Having a Ball

News flash! The sum of the measures of a triangle's interior angles is not always 180˚. In this module, you discover how—and why—this can occur.

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Introduction

Imagine yourself on board an airplane flying from Los Angeles to Tokyo, as illustrated in Figure **1**. What route should the pilot choose in order to travel the shortest distance between these two cities?

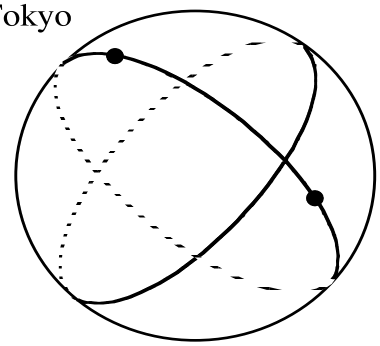


Figure 1: Flight path from Los Angeles to Tokyo

On a flat surface, the shortest distance between two points is along the line segment that connects the points. The earth, however, is shaped roughly like a sphere. A **sphere** is a set of points in space at a given distance from a fixed point. The fixed point is the **center**, the given distance is the **radius**. What is the shortest distance between two points on a sphere? How do other geometric properties on a sphere compare to the geometric properties of a flat surface, or a plane?

History Note

Although recognized for centuries, geometry on a flat surface was formalized by Greek mathematicians around 300 B.C. It is commonly known as **Euclidean geometry**, in honor of the Greek geometer Euclid. The flat surface of Euclidean geometry is the **Euclidean plane**.

Euclid's plane geometry was based on five **postulates**, statements assumed to be true without proof. The fifth postulate was the subject of controversy for over 20 centuries. According to this postulate, "if a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles."

The fifth postulate is illustrated in Figure **2**. If $m\angle 1 + m\angle 2 < 2 \bullet 90^\circ$, lines *l* and *m* will eventually meet on the same side of the vertical line *t* as $\angle 1$ and $\angle 2$.

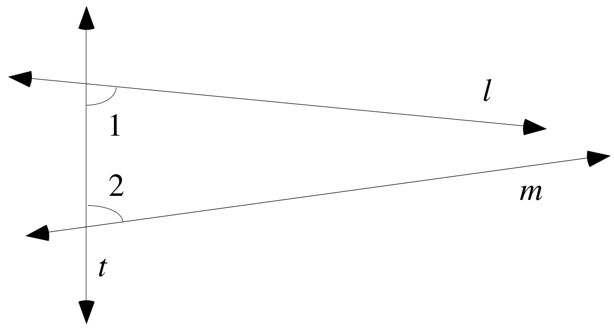


Figure 2: Geometrical representation of Euclid's fifth postulate

By modifying Euclid's fifth postulate and the concept of parallelism, mathematicians have developed other geometries, referred to as **non-Euclidean geometries**. One example is **spherical geometry**, the study of geometric figures constructed on the surface of a sphere. In spherical geometry, Euclid's fifth postulate is not satisfied.

Activity 1

In this activity, you determine whether or not each of the Euclidean concepts listed below also is true in spherical geometry.

- Two points determine a unique line.
- The length of the segment determined by two points is the shortest distance between the points.
- Lines cannot be measured because they are of infinite length.
- For any three points on a line, exactly one point is between the other two.
- Two lines either intersect or are parallel.

Exploration 1

In this exploration, you investigate the properties of lines on a sphere.

a. Obtain a sphere. Cut two pieces of string that are slightly longer than the circumference of the sphere.

b. **1.** Mark two points on the sphere that are not endpoints of a diameter of the sphere.

2. Determine whether it is possible to wrap a string around the sphere so that it passes through the two points marked in Step **1** and forms a circle with circumference equal to the circumference of the sphere.

3. If it is possible to complete Step **2**, determine whether it is possible to wrap a second string as described in Step **2** so that it does not coincide with the first string.

c. Repeat Part **b** for two additional pairs of points. Record your observations.

d. Locate the endpoints of a diameter of the sphere. Repeat Part **b** using these two points and record your observations.

Discussion 1

a. If the sphere were cut along the paths of the strings in Exploration **1**, what characteristics would be shared by all the cross sections?

Mathematics Note

A **great circle** of a sphere is a set of points determined by the intersection of the sphere and a Euclidean plane that contains the center of the sphere. Any two points on the surface of the sphere and contained in such an intersection determine a great circle. In spherical geometry, a **line** is defined as a great circle.

In Figure **3**, for example, points *A* and *B* lie on line *l*, a great circle of the sphere, and are not endpoints of a diameter of the sphere. These two points divide line *l* into **major** and **minor arcs**. The minor arc is the shorter of the two. Minor arcs typically are named by two letters, such as *AB*, while major arcs are named with three letters, such as *ADB*. If *A* and *B* are endpoints of a diameter of the sphere, then the two arcs are equal in length, and neither is a major nor minor arc.

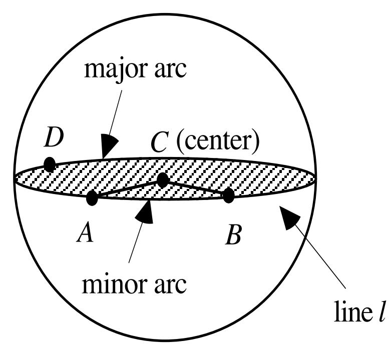


Figure 3: A line on a sphere

On a sphere, the **distance** between two points that are not endpoints of a diameter is the length of the minor arc of the great circle determined by those two points. The distance between two points that are endpoints of a diameter is half the circumference of the sphere. In Figure **3**, for example, the distance between points *A* and *B* is equal to the length of arc *AB*.

b. Is it possible for more than one great circle to pass through two given points on a sphere?

c. **1.** In spherical geometry, what is the least number of points needed to determine a unique line?

2. Will this number of points always determine a unique line?

d. Locations on Earth are often described using lines of longitude and latitude. How do lines of longitude and latitude compare to great circles?

e. Recall that the measure of an arc is the measure of its central angle and that the length of an arc can be found by multiplying the circumference of the circle by the fractional part of the circle represented by the arc.

1. In Figure **3**, what fractional part of the circle is represented by arc *AB*?

2. Given the length of arc *AB*, how could you determine $m\angle ACB$?

Mathematics Note

An **angle** on a sphere is defined by two minor arcs from different great circles that have a common endpoint. The intersection of the arcs is the vertex of the angle.

The measure of an angle on a sphere is defined by the planes that contain the great circles forming the angle. The tangents to the great circles at their point of intersection and in the planes of the circles intersect to form a planar angle. The measure of the angle on the sphere is the same as the measure of this planar angle. As in Euclidean geometry, angle measure is greater than or equal to 0° and less than 180°.

As shown in Figure **4**, for example, minor arc *AP* and minor arc *BP* form $\angle APB$. The measure of $\angle APB$ is the same as the measure of $\angle A'PB'$.

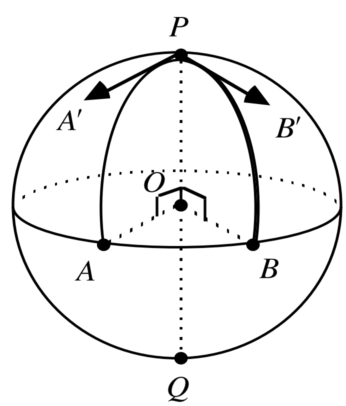


Figure 4: An angle on a sphere

f. In Figure **4**, points P and Q are endpoints of a diameter and the great circle containing A and B is perpendicular to $\overline{PQ}$.

Recall that in a plane, a line tangent to a circle is perpendicular to the radius that contains the point of tangency. Similarly, a plane tangent to a sphere is perpendicular to the radius containing the point of tangency.

Given these facts, what is the relationship between the plane tangent to the sphere at point P and the great circle that contains points A and B? Explain your response.

g.
1. Considering your response to Part **f** above, what is the relationship between $\angle A'PB'$ and $\angle AOB$?
2. What is the relationship between $\angle AOB$ and $\angle APB$?

h. If $m\angle APB = 90^\circ$, what is the length of arc AB in Figure **4**? Explain your response.

i. On a sphere, the endpoints of a diameter are equidistant from the great circle perpendicular to the diameter. How does the fact that central angles of equal measure intercept arcs of equal length prove this statement?

j. Describe how to determine the measure of $\angle ABC$ in Figure **5** below.

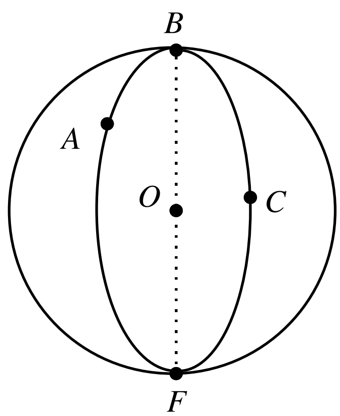


Figure 5: Angle *ABC* on a sphere

Exploration 2

Earth's north and south poles have a special relationship with the equator. In the following exploration, you examine similar relationships among points and lines on a sphere.

a.
1. Using a straightedge and a sheet of paper, determine the number of points of intersection possible for two distinct lines in a Euclidean plane.
2. Using rubber bands and a sphere, determine the corresponding number of points on the surface of a sphere.

b. 1. Determine the number of perpendicular lines that can be drawn from a point not on a line to that line in a Euclidean plane.

2. Determine the corresponding number on the surface of a sphere.

c. Construct two lines perpendicular to the same line in a plane. Repeat this construction on a sphere. Record any observations you make.

d. 1. Determine the number of points that are equidistant from all the points on a line in the Euclidean plane.

2. Determine the number of points that are equidistant from all the points on a line on a sphere.

Mathematics Note

A point is a **polar point** or **pole** of a given line on a sphere if is not on the given line and all lines perpendicular to the given line pass through the point. Every line on a sphere has two polar points.

In Figure **6**, for example, J and K are polar points of line l since every line perpendicular to line l passes through points J and K.

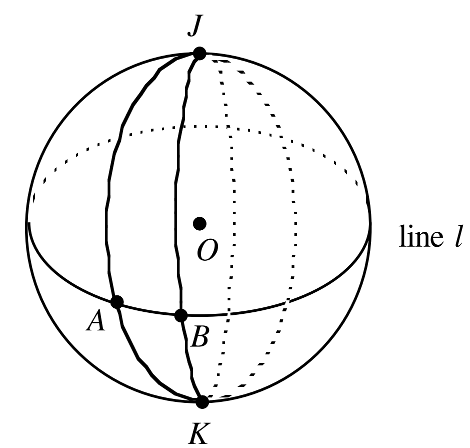


Figure 6: Sphere with line *l* and polar points *J* and *K*

Discussion 2

a. Does the term *pole* as defined in the mathematics note above also apply to Earth's north and south poles?

b. In Part **b** of Exploration **2**, you determined the number of perpendicular lines on a sphere that can be drawn from a point not on a line to that line. How would your response change if you do not consider polar points?

c. In Figure **6**, what is the relationship between the lengths of arc AJ and arc AK? Justify your response.

d. Describe how you could use perpendicular lines to locate the polar points of any line on a sphere.

e. Can any point on a sphere be a polar point? Explain your response.

f. Describe how you could find the measure of an angle on a sphere using an arc of the great circle for which the vertex of the angle is a polar point.

g. Is it possible for two lines on a sphere to be parallel? Explain your response.

Assignment

1.1 **a**. Are all of the circles that can be drawn on the surface of a sphere great circles? Explain your response.

b. What geometric figures can be formed by the intersection of a Euclidean plane and a sphere?

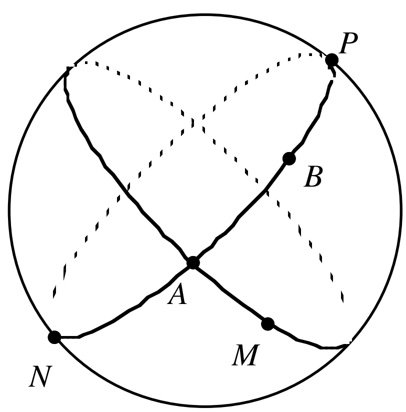
1.2 In Euclidean geometry, lines have infinite length. Is this also true of lines in spherical geometry? Explain your response.

1.3 The cities of Libreville, Gabon, and Quito, Ecuador, are very close to the equator. Libreville is located at approximately 0˚ N latitude and 9.5˚ E longitude. Quito is located at approximately 0˚ S latitude and 78.5˚ W longitude.

a. What is the measure of the angle formed by the great circles passing through the poles and each of the two cities? Explain your response.

b. Given that the diameter of Earth is approximately 12,756 km, what is the distance between Libreville and Quito?

1.4 As shown in the diagram below, P and N are poles of line AM.



a. Determine the number of lines that can be drawn perpendicular to line AM through each of the points listed below. Justify your responses.

1. point A
2. point B
3. point P

b. Summarize your findings in Part **a**.

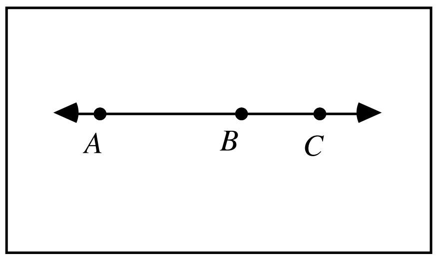
1.5 **a.** As mentioned in Problem **1.3**, Earth's diameter is approximately 12,756 km. Given this fact, what is the greatest possible distance between two cities on Earth?

b. City A is located on the same great circle as cities B and C, and is equidistant from both B and C. If you know the locations of B and C, how many locations are possible for A? Describe these locations.

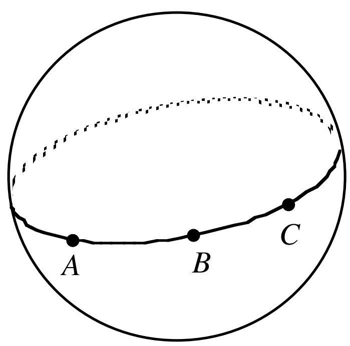
c. If city A is not located on the same great circle as cities B and C and is equidistant from both cities, how many locations are possible for A?

d. Describe how your response to Part **c** compares to the set of points in a Euclidean plane that are equidistant from the endpoints of a line segment.

1.6 On a line in a Euclidean plane, point B lies between points A and C if and only if $AB + BC = AC$.



As shown in the diagram below (not drawn to scale), points A, B, and C lie on the same great circle.



Using the Euclidean definition of "betweenness," when is B between A and C on a sphere?

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1.7 **a.** On Earth, which line of latitude is perpendicular to every line of longitude? Explain your response.

b. What is the distance from the north or south pole to any point on the equator?

1.8 Compare lines, collinear points, parallelism, and perpendicularity in Euclidean geometry with the same four concepts in spherical geometry.

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Research Project

Write a report on Euclid's contributions to the study of plane geometry, including an explanation of the significance of the work recorded in his *Elements*. Your report also should include descriptions of the five geometric postulates (or axioms) upon which Euclidean geometry is based, along with a geometrical illustration of each one.

Activity 2

In the previous activity, you discovered that lines in spherical geometry are very different from lines in Euclidean geometry. In this activity, you investigate triangles and other polygons on the surface of a sphere.

Exploration 1

In this exploration, you investigate the sum of the measures of the interior angles of a triangle on the surface of a sphere.

Mathematics Note

A **triangle** on a sphere is the union of three noncollinear points together with the minor arcs of the great circles defined by these points. For example, Figure **7** shows ΔABC on a sphere.

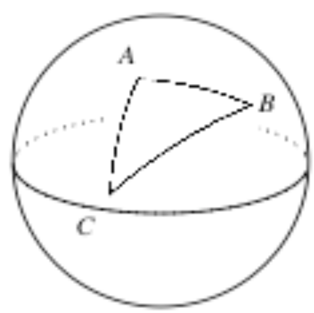

Figure 7: ΔABC on a sphere

For the purposes of this exploration, a triangle is any simple closed figure with three vertices connected by minor arcs of great circles.

a. Obtain a sphere and determine its circumference. Record this measurement to the nearest 0.1 cm.

b. Create three thin strips of paper: one 20 cm long, one 24 cm long, and one 27 cm long. Draw a line down the length of each strip, as shown in Figure **8**.

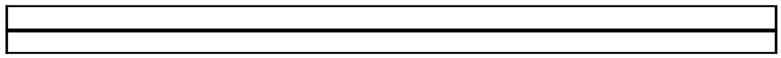

Figure 8: Paper strip with center line

c. Cut the 20-cm strip into three pieces that will form a triangle. Use the pieces to create a triangle *ABC* with a perimeter of 20 cm on the sphere.

Make sure that the center lines on the strips meet at each vertex, then tape the vertices securely.

d. Place three rubber bands around the sphere so that each rubber band lies along a side of the triangle and traces a great circle on the surface of the sphere.

e. As you discovered in Activity **1**, the measure of an angle on a sphere equals the measure of the minor arc of the great circle for which the vertex of the angle is a polar point.

1. Consider each vertex of the triangle as a pole of the sphere. Locate and mark the intersection of the great circles containing the sides of each angle (the rubber bands) with the great circle for which each vertex is a polar point.
2. Use the length of the minor arc determined by the points of intersection in Step **1** to determine the measure of each interior angle of the triangle. Record these measures in a table with headings like those in Table **1** below.

Table 1: Angle measures and sums

Perimeter of Triangle (cm)	$m\angle ACB$	$m\angle ABC$	$m\angle BAC$	Sum of Angle Measures
20				
24				
27				

f. Repeat Parts **c–e** using the 24-cm and 27-cm strips of paper.

g. Compare the results you recorded in Table **1** with those of your classmates.

Discussion 1

a. How do triangles constructed on the surface of a sphere differ from triangles constructed in a Euclidean plane?

b. How does the perimeter of a triangle constructed on the surface of a sphere appear to influence the sum of the measures of its interior angles?

c. How does the circumference of a sphere appear to influence the sum of the measures of a triangle's interior angles?

d. Is it possible to construct a triangle with two right angles on a sphere? Explain your response.

e. Is it possible to construct a triangle with three obtuse angles on a sphere? Explain your response.

f. Do you think that two-sided polygons can be constructed on a sphere? Explain your response.

Mathematics Note

In general, a polygon with *n* sides on the surface of a sphere is the union of *n* points that are the vertices of the polygon, with no three vertices collinear, together with the minor arcs of great circles defined by consecutive vertices. For example, Figure **9** shows a quadrilateral *ABCD* constructed on the surface of a sphere.

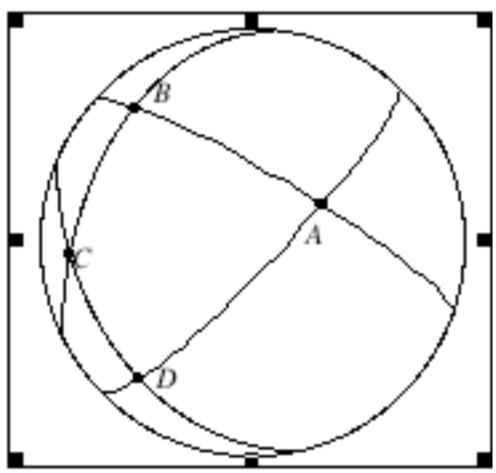

Figure 9: Quadrilateral *ABCD* on a sphere

Exploration 2

In this exploration, you examine the properties of quadrilaterals on a sphere.

a. Using paper strips and rubber bands, construct a quadrilateral on a sphere with a base of 6 cm and two sides of 3 cm perpendicular to the base.

b. Record the length of the side opposite the base and the measures of all interior angles.

c. Repeat Parts **a** and **b** for a quadrilateral with a base of 12 cm and perpendicular sides of 6 cm.

d. Construct a quadrilateral on the sphere with four sides of varying lengths. Record the measures of the interior angles.

e. For each quadrilateral constructed in Parts **a–d**, find the sum of the measures of the interior angles.

Discussion 2

a.
1. When two of the angles in a quadrilateral on a sphere are right angles, what are the measures of the other two angles?
2. If, as described in Part **a** of Exploration **2**, the two sides perpendicular to the base of a quadrilateral are congruent, the quadrilateral is a **Saccheri quadrilateral**. What can you determine about the non-right angles of a Saccheri quadrilateral?

b. If you constructed a quadrilateral with a base of 6 cm and two sides of 3 cm perpendicular to the base in a Euclidean plane,
1. what would be the length of the side opposite the base?
2. what would be the measures of the angles opposite the base?

c. In Parts **b** and **c** of Exploration **2**, how does the length of the base compare to the length of its opposite side? Do you think this relationship will always hold true on a sphere?

d. Do you think that rectangles can be constructed on the surface of a sphere? Explain your response.

e. In a Euclidean plane, the sum of the measures of a quadrilateral's interior angles is 360˚. Based on your results in Exploration **2**, suggest a general rule for the sum of the measures of a quadrilateral's interior angles on a sphere.

Assignment

2.1 **a.** What is the upper bound for the measure of an individual angle in a triangle on a sphere?

b. What is the upper bound for the sum of the measures of the angles of a triangle on a sphere?

2.2 In Euclidean geometry, the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Do you think this is also true in spherical geometry? Explain your response.

2.3 On the surface of a sphere, is there an upper bound for the sum of the lengths of a triangle's sides? Explain your response.

2.4 In Euclidean geometry, the sum of the measures of a triangle's interior angles is 180°. A quadrilateral can be divided into two triangles. Therefore, the sum of the measures of a quadrilateral's interior angles is $2 \bullet 180^\circ$ or 360°.

a. Use a similar argument to support your general rule for the sum of the measures of a triangle's interior angles on a sphere.

b. What do you think is the upper bound for the sum of the measures of a quadrilateral's interior angles on a sphere? Explain your reasoning.

2.5 In spherical geometry, what is the lower bound for the sum of the measures of a hexagon's interior angles? What is the upper bound for this sum? Explain your responses.

2.6 **a.** Determine a formula for finding the lower bound for the sum of the measures of the interior angles of an n-sided polygon on a sphere.

b. Determine a formula for finding the upper bound for the sum of the measures of the interior angles of an n-sided polygon on a sphere.

2.7 Describe the Euclidean quadrilaterals which cannot be constructed on the surface of a sphere and explain why they cannot exist.

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2.8 **a.** In spherical geometry, a triangle may have three right angles. Determine the dimensions of a triangle with three right angles on Earth's surface. (Earth's diameter is approximately 12,756 km.)

b. What portion of Earth's surface is represented by the area of the triangle?

2.9 Compare triangles and quadrilaterals in Euclidean geometry to triangles and quadrilaterals in spherical geometry.

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Activity 3

In Euclidean geometry, two polygons are **similar** if there is a one-to-one correspondence between their vertices so that corresponding sides are proportional and corresponding angles are congruent. In this activity, you compare similarity in a plane with similarity on a sphere.

Discussion

a. Is it possible for two triangles in a plane to be similar if the lengths of their sides are different? Explain your response.

b. On a sphere, what is the relationship between a triangle's perimeter and the sum of the measures of its interior angles?

c. Can two triangles on a sphere be similar if the lengths of their sides are different? Explain your response.

d. When is it possible for two triangles on a sphere to be similar? Explain your response.

Assignment

3.1 **a.** Are all right triangles in a plane similar? Explain your response.

b. Are all squares in a plane similar? Explain your response.

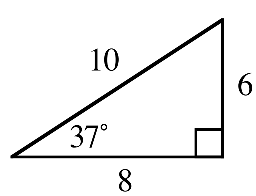
c. Are all rectangles in a plane similar? Explain your response.

3.2 **a.** Construct two rectangles that are similar but not congruent in a plane.

b. Is it possible to construct two quadrilaterals that are similar but not congruent on a sphere? Explain your reasoning.

3.3 Use similarity to describe the patterns on the surface of a soccer ball.

3.4 **a.** Consider a triangle in a plane similar to the one in the diagram below. Given that its longest side measures 12 cm, determine the lengths of the remaining sides and the measures of the triangle's three interior angles.



b. If the triangles in Part **a** were on a sphere, what would you expect to be true about the measures of their interior angles?

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3.5 A baseball diamond is a square that measures 90 feet between bases. The pitcher's mound is 60 feet 6 inches from home plate.

a. An artist wants to create a scale drawing of a baseball diamond on a baseball card. The card measures 6 cm by 9 cm. Describe how the artist can use similarity to create the scale drawing, including some possible dimensions (in centimeters) for the drawing.

b. Why would the process you described in Part **a** not produce an accurate scale drawing on a baseball?

3.6 **a.** In a Euclidean plane, how many triangles can be drawn that have angle measures of 45°, 55°, and 80°, but different perimeters? Explain your response.

b. On a sphere, how many triangles can be drawn that have angle measures of 50°, 65°, and 95°, but different perimeters? Explain your response.

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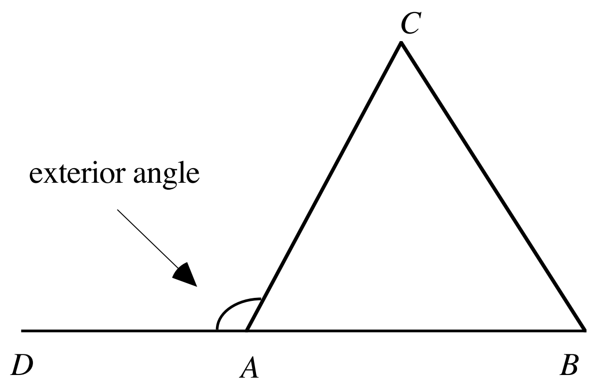
Summary Assessment

1. **a.** In Euclidean geometry, determine the relationship between the measure of an exterior angle of a triangle and the sum of the measures of two non-adjacent interior angles, as shown in the diagram below.



b. Does this same relationship hold true in spherical geometry? Justify your response.

2. The locations of points on the surface of a sphere can be described in a variety of ways.

a. Explain how degrees of latitude and degrees of longitude are determined for a point on Earth's surface.

b. Using longitude and latitude, describe a triangle on the Earth's surface for which the sum of the measures of the interior angles is 270˚.

c. Does a line of latitude satisfy the definition of a line in spherical geometry? Explain your response.

3. **a.** Use your knowledge of circles in a plane to define a circle on a sphere. Draw an example of a circle that fits your definition.

b. Can a circle on a sphere also be a line? Explain your response.

Module Summary

- A **postulate** is a statement that is assumed to be true without proof.
- A **sphere** is a set of points in space at a given distance from a fixed point. The fixed point is the **center**, the given distance is the **radius**.
- **Spherical geometry** is the study of geometric figures constructed on the surface of a sphere. In spherical geometry, Euclid's fifth postulate is not satisfied.
- A **great circle** of a sphere is a set of points determined by the intersection of the sphere and a Euclidean plane that contains the center of the sphere. Any two points on the surface of the sphere and contained in the intersection determine a great circle.
- In spherical geometry, a **line** is defined as a great circle.
- On a sphere, the **distance** between two points that are not endpoints of a diameter is the length of the minor arc of the great circle determined by those two points. The distance between two points that are endpoints of a diameter is half the circumference of the sphere.
- An **angle** on a sphere is defined by two minor arcs from different great circles that have a common endpoint. The intersection of the arcs is the vertex of the angle.
- The measure of an angle on a sphere is defined by the planes that contain the great circles forming the angle. The tangents to the great circles at their point of intersection and in the planes of the circles intersect to form a planar angle. The measure of the angle on the sphere is the same as the measure of this planar angle. It also equals the measure of the minor arc of the great circle for which the vertex of the angle is a polar point. As in Euclidean geometry, angle measure is greater than or equal to 0° and less than 180°.
- Point J is a **polar point** of the line l on a sphere if J is not on line l, and all lines perpendicular to l pass through J. Every line on a sphere has two polar points or **poles**.
- A **triangle** on a sphere is the union of three noncollinear points together with the minor arcs of the great circles defined by these points.
- In general, a polygon with n sides on the surface of a sphere is the union of n points that are the vertices of the polygon, with no three vertices collinear, together with the minor arcs of great circles defined by consecutive vertices.

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