Reinvent the Wheel

If wheels weren't round, would they still roll smoothly? In this module, you investigate some other types of curves that might supply a smooth ride.

Janet Higgins • Roger Patterson • Mike Trudnowski

Reinvent the Wheel

Introduction

The invention of the wheel took place several thousand years ago, sometime during the Bronze Age (about 3500–1000 B.C.). Since then, circular shapes have been incorporated in an almost endless list of useful inventions, from drill bits to medicine bottles. But have you ever stopped to wonder if a circle provides the best design for these products?

The circle's popularity may have discouraged inventors from examining the usefulness of other shapes. Are there other curved shapes that might work as well—or better—in some applications?

Activity 1

When building the pyramids, the ancient Egyptians may have transported the immense blocks of stone from quarry to construction site using a system of rollers. Today, cylindrical rollers help workers move heavy loads in a wide variety of situations—everywhere from shipyards to supermarkets. In this activity, you model the movement of an object using rollers with different shapes.

Exploration 1

a. Place a flat object on two cylinders with the same radius, as shown in Figure **1**. Roll the object and cylinders along a level surface.

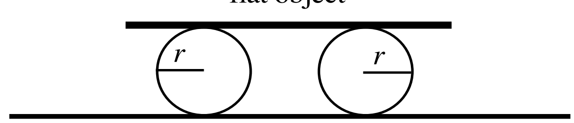


Figure 1: Two cylindrical rollers between object and surface

b. Observe the ride of the object on the rollers. Identify this motion as either "level" or "uneven."

c. Cut out each of the shapes in template A (seven in all) as accurately as possible. Predict which of the shapes, when used as a roller, might provide a level ride.

Mathematics Note

A **line of support** is a line that intersects a curve such that all points of the curve not on the line lie on one side of the line. The perpendicular distance between a pair of parallel lines of support is a **width** (w) of the curve.

A **curve of constant width** has the same width between every pair of parallel lines of support.

Figure **2** shows one example of a curve of constant width w with two parallel lines of support, line n and line m.

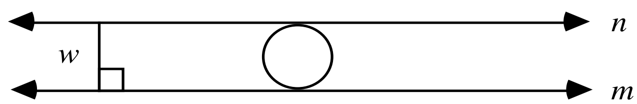


Figure 2: Curve of constant width with lines of support

d. Imagine that each of the seven shapes from template A represents a cross section of a roller. To simulate the motion of an object using rollers of each type, complete the following steps.

1. On a sheet of paper, draw a line m to represent a level surface (see Figure **1**).
2. Position shape A from the template so that line m is a line of support for it.
3. Draw a line parallel to line m to represent a flat object to be moved by rolling shape A. Label this line n; it should also be a line of support for shape A.
4. Simulate the motion of the flat object by "rolling" shape A along line m. If both m and n remain lines of support for shape A at all times (in other words, if shape A is always between, and touching, both lines), identify the motion as "level." If not, identify the motion as "uneven."
5. Repeat Steps **1–4** using shapes B through G.

e. Determine which of the seven shapes have constant widths.

Discussion 1

a. Is a line of support always a tangent line? Explain your response.

b. In Part **d** of Exploration **1**, why are lines m and n drawn parallel to each other?

c. Which shapes are curves of constant width? Justify your response.

d. **1.** What characteristics are common to the shapes that provided "level" rides?

2. What characteristics can cause shapes to provide "uneven" rides?

e. Is every circle a curve of constant width? Explain your response.

Exploration 2

In Exploration **1**, one of the figures that has a constant width is shape E. This shape is a **Reuleaux** (pronounced "rue-low") **triangle**, named after the German mathematician and high school teacher Franz Reuleaux (1829–1905). Figure **3** shows another example of a Reuleaux triangle.

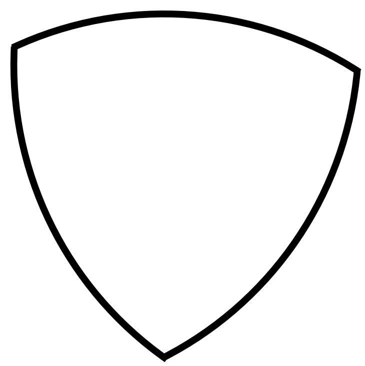


Figure 3: A Reuleaux triangle

In this exploration, you construct several **Reuleaux polygons**, then determine whether or not each one is a curve of constant width.

a. Construct a Reuleaux triangle by completing the following steps.

1. Construct an equilateral triangle with sides measuring 5 cm.

2. Construct an arc that has its center at one vertex and its endpoints at the two other vertices, as shown in Figure **4**.

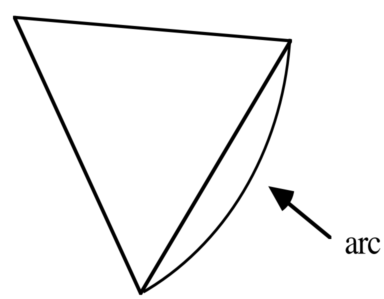


Figure 4: Arc connecting two vertices with center at third vertex

3. Construct similar arcs with centers at the other two vertices. The figure formed by the three arcs is a Reuleaux triangle.

4. Carefully cut out the shape of the Reuleaux triangle.

b. The constructions described in Part **a** are not possible for all regular polygons.

1. Attempt to repeat Part **a** using a square, a regular pentagon, a regular hexagon, and a regular heptagon.
2. Record which regular polygons did not allow the constructions described in Part **a**.

c. Using the method described in Part **d** of Exploration **1**, determine which of the Reuleaux polygons you created are curves of constant width.

d. Record the width of each Reuleaux polygon that is a curve of constant width.

Discussion 2

a.

1. Which regular polygons do not allow the constructions described in Part **a** of Exploration **2**? Explain your response.
2. In general, what types of polygons can be used to construct Reuleaux polygons?

b. Do all Reuleaux polygons appear to be curves of constant width? Explain your response.

c. How many pairs of parallel lines of support are there for a Reuleaux polygon? Explain your response.

Mathematics Note

A **Reuleaux polygon** is a curve of constant width constructed from a regular polygon with an odd number of sides. For example, Figure **5** shows a Reuleaux triangle with a constant width w.

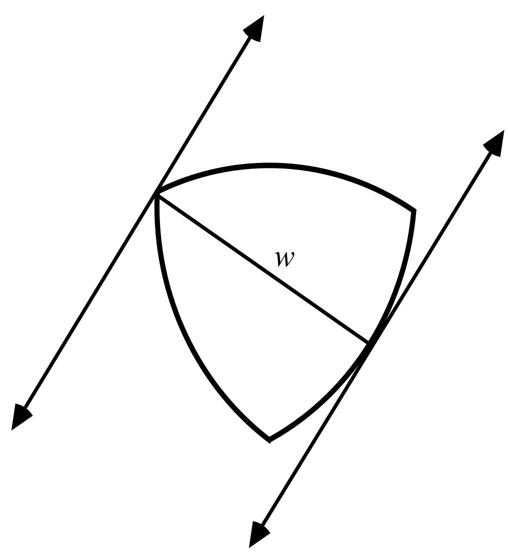


Figure 5: A Reuleaux triangle and two parallel lines of support

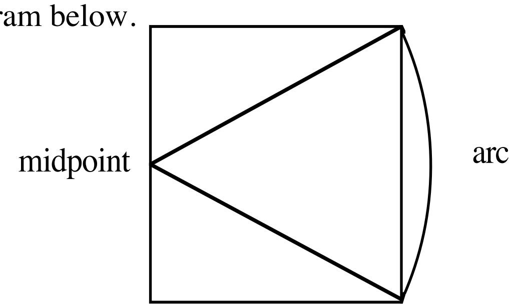
Assignment

1.1 Does a Reuleaux triangle fit the definition of a triangle in Euclidean geometry? Explain your response.

1.2 Use the following algorithm to construct a Reuleaux-like polygon from a regular polygon with an even number of sides.

a. Construct a regular polygon with an even number of sides, each measuring 5 cm.

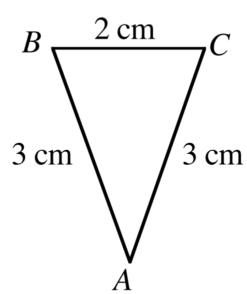
b. Construct an arc that has its center at the midpoint of one side and its endpoints at the vertices on the opposite side, as shown in the sample diagram below.



c. Construct similar arcs with centers at the midpoints of the remaining sides. Erase the original regular polygon.

d. Is the resulting figure a curve of constant width? Justify your response.

1.3 Two students, Anton and Jennifer, think that they can make a curve of constant width from isosceles triangle *ABC*, shown below.



a. If Anton and Jennifer are right, predict the measure of the constant width for the curve generated by ΔABC.

b. To test Anton and Jennifer's prediction, complete Steps **1–8**.

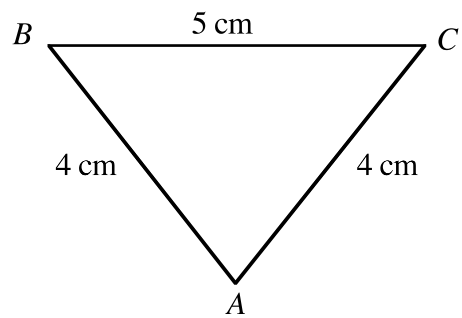
1. Create a triangle with the same dimensions as ΔABC above.
2. Draw $\overrightarrow{BC}$ about 2 cm past *C*.
3. Draw an arc with its center at *B* and one endpoint at *A* so that it intersects $\overrightarrow{BC}$. Label the point where the arc intersects $\overrightarrow{BC}$ as point *F*.
4. Draw $\overrightarrow{CB}$ about 2 cm past *B*.

5. Draw an arc with its center at C and one endpoint at A so that it intersects $\overrightarrow{CB}$. Label the point where the arc intersects $\overrightarrow{CB}$ as point E.
6. Construct a circle with its center at C that contains F. Construct another circle with its center at B that contains E.
7. Construct $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Label a point on $\overrightarrow{AB}$ that is not between A and B as point K. Label a point on $\overrightarrow{AC}$ that is not between A and C as point L.

   Label the point of intersection of circle B and $\overrightarrow{BK}$ as point G. Label the point of intersection of circle C and $\overrightarrow{CL}$ as point H.
8. Construct an arc with its center at A and endpoints at G and H.

c. Does figure *AFHGE* represent a curve of constant width? Were Anton and Jennifer correct? Explain your responses.

1.4 After attempting to create a curve of constant width from an isosceles triangle whose congruent sides are longer than the base side, Anton and Jennifer decided to try using an isosceles triangle whose base side is longer than the congruent sides. They started with ΔABC shown below.



a. First, the two friends constructed a line perpendicular to $\overline{BC}$ through point A. Then, they labeled a point A' on the perpendicular line so that $\Delta A'BC$ was equilateral.

Beginning with these same two steps, recreate Jennifer and Anton's construction.

b. Is the result in Part **a** a curve of constant width? If so, describe its width. If not, explain why not.

* * * * *

1.5 Many compact disc (CD) players feature a disc changer that holds more than one disc. Disc changers typically come in one of two formats: a stack or a single drawer.

a. Why might the single drawer of a three-disc changer be shaped like a Reuleaux triangle, rather than an equilateral triangle or a circle?

b. Why would the single drawer of a four-disc changer not be shaped like a square?

1.6 Describe a curve of constant width that might provide the best shape for a single-drawer CD changer that holds five discs.

* * * * * * * * * *

Activity 2

In addition to their applications in compact disc players, Reuleaux triangles also provide the basic design for drill bits used to create "square" holes. In this activity, you examine how and why these drill bits work.

Exploration

In this exploration, you model the process of drilling a square hole.

a. Create a square cardboard frame with outside edges that measure 12 cm each, as shown in Figure **6**. The inside of the frame should measure 8 cm on each side.

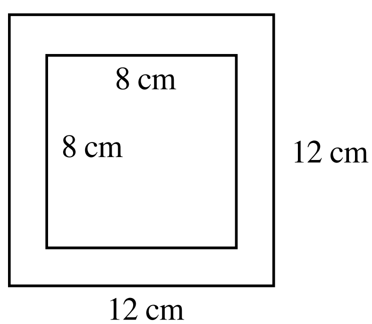


Figure 6: Square cardboard frame

b. Tape the frame to a sheet of white paper, then tape the white paper to a hard surface.

c. On another sheet of cardboard, construct a Reuleaux triangle using an equilateral triangle with sides measuring 8 cm.

Mathematics Note

A **median** is a segment that connects the vertex of a triangle with the midpoint of the opposite side. The intersection of the medians of a triangle is the **centroid**.

Consider an equilateral triangle inscribed in a Reuleaux triangle, as shown in Figure **7**. If the medians of the inscribed triangle are extended to intersect the Reuleaux triangle, they form the medians of the Reuleaux triangle. Since the intersection of the two sets of medians is the same, the centroid of the inscribed triangle is also the centroid of the Reuleaux triangle.

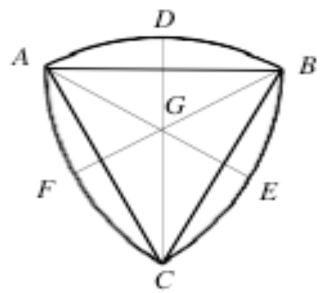


Figure 7: Reuleaux triangle with medians and centroid

In Figure **7**, the three medians of Reuleaux triangle *ABC* are $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. The centroid is located at point *G*:

d. Construct the three medians and centroid of your Reuleaux triangle from Part **c**. Punch a small hole through the centroid.

e. Cut out the Reuleaux triangle and place it in the square frame, as shown in Figure **8**.

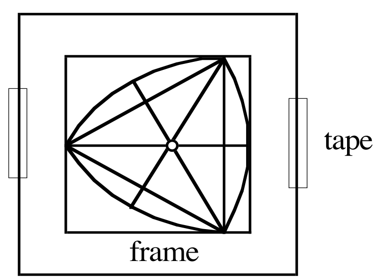


Figure 8: Reuleaux triangle in square frame

f. Place the point of a pencil in the hole at the centroid.

g. With the tip of the pencil, trace the path of the centroid as you rotate the Reuleaux triangle inside the square frame. (This may require two people: one to rotate the triangle and another to hold the pencil.)

Discussion

a. How do the results of the exploration confirm that a drill bit in the shape of a Reuleaux triangle can create a "square" hole?

b. Describe the path of the centroid in Part **g** of the exploration.

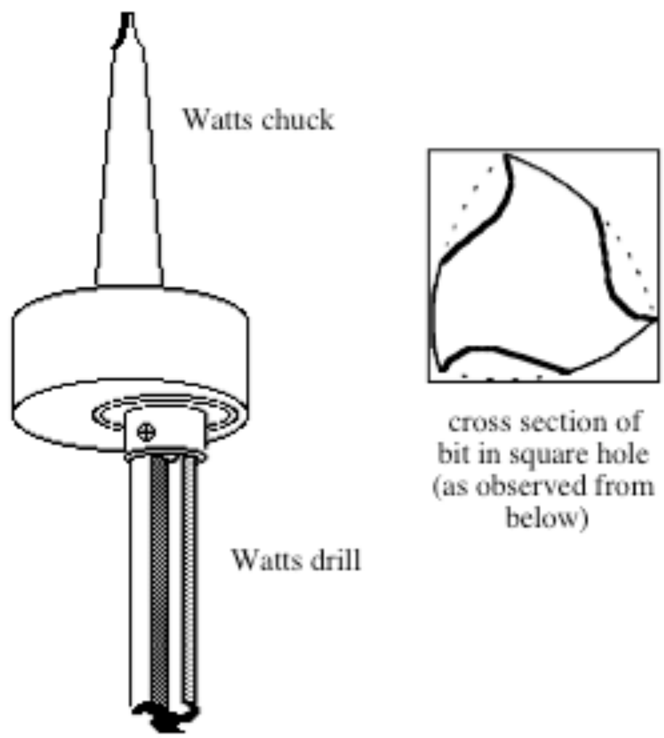
c. What does the path of the centroid represent for a drill bit?

d. Suppose that the path of the centroid were a single point. What is the shape of the hole produced by this drill?

e. Why do you think that Reuleaux polygons are not used as wheels on vehicles?

Assignment

2.1 The actual bit that drills square holes was invented by Harry Watts in 1914. The figure below shows a Watts drill, the Watts chuck that allows the unusual motion of the centroid, and a cross section of the bit. (A metal guide is necessary for accurate drilling.)



a. In which direction should this bit rotate to drill a hole?

b. Explain why the cross section of this bit does not show the shape of an entire Reuleaux triangle.

2.2 For a woodworking project, Ebdul wants to drill a square hole 4 cm on each side. Describe the size of the drill bit required to drill this hole.

2.3 **a.** Consider a drill bit based on the shape of a Reuleaux pentagon. If the centroid follows a path similar to the one in Part **g** of the exploration, what will be the shape of the hole? Hint: You may want to repeat the exploration using a Reuleaux pentagon.

b. As the number of sides increases in the Reuleaux polygon that determines the shape of the bit, what shape will the hole approach?

2.4 **a.** Draw all the lines of symmetry for an equilateral triangle.

b. Is every median of an equilateral triangle contained in a line of symmetry? Justify your response.

c. Is every median of a Reuleaux triangle contained in a line of symmetry? Justify your response.

* * * * *

2.5 **a.** Draw several regular polygons, other than equilateral triangles, each with a different number of sides. For each regular polygon, draw all the lines of symmetry.

b. Do you think the lines of symmetry for every regular polygon always intersect at one point? Justify your response.

c. Do you think that the lines of symmetry for every Reuleaux polygon always intersect at one point? Justify your response.

2.6 **a.** Draw several triangles, including at least one scalene, one isosceles, and one equilateral. For each triangle, draw the perpendicular bisector of each side. Extend the perpendicular bisectors until they intersect. This point of intersection is the **circumcenter**.

b. For each triangle, draw a circle with its center at the circumcenter and radius equal to the distance from the circumcenter to one of the triangle's vertices.

c. Based on your observations in Parts **a** and **b**, suggest a definition for a circumcenter.

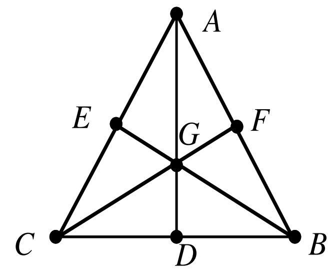
d. Use your responses to Parts **a–c** to explain why the following statement is true: "For any three noncollinear points in a plane, there is a circle that contains all three of them."

2.7 **a.** Draw several triangles, including at least one scalene, one isosceles, and one equilateral. For each triangle, draw the angle bisector of each angle. Extend the angle bisectors until they intersect. This point of intersection is the **incenter**.

b. For each triangle, draw a circle with its center at the incenter and radius equal to the perpendicular distance from the incenter to one of the triangle's sides.

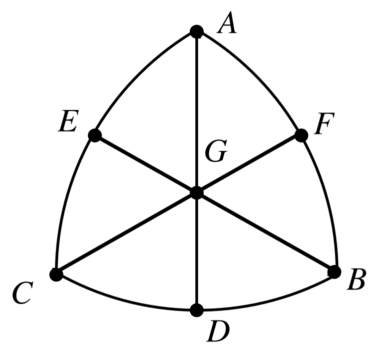
c. Based on your observations in Parts **a** and **b**, suggest a definition for an incenter.

2.8 **a.** The diagram below shows an equilateral triangle *ABC*, its medians, and its centroid. What do you think is the ratio of *AG* to *AD*? Use a geometry utility to test your prediction.



b. Prove that the centroid of an equilateral triangle is $2/3$ the distance from any vertex to the opposite side. Hint: Show that $AG/AD = 2/3$, $BG/BE = 2/3$, and $CG/CF = 2/3$.

2.9 The diagram below shows a Reuleaux triangle *ABC*, its medians, and its centroid. Determine the ratio AG/AD for a Reuleaux triangle.



* * * * * * * * * *

Research Project

Reuleaux triangles have been incorporated in a number of inventions, including the rotary engine. Developed by Felix Wankel in the 1950s, the rotary engine weighs about one-third as much as a piston engine of comparable power. Figure **9** shows a diagram of the rotor in this engine.

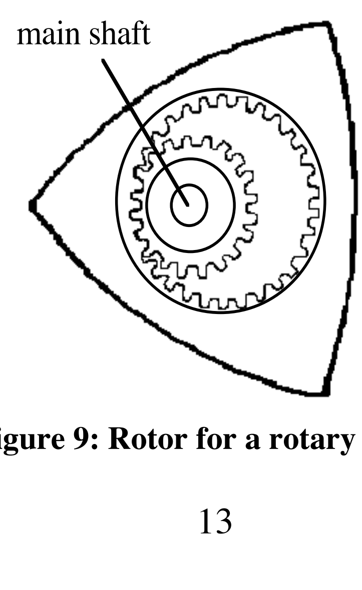


Figure 9: Rotor for a rotary engine

In the 1970s, the Mazda Company introduced two automobiles powered by rotary engines: the RX-2 and the RX-3. The popularity of these cars was limited, due in part to their poor fuel economy. By 1995, Mazda's RX-7 was one of the few automobiles with a rotary engine remaining on the U.S. market.

Investigate the development of the rotary engine. Your report should include:

- a description of the engine's origins
- an explanation of how it works
- a comparison with piston-powered engines
- a summary of its applications.

Activity 3

Since the sides of a Reuleaux polygon are curved, they cannot be measured in the same way as the sides of an ordinary polygon. In this activity, you derive a formula for the perimeter of a Reuleaux triangle, then generalize this formula for all Reuleaux polygons. You also determine how to calculate the area of Reuleaux polygons.

Exploration 1

Each side of a Reuleaux polygon is an arc of a circle. In this exploration, you determine a method for finding the length of an arc given the measure of its central angle and the radius of the circle.

Figure **10** below shows a regular polygon, along with one side ($\overset{\frown}{AB}$) of the corresponding Reuleaux polygon. This construction involves two different central angles: the central angle of the regular polygon ($\angle ADB$) and the central angle that determines the side of the Reuleaux polygon ($\angle ACB$). The measures of these central angles depend on the number of sides in the Reuleaux polygon.

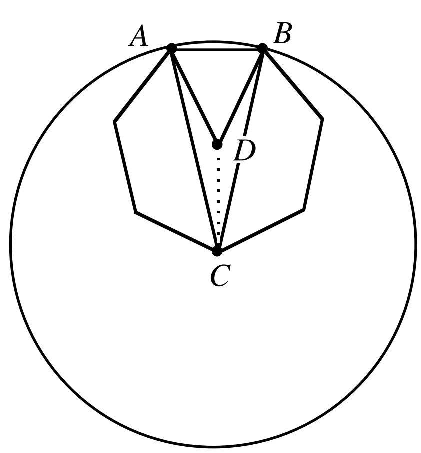


Figure 10: Construction of a Reuleaux polygon

a. Create a table with headings like those in Table **1** below.

Table 1: Angles used to create a Reuleaux polygon

Reuleaux Polygon	$m\angle ADB$	$m\angle ADC$	$m\angle ACB$	$\frac{m\angle ACB}{360^\circ}$
triangle	120°	120°	60°	1/6
pentagon		144°		
heptagon			$25\frac{5}{7}^\circ$	
nonagon				
n-gon				

b.
1. Create a regular pentagon and one arc of the corresponding Reuleaux polygon.
2. Construct $\angle ADB$, $\angle ADC$, and $\angle ACB$, as shown in Figure **10**.
3. Use your construction to complete the appropriate row in Table **1**.

c. Repeat Part **b** for a regular heptagon and a regular nonagon.

d. Use any patterns you observe to complete Table **1** for a regular polygon with n sides.

Mathematics Note

The **measure of an arc** is the measure of its central angle.

The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circumference of the circle by the fractional part of the circle that the arc represents.

For example, Figure **11** shows a circle with center at O and a radius of 5 cm.

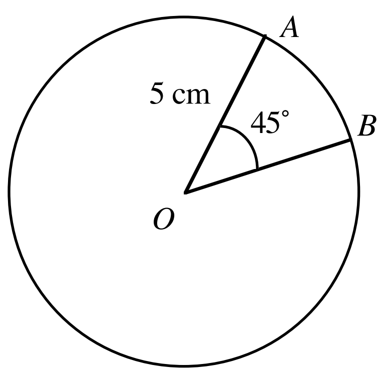


Figure 11: An arc of a circle

The measure of $\overset{\frown}{AB}$ is 45°. Since the central angle of the entire circle is 360°, the length of $\overset{\frown}{AB}$ can be found as follows:

$$\frac{45}{360} \bullet 2\pi(5) \approx 4 \text{ cm}$$

e. Given a constant width of 5 cm for each Reuleaux polygon, complete Table **2** below.

Table 2: Perimeter of Reuleaux polygons with $w = 5$ cm

Reuleaux Polygon	**Side Length**	**Perimeter**
triangle		
pentagon		
heptagon		
nonagon		
n-gon		

Discussion 1

a. When considering the arcs that form the sides of Reuleaux polygons, how is the radius of the circle that contains the arc related to the width of the Reuleaux polygon?

b. What is the relationship between the number of arcs used to construct a Reuleaux polygon and its perimeter?

c. Suggest a formula for the perimeter of a Reuleaux polygon with n sides and a constant width w.

Mathematics Note

The perimeter (P) of any curve of constant width w can be calculated by the formula $P = \pi w$.

For example, a Reuleaux triangle with a constant width of 10 cm has a perimeter of 10π cm.

d. Why doesn't the formula for the perimeter of a Reuleaux polygon include the number of sides?

e. A circle is a curve of constant width. Does the formula for the perimeter of a curve of constant width apply to a circle? Explain your response.

f. Recall that a central angle is an angle whose vertex is at a circle's center. An **inscribed angle** is an angle whose vertex is on the circle. For example, Figure **12** shows three different inscribed angles.

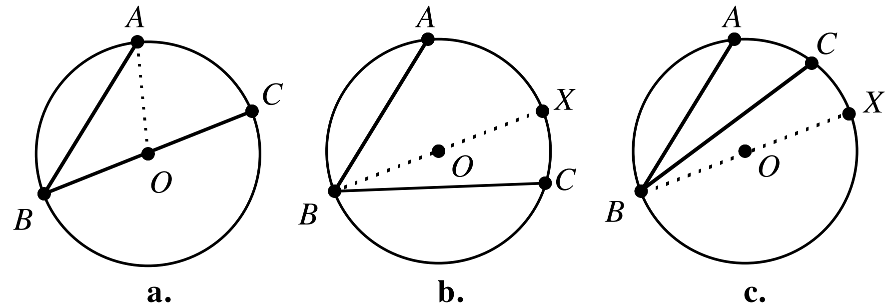


Figure 12: Three inscribed angles

1. In Figure **12a**, one of the sides of inscribed angle *ABC* passes through the circle's center *O*. What is the relationship between an inscribed angle of this type and the measure of its intercepted arc? Explain your response.
2. Is the relationship you described in Part **f1** also true when the center *O* lies in the interior of inscribed angle *ABC*, as shown in Figure **12b**? Justify your response.
3. Does this relationship also hold true when *O* lies outside inscribed angle *ABC*, as shown in Figure **12c**?

g. Figure **13** below shows part of the construction of a Reuleaux polygon.

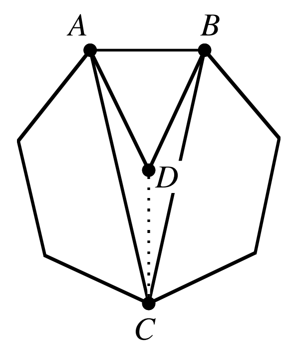


Figure 13: Construction of a Reuleaux polygon

In Exploration **1**, you discovered that

$$m\angle ACB = \frac{1}{2} m\angle ADB$$

How are these two angles related in terms of the circle with center at *D* and radius *DA*?

Exploration 2

In this exploration, you use your results from Exploration **1** to develop a method for calculating the areas of Reuleaux polygons.

Mathematics Note

A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. In Figure **12**, for example, the shaded region represents the sector defined by $\angle AOB$.

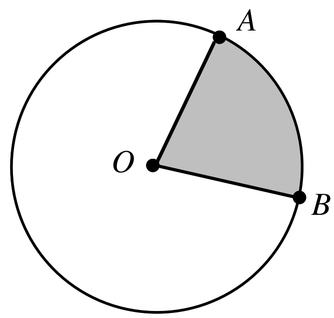


Figure 12: A sector of circle *O*

A **segment of a circle** is a region bounded by a chord and the arc that shares the same endpoints as the chord. For example, Figure **13** shows the segment defined by chord AB and $\overset{\frown}{AB}$.

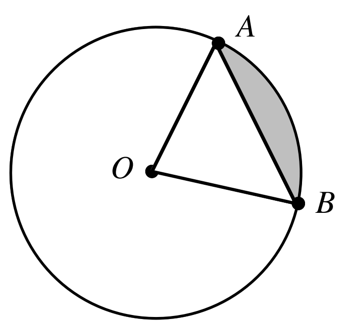


Figure 13: A segment of circle *O*

a. Create a table with headings like those in Table **3**.

Table 3: Area of segments for Reuleaux polygons with w = 5 cm

Reuleaux Polygon	Area of Sector	Area of Segment
triangle		
pentagon		
heptagon		
nonagon		
n-gon		

b. The area of a sector can be found by multiplying the area of a circle by the fractional part of the circle that the sector represents. Use the values you recorded in the appropriate column of Table **1** to find the area of the sector that corresponds with each Reuleaux polygon in Table **3**.

c. Use the area of each sector in Part **b** to determine the area of the corresponding segment of a circle for each Reuleaux polygon. Record these values in Table **3**.

Hint: Use the area of the triangle formed by the chord and the central angle to help determine the area of the segment. You may need to use trigonometry to find the lengths of sides.

d. Given a set of Reuleaux polygons with the same constant width, which one do you think will have the least area?

Discussion 2

a. Describe how you found the area of the sector that corresponds to a Reuleaux triangle for which $w = 5$ cm .

b. Describe how you found the area of the segment that corresponds to a Reuleaux triangle for which $w = 5$ cm .

c. In general, how does the area of a Reuleaux polygon compare with the area of the inscribed regular polygon?

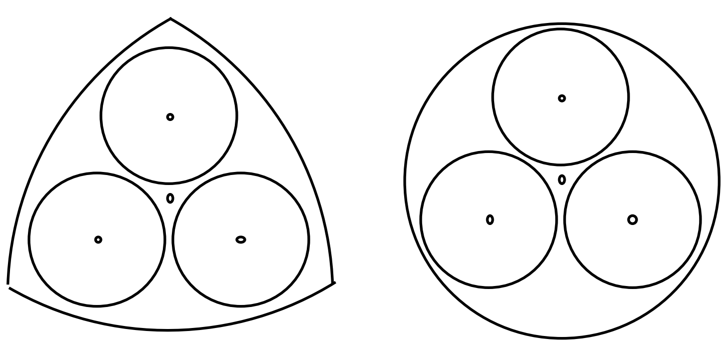
Assignment

3.1 Calculate the perimeter of each of the following Reuleaux polygons:

- **a.** a Reuleaux triangle with a constant width of 11 cm
- **b.** a Reuleaux 17-gon with a constant width of 15 cm
- **c.** a Reuleaux pentagon with a constant width of 10 cm.

3.2 Calculate the area of a Reuleaux triangle with constant width of 7 cm.

3.3 The diagram below shows the changer trays for two different CD players. The width of the tray shaped like a Reuleaux triangle is 26.1 cm. The radius of the tray shaped like a circle is 13.8 cm. Compare the areas of the two trays.



3.4 Determine the area of each of the following Reuleaux polygons:

a. a Reuleaux pentagon with a constant width of 10 cm

b. a Reuleaux heptagon with a constant width of 10 cm.

* * * * *

3.5 Describe a method for determining the area of any Reuleaux polygon.

3.6 For its new conveyor system, a manufacturing company is considering rollers shaped like Reuleaux heptagons. An engineer says this design will reduce costs because these rollers require less material than circular rollers. What percentage of material will be saved by using rollers shaped like Reuleaux heptagons instead of circular rollers?

3.7 Describe the three-dimensional surface that results when a Reuleaux triangle is rotated around an axis of symmetry. Include a sketch in your description, and suggest some possible applications for the surface.

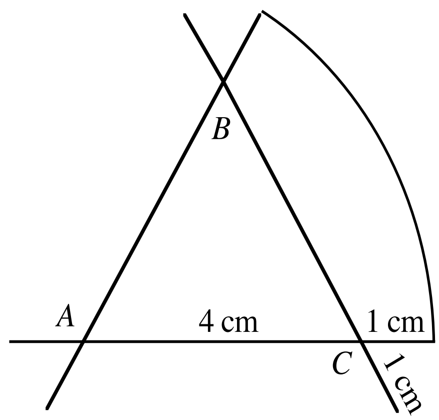
* * * * * * * * * *

Summary Assessment

1. In some applications, it may be desirable to modify the shape of a Reuleaux polygon to obtain more "rounded" corners.

a. To construct a Reuleaux triangle with rounded corners, complete the following steps.

- Construct an equilateral triangle *ABC* with sides measuring 4 cm each.
- Extend the sides of the triangle 1 cm past each vertex.
- Construct an arc with its center at one vertex of the triangle and a radius of 5 cm. Use the endpoints of two extended sides as the endpoints of the arc, as shown below.



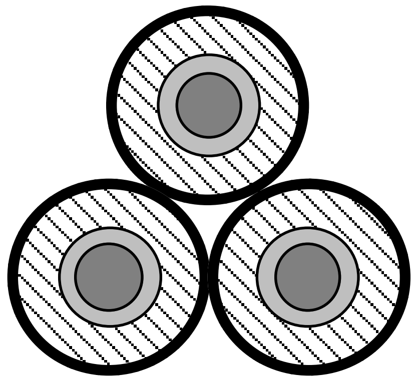
- Repeat the previous step for the two other vertices.
- To connect the endpoints of each pair of adjacent arcs, construct a smaller arc with its center at a vertex of the original triangle and a radius of 1 cm.

b. Is the shape you drew in Part **a** a curve of constant width? Justify your response.

c. Determine the perimeter of the shape.

d. Determine the area of the shape.

2. The circular cutting blades on a Norelco™ electric razor are arranged as shown in the diagram below.



What is the best shape for the head of this razor? Explain your response.

Module Summary

- A **line of support** is a line that intersects a curve such that all points of the curve not on the line lie on one side of the line.
- The perpendicular distance between a pair of parallel lines of support is a **width** (w) of the curve.
- A **curve of constant width** has the same width between every pair of parallel lines of support.
- A **Reuleaux polygon** is a curve of constant width constructed from a regular polygon with an odd number of sides.
- A **median** is a segment that connects the vertex of a triangle with the midpoint of the opposite side. The intersection of the medians of a triangle is the **centroid**.
- An **arc** of a circle is a part of the circle whose endpoints are the intersections, with the circle, of the sides of a central angle. The **measure of an arc** is the measure of its central angle.
- The **length of an arc** is the distance on the circle between the arc's endpoints. This distance can be found by multiplying the circle's circumference by the fractional part of the circle that the arc represents.
- The perimeter (P) of any curve of constant width w can be calculated by the formula $P = \pi w$.
- A **sector** of a circle is a region bounded by the sides of a central angle and an arc of the circle. The **area of a sector** can be found by multiplying the circle's area by the fractional part of the circle that the sector represents.
- A **segment of a circle** is a region bounded by a chord and the arc that shares the same endpoints as the chord.

Selected References

Benson, R. V. *Euclidean Geometry and Convexity*. New York: McGraw-Hill, 1966.

Cadwell, J. H. *Topics in Recreational Mathematics*. New York: Cambridge University Press, 1966.

Dossey, J. A. "What?—A Roller with Corners?" *Mathematics Teacher* 65 (December 1972): 720–724.

Gardner, M. "Mathematical Games." *Scientific American* 208 (February 1963): 148–156.

Johnson, M. "What Is a Curve of Constant Width?" In Evan M. Maletsky, ed., *Teaching with NCTM Student Math Notes 2*. Reston, VA: National Council of Teachers of Mathematics, 1993. pp. 81–86.

Klee, V. "Shapes of the Future." *Two-Year College Mathematics Journal* (Fall 1971): 14–27.

Lyusternik, L. A. *Convex Figures and Polyhedra*. Boston: D.C. Heath and Co., 1966.

Smith, S. "Drilling Square Holes." *Mathematics Teacher* 86 (October 1993): 579–583.