Fly the Big Sky
with Vectors

What do swimming across a river, landing a helicopter, and refueling a plane in midair have in common? In this module, you explore how vectors can be used to model these situations and more.

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Introduction

While camping with their parents, Alee and Christopher wander off to fish in a nearby river. They walk downstream from camp for a kilometer or two, cross the river on a bridge, then start walking back upstream, fishing along the way.

When they reach a point on the bank directly opposite their camp, they see their parents sitting down to dinner. Since the afternoon is warm and they are both strong swimmers, they decide to swim across the river. Figure **1** shows a diagram of this situation.
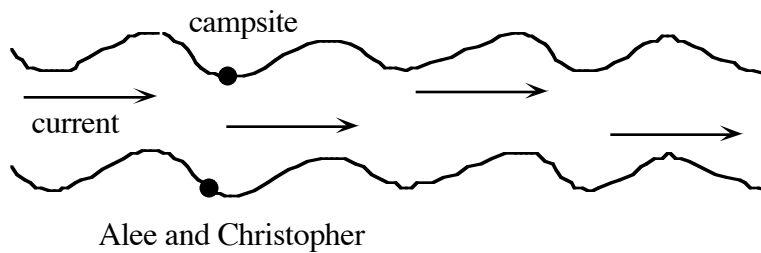
Figure 1: Christopher and Alee on the riverbank

Both Christopher and Alee know that if they try to swim straight to camp, the current will carry them far downstream. In what direction should they swim in order to land near their parents?

Navigators and pilots confront similar predicaments every day. Ocean currents can carry ships off course, while strong winds can force helicopters and planes to stray far from their destinations. In this module, you investigate one method for analyzing such situations.

Exploration

In this exploration, you use a model helicopter to observe how wind affects flight.

a. Using the paper template provided by your teacher, construct a model helicopter like the one shown in Figure **2**.
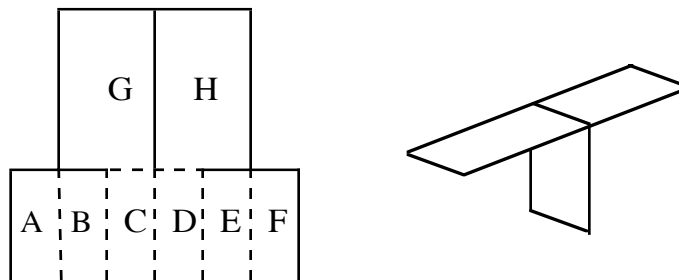
Figure 2: Template and helicopter

b.
    1. Place a target on the floor. Drop your helicopter from a height of at least 175 cm. Adjust your position (or that of the target) until the helicopter consistently lands on target.

    2. Record your observations of the helicopter's path for three flights, including the horizontal and vertical distance traveled in each flight.

c. Draw a diagram to represent the flight of the helicopter in Part **b**. Use one arrow to indicate the actual path of the helicopter. Use two more arrows to indicate the mean horizontal and vertical distances traveled during the three recorded flights.

d. Use a fan to create a light breeze. Repeat Parts **b** and **c**, recording your observations of the helicopter's flight in windy conditions.

Discussion

a. When you drop the helicopter in Part **b** of the exploration, what forces act on it?

b. If the air were perfectly still, what would you expect the helicopter's path to look like?

c. What causes the helicopter to move in a horizontal direction?

d. How would turning the fan on high speed affect the helicopter's path?

Activity 1

Imagine that you are a passenger on a flight from Butte, Montana, to Salt Lake City, Utah. Midway through the flight, the plane encounters a thunderstorm. To avoid the bad weather, the pilot must change course. What information does the pilot need to dodge the storm, then continue on to Salt Lake City?

An understanding of **vectors** can help you analyze this situation. Vectors can be used to model many types of quantities, including displacement, velocity, and force.

Mathematics Note

A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically represented by a bold, lowercase letter, such as vector **m**. In handwritten work, the same vector can be symbolized by $\vec{m}$. The magnitude of a vector **m** is denoted by |**m**|.

Displacement is a change in position in a particular direction. For example, a displacement of 5 km to the north can be represented by a vector with a magnitude of 5 km and a direction of north.

The **velocity** of an object is its speed in a specific direction. For example, a car driving to the east at 80 km/hr can be represented by a vector with a magnitude of 80 km/hr and a direction of east.

Force is a push or a pull in a particular direction. One metric unit of force is the newton (N). The force of gravity on a mass of 1 kg, for example, has a magnitude of about 9.8 N and is directed towards the center of the earth.

Exploration

In this exploration, you use vectors to represent flight paths between U.S. cities.

a. On a map provided by your teacher, draw displacement vectors to represent the flights listed below. Each vector should begin at the city in which the flight originates and end at the appropriate destination.

   1. Butte, Montana, to Salt Lake City, Utah
   2. Butte, Montana, to Seattle, Washington
   3. Atlanta, Georgia, to Knoxville, Tennessee
   4. Los Angeles, California, to Minneapolis, Minnesota
   5. Chicago, Illinois, to Denver, Colorado
   6. Tallahassee, Florida, to Atlanta, Georgia
   7. Knoxville, Tennessee, to Cincinnati, Ohio
   8. Salt Lake City, Utah, to Butte, Montana
   9. Tallahassee, Florida, to Cincinnati, Ohio

b. Use appropriate tools to determine the magnitude and direction of each vector in Part **a**.

c. Describe each vector using the measurements you made in Part **b**.

d. Describe a trip from Salt Lake City to Seattle using at least two vectors from Part **a**.

Discussion

a. Why would an instruction such as "fly 3 km" be of little use to a pilot?

b. How do the descriptions of vectors that you wrote in Part **c** of the exploration compare with those of others in your class?

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c.   A common navigational system defines due north as 0° and measures directions clockwise from north. These measures are referred to as **bearings**. In Figure **3**, for example, vector **m** has a bearing of 100°.
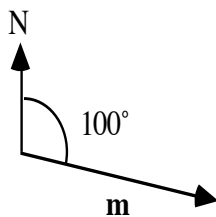

Figure 3: Bearing of 100°

Describe each vector in Part **a** of the exploration with a magnitude and a bearing.

Mathematics Note

Equivalent vectors have the same magnitude and direction. **Opposite vectors** have the same magnitude, but their directions differ by 180°.

In Figure **4**, for example, equivalent vectors **m** and **n** are represented by arrows with the same magnitude and direction. Opposite vectors **p** and **m** are represented by arrows with the same magnitude, but their directions differ by 180°. In this case, the following statements are true: **n = m** and **p = −m**.

100 km

Figure 4: Equivalent and opposite vectors

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d.   Are there any equivalent vectors in your list from Part **a** of the exploration? Explain your response.

e.   Are the vectors from Butte to Salt Lake City and from Salt Lake City to Butte opposite vectors? Justify your response.

Mathematics Note

One way to add vectors is the **tip-to-tail method**. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.

The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.

For example, Figure **5** shows the addition of vectors **m** and **n**. Since translation preserves the magnitude and direction of vectors, vector **n** is translated so that its tail coincides with the tip of vector **m**. The resultant vector **r** joins the tail of vector **m** to the tip of vector **n**. This sum can be written as **m** + **n** = **r** .
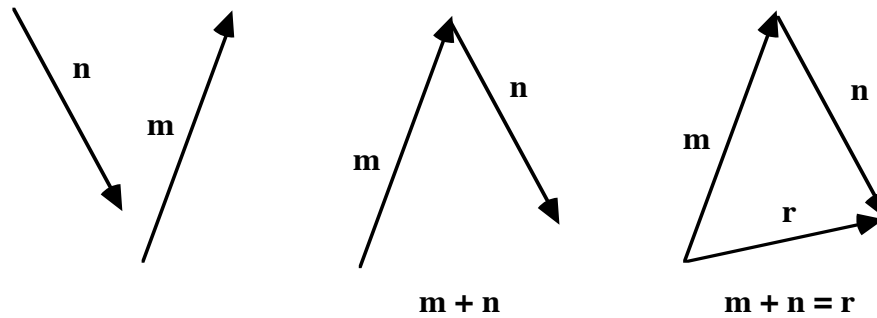
Figure 5: The tip-to-tail method of vector addition

>   **f.**   Describe how you could use vector addition to represent a trip from Salt Lake City to Seattle.

Mathematics Note

A **scalar** is a real number. For example, $5, -7, \pi, \sqrt{3}$, and $3/5$ are all scalars.

The operation $k\mathbf{m}$ denotes the **scalar multiplication** of **m** by the scalar k. As shown in Figure **6**, for example, 5**m** describes a vector in the same direction as **m** with a magnitude 5 times that of **m**. Similarly, –5**m** is a vector 5 times as long as **m**, but in the direction opposite to **m**.
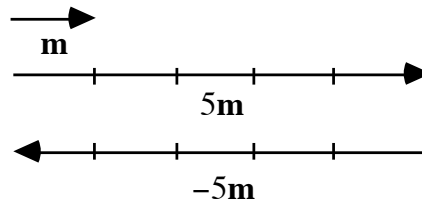
Figure 6: Scalar multiplication

>   **g.**   How could you use scalar multiplication to describe the vector from Tallahassee to Cincinnati in terms of the vector from Tallahassee to Atlanta?

Assignment

1.1   **a.** Describe three quantities that can be modeled by vectors.

   b. Describe three quantities that cannot be modeled using vectors.

1.2   In the game of chess, two players move pieces on a checkerboard according to a set of rules. One of the chess pieces, the knight, can be moved two spaces horizontally then one space vertically, or two spaces vertically then one space horizontally. As shown in the diagram below, a knight's change in position can be modeled by a resultant vector.

2 down, 1 right   2 right, 1 up
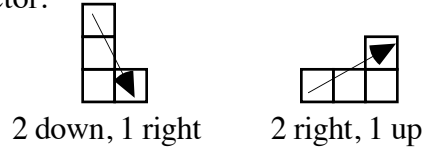
Imagine that a knight is positioned somewhere near the middle of the board. How many different moves are possible? Draw a diagram to support your response, using a resultant vector to represent each possible move.

1.3   **a.** Use the tip-to-tail method to add the vectors below in alphabetical order. Label the resultant vector **e**.
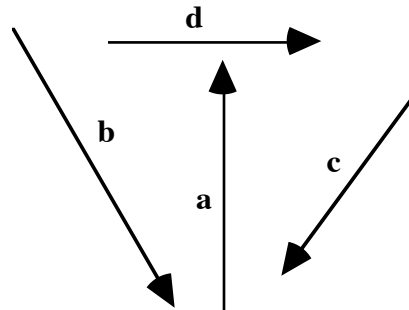

   b. Add the vectors in Part **a** in any order you choose. Does changing the order in which vectors are added change the resultant vector?

1.4   The diagram below shows a vector **m**.
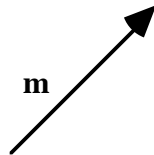
   a. Using the tip-to-tail method, add **m** to as few other vectors as possible to obtain a resultant that equals each of the following:

   1. 3**m**

   2. 1**m**

   3. –2**m**

   b. Describe the vector(s) you added to **m** in each part of Problem **1.4a**.

   c. Describe the vector you would add to **m** to obtain a resultant with a magnitude of 0.

1.5   An airplane is flying to the east at 500 km/hr. The plane encounters a
wind blowing to the north at 35 km/hr.

    a. Draw a vector diagram of this situation.

    b. How much is the airplane's velocity affected by the wind?

    c. How much is the plane's direction affected by the wind?

1.6   As mentioned in the introduction, Christopher and Alee want to swim
across a river to a point directly opposite them on the other bank. The
river has a current of 2 km/hr. In still water, Christopher and Alee can
swim at a speed of 2.2 km/hr. Sketch a diagram of this situation and
use trigonometry to determine the direction in which they should
swim.

1.7   Two children are playing with a red wagon. One child pulls on the
front of the wagon with a force of 32 N at a bearing of 270°. The other
child pushes on the side of the wagon with a force of 38 N at a bearing
of 0°. What is the resultant force on the wagon and at what bearing is
it applied? Use vector addition to support your response.

1.8   Without current, a ferry can cruise at 30 km/hr. While crossing an
ocean channel, the ferry encounters a current of 5 km/hr moving at a
right angle to its motion.

    a. What is the ferry's resultant velocity?

    b. How many degrees off its original course does the current push
the ferry?

1.9   An airplane is flying 250 km/hr on a bearing of 225°. The plane
encounters a wind blowing at 30 km/hr on a bearing of 180°. Use a
ruler and protractor to create a scale drawing of this situation. Find the
magnitude and bearing of the plane's resultant velocity using the
tip-to-tail method of vector addition.

<p align="center">* * * * *</p>

1.10   During its morning hunt, an eagle flies 15 km on a bearing of 55°.
This portion of its flight can be represented by the vector **m**.

    a. What would 5**m** represent in this situation?

    b. What would –**m** represent in this situation?

1.11 The diagram below shows two displacement vectors. Vector **m** represents a displacement of 10 km due east. Vector **n** represents a displacement of 17 km at a bearing of 15°. Add these two vectors and describe the resultant displacement.

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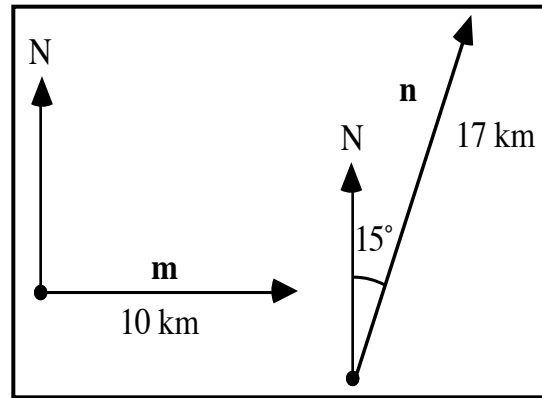
Activity 2

In Activity **1**, you used scale drawings and the tip-to-tail method to add vectors. In this activity, you discover how the properties of triangles can help you determine the magnitude and bearing of a resultant vector algebraically.

Exploration 1

The Pythagorean theorem states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. As you have seen in previous modules, this relationship can be represented by constructing a square on each side of a right triangle, then comparing the areas of the three squares.

In this exploration, you create a similar model to explore the relationship among these areas when the triangle is not a right triangle.

a.
   1. Using a geometry utility, construct a circle with center at point C.

   2. Create a moveable point on the circle. Label the point A.

   3. Construct $\overline{AC}$. Since point A is on the circle, AC remains constant as A moves around the circle.

   4. Construct a right triangle ABC in which $\overline{AB}$ is the hypotenuse.

b.   On each side of $\triangle ABC$, construct a square whose sides are congruent to the side of the triangle. Create the squares so that when the sides of the triangle are moved, the squares remain squares.

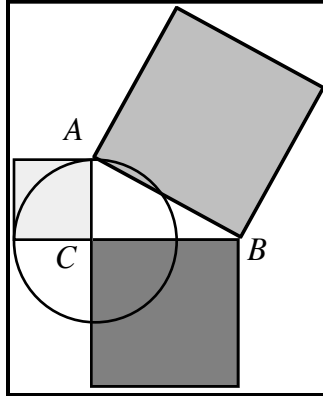Your construction should now resemble the one shown in Figure **7**.

Figure 7: Right triangle with squares on its sides

c.   Record the measure of $\angle ACB$, along with the area of each square created in Part **b**.

d.   Compare the area of the square constructed on $\overline{AB}$ to the sum of the areas of the other two squares.

e.   Move point A to several different locations on the circle and repeat Parts **c** and **d** at each location. **Note:** Save this construction for use in Exploration **2**.

Discussion 1

a.   What is the general relationship between the area of the square constructed on $\overline{AB}$ and the sum of the areas of the other two squares in each of the following situations:

1. when $m\angle ACB > 90°$?

2. when $m\angle ACB < 90°$?

b.   Describe your generalizations in Part **a** above in terms of the type of triangle—right, obtuse, or acute—and the lengths of its sides.

c.   The converses of the generalizations you made in Parts **a** and **b** are also true. How could you use these converses, along with the converse of the Pythagorean theorem, to classify a triangle as acute, obtuse, or right knowing only the lengths of its sides?

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Exploration 2

In Exploration **1**, you found that in an obtuse triangle, the square of the length of the side opposite the obtuse angle is greater than the sum of the squares of the lengths of the other two sides. You also found that in an acute triangle, the square of the length of each side is less than the sum of the squares of the lengths of the other two sides. In this exploration, you discover a more precise way to describe these relationships.

a.   **1.** Delete the squares from the sides of the triangle in your construction from Exploration **1**.

           2. Move point A so that $\angle BAC$ is obtuse.

b.   Let a represent the length of the side opposite $\angle BAC$, b represent the length of the side opposite $\angle CBA$, and c represent the length of the side opposite $\angle BCA$, as shown in Figure **8** below.

c.   Construct an altitude from A to $\overline{BC}$ to form two right triangles. Let h represent the altitude, x represent the length of $\overline{CD}$, and y represent the length of $\overline{DB}$, as shown in Figure **8**.
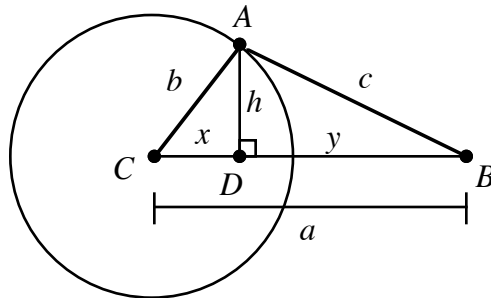

Figure 8: Labeled construction

d.   **1.** Use trigonometry to express x in terms of b and $m\angle BCA$.

           2. Use this expression for x, along with the Pythagorean theorem, to write an expression for h^2 in terms of b and $m\angle BCA$.

           3. Use the Pythagorean theorem to express h^2 in terms of c and y.

           4. Express y in terms of x and a.

           5. To describe h^2 in terms of a, b, c, and $m\angle BCA$, substitute the expression for y from Step **4** and the expression for x from Step **1** into the expression for h^2 from Step **3**.

e.   **1.** Since the expressions from Steps **2** and **5** of Part **d** both equal h^2, they are also equal to each other. Use these two expressions to solve for c^2.

           2. How does the resulting equation compare with the Pythagorean theorem?

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Discussion 2

a. The expression you wrote in Part **e** of Exploration **2** involves the cosine of $\angle BCA$. What is the sign of the cosine in each of the following situations:

    1. when $\angle BCA$ is acute?

    2. when $\angle BCA$ is obtuse?

Mathematics Note

The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of these lengths and the cosine of the included angle. In Figure **9**, for example, $c^2 = a^2 + b^2 - 2ab\cos\angle C$.
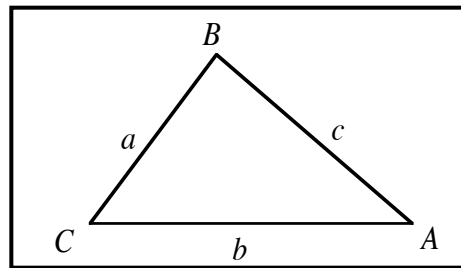

Figure 9: Triangle ABC

The law of cosines can be used to determine an unknown length or angle measure in a triangle. For example, if $a = 10$ cm, $b = 12$ cm, and $m\angle C = 35°$, then the law of cosines can be used to find c as follows:

$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$
$$c^2 = 10^2 + 12^2 - 2(10)(12)\cos35°$$
$$c^2 \approx 47.40$$
$$c \approx 6.89 \text{ cm}$$

b. How does the equation you found in Part **e** of Exploration **2** compare with the law of cosines?

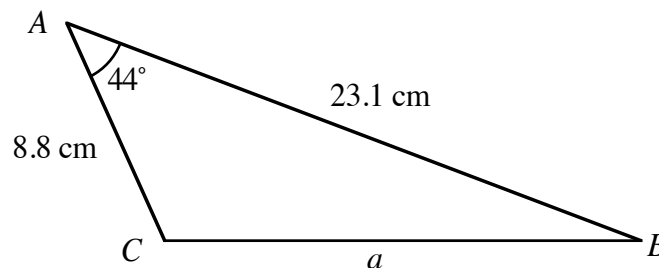**c.** Describe how you could use the law of cosines to find the value of a in Figure **10** below.

Figure 10: Triangle with an unknown length

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d. Once you have found the value of *a* in Figure **10**, how could you use the law of cosines to determine the measure of one of the other angles?

e. The law of cosines is true for both acute triangles and obtuse triangles. It is also true for right triangles? Justify your response.

f. Figure **11** below shows the addition of two vectors, one with a bearing of 20° and one with a bearing of 120°. The bearing of the resultant vector is 105°.
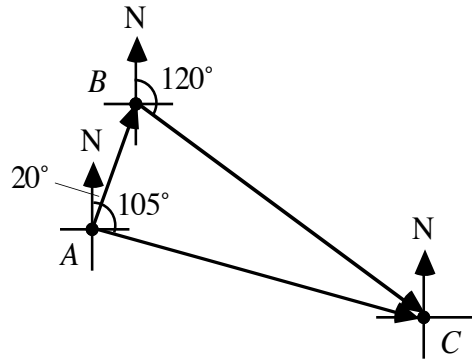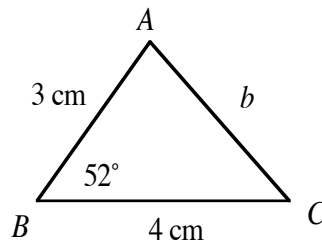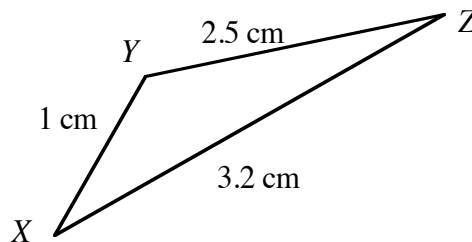

Figure 11: Vector sum with bearings

These three vectors form a triangle. Describe how to determine the measures of the angles in this triangle.
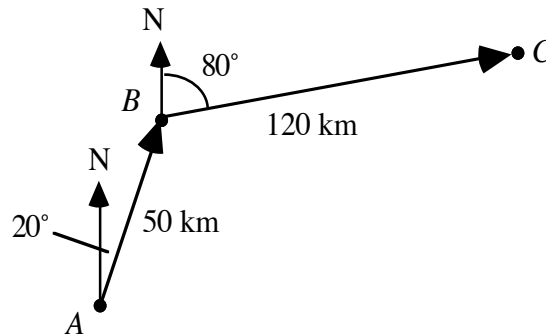
Assignment

2.1 **a.** Find the value of *b* in the following diagram.

b. Determine the measures of all the angles in the triangle below. Hint: You will need to use inverse trigonometric functions.


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2.2   Consider an airplane that flies 50 km at a bearing of 20°, then 120 km at a bearing of 80°. The corresponding displacement vectors are shown in the diagram below.

a. Make a scale drawing of this situation and construct the resultant vector.

b. Determine the magnitude and bearing of the resultant using a ruler and protractor.

c. Determine the magnitude and bearing of the resultant using the law of cosines.

2.3   The speed of a plane or helicopter with respect to the air is its airspeed. But pilots seldom fly in still air. When determining the proper bearings for their destination, they must consider the effect that wind will have on their speed and direction with respect to the ground.

a. An emergency helicopter leaves the hospital on a rescue mission. The patient is located 40 km due east. In still air, the helicopter averages 200 km/hr. If there is no wind, how long will it take the helicopter to reach the patient?

b. As the helicopter lifts off, the wind begins blowing at 15 km/hr at a bearing of 200°.

   1. In order to maintain a ground speed of 175 km/hr due east, what should the helicopter's airspeed be?

   2. At what bearing should the pilot fly to reach the patient?

   3. How long will it take for the helicopter to reach the patient?

c. After the paramedics load the patient on board, the wind changes again. To return to the hospital, the helicopter maintains an airspeed of 200 km/hr at a bearing of 280°. Its ground speed is 210 km/hr due west. What is the wind's magnitude and bearing?

* * * * *

2.4   A construction company plans to build a bridge from point *A* to point *B*, as shown on the map below. From point *C*, a survey crew has measured distances of 5 km to point *A* and 8 km to point *B*. The angle between these two measurements is 20°. How long will the bridge be?
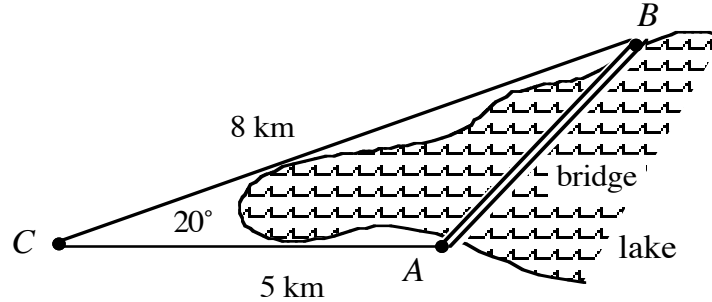
2.5   The vertices of Δ*FUN* are *F*(–3,–1), *U*(3,4), and *N*(5,–2). Make a sketch of this triangle and determine the lengths of its sides and the measures of its angles.

<center>* * * * * * * * * *</center>

Activity 3

Imagine that an F-16 fighter jet and an air tanker are both flying due north, 100 km apart. To rendezvous for a midair refueling, the tanker changes its course to a bearing of 32°, while the F-16 turns to a bearing of 340°, as shown in Figure **12**.
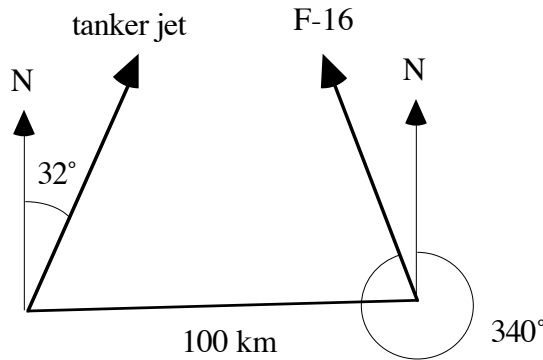

Figure 12: Bearings of F-16 and tanker jet

At the moment when they changed bearings, the two planes were 100 km apart. The pilots must now determine how far each plane must fly to reach the rendezvous point. Without more information, the pilots cannot use the law of cosines to identify these distances. In this activity, you investigate another relationship that could help them analyze this situation.

Exploration

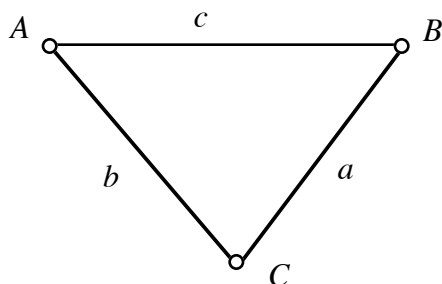**a.** Using a geometry utility, construct $\triangle ABC$ as shown in Figure **13** below.

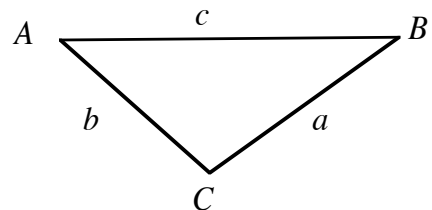**Figure 13: Triangle ABC**

b. Record the lengths of its sides and the measures of its angles.

c. Construct an altitude from vertex C to the opposite side $\overline{AB}$. Label the altitude h. Notice that the altitude divides the original triangle into two right triangles.

d. Write equations for $\sin \angle CAB$ and $\sin \angle ABC$ in terms of h, b, and a.

e. Solve for h in each equation from Part **d**.

f. Use the equations from Part **e** to calculate h. Compare this value to the distance h measured by the geometry utility.

g. Drag any vertex of the triangle and observe what happens to the value of h in each equation and to the measured altitude.

h. Set the two equations in Part **e** equal to each other and solve for the following ratio:

$$\frac{a}{\sin \angle CAB}$$

i. Using the appropriate sides and angles, repeat Parts **c–h** for an altitude k drawn from vertex B to the opposite side $\overline{AC}$. Compare your results with those obtained for altitude h.

Discussion

a. How does your equation from Part **h** of the exploration compare to those obtained by others in the class?

b. What is the relationship among the ratios you found in Parts **h** and **i** of the exploration?

Mathematics Note

The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles. In the triangle in Figure **14**, for example,

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}$$

Figure 14: Triangle *ABC*

Like the law of cosines, this relationship can be used to determine unknown lengths or angle measures in triangles. For example, if $a = 7.1$, $m\angle A = 50°$, and $m\angle B = 60°$, then b can be found as follows. By the law of sines,

$$\frac{7.1}{\sin 50°} = \frac{b}{\sin 60°}$$

In this case, $b \approx 8.0$.

c.   Describe how the law of sines can be used to find the unknown values in Figure **15**.
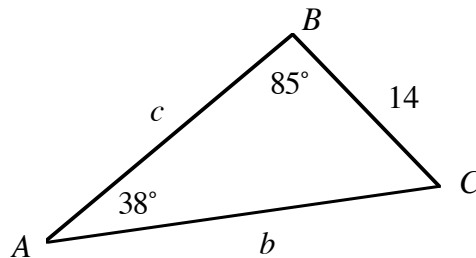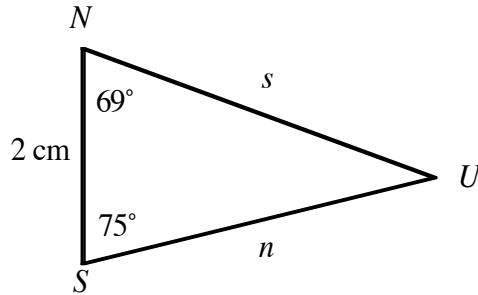

**Figure 15: A triangle with some unknown values**

d.   Describe a triangle with some unknown lengths or angle measures which cannot be determined by the law of sines.

e.   Describe a triangle with some unknown lengths or angle measures which can be determined by the law of sines but not by the law of cosines and vice versa.

Assignment

3.1 Find the values of *s* and *n* in Δ*SUN* below.

3.2 Using a geometry utility, make a scale drawing of the rendezvous of the F-16 and the air tanker described in the introduction to this activity.

    a. Measure the distance that each plane must fly to the rendezvous point. Verify these measurements using the law of sines.

    b. If the F-16 maintains an average speed of 500 km/hr after the change in bearings, how long will it take the plane to reach the rendezvous point?

    c. At what average speed should the tanker fly to meet the F-16?

3.3 After dinner, Christopher and Alee decide to explore the area around their campsite. To keep in touch with each other, they both carry hand-held radios. Christopher leaves camp at a bearing of 34°, while Alee follows a bearing of 300°. Both hike at about the same speed of 4 km/hr. Their radios have a range of 10 km. How long can they hike without losing radio contact with each other? Use diagrams and calculations to support your response.

3.4 On their way up a mountain, two hikers pause to rest by a lake at an elevation of 3600 m. They have an old map of the area, but the section that describes the elevation of the mountain peak has been torn away. As shown in the diagram below, they estimated an angle of 10° to the peak from the far side of the lake. From the near side, they now estimate an angle of 15° to the peak. The lake is 1 km across. Determine the elevation of the mountain peak.

* * * * *
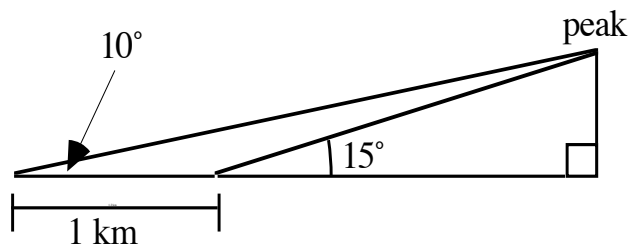
3.5   As part of a training mission, a U.S. Coast Guard pilot must fly a helicopter from Kalamazoo to Flint, Michigan.

a. On a map of Michigan, draw the displacement vector from Kalamazoo to Flint. Measure the magnitude and bearing of this vector.

b. In still air, the helicopter can maintain an average speed of 190 km/hr. If there is no wind, how long will it take to complete the flight?

c. As the helicopter leaves Kalamazoo, the wind begins blowing at 40 km/hr on a bearing of 120°. If the pilot wants to reach Flint in the same amount of time as calculated in Part **b**, at what airspeed and bearing should she fly?

* * * * * * * * * *

Research Project

In previous activities, you used the law of cosines and law of sines to determine unknown lengths or angle measures in triangles. Figure **16** shows ΔGHK. If you knew h, k, and $m\angle K$, could you find g? Or is more information required?
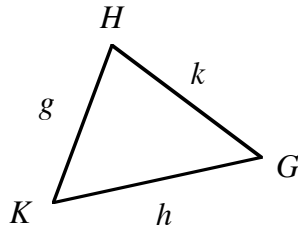
Figure 16: Triangle *GHK*

a.   Given ΔGHK, use the law of cosines and the law of sines to explore each of the following situations:

1. $h = 5$ cm, $k = 4.3$ cm, and $m\angle K = 55°$

2. $h = 5$ cm, $k = 2$ cm, and $m\angle K = 55°$

3. $h = 5$ cm, $k = 6$ cm, and $m\angle K = 55°$

4. $h = 5.00000$ cm, $k = 4.09576$ cm, and $m\angle K = 55.0000°$

b.   Based on your results in Part **a**, make a conjecture about the possibility of determining the remaining parts of a triangle given the lengths of two sides and the measure of a non-included angle.

Activity 4

Many situations require the addition of multiple vectors to determine a resultant. For example, an airplane in flight is simultaneously acted on by forces of lift, thrust, weight, and drag. To find the resultant force on the plane, you must add the vectors that correspond to the individual forces. In previous activities, you used the tip-to-tail method to find a resultant graphically. In this activity, you investigate a method for adding vectors algebraically.

Exploration 1

a. On a two-dimensional coordinate system, construct a vector **r** whose tail is located at the origin O and whose tip is located at a point P in the first quadrant.

b. Determine the vector's magnitude and direction in terms of the angle formed by the vector and the *x*-axis.

c. Construct a horizontal vector and a vertical vector whose sum is vector **r**. These vectors, shown in Figure **17** below, are the **horizontal component** and the **vertical component** of vector **r**. Label the intersection of the horizontal and vertical components as point F.
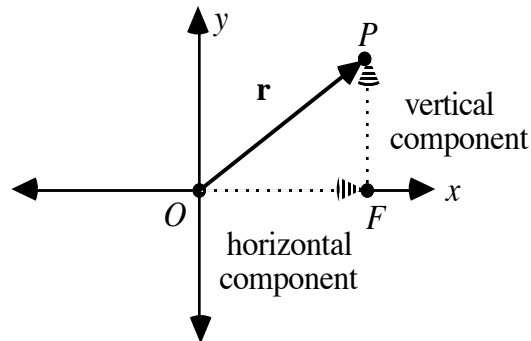
Figure 17: Vector r and its components

d. The three vectors—the horizontal component, the vertical component, and the resultant vector—form a right triangle. Determine the measure of $\angle POF$.

e. Use trigonometry to determine the magnitude of the horizontal and vertical components of vector **r**.

f. Repeat Parts **b–e** for a vector whose tail is located at the origin and whose tip is located in:

1. the second quadrant.

2. the third quadrant

3. the fourth quadrant.

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Discussion 1

a. Describe how you found the measure of $\angle POF$ in Part **d** of Exploration **1**.

b. Describe how you found the magnitudes of the horizontal and vertical components in Part **e** of Exploration **1**.

Mathematics Note

The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **m** is denoted by $\mathbf{m}_x$ (read "**m** sub x"), while its vertical component is denoted by $\mathbf{m}_y$.

For example, Figure **18** shows a graph of a vector **m** with its tail located at the origin, along with its components.
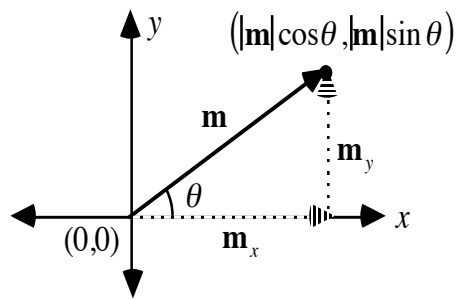

Figure 18: A vector and its components

Recall that the magnitude of vector **m** can be written as $|\mathbf{m}| = \sqrt{\left(\mathbf{m}_x\right)^2 + \left(\mathbf{m}_y\right)^2}$. Using trigonometric ratios, $\mathbf{m}_x = |\mathbf{m}| \bullet \cos \theta$ and $\mathbf{m}_y = |\mathbf{m}| \bullet \sin \theta$, where θ is an angle measured counterclockwise from the positive x-axis.

For example, Figure **19** shows a velocity vector **v** with a magnitude of 10 m/sec and a direction of 240° measured counterclockwise from the positive x-axis.
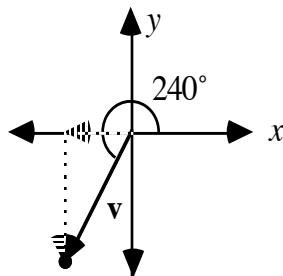
Figure 19: Vector v and its components

The components of vector **v** can be found as shown below:

$$\mathbf{v}_x = |\mathbf{v}| \bullet \cos \theta \qquad \mathbf{v}_y = |\mathbf{v}| \bullet \sin \theta$$
$$= 10 \bullet \cos 240° \qquad = 10 \bullet \sin 240°$$
$$= -5 \text{ m/sec} \qquad \approx -8.7 \text{ m/sec}$$

c. When a vector's tail is located at the origin, how are the coordinates of its tip related to its horizontal and vertical components?

d. Does locating a vector's tail somewhere other than at the origin affect its horizontal and vertical components? Explain your response.

e. The magnitude of a vector is always positive. The components of vector **v** in Figure **19**, however, are $v_x = -5$ m/sec and $v_y \approx -8.7$ m/sec. In this situation, what do the negative signs indicate?

f. A vector representing the flight of an airplane has a bearing of 260° and a magnitude of 355 km/hr. Describe how you would find the horizontal and vertical components of this vector.

Exploration 2

In Exploration **1**, you learned how to express any vector in terms of its horizontal and vertical components. In this exploration, you observe how these components can be used to add vectors algebraically.

a. Graph a vector **m** with its tail located at the origin of a two-dimensional coordinate system. Add another vector **n** to vector **m** using the tip-to-tail method. Graph the resultant vector **r**.

b. Determine the components of each vector in the sum $\mathbf{m} + \mathbf{n} = \mathbf{r}$. Record the value of each component in a table with headings like those in Table **1** below.

Table 1: Vectors and their components

Vector	Horizontal Component	Vertical Component
m		
n		
r		

c. **1.** Describe the relationship between the horizontal components of **m** and **n** and the horizontal component of **r**.

2. Describe the relationship between the vertical components of **m** and **n** and the vertical component of **r**.

d. Repeat Parts **a–c** for a resultant vector **r** that is the sum of three vectors **u**, **v**, and **w**.

Discussion 2

a. What relationship appears to exist between the components of the vectors in a sum and the components of the resultant vector?

b. Describe how you could use this relationship to determine a resultant vector without using the tip-to-tail method of vector addition.

c. Once the components of the resultant vector have been identified, how could you find its magnitude and direction?

Assignment

4.1 Describe the components of a vector with a magnitude of 75 km/hr and a bearing of 50°.

4.2 The forces represented by the following vectors, where direction is measured counterclockwise from the positive *x*-axis, are acting simultaneously on an object: 20 N at 45°, 15 N at 105°, 35 N at 200°, and 45 N at 300°.

   a. What is the sum of the *x*-components of these vectors?

   b. What is the sum of the *y*-components of these vectors?

   c. If the tail of the resultant vector is placed at the origin of a two-dimensional coordinate system, what would be the coordinates of its tip?

4.3 When graphed on a coordinate plane with their tails located at the origin, the tips of four vectors have the following coordinates: (4,5), (–4,13), (7,–4),and (–6,8), respectively. If the tail of the resultant vector is also located at the origin, what are the coordinates of its tip?

4.4 The components of a velocity vector **m** are $m_x = 15$ m/sec and $m_y = 20$ m/sec . What is the magnitude and direction of vector **m**?

4.5 In an orienteering competition, participants race across an unfamiliar course using a map and compass to navigate between checkpoints. During one race, your path can be described by the following displacement vectors: 350 paces at a bearing of 35°, 1250 paces at a bearing of 275°, and 1000 paces at a bearing of 140°. Use vector addition to describe your final change in position.
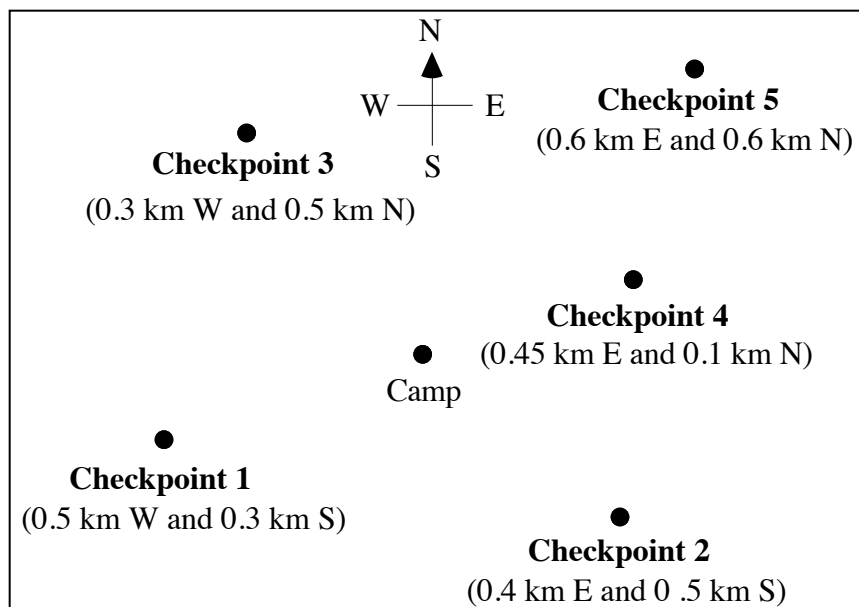
* * * * *

4.6   The forces represented by the following vectors, where direction is measured counterclockwise from the positive x-axis, are acting simultaneously on an object: 25 N at 65°, 30 N at 340°, and 10 N at 150°. What is the resultant force on the object?

4.7   The instructions in a treasure hunt include the following displacement vectors: 50 m at a bearing of 30°, 14 m at a bearing of 260°, 15 m at a bearing of 175°, and 25 m at a bearing of 105°.

   a. What is the sum of the x-components of these vectors?

   b. What is the sum of the y-components of these vectors?

   c. What are the magnitude and bearing of the resultant vector?

* * * * * * * * * *

1.   Imagine that you are a counselor at a summer camp. Your responsibilities include designing an orienteering course. The race begins at camp, proceeds to each of five checkpoints in numerical order, then returns to camp. At each checkpoint, campers should receive instructions on how to find the next checkpoint.

   The distances given in the diagram below are the distances from camp. Write a set of instructions for the campers that describes the distance and bearing from each point in the race to the next one.

N

W ─── E

S

● **Checkpoint 5**
(0.6 km E and 0.6 km N)

● **Checkpoint 3**
(0.3 km W and 0.5 km N)

● **Checkpoint 4**
(0.45 km E and 0.1 km N)

● Camp

● **Checkpoint 1**
(0.5 km W and 0.3 km S)

● **Checkpoint 2**
(0.4 km E and 0 .5 km S)

2.   An airplane leaves the airport at a speed of 240 km/hr with a bearing of 160°. To compensate for the wind, the pilot changes course to a bearing of 165° and a speed of 255 km/hr. What is the velocity of the wind?

3.   When an earthquake occurs, the movement generates two kinds of waves: primary waves and secondary waves. The speeds at which these waves travel through the earth differ by approximately 3.5 km/sec. With the help of seismographs, geologists can measure the difference in the arrival times of primary and secondary waves. By comparing the differences in arrival times at two or more locations, they can then determine the location of the quake's epicenter.

   In the state of Montana, there are seismographs in Missoula and Bozeman. If the seismograph in Missoula shows a difference in arrival times of 146 sec, while the one in Bozeman shows a difference of 122 sec, where is the epicenter of the earthquake? Use the map of Montana supplied by your teacher to justify your response.

- A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically represented by a bold, lowercase letter, such as vector **m**. In handwritten work, the same vector can be symbolized by $\vec{m}$. The magnitude of a vector **m** is denoted by |**m**|.

- **Displacement** is a change in position in a particular direction.

- The **velocity** of an object is its speed in a specific direction.

- **Force** is a push or a pull in a particular direction. The metric unit of force is the newton (N).

- **Equivalent vectors** have the same magnitude and the same direction.

- **Opposite vectors** have the same magnitude, but their directions differ by 180°.

- One way to add vectors is the **tip-to-tail method**. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.

- The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.

- A **scalar** is a real number.

- The operation k**m** denotes the **scalar multiplication** of **m** by the scalar k.

- The **law of cosines** states that the square of the length of any side of a triangle is equal to the sum of the squares of the lengths of the other sides minus twice the product of these lengths and the cosine of the included angle. In a triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

- The **law of sines** states that the lengths of the sides of a triangle are proportional to the sines of the opposite angles. In a triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **m** is denoted by $\mathbf{m}_x$ (read "**m** sub x"), while its vertical component is denoted by $\mathbf{m}_y$. Using trigonometric ratios, $\mathbf{m}_x = |\mathbf{m}| \cdot \cos\theta$ and $\mathbf{m}_y = |\mathbf{m}| \cdot \sin\theta$, where θ is a directional angle measured counterclockwise from the positive x-axis.

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