Game of Life

What do two children playing "Rock, Paper, Scissors" have in common with two businesses competing for customers? In both situations, game theory can be used to make decisions and determine strategies.

Masha Albrecht • Verne Schlepp • Steve Yockim

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Introduction

While Raoul and Colleen are driving to work on a rainy afternoon, a tire on their car goes flat. Neither one wants to go outside to change it. How should Raoul and Colleen decide who will perform this chore?

The method selected for resolving a conflict often depends on the consequences of the decision. In this case, simply flipping a coin might settle who changes the tire. If the consequences of the decision were more critical, then a more sophisticated method of decision making might be more appropriate.

Game theory is a branch of mathematics used to analyze decision making in situations that involve conflicting interests. While game theory also applies to some recreational games, it was developed during World War II to analyze competitive situations in business, warfare, and society.

In this module, you will analyze the decision-making process using the simplest of all games: the two-person, **zero-sum** game. In a two-person, zero-sum game, one player's loss is the other player's gain. In such situations, it is assumed that each player makes a decision without knowing what the other player will do. In addition, each player is assumed to believe that the other player is an intelligent opponent who will make the best possible move.

Exploration

The game of Odds or Evens can be used to make a decision when two people have conflicting interests. (This is one version of the ancient Italian game Morra.) The rules for Odds or Evens are described below.

- One person chooses odds; the other person chooses evens.
- At the count of three, each person holds out either one or two fingers.
- If the total number of fingers showing is odd, then the person who selected odds wins the game. If the total number of fingers showing is even, then the person who selected evens wins the game.

a. Play Odds or Evens 10 times with a partner. Keep a record of the number of games won by each player, and whether the winner chose odds or chose evens.

b. Play Odds or Evens once more to determine which player will report the results to the class.

c. Report the results to the class.

Discussion

a. How did you decide the number of fingers to show when playing Odds or Evens?

b. Based on the class results, do you think it is better to select odds or evens if you want to win the game?

Activity 1

When two people have conflicting interests, each should develop a plan of action, or **strategy**. In game theory, it is assumed that both parties use strategies that serve their own best interests.

Exploration

In this exploration, you and a partner will investigate three different versions of the game Odds or Evens (Morra). For each version, you will determine if there is a strategy you can use to win every time. The following rules apply to all three versions of the game.

- Randomly decide which person is player A and which is player B. Use this same assignment for all games.
- At the count of three, each player holds out one or two fingers.
- The winner receives a number of points equivalent to the total number of fingers shown by both players, while the loser relinquishes this same number of points. **Note:** It is possible for a player to have a negative number of points.

a. In the game Total, player A wins if the total number of fingers showing is odd. Player B wins if the total number of fingers showing is even.

1. Based on the rules of Total, predict who will win in this game.
2. Play Total 10 times. Record the points won by each player in each game, then determine the total points won by each player in 10 games.
3. Collect the class data.
4. Determine whether or not one player has an advantage in this game.
5. Suppose that players A and B randomly hold out one or two fingers with equal likelihood. Determine the expected value (in points) for each player and compare it to the class data.

b. In the game More, the player showing the most fingers wins. In the case of a tie, player A wins if both players show ones and player B wins if both players show twos.

Repeat Part **a** for the game More.

c. In the game Less, the players showing fewer fingers wins. In the case of a tie, player A wins if both players show ones and player B wins if both players show twos.

Repeat Part **a** for the game Less.

Discussion

a.
1. Is there a strategy for the game Total which ensures that player A always wins? If so, describe this strategy. If not, is there a strategy that minimizes player A's losses?
2. Is there a strategy for Total which ensures that player B always wins? If so, describe this strategy. If not, is there a strategy that minimizes player B's losses?

b. Repeat Part **a** for the game More.

c. Repeat Part **a** for the game Less.

d. For each game, how did the class results compare with the expected value?

e. Once players A and B recognize each other's strategy in a game, what should they do?

Mathematics Note

An **optimal strategy** for a player results in maximizing the winnings or minimizing the losses for that player. In the game More, for example, the optimal strategy for player B is to show two fingers in every game. This will always results in a win for player B. The optimal strategy for player A is to show one finger. This will minimize player A's losses.

In a **pure strategy**, a player makes the same choice each time the game is played. For example, the optimal strategies for both players in More are pure strategies.

In a **mixed strategy**, a player's choice of how to play varies from game to game. For example, the game Total offers no pure strategy that is an optimal strategy for either player. In this case, players should use mixed strategies. **Note:** You will investigate the choice of a mixed strategy later in this module.

f. By showing one finger each time, player A can always win the game of Less. This optimal strategy is a pure strategy. Is player B's optimal strategy also a pure strategy?

g. Why would it be unwise for either player to use a pure strategy in Total?

h. Recommend a mixed strategy for players of Total.

i.
1. If each player in the game More uses an optimal pure strategy, what is the expected value for each?
2. Compare this expected value to the class data from the exploration.

j.
1. If each player uses an optimal pure strategy in the game Less, what is the expected value for each?
2. Compare this expected value to the class data from the exploration.

Mathematics Note

In a two-player zero-sum game, the consequences of each player's choices can be summarized in a **payoff matrix**, where a payoff is the amount won or lost by a single player in one game. The rows of the matrix represent the choices of one player (the row player), while the columns represent the choices of the second player (the column player).

In this module, the entries in each cell of a payoff matrix always represent the payoffs for the row player. A positive payoff represents a win for the row player, while a negative payoff represents a loss for the row player. The payoff for the column player is the additive inverse of the payoff for the row player.

For example, Figure **1** shows a payoff matrix for player A in Total. Each player has two choices: show one finger or show two fingers.

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cc} \text{one} & \text{two} \end{array} \\ \text{Player A} \quad \begin{array}{c} \text{one} \\ \text{two} \end{array} & \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \end{array}$$

Figure 1: Payoff matrix for player A in Total

The entry in row 2, column 2 of the matrix indicates that player A loses 4 points and player B wins 4 points when both show two fingers.

In a **strictly determined** game, the optimal strategy for each player is a pure strategy. Since players make the same choices each time the game is played, the payoffs are always the same.

For example, consider the payoff matrix in Figure **2**. In this case, R_1 and R_2 represent the choices for the row player and C_1 and C_2 represent the choices for the column player. In this game, the optimal strategy for player A is to play row 2. This always ensures a win. Player B's optimal strategy is to play column 1. This strategy minimizes losses. Because both players use pure strategies, the game is strictly determined.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & 5 \\ 2 & 3 \end{bmatrix} \end{array}$$

Figure 2: A payoff matrix for a strictly determined game

When both players use these pure strategies, player A wins 2 points in every game. This constant payoff is the **value of the game** for player A.

k. What does an entry of –2 represent in the payoff matrix in Figure **2**?

l. Which of the games in the exploration are strictly determined? Explain your response.

m. Describe how you could use the payoff matrix for a strictly determined game to find each of the following:

1. the best strategy for the row player
2. the best strategy for the column player.

Assignment

1.1 According to the rules of Total described in the exploration, player B wins when the total number of fingers is even.

a. The payoff matrix for Total when player A is the row player is shown in Figure **1**. Determine a payoff matrix for Total if player B is the row player.

b. Is there a pure strategy for either player? Explain your response.

1.2 **a.** Construct a payoff matrix for the game More if player B is the row player.

b. Explain why the matrix indicates that player B's optimal strategy is a pure strategy.

c. Explain why the matrix indicates that player A's optimal strategy is a pure strategy.

d. Construct a payoff matrix for More if player A is the row player. Does this matrix indicate the same results as the one from Part **a**?

e. What is the value of the game for player B? for player A?

1.3 **a.** Construct a payoff matrix for the game Less if player A is the row player.

b. Explain why the matrix indicates that player A's optimal strategy is a pure strategy.

c. Explain why the matrix indicates that player B's optimal strategy is a pure strategy.

d. What is the value of the game for player A? for player B?

1.4 Consider the following payoff matrix for player A of a two-person, zero-sum game.

$$\text{Player A}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{cc} C_1 & C_2 \\ \end{array}} \begin{bmatrix} -5 & 6 \\ 2 & -4 \end{bmatrix}$$

a. Determine the payoff matrix if player B is the row player.

b. Describe the relationship between the two payoff matrices.

1.5 For each game described by payoff matrices in Parts **a–c** below, find the optimal pure strategy for player A and for player B. Then determine the value of the game for the winning player.

a.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \end{array}$$

b.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 62 & 39 \\ 78 & 43 \end{bmatrix} \end{array}$$

c.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 7 & 3 \\ -5 & 0 \end{bmatrix} \end{array}$$

Mathematics Note

The value of a strictly determined game can be determined from the payoff matrix by identifying the entry that is less than or equal to all entries in its row and greater than or equal to all entries in its column. This is the **saddle point** of the matrix, and is the value of a strictly determined game.

For example, consider the payoff matrix in Figure **3**. It is the payoff matrix for player B in More. The saddle point of the matrix is 3, since it is less than 4 and greater than –2. This is the value of the game for player B.

$$\begin{array}{cc} & \text{Player A} \\ & \begin{array}{cc} \text{one} & \text{two} \end{array} \\ \text{Player B} \begin{array}{c} \text{one} \\ \text{two} \end{array} & \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \end{array}$$

Figure 3: Payoff matrix for player B in More

A **strictly determined game** is a fair game if the saddle point is 0.

1.6 **a.** Design an algorithm for determining if a matrix has a saddle point.

b. If a payoff matrix for a game does not have a saddle point, what can you conclude about the game?

c. If a payoff matrix has more than one saddle point, what can you conclude about the game?

1.7 **a.** Design a 2×2 payoff matrix with no saddle point.

b. Design a 3×3 payoff matrix with no saddle point.

1.8 Bill and Ann are running against each other for a single seat on the student council. One of the key issues in the campaign is a proposal to renovate the student lounge. If the proposal is approved, new furniture will be purchased using the extracurricular activities budget. This will make fewer dollars available for other activities.

To predict how his position on this issue will affect undecided voters, Bill conducts a poll. Bill decides to use game theory to find his optimal strategy. The results of the poll are represented in the following payoff matrix, where F represents "favors," N represents "neutral," and O represents "opposes."

$$\begin{array}{cc} & \text{Ann} \\ & \begin{array}{ccc} \text{F} & \text{N} & \text{O} \end{array} \\ \text{Bill} \begin{array}{c} \text{F} \\ \text{N} \\ \text{O} \end{array} & \begin{bmatrix} -80 & -60 & 60 \\ 90 & 70 & 90 \\ -105 & -50 & 100 \end{bmatrix} \end{array}$$

In this matrix, the entry in row F, column O corresponds with the situation in which Bill is in favor of the new lounge and Ann opposes it. The value of this entry (60) indicates that Bill gains 60 votes and Ann loses 60 votes.

a. Describe the meaning of the matrix entry –80.

b. Which entry in the matrix shows the best outcome for Ann?

c. Bill finds out that a classmate has shown the data to Ann. What is the best outcome for Bill? Explain your response.

d. **1.** What is Bill's optimal strategy?

2. What is Ann's optimal strategy?

e. **1.** What is the saddle point of this payoff matrix?

2. What does this value mean to the two candidates?

* * * * *

1.9 Given the following payoff matrix for a two-person, zero-sum game, determine values of x and y so that the game is not strictly determined:

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}$$

1.10 In the two-person game "Rock, Paper, Scissors," each player holds out one hand to indicate the choice of one of three objects (rock, paper, or scissors). A winner is determined by the following rules: scissors cut paper, rock smashes scissors, and paper covers rock. In the case of a tie, neither player wins.

a. Write a matrix to represent this game.

b. Describe whether or not this game is strictly determined.

* * * * * * * * * *

Activity 2

In games that do not have pure optimal strategies, players should use mixed strategies. In this activity, you investigate this type of game.

Exploration 1

In games that are not strictly determined, the amount won or lost can vary from game to game, depending on the choices made by players. As a result, the optimal strategy for a particular player is one that results in the greatest average winnings per game or the least average losses per game, regardless of the other player's strategy.

a. Consider the following payoff matrix for a game.

$$\text{Player A}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \end{array}$$

Since this payoff matrix does not have a saddle point, the game is not strictly determined. In this case, it would not be wise for either player to use a pure strategy. For example, if player A always chose R_1, then player B would eventually recognize this strategy. By always choosing C_1, player B would win 1 point in every remaining game.

1. Devise a mixed strategy that you think will maximize player A's average winnings when the game is repeated a large number of times. Express your strategy in terms of the percentage of games in which player A should choose each row. For example, one strategy for player A would be to play R_1 10% of the time and R_2 90% of the time.
2. Devise a mixed strategy that you believe will maximize player B's average winnings when the game is repeated a large number of times. Express your strategy in terms of the percentage of games in which player B should choose each column.
3. Since the choices made by each player are independent events, the probability of each payoff is the product of the probabilities of the player's respective choices. Create a tree diagram showing the probability of each payoff for player A.

b. Use the tree diagram that you created in Part **a** to complete Table **1**.

Table 1: Determining the expected value for player A

Player A's Choice	Player B's Choice	Payoff	Probability of Payoff	Probability ∞ Payoff
R_1	C_1	–1		
R_2	C_1	1		
R_1	C_2	2		
R_2	C_2	0		

c. Find the sum of the four products in the right-hand column of Table **1**. This is the expected value of the game for player A.

d. Imagine that player B uses a pure strategy and always plays C_1. In this case, the probability that player B chooses C_1 is 1 while the probability that player B chooses C_2 is 0.

Table **2** shows the game that results when player B uses this pure strategy. The probability that player A chooses R_1 is represented by r_1, while the probability that player A chooses R_2 is represented by $1 - r_1$.

Table 2: Game when player B always chooses C_1

Player A's Choice	Player B's Choice	Payoff	Probability of Payoff	Probability ∞ Payoff
R_1	C_1	–1	$r_1 \bullet 1$	$-r_1$
R_2	C_1	1	$(1-r_1)\bullet 1$	$1-r_1$
R_1	C_2	2	$r_1 \bullet 0$	0
R_2	C_2	0	$(1-r_1)\bullet 0$	0

Write an expression for the expected value for player A for the game illustrated in Table **2**.

e. Assume that player B changes strategy and always chooses C_2. Write an expression for the expected value of the game for player A. As in Part **d**, represent the probability that player A chooses R_1 as r_1 and the probability that player A chooses R_2 as $1-r_1$.

f. The optimal mixed strategy for the row player does not depend on the strategy employed by the column player. This optimal strategy can be determined by analyzing games in which the column player uses pure strategies.

There are an infinite number of strategies that player A could employ. Table **3** shows 11 of those strategies. Recall that the expected value of the game for player A changes if player A changes the percentages assigned to each choice in a mixed strategy. Use the results from Parts **d** and **e** to complete a table with headings like those in Table **3**.

Table 3: Spreadsheet for various values for r_1

1	**2**	**3**	**4**
Probability that player A selects R_1 (r_1)	Probability that player A selects R_2 $(1-r_1)$	Expected value if player B always selects C_1	Expected value if player B always selects C_2
0.0	1.0		
0.1	0.9		
0.2	0.8		
0.3	0.7		
0.4	0.6		
0.5	0.5		
0.6	0.4		
0.7	0.3		
0.8	0.2		
0.9	0.1		
1.0	0.0		

g. Column 3 of Table **3** contains the expected values that result when player B always chooses C_1 and player A uses 11 different strategies. Column 4 contains the expected values that result when player B always chooses C_2 and player A uses 11 different strategies. Player A's optimal mixed strategy is the strategy that will result in the same expected value in both columns.

Using the entries in Table **3**, estimate the strategy that will result in the same expected value in both columns.

h. Another way to determine the strategy that results in the same expected value in both columns involves analyzing the situation graphically.

1. Write an equation that describes the relationship between the entries in columns 1 and 3 of Table **3**. Use this equation to create a graph of expected value versus the probability that player A selects R_1.
2. Write an equation that describes the relationship between the entries in columns 1 and 4 of Table **3**. Use this equation to create another graph of expected value versus the probability that player A selects R_1. Plot this graph on the same coordinate system as in Step **1**.
3. Use the graphs from Steps **1** and **2** to estimate the optimal strategy for player A.
4. Solve the system of equations from Steps **1** and **2** to determine the optimal strategy for player A.
5. Determine the expected value of the game when player A uses this optimal strategy.

Discussion 1

a. Compare the optimal strategy determined in Part **g** of Exploration **1** with the strategy you identified in Part **h**.

b. How do the coordinates of the point of intersection of the graphs in Part **h** relate to the player's strategy?

c.
1. When player A uses the optimal strategy, what is the expected value of the game?
2. Explain what this value means for player A.
3. If the game was played 1000 times using the optimal strategy, how many points would player A expect to win?

d.
1. If the game was played 1000 times using the optimal strategy, how could you determine player A's average payoff per game?
2. Will the mean payoff per game equal the expected value of the game? Explain your response.

e. **1.** Use Table **3** to describe the range of expected values of the game for player A if different strategies are employed.

2. Why would it be risky for player A to use a strategy other than the optimal one?

f. Assume that the optimal strategy for player A in a game is to select R_1 25% of the time and R_2 75% of the time.

1. Explain why these selections must be made in a random manner.

2. Describe a method for making these selections randomly.

g. Suppose a game is not strictly determined. How can you use algebra to find the optimal strategy for the row player?

Mathematics Note

In a game that is not strictly determined, it can be shown that an optimal mixed strategy exists for each player.

The optimal mixed strategy for a player is the strategy that results in the same expected value regardless of the strategy used by the other player. In other words, the expected value of the game for a player using an optimal strategy is independent of the strategy used by the other player.

For example, consider the payoff matrix in Figure **4**.:

$$\text{Row player}\quad \begin{array}{cc} & \text{Column player} \\ & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix} \end{array}$$

Figure 4: Payoff matrix

Table **4** shows the expected value of the game for the row player when the column player always chooses C_1, where the probability that the row player chooses R_1 is represented as r_1.

Table 4: Expected value when column player always chooses C_1

Payoff	Probability	Payoff • Probability
4	$r_1 \bullet 1$	$4 \bullet r_1$
–3	$1 \bullet (1 - r_1)$	$(-3) \bullet (1 - r_1)$
–2	$r_1 \bullet 0$	$(-2) \bullet 0$
5	$0 \bullet (1 - r_1)$	$5 \bullet 0$
	Expected Value	$4r_1 + (-3)(1 - r_1)$

Table **5** shows the expected value for the row player when the column player always chooses C_2.

Table 5: Expected value when column player always chooses C_2

Payoff	Probability	Payoff • Probability
4	$r_1 \bullet 0$	$4 \bullet 0$
–3	$0 \bullet (1-r_1)$	$(-3) \bullet 0$
–2	$r_1 \bullet 1$	$(-2) \bullet r_1$
5	$1 \bullet (1-r_1)$	$5 \bullet (1-r_1)$
	Expected Value	$-2r_1 + 5(1-r_1)$

The optimal strategy for the row player occurs when the two expected values are equal:

$$4r_1 + (-3)(1-r_1) = -2r_1 + 5(1-r_1)$$
$$r_1 = \frac{4}{7}$$

In this game, the optimal strategy for the row player is to choose R_1 an average of 4 times in every 7 games. The expected value of the game for the row player when this optimal strategy is used can be determined using either column of the payoff matrix:

$$\frac{4}{7} \bullet 4 + \frac{3}{7} \bullet (-3) = 1 \text{ or } \frac{4}{7} \bullet (-2) + \frac{3}{7} \bullet 5 = 1$$

The value of the game for the row player is the expected value that results when the row player uses the optimal strategy. The value of this game for the row player is 1.

h. What must be true of the value of a game if the game is a fair one?

i. What equation would you solve to determine the expected value for the row player in a game having the following payoff matrix? (Let r_1 represent the probability that the row player chooses R_1.)

$$\begin{array}{ccc} & & \text{Column player} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Row player} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

j. Consider the strictly determined game whose payoffs are represented in the following matrix:

$$\begin{array}{ccc} & & \text{Column player} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Row player} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \end{array}$$

In Activity **1**, the value of a strictly determined game was defined as the constant amount won or lost by the row player when both players use pure strategies. Could this value also be defined as the expected value of the game for the row player when both players use pure strategies?

Exploration 2

In this exploration, you investigate the expected values that result when players A and B use a variety of mixed strategies in the game from Exploration **1**. The payoff matrix for player A is shown below:

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \end{array}$$

a. Based on the results from Exploration **1**, what is the value of the game for player A?

b. Write the payoff matrix for player B.

c.
1. Determine the optimal strategy for player B.
2. Determine the value of the game for player B.
3. Compare the value of the game for player B with the value of the game for player A.

d. Obtain a simulation of this game from your teacher and complete the following steps.
1. Using the optimal strategies for each player, run the simulation 100 times. Determine the average payoff per game for player A. Compare the average payoff with the value of the game for player A.
2. Run the simulation 1000 times and compare the average payoff per game for player A with the value of the game for player A.

e.
1. Run the simulation 300 times using the optimal strategy for player A and a strategy for player B that is not the optimal one. Compare the average payoff per game for player A with the value of the game for player A.
2. Repeat the process described in Step **1** using several other non-optimal strategies for player B.

f. Run the simulation 300 times using strategies for both players that are not optimal. Compare the average payoff per game for player A with the value of the game for player A. Repeat this process using several other non-optimal strategies for both players.

Discussion 2

a.
1. Describe the results of the simulation of 1000 games in which both players used their optimal strategies.
2. How did these results compare with the results you obtained by simulating 100 games in which both players used their optimal strategies?

b.
1. Describe the results you obtained when using the optimal strategy for player A and a non-optimal strategy for player B.
2. Explain why these results occurred.

c. What happens when both players fail to use their optimal strategies?

d.
1. Could using a strategy other than the optimal one be advantageous for player A? Explain your response.
2. Could using a strategy other than the optimal one be unfavorable to player A? Explain your response.

Assignment

2.1 Determine the optimal strategy and value of the game for each player in the games represented by the following payoff matrices.

a.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \end{array}$$

b.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C_1 & C_2 \end{array} \\ \text{Player A} & \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix} \end{array}$$

2.2 In the game Total from Activity **1**, the payoff for the winner is the number of fingers showing.

a. Determine the optimal strategy for player B.

b. What is the value of the game for player B?

c. Is this a fair game? Explain your response.

2.3 When Royal Construction and its chief competitor, Chicago Remodeling, determine their bids for building projects, they have two options. They can emphasize cost or emphasize quality. If they emphasize cost, they use less expensive materials and make lower bids. When they emphasize quality, they use higher quality materials. However, this strategy also raises the amounts of the bids.

Based on Royal's past records, the projects are awarded as shown in the following table.

Strategy	Percentage of Projects Awarded to Royal
Both companies emphasize cost	60%
Royal emphasizes cost, Chicago emphasizes quality	40%
Royal emphasizes quality, Chicago emphasizes cost	45%
Both emphasize quality	55%

a. **1.** Construct a payoff matrix for Royal Construction.

2. Determine if Royal should always emphasize cost, always emphasize quality, or use a mixed strategy.

b. Write an expression to determine the expected value for Royal if the Chicago company always emphasizes cost.

c. Write the expression that will determine the expected value for Royal if Chicago Remodeling always emphasizes quality.

d. Determine the optimal strategy for Royal.

e. If Royal uses this optimal strategy, what percentage of the building projects should the company expect to win?

2.4 Consider a zero-sum game defined by the following payoff matrix:

$$\text{Row player}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Column player}}{\begin{array}{cc} C_1 & C_2 \\ \multicolumn{2}{c}{\begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix}} \end{array}}$$

a. Complete a table like the one below for each of the strategies described in Steps **1–3**.

Row Player	Column Player	Payoff	Probability	Expected Payoff
R_1	C_1			
R_1	C_2			
R_2	C_1			
R_2	C_2			
			Total	

1. The row player chooses R_1 40% of the time and the column player chooses C_1 30% of the time.
2. The row player chooses R_1 70% of the time and the column player chooses C_1 50% of the time.
3. The row player uses the optimal strategy for the row player and the column player uses the optimal strategy for the column player.

b. Compare the three total expected payoffs obtained in Part **a**. What are the advantages and disadvantages of the row player using the optimal strategy?

2.5 Consider the following payoff matrix for a game.

$$\text{Player A}\quad \begin{array}{c} \\ R_1 \\ R_2 \end{array} \overset{\text{Player B}}{\begin{array}{cc} C_1 & C_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 5 & 4 \\ -3 & 7 \end{bmatrix}} \end{array}}$$

a. Is this game strictly determined? Explain your response.

b. If player A would like to use a pure strategy, what should this strategy be?

c. Using the pure strategy from Part **b**, what is the expected value of the game for player A?

d. If player A uses a mixed strategy, player A will not win every game. Use expected values to explain why it is still advantageous for player A to use a mixed strategy.

2.6 One row in a payoff matrix **dominates** another row if each of its entries is greater than the corresponding entries in the other row. When making a choice between the two rows, the row player should always select the dominant row in order to maximize winnings.

Similarly, one column **dominates** another column if each of its entries is less than the corresponding entries in the other column. When choosing between the two columns, the column player should always select the dominant column in order to maximize winnings.

Finding dominant rows and columns allows you to identify the rows and columns that should never be chosen. Once these are identified, they can be eliminated to simplify the payoff matrix. Remove any such rows and columns from the following payoff matrices. Using the simplified matrices, find the optimal strategies for each player and the value of the game.

a.

$$\begin{bmatrix} -2 & 8 \\ 2 & -5 \\ 9 & 7 \end{bmatrix}$$

b.

$$\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 0 \end{bmatrix}$$

2.7 A university's decision to change the school's colors has upset some alumni. To urge the administration to reconsider, they have begun a petition drive. Meanwhile, a second group has organized to promote the change. Both sides are campaigning by sending out mailings, making telephone calls, and visiting graduates in their homes.

A consulting firm has estimated the number of signatures that the group opposed to the change can expect to collect with each combination of strategies. This information is shown in the matrix below.

$$\begin{array}{ll} & \text{Group for change} \\ & \begin{array}{ccc} \text{mail} & \text{phone} & \text{visit} \end{array} \\ \text{Group against change} \begin{array}{r} \text{mail} \\ \text{phone} \\ \text{visit} \end{array} & \begin{bmatrix} 300 & 125 & 200 \\ 700 & 500 & 400 \\ 900 & 200 & 700 \end{bmatrix} \end{array}$$

a. Use dominance to eliminate a row and a column from the matrix.

b. What are the best strategies for each group?

c. The university has stated that if the opposing group can obtain 500 signatures against the change, it will not change the school colors. Will the university change its colors? Explain your response.

* * * * *

2.8 The doctors at a local hospital are faced with an epidemic of throat infections. Because the results of their laboratory tests are inconclusive, they cannot determine which of two bacterial strains is responsible. Two medicines are available to treat the infections. Medicine 1 is 85% effective against bacterial strain 1 and 70% effective against strain 2. Medicine 2 is 60% effective against strain 1 and 95% effective against strain 2. These medicines cannot be used in combination on any single patient.

- **a.** Use game theory to determine which medicine doctors should use.
- **b.** How effective can doctors expect this treatment to be?

2.9

- **a.** Using the information given in Problem **2.3**, construct a payoff matrix for Chicago Remodeling.
- **b.** Write the expression that will determine the expected value for Chicago if Royal always emphasizes cost.
- **c.** Write the expression that will determine the expected value for Chicago if Royal always emphasizes quality.
- **d.** Determine the optimal strategy for Chicago Remodeling.
- **e.**
  - **1.** If Chicago Remodeling uses this optimal strategy, what percentage of the building projects should the company expect to win?
  - **2.** How does this percentage compare to the percentage you obtained in Problem **2.3e**?

* * * * * * * * * *

Summary Assessment

1. In baseball, the contest between batter and pitcher can be thought of as a two-person game. Consider a situation in which the pitcher can throw either a fastball or a curve ball. The batter's chances of getting a hit depend not only on the type of pitch thrown, but on whether the batter is anticipating that type of pitch.

This situation is represented in the following matrix, where F represents fastball and C represents curve ball. If the pitcher throws a fastball and the batter guesses fastball, the batter's probability of getting a hit is 0.315.

$$\begin{array}{ccc} & & \text{Pitcher} \\ & & \begin{array}{cc} \text{F} & \text{C} \end{array} \\ \text{Batter} & \begin{array}{c} \text{F} \\ \text{C} \end{array} & \begin{bmatrix} 0.315 & 0.250 \\ 0.100 & 0.565 \end{bmatrix} \end{array}$$

a. Should the players use pure strategies or mixed strategies in this situation?

b. What is the optimal strategy for the batter?

c. Using the strategy described in Part **b**, what is the expected outcome for the batter?

2. Imagine that you are a doctor. You have two medicines available to treat a bacterium with two known strains. One medicine is 55% effective against the first strain and 45% effective against the second. The other medicine is 40% effective against the first strain and 60% effective against the second. Decide which medicine you should use and what kind of results you can expect if no one patient can receive both medicines.

3. The Cougars are playing the Falcons for the conference basketball championship. The coach of the Cougars believes that if her team can control the Falcons' leading scorer, Rhonda Allen, then they have a good chance of winning the game. According to the scouting report, Rhonda has three basic moves: she drives to her left, she drives to her right, or she pulls up and shoots a jump shot.

After comparing Rhonda's strengths with the team's defensive skills, the coach constructs the following matrix for Rhonda's probability of scoring a basket.

In this matrix, L represents either a drive to the left for Rhonda or defense to the left, R represents a drive right or defense to the right, and J represents a jump shot or defense against the jump shot.

$$
\begin{array}{ccc}
 & & \text{Defender} \\
 & & \begin{array}{ccc} \text{L} & \text{R} & \text{J} \end{array} \\
\text{Rhonda} & \begin{array}{c} \text{L} \\ \text{R} \\ \text{J} \end{array} & \begin{bmatrix} 10\% & 60\% & 35\% \\ 85\% & 20\% & 70\% \\ 50\% & 60\% & 40\% \end{bmatrix}
\end{array}
$$

a. Which move will Rhonda be least likely to choose? Explain your response.

b. Considering your response to Part **a**, which defense will the Cougars use least? Explain your response.

c. Delete the column and row indicated by your responses to Parts **a** and **b**. Using the remaining 2×2 matrix, determine the optimal strategy for Rhonda.

d. How should the Cougars defend Rhonda?

e. On average, how often will Rhonda score a basket?

Module Summary

- **Game theory** is a branch of mathematics used to analyze decision making in situations that involve conflicting interests.
- An **optimal strategy** for a player results in maximizing the winnings or minimizing the losses for that player.
- In a **pure strategy**, a player makes the same choice each time the game is played.
- In a **mixed strategy**, a player's choice of how to play varies from game to game.
- A **payoff** is the amount won or lost by a single player in one play of a game.
- The consequences of each player's choices can be summarized in a **payoff matrix**. The rows of the matrix represent the choices of one player (the row player), while the columns represent the choices of the second player (the column player).
- In a **strictly determined** game, the optimal strategy for each player is a pure strategy.
- The **value** of a strictly determined game can be determined from the payoff matrix. It is less than or equal to all entries in its row and greater than or equal to all entries in its column. This value is the **saddle point** of the matrix.
- A strictly determined game is fair if the saddle point is 0.
- In a game that is not strictly determined, the optimal strategies are mixed strategies. When the row player uses an optimal strategy, the expected value of the game for that player is the same, regardless of the choice made by the column player.
- One row in a payoff matrix **dominates** another row if each of its entries is greater than the corresponding entries in the other row. When making a choice between the two rows, the row player should always select the dominant row in order to maximize winnings.

  Similarly, one column **dominates** another column if each of its entries is less than the corresponding entries in the other column. When choosing between the two columns, the column player should always select the dominant column in order to maximize winnings.

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