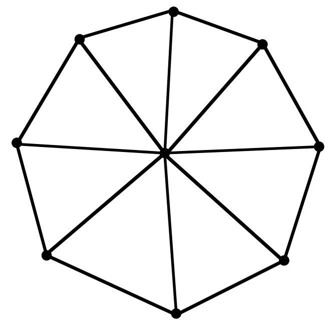
b. On the U.S. map, outline a subset of states whose graph is topologically equivalent to the graph of Australia.

c. Is it possible to outline a subset of states whose graph is topologically equivalent to the graph of South America? Justify your response.

* * * * *

3.4 Describe an example of a one-to-one correspondence between the map of the subset of states in Problem **3.3b** and its corresponding graph.

3.5 **a.** On a map of the United States, outline a subset of states whose graph is topologically equivalent to the graph shown below.



b. On a copy of the graph in Part **a**, label each vertex using the two-letter abbreviation of the corresponding state.

3.6 Describe several ways that a graph of a map of the United States, including Alaska and Hawaii, differs from a graph of a map of South America.

* * * * * * * * * *

Activity 4

In the following activity, you investigate the chromatic numbers of various graphs. You also examine why it was so difficult to prove that a chromatic number of 4 is sufficient for any map drawn on a plane.

Exploration 1

In some computer video games, when an object moves off one side of the screen, it reappears on the opposite side. Likewise, when the object moves off the top of the screen, it reappears on the bottom. In this exploration, you determine the chromatic number for such computer screens by examining **complete graphs**.

Mathematics Note

In a **complete graph**, every pair of vertices is joined by exactly one edge. For example, the graph in Figure **7** is complete because every vertex is connected to every other vertex by exactly one edge.

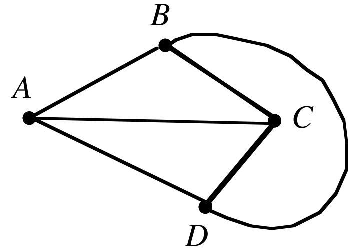


Figure 7: A complete graph

a. **1.** Use pieces of string to create a complete graph with three vertices.

2. Determine whether or not your complete graph is planar.

3. Is every complete graph with this number of vertices planar? Justify your response.

4. Is it possible to create a map whose graph is equivalent to your complete graph? If so, draw one example.

5. Determine the chromatic number of the resulting map, if applicable.

b. Repeat Part **a** for a complete graph with four vertices.

c. Repeat Part **a** for a complete graph with five vertices.

d. The map in Figure **8** appears on a computer game screen. Players can move off the screen from region B and reappear at region A. They also can move off the screen from region C and reappear at region D.

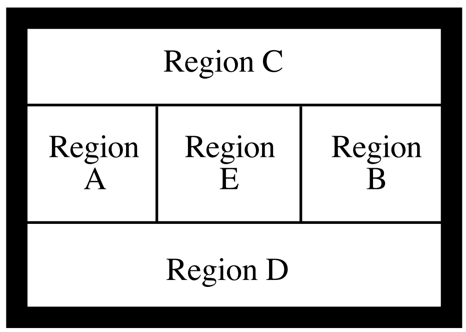


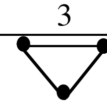
Figure 8: Map on a computer game screen

Since regions A and B have a common border, they cannot share the same color. Likewise, since regions C and D have a common border, they cannot share the same color. What is the chromatic number of this map?

e. Use pieces of string to create a complete graph of the map in Figure **8** with no overlapping edges.

f. Table **1** displays some characteristics of a complete graph with three vertices. The column heading "Possible Surfaces" refers to the surfaces on which it is possible to draw a particular graph without overlapping edges. Complete this table for graphs with one, two, four, and five vertices.

Table 1: Characteristics of complete graphs

No. of Vertices	1	2	3	4	5
Sample Graph					
Chromatic Number			3		
Possible Surfaces			any surface		

g. Draw a map that represents each sample graph you created in Table **1**.

Discussion 1

a. Describe any difficulties you encountered in Part **c** of Exploration **1**.

b. If the graph of a map is complete, what do you know about the countries on that map?

c. When is the graph of any flat map a planar graph?

d. What characteristics describe complete graphs that are also planar graphs?

e. Why can you model a complete graph with five vertices and no intersecting edges on a computer screen, but not on a flat surface?

f. Is there any relationship between the number of vertices in a complete graph and the chromatic number of the corresponding map? Explain your response.

g. Describe how your work in Exploration **1** helps show that four colors will always be sufficient to color any flat map.

Mathematics Note

The **four-color theorem** states that a chromatic number of 4 is sufficient to color any map drawn on a flat surface in which each region is contained in one continuous border and where regions that share a border must be a different color. In other words, planar maps never require more than four colors.

To prove this theorem, mathematicians searched for a planar graph that required five colors. After proving that a complete graph with five vertices could never be planar, however, they still had to consider whether or not other graphs that require five colors could be drawn in a plane.

In 1976, Kenneth Appel and Wolfgang Haken, two mathematicians from the University of Illinois, used a computer to check all the other possible graphs. The computer verified that no planar graph has a chromatic number greater than 4. It took thousands of diagrams and 1200 computer hours to prove the theorem.

Because the computer algorithms used in the proof have been questioned, Appel and Haken's work remains controversial. Other mathematicians are still searching for a simpler, more elegant, proof.

Exploration 2

In this exploration, you investigate the chromatic number of maps drawn on spherical surfaces.

a. Predict the chromatic number sufficient to color any map drawn on a globe in which each region is contained in one continuous border.

b.

1. Use an inflated balloon to model a globe. On your balloon, draw a complete graph with four vertices and no intersecting edges.
2. Determine the graph's chromatic number.
3. Draw a map on your balloon that corresponds to the graph you constructed in Step **1**.

c. Is it possible to draw a complete graph with five vertices and no overlapping edges on a balloon? If so, draw one example.

Discussion 2

a. Describe some surfaces that are topologically equivalent to a balloon.

b. Describe any difficulties you encountered when you tried to draw a complete graph with five vertices on the balloon.

c. How do you think the chromatic number of maps drawn on a spherical surface compare to the chromatic number of maps drawn on a flat surface?

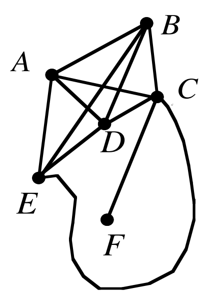
d. On a sphere, when are complete graphs of maps planar?

e. Explain why a sphere does not model a computer game screen like the one described in Exploration **1**.

f. What chromatic number is sufficient for any map drawn on a surface that is topologically equivalent to a sphere? Explain your response.

Assignment

4.1 **a.** Is the graph shown below a planar graph? Explain your response.



b. A non-empty subset of a graph's vertices and edges is a **subgraph**. Identify a subgraph of the graph in Part **a** that forms a complete graph with four vertices.

c. If a subgraph of a graph is not planar, then the graph cannot be planar.

1. Identify a subgraph of the graph in Part **a** that forms a complete graph with five vertices.
2. Is the subgraph in Part **c1** planar? Explain .

3. How can this subgraph help you to explain your response to Part **a**?

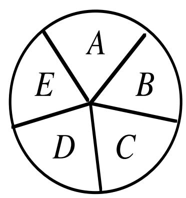
4.2 **a**. Draw a complete planar graph with four vertices.

b. Determine the degree of each vertex.

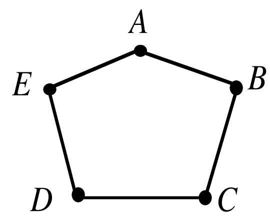
c. Predict the degree of each vertex in a complete graph with n vertices.

4.3 Identify the maximum number of vertices for any complete graph with no overlapping edges drawn on an egg and on a cylindrical surface with both bases. Justify your responses.

4.4 The "pie map" in the diagram below requires three colors—yet its graph does not have a complete subgraph with three vertices.



Pie Map



Graph of Pie Map

Create a map that requires four colors but whose graph does not have a complete subgraph with four vertices.

* * * * *

4.5 Draw any polygon. Explain whether or not the polygon and all of its diagonals can be considered a planar graph.

4.6 **a**. What is the difference between a planar graph figure that is not planar graph. Explain your response.

b. Give an example of a planar figure that is not a planar graph.

* * * * * * * * * *

Research Project

Mathematicians have extended the drawing of maps to surfaces other than planes and spheres. Included in these surfaces are doughnut shapes and pretzel shapes. Research the chromatic numbers of maps drawn on a **torus** (a doughnut shape). Your report should include a description of the equation used to find the chromatic number for a torus, as well as an example of a map that requires the maximum chromatic number for this surface.

Summary Assessment

1. The diagram below shows the arrangement of desks in a classroom.

1	6	11	16
2	7	12	17
3	8	13	18
4	9	14	19
5	10	15	20

To administer a test to a class, a teacher would like to create enough different versions so that no two adjacent students—either vertically, horizontally, or diagonally—have exactly the same test.

a. If one student sits at each desk, how many different versions of the test should the teacher create? Justify your response.

b. On a copy of the diagram above, show which version of the test should be placed at each desk.

Module Summary

- The **chromatic number** of a map is the least number of colors required to color the map.
- A **graph** is a non-empty set of points or **vertices** V, along with a set of **edges** E of one- or two-element subsets of V.
- The **degree** of a vertex is the number of edges that meet at that vertex.
- **Topology** is the branch of geometry concerned with the properties of shapes.
- If a graph, surface, or shape can be stretched, shrunk, or otherwise distorted into another graph, surface, or shape without cutting, tearing, folding, or overlapping, the two graphs are **topologically equivalent**.
- A graph is **planar** if it is topologically equivalent to a graph drawn on a plane with no overlapping edges.
- In a **complete graph**, every pair of vertices is joined by exactly one edge.
- The **four-color theorem** states that a chromatic number of 4 is sufficient to color any map drawn on a flat surface in which each region is contained in one continuous border and regions that share a border must be a different color. In other words, planar maps never require more than four colors.

Selected References

Appel, K., and W. Haken. “The Four Color Proof Suffices.” *The Mathematical Intelligencer* 8.1 (1986): 10–20.

——. “The Solution of the Four-Color-Map Problem.” *Scientific American* September 1977: 108–121.

Francis, R. “The Mathematician’s Coloring Book.” High School Mathematics and Its Applications Project (HiMAP). Module 13. Arlington, MA: Consortium for Mathematics and Its Applications (COMAP), 1989.

Glashow, S. L. “The Mapmaker’s Tale.” *Quantum* May/June 1993: 46–47.

Haas, R. “Three-Colorings of Finite Groups or an Algebra of Nonequalities.” *Mathematics Magazine* October 1990: 212–225.

Heawood, P. J. “Map Coloring Theorem.” *Quarterly Journal of Pure and Applied Mathematics* 24 (1890): 332–338.

Hutchinson, J. P. “Coloring Ordinary Maps, Maps of Empires, and Maps of the Moon.” *Mathematics Magazine* October 1993: 211–226.

Malkevitch, J. “Applications of Vertex Coloring Problems for Graphs.” Undergraduate Mathematics and Its Applications Project (UMAP). Module 442. Arlington, MA: COMAP, 1979.

Nelson, R., and R. J. Wilson. *Graph Colourings*. New York: John Wiley and Sons, 1990.

Pedersen, K. “Topology.” In *Projects to Enrich School Mathematics: Level 3*. Ed. by L. Sachs. Reston, VA: National Council of Teachers of Mathematics (NCTM), 1988. pp. 27–33.

Peterson, I. *The Mathematical Tourist*. New York: W. H. Freeman and Co., 1988.

Rouvray, D. H. “Predicting Chemistry from Topology.” *Scientific American* September 1986: 40–47.

Stahl, S. “The Other Map Coloring Theorem.” *Mathematics Magazine* 58.3 (May 1985): 131–145.

Stewart, I. “The Rise and Fall of the Lunar M-pire.” *Scientific American* April 1993: 120–121.

Williams, J. “Graph Coloring Used to Model Traffic Light.” *The Mathematics Teacher* March 1992: 212–214.

Williams, R. “Building Blocks for Space and Time.” *New Scientist* 12 June 1986: 48–51.