Motion Pixel Productions

Thanks to computer animation, you can now practice landing an airplane, synthesizing dangerous chemicals, even performing medical surgery. And all without fear of mistakes or injury. In this module, you use matrices to explore the applications of transformational geometry in computer graphics.

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Introduction

Many industries use computer animation to simulate unsafe, unreal, or otherwise extraordinary situations. For example, flight schools use computer simulators to allow pilots to practice dangerous landing situations. Chemical engineers use animation to study the molecular structure of experimental drugs. And Hollywood producers create everything from dinosaurs to starships with computer-generated special effects.

Computer animation requires more than just an understanding of art and motion. Animators must use computer software to describe each frame of a scene in a language the computer can understand. In some types of software, each point in a frame is given coordinates in three dimensions; the entire frame is then represented by matrices. The computer manipulates these matrices to build a vivid, three-dimensional world in the two-dimensional space of a video screen.

Discussion

a. What are some recreational uses of computer animation?

b. Describe some computer-generated graphics that you have seen recently on television or in magazines.

c. Computer-aided design (or CAD) software helps individuals create graphics on computers. What types of businesses might use CAD in designing or marketing their products?

Activity 1

To create the illusion of motion, animators once produced thousands of frames of film by hand, changing the positions of each figure slightly from frame to frame. When these frames were displayed in sequence at 24 frames per second, they produced the impression that the figure actually moved.

Each change in position can be thought of as a transformation. Recall that a **transformation** is a function, a one-to-one correspondence whose domain is a plane and whose range is the same plane. Under a transformation, each point in a **preimage** is paired with a point in the **image**. In order for a computer to display these images, the transformations may be described using matrices.

Discussion 1

a. Consider an animated scene that contains a figure shaped like a triangle. The coordinates of the vertices of the original figure, or preimage, can be represented in a matrix. Describe the possible dimensions of this matrix.

b. The figure created by a transformation is an image. If you performed a transformation on the triangle in Part **a**, what would be the dimensions of the image matrix?

c. Since multiplication of matrices is not commutative, order must be considered. If a preimage is represented in a 2×3 matrix, describe the order of multiplication with a 2×2 transformation matrix that will result in the desired image.

Mathematics Note

A **dilation** is a transformation that pairs a point *C*, the center, with itself and any other point *P* with a point P' on ray *CP* so that $CP'/CP = r$, where *r* is the **scale factor**. A dilation with center at point *C* and a scale factor of *r* is denoted as $\mathrm{D}_{C,r}$. For example, a dilation with center at point *E*(1,4) and a scale factor of 2 can be represented as $\mathrm{D}_{E,2}$.

A **rotation** is a transformation that pairs one point *C*, the center, with itself and every other point *P* with a point P' that lies on a circle with center *C* such that $m\angle PCP'$ is the magnitude of the rotation. A rotation of θ degrees with center at point *P* is denoted as $\mathrm{R}_{P,\theta}$. The value of θ is positive for counterclockwise rotations and negative for clockwise rotations. For example, a counterclockwise rotation of 90° with center at point *C*(–3,2) can be represented as $\mathrm{R}_{C,90^\circ}$. A clockwise rotation of 45° with the same center can be denoted by $\mathrm{R}_{C,-45^\circ}$.

A **reflection** in a line *m*, denoted by r_m, is a transformation that pairs each point on the line with itself and each other point *P* with a point P' so that *m* is the perpendicular bisector of $\overline{PP'}$. For example, a reflection in the line *l* whose equation is defined as $y = -x$ can be represented as r_l.

A **translation** is a transformation that pairs every point $P(x,y)$ with an image point $P'(x+h, y+k)$. A translation with point *P* as the preimage and point P' as the image is denoted as $\mathrm{T}_{P,P'}$.

For example, a translation with *B*(–5,2) as the preimage and $B'(1,-2)$ as the image can be represented as $\mathrm{T}_{B,B'}$. This translation represents a horizontal movement of $1-(-5)=6$ units and a vertical movement of $-2-2=-4$ units. Using the Pythagorean theorem and right-triangle trigonometry, this is equivalent to a movement of $\sqrt{6^2+(-4)^2} = \sqrt{52}$ units at a direction angle of $\tan^{-1}(-4/6) \approx -33.7^\circ$.

d. Describe how the preimage is changed under each of the four transformations described in the mathematics note. (You examined these four transformations in the Level 2 module "Crazy Cartoons.")

e. A preimage and its image have the same **orientation** if they have the same sequence of corresponding vertices when read either clockwise or counterclockwise.

In Figure **1**, for example, ΔABC and $\Delta A'B'C'$ do not have the same orientation. If the vertices of ΔABC are read counterclockwise starting with vertex A, the resulting sequence is A, B, C. Following the same procedure for $\Delta A'B'C'$ provides the sequence A', C', B'. The two figures ΔABC and $\Delta A''B''C''$, however, do have the same orientation.

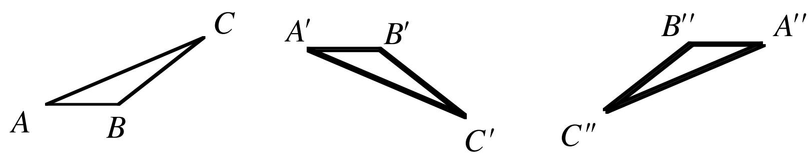

Figure 1: Three triangles

For which transformations is the orientation of the image the same as the orientation of the preimage?

f. Consider a transformation in which the image and the preimage are identical in both shape and size and located in the same position. What would be an appropriate name for a transformation of this kind?

Exploration 1

To generate a transformation using matrix multiplication, you must first determine the necessary **transformation matrix**. To find this transformation matrix, it may help to examine the transformation of a simple shape with easily identifiable coordinates.

For example, suppose that you wanted to perform a reflection, rotation, or dilation on a figure. One convenient method for determining the desired transformation matrix involves performing the transformation on a triangle like the one shown in Figure **2**.

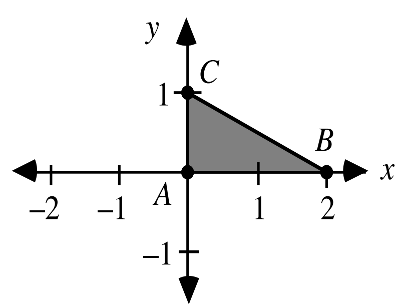

Figure 2: Right triangle in the first quadrant

This triangle can be represented by the following matrix:

$$\begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The transformation matrix can be represented as shown below:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The product of these two matrices represents the image under the desired transformation:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$$

a. **1.** Graph and label ΔABC from Figure **2** on a sheet of graph paper.

2. Place a sheet of clear acetate film over the graph paper and trace ΔABC. Label the corresponding vertices A', B', and C'. This will serve as the image of ΔABC.

b. Select a transformation from the following list:

- r_x where x is the x-axis
- r_y where y is the y-axis
- r_l where l is the line $y = x$
- r_m where m is the line $y = -x$.

c. Apply the chosen transformation to ΔABC using the sheet of acetate. Determine the coordinates of the image by locating $\Delta A'B'C'$ in the appropriate position.

d. Represent the coordinates of the image in a matrix.

e. Write a matrix equation for the transformation of the preimage ΔABC to its image $\Delta A'B'C'$ in the following form, where the product matrix is the matrix you wrote in Part **d**:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$$

f. Determine the values of a, b, c, and d. These are the elements of the 2×2 transformation matrix that will transform any figure in the manner chosen in Part **b**.

g. Verify your solution to Part **f** by multiplying the transformation and the preimage matrices.

h. Repeat Parts **c–g** using a different transformation from Part **b**.

Discussion 2

a. When any point (or figure) is transformed in a plane, the entire plane is transformed in the same manner. How is this demonstrated in Part **c** of Exploration **1**?

b. Describe how you determined the values of a, b, c, and d in Part **f**.

c. Describe how to use ΔABC to determine the transformation matrix for any of the transformations listed in Part **b**.

d. How would you determine the matrix equation necessary to reflect the figure represented by the matrix below in the x-axis?

$$\begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & 5 \end{bmatrix}$$

Exploration 2

In Exploration **1**, you determined the 2×2 transformation matrices necessary to reflect a figure in four different lines. In this exploration, you use a similar process to determine the matrix necessary to rotate an object a desired number of degrees. You also examine the relationship between the elements of this matrix and the sine and cosine functions.

a. Using graph paper and acetate as in Exploration **1**, transform the triangle in Figure **2** using one of the following rotations: $\mathrm{R}_{O,90°}$, $\mathrm{R}_{O,180°}$, or $\mathrm{R}_{O,270°}$, where O is the origin. Record the coordinates of the vertices of the image in a matrix.

b. Using the process described in Exploration **1**, determine the elements of the transformation matrix required to accomplish the selected rotation.

c. The 2×2 transformation matrix that produces a rotation about the origin is of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. Verify that the transformation matrix found in Part **b** is of this form.

d. Repeat Parts **a–c** using any angle of rotation.

Discussion 3

a. How did the elements in the rotation matrices compare to the sine and cosine of the angles of rotation?

b. Would the rotation matrix be affected if the angle of rotation were measured in radians rather than degrees? Explain your response.

Mathematics Note

The 2×2 transformation matrix that results in a rotation of angle r about the origin can be represented as shown below:

$$\begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix}$$

For example, the image of the triangle in Figure **2** under a rotation of $\pi/3$ radians about the origin can be determined as follows:

$$\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0 & 1.0 & -0.9 \\ 0 & 1.7 & 0.5 \end{bmatrix}$$

c. Consider the rotation of a figure about the origin. How could you determine the angle of rotation given only the coordinates of the preimage and the image?

Assignment

1.1 **a.** Determine the transformation matrix that produces a rotation of 90° about the origin.

b. Write a matrix expression that results in a rotation of 90° about the origin of the triangle with vertices $A(1,1)$, $B(1,5)$, and $C(4,1)$.

c. Determine the vertices of the image $A'B'C'$.

d. Verify that the image is correct by performing the rotation on graph paper.

1.2 Determine a matrix that produces each of the following transformations. Use appropriate notation to represent each one.

a. a reflection in the line x, the x-axis

b. a rotation of 270° about the origin, O

c. a rotation of -270° about the origin, O

d. a reflection in the line y, the y-axis

e. a reflection in the line l with equation $y = -x$

f. a rotation of 26° about the origin, O

g. a rotation of -78° about the origin, O

1.3 **a.** Use a series of at least four transformations to spin and flip a simple figure. Record the transformation matrix necessary to create each image.

b. Graph the result of each individual transformation on a separate index card. Use the same coordinate system on each card.

c. Arrange the cards in the appropriate order. Flip through the cards to create a simple animation of your figure.

1.4 **a.** Graph a simple figure on a sheet of graph paper. Plot and record the coordinates of the image under a dilation by a scale factor of 2 with center at the origin.

b. Using the method described in Explorations **1** and **2**, determine the 2×2 transformation matrix required to perform this dilation by matrix multiplication.

1.5 Computer animators and special-effects artists often use transformations to create the illusion of motion. For example, when objects on the screen that appear to be small and distant quickly become very large, viewers may feel the sensation of rapid flight.

a. Describe the geometric transformations that a special-effects artist might perform to make an asteroid appear to move toward the viewer.

b. Describe the geometric transformation that a special-effects artist might perform to make an asteroid appear to move away from the viewer.

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1.6 As part of an advertising presentation, a graphic artist must enlarge a 4×6 photograph to 12×18. Using the lower left-hand corner of the photograph as the origin, the location of a book in the original photo can be denoted by the vertices (1.5,2), (1.9,3.1), (2.5,2), and (2.9,3.1).

a. Create a matrix that will enlarge the 4×6 photograph appropriately.

b. Represent the vertices of the book in the enlargement in a matrix.

c. What is the ratio of the area of the book in the enlargement to the area of the book in the original?

1.7 When a slide projector displays an image on a screen, the image is a dilation by a designated scale factor with center located at the projector. Consider the preimage of a picture that can be denoted, in part, by the coordinates (–3,0), (–1.5,7), (0,6), (1.5,7), (3,0), and (0,3). If the scale factor is 15, what are the corresponding coordinates of the image?

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Activity 2

To make a character's movements appear life-like, computer animation programs require an efficient means of representing transformations and calculating new coordinates. In Activity **1**, you used 2×2 matrices and matrix multiplication to perform reflections, rotations, and dilations. It is not possible, however, to use this method to perform translations. In this activity, you examine a technique that allows the use of matrix multiplication to represent all transformations.

Exploration

The coordinates of the point (x, y) can be represented in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

If this point is translated *a* units horizontally and *b* units vertically, its image can be represented by the matrix below:

$$\begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

a. Determine the dimensions of the transformation matrix, denoted by $[\ \ ]$, required to perform the multiplication in the equation below:

$$[\ \ ] \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

b. Verify that a matrix with the dimensions you determined in Part **a** cannot be used to produce a translation of *a* units horizontally and *b* units vertically.

Mathematics Note

When represented as a matrix, the **homogeneous form** of a point (x, y) is:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This form allows the coordinates of a point in two dimensions to be represented in a 3×1 matrix. For example, the homogeneous form of the point (2,3) can be represented in the following matrix:

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

c. When represented in homogeneous form, the image of a point (x, y) translated a units horizontally and b units vertically results in the following matrix:

$$\begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

Determine the dimensions of the transformation matrix, denoted by $[\ \ ]$, required to perform the multiplication in the equation below:

$$[\ \ ] \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

d. Find the transformation matrix that satisfies the equation in Part **c**.

e. When the point (x, y) is reflected in the x-axis, the coordinates of the image are $(x, -y)$.

1. Use the homogeneous form of the points to write a matrix equation that represents this transformation.
2. Determine the transformation matrix that produces the desired image matrix.

f. When the point (x, y) is dilated by a factor of n with center at the origin, the coordinates of the image are (nx, ny). Repeat Part **e** for this transformation.

Discussion

a. Describe the 3×3 matrix that allows a translation to be represented using matrix multiplication.

b. In the translation matrix you described in Part **a** of the discussion, what is the significance of the portion below?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. How does the 3×3 matrix that produces a reflection in the x-axis compare with the 2×2 matrix for a reflection in the x-axis?

d. How does the 3×3 matrix that produces a dilation by a factor of 2 with center at the origin compare to the 2×2 matrix that produces the same result?

e. What 3×3 transformation matrix do you think rotates a figure θ degrees about the origin? Defend your response.

f. The **identity transformation** preserves the position of a figure. What 3×3 matrix produces the identity transformation?

g. What advantages are there in representing transformations using 3×3 matrices?

Assignment

2.1 **a.** Consider ΔABC with vertices at $A\ (0,0)$, $B\ (2,0)$, and $C\ (0,1)$. Use a matrix to represent the vertices of the triangle in homogeneous form.

b. Determine the 3×3 transformation matrix that produces a dilation of the triangle by a scale factor of 4 with center at the origin.

c. Compare this transformation matrix with the one you created in Part **f** of the exploration for a point (x,y).

2.2 The 3×3 transformation matrix that produces a rotation of angle θ with center at the origin is shown below:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. Use this matrix to determine the image of the triangle described in Problem **2.1** under a rotation of 30° about the origin.

b. Verify your results in Part **a** by performing the transformation on a sheet of graph paper.

2.3 Determine the 3×3 matrix that produces each of the following transformations:

a. a dilation by a scale factor of k with center at the origin

b. a reflection in the line $y = x$

2.4 Determine the 3×3 transformation matrix for each of the following transformations:

a. reflection in the y-axis

b. reflection in the line $y = -x$

2.5 **a.** Consider the triangle with vertices (0,0), (2,0), and (0,1). On a sheet of graph paper, reflect this preimage in the line $y = 1$.

b. Determine a 3×3 transformation matrix that results in a reflection in the line $y = 1$.

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2.6 **a.** Consider the triangle from Problem **2.5**. Use a 3×3 transformation matrix to reflect this preimage in the x-axis.

b. Write an expression using matrix multiplication that describes a translation of the image from Part **a** 2 units vertically.

c. **1.** Multiply the transformation matrix from Part **a** on the right by the transformation matrix from Part **b**.

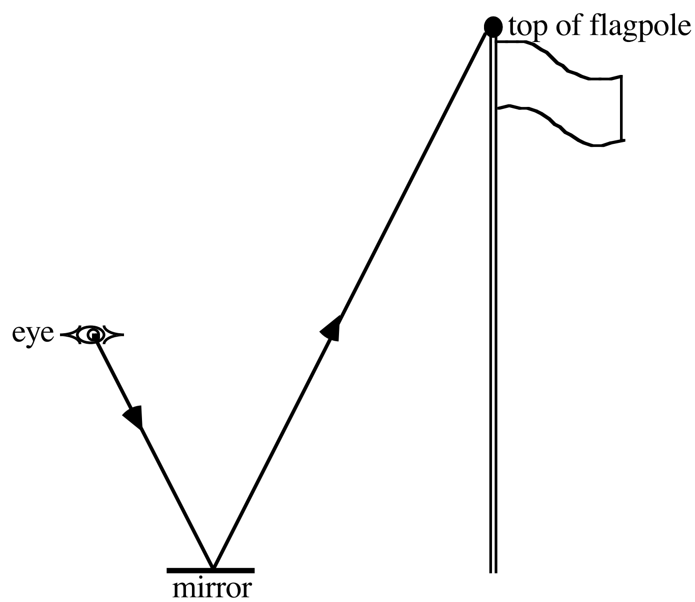
2. The resulting matrix represents the same transformation matrix as the one obtained in Problem **2.5**. Why do you think this occurs?

2.7 Computer animators frequently work in three-dimensional space, where the coordinates of points are represented by ordered triples in the form (x, y, z).

a. What do you think the homogeneous form of the point (x, y, z) would look like?

b. Based on your understanding of the 3×3 translation matrix for two dimensions, determine a matrix that would result in a translation of a units along the x-axis, b units along the y-axis, and c units along the z-axis.

2.8 To find the height of a flagpole, a student positioned a mirror on the ground. She then stood so that she could see the reflection of the top of the flagpole when she looked in the mirror, as shown below.



a. The line of sight from the top of the flagpole to the mirror and the line of sight from the student's eye to the mirror represent the hypotenuses of two similar right triangles. Sketch and label these triangles.

b. What transformations of the triangle in which the student's eye is a vertex result in the triangle in which the flagpole is a vertex?

c. Suppose that the mirror is placed at the origin of a rectangular coordinate system and the coordinates of the location of the student's eye are $(-2,1.7)$. The distance from the mirror to the flagpole's base is 3 times the distance from the mirror to the student's feet.

1. Use 3×3 matrix multiplication to perform one of the transformations identified in Part **a** on the triangle in which the student's eye is a vertex.

2. Perform the second transformation identified in Part **a** on the image created in Step **1**. Find the coordinates representing the top of the flagpole.

3. Multiply the 3×3 transformation matrix used in Step **1** on the left by the 3×3 transformation matrix from Step **2**.

4. Multiply the preimage by the matrix found in Step **3**. Compare the coordinates for the top of the flagpole to those found in Step **2**.

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Activity 3

An image on a video screen is made up of many small squares or dots called **pixels**. To simulate motion, the color or brightness of each pixel changes as the need arises. For each change in an animated scene, the computer must complete an enormous number of calculations. Because this process takes time, some computer animation may produce movements that are jerky and unrealistic.

To perform the thousands of transformations required for complicated animations, a computer may use multiple matrix operations. In the previous activity, you used 3×3 matrices to perform all transformations by matrix multiplication. In this activity, you examine another way to enhance the efficiency of the computer by combining multiple transformations into a single matrix.

Exploration

In many cases, a single transformation can produce the same image as a combination of two or more transformations. When more than one transformation is performed on a figure, the result is a composite transformation, or a **composition** of transformations. In this exploration, you investigate the composition of reflections in intersecting lines.

a.

1. Use appropriate technology to create an x-axis and a y-axis that intersect at the origin, O.
2. Create the lines $y = x$ and $y = -x$.
3. To represent a preimage, draw a scalene triangle somewhere on the coordinate plane. Label the vertices A, B, and C.
4. Reflect ΔABC in the line $y = x$ to obtain its image, $\Delta A'B'C'$.
5. Reflect $\Delta A'B'C'$ in the y-axis to obtain $\Delta A''B''C''$.
6. Compare $\Delta A''B''C''$ to the original preimage, ΔABC.
7. Record the measures of $\angle AOA''$, $\angle BOB''$, and $\angle COC''$.
8. Find a single transformation that maps ΔABC onto $\Delta A''B''C''$. Use technology to test your conjecture.

Mathematics Note

A **composition** B of transformations B_1 and B_2 is a function whose domain is the domain of B_1 and whose range is the range of B_2. In other words, the composition B is a one-to-one correspondence that maps a preimage point P in the domain of B_1 to an image point P'' in the range of B_2. This composition can be denoted by $\mathrm{B} = \mathrm{B}_2 \circ \mathrm{B}_1$, (read "B equals B_2 composed with B_1").

For example, if l represents the line $y = x$ and y represents the y-axis, the transformation $\mathrm{C} = \mathrm{r}_y \circ \mathrm{r}_l$ is the composition of a reflection in the line $y = x$ followed by a reflection in the y-axis.

b.

1. Record the 3×3 transformation matrix that produces each reflection in Part **a**. Use appropriate notation to represent each reflection.
2. Using appropriate notation, record the 3×3 matrix that represents the single transformation which maps ΔABC onto $\Delta A''B''C''$.
3. Multiply the two transformation matrices recorded in Step **1** in the order that represents the composition in Part **a**.
4. Compare the product in Step **3** with the matrix you identified in Step **2**.

c.

1. Delete the two images created in Part **a**. Your graph should now consist of the x- and y-axes, the lines $y = x$ and $y = -x$, and the preimage ΔABC.
2. Reflect ΔABC in one of the four lines to make its image, $\Delta A'B'C'$.
3. Reflect $\Delta A'B'C'$ in one of the remaining lines to get $\Delta A''B''C''$.
4. Compare $\Delta A''B''C''$ to the original preimage, ΔABC.
5. Record the measures of angles $\angle AOA''$, $\angle BOB''$, and $\angle COC''$.
6. Find a single transformation that maps ΔABC onto $\Delta A''B''C''$. Use technology to test your conjecture.

d. **1.** Record the 3×3 transformation matrix that produces each reflection in Part **c**. Use appropriate notation to represent each reflection.

2. Using appropriate notation, record the 3×3 transformation matrix that represents the single transformation which maps ΔABC onto $\Delta A''B''C''$.

3. Multiply the two transformation matrices recorded in Step **1** in the order that represents the composition in Part **c**.

4. Compare the product in Step **3** with the matrix you identified in Step **2**.

e. Use the method described in Parts **c** and **d** to investigate the transformations that result from the following compositions:

1. any two dilations with the same center

2. any two rotations with the same center

3. any two translations.

Discussion

a. Do you believe that order is important in the composition of reflections? Explain your response.

b. In the exploration, you observed that the composition of two reflections in intersecting lines is equivalent to a single rotation, where the center of rotation is the point of intersection.

1. Do you think that this will hold true for any even number of reflections in intersecting lines? Explain your response.

2. Do you think this will hold true for an odd number of reflections? Explain your response.

c. **1.** What is the result of a composition of two reflections in the same line? What single transformation is the same as this composition?

2. What transformation would result from a composition of two reflections in parallel lines?

d. **1.** What transformation results from the composition of two translations?

2. Is order important in the composition of translations? Explain your response.

e. **1.** What transformation results from the composition of two dilations with the same center?

2. Is order important in the composition of dilations? Explain your response.

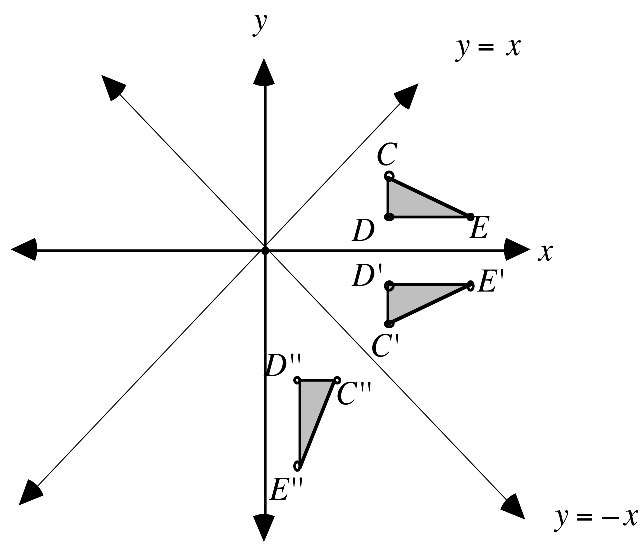
f. **1.** What transformation results from the composition of two rotations with the same center?

2. Is order important in the composition of rotations? Use an example to justify your response.

g. How do you think the composition of functions might affect the efficiency of computer animation programs?

Assignment

3.1 The diagram below shows the transformation of a scalene right triangle represented by ΔCDE.



a. What two consecutive transformations map ΔCDE onto $\Delta C''D''E''$? Justify your response by assigning coordinates to the vertices of ΔCDE and evaluating the appropriate matrix equation to find the coordinates of $\Delta C''D''E''$.

b. What single transformation maps ΔCDE onto $\Delta C''D''E''$? Justify your response by evaluating the appropriate matrix equation to find the coordinates of $\Delta C''D''E''$.

c. Write a matrix equation that shows that the single transformation you found in Part **b** equals the composition of two transformations described in Part **a**.

3.2 Complete the following chart by writing the single transformation that is equivalent to the composition of $B_2 \circ B_1$ in the appropriate cell. Let O represent the origin, x represent the x-axis, y represent the y-axis, l represent the line $y = x$, and m represent the line $y = -x$.

For example, the correct entry for $r_x \circ r_l$ is shown below. Note that the first transformation performed is drawn from the row headings, while the second transformation is drawn from the column headings. Be sure to use correct notation.

B_2

	$\circ$	$\mathbf{r}_x$	$\mathbf{r}_l$	$\mathbf{R}_{O,90°}$	$\mathbf{R}_{O,180°}$	$\mathbf{R}_{O,270°}$	
	$\mathbf{r}_x$						
	$\mathbf{r}_l$	$R_{O,270°}$					
B_1	$\mathbf{R}_{O,90°}$						
	$\mathbf{R}_{O,180°}$						
	$\mathbf{R}_{O,270°}$						

3.3 Consider the figure described by the matrix **M** below.

$$\mathbf{M} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

a. Find the product of the following three matrices using the order of multiplication indicated below:

$$\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \bullet \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

b. Find the product of the same three matrices using the following order of multiplication:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bullet \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

c. Compare your results in Parts **a** and **b**. What conjecture might be made concerning the grouping of matrix multiplication?

d. In Parts **a** and **b**, what composition of transformations was performed on matrix **M**?

3.4 **a.** Create a simple, non-symmetric "stick" figure to represent a cartoon character.

b. Graph several points of your cartoon character and determine the coordinates of each point.

c. If points A, B, C, and D have coordinates (0,0), (4,2), (6,7), and (7,4), respectively, what transformation results from $\mathrm{T}_{A,B} \circ \mathrm{T}_{C,D}$?

d. Write a matrix expression that represents this composition.

e. Determine the image matrix of your cartoon character under the transformation $\mathrm{T}_{A,B} \circ \mathrm{T}_{C,D}$.

3.5 Find the image matrix of your original cartoon character from Problem **3.4** under the transformation $\mathrm{r}_x \circ \mathrm{R}_{O,285^\circ}$.

3.6 **a.** Consider ΔABC with vertices $A(3,1)$, $B(6,1)$, and $C(6,2)$. Using ΔABC as the preimage, determine the coordinates of the image $\Delta A'''B'''C'''$ under three transformations of your choice.

b. Determine a single 3×3 matrix that represents the composition of the three transformation matrices in Part **a**. Use matrix operations to verify that your matrix is the correct one.

3.7 A reflection can be performed in any line, not just the x- and y-axes and the lines $y = x$ and $y = -x$. One method for determining the coordinates of an image under such a reflection involves transforming the plane so that the desired line of reflection coincides with one of the axes. After the reflection matrix for the axis is determined, the plane is transformed so that the line is returned to its original position. The composition of these three transformations results in the reflection of the object in the desired line.

a. Consider ΔABC with vertices at (0,0), (2,0), and (0,1), respectively. Suppose you wish to reflect ΔABC in the line $y = 2x$. To transform the plane so that the line $y = 2x$ coincides with the x-axis, determine the measure of the angle formed by the line $y = 2x$ and the x-axis using right-triangle trigonometry.

b. Determine the 3×3 transformation matrix that will rotate the line $y = 2x$ so that it coincides with the x-axis. This is the first transformation.

c. Since the line $y = 2x$ now coincides with the x-axis, the second transformation is a reflection in the x-axis. Determine the matrix for this reflection.

d. The final transformation should return the plane to its original position. Determine the rotation matrix that will result in this transformation.

e. Find the single 3×3 transformation matrix that represents the composition of the transformations in Parts **b–d**. Multiply this transformation matrix by the preimage matrix to determine the location of the image.

f. Verify your solution by reflecting ΔABC in the line $y = 2x$ on a sheet of graph paper and estimating the coordinates of the image.

3.8 A rotation can have any point as its center, much like a reflection can be in any line. The process for identifying the required transformation matrix and the coordinates of the image is very similar to the one described in Problem **3.7**. In this case, however, you must translate the plane so that the center of rotation coincides with the origin, perform a rotation of the desired angle, then translate the plane back to its original position.

a. Use ΔABC from Problem **3.7** to determine the composite matrix that results in a 45° rotation about the point (–2,3).

b. Determine the coordinates of the image under the composition in Part **a**.

c. Verify your solution using graph paper.

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3.9 Reflecting an object in a line that does not pass through the origin, such as $y = -3x + 5$, requires an additional transformation to get the line to coincide with the x-axis. In this case, the line must first be translated –5 units vertically, then rotated to coincide with the axis.

a. Use ΔABC, with vertices at (0,0), (2,0), and (0,1), respectively, to determine the composite matrix that results in a reflection in the line $y = -3x + 5$.

b. Determine the coordinates of the vertices of the image under the composition in Part **a**.

c. Verify your response using graph paper.

3.10 Designer patterns on wallpaper, floor tiles, and kitchen countertops often use transformational geometry. Use a geometry utility to create your own tile or wallpaper pattern. Print a copy of your design, and write a short paragraph describing the geometric transformations you used to create it.

* * * * * * * * * *

Activity 4

In previous activities, you examined the use of compositions to find a single transformation matrix that produces the same result as multiple transformations. One type of transformation that you have not yet examined is itself a composition. This transformation is a **glide reflection**.

Mathematics Note

A **glide reflection** is the composition of a reflection and a translation parallel to the line of reflection. For example, Figure **3** shows the image $\Delta A'B'C'$ of ΔABC under a glide reflection. It is the result of a reflection in line *m* and a translation parallel to line *m* along vector **t**.

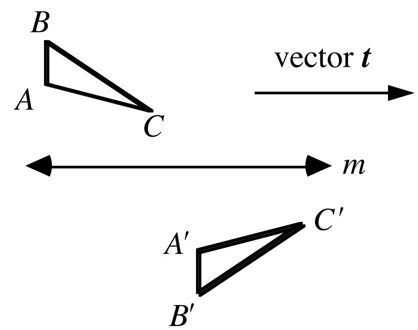

Figure 3: Glide reflection of ΔABC

Exploration 1

In this exploration, you examine one of the properties of the glide reflection.

a. Create ΔABC. Transform the triangle by translating it, then reflecting the image in a line parallel to the translation.

b. Find the image of ΔABC using the same transformations in Part **a**, but in reverse order. In other words, reflect the triangle in the line, then translate the image parallel to the line of reflection. Compare the result to the one obtained in Part **a**.

Discussion 1

a. In a glide reflection, does the order of the translation and reflection affect the image? Explain your response.

b. How does the orientation of the preimage compare to that of the image under a glide reflection? Why is this true?

c. The translation in a glide reflection can be expressed as the composition of two reflections.

1. Describe the relationship between these two lines of reflection.
2. Describe the relationship between these two lines and the original line of reflection in the glide reflection.

Exploration 2

In the previous discussion, you found that a glide reflection could be expressed as a composition of reflections. Is this true only for glide reflections? In this exploration you will attempt to determine if it is possible to express other transformations as compositions of reflections.

a. Create a quadrilateral *ABCD*.

b. Select and perform one of the following transformations on quadrilateral *ABCD*:

1. a reflection in a line
2. a rotation about a point
3. a translation
4. a glide reflection.

c. Figure **4** shows several of the steps used in the process of determining if a selected transformation can be expressed as a composition of reflections. In this case, quadrilaterals $ABCD$ and $A'B'C'D'$ represent the preimage and image under a glide reflection.

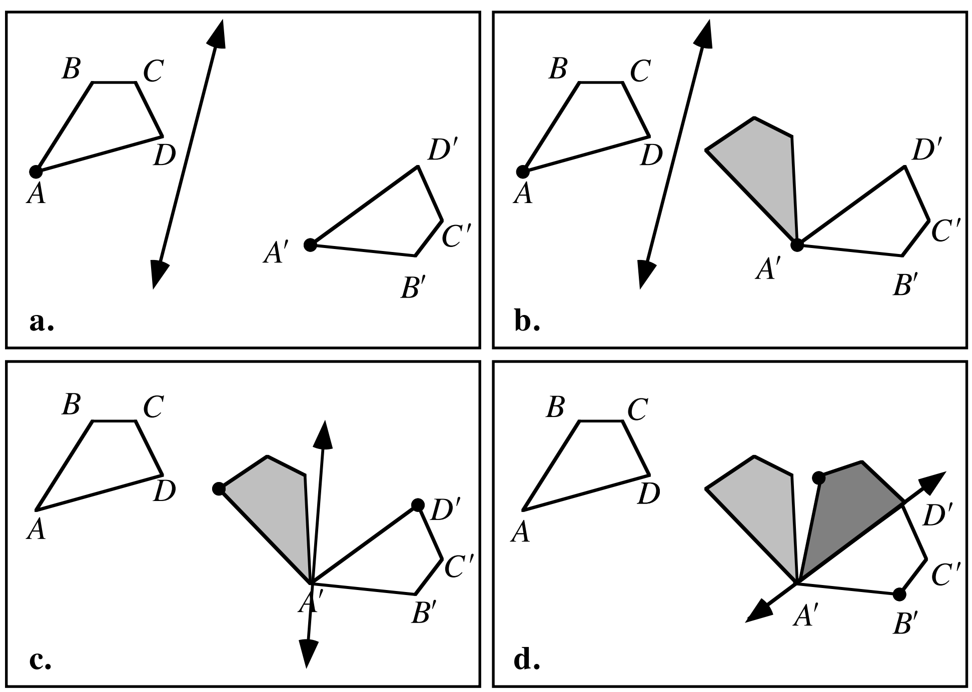

Figure 4: Quadrilateral $ABCD$ under a composition of reflections

Complete the following steps to determine if your selected transformation can be expressed as a composition of reflections.

1. Select a vertex in the image and its corresponding vertex in the preimage. Determine the line of reflection for these two vertices. For example, Figure **4a** shows the line of reflection for A and A'.
2. Reflect the preimage quadrilateral $ABCD$ in this line of reflection. (In Figure **4b**, the resulting figure is shaded.) If this reflection coincides with image quadrilateral $A'B'C'D'$, then the process is complete.
3. If the process is not complete, select another vertex on image quadrilateral $A'B'C'D'$ and its corresponding vertex on the image created in Step **2**. For example, the vertices chosen in Figure **4c** are D' and its corresponding vertex on the shaded quadrilateral. Repeat Steps **1** and **2** for these two points. (The result in Figure **4d** is the more densely shaded quadrilateral.)
4. If the process is not complete, repeat Step **3**, choosing a new vertex in the image quadrilateral $A'B'C'D'$ each time. In Figure **4d**, for example, the vertices chosen are B' and its corresponding point on the more densely shaded quadrilateral. The line of reflection between these two points contains A' and D'.

   If the last image created after repeating Step **3** for each of the remaining vertices does not coincide with the image quadrilateral $A'B'C'D'$, it is not possible to express the transformation as a composition of reflections.
5. If it is possible to express your chosen transformation as a composition of reflections, record the number of reflections necessary.

d. Repeat Parts **b** and **c** for a different transformation.

e. Graph a congruent image of your quadrilateral in any location on the plane and with any orientation. Repeat Part **c** for this new transformation.

Discussion 2

a. Which transformations can be expressed as a compositions of reflections?

b. Was it possible to express the transformation in Part **e** of Exploration **2** as a composition of reflections?

c. If a preimage and its image under a transformation in a plane are congruent, the transformation is either a rotation, a reflection, a translation, or a glide reflection. Given this fact, make a general statement concerning the expression of transformations as a composition of reflections and the number of reflections necessary.

d. Can all transformations be expressed as a composition of reflections?

e. Why might it be helpful for a computer programmer who creates software that simulates motion to express various transformations as a composition of reflections?

Assignment

4.1 **a.** Consider the triangle with vertices $A(-1,-3)$, $B(4,0)$, and $C(2,1)$. Graph and label this figure on a coordinate plane.

b. On the same set of axes, graph the image of ΔABC under the reflection r_y, where y is the y-axis. Label the image $\Delta A'B'C'$.

c. On the same set of axes, graph the image of $\Delta A'B'C'$ under the translation $T_{P,P'}$, where point P has coordinates (13,5) and P' has coordinates (12,8). Label the resulting image $\Delta A''B''C''$ and record the coordinates of its vertices.

d. Use composition notation to describe the transformation of ΔABC to $\Delta A''B''C''$.

e. Write a matrix equation that illustrates the relationship between the matrix for ΔABC, the transformation matrices, and the matrix for $\Delta A''B''C''$. Compare the resulting matrix with the coordinates found in Part **c**.

4.2 **a.** Use graph paper and a straightedge to complete Steps **1–5** below.

1. Graph a simple, non-symmetrical stick figure and label it C.
2. Determine the coordinates of several vertices of the figure.
3. Graph the image of the stick figure under the composition $T_{R,R'} \circ r_x$, where x is the x-axis, point R has coordinates (–5,10), and point R' has coordinates (–3,11). Refer to the image as C′.
4. Graph the image of the original figure C under the composition $r_x \circ T_{R,R'}$. Refer to this image as C″.
5. Explain whether or not the order of the transformations makes a difference in the final image.

b. Is the composition $T_{R,R'} \circ r_x$ a glide reflection? Justify your response.

4.3 **a.** Graph the image of your stick figure from Problem **4.2** under the composition $T_{S,S'} \circ r_x$, where x is the x-axis, point S has coordinates (–5,5), and point S' has coordinates (–3,5).

b. Graph the image of the stick figure under the composition $r_x \circ T_{S,S'}$.

c. Explain whether or not the order of the transformations makes a difference in the final image.

d. Write matrix equations that generate matrices for the final images obtained in Parts **a** and **b**.

4.4 As you found in the exploration, a glide reflection can be expressed as a composition of three reflections.

a. Make a sketch of three lines of reflection for which the final image is a glide reflection of the preimage.

b. Describe the properties of these three lines and explain how these properties guarantee that the resulting transformation is a glide reflection.

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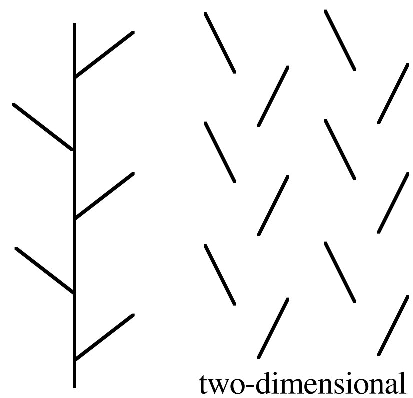
4.5 Some two-dimensional patterns in crystals or the twigs of trees have an especially appealing sense of symmetry. Use glide reflections and a geometry utility to recreate one of the following patterns. Include a short paragraph describing the geometric transformations in your design.



tree twig

two-dimensional crystallographic pattern

4.6 Interior decorators often use a patterned border to accent the walls or ceiling of a room. Design a pattern for a border using glide reflections. Create a copy of your design using a geometry utility, then write a short paragraph describing the geometric transformations in your design.

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Summary Assessment

1. Create a flip-card animation of a simple figure by plotting each frame onto a coordinate system that can be taped or glued to a 3×5 index card. Your flip cards should use all of the transformations studied in this module.
2. Express each transformation used as a 3×3 matrix.
3. Identify the coordinates of each image.
4. Determine a single 3×3 transformation matrix that will transform the initial preimage into the final image.

Module Summary

- An **image** is the figure created by a transformation of a **preimage**.
- A **dilation** with center at point *P* and a scale factor of *r* is denoted as $\mathrm{D}_{P,r}$.
- A **rotation** of θ degrees with center at point *P* is denoted as $\mathrm{R}_{P,\theta}$. The value of θ is positive for counterclockwise rotations and negative for clockwise rotations.
- A **reflection** in the line *m* is denoted as r_m.
- A **translation** with *P* as the preimage and P' as the image is denoted as $\mathrm{T}_{P,P'}$.
- The transformation B that produces the same image as a transformation B_1 followed by a transformation B_2 is the **composition** of B_1 and B_2. This composition is denoted $\mathrm{B} = \mathrm{B}_2 \circ \mathrm{B}_1$, (read "B equals B_2 composed with B_1"). This notation implies that transformation B_2 is performed after transformation B_1.
- A preimage and its image have the same **orientation** if they have the same sequence of corresponding vertices when read either clockwise or counterclockwise.
- When represented as a matrix, the **homogeneous form** of a point (x,y) is:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- The **identity transformation** preserves the position of a figure.
- A **glide reflection** is the composition of a reflection and a translation parallel to the line of reflection.
- If a preimage and its image under a transformation in a plane are congruent, the transformation is either a rotation, a reflection, a translation, or a glide reflection.
- If a preimage and its image under a transformation in a plane are congruent, the transformation can be expressed as a composition of reflections in no more than three lines.

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Wedge, J. C. "Role of the Animator in the Generation of Three-Dimensional Computer-Generated Animation." ERIC, 1985. ED 282512.