It's All in the Family

When trying to fit a model to a set of data, it can help you to know the characteristics of different types of mathematical functions. In this module, you examine and compare the graphs of several families of functions.

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Introduction

Biologists and botanists have established a classification system that can be used to describe any animal or plant. In this system, organisms with common characteristics are grouped together. For example, crocodiles, lizards, turtles, and snakes belong to the class Reptilia. These animals are all classified as reptiles based on, among other things, the fact that their bodies are covered with scales or bony plates.

As the number of shared characteristics increase, animals can be placed in more restrictive categories, such as orders and families. For example, the several different kinds of tropical crocodiles all belong to the family Crocodilidae.

Relationships among mathematical functions also can be described in terms of shared characteristics or common behaviors. In this module, you investigate some families of functions.

Exploration

Figure **1** shows a bicycle tire with a pebble stuck in its tread. If the tire is rolled to the right on a flat surface, so that the pebble rotates in a clockwise fashion, what would the path of the pebble look like?

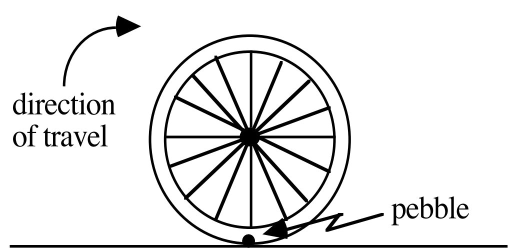

Figure 1: Bicycle tire with pebble

a. Use a cardboard or plastic disk to simulate a bicycle tire.

b. Cut a small notch in the disk and place a piece of chalk in the notch as shown in Figure **2**.

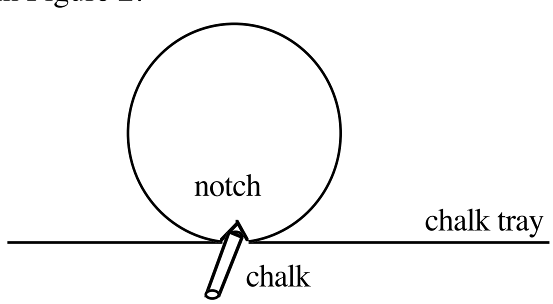

Figure 2: Disk with notch and chalk

c. Place the disk in the chalk tray of a blackboard, with the chalk at the lowest point of the disk and the end of the chalk resting against the board. Holding the chalk carefully in the notch, roll the disk clockwise along the tray. The chalk should trace a path that simulates the path of a pebble imbedded in a tire. Roll the disk until the chalk has completed two full revolutions.

d. To observe how the tire's radius affects the path of the pebble, repeat Parts **b** and **c** using a disk with a radius half that of your original disk.

Discussion

a. Do you think that the curves sketched in the exploration could represent the graphs of functions? Explain your response.

b. In the module "Can It," you used the mathematical terms *period* and *amplitude* to describe circular functions. Use these terms to describe the graphs from the exploration.

c. What characteristics would you use to describe the family of functions which includes the graphs from the exploration?

d. Would you place these graphs in the same class of functions as the sine and cosine functions? Explain your response.

Mathematics Note

A **family** of functions is a set of functions that have a common **parent**. Each family member is generated by performing one or more transformations on the parent function.

For example, consider the family of sine functions, a subset of the periodic functions. The parent function of the family of sine functions is $y = \sin x$. This relationship can be illustrated using the Venn diagram shown in Figure **3**.

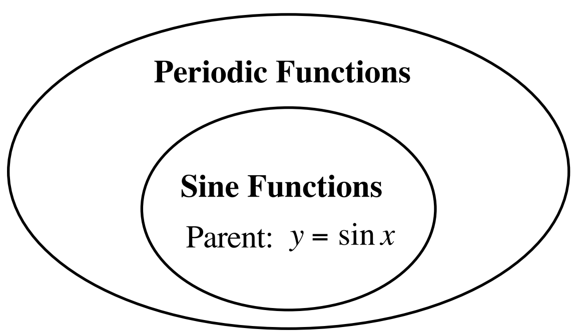

Figure 3: Venn diagram of periodic functions

The family of sine functions has an infinite number of members, all of which represent transformations of $y = \sin x$. Three of these functions are $y = 2\sin x$, $y = \sin x + 3$, and $y = \sin(x - \pi)$.

Activity 1

In the module "Can It," you modified the shapes and locations of graphs of the sine and cosine functions to model real-world data. In this activity, you observe how these modifications affect the graphs of several other families of functions.

Exploration

a. Consider the parent function $y = 2^{-x^2}$. Use a graphing utility to create a graph of this function.

b. Sketch a copy of the graph on a sheet of graph paper.

c. On the same coordinate system as in Part **b**, sketch a graph of the family member that results when the parent function is translated horizontally 1 unit to the right.

d. Use a graphing utility to determine which of the following equations represents the family member you sketched in Part **c**.

1. $y = 2^{-x^2} + 1$

2. $y = 2^{-x^2} - 1$

3. $y = 2^{-(x-1)^2}$

4. $y = 2^{-(x+1)^2}$

e. Predict the equation of the family member that results when the parent function is translated 2 units to the left. Verify your equation using a graphing utility.

f. Compare the graphs of each of the following pairs of functions:

1. $y = 2^{-x^2}$ and $y = 2^{-(3x)^2}$

2. $y = 2^{-x^2}$ and $y = 3 \bullet 2^{-x^2}$

g. Use a graphing utility to investigate the transformations of the parent function created by each of the following forms. In each case, experiment with both negative and positive values for the constant.

1. $y = 2^{-x^2} + d$

2. $y = 2^{-(x-c)^2}$

3. $y = 2^{-(bx)^2}$

4. $y = a \bullet 2^{-x^2}$

Discussion

a. The transformations you examined in the exploration can be applied to many other families of functions. For example, consider the family based on the parent function $y = x^2$.

1. What are the domain and range of the function $y = x^2$?
2. How would you modify this equation to obtain a function that is a vertical translation of the parent by 2 units?
3. How do the domain and range of the transformed function compare with those of the parent?

b.

1. How would you modify the equation $y = x^2$ to obtain a function that is a horizontal translation of the parent by 2 units?
2. How do the domain and range of the transformed function compare with those of the parent?

c.

1. In a function of the form $y = a \bullet x^2$, what transformation of the parent function $y = x^2$ results when $a = -1$?
2. How do the domains and ranges of the two functions compare?

d.

1. How would you modify the equation $y = x^2$ to obtain a function that is a vertical "stretch" of the parent?
2. Select a specific vertical stretch of the parent. How do the domains and ranges of the two functions compare?

e.

1. How would you modify the equation $y = x^2$ to obtain a function that is a horizontal "stretch" of the parent?
2. Select a specific horizontal stretch of the parent. How do the domains and ranges of the two functions compare?

Mathematics Note

For any point (x,y) of a function $f(x)$, adding a nonzero real number d to y results in an image point $(x,y+d)$. When this occurs, the graph is translated d units vertically. The equation of the resulting graph is $y = f(x) + d$.

For any point (x,y) of a function $f(x)$, subtracting a nonzero real number c from x results in an image point $(x-c,y)$. When this occurs, the graph is translated c units horizontally. The equation of the resulting graph is $y = f(x-c)$.

For any point (x,y) of a function $f(x)$, multiplying y by a nonzero real number a results in an image point (x,ay). When this occurs, the graph appears to be vertically "stretched" or "shrunk," depending on the value of a. The equation of the resulting graph is $y = a \bullet f(x)$.

For any point (x,y) of a function $f(x)$, multiplying x by a nonzero real number b results in an image point (bx,y). When this occurs, the graph appears to be horizontally "stretched" or "shrunk," depending on the value of b. The equation of the resulting graph is

$$y = f\left(\frac{1}{b}x\right).$$

For example, the graphs in Figure **4** show four transformations of the parent $f(x) = \sin x$.

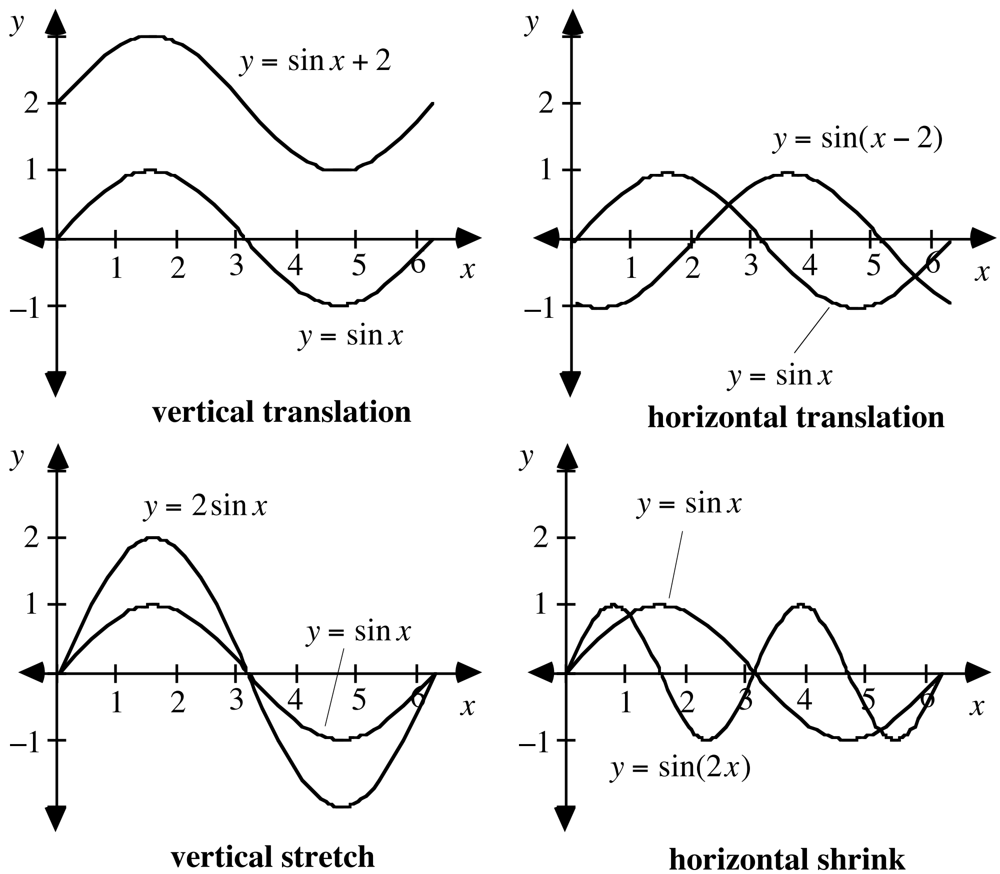

Figure 4: Four transformations of $y = \sin x$

f. As mentioned in the mathematics note, multiplying each x in an ordered pair (x,y) of a function $f(x)$ by a nonzero real number b results in a new function whose points are of the form (bx,y). Select a function f and use ordered pairs to show why the equation of the new function is

$$y = f\left(\frac{1}{b}x\right).$$

Assignment

1.1 **a.** **1.** Sketch a graph of the parent function $y = \cos x$.

2. Identify the domain and range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is reflected in the x-axis.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is translated horizontally 1 unit to the left.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

d. **1.** Sketch a graph of the family member that results when every y-value of the parent function is multiplied by 3.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

e. **1.** Sketch a graph of the family member that results when every x-value of the parent function is multiplied by $1/2$.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.2 **a.** **1.** Sketch a graph of the parent function $y = 1/x$ over the domain of nonzero real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is translated –3 units vertically.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is reflected in the x-axis.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.3 **a.** **1.** Sketch a graph of $y = 3^x$ over the domain of real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the *y*-values of the parent function are multiplied by 0.25.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when the parent function is translated –5 units horizontally.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

1.4 **a.** **1.** Sketch a graph of $y = x^3$ over the domain of real numbers.

2. Describe the range of the function.

b. **1.** Sketch a graph of the family member that results when the parent function is translated –4 units horizontally.

2. Determine the domain and range of the resulting function.

3. Write the equation of the new graph.

c. **1.** Sketch a graph of the family member that results when every *x*-value of the parent function is multiplied by 10.

2. Determine the domain and range of the resulting function.

3. Write the equation for the new function.

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1.5. **a.** Sketch a graph of the parent function $y = \log x$.

b. Sketch a graph of the parent function translated 5 units horizontally.

c. Sketch a graph of the family member that results when the *x*-values of the parent function are multiplied by $1/3$.

1.6 **a.** Sketch a graph of the parent function $y = \sqrt{x}$.

b. Sketch the graph of the parent translated 3 units vertically.

c. Sketch a graph of the family member that results when every *y*-value of the parent function is multiplied by 0.2.

1.7 **a.** Sketch a graph of the parent function $y = |x|$.

b. Sketch the graph of the family member that results when the *x*-values of the parent function are multiplied by 0.25.

c. Sketch the graph of the family member that results when the parent function is translated 4 units horizontally.

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Research Project

The terms *pitch*, *frequency*, *period*, and *amplitude* are associated with both the properties of sound waves and a particular family of functions. If you have access to an oscilloscope, record the graphs of some musical notes. Prepare a brief summary of your observations for the class, including answers to the following questions:

a. What does the graph of a musical note look like?

b. How does raising the pitch of a sound affect the period of its graph?

c. How does the intensity of a sound affect the amplitude of its graph?

Activity 2

In Activity **1**, you performed transformations on a parent function to obtain new family members, then determined the equations of the resulting functions. In this activity, you use the equations of family members to determine the transformations performed on the parent.

Exploration

Table **1** shows two parent functions along with seven members of their corresponding families. Complete Parts **a** and **b** for the functions in each column.

a. Predict how the graph of each transformed function in Table **1** differs from the graph of its parent function.

Table 1: Parent functions and family members

	Parent $y = 1/x$	Parent $y = x^2$
1.	$y = \dfrac{1}{x-3}$	$y = (x-3)^2$
2.	$y - 2 = 1/x$	$y - 2 = x^2$
3.	$y = -(1/x)$	$y = -(x^2)$
4.	$y - 4 = -\left(\dfrac{1}{x+1}\right)$	$y - 4 = -(x+1)^2$
5.	$y = \dfrac{1}{2x}$	$y = (2x)^2$
6.	$y = \dfrac{1}{(0.5x)}$	$y = (0.5x)^2$
7.	$y + 1 = -\left(\dfrac{1}{3(x-2)}\right)$	$y + 1 = -\left(3(x-2)^2\right)$

b. Graph each transformed function in Table **1** on the same set of axes as its parent. Record your observations.

c. Compare your observations for the transformations of $y = 1/x$ to the transformations of $y = x^2$.

d. Record any generalizations that appear to apply to both families of functions.

Discussion

a. **1.** Given an unknown function $y = f(x)$, how would you write the equation of the function whose graph represents $y = f(x)$ translated 3 units horizontally and –5 units vertically?

2. How would you write the equation of the function whose graph represents $y = f(x)$ reflected in the x-axis, stretched vertically by 4, and stretched horizontally by 2?

b. Consider an equation of the form $(y - d) = f(b(x - c))$. How does each of the constants b, c, and d transform the graph of $y = f(x)$?

Assignment

2.1 Describe how the graph of $y - 5 = 3(x - 2)$ is related to the graph of $y = x$.

2.2 In Parts **a–c** below, you determine how the transformation rules developed in the discussion apply to a family of exponential functions based on a parent function of the form $y = a^x$.

a. Select a parent for a family of exponential functions and sketch its graph.

b. Determine the equation of the function that results from the following transformations of the parent: a translation 6 units vertically, a translation –3 units horizontally, and a reflection in the x-axis.

c. Verify your equation in Part **b** using a graphing utility.

2.3 For each equation below, write the equation of a possible parent function and describe the corresponding transformations from the parent.

a. $y = \sin\left[3\left(x - \frac{\pi}{2}\right)\right]$

b. $y = 2x^3 - 90$

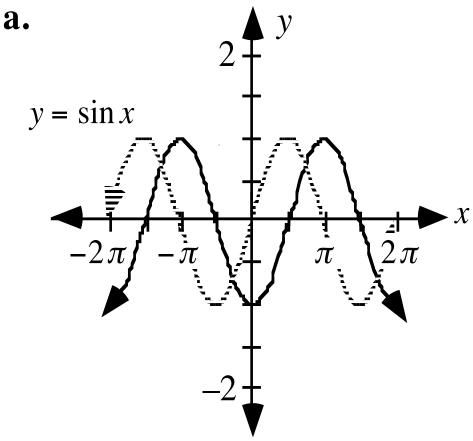
c. $y = 10^{(x-8)}$

d. $y = \dfrac{-100}{(x+100)^2}$

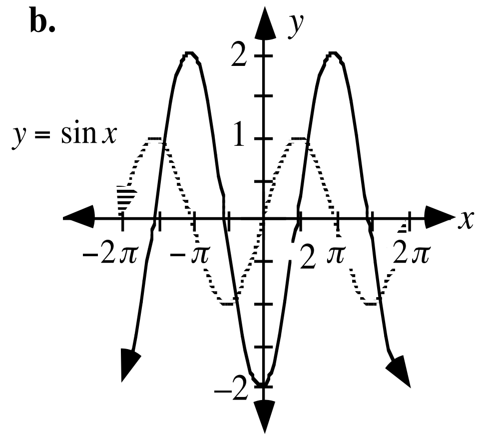
e. $y = \log(-x + 2.3) - 4.7$

2.4 On each set of axes below, the graph of the parent function $y = \sin x$ is represented by a dotted line, while the graph of the transformed function is represented by a solid line. Write an equation for the graph of each family member in terms of the parent function.

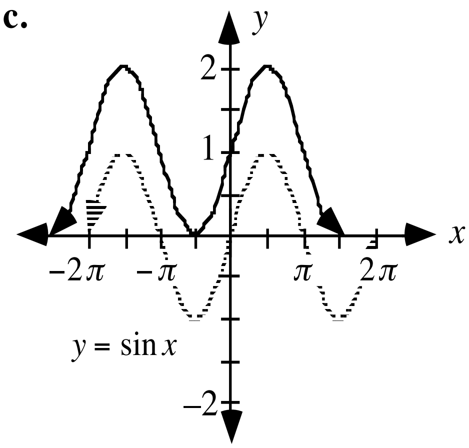
a.



b.



c.



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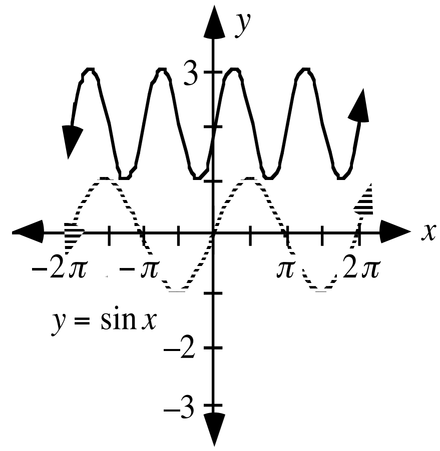
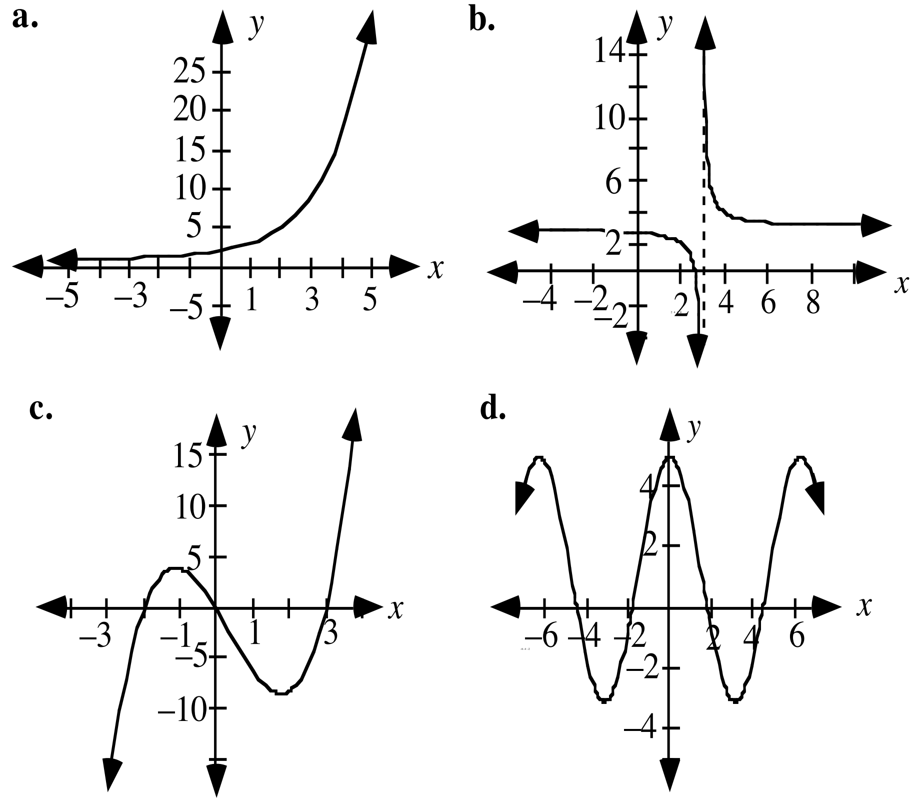


2.5 A horizontal translation of a periodic function is often referred to as a **phase shift**.

a. Are there any phase shifts in the graphs for Problem **2.4**? If so, describe the magnitude and direction of each one.

b. If the function $y = 3\cos x$ undergoes a phase shift of $-\pi/6$ radians, what is the equation for the transformed function?

2.6 Write a possible equation for the function in each of the following graphs and identify its corresponding parent.



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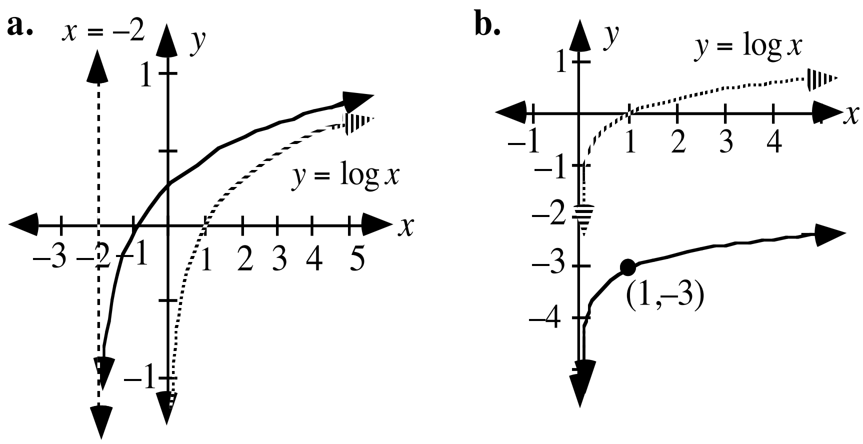
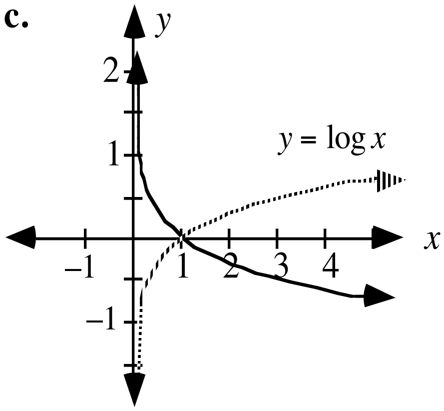
2.7 List at least five members of the family of second-degree polynomials. Describe how each one is related to the parent function.

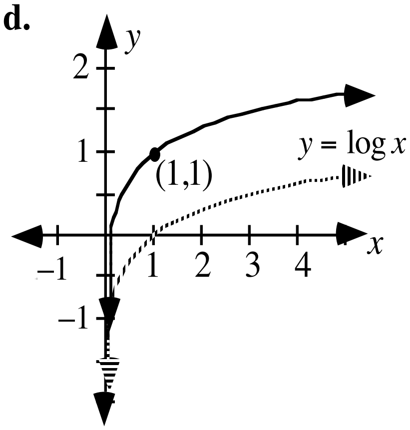
2.8 On each set of axes below, the graph of the parent function $y = \log x$ is represented by a dotted line, while the graph of the transformed function is represented by a solid line. Write an equation for the graph of each family member in terms of the parent function.



c.



d.



2.9 Describe how the constants a, b, c, and d transform the graph of each parent function below.

	Parent Function	Family Member
a.	$y = \tan x$	$(y - d) = a\tan[b(x - c)]$
b.	$y = 2^x$	$(y - d) = a \bullet 2^{(b(x-c))}$
c.	$y = \lvert x \rvert$	$(y - d) = a \bullet \lvert b(x - c) \rvert$

2.10 Write the equation of a parent function, if any, that illustrates each of the following characteristics:

a. has a vertical asymptote

b. has a horizontal asymptote

c. can be drawn without lifting a pencil

d. is periodic

e. has a discontinuity

f. contains the origin.

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Activity 3

Your knowledge of families of functions may prove especially helpful when modeling data sets. Selecting the best function with which to model a data set can be a difficult task. Any finite number of data points, for example, can be modeled by an infinite number of polynomial functions of different degrees. Although each of these polynomial functions may fit the points perfectly, they may not prove useful for making predictions.

A good model must do more than give a reasonable approximation of the data. It should also provide some insight into the relationships that exist in the real-world situation. For example, Figure **5** below shows two curves, $f(x)$ and $g(x)$, that fit a set of data points. One of the curves is a graph of a fifth-degree polynomial function, the other is the graph of a linear function. Which is the better model? That depends on the situation in which the data was collected.

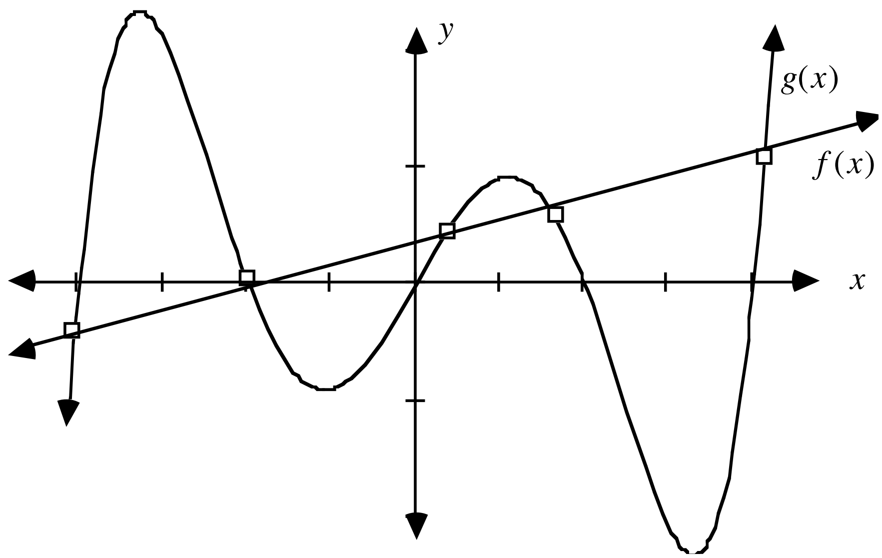

Figure 5: Two models for a set of data

Exploration

In this exploration, you investigate the relationship between light intensity and distance.

a. Obtain the following equipment from your teacher: a lamp with a 40-watt bulb, a science interface device with light sensor, and a meterstick.

b. As shown in Figure **6**, place the lamp on a flat surface. Position the meterstick 30 cm from the lamp.

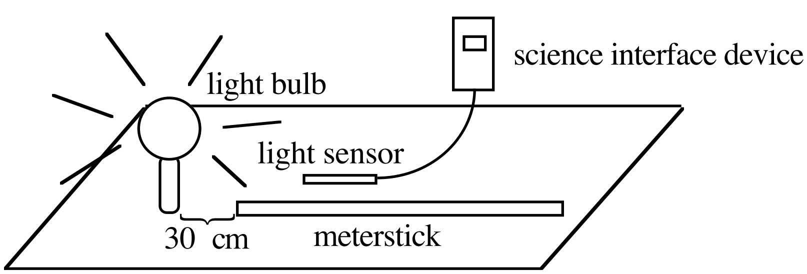

Figure 6: Lamp experiment

Darken the room. For best results, the room should contain no sources of light other than the lamp.

c. 1. Hold the sensor 30 cm from the lamp. Record both the distance from the lamp and the light intensity.

2. Move the sensor 5 cm farther from the lamp. Record both the distance and the intensity.

3. Repeat Step **2** until you have collected at least 15 data points.

d. Make a scatterplot of the data.

Discussion

a. Describe the shape of the scatterplot you created in Part **d** of the exploration.

b. What group of functions might be used to model this data? Explain your response.

c. Identify a possible parent function for a model. Defend your choice.

Assignment

3.1 The following table shows the frequencies (in beats per minute) of some musical notes. The first note in the table is concert A, the note an orchestra tunes to at the beginning of a concert.

Note	Frequency (beats/min)
A	440
B	494
C	523
D	587
E	659
F	698
G	784
A	880
B	987
C	1109
D	1245

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.2 The following data was collected during an experiment in which a ball was rolled up an inclined ramp. The ball eventually came to a stop, then rolled back down the incline. The distances in the table below were measured from a sonar range finder at the top of the ramp.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	2.1	1.4	0.9
0.2	2.1	1.6	1.0
0.4	1.9	1.9	1.1
0.6	1.5	2.2	1.6
0.8	1.2	2.4	2.0
1.0	1.0	2.6	2.2
1.2	0.9	2.8	1.3

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.3 Doctors use radioactive tracers to detect some diseases and injuries in patients. Since radioactive materials decay over time, the amount of tracer left in the patient decreases each day. The following table shows the percentage of tracer remaining at the end of each day.

Day	Percentage Remaining	Day	Percentage Remaining
0	100	5	33
1	81	6	26
2	65	7	21
3	51	8	16
4	41	9	13

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.4 The tide in the area near Boston, Massachusetts, varies by approximately 2.9 m from high tide to low tide. The tide changes from high to low and back to high approximately every 12.4 hr. In the following table, height represents distance in meters above low tide.

Time (hr)	**Height (m)**	**Time (hr)**	**Height (m)**
0	2.9	7	0.1
1	2.7	8	0.6
2	2.2	9	1.2
3	1.5	10	2.0
4	0.8	11	2.6
5	0.3	12	2.9
6	0.0	13	2.8

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

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3.5 The following data was collected as a pendulum swung back and forth in front of a motion detector.

Time (sec)	**Distance (m)**	**Time (sec)**	**Distance (m)**
0.0	1.393	2.4	1.856
0.4	1.145	2.8	1.838
0.8	0.851	3.2	1.549
1.2	0.682	3.6	1.308
1.6	0.859	4.0	0.841
2.0	1.328		

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.6 The table below shows some information on the eggs of 12 birds.

Type of Egg	Length (cm)	Mass (g)
hummingbird	1.3	0.5
black swift	2.5	3.5
dove	3.2	6.4
partridge	3.0	8.7
Arctic tern	4.2	18.0
grebe	4.3	19.7
Louisiana egret	4.5	27.5
mallard duck	6.2	80.0
great black-backed gull	7.6	111.0
Canada goose	8.9	197.0
condor	11.0	270.0
ostrich	17.0	1400.0

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

3.7 The data in the table below was collected during an experiment in which 11 mL of basic solution were added to an acidic solution, 1 mL at a time.

Basic Solution Added (mL)	pH of Solution
1	3.57
2	4.98
3	5.54
4	5.79
5	5.97
6	6.06
7	6.16
8	6.25
9	6.33
10	6.38
11	6.44

a. Create a graph of the data.

b. Identify a family of functions that could be used to model the data.

c. Determine a function that models the data reasonably well.

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Summary Assessment

1. Explain why it is possible to consider $y = \sin x$ as the parent function of $y = \cos x$.

2. In almost every presidential election since 1936, there has been at least three candidates on the ballot: one Democrat, one Republican, and at least one representative from a third party. The table below shows the number of votes (in thousands) that candidates from each party won in the presidential elections from 1976 to 1992.

Elections Since 1976	Democratic Party	Republican Party	Other Major Parties
1	40,831	39,148	910
2	35,484	43,904	6,641
3	37,577	54,455	307
4	41,809	48,886	649
5	44,909	39,104	20,034

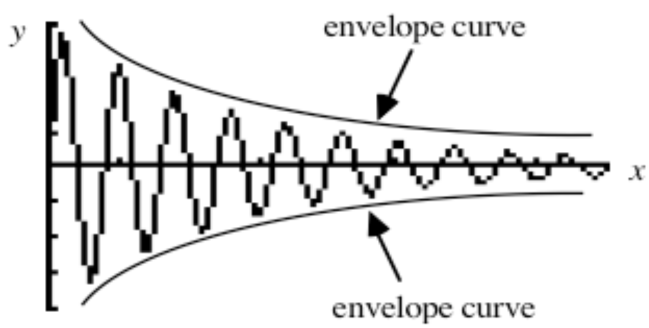
a. Create a separate scatterplot of the data for each political party.

b. Determine the equation of a function that could be used to model each data set. Graph each equation on the same coordinate system as the corresponding scatterplots from Part **a**.

c. Using your equations from Part **b**, predict which party will win the next presidential election, then use the same models to predict the fate of each party in future elections.

d. Write a paragraph describing the dangers of using your models to predict the outcomes of presidential elections.

3. Musical instruments like the piano cannot sustain notes for long periods of time. For example, the graph below represents the sound wave of a single note that was played then allowed to fade. The two curves tangent to the graph at its maximum and minimum values represent the **envelope** of the graph. What pair of equations might describe these envelope curves?



Module Summary

- A **family** of functions is a set of functions that have a common **parent**. Each family member is generated by performing one or more transformations on the parent function.
- For any point (x,y) of a function $f(x)$, adding a nonzero real number d to y results in an image point $(x,y+d)$. When this occurs, the graph is translated d units vertically. The equation of the resulting graph is $y = f(x) + d$.
- For any point (x,y) of a function $f(x)$, subtracting a nonzero real number c from x results in an image point $(x-c, y)$. When this occurs, the graph is translated c units horizontally. The equation of the resulting graph is $y = f(x-c)$.
- For any point (x,y) of a function $f(x)$, multiplying y by a nonzero real number a results in an image point (x,ay). When this occurs, the graph appears to be vertically "stretched" or "shrunk," depending on the value of a. The equation of the resulting graph is $y = a \bullet f(x)$.
- For any point (x,y) of a function $f(x)$, multiplying x by a nonzero real number b results in an image point (bx, y). When this occurs, the graph appears to be horizontally "stretched" or "shrunk," depending on the value of b. The equation of the resulting graph is

$$y = f\left(\frac{1}{b}x\right)$$

- The horizontal translation of a periodic function is often referred to as a **phase shift**.

Selected References

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