Can It!

How can you model such diverse phenomena as the height of tides in the ocean, the flow of electricity in a circuit, and the hours of daylight in a year? In this module, you explore functions that can model events which occur over and over again.

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Can It!

Introduction

Mathematicians have long used functions to model real-world situations. Many of these situations involve events that repeat over time. For example, the positions of the hands on the face of a clock, the rise and fall of ocean tides, and the hours of sunlight in the days of the year all occur in predictable cycles. In this module, you investigate the functions required to model these and other **cyclic** events.

Activity 1

Consider the cyclic behavior of the second hand as it moves around the face of a clock. As the hand sweeps around the face, every point on the dial is visited again and again. Given the distance that the second hand moves in each interval of time, it is possible to model its motion and predict its location at any particular time.

Exploration

Although time is measured on a linear scale, its passing can also be represented by the circular motion of a second hand. In this exploration, you simulate the "wrapping" of a linear scale onto a circle.

a.
1. Use a can to trace a circle on a sheet of graph paper.
2. Identify the center of the circle.
3. As shown in Figure **1**, create a coordinate system with its origin at the center of the circle. Let the radius of the circle represent 1 unit. A circle with a radius of 1 unit is a **unit circle**.

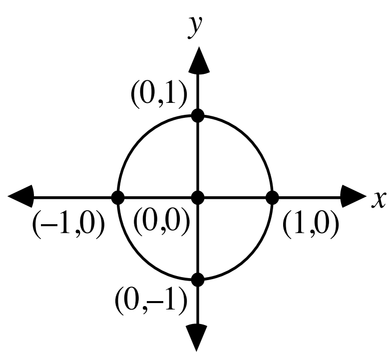

Figure 1: Coordinate system and unit circle

b. **1.** Cut a paper strip with a length of approximately 10 units, where 1 unit equals the radius of the can in Part **a**.

2. Create a number line on one side of the strip, beginning with 0 on the left. Label the points on the number line that represent the whole numbers from 0 to 9, where 1 unit equals the radius of the can.

3. Label the approximate location of each of the following real numbers on the number line: $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$, $7\pi/4$, 2π, $9\pi/4$, $5\pi/2$, and 3π.

4. Label the back of your paper-strip number line with the additive inverses of the labels on the front. Align the labels so that 0 is directly behind 0, $-\pi/4$ is directly behind $\pi/4$, –1 is directly behind 1, and so on.

5. Tape the left-hand end of your number line to the bottom of the can, as shown in Figure **2**.

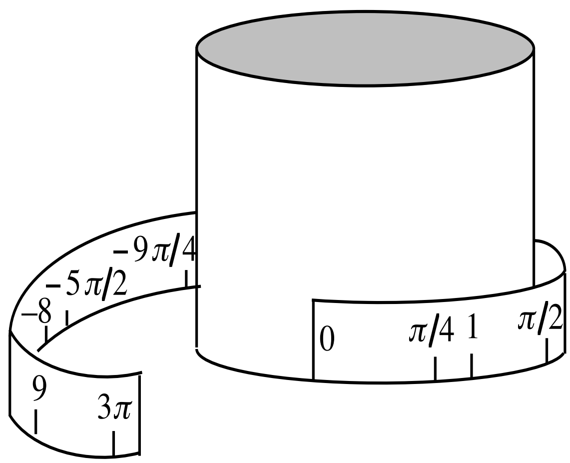

Figure 2: Placement of number line

c. Position the can on your drawing of a unit circle from Part **a** so that the 0 on the number line corresponds with the point (1,0). Use the number line to mark the points on the circle that correspond to labeled points on the number line. This process simulates the function that pairs each point on the real number line with a location on the unit circle, often referred to as a **wrapping function**.

Label the point on the circle that corresponds with 0 on the number line as $W(0)$, the point that corresponds with $\pi/4$ as $W(\pi/4)$, and so on. In this case, *W* indicates that the point has been assigned to the real number by the wrapping function *W*. If any point on the circle corresponds to more than one point on the number line, mark it with all corresponding labels.

d. The location of each point on the unit circle is identified by an ordered pair of coordinates. Approximate the ordered pair that corresponds to each labeled point on the circle. Record them in a spreadsheet with headings like those in Table **1** below.

Table 1: Real numbers and corresponding *xy*-coordinates

Real Number	*x*-coordinate	*y*-coordinate
0	1	0
$\pi/4$	0.7	0.7
1	0.5	0.8
⋮	⋮	⋮

e. Draw a ray from the origin through the point that corresponds to $W(1)$.

f. Leaving the left-hand end of the number line taped to the can, wrap the number line so that the negative values are showing. Repeat Parts **c** and **d** for these values.

g. Create two scatterplots: one in which the domain is the real numbers and the range is the *x*-coordinates, and another in which the domain is the real numbers and the range is the *y*-coordinates.

Note: Save your number line, drawing of the unit circle, and spreadsheet for use throughout the module.

Discussion

a. In the exploration, each labeled point on your number line was matched with an ordered pair on the circle. If the number line were continued indefinitely, how many real numbers would you expect to be paired with each point on the circle?

b.
1. What is the domain of the wrapping function used in the exploration?
2. What is the range of this function?

c. Describe the scatterplots you created in Part **g** of the exploration.

d. Using the wrapping function, what are the greatest and least values that result for each of the following:

1. the *x*-coordinate?
2. the *y*-coordinate?

e. Consider the length of the arc from (1,0) to the point $W(1)$. What is the ratio of this arc length to the radius of the unit circle?

Mathematics Note

On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**. In general, the measure of a central angle in radians is the ratio of the length of the intercepted arc to the radius of the circle.

In Figure **3**, for example, the measure of angle θ is 1 radian because the ratio of the arc length to the radius is $r/r = 1$.

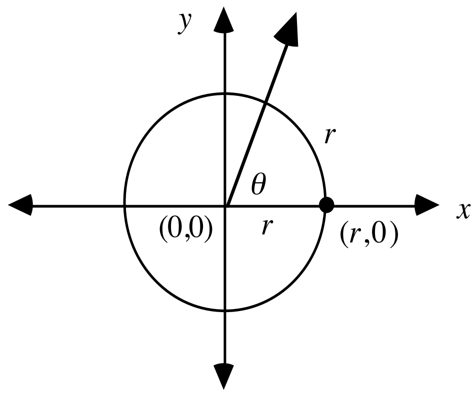

Figure 3: An angle with a measure of 1 radian

f. **1.** What is the measure, in radians, of the central angle that defines the circumference of a circle?

2. Why doesn't this value change as the size of the circle changes?

g. Based on your responses to Part **f** of the discussion, how are radian measures and degree measures related?

Assignment

1.1 **a.** Describe the location of the point (0,1) on the unit circle using a real number.

b. Describe the measure of the central angle whose sides intercept the unit circle at (1,0) and (0,1) in each of the following units:

1. degrees

2. radians

1.2 Consider a point on the unit circle that is paired with the real number 3. Identify a second real number that is paired with this point and describe how you determined this number.

1.3 Describe the radian measures of two central angles, one positive and one negative, whose sides intercept the unit circle at (1,0) and each of the following points:

a. (1,0)

b. (–1,0)

c. a point one-fourth of the way cloclwise around the circle

d. a point four-thirds of the way clockwise around the circle

1.4 Two children are sitting on a merry-go-round. One child is 1 m from the center and the other is 1.5 m from the center.

a. How many radians has each child rotated through after 3 complete rotations of the mery-go-round?

b. Determine the total distance traveled by each child after 3 complete rotations of the merry-go-around.

1.5 Consider a circle with center at the origin and a radius of 5 units. Assume a point starts at (5,0) and moves around the circle at a constant rate of 5 units per second.

a. How many seconds will it take the point to complete one revolution?

b. How long will it take the point to reach (–5,0)?

1.6 The earth's orbit around the sun is roughly circular, with a radius of approximately $1.496 \bullet 10^8$ km.

a. Assuming that the earth moves at a constant speed, how far does it travel in 1 hr?

b. How far does the earth travel between April 1 and July 1?

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1.7 **Angular velocity** describes the rate of change in a central angle over time. Consider a wheel that makes 150 revolutions per minute. What is the wheel's angular velocity in radians per second?

1.8 A rotating water sprinkler makes one revolution every 20 sec. The water reaches a distance of 10 m from the sprinkler.

a. What is the arc length of the sector watered when the sprinkler has rotated through $5\pi/2$ radians?

b. What is the area of the sector watered when the sprinkler has rotated through $\pi/4$ radians?

1.9 A clock maker is designing the face of a watch. The watch face is a circle with a radius of 1 cm. To help specify the location of each element of the design, the clock maker has placed the center of the face at the origin of a two-dimensional coordinate system.

a. Determine the coordinates of the points where the hours 3, 6, 9, and 12 are located. Include a diagram with your response.

b. **1.** Determine the coordinates of the point where 11 o'clock is located.

2. Describe three other points on the face of the watch whose coordinates have the same absolute values as the coordinates of the point associated with 11 o'clock. Identify the hour that corresponds with each pair of coordinates.

c. Determine the coordinates of the hour 33 hours after 12 o'clock.

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Activity 2

In Activity **1**, you used a wrapping function to assign each real number to a location on a unit circle with center at the origin. This allowed you to determine a corresponding
x-and *y*-coordinate for each real number. This wrapping function provides the basis for several other mathematical functions. Because of their association with the unit circle, these functions are often referred to as **circular functions**. In this activity, you examine the graphs and identify some of the characteristics of three circular functions.

Discussion 1

a. Figure **4** shows a right triangle *ABC* constructed in a unit circle with center at the origin.

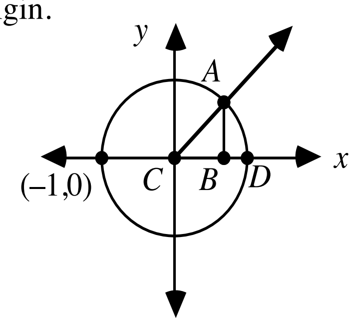


Figure 4: Construction of a triangle in a unit circle

1. Use right-triangle trigonometry to describe the relationship between *AB* and $\angle ACB$. Remember $AC = 1$.

2. Use right-triangle trigonometry to describe the relationship between *BC* and $\angle ACB$.

Mathematics Note

The **sine function**, $f(t) = \sin(t)$, uses the wrapping function to assign a real number t to the y-coordinate of the corresponding point on a unit circle with center at the origin.

The **cosine function**, $f(t) = \cos(t)$, uses the wrapping function to assign a real number t to the x-coordinate of the corresponding point on a unit circle with center at the origin.

For example, the point on the unit circle assigned to $t = 0$ by the wrapping function has coordinates (1,0). Since the x-coordinate of this point is 1, $\cos(0) = 1$. Since the y-coordinate of this point is 0, $\sin(0) = 0$.

b. When a real number is assigned by the wrapping function to a point on the unit circle, the point can be located in one of four possible quadrants.

1. In which quadrants are values of $\sin(t)$ positive? In which quadrants are they negative?
2. In which quadrants are values of $\cos(t)$ positive? In which quadrants are they negative?

c. Figure **5** shows the line tangent to the unit circle at point D.

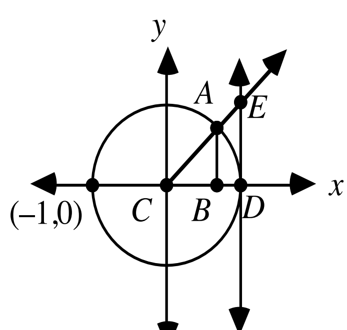


Figure 5: Unit circle with a tangent line

1. What is the measure of $\angle CDE$? Explain your response.
2. Why is $\Delta ABC \sim \Delta EDC$?
3. Using similar triangles, how could you show that $DE = AB/BC$?

d. Considering your responses to Part **a** of the discussion, what would you expect to be the relationship between DE and $\tan \angle ACB$?

Mathematics Note

The **tangent function**, $f(t) = \tan(t)$, where t is any real number except an odd multiple of $\pi/2$, is the ratio of the y-coordinate to the x-coordinate of the point assigned to t by the wrapping function of the real number line around a unit circle with center at the origin.

For example, the point on the unit circle assigned to $t = 0$ by the wrapping function has coordinates (1,0). Since the x-coordinate for this point is 1 and the y-coordinate is 0, $\tan(0) = 0/1 = 0$.

e. In the mathematics note, values of t that are odd multiples of $\pi/2$ are excluded from the domain of the tangent function. Explain why this restriction exists.

f. In which quadrants are values of $\tan(t)$ positive? In which quadrants are they negative?

Exploration

In Activity **1**, you used a number line to mark points on a unit circle. In this exploration, you use these points to create a graph of one of the circular functions.

a. Choose one of the three circular functions: sine, cosine, or tangent.

b.
1. Draw a two-dimensional coordinate system on a sheet of freezer paper.
2. Use your number line from Activity **1** to label the x-axis for each marked point on the number line from -3π to 3π.
3. Label the following additional points on your number line from Activity **1**: 0.25, 0.5, 0.75, 1.25, 1.5, 1.75.
4. Use the number line to label the y-axis in increments of 0.25 for at least 2 units above and 2 units below the origin.

c. On your drawing of the unit circle from Activity **1**, construct a line tangent to the circle at the point $(1,0)$ and $(-1,10)$.

d.
1. Draw a ray from the origin to the first mark on the unit circle (the point that corresponds with $\pi/4$).
2. Construct a perpendicular to the x-axis from the point where the ray intersects the circle. Your drawing should now resemble Figure **6**.

   Note: If you have chosen the tangent function, you must use the intersection of the ray and the tangent through $(-1,0)$ for measures greater than $\pi/2$ radians and less than $3\pi/2$ radians.

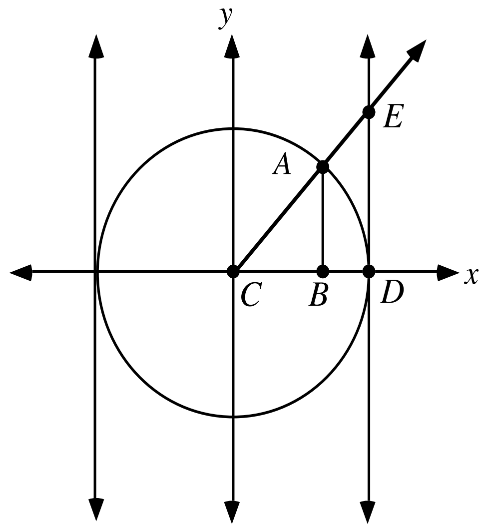


Figure 6: Construction on a unit circle

3. Cut a paper strip whose length corresponds to $f(\pi/4)$, where f represents the circular function you have chosen to plot. (Recall that in Figure **6**, CB corresponds with $\cos\angle ACB$, BA with $\sin\angle ACB$, and DE with $\tan\angle ACB$.)
4. Glue the paper strip onto the coordinate system you created in Part **b**, so that its vertical axis of symmetry aligns with the point on the x-axis that corresponds to $\pi/4$.

   If the value of the function is positive, glue the strip above the x-axis. If the value is negative, glue the strip below the x-axis.

   For example, Figure **7** shows a paper strip representing $\sin(\pi/4)$ glued in the appropriate position.

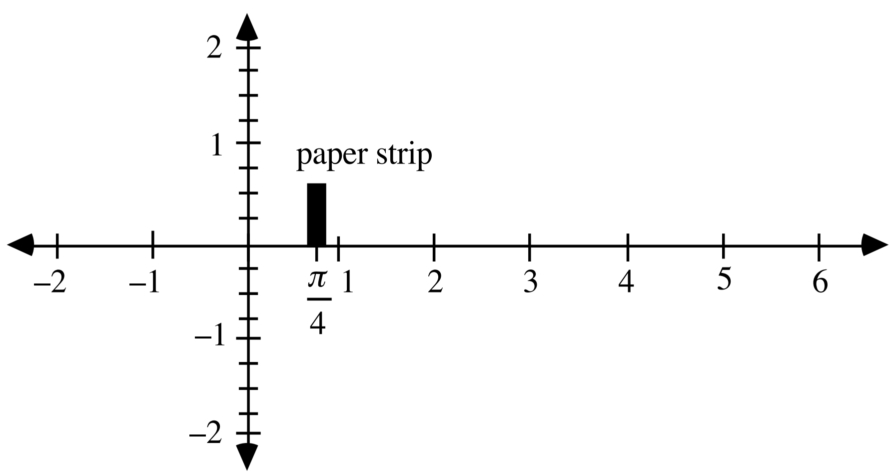


Figure 7: Position of paper strip on coordinate system

e. Repeat the process described in Part **d** for each mark on the unit circle.

f. The center of the top of each paper strip represents a point on the graph of the function. Sketch a smooth curve connecting these points, including points where the value of the function is 0.

g. Using a graphing utility, graph each of the following functions on a separate coordinate system. Describe the characteristics of each graph.

1. $y = \sin x$
2. $y = \cos x$
3. $y = \tan x$

Discussion 2

a. Display your paper-strip graph to the class and describe its characteristics.

b. What do the coordinates of each point on the smooth curves drawn in Part **f** represent?

c. Compare the paper-strip graphs, the scatterplots created in Activity **1**, and the graphs created using a graphing utility in Part **g** of the exploration.

d.
1. What interval of the domain was graphed before the values of each function began to repeat?
2. How are these intervals related to the unit circle?

e.
1. Identify the domain and range for each of the three circular functions: sine, cosine, and tangent.
2. How does the unit circle determine the range values for each of these functions?

Mathematics Note

A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest interval of the domain over which the function repeats.

The **absolute maximum** of a function is the greatest value of the range. The **absolute minimum** is the least value of the range.

If both an absolute maximum and an absolute minimum exist in a periodic function, the **amplitude** of the function is half the distance between them. If M represents the absolute maximum and m represents the absolute minimum, the amplitude can be found by the following formula:

$$\left|\frac{M - m}{2}\right|$$

For example, Figure **8** shows a periodic function with a period of 2. In other words, the function completes 1 cycle for every interval of 2 units in the domain.

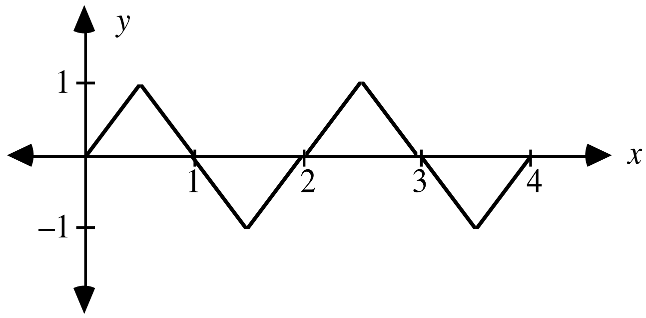


Figure 8: A periodic function

Since the absolute maximum is 1 and the absolute minimum is –1, the amplitude of this function can be found as follows:

$$\left|\frac{1-(-1)}{2}\right| = 1$$

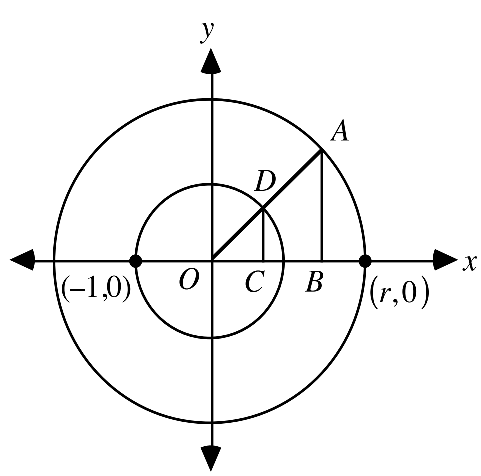
f. Identify the period and amplitude of the sine, cosine, and tangent functions.

g. How does the relationship below explain the characteristics of the graph of the tangent function?

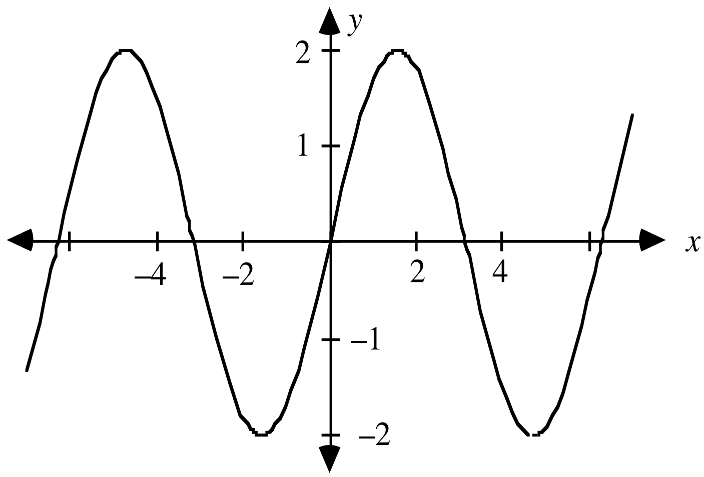
$$\tan x = \frac{\sin x}{\cos x}$$

Assignment

2.1 The figure below shows a unit circle with center at the origin and a circle of radius r with center at the origin. As you have seen in the previous activity, the coordinates of point D are $(\cos\angle DOC, \sin\angle DOC)$. What are the coordinates of point A? Justify your response using similar triangles.



2.2 Use the graph below to answer the following questions.



a. What is the period of the curve?

b. What is the amplitude of the curve?

c. Sketch a curve that has the same period as the given curve but twice its amplitude.

d. Sketch a curve that has the same amplitude as the given curve but twice its period.

e. Sketch a curve that has twice the amplitude and twice the period of the given curve.

2.3 **a.** Graph $y = \sin x$ over the interval $[0, 4\pi]$ and $y = 0.5$ on the same coordinate system.

b. Determine the values of x for which $\sin x = 0.5$ and explain how you identified these values.

c. If you had graphed $y = \sin x$ over the interval $[-4\pi, 4\pi]$, for how many values of x would $\sin x = 0.5$? Justify your response.

2.4 **a.** An **identity** is an equation that is true for all real numbers. For example, the equation $0 \bullet x = 0$ is an identity since it is true for any real number x. There are many identities involving circular functions, including the three shown below. Verify that each of these equations is true for several values of θ.

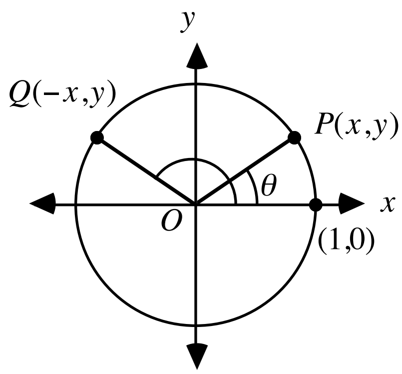
1. $\sin\theta = \sin(\pi - \theta)$

2. $\cos\theta = \cos(-\theta)$

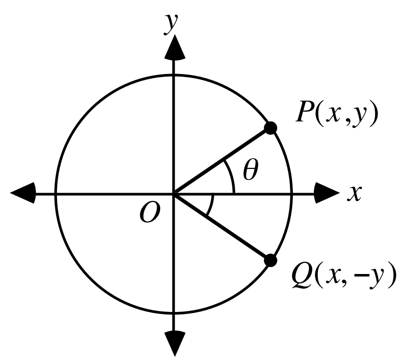
3. $\tan\theta = \tan(\pi + \theta)$

b. The diagrams below show a unit circle with center at the origin and two points on the circle, P and Q. Also, $\angle QOR$ is equal to θ in each diagram. Use these figures to explain why each identity is true.

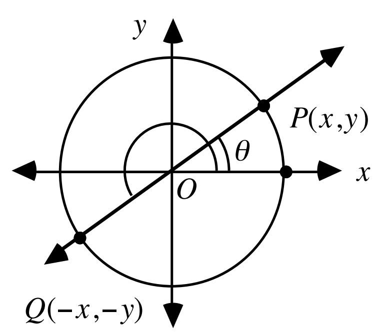
1. $\sin\theta = \sin(\pi - \theta)$



2. $\cos\theta = \cos(-\theta)$



3. $\tan\theta = \tan(\pi + \theta)$



c. How could you use the identities listed in Part **a** to determine two real numbers that have the same sine, cosine, or tangent?

2.5 Chronobiology is the study of biological rhythms. Scientists in this field have found that seasonal changes in the length of the day can influence the behavior and metabolism of animals. The following table lists the number of daylight hours for specified dates in Boston, Massachusetts.

Date	Day	Daylight Hours	Date	Day	Daylight Hours
12/01/91	1	9.30	06/28/92	211	15.25
12/15/91	15	9.10	07/01/92	214	15.23
12/28/91	28	9.10	07/15/92	228	14.98
01/01/92	32	9.13	07/28/92	241	14.60
01/15/92	46	9.43	08/01/92	245	14.47
01/28/92	59	9.85	08/15/92	259	13.92
02/01/92	63	10.00	08/28/92	272	13.33
02/15/92	77	10.58	09/01/92	276	13.17
02/28/92	90	11.18	09/15/92	290	12.50
03/01/92	92	11.23	09/28/92	303	11.88
03/15/92	106	11.88	10/01/92	306	11.75
03/28/92	119	12.52	10/15/92	320	11.10
04/01/92	123	12.72	10/28/92	333	10.52
04/15/92	137	13.37	11/11/92	337	10.35
04/28/92	150	13.93	11/15/92	351	9.80
05/01/92	153	14.07	11/28/92	364	9.40
05/15/92	167	14.60	12/01/92	367	9.30
05/28/92	180	14.98	12/15/92	381	9.10
06/01/92	184	15.07	12/28/92	394	9.10
06/15/92	198	15.27			

Source: *The Old Farmer's Almanac*, 1992.

a. Use a graphing utility to create a scatterplot of the data.

b. Does the resulting graph appear to be related to the sine function? Explain your response.

c. What is the amplitude of the curve? Explain your response.

d. What is its period? Explain your response.

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2.6 In Problem **2.4**, you examined three trigonometric identities. There are many, many more. The three identities listed in Part **a**, for example, show relationships between the sine and cosine functions.

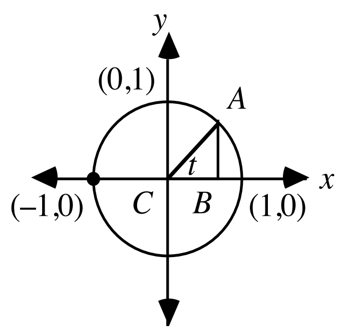
a. Verify that the following equations are true for several values of t.

1. $(\sin t)^2 + (\cos t)^2 = 1$

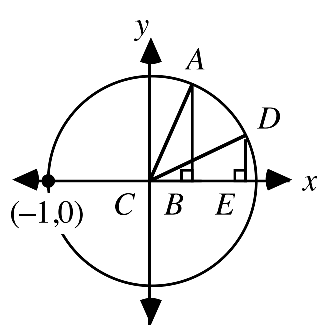
2. $\sin t = \cos\left(\frac{\pi}{2} - t\right)$

3. $\cos t = \sin\left(\frac{\pi}{2} - t\right)$

b. Use right-triangle trigonometry, the Pythagorean theorem, your knowledge of the circular functions, and the figure below to argue that $(\sin t)^2 + (\cos t)^2 = 1$ is an identity for all real numbers that can be represented in the first quadrant on a unit circle.



c. In the following diagram, $m\angle DCE = (\pi/2) - m\angle ACB$. Use this relationship, along with right-triangle trigonometry, similar triangles, and your knowledge of the circular functions, to argue that $\sin t = \cos((\pi/2) - t)$ is an identity for all real numbers that can be represented in the first quadrant on a unit circle.



2.7 A carousel has a radius of 6 m and completes 6 revolutions in 1 min.

a. Through how many radians does the carousel turn in 1 min?

b. A horse named Moonlight is located 5 m from the center of the carousel. Assuming that the center of the carousel is located at the origin of a two-dimensional coordinate system, and that Moonlight's initial position is (5,0), what are Moonlight's coordinates after 3 min?

c. A horse named Red Ribbon is located 3.5 m from the center of the carousel, next to Moonlight. What are Red Ribbon's coordinates after 3 min?

d. Determine the coordinates for the locations of Moonlight and Red Ribbon after $13\pi/4$ radians.

2.8 Some people believe that biological rhythms influence our personal lives. In its simplest form, the biorhythm theory states that from birth to death, each of us is influenced by three internal cycles: the physical, the emotional, and the intellectual.

The 23-day physical cycle affects a broad range of bodily functions, including resistance to disease and strength. The 28-day emotional cycle governs creativity, sensitivity, and mood. The 33-day intellectual cycle regulates memory, alertness, ability to learn, and other mental processes.

According to this theory, each of the cycles starts at a neutral baseline or zero point on the day of birth. From that zero point, the three cycles enter a rising, positive phase, during which the energies and abilities associated with each one are high, then gradually decline. Each cycle crosses the zero point midway through its period, entering a negative phase in which physical, emotional, and intellectual capabilities are somewhat diminished, and energies are recharged.

Since the three cycles have different periods, the highs, lows, and baseline crossings rarely coincide. As a result, people are usually subject to a mix of biorhythms and their behavior—from physical endurance to academic performance—is a composite of these varying influences.

a. Sketch your first physical, emotional, and intellectual cycles on the same set of axes. On this graph, let the origin represent birth.

b. What is the period for each of the three cycles?

c. How does the graph of a sine curve compare with the graph of the physical cycle? the emotional cycle? the intellectual cycle?

d. According to biorhythm theory, the most vulnerable days occur when each cycle crosses the baseline, switching from positive to negative or vice versa. These are "critical days." Mark and label the critical days for each of the three cycles on the graph.

e. Determine your biological rhythms for today.

f. In an average human life span of 70 years, when will the three cycles coincide on the baseline?

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Activity 3

In order to approximate cyclic events with circular functions, it is often necessary to modify $y = \sin x$ or $y = \cos x$. In this activity, you investigate the general forms of the circular functions and explore how to transform the shapes of their graphs.

Exploration

The general form of the sine function is $y = a(\sin b(x + c)) + d$, where *a*, *b*, *c*, and *d* are real-number parameters. In this exploration, you investigate the transformations that result from using different values for these four parameters.

a.
1. Using a graphing utility, graph the functions $y = \sin x$, $y = 2\sin x$, $y = 4\sin x$ where *x* is a real number, on the same coordinate system. **Note:** Set your graphing utility to report angle measures in radians.
2. On a second coordinate system, graph $y = \sin x$, $y = -2\sin x$, and $y = -4\sin x$.
3. Make a prediction about the role of *a* in the graph of $y = a\sin x$.
4. Check your prediction by graphing $y = \sin x$ and $y = a\sin x$ for various values of *a*.

b.
1. On the same coordinate system, graph $y = \sin x$ and $y = \sin bx$ for a chosen value of *b*.
2. Repeat Step **1** for two other values of *b*.
3. Make a prediction about the role of *b* in the graph of $y = \sin bx$
4. Check your prediction by graphing $y = \sin x$ and $y = \sin bx$ for various values of *b*.

c. Using a process similar to the one described in Part **b**, determine the role of *c* in the graph of $y = \sin(x + c)$. Hint: Examine the graphs for the following values of *c*: $\pi/4$, $\pi/2$, π, $-\pi/4$, $-\pi/2$, and $-\pi$.

d. Using a process similar to the one described in Part **b**, determine the role of *d* in the graph of $y = \sin x + d$.

e. **1.** Based on your results in Parts **a–d**, sketch a graph of $y = 3\sin(2(x - \pi)) - 5$ on graph paper.

2. Check the accuracy of your sketch by graphing the equation on a graphing utility.

Discussion

a. Describe the role of each of the four parameters in the graph of the equation $y = a(\sin b(x + c)) + d$.

b. Defend the following statement: "The graph of the cosine function is a transformation of the graph of the sine function."

Mathematics Note

Given a function of the form $y = a(\sin b(x + c)) + d$, the parameters a, b, c, and d transform the graph of $y = \sin x$ in the following manner:

- The amplitude of the graph is $|a|$.
- If a is negative, the graph is a reflection of $y = \sin x$ in the x-axis.
- The period of the graph is $2\pi/b$.
- If b is negative, the graph is a reflection of $y = \sin x$ in the y-axis.
- The graph is a horizontal translation of $y = \sin x$ of $-c$ units.
- The graph is a vertical translation of $y = \sin x$ of d units.

For example, the graph of $y = -3(\sin 2(x + \pi/2)) + 1$ has an amplitude of 3 and a period of π. As shown in Figure **9**, the graph also represents a horizontal translation of $-\pi/2$ units, a vertical translation of 1 unit, and a reflection in the x-axis of a graph of $y = \sin x$.

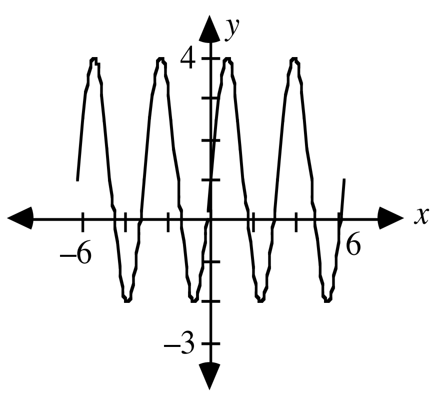
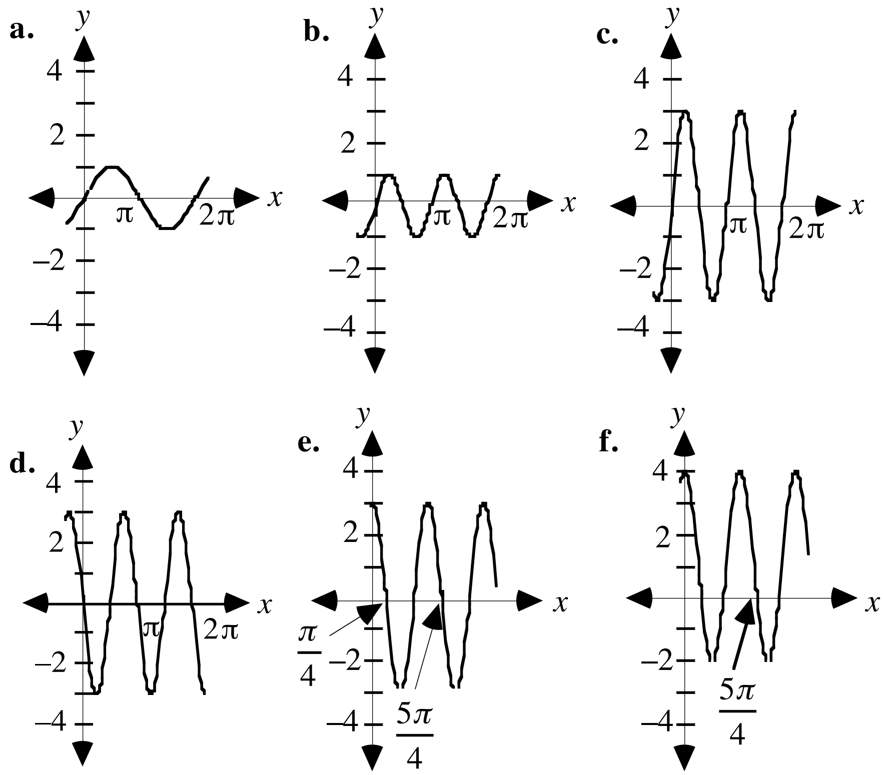


Figure 9: Graph of $y = -3\sin(2(x + \pi/2)) + 1$

These parameters play the same roles in the graph of $y = a(\cos b(x + c)) + d$.

Assignment

3.1 The graph of $y = -3\sin(2(x - \pi/4)) + 1$ can be obtained from the graph of $y = \sin x$ using the sequence of transformations shown in Parts **a–f** below. Determine the equation of each graph.

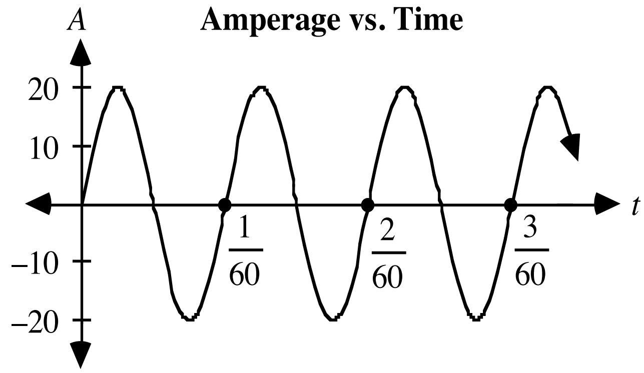


3.2 The flow of electrical current is measured in **amperes** (A). In the United States, ordinary household circuits use alternating current of 20 A. An alternating current flows in one direction during part of a generating cycle, and in the opposite direction during the rest of the cycle.

The rate at which the current alternates is its **frequency**. Frequency is measured in cycles per second. A frequency of 1 cycle per second equals 1 **hertz** (Hz). The frequency of ordinary household current is 60 Hz.

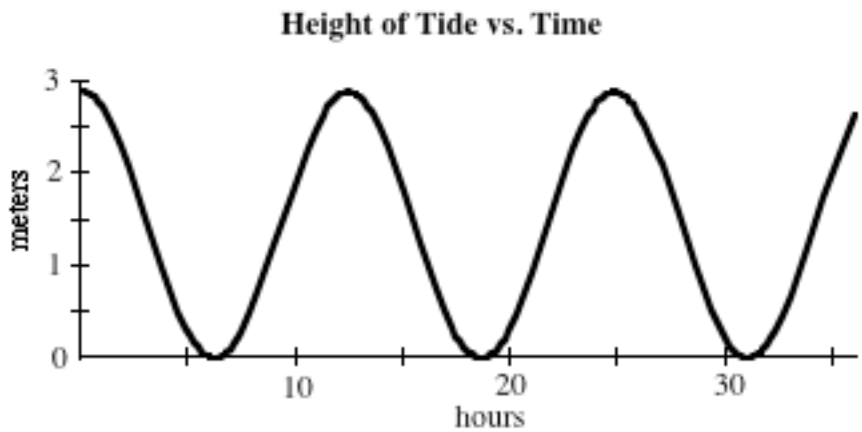
Because of the way in which the flow of alternating current changes over time, it can be modeled by a sine function, as shown in the following graph.

Determine the equation of the curve below using A to represent amperage and t to represent time. (Hint: The period of this function equals the reciprocal of the frequency.)



3.3 In the ocean near Boston, Massachusetts, the average high tide exceeds the level of the water at low tide by 2.9 m. The tide comes in and goes out every 12.4 hr. This fluctuation in height can be roughly approximated by a cosine curve, as shown in the following graph.

Determine an equation for the curve below, using h to represent the height of the water above low tide and t to represent the number of hours since high tide. Assume that the water is at high tide at $t = 0$.



3.4 The Bay of Fundy in eastern Canada has the highest tides in the world. The tides there rise and fall by as much as 15 m. The average high tide is approximately 8 m and the tidal cycle takes 12.4 hr.

a. **1.** Determine the equation of a cosine curve that models the tides at this bay if the water is at high tide at $t = 0$.

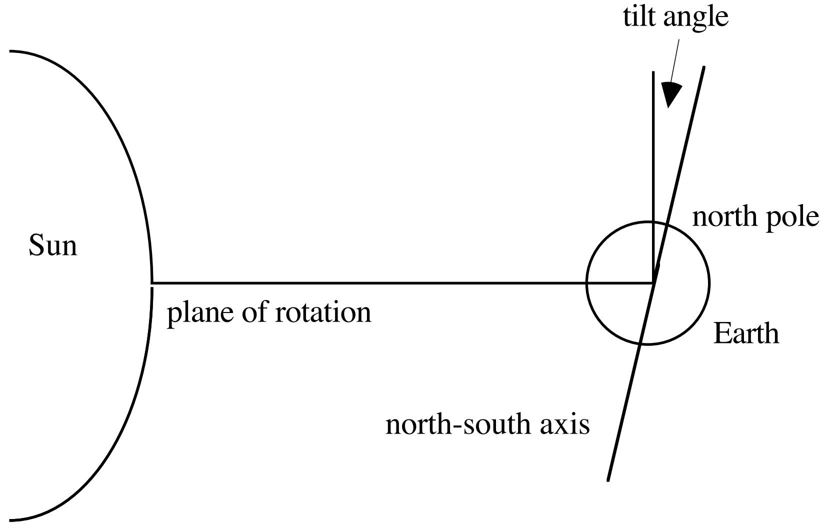
2. Calculate the change in the height of the tide from $t = 3$ to $t = 4$, where t represents hours after high tide.

b. **1.** Determine the equation of a sine curve that models the tides at this bay if the water is 4 m above low tide and rising at $t = 0$.

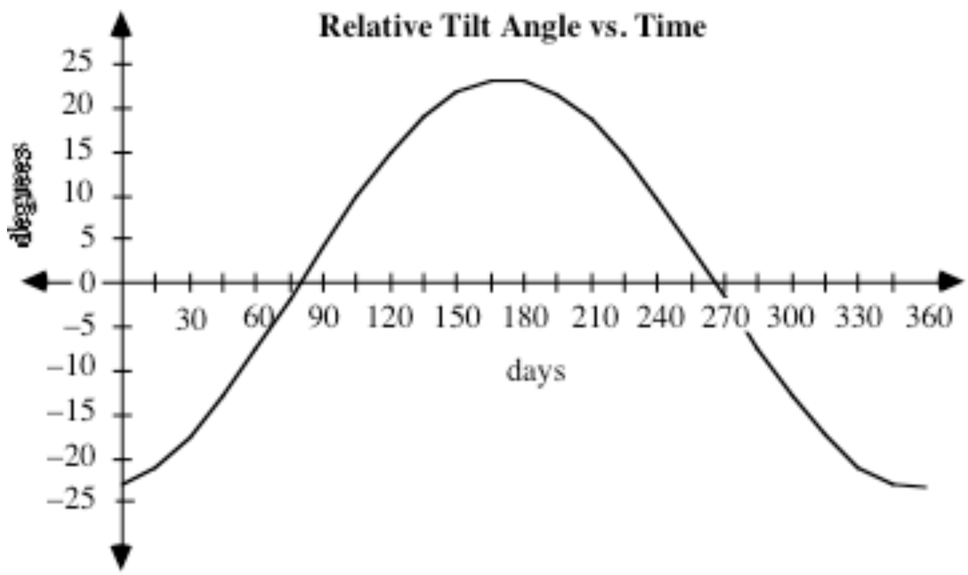
2. Calculate the change in the height of the tide from $t = 3$ to $t = 4$

.

3.5 As shown in the following diagram, the earth's north-south axis is tilted relative to a line perpendicular to its plane of rotation about the sun.



As the earth rotates about the sun, this relative angle changes. The following graph shows the change in the earth's relative 'tilt' for 360 days.



a. What are the period and amplitude of this graph?

b. Which curve, the sine or the cosine, would you use to model this graph? Justify your response.

c. How would you alter the function selected in Part **b** to model the graph more closely?

* * * * *

3.6 The original Ferris wheel was built in Chicago in 1893. Named after its inventor, George W. Ferris, the wheel had 36 cars, each of which could seat 60 people. The diameter of the wheel was about 76 m. It completed 5 revolutions every 6 min.

a. Imagine that you are riding the original Ferris wheel. At $t = 0$, the height of your chair above the ground is 0 m. Determine a circular function that models the vertical motion of your chair on the wheel.

b. How are the values for the period and amplitude represented in your model equation?

c. Describe the vertical or horizontal translations, if any, in your model equation.

d. Near the end of the ride, the operator slows the Ferris wheel to 0.5 revolutions per minute. Determine a circular function that best models the motion of your chair at this rate.

3.7 A home furnace turns on whenever the room temperature drops below the thermostat setting. The furnace stays on until the temperature reaches a specific number of degrees above the thermostat setting.

a. Estimate how often this cycle of heating and cooling occurs and select reasonable values for the maximum and minimum room temperatures.

b. Model the room temperatures over time with a circular function.

c. What errors are possible when using a sine or cosine function to model the temperatures in a room?

* * * * * * * * * *

Activity 4

One of the real-world events that can be modeled by a circular function is the motion of a mass on a spring. For example, Figure **10** shows an object suspended from a spring. When the object is not moving, its position is referred to as its equilibrium point. As the object bounces up and down, its distance in centimeters above or below the **equilibrium point** can be modeled by the equation $d = -3\cos t$, where t represents time in seconds.

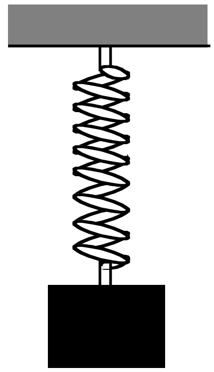


Figure 10: Object on a spring

Given what you have learned about the cosine function, you can use $d = -3\cos t$ to determine the object's distance from the equilibrium point for any time t. However, what if you had to determine the times at which the mass would be 2 cm above its equilibrium point? To do this, you must identify the values of t that correspond with $d = 2$. To find these values, it may help to consider the concept of **inverses**.

Mathematics Note

The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.

For example, consider the relation $\{(0,2), (1,3), (4,-2), (-3,-2)\}$. The inverse relation is $\{(2,0), (3,1), (-2,4), (-2,-3)\}$.

If the inverse of a function f is also a function, it is denoted by f^{-1}.

For example, consider the function $g = \{(0,2),\ (1,3),\ (4,-2)\}$. The inverse of this function is the function $g^{-1} = \{(2,0),\ (3,1),\ (-2,4)\}$.

Exploration 1

In this exploration, you examine the inverse of a circular function.

a. In Activity **2**, you learned that the sine of a real number is the y-coordinate of a point on a unit circle with center at the origin assigned to that real number by a wrapping function. Likewise, the cosine is the x-coordinate and the tangent is the ratio of the y-coordinate to the x-coordinate.

1. Modify the headings of your spreadsheet from Part **d** of the exploration of Activity **1**, replacing "y-coordinate" with "sine" and "x-coordinate" with "cosine."
2. Add another column to the spreadsheet to calculate the tangent of each real number you marked on the unit circle.

b. Select either the sine or cosine function. Create a scatterplot of the inverse of the chosen function.

c. Using the scatterplot, determine if the inverse is also a function.

Discussion 1

a. Describe how the scatterplot of the function you selected in Exploration **1** compares with the scatterplot of its inverse.

b.
1. Judging from the values in the spreadsheet, what do you think a graph of the inverse of the tangent function will look like?

2. Do you think the inverse tangent relationship is a function? Explain your response.

c. Are the inverses of the sine and cosine functions also functions?

d. The graph in Figure **11** shows the distances above and below the equilibrium point, over time, for the mass in Figure **10**. Describe what each function in the graph represents and how you would use the graph to approximate the times when the object is 2 cm above and 2 cm below its equilibrium point.

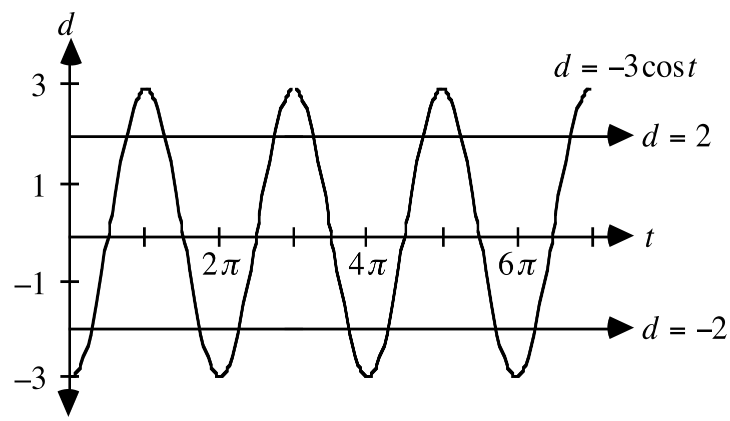


Figure 11: Graph of distance from equilibrium point versus time

Exploration 2

a. Select one of the three circular functions: sine, cosine, or tangent. Restrict its domain so that the inverse of the restricted relationship represents a function. This inverse function should include all range values of the original circular function.

b. The inverse functions for sine, cosine, and tangent can be denoted as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively. Using a graphing utility, graph the inverse of the function selected in Part **a**.

c.
1. Identify the range of the inverse function graphed by the utility.

2. Determine the corresponding restrictions made on the domain of the original function.

d. 1. Select a value, x_1 within the restricted domain identified in Part **c** for the sine function.

2. Evaluate $\sin(x)$.

3. Evaluate the inverse $\sin(x)$ noted as $\sin^{-1}(\sin x)$

4. Compare $\sin^{-1}(\sin x)$ to the original x value selected in Step **1**.

5. Repeat Steps **2–4** for a value outside the domain identified in Part **c** and record your observations.

6. Repeat Steps **1-5** for cosine and tangent.

e. An inverse function can be used to determine when the mass in Figure **10** will be 2 cm below its equilibrium point by solving the equation $-2 = -3\cos t$ for t.

After dividing both sides of the equation by –3, the value of t can be found using the inverse cosine as shown below:

$$-2 = -3\cos t$$
$$-2/-3 = \cos t$$
$$\cos^{-1}(2/3) = \cos^{-1}(\cos t)$$

The inverse function "undoes" the original function, the resulting equation is:

$$\cos^{-1}(2/3) = t$$

1. Determine the value of t in the equation $\cos^{-1}(2/3) = t$.

2. Determine when the mass in Figure **10** will be 2 cm above its equilibrium point by solving the equation $2 = -3\cos t$ for t.

Discussion 2

a. Describe how you determined the appropriate restrictions on the domain of the function in Part **a** of Exploration **2**.

b. How did these restrictions compare with the restrictions used by the graphing utility in Part **c** of Exploration **2**?

c. In Part **d** of Exploration **2**, you used the inverse of a function to "undo" the function.

1. What results did you observe using values within the restricted domain of the function?

2. What results did you observe using values outside the restricted domain?

3. Explain why you think these results occurred.

d. In Part **d** of Discussion **1**, you used the graph of $d = -3\cos t$ to determine when the mass on the spring would be 2 cm above or below its equilibrium point. How did the solutions found in Part **e** of Exploration **2** compare to those determined in Discussion **1**? Explain why this occurs.

e. Using the inverse cosine function to solve $2 = -3\cos t$ for t yields only a single solution. In Problem **2.4**, you were introduced to the following trigonometric identities: $\sin\theta = \sin(\pi - \theta)$, $\cos\theta = \cos(-\theta)$, and $\tan\theta = \tan(\pi + \theta)$.

Describe how these identities, along with the periodic nature of the circular functions, can help you determine all possible solutions for $2 = -3\cos t$, including those not found using the inverse cosine function.

f. What is the result of $\sin^{-1}(\sin(3x + 1))$?

Assignment

4.1 **a.** Solve each of the following equations without using a symbolic manipulator, identifying at least two possible values of x.

1. $-\sin(x) = 0.75$

2. $2\tan(x) = 6.5$

3. $\cos(3x) = 0.15$

4. $4\sin(2x) + 1 = 1.75$

5. $-2\cos(3(x - 1)) + 5 = -4.32$

b. Use technology to check your responses in Part **a**.

4.2 The hours of light in each day changes with the seasons. At locations near 40° N latitude, the hours of daylight range from a minimum of about 9 hr to a maximum of approximately 15 hr.

Assuming that the mean number of daylight hours occurs on March 21 (and that it is not a leap year), the number of daylight hours on any given day can be modeled by the following equation:

$$h = 12 + 3\sin\left(\frac{2\pi}{365}d\right)$$

where h represents the number of daylight hours and d represents the number of days after March 21.

a. **1.** Based on this model, what is the mean number of daylight hours in a year?

2. When do the longest and shortest days of the year occur?

b. During what dates would you expect to have at least 12 hr of daylight?

c. The model described above is based on the sine function. To apply a similar model to a location at 40° S latitude, the graph must be reflected in the line $y = 12$. What equation would you use to model the daylight hours for a location at 40° S latitude?

d. Use your model from Part **c** to determine when you would expect to have at least 12 hr of daylight in a location at 40° S latitude.

4.3 As the harbormaster at a seaport, you must be aware of the change in depth that occurs due to the rise and fall of the tides. To enter or leave the port, each ship requires a minimum depth of water. The required depth varies with each ship and depends on whether it is loaded with cargo or not. The depth of the entrance can be modeled by the equation $d = \cos(0.51t) + 5.2$, where d is the depth of the water and t is the number of hours after 12:00 noon today.

a. At 1:30 P.M., a ship that requires 4.4 m of water asks to enter the harbor. How much time does the ship have before the entrance becomes too shallow?

b. If the ship enters the harbor at the latest possible time identified in Part **a**, how soon after this time could the ship leave the port?

c. The ship in Part **a** must be back at sea no more than 26 hours after entering the harbor. Determine the latest possible time the ship can leave the port.

4.4 As a playground swing moves back and forth, its horizontal velocity can be modeled by a circular function. Consider a swing that has a horizontal velocity of 0 twice in each second and reaches a maximum horizontal velocity of 5 m/sec.

a. Determine an equation that models the swing's horizontal velocity over time.

b. On a graph of your model, identify the times when the swing's height above the ground is the greatest. Justify your choices.

c. At what times will the swing be traveling at a horizontal velocity of 2 m/sec? Justify your response.

* * * * *

4.5 The horizontal distance between a pendulum and a motion detector can be modeled by the function $d = 1.1\cos(20.9(t - 8.5)) + 3.2$, where d is the distance in centimeters and t is the time in seconds.

a. What is the greatest horizontal distance between the pendulum and the motion detector? Explain your response.

b. At what times will the pendulum be as far away from the motion detector as possible? Explain your response.

c. At what times during its swing will the pendulum be hanging straight down? Explain your response.

4.6 As noted in Problem **3.2**, ordinary household circuits carry alternating current. This current can be modeled by the function $i = I_{max}\sin\theta$, where i is the current, I_{max} is the maximum current, and θ is the angle of rotation measured in the generator.

a. Write a function that could be used to model alternating current with a maximum of 20 A.

b. Graph the function in Part **a** over the interval $[0,\ 4\pi]$.

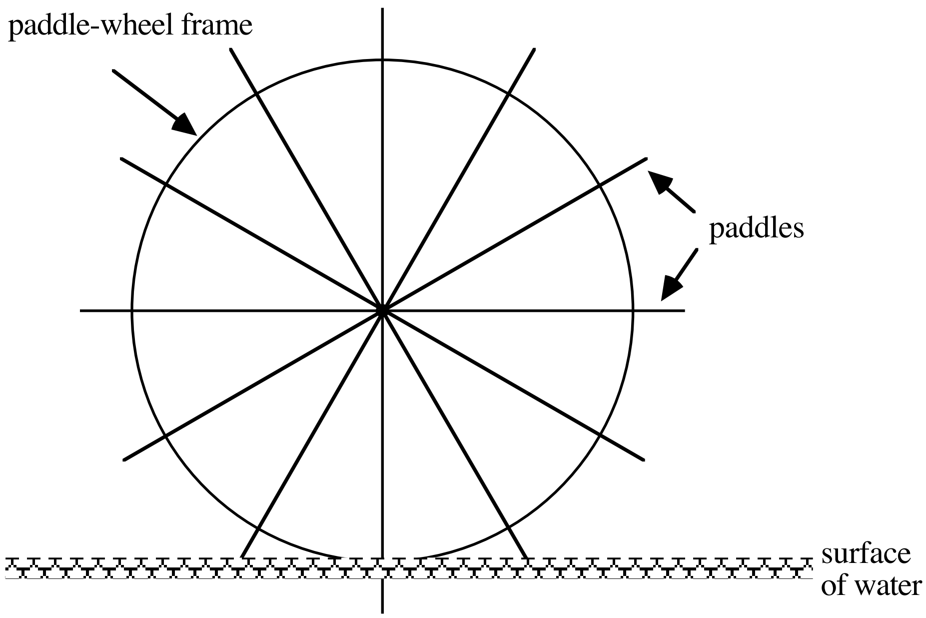
c. The effective value of an alternating current equals $0.707 \bullet I_{max}$. Determine the values of θ when i equals the effective current.

* * * * * * * * * *

Summary Assessment

A paddle wheel on the back of a riverboat measures 1.91 m from the center of the wheel to the end of each paddle. The circular frame of the wheel has a diameter of 3.3 m. The wheel rotates so that the circular frame is tangent to the water level. It completes one rotation every 36 sec.

1. Graph the distance (height) from the surface of the water to the tip of one of the paddles with respect to time. Indicate which paddle you used and any assumptions you made.



a. What is the amplitude of the graph?

b. What is the period of the graph?

c. Write the cosine function that models the graph.

d. Write the sine function that models the graph.

2. **a.** How many radians does one paddle rotate through in 6 sec?

b. How long does it take one paddle to rotate through $15\pi/4$ radians?

c. Determine the angular velocity of a paddle in radians per minute.

3. The paddles on the wheel are spaced so that when one paddle is entering the water, a second paddle is in the water, while a third is exiting the water. Determine the number of evenly-spaced paddles on the paddlewheel.

Module Summary

- A **unit circle** has a radius of 1 unit.
- On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**.
- In general, the measure of a central angle in radians is the ratio of the length of the intercepted arc to the radius of the circle.
- The **wrapping function** pairs each point on the real number line with a location on the unit circle.
- The **sine function**, $f(t) = \sin(t)$, uses the wrapping function to assign a real number t to the y-coordinate of the corresponding point on a unit circle with center at the origin.
- The **cosine function**, $f(t) = \cos(t)$, uses the wrapping function to assign a real number t to the x-coordinate of the corresponding point on a unit circle with center at the origin.
- The **tangent function**, $f(t) = \tan(t)$, where t is any real number except an odd multiple of $\pi/2$, is the ratio of the y-coordinate to the x-coordinate of the point assigned to t by the wrapping function of the real number line around a unit circle with center at the origin.
- The tangent function is equivalent to the ratio of the sine function to the cosine function:

$$\tan x = \frac{\sin x}{\cos x}$$

- A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest interval of the domain over which the function repeats.
- The **absolute maximum** of a function is the greatest value of the range. The **absolute minimum** is the least value of the range.
- If both an absolute maximum and an absolute minimum exist in a periodic function, the **amplitude** of the function is half the distance between them. If M represents the absolute maximum and m represents the absolute minimum, the amplitude can be found by the following formula:

$$\left|\frac{M - m}{2}\right|$$

- Given a function of the form $y = a(\sin b(x + c)) + d$, the parameters a, b, c, and d transform the graph of $y = \sin x$ in the following manner:
  - The amplitude of the graph is $|a|$.
  - If a is negative, the graph is a reflection of $y = \sin x$ in the x-axis.
  - The period of the graph is $2\pi / b$.
  - If b is negative, the graph is a reflection of $y = \sin x$ in the y-axis.
  - The graph is a horizontal translation of $y = \sin x$ of $-c$ units.
  - The graph is a vertical translation of $y = \sin x$ of d units.

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