Transmitting Through Conics

What does watching television have to do with conic sections? In this module, you'll see how conics influence your favorite telecasts—from signal to satellite dish.

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Introduction

Conic sections are geometric figures that play important roles in satellite, radio, and microwave communications. In this module, you investigate how conic sections can be described both geometrically and algebraically.

Exploration

In this exploration, you use a pencil to visualize a cone.

a. Hold a pencil at its midpoint between your thumb and finger, as illustrated in Figure **1**. While keeping the midpoint of the pencil stationary, move the pencil so that its tip draws a circle.

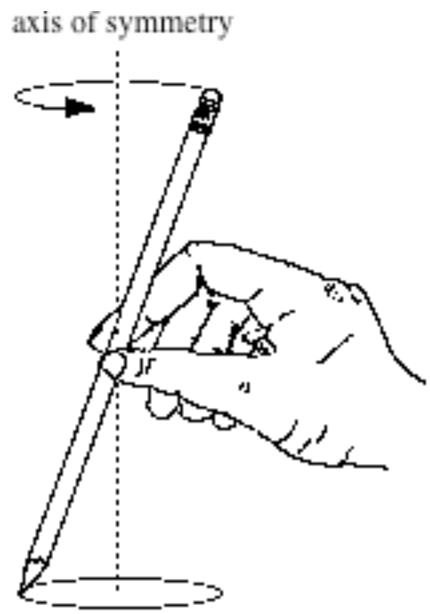

Figure 1: Pencil drawing a circle

b. Note the positions of the pencil as its tip draws a circle. Record your observations.

> ### Mathematics Note
> The set of all points that satisfy a given geometric condition is a **locus** (plural **loci**). For example, a circle is the locus of all points in a plane that are a given distance, the radius, from a fixed point, the center.
>
> An equation defining a locus is satisfied by the coordinates of the points belonging to the locus and by no other points. In a coordinate plane, for example, the locus of the equation $y = 2x + 3$ is the set of all points on a line with a slope of 2 and a y-intercept of 3.

Discussion

a. Describe the locus of points traced by the tip of the pencil. How is this locus related to the one traced by the other end of the pencil?

b. As the pencil's tip traces a circle, describe the locus of the pencil's midpoint. This is the **apex** of the cone.

c. The pencil in the exploration represents a line segment. Describe the locus of points generated by a line moving in the same way as this pencil.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a cone. In a right circular cone, the conic section formed depends on the slope of the intersecting plane. The intersection may be a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**, as shown in Figure **2** below.

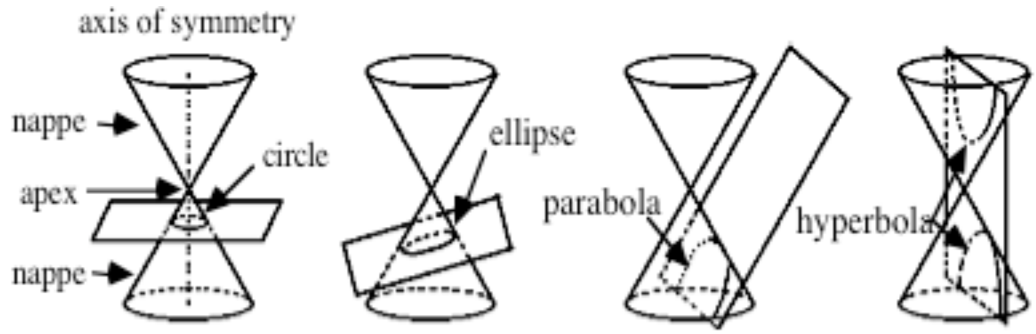

Figure 2: Conic sections

To visualize the four conic sections, imagine a plane whose slope gradually changes as it slices a cone. When the plane is perpendicular to the cone's axis of symmetry and intersects the cone in more than one point, the intersection is a circle. As the slope of the plane gradually changes, the intersection is an ellipse. When the plane is parallel to a line generating the cone, a parabola is formed. When the plane intersects both nappes, a hyperbola is formed.

d. Describe some familiar items that contain objects shaped liked conic sections.

e. How would you describe the shapes of the conic sections to someone who had never seen them before?

f. When the intersection of a plane and a cone contains the cone's apex, geometric figures other than a circle, an ellipse, a parabola, or a hyperbola are formed. These other intersections are **degenerate conic sections**.

There are three degenerate conic sections. Describe the shape of each one and the location of the plane relative to the cone.

Activity 1

In this activity, you investigate the locus of points that forms a circle.

Exploration

In this exploration, you use the Pythagorean theorem to develop an equation that describes a circle.

a. Select a point at an intersection of grid lines as the center of the circle.

b. Select a distance for the radius, r. Draw the locus of points that lie this distance from the center of the circle.

c. As you may recall from previous modules, the distance d between two points (x_1, y_1) and (x_2, y_2) can be found using a formula derived from the Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Use the distance formula to determine an equation for the circle you drew in Part **b**.

d. Repeat Parts **a–c** four more times, using a different radius for each circle and placing the center of each circle in a different quadrant.

Discussion

a. Consider a circle with center at (h,k) and a radius of r. Given an arbitrary point (x,y) on the circle, how could you use the distance formula to write an equation for the circle?

b. To write an equation without a radical, you must isolate the radical on one side of the equation, then square both sides of the equation. For example,

$$\sqrt{(x^2 + y^2)} = 4$$

$$\left(\sqrt{(x^2 + y^2)}\right)^2 = 4^2$$

$$x^2 + y^2 = 16$$

Square both sides of the equation you wrote in Part **a** of the discussion.

Mathematics Note

The **standard form** of the equation of a circle with center at (h,k) and radius r is:

$$(x-h)^2+(y-k)^2=r^2$$

For example, the standard form of the equation for a circle with center at (–2,4) and a radius of 5 is $(x+2)^2+(y-4)^2=25$.

c. How does the standard form of the equation of a circle compare to the equation you determined in Part **a** of the discussion?

d. How could you determine whether or not any point in the coordinate plane is a point on a given circle?

e. If $x^2=16$, is it always true that $\sqrt{x^2}=4$? Justify your response.

f. Some graphing utilities graph only functions. Since a circle does not represent a function, its graph may have to be produced by graphing two functions that represent halves of the circle.

1. To find these two functions for the circle described by $x^2+y^2=16$, for example, the equation can be solved for y. Identify these two functions.
2. Determine the appropriate domain and range for each function in Step **1**.
3. Do you believe that a combined graph of the two functions in Step **1** includes all the points of the circle defined by $x^2+y^2=16$? Explain your response.

Assignment

1.1 Write the equation in standard form of each of the following:

a. a circle with center at the origin and a radius of 3

b. a circle with center at (–1.3,8.9) and a radius of $\sqrt{21}$

c. a circle with center at (a,b) and a radius of $\sqrt{c}$.

1.2 Radio signals may be thought of as concentric circles (circles with the same center) emitted from a transmitter. Write the equations of three concentric circles.

1.3 Identify the centers and radii of the circles described by the following equations:

a. $(x-30)^2+(y-120)^2=289$

b. $(x+21)^2+(y-73)^2=141$

c. $x^2+y^2=121$

1.4 **a.** For each circle described by an equation in Problem **1.3**, list the coordinates of:

1. a point on the circle

2. a point inside the circle

3. a point outside the circle.

b. When the coordinates of a point outside a circle are substituted into its equation in standard form, what must be true of the result?

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1.5 The center of a circle is located at (–2,–8). The coordinates of one point on the circle are (2,6). Find the equation of the circle in standard form.

1.6 Zhang listens to radio station KIZY, 105.3 on the FM dial. Zhang's home is located 32 km north and 75 km east of the station's transmitter, on the edge of KIZY's maximum broadcast range.

a. Determine the distance traveled by the signal when it is received by Zhang's home radio and make a sketch of the station's listening area.

b. If the radio station has coordinates (0,0), find the equation that represents the locus of KIZY's maximum broadcast range.

c. Determine the approximate coordinates of four locations—other than Zhang's home—that lie on the locus of KIZY's maximum broadcast range.

1.7 A new radio station, KZME, is building a transmitter 23 km east and 57 km north of Zhang's house.

a. Determine the location of KZME relative to KIZY.

b. Zhang's home also happens to be on the edge of KZME's maximum broadcast range. Determine the distance traveled by the signal when it is received by Zhang's home radio.

c. Determine an equation that represents the locus of KZME's maximum broadcast range.

d. Find the approximate coordinates of two locations other than Zhang's home that lie on the locus of KZME's maximum broadcast range.

e. Make a sketch of the listening areas for both KZME and KIZY.

f. Estimate the coordinates of the points where the two maximum broadcast ranges intersect.

g. Describe the location of Zhang's home relative to the intersection points of the two maximum broadcast ranges.

h. What does the region formed by the intersection of the two listening areas represent?

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Activity 2

In 1609, German astronomer Johannes Kepler (1571–1630) hypothesized that the planets in our solar system orbit the sun in paths that are not circular. His theory provided the best explanation for years of observations and revolutionized the science of astronomy. In this activity, you examine the characteristics of planetary orbits.

Exploration 1

In this exploration, you construct various loci using a piece of string and a pencil. You then examine how the length of string is related to a particular locus.

a.

1. Obtain a piece of string at least 15 cm long. Tie a knot at each end of the string.
2. Tape a sheet of paper to a sheet of cardboard. Near the center of the paper, stick a tack through both knots and into the cardboard.
3. Place the tip of a pencil inside the loop formed by the string. Use the pencil to pull the string taut, then place the pencil's tip on the paper, as shown in Figure **3**. Keeping the string taut, use the pencil to trace a locus of points.

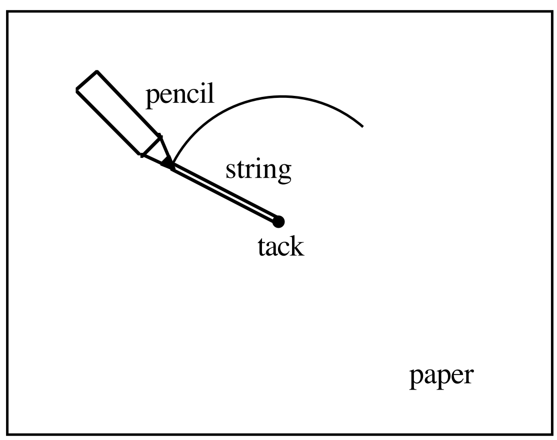


Figure 3: Construction of a locus of points using one tack

4. Describe the figure defined by the locus of points. Observe how the length of the string determines the size of the figure.

b.

1. Remove the tack and the paper from the cardboard. Tape another sheet of paper to the cardboard. Stick one tack into one of the knots in the string and another tack into the second knot. Stick the two tacks into the cardboard so that the distance between them is less than the length of the string.

2. Use the pencil to pull the string taut and place the pencil's tip on the paper, as shown in Figure **4**. Keeping the string taut, use the pencil to trace a locus of points until the tacks interfere with the construction of the locus. When this occurs, lift the pencil and string to the other side of the tacks and continue creating the locus of points. The locus is complete when the pencil begins retracing its path.

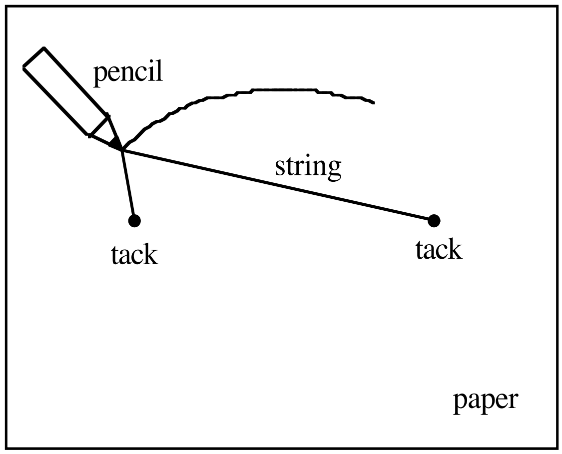


Figure 4: Construction of a locus using two tacks

c. The locus of points created in Part **b** is an **ellipse**. The location of each tack is a **focus** (plural **foci**) of the ellipse. Move the pencil to a location that is equidistant from both foci.

When the pencil is in this location, what shape is formed by the string and the segment joining the foci?

d.
1. Remove the tacks and label the foci F_1 and F_2.
2. The midpoint of the segment joining the foci is the **center** of an ellipse. To find the center of the ellipse drawn in Part **b**, construct the perpendicular bisector of $\overline{F_1F_2}$. Label the center of the ellipse O, as shown in Figure **5**.

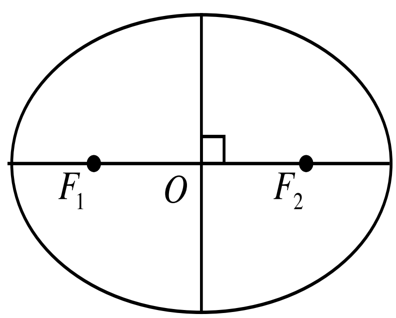


Figure 5: An ellipse, its center, and its foci

3. Compare the length of the segment that contains the foci and whose endpoints are on the ellipse to the distance between the knots in the string.

e. **1.** Remove your drawing of an ellipse from the cardboard and tape down another sheet of paper. Position the two tacks so that the distance between them is different than the distance in Part **b**.

2. Repeat Parts **b–d**. Describe the changes in the ellipse produced by moving the tacks.

Discussion 1

a. **1.** Describe the locus of points created using one tack in Part **a** of Exploration **1**.

2. Describe the lines of symmetry for this locus of points.

b. How is the distance between the knots in the string related to the size of the locus of points created using one tack?

Mathematics Note

An **ellipse** is a locus of points in a plane such that the sum of the distances from two fixed points, the foci, is a constant.

For example, Figure **6** shows an ellipse with its center at the origin of a two-dimensional coordinate system. Point P is on the ellipse because $d_1 + d_2 = 2a$, where $2a$ is a constant.

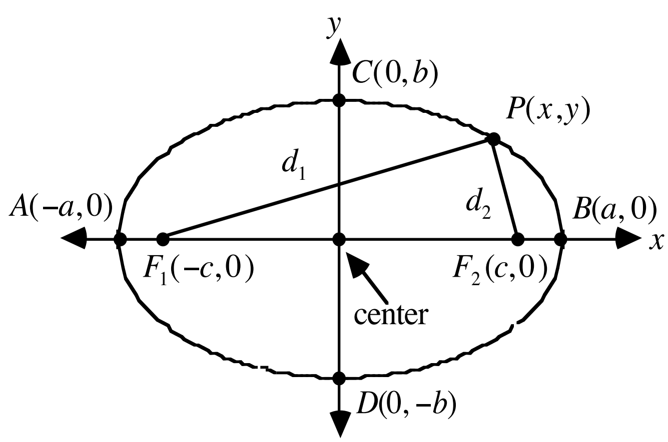


Figure 6: An ellipse and its foci

In Figure **6**, the points $F_1(c, 0)$ and $F_2(-c, 0)$ are the **foci** of the ellipse. Points A, C, B, and D are the **vertices** of the ellipse. The **major axis** of the ellipse is $\overline{AB}$; its length is $2a$. The **minor axis** is $\overline{CD}$, which has a length of $2b$. The major axis is the longer of the two and always contains the foci. The intersection of the major and minor axes is the **center** of the ellipse.

c. How does the locus of points created in Part **b** of Exploration **1** satisfy the definition of an ellipse given in the mathematics note?

d. Describe the symmetries found in an ellipse.

e. **1.** Describe the distance between the knots in the string in Part **b** of Exploration **1** in terms of a in Figure **6**.

2. Assuming that the distance between the knots in the string remains unchanged, how does the distance between the tacks affect the shape of the ellipse created?

f. In Part **c** of Exploration **1**, you placed the pencil at the location indicated by point P in Figure **7**. At this point, the distance from P to each focus is the same. What equation describes OP in terms of PF_1 and OF_1?

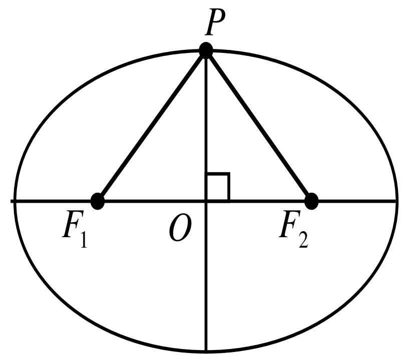


Figure 7: A point *P* on an ellipse

g. If P is any point on the ellipse, how does the distance between the knots in the string compare to $PF_1 + PF_2$?

h. As mentioned in the mathematics note, the sum of the distances from the foci to a point P is the constant $2a$, the length of the major axis. Using this fact, along with the distance formula, determine an equation for the ellipse in Figure **6**.

Exploration 2

In Exploration **1**, you created circles and ellipses using a piece of string, tacks, and a pencil. In one sense, it can be argued that an ellipse is formed when the segment between two tacks at the center of a circle is "stretched." In this exploration, you examine how this "stretching" transformation can be used to find another form of the equation for an ellipse.

a. Write an equation for a circle with its center at the origin.

b. In the Level 4 module "It's All in the Family," you observed that a function can be stretched horizontally by m if each point in the function is transformed from (x,y) to (mx,y), where $m \neq 0$. The equation which results in this transformation can be found by replacing x with x/m in the original equation.

Similarly, a function can be stretched vertically by n if each point in the function is transformed from (x,y) to (x,ny), $n \neq 0$. The equation which results in this transformation can be found by replacing y with y/n in the original equation.

Write the equation that results when the equation for the circle in Part **a** is stretched horizontally by 3 and stretched vertically by 2.

c. **1.** Rewrite the equation from Part **b** by multiplying both sides of the equation by $1/r^2$, where r represents the radius of the circle.

2. Simplify this equation and write it in the form below:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

d. **1.** Create a graph of the equation from Part **c**.

2. Identify the points where the graph intersects the x- and y-axes.

e. Repeat Parts **b–d** when the equation for the circle in Part **a** is stretched horizontally by 2 and stretched vertically by 3.

Discussion 2

a. Compare the ellipses you created in Parts **d** and **e** of Exploration **2**.

b. How do the intersections of each ellipse and the x- and y-axes relate to its equation in the form below?

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

c. How could you use an equation in the form above to identify the lengths of the major and minor axes of the ellipse?

d. How could you use such an equation to identify the foci of the ellipse?

Mathematics Note

The **standard form** of the equation of an ellipse with center at the origin and foci on the x-axis is written as follows:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $b^2 = a^2 - c^2$ and c is the distance from the center to each focus.

The equation in standard form can be used to quickly sketch the graph of an ellipse. As shown in Figure **6**, the x-intercepts of the ellipse are $(a,0)$ and $(-a,0)$ and the length of the major axis is $2a$. The y-intercepts of the ellipse are $(b,0)$ and $(-b,0)$ and the length of the minor axis is $2b$.

Assignment

2.1 **a**. Sketch a graph of the ellipse described by the equation below:

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

b. Find the foci of the ellipse in Part **a**.

c. Write an equation for this ellipse in the form based on the distance formula (described in Part **h** of Discussion **1**).

2.2 The following equation describes an ellipse with center at the origin:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

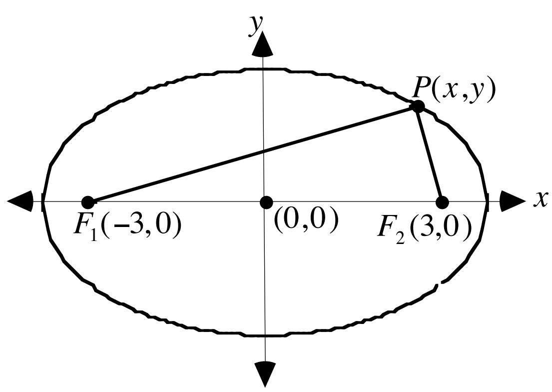
a. Does this equation represent a function?

b. Describe any adjustments you would have to make in order to graph this equation on a graphing utility.

c. The vertices of this ellipse occur at the *x*- and *y*-intercepts. Determine the coordinates of the vertices.

d. If *P* is any point on the ellipse and F_1 and F_2 are the foci, what is the value of $PF_1 + PF_2$?

2.3 The foci of the ellipse in the diagram below are located at (–3,0) and (3,0). Its locus of points is described by the sum $PF_1 + PF_2 = 16$.



a. Use the distance formula to express PF_1 and PF_2 in terms of *x* and *y*.

b. Use your response to Part **a** to write an equation for the ellipse based on the distance formula.

c. Determine the coordinates of the vertices that lie on the *y*-axis.

d. Determine the coordinates of the vertices that lie on the *x*-axis.

e. Determine the equation of the ellipse in standard form.

Mathematics Note

When an equation contains two radicals, the following process may be used to eliminate them:

- Given: $\sqrt{x} + \sqrt{y} = k$
- Isolate one of the radicals. $\sqrt{x} = k - \sqrt{y}$
- Square both sides and simplify. $\left(\sqrt{x}\right)^2 = \left(k - \sqrt{y}\right)^2$

$$x = k^2 - 2k\sqrt{y} + \left(\sqrt{y}\right)^2$$

$$x = k^2 - 2k\sqrt{y} + y$$

- Isolate the next radical. $\dfrac{x - k^2 - y}{-2k} = \sqrt{y}$
- Square both sides and simplify. $\left(\dfrac{x - k^2 - y}{-2k}\right)^2 = \left(\sqrt{y}\right)^2$

$$\left(\frac{x - k^2 - y}{-2k}\right)^2 = y$$

If the equation contains more than two radicals, the process of isolating one radical at a time is continued until all radicals are removed.

For example, to remove the radicals from the equation $\sqrt{x} + \sqrt{y} = 10$, you could use the following steps:

$$\sqrt{x} + \sqrt{y} = 10$$

$$\sqrt{x} = 10 - \sqrt{y}$$

$$\left(\sqrt{x}\right)^2 = \left(10 - \sqrt{y}\right)^2$$

$$x = 100 - 20\sqrt{y} + y$$

$$\frac{x - 100 - y}{-20} = \sqrt{y}$$

$$\left(\frac{x - 100 - y}{-20}\right)^2 = \left(\sqrt{y}\right)^2$$

$$\left(\frac{x - 100 - y}{-20}\right)^2 = y$$

2.4 **a**. Demonstrate that the following equation of an ellipse,

$$\sqrt{(x+2)^2 + (y-0)^2} + \sqrt{(x-2)^2 + (y-0)^2} = 10$$

is equivalent to its equation in standard form:

$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

b. Verify that the equation involving radicals defines the same set of points as the equation in standard form by solving each one for y.

2.5 Consider the ellipse described by the equation below:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

a. Write an equation that describes the image of this ellipse translated 8 units horizontally and 6 units vertically. List the coordinates of its center.

b. Using the equation of the translated ellipse from Part **a**:

1. determine the coordinates of the endpoints of the major axis
2. determine the coordinates of the endpoints of the minor axis.

c. What are the coordinates of the foci of the translated ellipse?

d. Write the standard form of the equation of an ellipse with center at (h,k) and foci on the line $y = k$.

2.6 Earth's orbit is an ellipse with the sun located at one focus. The sun is $2.99 \bullet 10^6$ km from the center of the ellipse. The length of the major axis is $2.99 \bullet 10^8$ km.

a. As the earth travels its elliptical path, what is the shortest distance between the earth and the sun?

b. What is the greatest distance between the earth and the sun?

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2.7 Demonstrate that the following equation of an ellipse,

$$\sqrt{(x+4)^2 + (y-0)^2} + \sqrt{(x-4)^2 + (y-0)^2} = 34$$

is equivalent to its equation in standard form:

$$\frac{x^2}{289} + \frac{y^2}{273} = 1$$

2.8 A communications satellite has been placed in an elliptical orbit around Earth. The satellite's orbital path is directly above the equator. One focus is at Earth's center, and each focus is 410 km from the center of the ellipse. The length of the major axis is 13,960 km.

a. Write an equation that describes the satellite's orbit.

b. How does the satellite's orbit compare to a circular orbit?

c. Earth's radius at the equator is about 6400 km. At its lowest point, how close is the satellite to Earth's surface?

d. At its highest point, how far is the satellite from Earth's surface?

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Activity 3

As you discovered in the previous activity, an ellipse is the locus of points in a plane for which the sum of the distances from two foci is a constant. In this activity, you use technology to continue your investigations of the ellipse, then explore the geometry of another conic section.

Exploration 1

a. Construct a circle and label its center O. Construct P_1 on the circle.

b. Construct P in the interior of the circle, but not at the center.

c. Construct $\overline{PP_1}$.

d. Construct the perpendicular bisector of $\overline{PP_1}$.

e. Construct $\overleftrightarrow{OP_1}$.

f. Mark the intersection of the perpendicular bisector from Part **d** and $\overleftrightarrow{OP_1}$. Label this point X. Your construction should now resemble the one shown in Figure **8**.

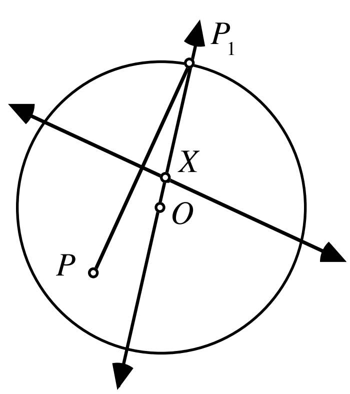


Figure 8: Completed construction

g. Trace the path of X as P_1 is moved around the circle. Note which conic is formed by the path.

h. Move P to a different location inside the circle and repeat Part **g**. Experiment with several other locations of P, including the center of the circle. **Note:** Save this construction for use in Exploration **2**.

Discussion 1

a. Describe how the location of point P affects the shape of the figure generated by the construction in Exploration **1**.

b. The figure generated when P is located inside the circle appears to be an ellipse, but appearance alone is not enough to prove that it is one. For this to be true, the figure must satisfy the definition of an ellipse.

In Figure **9**, P is any point within the circle that is not the center. The point X is a point on the figure.

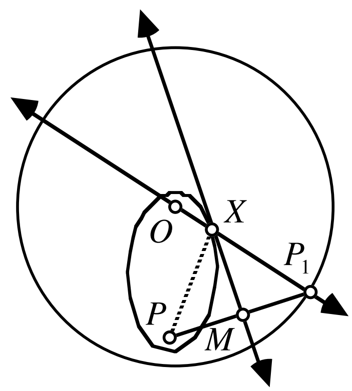


Figure 9: Construction from Exploration 1

1. Which points appear to be the foci of the figure?
2. If the figure generated is an ellipse, what sum must remain constant?
3. Since $\overline{MX}$ is the perpendicular bisector of $\overline{PP_1}$, what is the relationship between P_1X and PX?
4. Why is the length of $\overline{OP_1}$ always the same, no matter where on the circle P_1 is located?
5. Why is the length of $\overline{OP_1}$ equal to $OX + XP$?
6. How does the fact that $OX + XP = OP_1$ prove that the generated figure is indeed an ellipse?

Exploration 2

In the previous exploration, you created a construction that generated two conic sections: an ellipse and a circle. In this exploration, you use the same construction to generate another conic section, the hyperbola.

a. Using the construction from Exploration **1**, locate P outside the circle. Trace the path of point X as P_1 moves around the circle.

b. Move P to several different locations outside the circle and at least one point on the circle. Repeat Part **a** for each location. Note the differences in the figures generated.

Mathematics Note

A **hyperbola** is the locus of points in a plane for which the positive difference of the distances from two designated foci is a constant. In Figure **10**, for example, points F_1 and F_2 represent the foci. The hyperbola formed is the set of all points H in a plane for which the difference between HF_1 and HF_2 is a constant.

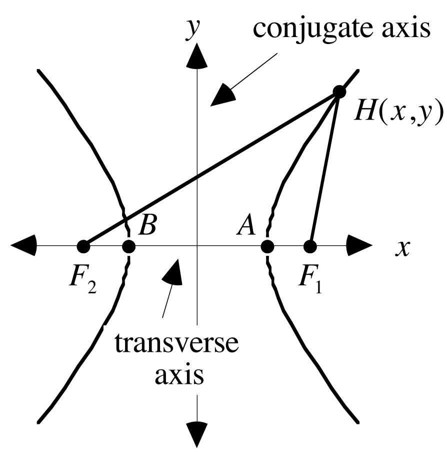


Figure 10: A hyperbola

The midpoint of $\overline{F_1F_2}$ is the **center** of the hyperbola. The **vertices** of the hyperbola occur at the intersections of the **branches** and $\overline{F_1F_2}$. In the hyperbola in Figure **10**, the vertices occur at points A and B.

The line segment joining the vertices is the **transverse axis**. The perpendicular bisector of the transverse axis lies on the **conjugate axis**.

Discussion 2

a. Describe how the location of point *P* affects the shape of the figure generated by the construction in Exploration **2**.

b. The figure generated when *P* is located outside the circle appears to be a hyperbola, but as with the ellipse, appearance alone is not enough to prove this conjecture. In Figure **11**, *P* is any point outside the circle. The point *X* is a point on the figure.

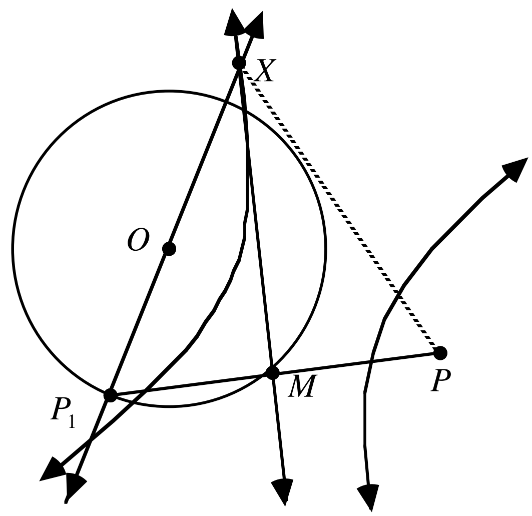


Figure 11: Construction from Exploration 2

1. Which points appear to be the foci of the figure?
2. As P_1 moves along the circle, how is the length of $\overline{OP_1}$ affected? Explain why this occurs.
3. Since $\overline{MX}$ is the perpendicular bisector of $\overline{PP_1}$, what is the relationship between P_1X and PX?
4. Why is the length of $\overline{PX}$ equal to $OX + OP_1$?
5. Why does $PX - OX = OP_1$?
6. How does the fact that $PX - OX = OP_1$ prove that the generated figure is indeed a hyperbola?

c. Describe the symmetries that exist within a hyperbola.

d.

1. The hyperbola shown in Figure **10** has its center at the origin. If the coordinates of F_1 are $(c, 0)$, what are the coordinates of F_2?
2. If the coordinates of vertex *A* are $(a, 0)$, what is the difference between AF_1 and AF_2?
3. Is the difference between BF_1 and BF_2 the same as the difference between AF_1 and AF_2? Why or why not?
4. What can you conclude about the constant difference for the hyperbola?

Mathematics Note

The **standard form** of the equation of a hyperbola with center at the origin and foci on the *x*-axis is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = c^2 - a^2$, c is the distance from the center to each focus, and a is the distance from the center to each vertex.

For example, suppose that the coordinates of F_1 in Figure **10** are $(-5,0)$, the coordinates of F_2 are $(5,0)$, and the constant difference $HF_1 - HF_2 = 6$ for any point *H* on the hyperbola. It follows that $a = 3$, $c = 5$, and $b = 4$. The equation, in standard form, of this hyperbola would be:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

e. **1.** If a graphing utility graphs only functions, it will not graph the equation of a hyperbola in standard form. Explain why this occurs.

2. How could you create a graph of a hyperbola on such a graphing utility?

Assignment

3.1 **a.** The positive difference of the distances from any point $H(x,y)$ on a hyperbola to each focus is a constant, $2a$. Use this fact, along with the distance formula, to write an equation for the hyperbola with center at the origin and foci at (4,0) and (–4,0), with $a = 3$.

b. Use appropriate technology to graph the hyperbola in Part **a**. What considerations, if any, must be addressed in order to create the graph?

3.2 **a.** How does the equation of a hyperbola compare to the equation of an ellipse?

b. Use appropriate technology to demonstrate that the following equation for a hyperbola,

$$\sqrt{(x-4)^2 + (y-0)^2} - \sqrt{(x+4)^2 + (y-0)^2} = 6$$

is equivalent to the equation in standard form below:

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

3.3 Consider the hyperbola described by the equation below:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

a. Identify the vertices of the hyperbola.

b. **1.** Use a symbolic manipulator to solve the equation of the hyperbola for y.

2. Graph the equations from Step **1**.

3. As $|x|$ increases, what does the value of $\sqrt{(x^2 - 4)}$ approach?

4. Rewrite the equations from Step **1** for large $|x|$. What kind of function is described by these equations?

5. Do the expressions that you wrote in Step **4** describe the actual values of y for the hyperbola?

c. In the Level 4 module "Big Business," an **asymptote** to a curve is described as a line such that the distance from a point P on the curve to the line approaches 0 as the distance from point P to the origin increases without bound.

Are the lines $y = (3/2)x$ and $y = -(3/2)x$ asymptotes of the hyperbola? Explain your response.

d. Describe how the denominators of the equation of the hyperbola can be used to find the equations of the asymptotes.

e. Identify the asymptotes for a hyperbola described by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3.4 Consider the hyperbola described by the following equation:

$$\frac{x^2}{9} - \frac{y^2}{144} = 1$$

a. Find the vertices and foci for the hyperbola.

b. Determine the equations of the asymptotes.

c. Sketch a graph of the equation including the asymptotes.

d. Determine two points on the curve where the x-coordinate equals 6.

3.5 Consider a hyperbola with the equation below:

$$\frac{x^2}{25} - \frac{y^2}{4} = 1$$

a. Write the equation of the image obtained by a horizontal translation of –4 units and a vertical translation of 3 units.

b. **1.** Determine the coordinates of the center of the translated hyperbola.

2. Determine the coordinates of the foci.

3. Determine the equations of the asymptotes for the translated hyperbola.

3.6 Miguel is designing a stained glass window 1.5 m high and 1 m wide. He plans to incorporate the distinct curve of a hyperbola in the window. In order to accomplish this, Miguel must decide how the hyperbola should appear, then determine an equation that can be used to make a template to cut the glass.

a. Design a stained glass window that includes a hyperbola.

b. Determine an equation that describes the hyperbola.

c. Write a paragraph that explains how you determined the equation for the hyperbola. Be sure to mention the constant difference and include the coordinates of the foci and vertices.

* * * * *

3.7 Graph the following hyperbola:

$$\frac{(x+1)^2}{4} - \frac{(y+3)^2}{1} = 1$$

On your graph, list the equations of the asymptotes and the coordinates of the foci, center, and vertices.

3.8 Demonstrate that the following equation for a hyperbola,

$$\sqrt{(x-5)^2 + (y-0)^2} - \sqrt{(x+5)^2 + (y-0)^2} = 8$$

is equivalent to the equation below:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

* * * * * * * * * *

Research Project

In this module, you have encountered equations for ellipses and hyperbolas, with center at the origin and foci located on the x-axis, in two very different forms: the standard form and a form based on the distance formula. For this research project, create algebraic proofs that show that these forms are equivalent.

a. For an ellipse, prove that $\sqrt{(x-c)^2+(y-0)^2}+\sqrt{(x+c)^2+(y-0)^2}=2a$ and the equation below are equivalent:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

b. For a hyperbola, prove that $\sqrt{(x-c)^2+(y-0)^2}-\sqrt{(x+c)^2+(y-0)^2}=2a$ and the equation below are equivalent:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Activity 4

As you observed in the previous activity, circles, ellipses, and hyperbolas can all be generated in a similar manner. The remaining conic section, however, has different restrictions than the others. In this activity, you examine the geometric properties of a **parabola**.

Mathematics Note

A **parabola** is the set of all points in a plane equidistant from a line and a point not on the line. The line is the **directrix** of the parabola. The point is the **focus** of the parabola. In Figure **12**, for example, the directrix of the parabola is line l and the focus is point F. The distances from any point P on the parabola to F or l are equal.

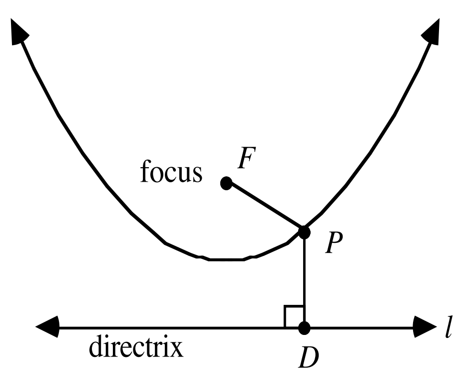


Figure 12: A parabola

Exploration

In this exploration, you create a construction for generating a parabola and determine an equation that describes its locus of points.

a. Using a geometry utility, construct a horizontal line to represent the directrix of a parabola. Construct point D on the line.

b. Construct a point F not on the line. This point represents the focus of the parabola.

c. Construct $\overline{FD}$.

d. Construct the perpendicular bisector of $\overline{FD}$.

e. Construct a line perpendicular to the horizontal line through D.

f. Construct a point at the intersection of the line from Part **e** and the perpendicular bisector of $\overline{FD}$. Label this intersection P. This point represents a point on a parabola.

Your construction should now resemble the one shown in Figure **13**.

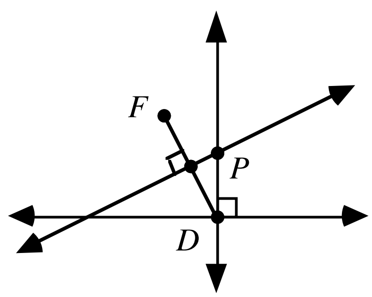


Figure 13: Completed construction

g. Trace the path of P as D is moved along the horizontal line. As you trace the path of P, record its coordinates for at least 10 different locations.

h.
1. Graph the coordinates of each point you recorded in Part **g** in a scatterplot.
2. As with the other conic sections, the equation of a parabola contains a second-degree equation. Determine an equation that models the scatterplot.
3. Graph the equation found in Step **2** on the same coordinate system as the scatterplot.

i. Move point F to at least two other locations and repeat Parts **g** and **h**. Locate point F below the line through D at least once.

Discussion

a. How could you prove that the construction in the exploration produces a parabola?

b. Describe any symmetries you observe in the parabola in Figure **14** below.

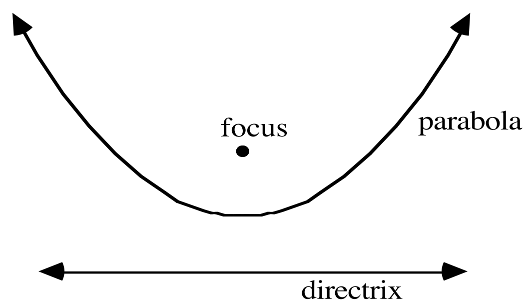


Figure 14: A parabola, its focus, and its directrix

c. The **vertex** of a parabola occurs at the point where the axis of symmetry intersects the parabola.

Describe how you could use the construction in the exploration to find the vertex of the parabola.

d. How does changing the distance from the focus to the directrix change the shape of the resulting parabola?

e.
1. How did locating the focus below the directrix affect the resulting parabola?
2. How did it affect the equation that defines the parabola?

f. How would you change the construction in the exploration to draw a parabola that opened to the left? to the right?

g. In the Level 4 module "It's All in the Family," you explored several transformations of the function $y = x^2$. Suppose that the parabola in Figure **14** is a graph of $y = x^2$.

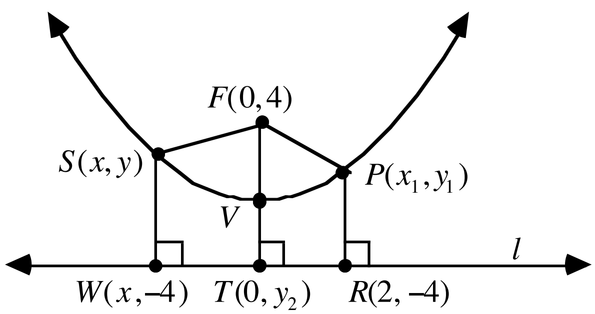
1. Describe where the *x*- and *y*-axes are located.
2. Describe the graph of $y = -x^2$.
3. Describe the graph of $y = x^2 + 3$.
4. Describe the graph of $y = (x - 2)^2$.

h. The shapes of the transformed graphs in Part **g** are identical to the shape of the parent. How would you modify the equation $y = x^2$ to change the shape of the parabola?

i. Describe how the graph of $y = x^2$ is transformed in the graph of $y = 3(x - 4)^2 + 5$.

Assignment

4.1 Use the diagram of the parabola below to complete Parts **a–e**.



a. Find y_2.

b. What are the coordinates of the vertex?

c. Find x_1.

d. Use the distance formula to describe the relationship between *PF* and *PR*.

e. Use the relationship you described in Part **d** to determine y_1.

4.2 Use the parabola shown in Problem **4.1** to complete Parts **a–d**.

a. Why are the *x*-coordinates of points *S* and *W* equal?

b. Use the distance formula to show that $FS = SW$.

c. Use the relationship in Part **b** to determine the equation of the parabola in the form $y = ax^2$. This is the **standard form** of the equation for a parabola with vertex at the origin.

d. Verify your equation using the coordinates of *P* from Problem **4.1**.

Mathematics Note

The **standard form** of the equation for a parabola with a horizontal directrix and vertex $V(h,k)$ is:

$$y = a(x - h)^2 + k$$

When a is positive, the parabola opens upward. When a is negative, the parabola opens downward.

The distance p from the vertex to the focus is the same as the distance from the vertex to the directrix. As shown in Figure **15**, the coordinates of the focus are $(h, k + p)$ and the equation of the directrix is $y = k - p$.

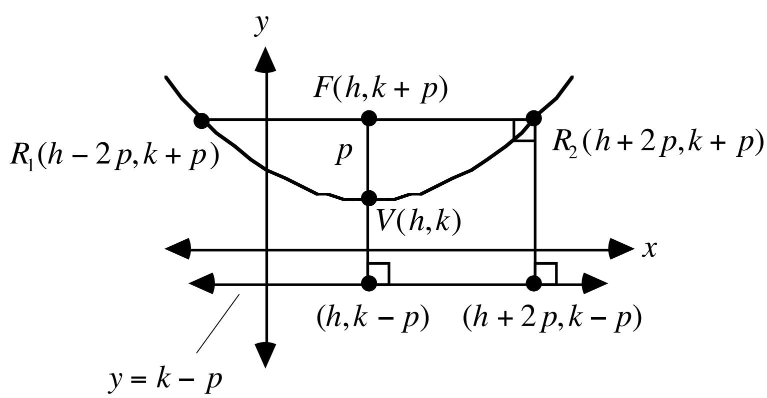


Figure 15: A parabola with horizontal directrix

4.3 **a.** What transformations of the parent function $y = x^2$ are described by $y - k = (x - h)^2$?

b. What additional transformation of the parent function is described by $y - k = a(x - h)^2$?

c. How is R_1R_2 in Figure **15** related to a in $y - k = a(x - h)^2$?

d. Write an equation that relates a to R_1R_2.

e. Since $R_1R_2 = 4p$, write an equation that relates a to $4p$.

Mathematics Note

Given the equation of a parabola in standard form, $y = a(x - h)^2 + k$, then

$$|a| = \frac{1}{4p}$$

where p is the distance from the focus to the vertex or the vertex to the directrix.

For example, Figure **16** shows the graph of the equation

$$y - 4 = -\frac{1}{8}(x - 8)^2$$

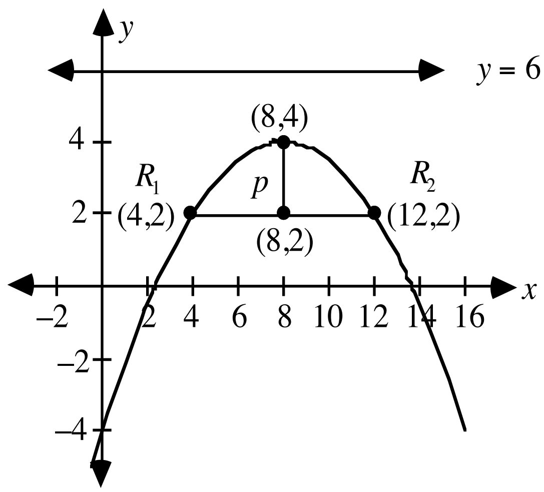


Figure 16: The graph of $\mathbf{y - 4 = -\frac{1}{8}(x - 8)^2}$

For this parabola, $R_1R_2 = 8$, $p = 2$, and

$$|a| = \frac{1}{4 \bullet 2} = \frac{1}{8}$$

4.4 Consider the parabola with focus at (2,6) and directrix described by the equation $y = 4$.

a. Plot the focus and vertex and draw the directrix.

b. Determine the equation of the parabola.

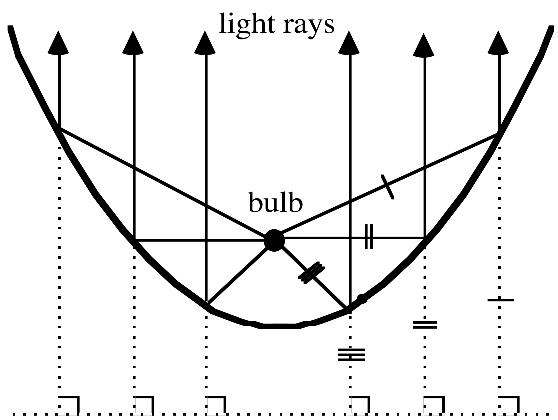
c. By inspection, determine two points on the parabola (other than the vertex) which have coordinates that are integers.

d. Verify that the points you identified in Part **c** are on the parabola by substituting their coordinates in the equation from Part **b**.

* * * * *

4.5 Determine the equation, in standard form, of the parabola with focus at (–1,2) and directrix $y = 5$.

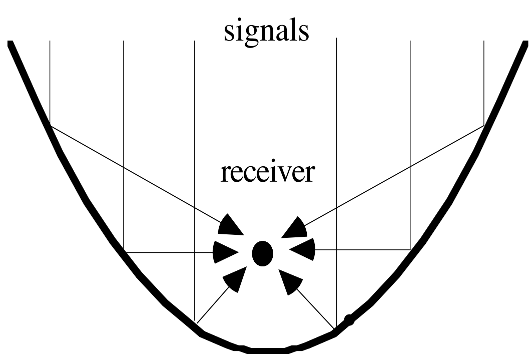
4.6 A paraboloid is a three-dimensional figure whose cross section is a parabola. The reflectors used in searchlights and satellite dishes, for example, are paraboloids. In a searchlight, the bulb is placed at the focus of the paraboloid. As shown in the diagram below, all light rays emitted from the bulb are reflected parallel to the axis of symmetry of the parabola that generated the shape of the reflector.



a. Consider a parabolic reflector formed by revolving the portion of the parabola $20y = x^2$ between $x = -20$ and $x = 20$ about the y-axis. Determine the coordinates of the location where the bulb should be placed.

b. If each unit on the coordinate system represents 1 cm, what is the maximum depth of the reflector?

4.7 Broadcast signals traveling into the opening of a parabolic reflector parallel to its line of symmetry are reflected through the focus of the paraboloid. This is an important property for collecting and receiving satellite television broadcasts.



a. Imagine that you have been asked to design a television satellite dish at least 180 cm in diameter at its opening, but no more than 60 cm deep. Determine an equation for a parabola that can be used to generate a suitable parabolic reflector.

b. If each unit on the coordinate system represents 1 cm on the satellite dish, where should the receiver be located?

4.8 Consider the parabola with focus at (2,6) and directrix $x = 0$.

a. Determine the equation of the parabola.

b. Graph the focus, vertex, and directrix on a sheet of graph paper.

c. By inspection, determine two points on the parabola (other than the vertex) which have coordinates that are integers.

d. Using the coordinates of the points identified in Part **c**, verify the equation you determined in Part **a**.

* * * * * * * * * *

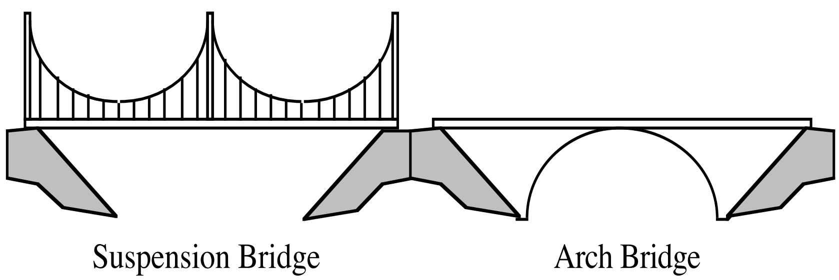
Summary Assessment

1. Imagine that you have just observed a comet that you cannot find in any star chart or atlas. So that others may observe the comet and verify your discovery, you must describe its orbit. If the comet is indeed a new discovery, it will be named after you.

From Kepler's laws of planetary motion, you know that the comet's orbit is an ellipse with the sun at one focus. According to your observations, the comet's closest approach to the sun is $9.00 \bullet 10^6$ km, while its farthest distance from sun is $1.79 \bullet 10^9$ km.

Using an appropriate coordinate system, write an equation that describes the comet's elliptical orbit. On a graph of the equation, label all important points and measurements.

2. Many famous bridges around the world contain conic sections in their design. Others contain curves that closely resemble conic sections. For example, the diagram below shows a suspension bridge and an arch bridge.



a. Find pictures of a suspension bridge and an arch bridge. Identify the conic sections that most closely approximate the curves found in the designs. Justify your selections.

b. Determine the equations of the conic sections identified in Part **a**.

Module Summary

- The set of all points that satisfies one or more given conditions is a **locus** (plural **loci**).
- A **conic section** can be formed by the intersection of a plane with a cone. In a right circular cone, the conic section formed depends on the slope of intersecting plane. The intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.

  It is also possible for the intersection of a plane and a cone to form a point, a line, or two intersecting lines. These intersections are called **degenerate conic sections**.
- A **circle** is the locus of points in a plane that are a given distance, the **radius**, from a fixed point, the **center**.
- The **standard form** of the equation of a circle with center at (h,k) and radius r is:

  $$(x-h)^2+(y-k)^2=r^2$$
- An **ellipse** is a locus of points in the plane such that the sum of the distances from two fixed points, the **foci**, is a constant.

  An ellipse is symmetric with respect to the lines containing its two axes. The **major axis** is the longer of the two and always contains the foci; the **minor axis** is the shorter of the two. The intersection of the major and minor axes is the **center** of the ellipse. The intersections of the ellipse and the major and minor axes are the **vertices** of the ellipse.
- The **standard form** of the equation of an ellipse with center at (h,k) is:

  $$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

  where $b^2=a^2-c^2$, $2a$ is the length of the major axis, and c is the distance from the center to each focus.
- A **hyperbola** is the locus of points in a plane for which the positive difference of the distances from two designated foci is constant.

  The midpoint of the segment joining the foci is the **center** of the hyperbola. The intersections of the **branches** and this segment are the **vertices** of the hyperbola. The line segment joining the vertices is the **transverse axis**. The perpendicular bisector of the transverse axis lies on the **conjugate axis**.

- The **standard form** of the equation of a hyperbola with center at (h,k) is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

  where $b^2 = c^2 - a^2$, $2a$ is the distance between the vertices, and c is the distance from the center to each focus.

- The **asymptotes** for a hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

  are $y = (b/a)x$ and $y = -(b/a)x$.

- A **parabola** is a locus of points in a plane for which each point is equidistant from a fixed line, the **directrix**, and a fixed point, the **focus**, not on the line. The **vertex** of a parabola is the midpoint of the line segment from the focus perpendicular to the directrix.

- The **standard form** of the equation of a parabola with a horizontal directrix and vertex $V(h,k)$ is:

$$y = a(x-h)^2 + k$$

  The coefficient a in the equation depends on the distance from the directrix to the focus. It can be shown algebraically that:

$$|a| = \frac{1}{4p}$$

  where p is the distance from the focus to the vertex or the vertex to the directrix.

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