Believe It or Not

How can you convince yourself—and others—that a statement is true or false? In this module, you'll explore some useful tools for developing a reasoned argument.

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Introduction

Children experience the burden of proof early in life. On the playground, friends challenge each other with cries of "Prove it!" In the classroom, teachers require students to support their responses with learned knowledge. Before granting adult privileges, many parents want proof of their teenagers' maturity and responsibility.

The ability to think and reason logically is an important asset in every occupation. Corporate executives, airplane mechanics, office managers, and emergency room nurses must all make decisions based on known facts and sensible assumptions. In this module, you examine the tools for developing convincing arguments.

Activity 1

Convincing arguments are especially vital in a court of law. Under the U.S. judicial system, the accused is presumed innocent until proven guilty. Before making an arrest, detectives examine the evidence to identify the suspect most likely to have committed the crime. In the courtroom, attorneys on both sides use logical arguments to convince the jury of a defendant's innocence or guilt based on certain facts. In this activity, you use logical arguments to defend your own conclusions in "The Murder of Sam Barone."

Exploration

You have been hired as an attorney in a murder case. The following story, "The Murder of Sam Barone," contains some evidence gathered about the crime. Assume that the guilty person is the one who owns the murder weapon and has sufficient motive for the crime. All the statements made by the people involved are true.

a. After reading the story, select the person who you think is guilty.

b. Organize the evidence in the story that supports your verdict.

c. Prepare an argument for presentation to a jury.

“The Murder of Sam Barone”

Sam Barone lived in a residential hotel at 1314 East Labor Street, Hillside, Illinois. A hotel employee found Sam’s body in his room on Sunday, January 4, 1997, at 8:00 P.M. The investigative report revealed that the murder weapon was a rare Egyptian sword purchased that day at an antique shop in Hillside. The estimated time of death was about 7:00 P.M. on the day the body was found.

Police eventually arrested three suspects: two men and one woman. The suspects were Bill and Hank Barone, the brothers of the dead man, and Paula Stewart, a friend of the Barones. Sam’s estranged wife testified that both Bill and Hank hated anybody who owed them money. This hatred was sufficient to motivate either one of them to kill Sam. She also testified that Sam never borrowed money from anyone except a brother. It was a family code.

The following is a transcript of the conversation between police, the three suspects, and the owner of the antique shop.

Bill: Sure I hated Sam, but it doesn’t mean I killed him.

Hank: Sam owed me $10,000, but that isn’t reason to kill a man.

Paula: You seem to be assuming that one of us is guilty. There were plenty of other people in the area on the day of the murder.

Hank: As for that Egyptian sword, if my name was on the bill of sale, I own it. If you’d look, you’d see my name is not on the bill of sale.

Shopkeeper: I could never tell those two brothers apart, but I know this: if it ain’t Bill that owns the sword, then it’s Hank. If it ain’t Hank, then it’s Bill. I know I sold it to one of them.

Paula: The owner of the sword must know a great deal about Egyptian history. Without realizing its true worth, nobody would pay so much money for it.

Bill: I wrote a book on Egyptian history.

Hank: I’m sure that the book is accurate, because I personally verified much of the information in it.

Shopkeeper: The sword was purchased on January 4, 1997, so obviously the owner was in my shop in Hillside on that day.

Hank: I was in Paris, France, on January 4, 1997, and it would take over 24 hours to reach Hillside from Paris.

Bill: I was in Aspen, Colorado, skiing on January 2, 1997. Because of a big snowstorm there, nobody in Aspen could leave town for at least three days before or after January 2.

Discussion

a. Present the argument you developed in the exploration to the class.

b. Was each argument believable? Why or why not?

c. Is it possible for any of the statements made by the people involved to be both true and false?

d. Was it necessary to make any invalid or untrue assumptions to support your conclusions?

e. Were any statements reworded to support a conclusion? Did this rewording change the meaning of the statement?

Mathematics Note

In mathematics, a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.

A **conditional statement** is one that can be written in **if-then form**. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "if p, then q," or by $p \to q$ (read "p implies q").

For example, consider the conditional statement "If an animal is a German shepherd, then the animal is a dog." In this case, the hypothesis is "an animal is a German shepherd." The conclusion is "the animal is a dog."

Conditionals are sometimes illustrated using Venn diagrams. In Figure **1**, the outer circle of the Venn diagram represents the conclusion, while the inner circle represents the hypothesis. If an animal can be placed within the inner circle (a German shepherd), then it is also included within the outer circle (a dog).

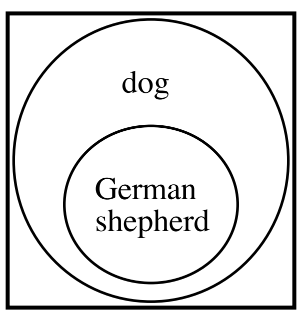


Figure 1: Venn diagram of a conditional statement

f.
1. Using the example given in the mathematics note, describe the information given in the hypothesis of a conditional statement.
2. Describe the information given in the conclusion of a conditional statement.

g. **1.** Write an if-then statement that describes the Venn diagram in Figure **2**.

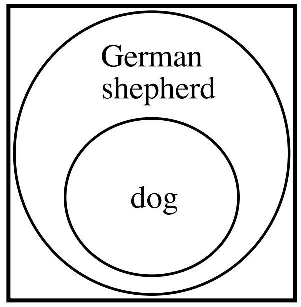


Figure 2: Venn diagram of another conditional

2. Is this conditional statement true or false?

Assignment

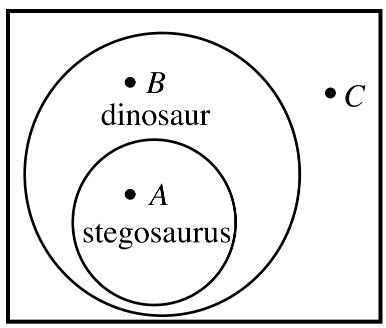
1.1 Use the Venn diagram below to complete Parts **a–d**.

a. What can you conclude about an animal at point *A*?

b. What can you conclude about an animal at point *B*?

c. What can you conclude about an animal at point *C*?

d. Write a conditional that is represented by this Venn diagram.



1.2 Draw a Venn diagram that represents the statement, "If students are in art class, then they draw pictures." Place the names of the students mentioned in Parts **a–d** in the appropriate locations in your diagram. In each case, explain why you chose a particular location.

a. Travis is in art class.

b. Natalie draws pictures.

c. Sydney does not draw pictures.

d. C.T. is not in art class.

1.3 Rewrite each of the statements below in if-then form. Underline the hypothesis and place parentheses around the conclusion.

a. All whales are mammals.

b. A triangle is a polygon.

c. In order to be guilty, Hank had to be at the scene of the crime.

d. Bill was not in Hillside if he was snowbound at Aspen.

1.4 Find four statements in "The Murder of Sam Barone" that are not in if-then form. Rewrite each one in if-then form. Underline the hypothesis and place parentheses around the conclusion.

1.5 Draw a Venn diagram for each of the statements in Problem **1.4**.

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1.6 Draw a Venn diagram for each of the statements below.

a. Every square is a quadrilateral.

b. All Irish setters are dogs.

c. The number 5 is an integer.

1.7 Rewrite each of the following statements in if-then form.

a. Isosceles triangles have two congruent sides.

b. The square of 3 is 9.

c. All people living in Hawaii live in the United States.

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Activity 2

Rewriting a conditional statement in if-then form does not always allow us to decide whether the statement is true or false. For example, consider the following conditional statement: "Students who have summer jobs will be able to go to the movies." This statement can be rewritten in if-then form as: "If a student has a summer job, then that student will be able to go to the movies."

Is a conditional statement true if its hypothesis is true? Is the statement false if its hypothesis is false? In this activity, you answer these questions by exploring some mathematical logic.

Exploration

One way to determine the truth value of a conditional statement is to treat it like a promise. For example, consider the conditional "If you study, then you will pass math." This statement can be rewritten as, "I promise you will pass math, if you study." If the promise is kept, or you have no valid reason to complain about the result, then the conditional is considered true. Otherwise, the conditional is false.

a. Write the following sentence as a promise: "If you finish this exploration, then you have permission to go to the dance."

b. In this situation, there are four possible cases to consider. Determine the truth value of each one and record the results in Table **1** below.

Table 1: Record of promises kept

You finished the exploration	You have permission to go to the dance	Is the promise kept? (truth value)
yes	yes	
yes	no	
no	yes	
no	no	

c. Describe any case that leads to a broken promise.

d. The four cases you examined in Part **b** are the same for all conditional statements. As shown in Table **2**, the hypothesis and conclusion of a conditional are evaluated using "true" or "false" instead of "yes" or "no." Use this table to determine the truth value of each case.

Table 2: Truth table for $p \rightarrow q$

Hypothesis (p)	Conclusion (q)	Conditional ($p \rightarrow q$)
true	true	
true	false	
false	true	
false	false	

e. Describe any case that gives a false conditional.

Discussion

a. Consider the case in Table **1** in which you did not finish the exploration and you had permission to go to the dance. In this case, why is the truth value "true"?

b. Describe the case(s) in which a conditional is a true statement.

c. Use Table **2** to summarize the possible truth values of a conditional.

Mathematics Note

A conditional is false only if its hypothesis is true and its conclusion is false. In all other cases, a conditional is true.

The **negation** of a statement p is the statement "It is not the case that p" or simply "not p." Symbolically, this is written as $\sim p$.

In "The Murder of Sam Barone," for example, Hank said, "My name is not on the bill of sale." The negation of this statement is "My name is on the bill of sale."

A statement and its negation have opposite truth values. For example, when statement p is true, its negation, not p, is false.

d. Consider the conditional, "If you finish this exploration, then you have permission to go to the dance."

1. What is the negation of the hypothesis?

2. What is the negation of the conclusion?

e. Given a statement p, what is the truth value of $\sim(\sim p)$?

f. Consider the conditional, "If it is raining, then it is cloudy." Based on this statement, on how many rainy days would you expect it to be cloudy?

Mathematics Note

Quantifiers are words or phrases that indicate the quantity of a particular subject referred to in a statement.

Existential quantifiers provide for the existence of at least one case. For example, *some*, *at least one*, *exactly one*, and *there exists* are existential quantifiers.

Universal quantifiers demand that all cases be considered. For example, *all*, *every*, and *none* are universal quantifiers.

Mathematical statements often contain quantifiers. For example, the following statement uses an existential quantifier: "For some real number x, $2x^2 + 7 = 10$." The statement, "Every rhombus is a parallelogram," uses a universal quantifier.

g. The quantifiers in conditionals are not always stated explicitly. In some cases, the quantifier is implied. What do you think is the implied quantifier in the statement, "A square is a rectangle"?

Assignment

2.1 In each of the following statements, a quantifier is implied. Rewrite each statement using the appropriate quantifier.

a. A mammal is an animal.

b. This person has red hair.

2.2 If possible, write a conditional with a false hypothesis so that:

a. the conditional is true

b. the conditional is false.

2.3 If possible, write a conditional with a true hypothesis so that:

a. the conditional is true

b. the conditional is false.

2.4 Does the negation of both the hypothesis and the conclusion make a true conditional false? Complete the following table, then use the results to justify your response.

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
true	true				
true	false				
false	true				
false	false				

2.5 Under what conditions would the negation of the hypothesis of a false conditional make the conditional true?

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2.6 In each of the following statements, identify the quantifier and explain its meaning.

a. Every rhombus is a parallelogram.

b. A possible solution to the equation $x^2 + 3x - 10 = 0$ is $x = -5$.

2.7 Let p represent the statement "the sky is not cloudy" and q represent the statement "the sun can be seen during the day."

a. Represent each of the following statements symbolically.

1. The sky is cloudy.
2. The sun cannot be seen during the day.
3. If the sky is not cloudy, then the sun can be seen during the day.
4. If the sun cannot be seen during the day, then the sky is cloudy.
5. If the sun can be seen during the day, then the sky is not cloudy.

b. Create a truth table for one of the conditionals in Part **a**.

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Activity 3

In this activity, you examine three forms of conditional statements: the **converse**, the **inverse**, and the **contrapositive**.

One of your goals will be to determine when a change in form changes the meaning of a statement, and when it does not. For example, consider the following excerpt from Lewis Carroll's *Alice in Wonderland*:

> "You should say what you mean," the March Hare went on.
>
> "I do," Alice hastily replied; "At least—at least I mean what I say—that's the same thing, you know."
>
> "Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
>
> "You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"
>
> "You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"
>
> "It is the same thing with you," said the Hatter, and here the conversation dropped, and the party sat silent for a minute.

Exploration

When Alice, the Hatter, the March Hare, and the Dormouse changed the order of the clauses in their sentences, they also changed the meaning of their statements. In this exploration, you examine how the truth value of a statement changes when the words are rearranged.

a. Write this statement in if-then form: "It is cloudy when it is raining."

b. Complete a truth table like the one in Table **3** for the conditional statement in Part **a**.

Table 3: Truth table for a conditional

Hypothesis	Conclusion	Statement

c. The **converse** of a conditional statement is formed by interchanging its hypothesis and its conclusion. Given the conditional "if p, then q," the converse is, "if q, then p." Symbolically, the converse of the conditional $p \rightarrow q$ is written $q \rightarrow p$.

1. Write the converse of the conditional in Part **a**.
2. Construct a truth table for the converse.

d. The **inverse** of a conditional statement is formed by negating its hypothesis and its conclusion. Given the conditional "if p, then q," the inverse is, "if not p, then not q." Symbolically, the inverse of the conditional $p \to q$ is written as $\sim p \to \sim q$.

1. Write the inverse of the conditional in Part **a**.
2. Construct a truth table for the inverse.

e. The **contrapositive** of a conditional is formed by interchanging its hypothesis and its conclusion and negating both of them. Given the conditional "if p, then q," the contrapositive is, "if not q, then not p." Symbolically, the contrapositive of the conditional $p \to q$ is written as $\sim q \to \sim p$.

1. Write the contrapositive of the conditional in Part **a**.
2. Construct a truth table for the contrapositive.

f. Compare the truth tables you created in Parts **b–e**. What appears to be the relationship among the truth values of this conditional and its converse, inverse, and contrapositive?

Discussion

a. Describe how rewriting the original statement in the exploration changed its truth values.

b. What is the contrapositive of the inverse of a conditional statement? Justify your response.

Mathematics Note

Two statements are **logically equivalent** if one statement is true (or false) exactly when the other statement is true (or false).

For example, Table **4** shows a truth table for the conditional statements $p \to q$ and $\sim q \to \sim p$. Since the two statements have exactly the same truth values, they are logically equivalent.

Table 4: Truth tables for two logically equivalent statements

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim q \to \sim p$
true	true	true	false	false	true
true	false	false	false	true	false
false	true	true	true	false	true
false	false	true	true	true	true

c. Using the truth tables you created in the exploration, which forms of a conditional statement appear to be logically equivalent? Justify your response.

d. In mathematics, a definition is a statement that has a logically equivalent converse. Considering this fact, what appears to be the relationship among a definition and its converse, inverse, and contrapositive? Defend your response.

Assignment

3.1 Choose a television, magazine, or newspaper advertisement that contains a product slogan.

a. Write the product slogan in if-then form and explain what it implies. Underline the hypothesis and put parentheses around the conclusion.

b. Draw a Venn diagram of the conditional.

c. Write the converse of the conditional.

d. Write the inverse of the conditional.

e. Write the contrapositive of the conditional.

f. Which form of the slogan do you think is better for selling the product: the conditional or the contrapositive? Justify your choice.

3.2 **a.** Write the following statement in if-then form, "An equilateral triangle is an isosceles triangle."

b. Complete truth tables for the conditional in Part **a** and its converse, inverse, and contrapositive.

c. Using the truth tables you completed in Part **b**, determine which of the following pairs of statements are logically equivalent.

1. the conditional and the converse
2. the conditional and the contrapositive
3. the converse and the contrapositive
4. the converse and the inverse
5. the conditional and the inverse
6. the inverse and the contrapositive

d. Summarize your findings from Part **c**.

3.3 Write a logically equivalent statement for each of the following statements.

a. If you do not live in the United States, then you do not live in New York.

b. A triangle is a polygon.

Mathematics Note

When a conditional and its converse are both true, they can be written as a single statement using the words **if and only if**. Symbolically, this can be represented as $p \leftrightarrow q$, or p iff q (read "p if and only if q"). All definitions may be written as statements using *if and only if*.

For example, consider the conditional statement for the definition of supplementary angles: "If two angles are supplementary, then the sum of their measures is 180°." The converse of this statement is: "If the sum of the measures of two angles is 180°, then the angles are supplementary." Since both statements are true, they can be combined in the following definition: "Two angles are supplementary if and only if the sum of their measures equals 180°."

3.4 Write a statement using *if and only if* for each definition below.

a. A googol is the numeral 1 followed by one hundred 0s.

b. A regular triangle is an equilateral and equiangular triangle.

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3.5 Consider the statement "The number π is an irrational number."

a. **1.** Write this statement in if-then form.

2. Write the converse of the conditional.

3. Write the inverse of the conditional.

4. Write the contrapositive of the conditional.

b. Determine which of the statements in Part **a** are logically equivalent.

3.6 If possible, write a statement using *if and only if* for each of the following conditionals.

a. If a quadrilateral is a rectangle, then it is a parallelogram.

b. If an animal is a canine, then it is a dog.

c. If $a = b$, then $b = a$.

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Activity 4

Photographs, videotapes, and computer simulations are often admitted as evidence in court. Suppose that the prosecuting attorney in "The Murder of Sam Barone" discovers a picture of one suspect at the crime scene, holding the murder weapon. Is this sufficient evidence to convict that person? Is the fact that a person does not appear in the picture enough to show that the person is innocent?

Exploration

In this exploration, you examine the relationships that exist among the angles formed by lines in the same plane.

a. Sketch all the possible situations that occur when two distinct lines lie in the same plane.

b. Make a conjecture about any of the relationships between the angles formed in your sketches.

c. Use appropriate technology to help support your conjecture.

d. Write generalizations in if-then form about the relationships you found in Part **b**.

e. Repeat Parts **a–d** for three distinct lines in the same plane.

Discussion

a. Compare the relationships you discovered among the angles in the exploration with those observed by your classmates.

b. Describe how your sketches helped you make conjectures.

c. Are your sketches sufficient to convince yourself that these conjectures are true?

d. Does your use of technology prove the conjectures made in Part **b** of the exploration? Explain your response.

Mathematics Note

To create convincing arguments, mathematicians use a method based on logical reasoning. This process is referred to as **proof**. Although diagrams and pictures may be used to support arguments, a conclusion based solely on observation may or may not be true. The method of examining *all* possibilities to prove a statement is **proof by exhaustion**.

For example, consider the statement, "If an integer is between 89 and 97, then it is not prime." To prove this statement by exhaustion, you must examine the factors of the following integers: 90, 91, 92, 93, 94, 95, and 96. Since each of these integers is divisible by a number other than 1 and itself, each is not prime. Thus, the statement is proven true.

Assignment

4.1 Mandy has a box of 100 marbles. She draws a single green marble from the box and sets it aside. She repeats the process 30 times, each time drawing a green marble.

a. Are you convinced that the next marble Mandy draws will be green?

b. Write a conjecture about the remaining marbles in the box.

c. How could you prove the conjecture in Part **b**?

4.2 Using proof by exhaustion, could you prove that vertical angles formed by two intersecting lines are congruent? Explain your response.

4.3 Consider the following statement: "If I count to 1,000,000 without stopping, then it will take more than a week."

a. Write a convincing argument to support or refute this statement.

b. Could this statement be proved by exhaustion? Explain your response.

c. Would it be practical to prove this statement by exhaustion? Explain your response.

4.4. On a 12-hour analog clock, addition can be performed by "counting on." For example, the sum of 8 and 5 on a clock can be found by starting at 8 o'clock and counting on 5 hours to 1 o'clock.

Recall that the sum of a number n and the additive identity is n. Using proof by exhaustion, prove that the additive identity for clock addition is 12. In other words, show for the numbers n, where $n \in \{1, 2, 3, \ldots, 12\}$, that $12 + n = n + 12 = n$.

Mathematics Note

A **counterexample** illustrates a contradiction to a statement. A statement can be proven false by finding only one counterexample.

For example, consider the following conditional: "If x is a real number, then $\sqrt{x^2} = x$." This statement can be proven false using the counterexample $x = -2$ because $\sqrt{(-2)^2} = \sqrt{4} = 2$ and $2 \neq -2$.

4.5 Consider the situation described in Problem **4.1**. What would it take to prove that the statement, "All the marbles in the box are green," is false?

4.6 Spencer examined 10 quadrilaterals, each with four right angles. He observed that all the quadrilaterals were squares. After his observations, he wrote, "I have proven the statement, 'If a quadrilateral has four right angles, then it is a square.' "

a. Explain to Spencer why his method of proof is not correct.

b. Give a counterexample that proves Spencer's statement is false.

c. Suggest a true statement that Spencer might have made.

d. Do Spencer's observations prove the statement you made in Part **c**? Explain your response.

4.7 Find counterexamples in "The Murder of Sam Barone" to prove that each of the following statements is not true.

a. Paula owned the murder weapon.

b. Bill liked Sam.

c. Bill was in Hillside on January 4, 1997.

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4.8 The covers of four books are each printed a different color: red, yellow, green, or blue. Prove that if these four books are arranged randomly on a shelf, the green book and the yellow book will be next to each other 50% of the time.

4.9 Identify a counterexample to prove that each of the following statements is false.

a. The sum of an odd number and an even number is always even.

b. Angles with the same measure are vertical angles.

c. If the sidewalk is wet, then it is raining.

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Activity 5

In Activity **4**, you learned that observations alone are not sufficient to prove a statement true unless you show that every case is true. "The Murder of Sam Barone" contains more than just observations, however. The story also introduces some known facts as evidence. Through logical reasoning, these facts can be used to establish the validity of the statements made by each suspect.

Mathematics Note

Deductive reasoning uses a logical sequence of valid arguments to reach a conclusion. In mathematics, these arguments are often written as a series of if-then statements, each supported by some justification, that yield another valid argument or a conclusion.

For example, suppose that you wanted to prove the statement: "If a triangle is an equilateral triangle, then it is an isosceles triangle." This statement can be proved using the following series of if-then statements.

- If a triangle is equilateral, then it has three equal sides.
- If a triangle has three equal sides, then it has at least two equal sides.
- If a triangle has at least two equal sides, then it is an isosceles triangle.
- In conclusion, if a triangle is an equilateral triangle, then it is an isosceles triangle.

This example illustrates the **transitivity** of if-then statements. Using transitivity, the true statements "if p, then q" and "if q, then r" may be used to form the valid conclusion: "if p, then r."

Exploration

Using the testimony of the characters in "The Murder of Sam Barone," investigators have created the following list of true statements.

- If Bill was snowbound, then he couldn't leave Aspen until January 5.
- If Bill didn't buy the sword, then he didn't own the murder weapon.
- If Bill couldn't leave Aspen until January 5, then he couldn't be in Hillside on January 4.
- If Bill didn't own the murder weapon, then he didn't commit the murder.
- If Bill was in Aspen on January 2, 1997, then he was snowbound.
- If Bill couldn't be in Hillside on January 4, then he couldn't buy the sword.

In the order given, these sentences do not make a convincing argument of Bill's innocence. In the following exploration, you use transitivity to develop a logical sequence for these statements.

a. Cut a copy of the list of statements into six separate slips of paper.

b. Rearrange the sentences so that they lead logically to the following statement: "If Bill was in Aspen on January 2, 1997, then he didn't commit the murder."

Discussion

a. Describe how you determined an order for the statements in Part **b** of the exploration.

b. Why is transitivity important in deductive reasoning?

c. Describe a series of if-then statements that could be used to prove the following conditional: "If $2x - 3 = 5x + 6$, then $x = -3$."

Mathematics Note

A series of if-then statements is only one way of presenting a mathematical proof. For example, the following paragraph is a proof of the conditional, "If *n* is an even integer, then n^2 is an even integer."

> The hypothesis is that *n* is an even integer. By the definition of an even integer, $n = 2b$, where *b* is an integer. When both sides of the equation $n = 2b$ are squared, the result is $n^2 = (2b)^2 = 4b^2 = 2(2b^2)$. By the definition of an even integer, n^2 is an even integer. Therefore, if *n* is an even integer, then n^2 is an even integer.

Notice that each step in this argument includes its own justification. In a mathematical proof, these justifications may consist of definitions, **axioms** or **postulates** (statements accepted as true), or **theorems** (statements that have previously been proven true).

Since it makes direct use of the hypothesis to arrive at the conclusion, the example given above is a **direct proof**. However, not all statements can be proven directly. **Indirect proofs** begin by assuming that the original statement is false. From this assumption, valid arguments are followed until a contradiction to a known fact is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

For example, the conditional "If n^2 is an odd integer, then *n* is an odd integer" cannot be proven using a direct proof. It can, however, be proven indirectly, as shown in the following paragraph.

> Assume that the statement "If n^2 is an odd integer, then *n* is an odd integer" is false. If this conditional is false, then "n^2 is an odd integer" is true and "*n* is an odd integer" is false. If "*n* is an odd integer" is false (and *n* is an integer), then *n* must be an even integer. By the definition of an even integer, $n = 2a$, where *a* is an integer. Squaring both sides of $n = 2a$ results in $n^2 = (2a)^2 = 4a^2 = 2(2a)^2$. By the definition of an even integer, n^2 is an even integer. Since this contradicts the known fact that n^2 is an odd integer, the assumption is false. Therefore, the original statement is true: "If n^2 is an odd integer, then *n* is an odd integer."

d. Why can't the statement, "If n^2 is an odd integer, then n is an odd integer," be proven directly?

e. Consider the conditional "If n is an even integer, then n^2 is an even integer." In an indirect proof of this conditional, what would be your first step?

Assignment

5.1 Prove each of the following conclusions to "The Murder of Sam Barone" using a series of if-then statements.

a. Hank owned the murder weapon.

b. Hank was guilty.

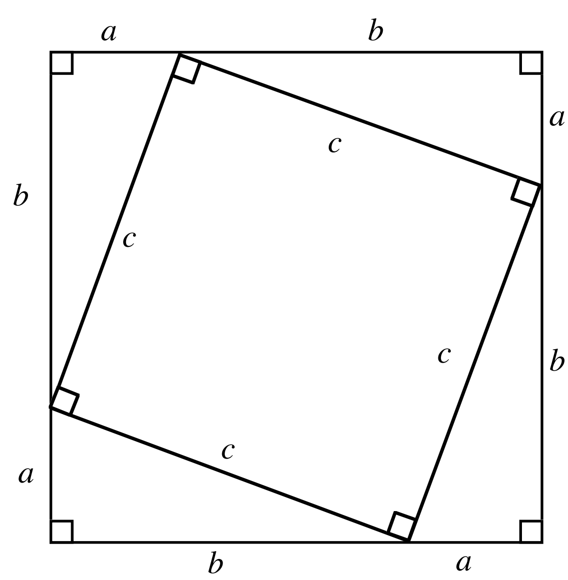
5.2 Prove each of the following using if-then statements. Include justification for each step in your argument.

a. The solution to $5(x-2)=-20$ is $x=-2$.

b. The measure of an angle of $\pi/3$ radians is 60°.

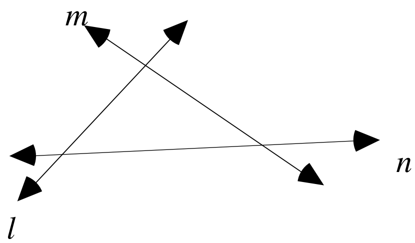
5.3 Changing a conditional to its contrapositive can sometimes make proving a statement easier. Change the following statement to its contrapositive, then prove it: "If $x^2 \neq 4$, then $x \neq 2$."

5.4 The area of a region can be expressed as the sum of the areas of its non-overlapping parts. Use this fact, along with the diagram below, to prove the Pythagorean theorem.



5.5 Use an indirect proof to prove each of the following statements.

a. If lines l, m, and n intersect in three different points, then it is not possible for both l and m to be perpendicular to n.



b. A triangle cannot have three interior angles each with a measure of 70°.

5.6 Using the "The Murder of Sam Barone," prove the following statement indirectly: "If Bill did not own the murder weapon, then he was innocent."

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5.7 Prove each of the following using a series of if-then statements. Include justification when necessary.

a. Hank had a motive to murder Sam Barone.

b. All equilateral triangles are isosceles.

5.8 Prove that the supplements of angles with equal measure are congruent.

5.9 Prove the following statement indirectly: "If x and y are integers and $3x + 12y = 450$, then x is an even integer."

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Summary Assessment

1. In the game "Prove It," players match a secret four-digit number by repeatedly guessing four-digit numbers and adjusting each successive guess based on the clues they receive. The clues are "hot" and "warm." Each hot clue indicates that a digit in the guess is correct and in the appropriate position. Each warm clue indicates that a digit in the guess is correct but improperly positioned.

   The table below shows the first seven guesses in one game of "Prove It." Determine the secret four-digit number in this game and explain the logic you used to find it.

Guess	Hot clues	Warm clues
0123	1	1
1234	1	1
2345	0	1
3456	0	0
7899	1	1
8999	1	0
9999	1	0

2. The definition of **divisibility** states that, "Given integers a and b, if b divides a (written $b|a$), then there is an integer c such that $a = b \bullet c$." If $3|15$, for example, then there is an integer 5 such that $15 = 3 \bullet 5$.

   Using this definition, classify each of the following statements as true or false. If the statement is true, prove it. If the statement is false, provide a counterexample. In all cases, m and n are integers.

   a. If $3|m$, then $3|nm$.

   b. If $3|(m+n)$, then $3|m$.

   c. For all integers m, $1|m$.

   d. If $d|m^2$, then $d|m$.

3. Prove the following statement: "If a figure is a triangle, then it does not have two angles each with a measure of 90°."

4. Prove the following statement: "The equation $2(3x - 5) = 6x + 3$ is never true."

Module Summary

- In mathematics, a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.
- A **conditional statement** is one that can be written in **if-then form**. A conditional consists of two parts: the **hypothesis** and the **conclusion**. The hypothesis is the "if" part of the conditional. The conclusion is the "then" part. A conditional statement can be represented symbolically by "If p, then q," or by $p \rightarrow q$ (read "p implies q").
- A conditional is **false** only if its hypothesis is true and its conclusion is false. In all other cases, a conditional is true.
- The **negation** of a statement p is the statement "It is not the case that p" or simply "not p." Symbolically, this is written as $\sim p$. A statement and its negation have opposite truth values.
- **Quantifiers** are words or phrases that indicate the quantity of a particular subject referred to in a statement.
- **Existential quantifiers** provide for the existence of at least one case. For example, *some*, *at least one*, *exactly one*, and *there exists* are existential quantifiers.
- **Universal quantifiers** demand that all cases be considered. For example, *all*, *every*, and *none* are universal quantifiers.
- The **converse** of a conditional statement is formed by interchanging its hypothesis and its conclusion. Given the conditional statement "if p, then q," the converse of the statement is, "if q, then p." This can be represented symbolically as $q \rightarrow p$.
- The **inverse** of a conditional statement is formed by negating its hypothesis and its conclusion. Given the conditional "if p, then q," the inverse of the statement is, "if not p, then not q." This can be represented symbolically as $\sim p \rightarrow \sim q$.
- The **contrapositive** of a conditional is formed by interchanging its hypothesis and its conclusion and negating both of them. Given the conditional "if p, then q," the contrapositive of the statement is, "if not q, then not p." This can be represented symbolically as $\sim q \rightarrow \sim p$.
- Two statements are **logically equivalent** if one statement is true (or false) exactly when the other statement is true (or false).

- When a conditional and its converse are both true, they can be written as a single statement using the words *if and only if*. Symbolically, this can be represented as $p \leftrightarrow q$, or p iff q (read "p if and only if q"). All definitions may be written as statements using *if and only if*.
- The method of examining all possibilities to prove a statement is **proof by exhaustion**.
- A **counterexample** illustrates a contradiction to a statement. A statement can be proven false by finding only one counterexample.
- **Deductive reasoning** uses a logical sequence of valid arguments to reach a conclusion.
- Using **transitivity**, the statements "if p, then q" and "if q, then r" may be used to form the valid conclusion: "if p, then r."
- In mathematical proofs, definitions, **axioms** or **postulates** (statements accepted as true), and **theorems** (statements that have previously been proven true) are used to justify arguments.
- A **direct proof** makes direct use of the hypothesis to arrive at a conclusion.
- **Indirect proofs** begin by assuming that the original statement is false. From this assumption, valid arguments are followed until a contradiction to a known fact is reached. Once a contradiction is reached, the assumption must be false. Therefore, the original statement is true.

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