Controlling the Sky with Parametrics

Air traffic controllers must know the precise locations of all nearby planes at any given time. In this module, you discover how mathematics can help model these situations.

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Introduction

An air traffic controller stares in surprise at the radar screen. Two airplanes are flying toward each other on a collision course! In order to avoid catastrophe, the controller must alter the flight path of one of the planes.

In this module, you explore the mathematics needed to help the controller handle this situation.

Activity 1

Minute Aviation sponsors an air show in which a large jet and a small plane fly by the control tower at the same time. In this activity, you use **parametric equations** to simulate the paths of these flights on a graphing utility.

Exploration 1

The small plane flies at a speed of 55 m/sec, while the large jet flies at a speed of 90 m/sec. They fly past the tower at the same moment and at the same altitude, on parallel courses from west to east. The small plane is on a course closer to the tower. The controller watches the flight paths on a radar screen, as shown in Figure **1**.

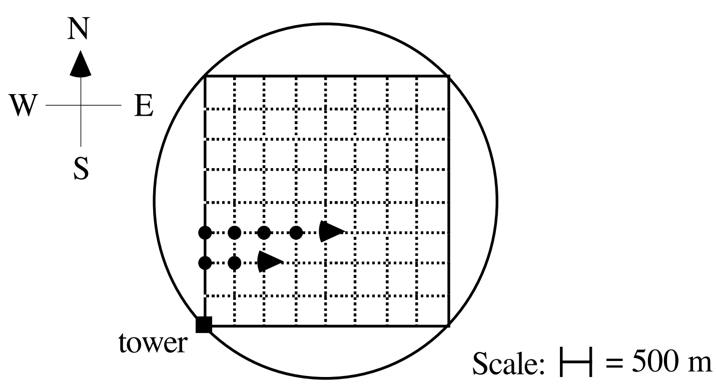

Figure 1: Large jet and small plane flying by control tower

Imagine that the control tower shown in Figure **1** is located at the origin of a rectangular coordinate system. Each side of the squares on the grid in Figure **1** represents a distance of 500 m.

a. Determine how far each airplane is from the tower when it first enters the radar screen.

b. Write a linear equation in slope-intercept form $(y = mx + b)$ to model the path of each plane on the grid.

c.
1. Determine the coordinates of each plane 1 sec after it passes the tower.
2. Determine the coordinates of each plane 2 sec after it passes the tower.
3. Determine the number of seconds each plane remains on the radar screen.

d.
1. Identify the corresponding domain and range for the segment of the line that models each path as it appears on the radar screen.
2. Use a graphing utility to graph both equations simultaneously over the appropriate intervals.
3. On the graph, indicate the location of each airplane 20 sec and 30 sec after passing the tower.

Discussion 1

a. If you traced values on either line segment graphed in Exploration **1**, what type of information would you get from the ordered pairs?

b. Federal regulations for air shows control a plane's speed, altitude, and distance from the crowd. The equations and graphs in Exploration **1** model the flight paths of two airplanes. If the controller used these equations and graphs to monitor the flights, what information would be missing?

c. What difficulties arose when you attempted to identify the planes' positions in Part **d** of Exploration **1**?

Exploration 2

In Exploration **1**, you wrote equations that described the path of each airplane as it flew by the tower. The small plane traveled at a speed of 55 m/sec, while the large jet traveled at a speed of 90 m/sec. In this exploration, you examine a mathematical method for visualizing this difference in speeds.

a. Copy and complete Table **1** for the small airplane. The values in the column with the heading "Time" represent the number of seconds after the small airplane passes the control tower.

Table 1: Position of small airplane at different times

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0		
1		
2		
3		
⋮		
10		
⋮		
t		

b. Copy and complete Table **2** for the large jet. The values in the column with the heading "Time" represent the number of seconds after the large jet passes the control tower.

Table 2: Position of large jet at different times

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0		
1		
2		
3		
$\vdots$		
10		
$\vdots$		
t		

Mathematics Note

Parametric equations allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. On an xy-plane, for example, both x and y can be expressed as functions of a third variable t:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

When parametric equations are used to model linear relationships, they can be written in the general form:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

where (a,c) are the coordinates of the graph when $t = 0$. The coefficients b and d represent the rates of change in the x- and y-components, respectively.

For example, the graph of the parametric equations below has coordinates (3,–4) when $t = 0$. The horizontal and vertical rates of change are –2 and 5 units, respectively, for every unit change in t.

$$\begin{cases} x = 3 - 2t \\ y = -4 + 5t \end{cases}$$

c. Write the parametric equations that describe the x- and y-coordinates of the small airplane at any given time t.

d. Write the parametric equations that describe the x- and y-coordinates of the large jet at any given time t.

e.
1. Set your graphing utility to graph parametric equations simultaneously. Using the same intervals for x and y as in Exploration **1** and the interval $[0, 73]$ for the parameter t, graph both pairs of equations from Parts **c** and **d**.
2. Experiment with different increments for t. Record your observations.

Discussion 2

a. Which graphs do you think give a more accurate depiction of the location of the two airplanes when they fly by the tower: the graphs in Exploration **1** or the graphs in Exploration **2**? Explain your response.

b.
1. How does changing the increment for the parameter affect the graph?
2. Given that the parameter represents time, what would you expect to occur as the size of the increment decreases?

c. Does tracing values on either line segment allow you to determine an airplane's position at a particular time? Explain your response.

d. What would you need to change to model the motion of the airplanes before they pass the control tower?

e. In the parametric equations below, which variables are represented in the domain and which variables are represented in the range?

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

f.
1. Suppose that the jet and the airplane are flying toward each other, with the jet entering the radar screen from the right, along the same path and at the same speed as in the explorations. How would you modify your parametric equations to describe this situation?
2. How could you determine when the two planes would pass each other?

Assignment

1.1 **a.** Use a graphing utility and the parametric equations from Exploration **2** to model the motion of the small plane and the large jet for 8 sec after they pass the tower.

b. **1.** Use your graph to estimate the time required (to the nearest 0.1 sec) for the small airplane to travel 100 m past the tower.

2. Use algebra to determine the time required (to the nearest 0.1 sec) for the small airplane to travel 100 m.

c. Repeat Part **b** for the large jet.

d. After 8 sec, what is the distance between the two airplanes?

1.2 **a.** The parametric equations below model the motion of two planes as they fly across the radar screen.

$$\begin{cases} x = 0 + 55t \\ y = 1000 + 0t \end{cases} \quad \begin{cases} x = 4000 - 90t \\ y = 1500 + 0t \end{cases}$$

b. Use the trace feature to estimate when the planes will pass each other.

c. Use algebra to determine when the planes will pass each other.

1.3 **a.** Write parametric equations to model the motion of a plane that enters the radar screen 1000 m east of the tower flying due north at 55 m/sec.

b. Write parametric equations to model the motion of a plane that leaves the radar screen 1500 m east of the tower flying due south at 90 m/sec.

1.4 The small plane and large jet described in Activity **1** also plan to stage a near collision in the air show. The planes will fly on perpendicular paths at the same altitude. The small airplane will fly east at 55 m/sec along a path 1000 m north of the control tower. The large jet will fly due north at 90 m/sec along a path somewhere to the east of the tower.

a. Write parametric equations to describe the paths that will produce an actual collision.

b. The pilots have decided that at the moment the small plane reaches the point where their paths intersect, the large jet should be 25 m away. Modify the parametric equations from Part **a** to model this situation.

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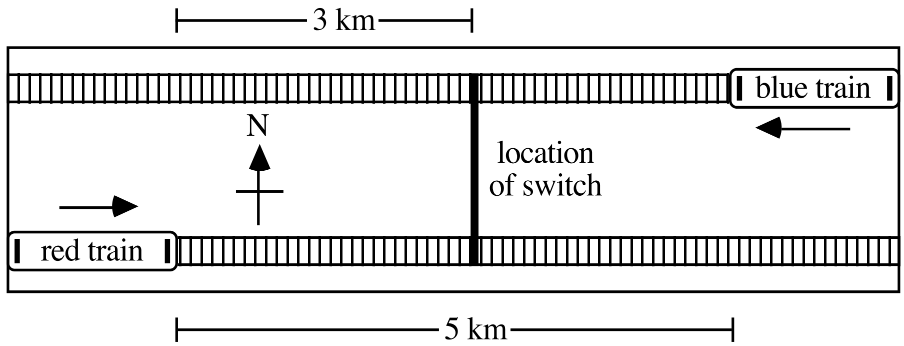
1.5 Shown in the diagram below, two subway trains are traveling toward a switch, where trains can change from one set of tracks to another. The red train is 3 km from the switching point, traveling east at 60 km/hr. At the same time, the blue train is 2 km from the switch, traveling west at 55 km/hr.

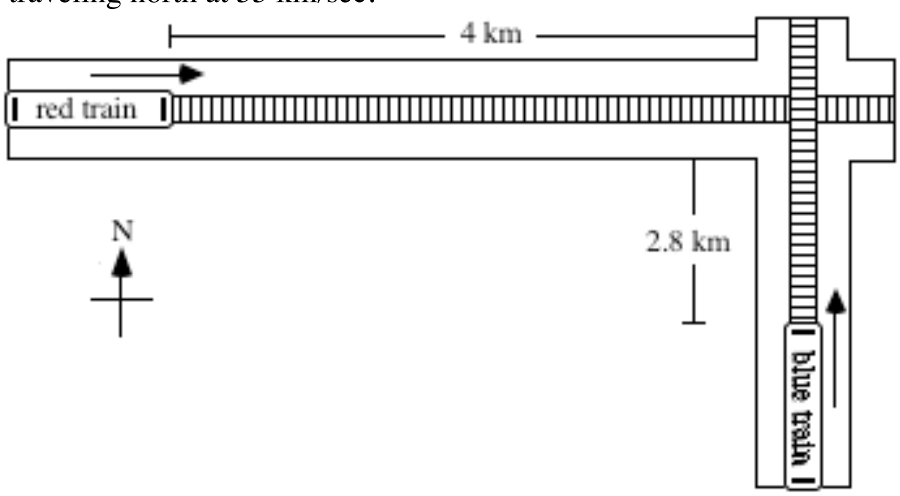


a. Write sets of parametric equations to model this situation.

b. Judging from a graph of the equations in Part **a**, will it be safe to switch the blue train to the south tracks if both trains continue at their present speeds? Justify your response.

c. If it is safe to switch the blue train, determine the red train's location when the blue train reaches the switch. If it is not safe, adjust the speed of one train so that the switch can occur safely and demonstrate algebraically that your adjustment will result in a safe switch of the blue train to the south track.

1.6 As shown in the following diagram, two trains are traveling toward an intersection. The red train is 4 km from the intersection and traveling east at 60 km/hr. The blue train is 2.8 km from the intersection and traveling north at 55 km/sec.



a. Model this situation parametrically.

b. Will the two trains collide? Justify your response algebraically.

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Activity 2

In the real world of air travel, flight paths seldom can be modeled as strictly horizontal or vertical line segments on a coordinate system. In this activity, you explore how to represent other types of linear motion in parametric form.

Exploration

Imagine that a Boeing 747 enters an air traffic controller's radar screen from the south. As shown in Figure **2**, the 747 appears at a point 2000 m east of the tower. At the same moment, a Piper Cub enters the screen at a point 500 m north of the tower.

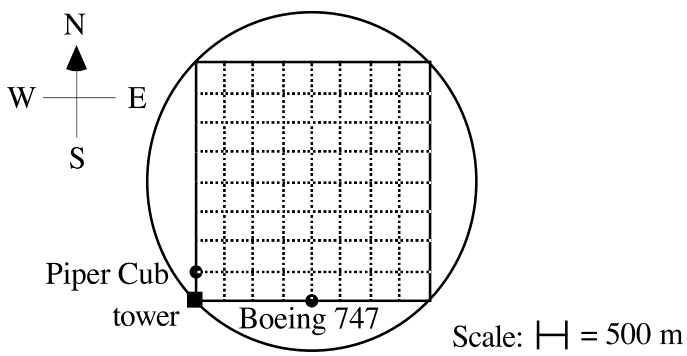


Figure 2: Initial positions of two planes on a radar screen

The radar reports the same altitude for both planes. After 1 sec, the 747 is 150 m west and 100 m north of its initial position. The Piper Cub, meanwhile, is now at a point 60 m east and 40 m north of its initial position. If both planes continue on these courses at constant speeds, will they collide?

a. Represent the grid on the radar screen in Figure **2** as a coordinate plane with the tower located at the origin.

1. Determine the coordinates for the location of each airplane as it enters the radar screen.
2. Find the coordinates for the location of each airplane after 1 sec.
3. Determine equations for the lines that represent the planes' flight paths.

b. The direction and speed of the Boeing 747 can be represented by a velocity vector.

1. Determine the horizontal component ($\mathbf{v}_x$) of the 747's velocity.
2. Determine the vertical component ($\mathbf{v}_y$) of the 747's velocity.
3. Determine the ratio $\mathbf{v}_y/\mathbf{v}_x$.
4. Compare the ratio $\mathbf{v}_y/\mathbf{v}_x$ to the slope of the line that represents the 747's flight path.

c. Create tables with headings like those in Tables **3** and **4**. Complete each table with the appropriate distances. The values in the columns with the heading “Time” represent the number of seconds after the air traffic controller began monitoring the planes.

Table 3: The Boeing 747’s position

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0		
1		
2		
3		
⋮		
7		
⋮		
t		

Table 4: The Piper Cub’s position

Time (sec)	*x*-coordinate (m)	*y*-coordinate (m)
0		
1		
2		
3		
⋮		
7		
⋮		
t		

d. Write parametric equations to model the path of each airplane.

e. Graph the equations simultaneously on a graphing utility.

f.
1. Graph the two linear equations from Part **a** on a coordinate grid with the same scales as in Part **e**.
2. Compare these graphs to the parametric graphs in Part **e**.

g. Determine the coordinates of the point where the paths of the two airplanes intersect.

h. Suppose that the air traffic controller did not begin monitoring the planes until the position of the 747 was (1700,200) and the position of the Piper Cub was (120, 580). Repeat Parts **c–f** for this situation.

Discussion

a. Describe what each term represents in the parametric equations that model the path of the Piper Cub.

b.
1. How did changing the positions of the planes at $t = 0$ affect the parametric equations you used to model their flight paths?

2. How did this affect the corresponding graphs?

c. Describe how you could determine the slope of a line from its parametric equations.

d. Why would you expect there to be infinitely many parametric equations that can model a single line?

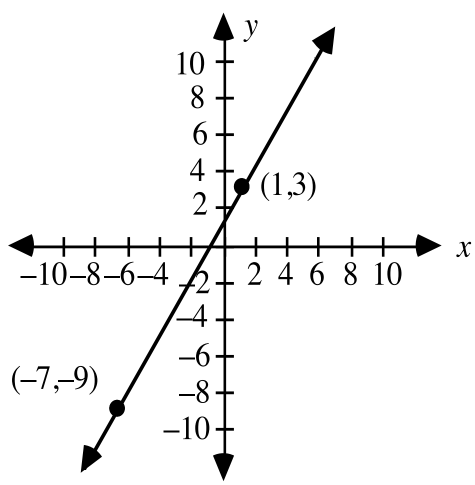
e.
1. Will the two airplanes in the exploration collide? Explain your response.

2. Which graphs better support your response: the ones from Part **e** of the exploration or the ones from Part **f**?

f. How could you determine the distance between the planes when the first plane reaches the intersection of the flight paths?

Assignment

2.1 The graph below shows the line that passes through (–7,–9) and (1,3).



a. Write a set of parametric equations that represents an object moving along the line segment from (–7,–9) to (1,3).

b. Write a second set of parametric equations that represents an object moving along a different segment of the line.

c. Write a third set of parametric equations that represents an object moving along the segment in Part **a** at a different velocity.

d. How many different sets of parametric equations could be used to describe a segment of this line?

2.2 **a.** Determine the distance between the planes in the exploration when the first plane reaches the intersection of their flight paths.

b. Determine the distance between the planes in the exploration when the second plane reaches the intersection of their flight paths.

2.3 The path of an airplane on a radar screen is represented by the parametric equations below, where *x* and *y* represent distances in meters and *t* represents time in seconds.

$$\begin{cases} x = 3500 - 30t \\ y = 1000 + 60t \end{cases}$$

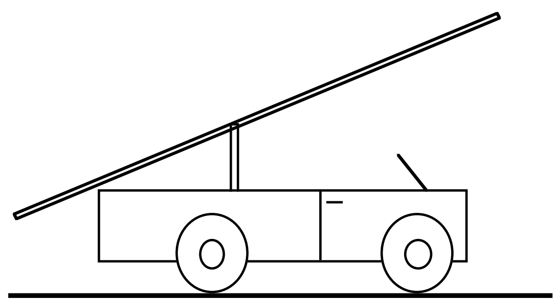
a. Graph the flight of the airplane for the interval from 0 sec to 50 sec using an appropriate increment.

b. At what location was the airplane first detected on the screen?

c. What is the slope of the line segment depicted by the parametric equations?

d. Determine the equation of the line that contains this segment in slope-intercept form.

2.4 To load baggage onto a plane, airline employees use a mechanized ramp. When the ramp is elevated as shown in the diagram below, the higher end is 4 m above the ground while the lower end is 1 m above the ground. The length of the ramp is 8.5 m.



a. Using the point on the ground directly below the lower end of the ramp as the origin of a coordinate system, determine a set of coordinates to represent:

1. the lower end of the ramp

2. the higher end of the ramp

b. What is the slope of the ramp?

c. Determine a set of parametric equations that could be used to model the path of a suitcase moving up the ramp.

d. If the suitcase falls off the ramp three-fourths of the way to the top, how far will it fall to the ground? Explain your response.

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2.5 The path of a moving object is described by the parametric equations below, where t represents time in seconds:

$$\begin{cases} x = 4 - 3t \\ y = -5 + 2t \end{cases}$$

a. Graph the path using the interval [0, 3] for the domain.

b. What is the position when $t = 0$?

c. What is the slope of the line depicted by the parametric equations?

d. Determine the equation of the line in slope-intercept form.

e. What is the distance between the object's locations at $t = 0$ and $t = 2.5$?

2.6 The motion of object A is defined by the following parametric equations, where x and y represent distances in meters and t represents time in seconds:

$$\begin{cases} x = -6 + 2t \\ y = -7 + 3t \end{cases}$$

The motion of object B is defined by the parametric equations below:

$$\begin{cases} x = 9 - 2t \\ y = -2 + 2t \end{cases}$$

a. Graph the parametric equations for both objects simultaneously.

b. At what point do the paths of the objects intersect?

c. Which object reaches the point of intersection first?

d. After the first object reaches the point of intersection, how long does it take the second object to reach that same point?

e. Modify the set of parametric equations for object B so that the two objects reach the point of intersection from Part **b** at the same time.

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Activity 3

The day-to-day operations of an airport feature many kinds of motion, including some that cannot be modeled with linear equations. An airplane waiting its turn to land, for example, might travel in a circle. The path of a helicopter blade also is circular, as is the motion of an anemometer, a device used to measure wind speed. In this activity, you use parametric equations to model circular paths.

Exploration

Airports use many different methods for delivering baggage to arriving passengers. In one popular system, luggage rotates around a carousel until claimed by the owners. Imagine that a lone suitcase is revolving on a circular luggage carousel, as shown in Figure **3**. Assume that the carousel has a radius of 1 unit and that the center of the carousel is located at the origin of a rectangular coordinate system.

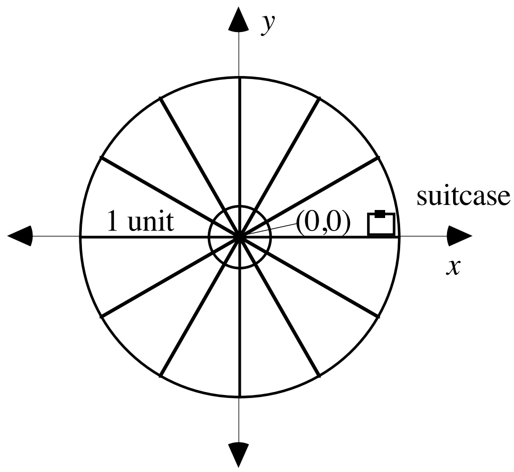


Figure 3: Suitcase on a luggage carousel

a. Represent the coordinates of any ordered pair (x,y) on the unit circle in terms of a central angle of the circle. Use radians to measure the angle.

b. Create a table with headings like those in Table **5** below. Use your results from Part **a** to complete the table for one revolution of the suitcase counterclockwise from the positive *x*-axis.

Table 5: Positions on a unit circle

Angle Measure (radians)	*x*-coordinate	*y*-coordinate
0		
$\pi/6$		
$\pi/3$		
$\pi/2$		
$2\pi/3$		
$5\pi/6$		
π		
$7\pi/6$		
$4\pi/3$		
$3\pi/2$		
$5\pi/3$		
$11\pi/6$		
2π		

c. Convert the relationships you wrote in Part **a** to parametric equations of the form below:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

d. Graph the parametric equations in Part **c**. Verify that the coordinates of the points on the graph agree with the values in Table **5**.

Mathematics Note

A circle with radius *r* and center at (h,k) can be represented by parametric equations in the following form:

$$\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases}$$

For example, the parametric equations below represent a circle with a radius of 3 units and center at the point (–2,5).

$$\begin{cases} x = -2 + 3\cos t \\ y = 5 + 3\sin t \end{cases}$$

e. Write parametric equations to describe a luggage carousel with a radius of 2 units and center at the origin of a two-dimensional coordinate system.

Graph these equations simultaneously with the parametric equations from Part **c**.

f. Many larger airports place two or more carousels side by side in the baggage claim area. To model this situation, write parametric equations for a luggage carousel with a radius of 1 unit and center at (2,3).

Graph these equations simultaneously with the parametric equations from Part **c**.

Discussion

a. In Activities **1** and **2**, the parameter *t* represented time. What does the parameter *t* represent in Parts **c–f** of the exploration?

b. How do the coefficients of $\cos t$ and $\sin t$ affect the graphs in Part **e** of the exploration?

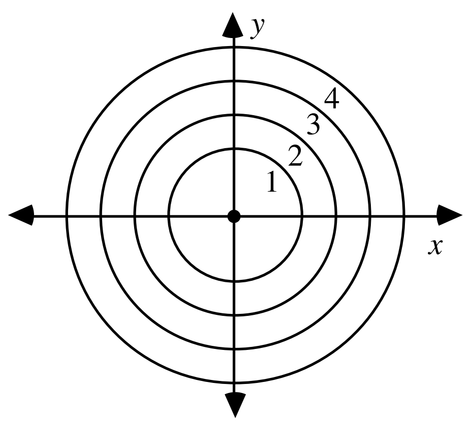
c. In Activities **1** and **2**, the constant terms represented the position of an object at $t = 0$. What do the constant terms represent in Part **f** of the exploration?

d. What are some advantages to graphing circles parametrically?

e. In the Level 4 module "Can It," you showed that $(\sin t)^2 + (\cos t)^2 = 1$ for any real-number value of t. Describe how to use this knowledge to write the standard form of the equation for a circle given its parametric equations. **Note:** The standard form of the equation for a circle with center at (h,k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$.

Assignment

3.1 A radar transmitter sweeps the surrounding area in a circular motion. The diagram below shows four concentric circles with a radar transmitter at the center. The innermost circle (1) has a radius of 40 km. The radius of each successive circle increases by 20 km. Assume that the transmitter is located at the origin of a two-dimensional coordinate system.

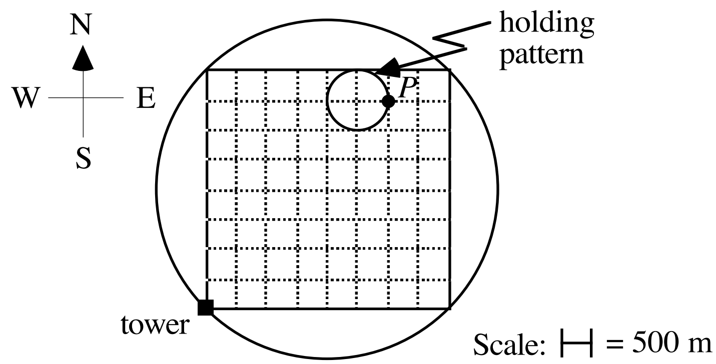


a. Write parametric equations for the circle farthest from the radar transmitter (circle 4).

b. What is the parameter for the equations you wrote in Part **a**?

c. What are the dependent variables in the equations you wrote in Part **a**? What are the independent variables?

3.2 While waiting for permission to land, a plane flies in a circular holding pattern as shown in the diagram below. The path has a diameter of 1 km and the plane completes one revolution every 5 min. The center of the circle is 2.5 km east and 3.5 km north of the tower.



a. Assume that the tower is located at the origin of a rectangular coordinate system and that point P is the initial location of the plane. Use parametric equations to model the airplane's flight path.

b. Determine the coordinates that describe the plane's location after it has traveled $1/5$ of a revolution counterclockwise from its initial position.

c. If the airplane began circling counterclockwise from point P, find the location of the plane after 9 min.

d. How far will the airplane have traveled after 9 min?

3.3 **a.** Graph the parametric equations below using appropriate technology:

$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$$

b. What was the first point plotted on the circle in Part **a**? In which direction was the circle plotted?

c. Change the parametric equations so that the circle will be plotted in the clockwise direction. (The starting point may be different than the starting point in Part **a**.)

d. Change the parametric equations so that the plotting starts on the positive x-axis and proceeds in a clockwise direction.

e. Change the parametric equations so that the plotting starts on the positive y-axis and proceeds in a counterclockwise direction.

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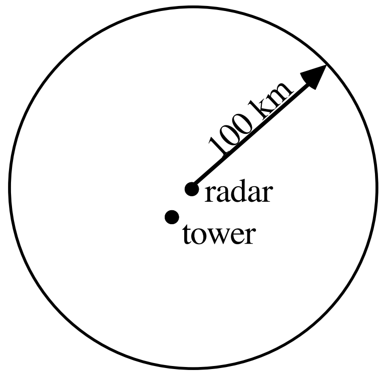
3.4 As shown in the diagram below, the radar transmitter for an airport is located 1.5 km east and 2 km north of the control tower. The radar has a range of 100 km.



a. How far (in km) is the radar transmitter from the control tower?

b. Assuming that the tower is located at the origin of a rectangular coordinate system, use parametric equations to model the outer perimeter of the radar.

c. **1.** The radar detects an airplane 75 km west and 10 km south of the tower. Write coordinates for the location of the plane.

2. Find the distance from the plane to the radar transmitter.

3. Write parametric equations to model a circle with its center at the radar transmitter and a radius equal to the distance in Step **2**.

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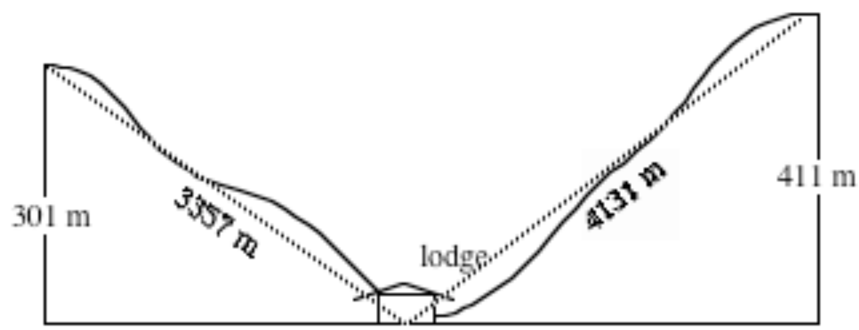
Summary Assessment

1. A ski lodge is located at the junction of two ski runs. As shown in the diagram below, the ski run to the left of the lodge is 3357 m long and has a vertical drop of 301 m. The run to the right of the lodge is 4131 m long and has a vertical drop of 411 m.



Jolene is at the summit of the run to the left, while Michael is at the summit of the run to the right. The two friends plan to meet at the lodge for lunch. Both skiers start toward the lodge at the same moment, and both travel downhill at a rate of 0.8 m of elevation every second.

Assume that the lodge is located at the origin of a rectangular coordinate system and that the skiers' paths are linear.

a. Write the ordered pairs that represent Jolene's and Michael's initial positions.

b. **1.** Determine the horizontal distance that Jolene covers in 1 sec.

2. Determine the horizontal distance that Michael covers in 1 sec.

c. Write parametric equations that model the each skier's trip down the mountain.

d. Using appropriate intervals for the domain and range, graph the parametric equations from Part **c** on a graphing utility.

e. Determine which skier arrives at the lodge first, and find the time required to descend.

f. What is the difference in the arrival times of the two skiers?

g. At the time the first skier arrives at the lodge, how far (to the nearest meter) is the other skier from the lodge?

2. Anton and Julia board the Ferris wheel at the local carnival. The diameter of the wheel is 16 m and the bottom of the wheel is 3 m above the ground. When the ride operator finishes loading the passengers, Anton and Julia's chair is at the highest point of the wheel. At that moment, the Ferris wheel starts to turn counterclockwise at a constant rate of 4 revolutions per minute. A complete ride lasts 4.75 min.

a. Use parametric equations to model the path of Anton and Julia's chair. (Remember that Anton and Julia start their ride at the top of the wheel.)

b. Determine how far Anton and Julia will travel after 30 sec.

c. Determine how far Anton and Julia will travel after one complete ride.

d. How far above the ground will the two friends be after 10 sec? after 20 sec? after 30 sec? (Hint: In this case, the parameter t represents radian measure.)

Module Summary

- **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. On an *xy*-plane, for example, both *x* and *y* can be expressed as functions of a third variable *t*:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

- When parametric equations are used to model linear relationships, they can be written in the general form:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

  where (a,c) are the coordinates of the graph when $t = 0$. The coefficients *b* and *d* represent the constant rates of change in the *x*- and *y*-components, respectively, in terms of *t*.

- A circle with radius *r* and center at (h,k) can be represented by parametric equations in the following form:

$$\begin{cases} x = h + r\cos t \\ y = k + r\sin t \end{cases}$$

Selected References

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