Pixel This

The ability to communicate effectively is an important skill in all walks of life. In this module, you discover how understanding algorithms can enhance your communication skills.

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Introduction

An image on a video screen is made up of many small discrete elements, or **pixels**. On some types of screens, each pixel can be either on or off. When a pixel is on, you can see it. For example, Figure **1** shows an image of the word *pixel* as it might appear on the screen of a graphing calculator.
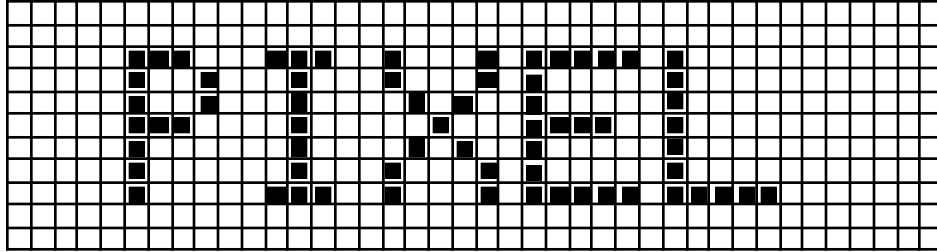

Figure 1: Pixels on a screen

Activity 1

In this activity, you create a picture on grid paper, then write step-by-step directions for recreating the image on a screen.

Exploration

a. **1.** Design and construct a simple picture by shading complete squares on a sheet of grid paper. **Note:** Do not allow anyone else to see your picture.

2. Without using diagrams, write a set of instructions to recreate your picture.

3. Exchange instructions with a classmate. Use the instructions that you receive to recreate the classmate's picture.

Mathematics Note

An **algorithm** is a step-by-step process for completing a task. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.

For example, the following steps provide an algorithm to find the distance between two points with coordinates (x_1, y_1) and (x_2, y_2) in the xy-plane.

1. Square the difference of x_1 and x_2.

2. Square the difference of y_1 and y_2.

3. Add the values from Steps 1 and 2.

4. Find the square root of the sum from Step 3.

5. Stop. The value for Step 4 is the distance between the two points with coordinates (x_1, y_1) and (x_2, y_2).

b. One way to identify the locations of pixels on an electronic screen is with ordered pairs. In Figure **2**, for example, the location of each pixel can be represented by an ordered pair (x, y), where the x-values increase from left to right and the y-values increase from bottom to top.
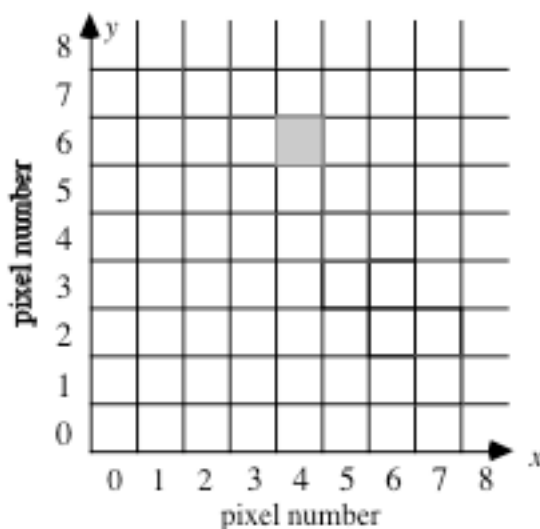

Figure 2: Simulated screen with pixel (4,6) turned on

Note that in this coordinate system, x and y are used to describe locations of pixels, not points. For example, the coordinates (0,0) describe the pixel in the lower left-hand corner of the screen, not the point at the intersection of the x- and y-axes.

1. Place an x-axis and a y-axis on the grid paper you used in Part **a**, and label the grid as shown in Figure **2**.

2. Write another algorithm to recreate your picture by representing each shaded square as an ordered pair.

3. Exchange your algorithm with a different classmate than in Part **a**. Use the algorithm you receive to recreate the classmate's picture.

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Discussion

a.   **1.** How does the picture you recreated in Part **a** of the exploration compare with its original version?

       2. How does the picture you recreated in Part **b** of the exploration compare with its original version?

b.   **1.** Is it possible to write more than one algorithm to recreate the same picture? Explain your response.

       2. Which algorithm do you think is better, the one from Part **a** or the one from Part **b**? Explain your response.

c.   The previous mathematics note describes a five-step algorithm for finding the distance between two points with coordinates (x_1, y_1) and (x_2, y_2). The mathematical formula for this distance is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

       1. Is this formula an algorithm? Explain your response.

       2. In general, do mathematical formulas represent algorithms?

d.   **1.** What assumptions did you make about the vocabulary and skills of the classmates with whom you exchanged your algorithms?

       2. How might you make your algorithms easier to understand for any potential audience?

       3. What characteristics of the audience should you consider when creating an algorithm?

e.   Describe some of the characteristics of a "good" algorithm.

Assignment

1.1   **a.** Design and construct another picture using the pixel numbering system shown in Figure **2**.

       b. Write an algorithm to recreate your picture.

       c. Ask someone who is not in your mathematics class to use the algorithm you wrote in Part **b** to recreate your picture.

       d. How does the recreated picture compare with your original?

       e. Describe how you could change your algorithm so that the recreated picture would more closely resemble the original.

1.2   In the early part of the 20th century, Russian peasants commonly used the algorithm described below to multiply two positive integers.

       1. Create a table with two columns. Write the first integer (x) in the left-hand column and the second integer (y) in the right-hand column.

2. Halve the number in the left-hand column. Disregarding any remainder, and write the result in the next row of the left-hand column. Double the number in the right-hand column, and write that result in the next row of the right-hand column.

3. Repeat Step 2 using the newly created row, until the result in the left-hand column is 1. Calculate the corresponding number for the right-hand column.

4. Find the sum of all the numbers in the right-hand column (including the original integer) that are paired with odd numbers in the left column.

5. Stop. This sum is the product of x and y.

   a. Use this algorithm to multiply $x = 19$ and $y = 12$.

   b. Use this algorithm to multiply a pair of integers of your choice.

   c. Do you think that this a good algorithm? Explain your response.

1.3 Recall that the set of natural numbers is $\{1, 2, 3, 4, \ldots \}$. A **prime number** is any natural number that has exactly two different positive factors. A **composite number** is any natural number that has more than two different positive factors. The number 1 is neither prime nor composite.

   For example, 17 is a prime number because it has exactly two positive factors: 1 and 17. The number 4 is composite because it has three positive factors: 1, 2, and 4.

   Create an algorithm that determines whether a natural number greater than 1 is prime or composite.

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1.4 Imagine that you have created a small business with the help of two friends. Although the company's earnings vary from month to month, you have agreed to share the profits as follows: you receive 50% of each month's profits, while your friends each receive 25%. Write an algorithm that describes a process by which you could divide the monthly profits.

1.5 **a.** Draw a triangle using paper, pencil, a compass and a ruler. Design an algorithm to recreate this triangle. Describe any assumptions you make about your target audience.

   b. Draw a triangle using a geometry utility. Design an algorithm to recreate this triangle.

   c. Compare the two algorithms you wrote in Parts **a** and **b**. Describe any similarities or differences you observe.

   d. Which algorithm do you think is the better one? Explain your response.

1.6   Create an algorithm that generates a random natural number in the interval [1, 10].

1.7   Create an algorithm that rounds any positive real number in decimal form to the nearest tenth.

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Activity 2

In Activity **1**, you wrote algorithms for your friends and classmates. As precise as you must be when writing algorithms for human audiences, it is even more important to be exact when writing algorithms for technology.

Exploration

In this exploration, you devise an algorithm that a computer could use to draw a line segment between two points.

a.   **1.** Use the pixel-numbering system from the exploration in Activity **1** to create a simulated video screen on a sheet of graph paper.

   2. Draw a line segment connecting the pixels labeled (2,8) and (7,1).

   3. Write an algorithm that activates all the pixels required to represent a line segment connecting (2,8) and (7,1).

b.   Try your algorithm using two pixels other than (2,8) and (7,1). If it does not correctly represent the line segment, revise the algorithm so that it will work for any line segment that connects pixels (x_1, y_1) and (x_2, y_2).

c.   Exchange your algorithm with a classmate. Use your classmate's algorithm to represent the line segment between pixels (2,8) and (7,1).

d.   Write down any suggestions you may have for improving your classmate's algorithm, then return it to its author.

e.   Revise your own algorithm using the suggestions you received. Verify that the revised version works.

Discussion

a.   Describe any problems you encountered when writing the algorithm in Part **a** of the exploration.

b.   **1.** Compare your revised algorithm with those of your classmates.

   2. Use the best features of these revisions to write one algorithm that represents a line segment that connects pixels (x_1, y_1) and (x_2, y_2).

c.   **1.** What advantages do you see in writing a general algorithm that represents a segment connecting pixels (x_1, y_1) and (x_2, y_2), instead of an algorithm that represents a segment that connects specific pixels like (2,8) and (7,1)?

   2. In what types of situations might it be better to write a specific algorithm?

Assignment

2.1   In the exploration, you wrote an algorithm that shaded pixels to represent a line segment between two given pixels. Write an algorithm that could be used to represent the line segment defined by a pixel with coordinates (x_1, y_1), a slope m, and a minimum length d.

2.2   On most calculators, the square root of any non-negative number n may be found by pushing the "$\sqrt{}$" button. Write an algorithm that describes how to find the square root, to the nearest hundredth, of a non-negative number without using this button.

2.3   The Pythagorean theorem may be stated as follows: Given any right triangle ABC, where a and b are the measures of the two legs and c is the measure of the hypotenuse, $a^2 + b^2 = c^2$.

   Write an algorithm that describes how to use this theorem to find the measure of the hypotenuse of any right triangle, given the measures of the two legs. Assume that your potential audience is not familiar with mathematical symbolism, such as exponents.

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2.4   Imagine that a friend has called for help with his mathematics homework. He would like to graph the equation:

$$V = \frac{4}{3}\pi r^3$$

where V is the volume of a sphere with radius r. Unfortunately, he has no idea how to use his new graphing utility. Write an algorithm that you could describe over the phone to help your friend graph this equation.

2.5   Recall that the greatest integer function $y = [x]$ always rounds x to the greatest integer less than or equal to x.

   a.   For integers n and d, where $d \neq 0$, determine when the following equation is true:

$$\frac{n}{d} = \left[\frac{n}{d}\right]$$

   b.   The algorithm outlined below can be used to identify leap years:

   1.   If the year is not divisible by 4, then it is not a leap year. Go to Step 5.

   2.   If the year is not divisible by 100, then it is a leap year. Go to Step 5.

   3.   If the year is divisible by 400, then it is a leap year. Go to Step 5.

   4.   The year is not a leap year.

   5.   Stop.

   Use this algorithm to identify all the leap years between 1895 and 1925.

   c.   Rewrite the leap-year algorithm using the greatest integer function. Use y to represent the year.

2.6   **a.**   Write an algorithm for using paper and pencil to solve an equation of the form $ax + by = c$, where $b \neq 0$, for y.

   b.   Write an algorithm for using a symbolic manipulator to solve an equation of the form $ax + by = c$, where $b \neq 0$, for y.

2.7   Some calculators allow the user to select the number of digits to be displayed on the screen after a calculation is performed. Some calculators even allow the user to select a floating decimal point or a fixed decimal point.

   a.   The table below shows how one calculator displays results for two different settings: "Fix 4" and "Float 4." Describe the difference between these two settings.

Calculation	Fix 4	Float 4
5.2894588 • 1	5.2895	5.289
5.2894588 •10,000	52894.5880	5.289E4
5.2894588 • 0.00001	5.2895E–5	5.289E–5

   b.   Which algorithm would be easier to write: the one for a setting of "Fix 4" or the one for "Float 4"? Explain your response.

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Activity 3

In the first two activities, you examined the characteristics of effective algorithms. In this activity, you focus on creating general algorithms that are useful in many situations.

Exploration

One method of displaying the graph of a function on a screen is the **scan-conversion algorithm**. In this exploration, you investigate how well the scan-conversion algorithm works for graphing several different types of functions.

Technology Note

The **scan-conversion algorithm** calculates ordered pairs of integers, then activates pixels that correspond to these ordered pairs. A general form of the scan-conversion algorithm is outlined below.

1. List a domain for x that consists only of integer values.

2. Use the x-values from Step 1 to determine corresponding values for $f(x)$.

3. Round all the values for $f(x)$ from Step 2 to the nearest integer.

4. Write the corresponding integer values for x and $f(x)$ as ordered pairs.

5. Activate the pixels that correspond to the ordered pairs from Step 4.

6. Stop.

For example, the scan-conversion algorithm can draw a line segment between any two points using the steps below:

1. Determine the equation of the line containing the pixels (x_1, y_1) and (x_2, y_2) using the formula $y = m(x - x_1) + y_1$, where m is the slope of the line and x_1, x_2, y_1, and y_2 are integers.

2. Use the equation from Step 1 to calculate the y-value for each integer x-value between x_1 and x_2, inclusive.

3. Round each y-value to the nearest integer.

4. Write the corresponding x- and y-values as ordered pairs.

5. Activate the pixels that correspond to the ordered pairs from Step 4.

6. Stop.

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a. Use the scan-conversion algorithm to create a pixel graph of the function below for integer values of x in the interval $[0, 10]$:

$$f(x) = \frac{12}{7}x + 2$$

b. Determine whether or not the algorithm generates a good approximation of the graph of the function.

c. Repeat Parts **a** and **b** using each of the following functions:

1. $f(x) = (x - 5)^2$

2. $f(x) = 2 + \dfrac{2}{x - 5}$

3. $f(x) = (x - 5)^3 + 10$

4. $f(x) = 3(2)^{(x-5)}$.

Discussion

a. Compare the scan-conversion algorithm to the algorithm you developed in the exploration in Activity **2**.

b. Why does the scan-conversion algorithm appear to approximate the graphs of some functions better than others?

c. Do you think that the scan-conversion algorithm is a general algorithm or a specific algorithm? Explain your response.

Assignment

3.1 The screen on one graphing calculator contains 95 columns and 63 rows of pixels. As shown in the diagram below, columns are numbered from left to right, while rows are numbered from top to bottom.
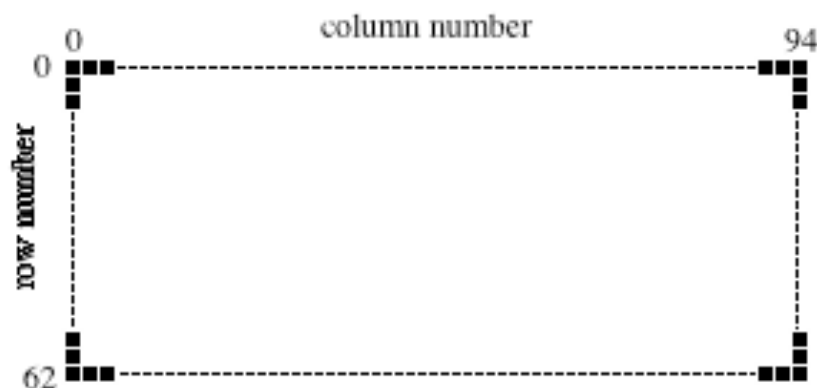
Suppose that the graphing window is set for a domain of $[-10, 10]$ and a range of $[-10, 10]$. In this case, the point $(-10, 10)$ corresponds to the pixel in column 0 and row 0.

a. Identify the pixel, by column number and row number, that corresponds to each of the following points:

1. $(10,-10)$

2. $(10,10)$

3. $(0,0)$

4. $(3,5)$

5. $(2,-2)$

b. Describe an algorithm for associating a point (x,y), where x and y are integers, with the appropriate pixel on this screen.

3.2 The formula below can be used to calculate the account balance A in a savings account after n years, where r is the annual interest rate and p is the initial principal.

$$A = p(1 + r)^n$$

a. Write an algorithm describing how to calculate A, given values for $p, n,$ and r.

b. What assumptions did you make about the skills and vocabulary of the audience for your algorithm in Part **a**?

c. Use your algorithm to find the value of A when $p = 1000$, $r = 0.06$, and $n = 7$. Describe what this value represents.

3.3 The diagram below shows the first several rows in the pattern of numbers known as Pascal's triangle. Named in honor of the French mathematician Blaise Pascal (1623–1662), the pattern actually was discovered centuries earlier by Chinese mathematicians.

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            1              row 0
          1   1            row 1
        1   2   1          row 2
      1   3   3   1        row 3
    1   4   6   4   1      row 4
```

a. Determine the numbers in row 5 of Pascal's triangle.

b. Develop a general algorithm for generating any row in Pascal's triangle, assuming that all the previous rows are known.

3.4 The diagram below shows a drawing of a quarter circle on a pixel grid. Do you think that the scan-conversion algorithm will produce a good approximation of this drawing? Explain your response.

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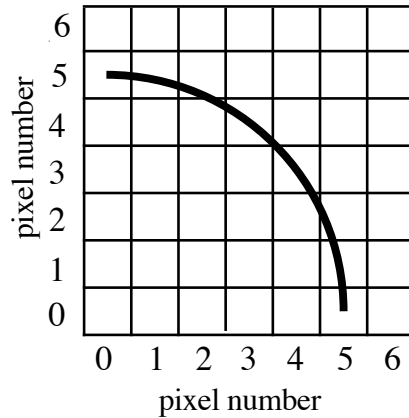
3.5 The solution for the system of linear equations below can be found either algebraically or graphically:

$$\begin{cases} y = \dfrac{3}{2}x + 5 \\ y = -2x - 9 \end{cases}$$

a. Write an algorithm for solving the system algebraically.

b. Write an algorithm for solving the system graphically.

c. Use your algorithms from Parts **a** and **b** to solve the following system of equations:

$$\begin{cases} y = \dfrac{3}{4}x + 7 \\ y = \dfrac{9}{12}x - 4 \end{cases}$$

d. Were your algorithms general enough to solve the system in Part **c**? If not, how can you modify them?

3.6 Write a general algorithm to solve any system of linear equations.

3.7 Consider the equations $3x^2 + y^2 = 15$ and $2y^2 - 3x^2 = 21$.

a. Write a specific algorithm to solve each equation for y.

b. Write a general algorithm that will solve either equation for y.

c. What must a person know in order to use your algorithms?

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Research Project

In 1965, J. E. Bresenham described a method for plotting line segments using technology. This method increased the speed of segment drawing. Find out more about the Bresenham method. Rewrite it as an algorithm that other students in your mathematics class can understand.

1. With the discovery of the Rhind papyrus, a mathematical document dating back to 1650 B.C., historians have determined the method by which the ancient Egyptians multiplied any two numbers without using calculators or multiplication tables. By this method, the product of 26 and 33 can be found using an algorithm like the one described below.

1. Make a two-column table.

2. Write 1 in the left-hand column. Write the second factor, 33, in the right-hand column.

3. Double the number in the left-hand column. Write the doubled number in the left-hand column of the next row only if it is smaller than the first factor, 26. Continue this process until the doubled number is greater than or equal to 26.

4. Double the number in the right-hand column and write the result in the right-hand column of the next row. Continue this process until you have written a number in the right-hand column for each number in the left-hand column.

    Identify and mark the numbers in the left-hand column whose sum is the first factor, 26. In this case, the numbers are 2, 8, and 16. Your completed table should resemble the one shown below.

1	33
2	66
4	132
8	264
16	528

5. Add the numbers from the right-hand column $(66 + 264 + 528)$ that correspond to the marked numbers in the left-hand column. The result, 858, is the product of 26 and 33.

6. Stop.

a. Write a general algorithm that uses the method described above to calculate the product of any two numbers.

b. Use your algorithm to find the product of 18 and 65. Show the completed table in your response.

c. Describe why this algorithm works.

2.   When a circle is inscribed in a triangle, the circle is tangent to each of the triangle's three sides. Write a general algorithm for using a geometry utility to draw a circle inscribed in a triangle.

    Hint: The center of the inscribed circle is located at the point of intersection of the three angle bisectors of the triangle.

- An **algorithm** is a step-by-step process for completing a task. An algorithm should be precise and produce a result that successfully completes the task or determines that the task cannot be done.

- **Scan-conversion** is a process for determining the pixels that closely represent a function. The **scan-conversion algorithm** calculates ordered pairs of integers, then activates the pixels that correspond to these ordered pairs. A general form of the scan-conversion algorithm is outlined below.

  1. List a domain for x that consists only of integer values.

  2. Use the x-values from Step 1 to determine corresponding values for $f(x)$.

  3. Round all values for $f(x)$ from Step 2 to the nearest integer.

  4. Write the corresponding integer values for x and $f(x)$ as ordered pairs.

  5. Activate the pixels that correspond to the ordered pairs from Step 4.

  6. Stop.

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