More or Less?

What do car engines and compact discs have in common? When manufacturing either product, quality control engineers use inequalities to represent tolerance and precision intervals.

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Introduction

Manufacturers monitor product quality by comparing performance with predetermined standards. If they find unacceptable variations from these standards, they may adjust the manufacturing process in order to improve the product. This process, known as **quality control**, affects the performance of everything from toasters to televisions.

Quality control does not mean that all products will be perfect, however. For example, all computer chips manufactured for a particular model are not exactly the same. Some of the variations that occur during manufacturing or in component materials are beyond physical control. Fortunately, slight differences in a product do not usually affect its performance. The acceptable error in a product is often expressed as a **tolerance interval**.

The tolerance interval, in turn, is determined by the accuracy or precision of the components that make up the product. In this module, this level of accuracy is referred to as a **precision interval**.

Discussion

a. Why is quality control important to consumers?

b. Why is quality control important to manufacturers?

c. What are some products that you associate with high quality?

d. How does competition between manufacturers of similar products result in better products?

e. What factors make it impossible for manufacturers to produce identical products all of the time?

f. What can consumers do when they are not satisfied with the performance of a product?

Activity 1

Most familiar products, from milk cartons to microwave ovens, are manufactured within certain tolerance levels for a variety of specifications. For example, when producing compact discs, the manufacturer must monitor the quality of the recording, the reliability of the materials, and the size of the disc itself. If the dimensions of each disc are not reasonably consistent, some may fail to work in a customer's compact disc player. Because it is not possible to produce hundreds of thousands of disks with exactly the same dimensions, the manufacturer must determine an acceptable tolerance interval.

Exploration

Imagine that you are the quality control manager for a manufacturer of compact discs. The desired circumference of your product is 37 cm.

a.
1. To model the manufacturing process, cut out five paper circles with the desired circumference of 37 cm.
2. By inspection, determine whether or not your five model discs are all exactly alike.
3. Use a piece of string to measure the circumference of each disc. Record your measurements.

Mathematics Note

An **interval** of real numbers is a set containing all numbers between two given points, the **endpoints** or **bounds** of the interval. An interval may contain one endpoint, both endpoints, or neither endpoint. In addition, some intervals may have only one endpoint, or no endpoints. For example, the set of real numbers can be represented as the interval $(-\infty,\ \infty)$, while the set of real numbers greater than or equal to a can be represented as $[a,\ \infty)$.

Some intervals may be expressed as conjunctions of inequalities. A **conjunction** combines two mathematical statements with the word *and*. For example, the conjunction $2.5 < x$ and $x < 3.9$ describes the interval of real numbers between 2.5 and 3.9, or (2.5, 3.9). This conjunction can be written as $2.5 < x < 3.9$. A conjunction is true only if both statements are true. If one or both of the statements is false, the conjunction is false.

Intervals and associated inequalities also can be expressed using set notation. For example, the inequality $-23.7 \geq x \geq -28.9$ may be written as $x \in [-28.9, -23.7]$, which means "x is an element of the set of numbers in the interval [–28.9, –23.7]."

b.

1. Determine the least interval that contains all five measurements from Part **a**, as well as the desired circumference of 37 cm.
2. Obtain a copy of template A from your teacher. The first number line on the template resembles Figure **1** below. Label this number line *c* to represent the circumference of the discs, then use it to illustrate the tolerance interval.

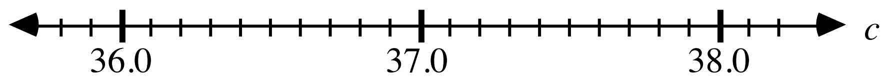

Figure 1: Number line for tolerance

3. Express the tolerance interval as an inequality and using set notation.

c.

1. Find the radius *r* that corresponds to the desired circumference of a compact disc.
2. Determine a precision interval for the radius of compact discs that corresponds to the tolerance interval from Part **b**.
3. The second number line on template A resembles Figure **2** below. Label this number line *r* to represent radius, then use it to illustrate the precision interval.

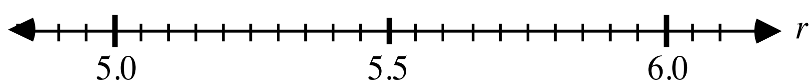

Figure 2: Number line for precision

4. Express the precision interval using set notation.

d. As shown in Figure **3**, draw an arrow from several values for the radius of a compact disc on line *r* to the corresponding values for the circumference on line *c*. This type of model is a **mapping diagram** showing a function from *r* to *c*.

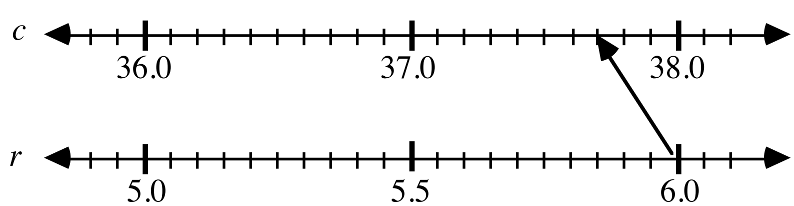

Figure 3: A mapping diagram

e. Compile the class data from Part **a**.

f.

1. Determine a tolerance interval for circumferences and a precision interval for radii that includes all the class data. Express these intervals as inequalities.
2. Represent these intervals graphically using the second set of number lines on template A.

Discussion

a. Were all of the model discs exactly the same size? If not, what factors might explain the differences in size?

b. What function relates the radius of a compact disc to its circumference?

c. Describe the tolerance interval and the precision interval created for the class data.

d. How do your intervals from Parts **b** and **c** of the exploration compare with the class values?

e. In Part **c** of the exploration, you determined the precision interval for the radius of your five compact discs using the tolerance interval of the desired circumference. Describe this relationship using an if-then sentence.

f. Given any tolerance interval for a desired circumference c, how would you determine the corresponding precision interval for the radius r?

g. Describe how to represent an interval indicating each of the following sets of numbers:

1. all non-negative real numbers
2. all negative real numbers.

Assignment

1.1 The tolerance interval for the mass M (in grams) of three cookies in a Snack Pack is $113.4 < M < 114.6$. If x represents the desired mass of each cookie, then $M = 3x$. The tolerance interval may then be expressed as the following conjunction of inequalities: $113.4 < 3x < 114.6$.

In order to determine the corresponding precision interval, it is necessary to solve this expression for x. To accomplish this, the conjunction of inequalities can be written as $113.4 < 3x$ and $3x < 114.6$.

a. Find the values of x for which $113.4 < 3x$ by solving for x. Describe the steps you used and substitute a few solution values into $113.4 < 3x$ as a partial check.

b. Find the values of x for which $3x < 114.6$ by solving for x. Describe the steps you used and substitute a few solution values into $3x < 114.6$ as a partial check.

c. Solve the two inequalities using a symbolic manipulator and compare the results with your solutions in Parts **a** and **b**.

d. Write the solutions found in Parts **a** and **b** as intervals.

e. Determine the values of x that satisfy both inequalities and express these values both as a conjunction of inequalities in the form $a < x < b$ and using interval notation.

1.2 As quality control manager of a compact disc manufacturer, you have received several complaints about discs that do not fit properly in some players. To answer these complaints, your assistant proposes a new tolerance interval of [36.94, 37.06] for the disc circumference.

a. Write a conjunction of inequalities that represents this interval in terms of r, the radius of the disc.

b. Express the precision interval for Part **a** in interval notation.

c. Explain how decreasing the size of the tolerance interval affects the accuracy necessary for the machines that produce compact discs.

1.3 A variety pack of breakfast cereal contains 6 boxes. The tolerance interval for the total mass of a variety pack, including 30 g of packaging material, is $237 \leq M + 30 \leq 243$, where M is the mass of the cereal in grams.

a. If x is the mass of the cereal in each box, express the tolerance interval in terms of x.

b. Express the precision interval for Part **a** in interval notation.

c. Describe the steps you used to solve the inequality in Part **a**.

1.4 **a.** **1.** Express the relationship between the numbers 3 and 5 as a true inequality.

2. Multiply both sides of the inequality by 5.

3. Determine whether or not the resulting inequality is true. If it is not, adjust the inequality sign so that the statement is true.

b. Repeat Part **a**, multiplying both sides of the inequality by –5.

c. Based on your observations in Parts **a** and **b**, what conjecture can you make about the effects of multiplying the terms of an inequality by a constant?

d. Use at least four different examples, two of which involve negative numbers, to test your conjecture.

e. In general, if $a < b$, then $a + c = b$ where c is a positive number. Given that n is a negative number, use algebra to show that $na > nb$.

1.5 **a.** Do you think that the generalization you made for multiplication in Problem **1.4c** would also be true for division? Explain your response.

b. Support your answer to Part **a** using at least four examples.

1.6 A tent manufacturer uses nylon rope for several components in its tents. One section of rope 630 cm long is cut into 7 pieces. Six of the pieces are used as guylines for each tent, while the remaining piece becomes the drawstring for the storage bag.

a. The drawstring must be between 30 cm and 36 cm long. Find an inequality that expresses the tolerance interval for the drawstring where x is the length of each guyline.

b. Determine the precision interval for the length of each guyline. Express your answer using both inequality notation and interval notation.

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1.7 The snow leopard is one of many species of mammals on the endangered list. Imagine that a wildlife biologist is nursing a sick snow leopard back to health. The animal currently weighs 43 kg.

A healthy leopard should weigh more than 70 kg. The biologist estimates that a special diet will allow the cat to gain an average of 0.5 kg per week. Write and solve an inequality expression to determine when the snow leopard's weight will exceed 70 kg.

1.8 Steel expands when heated and contracts when cooled. Because of this phenomenon, both bridges and railroad tracks contain expansion joints to allow for variations in the lengths of their steel beams.

As the temperature of a steel beam changes, the resulting length of the beam l, in meters, can be described by the following equation

$$l = l_0 + \left(1.05 \bullet 10^{-5} x\right) l_0$$

where l_0 is the original length and x is the change in temperature in degrees Celsius. The value of x may be either positive or negative.

a. Imagine that you are designing a steel bridge to be built in a region where the temperature generally varies between –15° C and 35° C. Express the possible change in temperature as an interval.

b. Express the possible length l of the beam as an inequality involving l_0.

c. If the original length l_0 of a steel beam is 300 m, what is the possible variation in the length of the beam?

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Activity 2

In Activity **1**, you investigated how a tolerance interval for the circumference of a compact disc affected the range of acceptable values for the radius of a disc. In manufacturing and other applications, tolerance intervals are often described in terms of the distance between each endpoint and the desired value.

For example, if the desired circumference of a CD is 37 cm, and e is the allowable error in the circumference, the tolerance interval can be written as $(37 - e, 37 + e)$. The corresponding precision intervals vary according to the error selected for the tolerance interval.

Exploration

In this exploration, you use a graphical model to examine how changing the allowable error in a tolerance interval affects the corresponding precision interval.

a. Figure **4** below shows line l, the graph of a linear function $f(x)$.

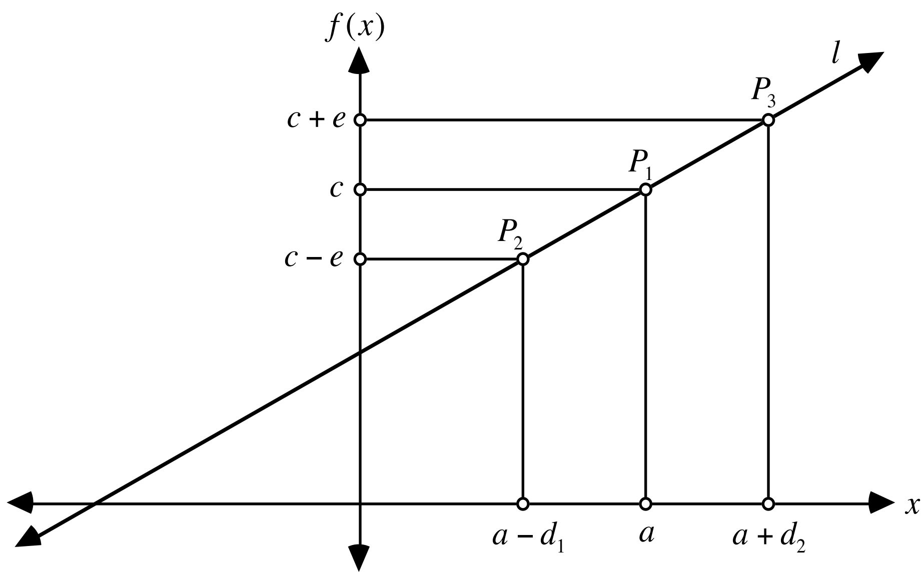

Figure 4: An arbitrary linear function

Using a geometry utility, create a construction like the one shown in Figure **4**. Your construction should meet the following conditions.

1. Line l is oblique to the $f(x)$-axis and represents a relationship between a desired measure c and its corresponding x-value, a.
2. The point with coordinates $(0, c - e)$ on the $f(x)$-axis is a moveable point. The interval $(c - e, c + e)$ represents a tolerance interval for c.

3. The segment from point P_1 to the point with coordinates $(0, c)$ is perpendicular to the $f(x)$-axis. The segment from point P_1 to the point with coordinates $(a, 0)$ is perpendicular to the x-axis.

4. The point with coordinates $(0, c + e)$ is the reflection of the point with coordinates $(0, c - e)$ in the segment from point P_1 to the point with coordinates $(0, c)$.

5. The segment from the point with coordinates $(0, c - e)$ perpendicular to the $f(x)$-axis intersects l at P_2. The segment from point P_2 to the point with coordinates $(a - d_1, 0)$ is perpendicular to the x-axis.

6. The segment from the point with coordinates $(0, c + e)$ is perpendicular to the $f(x)$-axis and intersects l at P_3. The segment from point P_3 to the point with coordinates $(a + d_2, 0)$ is perpendicular to the x-axis.

b. Measure the appropriate segments in order to determine the following distances:

1. e
2. d_1
3. d_2

c. While moving the point with coordinates $(0, c)$ on the $f(x)$-axis, observe the values of the three distances listed in Part **b**.

d.
1. Use an inequality to describe the tolerance interval in terms of $f(x)$.
2. Use an inequality to describe the precision interval in terms of x.

Discussion

a. In the exploration in Activity **1**, you modeled the manufacturing process for compact discs. In that setting, which quantity could be represented by c in Figure **4**? Which quantity could be represented by a in Figure **4**?

b. In Part **c** of the exploration, how did a change in e affect the values of d_1 and d_2?

c. Describe the effect that the size of the tolerance interval has on the size of the precision interval.

d. If $P(x_1, 0)$ is a point on the line segment with endpoints $(a - d_1, 0)$ and $(a + d_2, 0)$, where would you expect the point with coordinates $(0, f(x_1))$ to be located?

e. In this case, once a tolerance interval is selected, how would you determine the corresponding precision interval?

Mathematics Note

A limit is used to examine the behavior of a function close to, but not at, a particular point. For example, consider the value of $f(x)$ as x approaches a.

If the value of $f(x)$ gets arbitrarily close to c as x gets close to a, then c is the **limit** of the function as x approaches a. This can be denoted as follows:

$$\lim_{x \to a} f(x) = c$$

Mathematically, this is true if, for every real number e, there exists a corresponding positive real number d so that $c - e < f(x) < c + e$ whenever $a - d < x < a + d$.

In Figure **5**, c is the limit of $f(x)$ as x approaches a when, no matter how small the distance from $c - e$ to $c + e$, there is an interval from $a - d$ to $a + d$ small enough so that every point x in the interval $(a - d, a + d)$ has an $f(x)$ in the interval $(c - e, c + e)$.

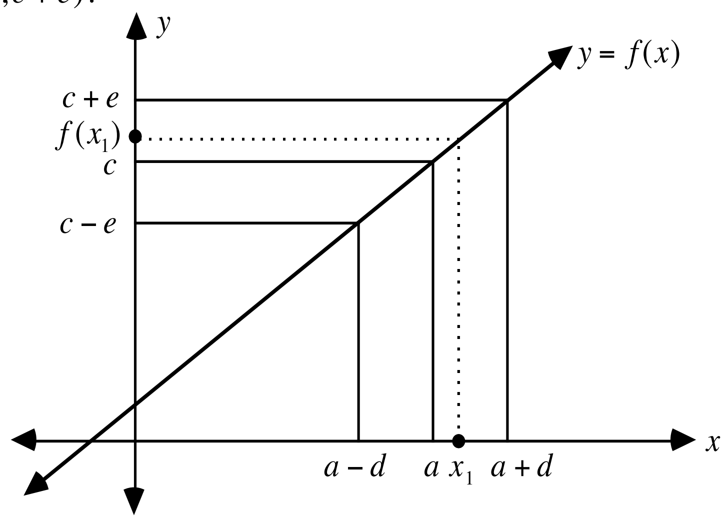

Figure 5: Graphical representation of a limit

In the exploration in Activity **1**, the function $f(r) = 2\pi r$ relates the radius r of a compact disc to its circumference. For example, suppose that $2\pi(5.9)$ cm is the desired circumference for a disk of radius 5.9 cm. If the tolerance interval is determined by an error of $e < 0.4$, then the precision interval can be found as follows:

$$2\pi(5.9) - 0.4 < 2\pi r < 2\pi(5.9) + 0.4$$

$$5.9 - \frac{0.4}{2\pi} < r < 5.9 + \frac{0.4}{2\pi}$$

In this case, d must be less than $0.4/2\pi$, or approximately 0.063. Because any value of r in the interval $(5.9 - 0.063, 5.9 + 0.063)$ yields a circumference in the required tolerance interval, then any other interval determined by $(5.9 - d, 5.9 + d)$, where $d < 0.063$, will also work. In other words, once a precision interval is determined, any “more precise” interval also is acceptable.

f. Describe some other situations, besides the manufacturing of compact discs, in which a product may have an infinite number of acceptable values within a tolerance interval.

Assignment

2.1 Consider the function $f(x) = 2x + 3$ with the set of real numbers as its domain.

a. Find a value for x so that $f(x) = 12$.

b. Determine the interval for x that corresponds with the following interval for $f(x)$: (11.8, 12.2).

c. Express the interval $11.8 < f(x) < 12.2$ in the form $(c - e,\ c + e)$.

d. Determine the interval for x that corresponds with the following interval for $f(x)$:

$$\left(c - \frac{e}{2},\ c + \frac{e}{2}\right)$$

2.2 Consider the function $f(x) = 4x - 7$.

a. What would you expect the limit c of $f(x)$ to be when x approaches 4.5?

b. Determine the interval for x that corresponds with the following interval for $f(x)$: $(c - 1.2,\ c + 1.2)$.

c. Draw a sketch that illustrates the relationship between the two intervals described in Part **b**.

2.3 Consider the function $f(x) = 3x + 1$. The limit c of $f(x)$ when x approaches a is 13.

a. Determine the interval for x that corresponds with the following interval: $13 - e < f(x) < 13 + e$.

b. Write the interval for x you determined in Part **a** in the form $(a - d,\ a + d)$.

2.4 A candy manufacturer markets chocolate candies in a paper package. Each package contains 56 candies. The packaging material has a mass of 2 g.

a. If x represents the mass in grams of each candy, determine a function $f(x)$ that represents the total mass of a package of candy.

b. The desired total mass of each package is 47.9 g, with an allowable error of no more than e grams. Write the tolerance interval for the total mass as an inequality.

c. Determine an inequality to express the largest possible precision interval meeting the conditions in Part **b**.

d. Find a value for a so that $f(a) = 47.9$.

2.5 A food processing company markets a box containing 8 whole-grain snack bars. The desired mass of the 8 bars, including the box, is 296 g, with an allowable error of less than 4 g. The box itself has a mass of 12 g.

a. If x represents the desired mass of each snack bar, write an expression for $f(x)$, the total mass of the product including 8 bars and the box.

b. Determine the tolerance interval for $f(x)$ that meets the above conditions.

c. Determine the largest possible precision interval for x yielding the tolerance interval in Part **b**.

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2.6 A commuter airplane flies 270 km from city A to city B, remains on the ground for 30 min, then returns to city A. To allow passengers to connect with other flights, the plane must complete the round-trip in less than 3.5 hr but more than 3.25 hr. Determine an acceptable interval for the average speed of the plane while it is in the air.

2.7 How close to 2 must x be for x^3 to be within 0.1 of 8? Explain your response.

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Activity 3

In previous activities, you described tolerance intervals and precision intervals using both interval notation and inequalities. A third way of representing intervals involves the use of **absolute values**.

Mathematics Note

The **absolute value** of a real number x, denoted by $|x|$, is defined as follows:

$$|x| = x, \text{ if } x \geq 0$$

$$|x| = -x, \text{ if } x < 0$$

For example, $|2| = 2$, $|-4| = 4$, $|-2/3| = 2/3$, $|0| = 0$ and $|\sqrt{2}| = \sqrt{2}$.

Exploration

In this exploration, you investigate some properties of the absolute-value function.

a. Graph $f(x) = |x|$.

b. Examine the relationship between the x- and y-coordinates of points on the graph.

c. Use the graph of $f(x) = |x|$ to solve each of the following:

1. $|x| = 5$

2. $|x| = 0$

3. $|x| = -5$

d. **1.** Solve the inequality $|x| < 5$ by inspecting the graphs of $f(x) = |x|$ and $g(x) = 5$. Use inequalities to describe the solution set.

2. Solve the inequality $|x| > 5$. Use inequalities to describe the solution set.

3. Express the solution to the inequality in Step **2** as the union of two intervals.

e. **1.** Solve the equation $|x - 2| = 5$ by inspecting the graphs of $h(x) = |x - 2|$ and $g(x) = 5$.

2. Graph these solutions on a number line.

3. Describe the locations of the solutions in terms of their distances from 2 on the number line.

4. Express the solution using set notation.

f. Repeat Part **e** for $|x - a| = b$, where *a* and *b* are positive integers, for several values of *a* and *b*.

g. **1.** Solve the inequality $|x - 2| < 5$.

2. Graph these solutions on a number line.

3. Describe the locations of the solutions in terms of their distances from 2 on the number line.

4. Describe the solution using interval notation.

h. Repeat Part **g** for $|x - a| < b$, where *a* and *b* are positive integers, for several values of *a* and *b*.

Discussion

a. Describe the graph of $f(x) = |x|$.

b. What are the domain and range of the absolute-value function $f(x) = |x|$?

c. The expressions $|x|$ and $|x - 0|$ represent the same distance. Describe this distance.

d. What distance is represented by $|x - 2|$? by $|x + 2|$?

e. Using absolute value, describe an expression that represents the interval $5 < x < 9$.

f. Describe the general solution to each of the following inequalities, where $a \geq 0$ and $b \geq 0$:

1. $|x - a| < b$

2. $|x - a| > b$

Assignment

3.1 Graph the solution to each of the following inequalities on a number line.

a. $|x - 11| < 4$

b. $|x - 7| > 4$

c. $|x + 15| < 10$

3.2 Express each of the following intervals using absolute values.

a. $-5 < x < 7$

b. $-7 < x < -4$

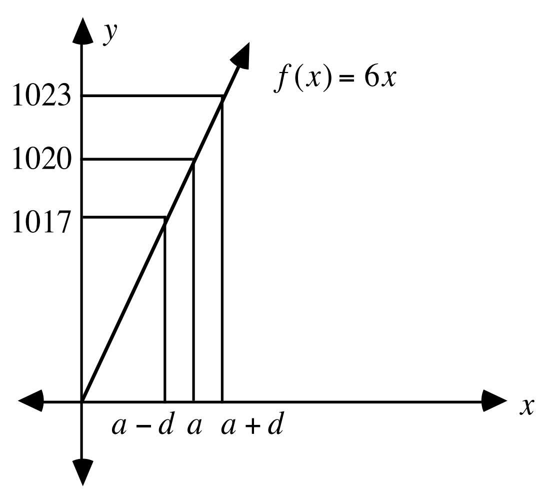
c. $x < 24$ or $x > 28$

d. $[-6,3]$

e. $(-\infty,12) \cup (14,\infty)$

3.3 A tolerance interval for the volume of soda in a soft-drink container can be expressed as $|x - 354| < 2$, where x is measured in milliliters. Solve this inequality.

3.4 The relationship between the tolerance interval for the total volume in milliliters of 6 bottles of fruit juice, and the largest corresponding precision interval for x, the volume of each bottle of juice, is illustrated below.



a. Express the tolerance interval as an inequality using absolute values.

b. Express the precision interval as an inequality using absolute values.

3.5 The total mass M in grams, of 8 slices of lunch meat, not including the package, must be less than 3 g from the desired mass of 224 g.

a. Express the tolerance interval for M in absolute-value form.

b. Determine a function $f(x)$ for the total mass of 8 slices of lunch meat if x is the mass of 1 slice.

c. Determine the largest possible precision interval for x.

d. Determine a, the desired mass of 1 slice, and d, the allowable error in the mass of 1 slice for this interval.

e. Using your responses to Part **d**, express the precision interval for x in absolute-value form.

f. Draw a diagram similar to the one in Problem **3.4** to illustrate the relationship between the tolerance interval and the precision interval.

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3.6 The total mass in grams, including packaging material, of a 4-pack of pudding is $4x + 6$, where x represents the mass of 1 serving of pudding.

a. The tolerance interval for the total mass can be described by the following inequality: $|(4x + 6) - 452| < 12$. Express this inequality without using absolute values.

b. Determine the largest possible precision interval for the mass of 1 serving of pudding.

3.7 If the solution to an inequality in the form $|x - a| \leq b$ is $[-13, 86]$, what is the solution to $|x - a| > b$?

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Activity 4

The precision intervals you explored in the first three activities were determined by solving inequalities involving linear functions. Many manufacturing processes, however, are modeled by nonlinear functions.

Exploration

In this exploration, you discover when the magnitude of the area of a geometric figure exceeds the magnitude of its perimeter.

a. Using a geometry utility, construct a square so that when you drag any vertex, the polygon remains a square.

b. Calculate the length of one side of the square, the perimeter of the square, and the area of the square.

c. Drag one of the vertices of the square until the numbers representing the perimeter and area have approximately the same value. Record the resulting length of the side of the square.

d. Let x represent one side of the square.

1. Write a function $f(x)$ that represents the area of the square. Identify the domain of the function.
2. Write a function $g(x)$ that represents the perimeter of the square and identify its domain.
3. Graph $f(x)$ and $g(x)$ on a graphing utility, using appropriate domains and ranges.
4. Write an inequality that describes the values of x for which the magnitude of the square's area is less than the magnitude of its perimeter.
5. Inspect the graphs from Step **3** to determine the interval for x in which $f(x) < g(x)$.

e.

1. Express the difference of $f(x)$ and $g(x)$ as an inequality in the form $f(x) - g(x) < 0$. Use a symbolic manipulator to factor the inequality.
2. Determine the values of x for which the factors equal 0.
3. How are these values related to the interval you determined in Part **d**?
4. The values found in Step **2** separate a number line into three intervals. Determine the sign of each of the expressions, x, $x - 4$, and $x(x - 4)$ for values of x in each of the intervals.

f. Repeat Parts **a–e** using a circle, along with its radius, circumference, and area.

Discussion

a.

1. What type of graph represents the perimeter of geometric figures?
2. What type of graph represents the area of geometric figures?

b. How does inspecting the graphs of the functions in Part **d** of the exploration help determine the intervals of x for which:

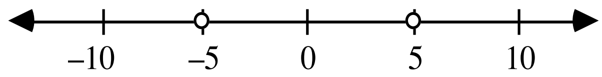
1. the magnitude of a figure's area exceeds the magnitude of its perimeter?
2. the magnitude of a figure's area is less than the magnitude of its perimeter?

c.

1. Describe how to use graphs to solve an inequality of the form $x^n > k$, where k is a constant and n is a positive integer.
2. Describe how to use graphs to solve an inequality of the form $x^n < k$, where k is a constant and n is a positive integer.

Assignment

4.1 Use a number line like the one shown below to complete Parts **a–e**.



a. **1.** Place a positive sign (+) above the portions of the number line where $x-5$ is positive.

2. Place a negative sign (–) above the portions of the number line where $x-5$ is negative.

b. Repeat Part **a** for $x+5$.

c. Identify values on the number line for which $(x-5)(x+5)>0$.

d. Identify values on the number line for which $(x-5)(x+5)<0$.

e. How do the values in Parts **c** and **d** relate to the solutions of the inequalities $x^2<25$ and $x^2>25$?

4.2 **a.** Solve the inequality $x^2-9<0$ and graph the solution set on a number line.

b. Use your response to Part **a** to write an inequality using absolute values that is equivalent to $x^2<9$.

c. Solve the inequality $x^2>9$. Graph the solution set on a number line.

4.3 **a.** Solve the inequality $x^2<16$. Graph the solution set on a number line.

b. Solve the inequality $4<x^2$. Graph the solution set on a number line.

c. Use your responses to Parts **a** and **b** to solve the conjunction of inequalities $4<x^2<16$. Graph the solution set on a number line.

d. **1.** Graph $f(x)=x^2$, $g(x)=4$, and $h(x)=16$ on a sheet of graph paper.

2. Use the graph to illustrate the relationship between $4<x^2<16$ and the solution set from Part **c**.

4.4 Solve $(x-2)(x-3)(x+4)<0$.

4.5 A container manufacturer is designing a small cylindrical jar made from cut glass. The jars will be used to package caviar. Because cut glass is expensive, the surface area of the jar for a given volume should be kept as small as possible.

a. If the jar's height is 4 cm, determine the minimum radius for the jar's base for which the magnitude of the volume exceeds the magnitude of the surface area. (The surface area of the jar does not include the lid.)

b. Determine the volume of the jar for which the magnitude of the volume is approximately equal to the magnitude of the surface area.

4.6 A packaging company plans to produce a cardboard milk carton. The carton will be a rectangular prism 20.1 cm high, with a square base. The desired volume of the carton is 1 L, or 1000 cm^3.

a. What is the desired measurement for the sides of the square base?

b. An acceptable container must hold within 5% of the specified capacity. What is the tolerance interval for the volume of the container?

c. If the manufacturing process can consistently produce a carton with a height of 20.1 cm:

1. determine an interval of acceptable measures for the sides of the square base

2. find the largest acceptable error for the sides of the square base.

4.7 Imagine that you are an engineer at a container company. A customer has requested a cardboard box with a rectangular base and a volume of 120 cm^3. The width of the box must be 3 cm and—for aesthetic reasons—the customer would like the ratio of length to height to be the golden ratio: $\left(1+\sqrt{5}\right)/2$, or approximately 1.618. The acceptable tolerance for the volume is less than 1%.

a. Write a function that describes the volume in terms of height.

b. What is the desired length of the box? What is the desired height?

c. What is the tolerance interval for the volume of the box?

d. What are the precision intervals for the length and height of the box?

e. If the tolerance for volume is reduced to less than 0.5%, how would this affect the precision intervals for length and height?

f. Describe what happens to the precision interval when the size of the tolerance interval is decreased.

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4.8 As a design engineer, you have been assigned to create a cylindrical container with a capacity of 300 L. The container's height must be 3 times the radius of its base. The volume of the container must be within 1% of the specified capacity.

a. Write a function for the volume in terms of the radius.

b. Find the desired radius of the base.

c. Determine acceptable intervals for each of the following:

1. volume

2. radius

3. height.

4.9 A skydiver leaps from a plane flying 3000 m above the earth's surface. Disregarding air resistance, her distance from the ground is described by the function $f(t) = 3000 - 4.9t^2$, where t is the time in seconds after the start of the jump. The skydiver would like to open her parachute when her distance from the ground is approximately 1200 m. Her skydiving instructor has given her a safety margin of less than 300 m. Determine the appropriate time interval in which she can open her parachute safely, and express your answer as an inequality.

4.10 Like most metals, aluminum expands when heated and contracts when cooled. The volume v of an aluminum container, in cubic meters, after a temperature change of x degrees Celsius is

$$v = v_0 + \left(6.9 \bullet 10^{-5} x\right) v_0$$

where v_0 is the volume before the temperature change.

a. An aluminum vat is used to store milk. Its maximum volume is 40 m^3 and temperatures vary by 36° C. Find the minimum volume of the vat.

b. Express the interval of volumes for the vat, to the nearest 0.001 m^3 in interval notation.

c. If the vat is a cylinder and its height equals its diameter, determine an interval, in centimeters, for the height of the vat. Express the heights to the nearest 0.1 cm.

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Summary Assessment

A packaging manufacturer is planning to retool its factory to produce cylindrical metal containers. As an engineer for the company, you have been asked to determine the precision necessary for the machinery which will stamp out the sides of the container. In order to make this determination, you first must identify an acceptable interval for the container's height.

Using the available equipment, you know that circular bases with the desired 8-cm radius can be produced within a 1% tolerance. The capacity or volume of each container should be 1 L (or 1000 cm^3) with a tolerance of less than 2.5%.

Write a report to the manager of the factory describing an appropriate precision interval for the height of the container. Explain how you determined this interval and verify that it will allow the production of containers within the desired 2.5% tolerance for volume.

Module Summary

- The process of measuring product performance, comparing those measurements with predetermined standards, and then acting on the difference is called **quality control**.
- A **tolerance interval** represents the acceptable error in a product.
- A **precision interval** represents the acceptable error in a component of a product.
- An **interval** of real numbers is a set containing all numbers between two given points, the **endpoints** or **bounds** of the interval. An interval may contain one endpoint, both endpoints, or neither endpoint. In addition, some intervals may have only one endpoint, or no endpoints. For example, the set of real numbers can be represented as the interval $(-\infty,\ \infty)$, while the set of real numbers greater than or equal to a can be represented as $[a,\ \infty)$.
- Some intervals may be expressed as a conjunctions of two inequalities. A **conjunction** combines two mathematical statements with the word *and*. A conjunction is true only if both statements are true. If one or both of the statements is false, the conjunction is false.
- Intervals and associated inequalities can be expressed using set notation.
- A limit is used to examine the behavior of a function close to, but not at, a particular point. If the value of $f(x)$ gets arbitrarily close to c as x gets close to a, then c is the **limit** of the function as x approaches a. Mathematically, this is true if, for every real number e, there exists a corresponding positive real number d so that $c - e < f(x) < c + e$ whenever $a - d < x < a + d$.
- The **absolute value** of a nonzero real number x, denoted by $|x|$, is defined as follows:777

$$|x| = x, \text{ if } x \geq 0$$

$$|x| = -x, \text{ if } x < 0$$

- The absolute value of $x - a$, where x and a are real numbers, represents the distance between x and a.
- The inequality $|x - a| < b$, where b is a positive number, is satisfied by the real numbers x whose distance from a is less than b units. These real numbers constitute the interval $(a - b, a + b)$.
- The inequality $|x - a| > b$, where b is a positive number, is satisfied by the real numbers x whose distance from a is more than b units. These real numbers constitute the interval $(-\infty, a - b) \cup (a + b, \infty)$.

Selected References

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