Big Business

How can mathematics help launch a new business? In this module, you'll explore how rational functions and nonlinear inequalities can offer direction to a fledgling cosmetics company.

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Big Business

Introduction

Americans spend hundreds of millions of dollars annually on products to keep themselves clean, conditioned, made-up, and trimmed down. To keep pace with the wants of consumers, cosmetics companies have entire departments dedicated to research and development. These and other firms also collaborate with the scientific community to create innovative and marketable products.

In the health and beauty industry, developing a new product involves a mix of disciplines, including chemistry, biology, and mathematics. Finding the financial support to launch such a product can be difficult. Most sources of capital require a business plan prior to investment. Business plans may include background information about the product, descriptions of the targeted consumers, sales projections, and advertising strategies.

Exploration

Imagine that you work for a company that wants to market a new home permanent. Permanents (or perms) alter straight hair in order to produce curls. Some types of perms use extremes in pH, either acidic or basic, to curl hair. The pH scale is shown in Figure **1**. A solution with a pH less than 7 is an acid; one with a pH greater than 7 is a base.

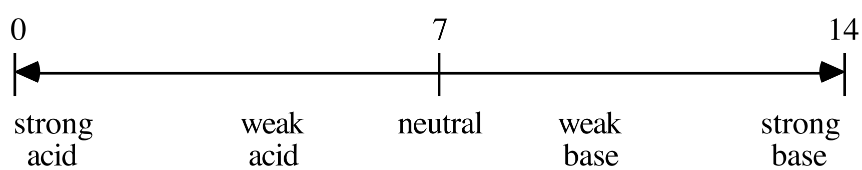

Figure 1: The pH scale

Because chemicals with extreme pH values can damage hair and skin, your company wants the pH of its permanent to be as close to neutral as possible. In this exploration, you simulate the pH balance of a home permanent using ammonia, an actual ingredient in many perms. In water, ammonia forms a solution of ammonium hydroxide, a mild base.

a. In the module “Log Jam,” you learned that the pH of a solution is determined by the negative log of the concentration of hydrogen ions in moles per liter.

In a solution of ammonia and water, the concentration of hydrogen ions depends on the concentration of ammonium hydroxide. Determine the pH of a solution of 6.80% ammonium hydroxide.

b. Concentration also may be expressed as a percentage using mass or volume, as shown below:

$$\text{percent concentration} = \frac{\text{amount of pure substance}}{\text{total amount of solution}} \bullet 100$$

One milliliter of 6.80% ammonium hydroxide solution has a mass of 0.993 g. Use this information to calculate the mass (in grams) of pure ammonium hydroxide in 1 mL of this solution.

c. Measure 1 mL of a solution of 6.80% ammonium hydroxide and pour it into a 1-L beaker. Record its pH (determined in Part **a**) in a table with headings like those in Table **1** below.

Table 1: Ammonium hydroxide solution data

Mass of Distilled Water (g)	Total Mass of Solution (g)	pH	Concentration of Ammonium Hydroxide
0	0.993		6.80%
50	50.993		
100			

d. Add 50 mL of distilled water to the ammonium hydroxide solution. At room temperature, 1 mL of water has a mass of 1 g. Therefore, the total mass of the new solution is now 50.993 g, as shown in Table **1**.

1. Measure the pH of the new solution and record it in Table **1**.
2. Calculate the percent concentration of ammonium hydroxide in the new solution and record this value in Table **1**.

e. Add another 50 mL of water to the solution. Record the total mass of the solution, its pH, and the percent concentration of ammonium hydroxide in Table **1**. Continue this process until you have added a total of 1 L of water.

f. Create two separate scatterplots of the data: one that shows pH versus the mass of distilled water added, and one that shows the percent concentration of ammonium hydroxide versus the mass of distilled water added.

g.
1. Write a function that relates percent concentration to the mass of distilled water added. (Hint: Use the relationship for percent concentration described in Part **b** as a guide.)
2. Considering the context, describe an appropriate domain for the function.

h.
1. Use a graphing utility to graph your function from Part **g** over the interval [–5, 5].
2. Describe the domain for this function in general.

Discussion

a. How much water did you have to add to the ammonium hydroxide solution to obtain a pH less than 8?

b. How much water would have to be added to the ammonium hydroxide solution to obtain a pH of 7? Explain your response.

c. How much water would you have to add to reduce the concentration of ammonium hydroxide to 0.01%? Explain your response.

d. Compare the two plots you graphed in Part **f** of the exploration.

e. How does the graph from Part **h** of the exploration compare to the graphs from Part **f**?

f. Compare the domain of the function found in Part **h** with the domain that applies to the context of the exploration.

g. What happens to the graph in Part **h** of the exploration when $x = -0.993$?

Activity 1

In the introduction, you investigated one situation in which mathematics might help the developers of a new home permanent. The function you used to model the concentration of ammonium hydroxide is a **rational function**. Rational functions can be used to model a variety of phenomena from fields as diverse as business, biology, physics, and social science. In this activity, you investigate some of the characteristics of rational functions and their graphs.

Mathematics Note

A **rational function** is in the form:

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomial expressions and $d(x) \neq 0$. The domain of $f(x)$ does not contain values of x for which $d(x)$ equals 0.

For example, consider the function below:

$$f(x) = \frac{2x^3 - 3x + 4}{x + 7}$$

This is a rational function where $n(x) = 2x^3 - 3x + 4$ and $d(x) = x + 7$, with $x \neq -7$. Since $d(x) = 0$ when $x = -7$, the function is undefined at $x = -7$ and the domain of $f(x)$ is the set of all real numbers except -7.

Exploration 1

In some cases, the graph of a rational function can resemble the graph of a polynomial function. This occurs when all the factors of the denominator are also factors of the numerator. Although the two graphs may appear identical, there are important differences. In this exploration, you compare a rational function and a polynomial function with similar graphs.

a. In the module "Drafting and Polynomials," you found that a polynomial function with all real roots can be expressed as the product of factors in the form:

$$p(x) = a(x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$$

where c_n is a real root of the polynomial function $p(x)$.

1. Graph the polynomial function $f(x) = x^3 + x^2 - 14x - 24$.
2. Determine the roots of $f(x)$.
3. Identify the domain of the function.
4. Write $f(x)$ as a product of first-degree factors.

b. Repeat Part **a** for the polynomial function $g(x) = x^2 + 5x + 6$. Graph $g(x)$ on the same coordinate system as $f(x)$.

c.
1. Create the rational function:

$$r(x) = \frac{f(x)}{g(x)}$$

2. Identify the domain of $r(x)$.
3. Graph $r(x)$ on the same coordinate system as $g(x)$ and $f(x)$.

d. The graph of $r(x)$ appears to be a line. Determine the equation of the line that appears to coincide with the graph of $r(x)$.

Mathematics Note

Two functions $f(x)$ and $g(x)$ are **equivalent** if and only if the domain of $f(x)$ is the same as the domain of $g(x)$ and $f(x) = g(x)$ for all values of x in the domain.

For example, $f(x) = x^2/1$ and $g(x) = x^2$ are equivalent functions since the domain of each is the set of all real numbers and $f(x) = g(x) = x^2$ for all x-values in the domain. The functions $g(x) = x^2$ and $h(x) = x^3/x$, however, are not equivalent, since their domains are not the same. The domain of $g(x)$ is the set of all real numbers; the domain of $h(x)$ is all real numbers except 0.

e. When working with fractions that contain integers in their numerators and denominators, it is possible to express them as equivalent fractions in reduced form. This can be done by expressing the numerator and denominator as the products of their factors, then dividing like factors, as shown below:

$$\frac{12}{6} = \frac{2 \bullet 2 \bullet 3}{2 \bullet 3} = 2 \bullet \frac{2}{2} \bullet \frac{3}{3} = 2 \bullet 1 \bullet 1 = 2$$

This process also can be used to rewrite a rational function whose numerator shares the factors of the denominator. For example, consider the following rational function:

$$\begin{aligned} \frac{x^2 - x - 6}{x - 3} &= \frac{(x+2)(x-3)}{x-3} \\ &= \frac{(x+2)\cancel{(x-3)}}{\cancel{x-3}} \\ &= x + 2, \text{ as long as } x \neq 3 \end{aligned}$$

Use this process to rewrite the rational function $r(x)$ as a function $t(x)$. Identify any value of x for which the division of like factors is not defined.

f. Use a symbolic manipulator to divide $f(x)$ by $g(x)$. Compare the result with the equation of the line you found in Part **d** and the equation of $t(x)$ from Part **e**.

Discussion 1

a. Where do the graphs of $f(x)$ and $g(x)$ intersect? Why does this occur?

b. Where does the graph of $r(x)$ intersect the x-axis? Does $r(x)$ also intersect with $f(x)$ or $g(x)$ at this point (or points)? Explain your response.

c. How could you use the graphs of two polynomial functions to determine if they have any common factors?

d. Consider a rational function $r(x)$ for which all the factors of the denominator are also factors of the numerator. Describe how to identify a polynomial function whose graph resembles that of $r(x)$.

e.
1. In Part **d** of Exploration **1**, you determined the equation of a line that appeared to coincide with the graph of $r(x)$. Are this line and $r(x)$ equivalent functions? Explain your response.
2. Are $r(x)$ and $t(x)$ equivalent functions? Explain your response.
3. Is the result of the division of $f(x)$ by $g(x)$ found in Part **f** of Exploration **1** equivalent to $r(x)$? Explain your response.

f. Describe how to use a symbolic manipulator to rewrite a rational function as described in Exploration **1**.

Exploration 2

Consider the following two functions: $f(x) = x + 3$ and

$$g(x) = \frac{x^2 + x - 6}{x - 2}$$

a. Identify the domains of $f(x)$ and $g(x)$.

b. Divide the numerator of $g(x)$ by the denominator.

c. Compare the graphs of $f(x)$ and $g(x)$ in a small interval near the value for which the denominator of $g(x)$ equals 0.

d. To investigate $f(x)$ and $g(x)$ near the value for which $g(x)$ is undefined, use appropriate technology to complete the following table.

Table 2: Values of $f(x)$ and $g(x)$

x	$f(x)$	$g(x)$
1.5		
1.6		
1.7		
1.8		
1.9		
2.0		
2.1		
2.2		
2.3		
2.4		
2.5		

e. Determine whether $f(x)$ and $g(x)$ are equivalent.

Discussion 2

a.
1. Which of the functions in Exploration **2**, if any, are polynomial functions?
2. Which, if any, are rational functions?

b. Describe another function that is equivalent to each of the following:
1. $f(x)$
2. $g(x)$

Mathematics Note

A function is **continuous** at a point c in its domain if the following conditions are met:

- the function is defined at c, or $f(c)$ exists
- the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
- the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$

A function is continuous over its domain if it is continuous at each point in its domain.

A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.

For example, a function is discontinuous at $x = c$ if the function is undefined at c, as shown in Figure **2**.

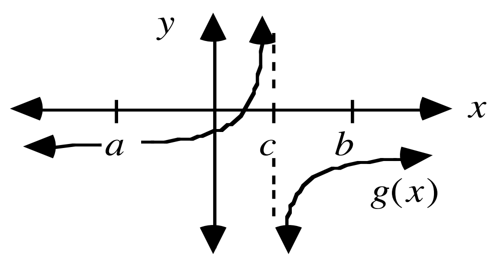

Figure 2: Graph of the discontinuous function $g(x)$

A function also is discontinuous at $x = c$ if the limit of the function does not exist at c, as shown in Figure **3**.

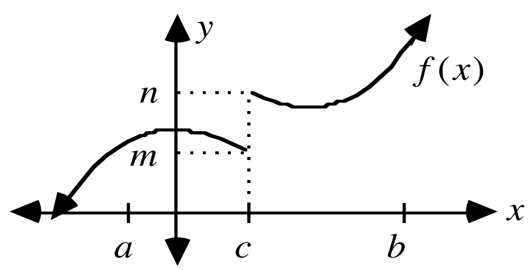

Figure 3: Jump discontinuity in the function $f(x)$

In this case, $f(x)$ approaches m as x approaches c from the left. As x approaches c from the right, $f(x)$ approaches n. Since $m \neq n$, the limit of $f(x)$ as x approaches c does not exist. This kind of discontinuity is referred to as a **jump discontinuity**.

A function also is discontinuous at $x = c$ if the value of the function at c does not equal the limit of the function as x approaches c, as shown in Figure **4**. In this case, the limit of $h(x)$ as x approaches c is m, while $h(c) = n$, and $m \neq n$.

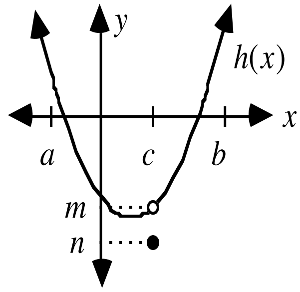

Figure 4: Hole in the graph of the function $h(x)$

c. Consider the rational function below:

$$h(x) = \frac{x^3 - 4x^2 + x + 6}{x - 2}$$

1. Is $h(x)$ a continuous function? Explain your response.
2. Identify the factors of the numerator and the denominator. Use these factors to explain why the graph of $h(x)$ would be similar to the graph of the polynomial function $p(x) = x^2 - 2x - 3$.
3. What differences would exist between the graphs of $h(x)$ and $p(x)$? Explain your response.
4. Identify a discontinuous rational function with a hole at $x = -3$ whose graph would be similar to the graph of $p(x) = x^2 - 2x - 3$. Justify your response.

Assignment

1.1 Identify the domain of each rational function below and describe the values of x at which the function is discontinuous.

a. $f(x) = \dfrac{(x + 2)(x - 7)}{(x - 7)}$

b. $g(x) = \dfrac{x^3 + 4x^2 - x - 4}{x^2 + 5x + 4}$

c. $h(x) = \dfrac{1}{x^2 + 1}$

1.2 Sketch the graph of a function with a hole at $x = -2$.

1.3 Consider the following two functions: $f(x) = x^2 + 2x - 3$ and

$$g(x) = \frac{x^3 + 5x^2 + 3x - 9}{x + 3}$$

a. Divide the numerator of $g(x)$ by the denominator.

b. Are $f(x)$ and $g(x)$ equivalent? Justify your response.

1.4 Identify the domain of each rational function in Parts **a–e**. Sketch a graph of each function and label any holes that occur.

a. $f(x) = (x + 2)(x - 2)$

b. $f(x) = \dfrac{x^3 - 5x^2 - 2x + 24}{x - 4}$

c. $f(x) = \dfrac{x^3 - 5x^2 - 2x + 24}{x^2 - x - 6}$

d. $f(x) = \dfrac{2x^2 + 1}{5}$

e. $f(x) = \dfrac{5}{2x^2 + 1}$

1.5 Explain why any polynomial function is also a rational function.

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1.6 Determine the domain of each rational function below. Sketch a graph of each function and label any holes that occur.

a. $f(x) = \dfrac{x^3 - 10x^2 - 23x + 132}{x - 11}$

b. $f(x) = \dfrac{-4}{x^2 + 3x + 13}$

c. $f(x) = \dfrac{x^5 + 3x^4 - 17x^3 - 15x^2 + 88x - 60}{x^2 + 3x - 10}$

d. $f(x) = \dfrac{x^2 + 0.15x - 0.45}{0.75 + x}$

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Activity 2

In the previous activity, you examined the graphs of rational functions with common factors in the numerator and the denominator. In this activity, you continue your investigation of rational functions and their graphs by examining functions whose numerators and denominators do not have common factors.

Exploration 1

Not all rational functions have numerators and denominators that share factors. In this exploration, you examine how to express rational functions in much the same way as mixed numbers are used to express improper fractions.

One way to determine the mixed number that is equivalent to an improper fraction is to rewrite the numerator as a sum of two rational numbers. For example, the improper fraction $12/5$ can be rewritten as the mixed number $2\frac{2}{5}$ as follows:

$$\frac{12}{5} = \frac{10+2}{5} = \frac{10}{5} + \frac{2}{5} = 2 + \frac{2}{5}$$

Using a similar process, any rational expression $n(x)/d(x)$ in which the degree of the numerator is greater than or equal to the degree of the denominator can be written as an expression of the form:

$$\frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder in the indicated division.

For example, consider the following rational expression:

$$\frac{5x+2}{x+2}$$

One method for rewriting this as a mixed expression is outlined below:

$$\begin{aligned}
\frac{5x+2}{x+2} &= \frac{5x+10-8}{x+2} \\
&= \frac{5(x+2)-8}{x+2} \\
&= \frac{5(x+2)}{x+2} + \frac{-8}{x+2} \\
&= 5 - \frac{8}{x+2}
\end{aligned}$$

a. If the resulting mixed expression is equivalent to the original rational expression, then $d(x) \bullet Q(x) + R(x) = n(x)$.

Verify that this relationship is true for the example described above.

b. Rewrite the rational expression below in the form $Q(x)+R(x)/d(x)$.

$$\frac{6x-5}{x+4}$$

c. Verify that the final expression in Part **b** is equivalent to the original expression.

d. Use a symbolic manipulator to determine an expression that is equivalent to the original expression in Part **b**. Compare the result with your solution in Part **b**.

e. Select another rational expression in which the degree of the numerator is greater than or equal to the degree of the denominator. Rewrite the expression in the form $Q(x)+R(x)/d(x)$.

Discussion 1

a. When a rational expression in which the numerator shares all the factors of the denominator is written in the form $Q(x)+R(x)/d(x)$, the value of $R(x)$ is 0. Explain why this is true.

b. **1.** When would the value of $Q(x)$ be 0? Explain your response.

2. When would the value of $Q(x)$ be a real number other than 0?

c. Describe the process required to express a rational expression as a mixed expression using a symbolic manipulator.

Exploration 2

In this exploration, you examine the graph of a rational function in which the numerator and denominator share no common factors.

a. Consider the rational function

$$f(x)=\frac{n(x)}{d(x)}$$

where $n(x)=3x+2$ and $d(x)=x-1$.

For what value(s) of x does $d(x)=0$?

b. Identify the domain of $f(x)$.

c. Divide $n(x)$ by $d(x)$. Write your answer in the following form:

$$f(x)=\frac{n(x)}{d(x)}=Q(x)+\frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder.

d. To explore the values of $f(x)$ near $x = 1$, use appropriate technology to complete the following table.

Table 3: Values of $f(x)$ near $x = 1$

x	$f(x)$
0.9	
0.95	
0.99	
0.999	
1.001	
1.01	
1.05	
1.1	

e. In "Drafting and Polynomials," you examined the graphs of polynomial functions as values in the domain became very large or very small. To explore the values of the rational function $f(x)$ as $|x|$ becomes large, use appropriate technology to complete the following table.

Table 4: Values of $f(x)$ as $|x|$ becomes large

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–10,000			
–1000			
–100			
100			
1000			
10,000			

f. Create a graph of $f(x)$.

g. Use the values in Table **3** to make conjectures about the graph of $f(x)$ as x approaches 1, the value for which $d(x) = 0$.

h. Use the values in Table **4** to make conjectures about the relationship among $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large.

i. Consider the rational expression you rewrote in Part **b** of Exploration **1**, shown below:

$$f(x) = \frac{6x - 5}{x + 4} = 6 - \frac{29}{x + 4}$$

Do the conjectures you made in Part **h** appear to hold true for this function?

Discussion 2

a. Does the graph of $f(x)$ in Part **a** of Exploration **2** have any discontinuities? If so, describe the values at which they occur.

b.
1. Describe the graph of $f(x)$ as x approaches 1 from the left.
2. Describe the graph of $f(x)$ as x approaches 1 from the right.

c. Describe the graph of $f(x)$ as $|x|$ becomes large.

d. What conjectures did you make about the relationship among $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large?

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line approaches 0 as the distance from point P to the origin increases without bound. Asymptotes, like holes, may be the result of discontinuities.

For example, consider the rational function below:

$$f(x) = \frac{x-3}{x-5}$$

By dividing the numerator by the denominator, this function can be rewritten as

$$f(x) = 1 + \frac{2}{x-5}$$

As x approaches 5, $(x-5)$ approaches 0 and $f(x)$ becomes increasingly large or increasingly small. Because $f(5)$ is not defined, a discontinuity occurs at $x = 5$. The result of that discontinuity is a vertical asymptote.

As $|x|$ becomes large, the quantity $2/(x-5)$ approaches 0 and $f(x)$ approaches 1. In this case, the line $y = 1$ is a horizontal asymptote of the function. Figure **5** shows a graph of $f(x)$ and its asymptotes.

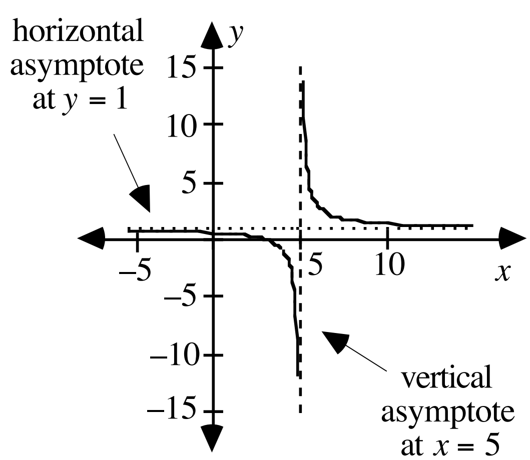

Figure 5: Graph of $f(x) = \dfrac{x-3}{x-5}$

e. Describe any asymptotes that exist in the graph of the rational function you investigated in Parts **a–f** of Exploration **2**.

f. Does a horizontal asymptote indicate a discontinuity? Explain your response.

g. **1.** Describe a rational function in the following form whose graph includes a vertical asymptote:

$$f(x) = Q(x) + \frac{R(x)}{d(x)}$$

2. Describe a rational function that includes the same expression for $d(x)$ as in Step **1** and has a hole in its graph.

3. Identify the values of x for which the discontinuities appear in each function you described above.

h. Based on your observations, how do rational functions whose graphs include holes compare with those whose graphs include vertical and horizontal asymptotes?

Assignment

2.1 Sketch the graph of a function with a vertical asymptote at $x = -3$ and a horizontal asymptote at $y = 2$.

2.2 Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x) = 2$ and $d(x) = x - 1$.

a. For what value(s) of x does $d(x) = 0$?

b. Identify the domain of $f(x)$.

c. Rewrite $f(x)$ in the following form:

$$Q(x) + \frac{R(x)}{d(x)}$$

d. **1.** Create a graph of $f(x)$ and describe its asymptotes.

2. How are these asymptotes related to your response to Part **c**?

2.3 Identify the domain of each rational function in Parts **a–e**. Sketch a graph of each function and label any holes or asymptotes that occur.

a. $f(x) = 1/x^2$

b. $f(x) = \dfrac{2x - 10}{x - 4}$

c. $f(x) = \dfrac{6}{4 - x^2}$

d. $f(x) = \dfrac{x^3 - x^2 - 8x + 12}{x - 2}$

e. $f(x) = \dfrac{x^3 + 4x^2 - 17x - 60}{x^3 + 11x^2 + 39x + 45}$

2.4 In the introduction tothe module, you explored the relationship between the percent concentration of ammonium hydroxide in a solution and the mass of distilled water added. This relationship can be modeled by the function below, where x represents the mass of distilled water in grams.

$$f(x) = \frac{0.0680}{x + 0.993} \bullet 100$$

a. Identify the domain of $f(x)$.

b. Sketch a graph of $f(x)$.

c. Write the equations of any asymptotes.

2.5 The production manager at your company has determined that the average cost per unit for manufacturing your home perm is approximated by the rational function below, where x is the number of perms produced.

$$C(x) = 27{,}000/(x + 50)$$

a. Find $C(5)$, $C(1000)$, and $C(10{,}000)$.

b. Why do you think that the average cost per unit decreases as the number of units produced increases?

c. Identify the domain of $C(x)$.

d. Sketch a graph of $C(x)$. Label any horizontal or vertical asymptotes.

e. There are usually fixed costs that are required before manufacturing begins. What are the fixed costs for the perm manufacturing process? Justify your answer.

f. Due to the cost of materials, the production manager claims that the lowest possible cost per unit is $2.45, regardless of the number of perms produced. After how many units would this cost be reached?

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2.6 Determine the domain of each rational function below. Sketch a graph of each function and label any holes or asymptotes.

a. $f(x) = 5x/(x - 6)$

b. $f(x) = \dfrac{2x^2 - 7x - 4}{4x + 2}$

c. $f(x) = 5/(x - 3)(x - 1)$

d. $f(x) = \dfrac{0.5x^3 + 2}{x + 1}$

e. $f(x) = \dfrac{x^2 - 3x - 4}{x^3 - 7x^2 + 8x + 16}$

2.7 In an electrical circuit, **resistance** is the opposition to the flow of current. When two resistors, r_1 and r_2, are connected in parallel, the total resistance R of the circuit (in ohms) is described by the following formula:

$$R = \frac{r_1 \bullet r_2}{r_1 + r_2}$$

Consider a circuit in which $r_1 = 10$ ohms and r_2 is unknown.

a. Let $r_2 = x$. Write a function that could be used to model the total resistance R in terms of x. Identify the domain of the function.

b. Represent the function in the form $Q(x) + R(x)/d(x)$. Identify any holes or asymptotes.

c. Graph your function from Part **a** and label any holes or asymptotes.

d. Find the value of r_2 that results in a total resistance of 3 ohms.

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Activity 3

In the previous activity, you examined the graphs of rational functions in the form

$$f(x) = Q(x) + \frac{R(x)}{d(x)}$$

and determined where holes, vertical asymptotes, and horizontal asymptotes occur. In this activity, you examine rational functions with **oblique asymptotes** (asymptotes that are neither horizontal nor vertical).

Exploration

Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x) = x^2 + 3x - 2$ and $d(x) = x + 5$.

a. Identify the domain of $f(x)$.

b. Divide $n(x)$ by $d(x)$. Write your answer in the following form:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder.

c. To explore the values of $f(x)$, $Q(x)$, and $R(x)/d(x)$ as $|x|$ becomes large, complete the following table.

Table 5: Values of $f(x)$ as $|x|$ becomes large

x	$f(x)$	$Q(x)$	$R(x)/d(x)$
–1000			
–500			
–100			
–50			
50			
100			
500			
1000			

d. Create a graph of $f(x)$ and label any discontinuities.

e. Use your graph from Part **d** and the values in Table **5** to suggest a possible equation for the oblique asymptote to the graph of $f(x)$. Test your equation by graphing it on the same coordinate system as $f(x)$.

Discussion

a. Describe the graph of $f(x)$, including any holes or asymptotes.

b. What happens to the value of $R(x)/d(x)$ as $|x|$ becomes large?

c. What happens to the values of $f(x)$ and $Q(x)$ as $|x|$ becomes large?

Mathematics Note

A rational function $f(x)$ can be rewritten in the following form by dividing the numerator by the denominator:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder. In this form, $Q(x)$, $R(x)$, $n(x)$, and $d(x)$ are polynomial functions. When the degree of $R(x)$ is less than that of $d(x)$, the value of $R(x)/d(x)$ approaches 0 as $|x|$ becomes very large. As a result, its effect on a graph of $f(x)$ can be ignored, producing three general cases.

- If $Q(x)$ equals some constant a, the graph of $f(x)$ has a horizontal asymptote at $y = a$.
- If $Q(x)$ is a linear function in the form $y = mx + b$ with $m \neq 0$, the graph of $f(x)$ has an oblique asymptote described by $y = mx + b$.
- If $Q(x)$ is a polynomial function of degree 2 or greater, $f(x)$ has no horizontal or oblique asymptotes.

For example, consider the rational function

$$f(x) = \frac{2x^2 - 5x + 3}{x - 2}$$

The domain of this function is the set of all real numbers except 2. This discontinuity is represented by a vertical asymptote described by the line $x = 2$. Dividing the numerator by the denominator yields:

$$f(x) = \frac{2x^2 - 5x + 3}{x - 2} = 2x - 1 + \frac{1}{x - 2}$$

In this case, $Q(x) = 2x - 1$, a linear function in the form $y = mx + b$ with $m \neq 0$. Therefore, $f(x)$ also has an oblique asymptote described by the equation $y = 2x - 1$. Figure **6** shows a graph of $f(x)$ and its asymptotes.

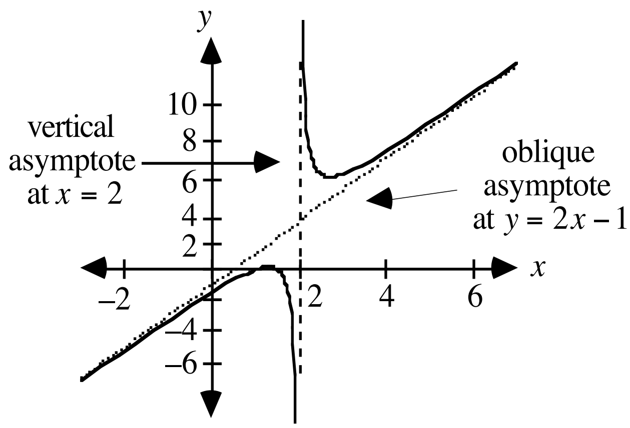

Figure 6: Graph of $f(x) = \frac{2x^2 - 5x + 3}{x - 2}$

Assignment

3.1 Sketch a graph of a function with a vertical asymptote at $x = 2$ and an oblique asymptote with the equation $y = x - 1$.

3.2 Write rational functions whose graphs have the characteristics described in Parts **a–c** below. Explain how you selected each function.

a. a hole at $x = -1$

b. a vertical asymptote at $x = -1$ and a horizontal asymptote

c. a vertical asymptote at $x = -1$ and an oblique asymptote

3.3 Identify the domain of each function in Parts **a–d**. Sketch a graph of each function and label all holes and asymptotes.

a. $f(x) = \dfrac{x^2 + 7x + 9}{x + 1}$

b. $f(x) = x + \dfrac{1}{x}$

c. $f(x) = \dfrac{-2x^2 - 11x + 25}{x + 7}$

d. $f(x) = \dfrac{x^3 + 5x^2 + 8x + 9}{x^2 + 2x + 1}$

3.4 The adult dosage for a particular prescription drug is 85 mg. To calculate the appropriate dosage for patients under 16 years old, a pharmacist uses the formula below, where x is the patient's age in years:

$$D(x) = \frac{85x}{x + 6.5}$$

a. Sketch a graph of $D(x)$ and label all asymptotes.

b. Determine the appropriate dosage for a 2-year-old child.

c. Determine the appropriate dosage for a 15-year-old patient.

* * * * *

3.5 Write rational functions whose graphs have the characteristics described in Parts **a–c** below. Explain how you selected each function.

a. a hole at $x = 5$

b. a vertical asymptote at $x = 5$ and a horizontal asymptote

c. a vertical asymptote at $x = 5$ and an oblique asymptote

3.6 Determine the domain of each function in Parts **a–d**. Sketch a graph of each function and label all holes and asymptotes.

a. $f(x) = \dfrac{-x^2 + 5x + 14}{x + 2}$

b. $f(x) = \dfrac{4x + 3}{2x - 1}$

c. $f(x) = \dfrac{2x^2 - 1}{3x^3 + 2x^2 + 1}$

d. $f(x) = \dfrac{2x^3 + 7x^2 - 4}{x^2 + 2x - 3}$

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Activity 4

After developing a safe and effective home permanent, your company must determine how to package it. Because cylindrical bottles are the most inexpensive form of plastic packaging, your production team has decided to use a 500-mL cylindrical container. (Since the cap does not contain any solution, its volume is not included in the 500 mL.)

Exploration

The formula for the volume of a cylinder is $V = \pi r^2 h$, where r represents the radius of the base and h represents the height.

a. Recall that 1 mL of water has a volume of 1 cm^3. Substitute 500 cm^3 for V in the formula above and solve the resulting equation for h in terms of r.

b. Using a value of r supplied by your teacher, determine the corresponding height of a 500-mL cylindrical container. Construct a model cylinder with these measurements. **Note:** Save your cylinder for use later in this activity.

c. Collect the corresponding values for r and h from the entire class and graph them as ordered pairs in the form (r,h).

d. Graph the function you determined in Part **a** using a reasonable domain.

e. Graph the function from Part **a** over the interval from –20 to 20. Identify any values for which the function is undefined.

Discussion

a. Which of the cylinders constructed by your class would not provide reasonable containers for a home permanent? Explain your response.

b. If you solve the equation $500 = \pi r^2 h$ for r in terms of h, is the result a rational function? Explain your response.

c. What would be a reasonable domain for a function that describes the relationship between the height and radius of a cylinder with a volume of 500 mL?

d. How does your graph in Part **e** of the exploration differ from the graph in Part **d**?

e.
1. Over the set of all real numbers, what value of r would not satisfy the function you wrote in Part **a** of the exploration?
2. What is the domain of this function?
3. Does this function have any discontinuities? Explain your response.

Assignment

4.1 An employee in the marketing department has suggested that the home permanent bottle's height equal its circumference.

a. Write a function that describes this relationship. What is an appropriate domain for the function in this context?

b. Graph the function over the set of all real numbers.

c. Use technology to find the coordinates (to the nearest 0.01) of the intersection of this function with the one you wrote in Part **a** of the exploration.

d. What is the significance of the point of intersection?

e. What would you conclude if the two functions did not intersect?

f. If your company decides to use a bottle whose height equals its circumference, what will its dimensions be?

4.2 A marketing survey has indicated that consumers prefer bottles that do not easily tip over. After experimenting with the bottle's dimensions, your team has decided that its height should not exceed twice the diameter.

a. Write this constraint on the bottle's height as an inequality.

b. The equation $h = 500/\pi r^2$, where h represents height and r represents the radius of the base, models another constraint on the bottle's height. Graph this equation along with the inequality from Part **a** as a system of relations on a coordinate plane.

c. What is the solution of the system? Explain your response.

d. Use the graph of the system to describe the possible dimensions of the bottle.

4.3 The chief financial officer of your company would like to reduce the cost of packaging the home permanent. In response, your production team has suggested that minimizing the bottle's surface area could help minimize the cost of materials.

a. Without its cap, the bottle can be represented as a cylinder. Write a formula for the surface area of a cylinder.

b. Substitute the expression $500/\pi r^2$, where r represents the radius of the base, for h in your formula for surface area. The resulting equation should describe the surface area of a 500-mL cylinder in terms of its radius.

c. Graph this function using an appropriate domain.

d. What is the minimum surface area for a 500-mL cylindrical bottle?

e. What are the dimensions of that bottle (to the nearest 0.01 cm)?

f. Does the bottle fit the condition that its height should not exceed twice its diameter?

g. Would you use a bottle with the dimensions from Part **e**? Explain your response.

* * * * *

4.4 A new restaurant in town has asked you to design a cylindrical bowl that will hold 300 mL of liquid.

a. Use the formula for the volume of a cylinder to write an equation for the height of the bowl in terms of its radius.

b. The height of the bowl should be one-third its diameter. Write an equation that describes this constraint on height in terms of the radius.

c. What is an appropriate domain for the system of equations in Parts **a** and **b**? Explain your response.

d. Graph this system of equations on a coordinate plane.

e. Find the dimensions of the bowl that satisfies these constraints.

* * * * * * * * * *

Activity 5

As you observed in the previous assignment, real-world problems may have more than one acceptable solution. A solution set for an inequality, for example, can consist of a half-plane or a region bounded by curves. Figure **7** shows a solution region bounded on one side by a curve.

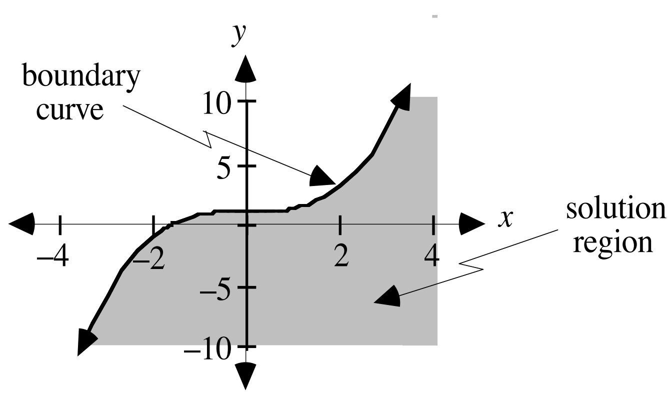

Figure 7: Graph of $y \leq 0.25x^3 + 1$

In this activity, you explore some of the difficulties that can arise in graphing solution sets when a boundary curve is described by a rational function.

Exploration

a. Graph the solution set for $y \geq -0.2x^2 + 3$ by completing Steps **1–4** below.

1. Graph the boundary curve.
2. Decide whether or not the boundary curve is part of the solution set.
3. Decide which side of the boundary curve represents the solution region.
4. Shade the solution region.

b. Repeat Part **a** to graph the solution region for $y < -0.2x^2 + 3$.

c. Describe how the solution regions for $y > -0.2x^2 + 3$ and $y \leq -0.2x^2 + 3$ are related.

Mathematics Note

When graphing inequalities that involve rational functions, a boundary function may contain discontinuities. In such cases, the graph should indicate those portions that are not included in the feasible region as a result of discontinuities.

For example, consider the rational inequality:

$$y \geq \frac{x^3 - 2x^2 - 4x + 8}{x - 2}$$

In this case, the function that defines the boundary of the solution set can be expressed as follows:

$$y = x^2 - 4 + \frac{0}{x - 2}$$

As shown in Figure **8**, there is a hole in the graph of the boundary function at $x = 2$ and the solution set for the inequality contains no points on the line $x = 2$.

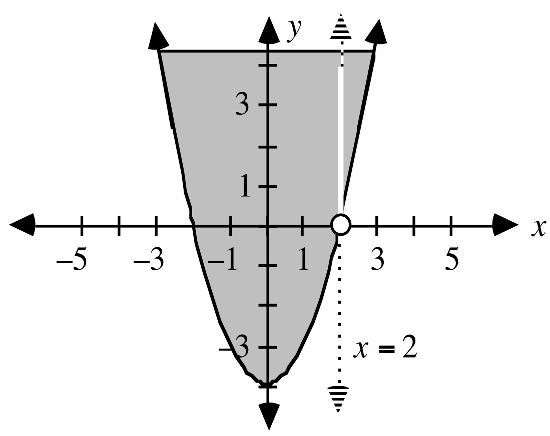

Figure 8: Graph of $y \geq x^2 - 4 + \dfrac{0}{x - 2}$

d. Use a graphing utility to graph the solution sets for each of the following rational inequalities. Label any holes or asymptotes that occur.

1. $y \geq \dfrac{11}{2x}$
2. $y < \dfrac{-2x^2 + 7x - 6}{x - 2}$
3. $y \leq \dfrac{0.3x^3 + x - 1.5}{x + 2}$
4. $y > \dfrac{15x + 6}{(x - 2)(x + 3)}$

Discussion

a. Describe the strategies you used to graph the solution regions in the exploration.

b. What portions of the coordinate plane are included in the graphs of the inequalities $y > -0.2x^2 + 3$ and $y \leq -0.2x^2 + 3$?

c. Why are there no solutions to a rational inequality on the vertical line marking a hole in the boundary function?

Assignment

5.1 To market its product to salons, your company has decided to offer larger bottles of its permanent solutions. The volume of these bottles should be between 500 mL and 1 L. As with the 500-mL size, the height of the bottle should not exceed twice its diameter.

a. Write a system of inequalities that models these constraints on the height of the bottle. (Recall that $V = \pi r^2 h$.)

b. Graph the system of inequalities.

c. Shade the solution to the system and describe its shape.

d. Describe the meaning of the solution set in terms of the bottle's dimensions.

e. Identify the dimensions of a bottle that satisfies these constraints.

5.2 Graph the solution set for each of the following inequalities. Label any discontinuities that exist.

a. $y \geq \dfrac{x^4 - x}{x-1}$

b. $y < \dfrac{x^4 - 7x^3 + 10x^2 + x - 2}{x-2}$

c. $y > 0.0625x^4 - x^3 + 4x^2 - 7$

5.3 Graph each of the following systems of inequalities. Shade the solution region and label any discontinuities that exist.

a. $\begin{cases} y > \dfrac{x^3}{10} \\ y \leq \dfrac{x^2 + 4x - 5}{x-1} \end{cases}$

b. $\begin{cases} y > \dfrac{15}{(x-1)(x+4)} \\ y \leq 0.5x^2 + 1.5x - 5 \end{cases}$

* * * * *

5.4 Graph the solution set for each of the following inequalities. Label any discontinuities that exist.

a. $y > \dfrac{x^5 - 8x^4 + 20x^3 - 25x^2 + 39x - 27}{(x-1)(x-3)}$

b. $y \leq x^4 - 3x^3 - 2x^2 - 9x - 8$

c. $y < \dfrac{3x^4 + 3x^3 - 25x - 25}{x+1}$

5.5 Graph the following system of inequalities. Shade the solution region and label any discontinuities that exist.

$$\begin{cases} y \geq \dfrac{3x^3 - 27}{5} \\ y \leq \dfrac{2x^2 + 6x - 21}{x-5} \end{cases}$$

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Summary Assessment

In this module, you have simulated the development of a home permanent and its packaging. However, you have not yet considered the financial aspects of the business. Following the completion of the research and development phases of the project, your company has determined that an additional investment of $65,000 will be required to bring the product to market.

Using some of its real estate as collateral, the company plans to borrow the money from a bank. The board of directors would like to repay the loan in 20 years, with maximum monthly payments of $565. One formula used by banks to calculate the size of monthly loan payments is:

$$P = \frac{Ar(1+r)^n}{(1+r)^n - 1}$$

where P is the size of the monthly payment in dollars, A is the amount borrowed, r is the monthly interest rate written as a decimal, and n is the total number of monthly payments.

1. Use the formula shown above to write a function that describes payment size in terms of the monthly interest rate. Identify the domain of this function.

2. Monthly interest rates are usually calculated by dividing an annual percentage rate (APR) by 12. If the APR does not exceed 24%, what is the maximum monthly rate?

3. What portion of the domain of the function in Problem **1** applies to this setting?

4. Are there any discontinuities in the graph of the function in Problem **1**? If so, how do they affect the company's situation?

5. Determine the interval of monthly interest rates (to the nearest 0.01) that will allow the company to meet its repayment goals.

6. What is the highest annual percentage rate that will allow the company to meet its repayment goals?

Module Summary

- A **rational function** is in the following form, where $n(x)$ and $d(x)$ are polynomial expressions, and $d(x) \neq 0$:

  $$f(x) = \frac{n(x)}{d(x)}$$

  The domain of $f(x)$ does not contain values of x for which $d(x)$ equals 0.

- Two functions $f(x)$ and $g(x)$ are **equivalent** if and only if the domain of $f(x)$ is the same as the domain of $g(x)$ and $f(x) = g(x)$ for all values of x in the domain.
- A function is **continuous** at a point c in its domain if the following conditions are met:
  - the function is defined at c, or $f(c)$ exists
  - the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
  - the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$
- A function is **continuous** over its domain if it is continuous at each point in its domain.
- A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.
- An **asymptote** to a curve is a line such that the distance from a point *P* on the curve to the line approaches 0 as the distance from point *P* to the origin increases without bound.
- An **oblique asymptote** is an asymptote that is neither horizontal nor vertical.

- A rational function $f(x)$ can be rewritten in the following form by dividing the numerator by the denominator:

$$f(x) = \frac{n(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

  where $Q(x)$ is the quotient and $R(x)$ is the remainder. In this form, $Q(x)$, $R(x)$, $n(x)$, and $d(x)$ are polynomial functions. When the degree of $R(x)$ is less than that of $d(x)$, the value of $R(x)/d(x)$ approaches 0 as $|x|$ becomes very large. As a result, its effect on a graph of $f(x)$ can be ignored, producing three general cases.

  - If $Q(x)$ equals some constant a, the graph of $f(x)$ has a horizontal asymptote at $y = a$.
  - If $Q(x)$ is a linear function in the form $y = mx + b$ with $m \neq 0$, the graph of $f(x)$ has an oblique asymptote described by $y = mx + b$.
  - If $Q(x)$ is a polynomial function of degree 2 or greater, $f(x)$ has no horizontal or oblique asymptotes.

- When graphing inequalities that involve rational functions, a boundary function may contain discontinuities. In such cases, the graph should indicate those portions that are not included in the feasible region as a result of discontinuities.

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