Catch a Wave

Catchin' a wave and listening to tunes—what a glorious summer day. In this module, you explore the mathematics of waves and musical notes using sine and cosine functions.

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Catch a Wave

Introduction

Some events in everyday life occur again and again over time: the phases of the moon, the crash of ocean waves on the beach, the seasons and their temperatures. Other occurrences also display periodic behavior, such as the vibrations of a guitar string or the motion of a ride at an amusement park.

Discussion

a. Cassandra is about to bungee jump off a 40-m platform. The bungee cord is 18 m long. Describe a graph of her height above the ground versus time from the moment she jumps until she stops bouncing.

b. Although Hannah is afraid of heights, her friends have talked her into riding the Ferris wheel. Describe a graph of Hannah's height above the ground versus time as the Ferris wheel makes three complete revolutions.

c. Compare the graphs you described in Parts **a** and **b**.

Activity 1

As the Ferris wheel turns, Hannah's height above the ground changes. A graph of her height above the ground versus time is **periodic**. In other words, the various values for height repeat over time. In this module, you investigate functions that can model periodic events.

Mathematics Note

A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest constant interval of the domain over which the function repeats. For example, Figure **1** shows a graph of a function which repeats itself every 3 units. In this case, $f(x+3)=f(x)$. The period of this function is 3.

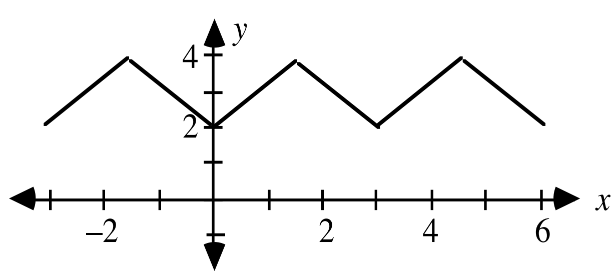

Figure 1: A periodic function

Exploration

Time is measured on a linear scale. When determining the height above ground of a point on a Ferris wheel, however, it can be helpful to associate each unit of time with a point on a circle. In this exploration, you simulate “wrapping” a linear scale onto a circle.

a.

1. Use a can to trace a circle on a sheet of paper.
2. Identify the center of the circle.
3. As shown in Figure **2**, create a coordinate system with its origin at the center of the circle. Let the radius of the circle represent 1 unit. A circle with a radius of 1 unit is a **unit circle**.

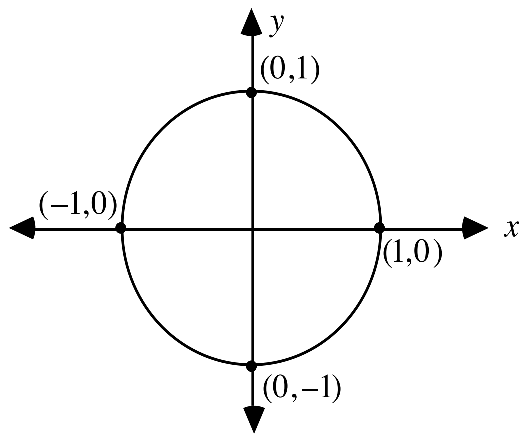

Figure 2: Coordinate system and unit circle

4. Using a paper strip slightly longer than the circumference of the can, create a number line on which 1 unit equals the radius of the circle.
5. Measuring as accurately as possible, label the locations of the points on the number line that correspond to the following real numbers: 0, 1, $\pi/2$, 2, 3, π, 4, $3\pi/2$, 5, 6, 2π, and 7.
6. Tape the beginning of your number line to the lower portion of the can, as shown in Figure **3**.

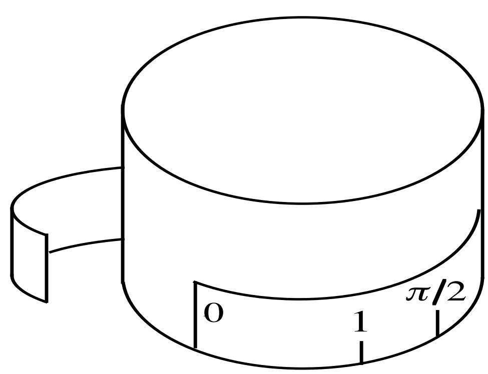

Figure 3: Placement of number line on can

b. Position the can on your drawing of the unit circle from Part **a** so that the 0 on the number line is located at the point (1,0). Use the number line to label every point on the circle that corresponds to a labeled point on the number line. If any point on the circle corresponds to more than one point on the number line, mark it with all corresponding labels.

c. Using your unit circle, find the approximate number of radii that correspond with each of the following lengths:

1. the circumference of the circle
2. three-fourths of the circumference of the circle
3. one-half of the circumference of the circle
4. one-fourth of the circumference of the circle.

d. Approximate the ordered pair that corresponds to each labeled point on the unit circle. Record these values in a table similar to Table **1** below. **Note:** Save this table for use in Activity **2**.

Table 1: Number-line values and ordered pairs

Number	*x*-coordinate	*y*-coordinate
0	1.0	0.0
1	0.5	0.8
$\pi/2$	0.0	1.0
$\vdots$	$\vdots$	$\vdots$

e. Draw a ray from the origin through the point that corresponds to 1 unit on the circumference of your unit circle. The angle formed by the non-negative portion of the x-axis and this ray has a measure of 1 **radian**. Find the approximate measure of this angle in degrees. Save your unit circle for use with Problem **1.6**.

Discussion

a.
1. Compare the number of radii that fit around the circumference of your circle with the values obtained by others in your class.
2. How does your answer compare to what you already know about the circumference of a circle?

b. Express each of the following lengths in terms of π:

1. the circumference of your circle
2. one-half the circumference of your circle
3. one-fourth the circumference of your circle

c. In the exploration, you matched each labeled point on your number line with a point on a unit circle.

1. Compare the coordinates of each labeled point on your unit circle with those of others in your class.
2. Describe how you could represent negative numbers from the number line on the unit circle.
3. The point $\pi/2$ on the number line is paired with the point (0,1) on the unit circle. If the number line was continued indefinitely, what other numbers would be paired with (0,1)?
4. Given a point on the unit circle, how many real numbers can be paired with its location?

d. In Figure **4**, arc *AB* has a length of 1 unit, and the two circles have the same center. The smaller circle has a radius of 1 unit and the larger circle has a radius of 2 units.

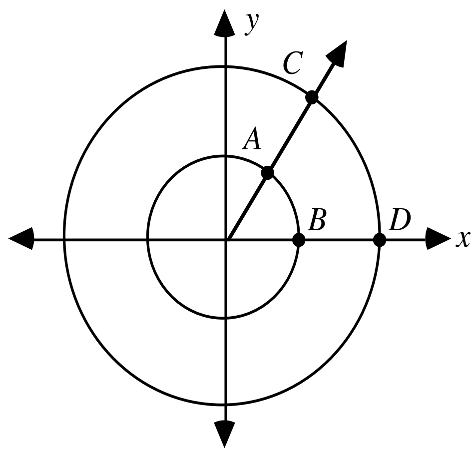

Figure 4: Two concentric circles

1. What is the length of arc *CD*?
2. What appears to be the relationship between the radius of a circle the length of an arc of that circle?

Assignment

1.1 Consider a Ferris wheel with a radius of 12 m.

a. Describe the circumference of the Ferris wheel in terms of its radius.

b. What is the approximate distance around the wheel in meters?

1.2 One revolution of a seat around a Ferris wheel sweeps out an arc with a measure of 2π radians. What radian measure corresponds to each of the following numbers of revolutions?

a. 2

b. 3

c. 8

1.3 Consider a circle with a radius of 12 m. If an arc of the circle has a measure of 4 radians, what is its length in meters?

Mathematics Note

The set of real numbers can be associated with the points on a unit circle by placing a number line so that 0 on the number line is tangent to the circle at the point (1,0), as shown in Figure **5a**. When the positive portion of the number line is wrapped around the unit circle in a counterclockwise direction, as illustrated in Figure **5b**, each positive number is paired with exactly one point, (a,b), on the circle. Positive numbers that differ by 2π units are paired with the same point on the circle.

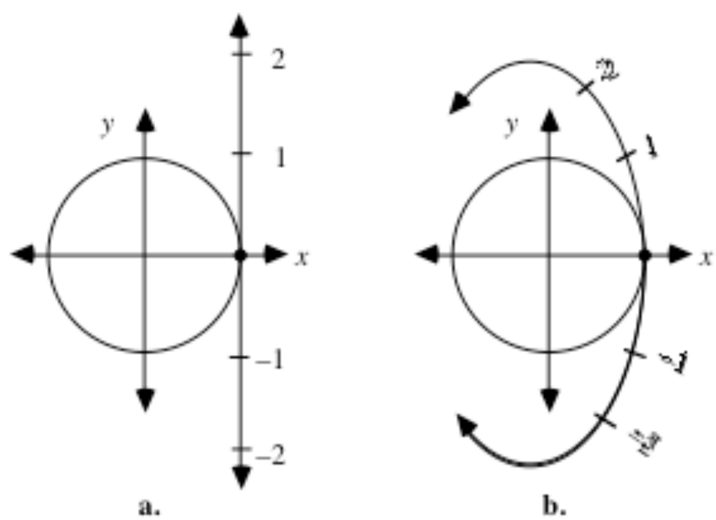

Figure 5: Wrapping function

There is a similar correspondence between the negative real numbers and points on the unit circle, found by wrapping the negative portion of the number line in a clockwise direction. Negative numbers that differ by 2π units are paired with the same point on the circle.

1.4 Consider a wrapping function that pairs each point on the real number line with a location on a unit circle with center at the origin. In this function, 0 on the number line corresponds with the point (1,0).

a. Identify a real number that corresponds with each of the following points on the unit circle:

1. $(-1,0)$

2. $(0,-1)$

3. $(0,1)$

b. Determine the coordinates of the points on the unit circle that are paired with the following real numbers:

1. π

2. $-3\pi/2$

3. $7\pi/2$

4. $-15\pi/2$

5. -11π

1.5 Consider the point on a unit circle with center at the origin that is paired with the real number 3.

a. Identify another positive real number paired with that point.

b. Identify a negative real number paired with that point.

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1.6 Use your circle from the exploration to model a Ferris wheel and your number line to model a string of 16 evenly spaced lights placed around the outer edge of the wheel. Mark the position representing the first light at the point corresponding to 0 on the number line. Mark the locations for the remaining 15 lights so that the lights are evenly spaced around the wheel.

a. What is the length of the arc between two consecutive lights?

b. What is the measure of the arc determined by two consecutive lights?

1.7 Consider a sector of a circle whose central angle measures $5\pi/4$ radians. If the circle has a radius of 10 units, what is the area of the sector?

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Activity 2

Shortly after boarding the Ferris wheel, Hannah reaches the top. A few seconds later, she is at the bottom again. The wheel then begins to carry Hannah back to the top. Hannah's motion on the Ferris wheel is periodic. The process you used in Activity **1** to pair real numbers with points on a circle also is periodic. Many functions pair their domains with their ranges in a similar fashion.

Exploration

In previous modules, you examined the trigonometric ratios *sine* and *cosine* in terms of the sides of a right triangle. In this exploration, you investigate the relationship between these ratios and the **sine** and **cosine functions**.

Recall that a circle is the set of all points in a plane that are the same distance from a given point in the plane. Figure **6** shows a circle with radius r centered at the origin of the xy-plane. The right triangle shows the relationship between the x- and y-coordinates of a point on the circle and the radius of the circle.

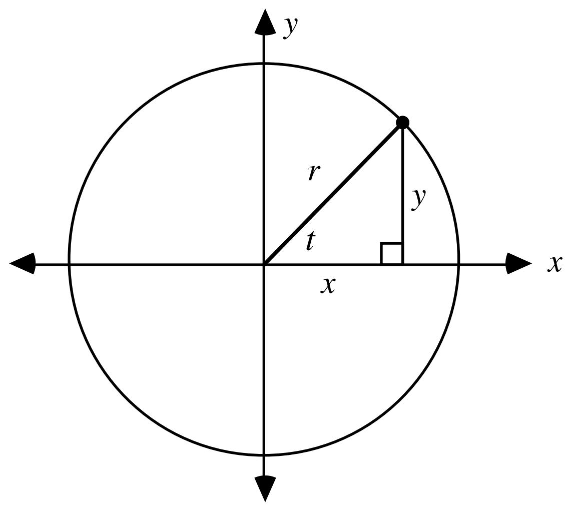


Figure 6: A circle with radius *r*

a. Use the circle in Figure **6** to complete Steps **1–4** below.

1. Write an equation for the sine of the angle t in terms of y and r.

2. Solve this equation for y.

3. Write an equation for the cosine of the angle t in terms of x and r.

4. Solve this equation for x.

b.

1. Use a geometry utility to create a unit circle with its center C located at the origin of a two-dimensional coordinate system. Label the intersection of the positive x-axis and the circle as point D.
2. Construct a point A on the circle so that it moves freely on the circle.
3. From A, construct a segment perpendicular to the x-axis. Label the intersection of the perpendicular and the x-axis as point B.
4. Form right triangle ABC by constructing segments CB, AB, and CA. Your construction should now resemble Figure **7**.

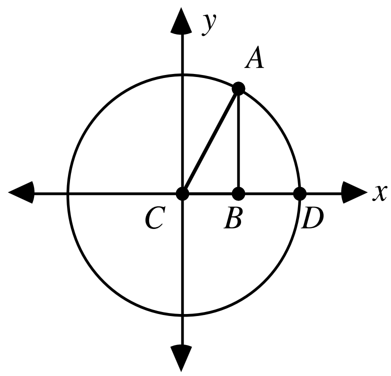


Figure 7: Construction of a triangle in a unit circle

c.

1. Move point A to a location in the first quadrant.
2. Determine the coordinates (x,y) of point A and record them in a table with headings like those in Table **2** below.

Table 2: Measurements on a unit circle for point $A(x,y)$

x	y	$m\angle DCA$	$\sin\angle DCA$	$\cos\angle DCA$

3. Determine the measure of $\angle DCA$ in radians and record it in the table.
4. Use this radian measure, along with the sine and cosine keys on your calculator, to determine the sine and cosine of $\angle DCA$. Compare the sine and cosine with the coordinates of point A.
5. Repeat Steps **1–4** for several more locations of point A in the first quadrant.

d. Make a conjecture about the coordinates of A in terms of $\sin\angle DCA$ and $\cos\angle DCA$.

e. For each location of point A that you used in Part **c**, record the coordinates of A', the reflection of A in the y-axis.

1. Determine the measure of $\angle DCA'$ in radians and record it in the table.
2. Use this radian measure, along with the sine and cosine keys on your calculator, to determine the sine and cosine of $\angle DCA'$. Compare the sine and cosine with the coordinates of point A'.

f. Repeat Part **e** for A'', the reflection of A in the x-axis.

g. Repeat Part **e** for A''', the reflection of A'' in the y-axis.

Mathematics Note

The pairing of each real number with exactly one point on the unit circle is the basis for the **circular functions**. Two of the circular functions, **sine** and **cosine**, can be defined in the following manner:

For any real number x, where x is paired with the point (a,b) on a unit circle with center at the origin, $\sin x = b$ and $\cos x = a$.

For example, the point on the unit circle paired with $\pi/3$ is $\left(1/2, \sqrt{3}/2\right)$. Therefore, $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$.

h.

1. Graph $y = \sin x$ for the domain $-2\pi \le x \le 2\pi$.
2. Describe your graph.
3. Graph $y = 0.5$ on the same coordinate system and approximate the value(s) of x for which $\sin x = 0.5$ over the domain $-2\pi \le x \le 2\pi$.
4. What is the range of the values for the sine function?
5. What is the period of the sine function?

Discussion

a. Describe the relationship you observed between the coordinates of a point in the first quadrant and the sine and cosine of $\angle DCA$. Why does this relationship occur?

b. Describe how to find the coordinates of a point on the unit circle in any quadrant.

c. Are your results from Parts **c–g** of the exploration consistent with the values you recorded in Table **1** in Activity **1**?

d. Explain how the range of the sine function is related to the unit circle.

e. Explain how the shape of the graph of the sine function is related to the unit circle.

f. How many solutions are there to the equation $\sin x = 0.5$ over the interval $-2\pi \le x \le 2\pi$? over the real numbers?

g. How many solutions are there to the equation $\sin x = 2$ over the real numbers?

h.
1. Describe the conditions necessary for the cosine of a number to be positive.
2. Describe the conditions necessary for the sine of a number to be positive.

i. The **tangent function** is equivalent to the ratio of the sine function to the cosine function:

$$\tan x = \frac{\sin x}{\cos x}$$

1. Given this fact, how must the domain of the tangent function be restricted?
2. Describe the conditions necessary for the tangent of a number to be positive.

Assignment

2.1 Use a unit circle to determine the sine, cosine, and tangent of each of the following real numbers:

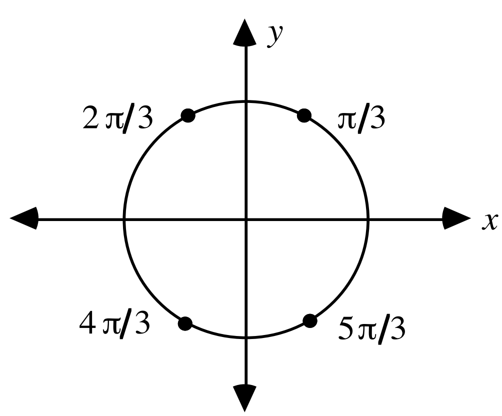
a. 0

b. $\pi/2$

c. π

d. $3\pi/2$

2.2 Consider a real number line wrapped around a unit circle as in Activity **1**. The following diagram shows the locations of points that correspond to four real numbers.



a. Find the x- and y-coordinates of each of the four points.

b. Describe how the coordinates that correspond to the four real numbers are related.

c. Describe how the points that correspond to the following real numbers are related to the coordinates in Part **a**: $7\pi/3$, $8\pi/3$, $10\pi/3$, and $11\pi/3$.

d. Determine four negative real numbers whose corresponding coordinates are the same as the four pairs of coordinates from Part **a**.

2.3 **a.** Graph $y = \cos x$ for $-2\pi \le x \le 2\pi$.

b. Describe your graph.

c. Graph $y = 0.5$ on the same coordinate system and determine the value(s) of x for which $\cos x = 0.5$ over the interval $-2\pi \le x \le 2\pi$.

d. What is the period of the cosine function?

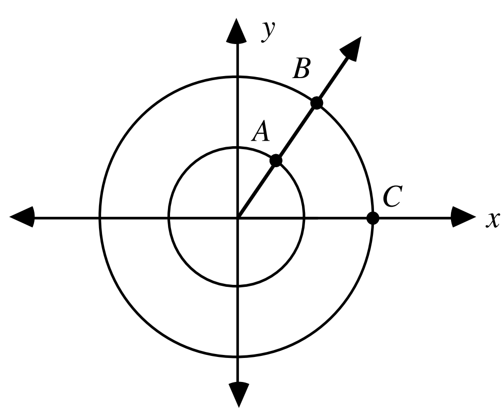
2.4 **a.** Graph $y = \tan x$ for $-2\pi \le x \le 2\pi$.

b. Describe your graph and explain why the graph has these properties.

c. Graph $y = 5$ on the same coordinate system and determine the value(s) of x for which $\tan x = 5$ over the interval $-2\pi \le x \le 2\pi$.

d. Is the tangent function periodic? If it is periodic, what is its period?

2.5 The two circles shown in the diagram below have the same center. The smaller circle has a radius of 1 and the larger circle has a radius of 2. The measure of arc BC is $\pi/3$ radians.



a. Determine the coordinates of point A.

b. Use the coordinates of point A and the properties of similar triangles to determine the coordinates of point B.

c. Describe the coordinates of the point where ray AB intersects a circle with radius r.

2.6 The Ferris wheel that Hannah is riding on has a radius of 12 m. Consider a coordinate system with its origin at the center of the Ferris wheel. The point at which Hannah begins her ride has the coordinates (0,–12).

a. If the wheel completes one revolution every 2.5 min, through how many radians does the wheel rotate in 1 sec?

b. After how many seconds will Hannah's distance above the ground be the greatest?

c. What is the length of the arc that Hannah moves through in 1 sec?

d. What are the coordinates of the point where Hannah will be 1 sec after she begins her ride?

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2.7 A watchmaker would like to design the face of a new watch. The watch face is a circle with a radius of 1 cm. The watchmaker locates the center of the face at the origin of a two-dimensional coordinate system and 3 o'clock at the point (1,0).

a. Determine the coordinates of the points where the hours 6, 9, and 12 are located.

b. **1.** Determine the coordinates of the point where 11 o'clock is located.

2. Identify the hours for which the absolute values of the coordinates equal the absolute values of the coordinates for 11 o'clock.

c. The hour hand is currently pointing at 12 o'clock. Identify the coordinates of the location that the hour hand will be pointing to after 33 hr.

2.8 **a.** Graph $y = \cos x$ and $y = \sin x$ for $-2\pi \le x \le 2\pi$ on the same coordinate system.

b. Determine the values of x in this interval for which $\cos x = \sin x$.

2.9 **a.** On the same coordinate system, graph the following two equations for $-2\pi \le x \le 2\pi$:

$$y = \cos x \text{ and } y = \sin\left(x + \frac{\pi}{2}\right)$$

b. Determine the values of x in this interval for which

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

2.10 **a.** On the same coordinate system, graph the following two equations for $-2\pi \le x \le 2\pi$: $y = 2(\sin x)\cos x$ and $y = \sin(2x)$.

b. Determine the values of x in this interval for which $2(\sin x)\cos x = \sin(2x)$.

2.11 Describe the graph of $y = \sin^2 x + \cos^2 x$ for $-2\pi \le x \le 2\pi$. What does this graph tell you about the sum of $\sin^2 x$ and $\cos^2 x$?

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Activity 3

When any sound is produced, it disturbs nearby air molecules, creating waves of high and low pressure. These variations in pressure over time can be recorded with a microphone and modeled by a sine curve, as shown in Figure **8**.

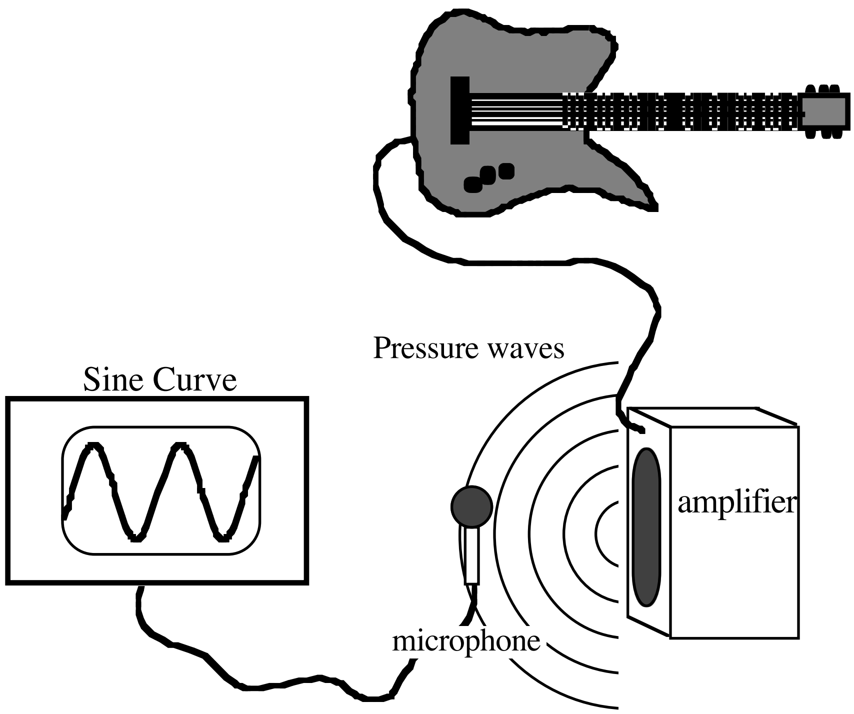


Figure 8: Sine curve modeling pressure waves from a speaker

In this activity, you examine two characteristics of sound waves: amplitude and frequency.

Exploration

a. Graph $y = \sin x$ over the interval $-2\pi \le x \le 4\pi$.

b. Graph $y = r\sin x$ over the domain in Part **a** for several different values of *r*. Describe how changing the value of *r* changes the maximum and minimum *y*-values of the graph.

Mathematics Note

If a periodic function has a maximum M and a minimum m, its **amplitude** is defined as:

$$\frac{M - m}{2}$$

For example, Figure **9** shows a graph of the function $y = 6\sin x$.

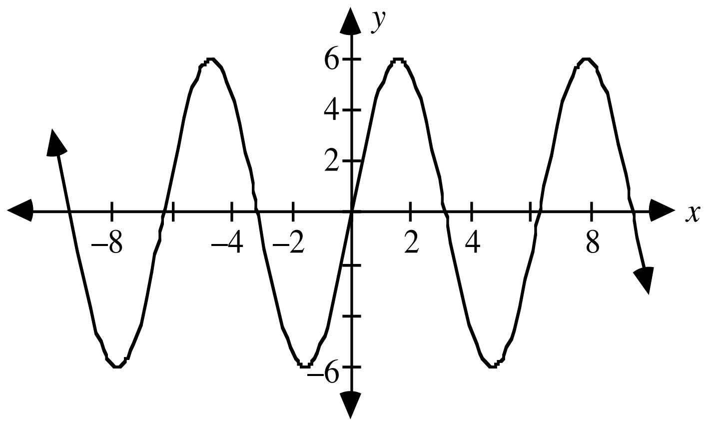


Figure 9: Sine curve with amplitude 6

Since the maximum is 6 and the minimum is –6, the amplitude can be found as follows:

$$\begin{aligned}\frac{M - m}{2} &= \frac{6 - (-6)}{2} \\ &= 6\end{aligned}$$

A **cycle** is the portion of a periodic function included in one period. The **frequency** of a periodic function is the number of cycles per unit on the x-axis. The frequency is the reciprocal of the period, as shown below:

$$\text{frequency} = \frac{1}{\text{period}}$$

For example, the sine function in Figure **9** has a period of 2π. In other words, it completes 1 cycle every 2π units. Its frequency is $1/(2\pi)$.

c. The frequency of sound waves is measured in **hertz (Hz)**, where 1 Hz represents 1 cycle per second. If x represents time in seconds, what is the frequency of the graph of $y = \sin x$?

d. Use a graphing utility to complete Table **3**.

Table 2: Cycles, periods, and frequencies

Function	Period	Frequency
$y = \sin x$	2π	$1/(2\pi)$
$y = \sin(2x)$		
$y = \sin(4x)$		
$y = \sin(\pi x)$		
$y = \sin(x/3)$		
$y = \sin(bx)$		
$y = \sin(2\pi x)$		
$y = \cos x$		
$y = \tan x$		

Discussion

a. Compare the graphs of $y = 8\sin x$ and $y = \sin x$.

b. How does changing b in $y = \sin(bx)$ affect the graph of the function?

c.
1. If x represents time in seconds, what function would create a cosine graph that has 50 cycles in 2π sec?
2. What function would create a sine graph that has 12 cycles in π sec?

d.
1. If x represents time in seconds, what function would create a sine graph that has an amplitude of 4 and a frequency of 1 Hz?
2. What function would create a cosine graph with an amplitude of 0.5 and a frequency of 10 Hz?

Assignment

3.1 Write a function that describes the graph of each of the following:

a. a sine curve with an amplitude of 8

b. a cosine curve with an amplitude of 0.4.

3.2 If x represents time in seconds, write a function that describes the graph of a cosine curve with each of the following frequencies:

a. 15 Hz

b. 25 Hz

c. 0.5 Hz

3.3 Write a function that describes the graph of a cosine curve with each of the following periods:

a. 15 units

b. 25 units.

3.4 The human ear can detect sounds with frequencies between 20 Hz and 20,000 Hz.

a. Write a function that describes the graph of a sound wave with each of the following frequencies:

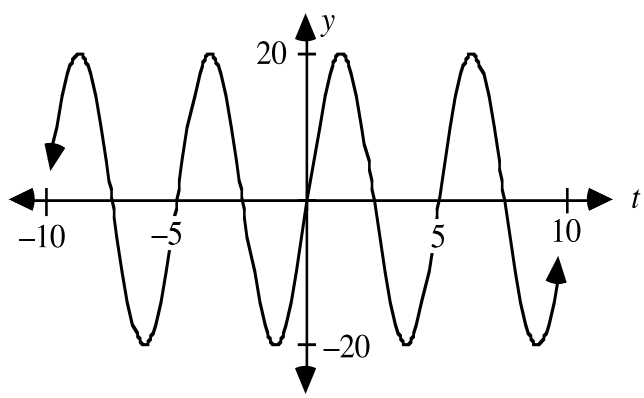
1. 20 Hz

2. 20,000 Hz

b. How would the graphs of all other audible sounds compare to the graphs from Part **a**?

3.5 Write the function that describes the graph of a sine curve with a period of 6 and an amplitude of 7. Graph the function for $0 \le x \le 12$.

3.6 Consider the graph below, where y represents centimeters and t represents seconds.



a. What is the amplitude of the graph?

b. What is the period?

c. What is the frequency?

d. Write an equation that describes this curve.

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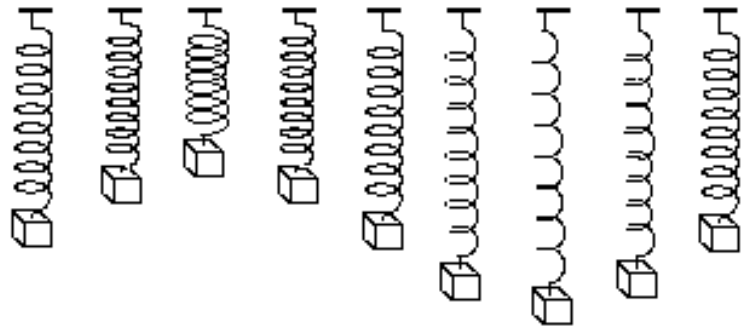
3.7 The diagram below shows the positions of a weight attached to the end of a spring at regular time intervals. The distance between the lowest point and the highest point reached by the weight is 20 cm. It takes the weight 0.4 sec to travel this distance.



a. As the weight bounces up and down, its displacement over time can be modeled with a sine function. Assume that at $t = 0$, the weight is halfway between its highest and lowest positions. If this position corresponds with a displacement of 0 cm, write an equation that models displacement with respect to time.

b. Graph the equation from Part **a** over a domain of 2 sec.

c. Use your equation to describe the position of the weight at each of the following times:

1. 0.1 sec
2. 0.5 sec
3. 0.95 sec
4. 1.75 sec.

d. Indicate the points on the graph from Part **b** that correspond to each of the times in Part **c**.

e. Identify the times when the weight is located 10 cm below its position at $t = 0$.

3.8 Graph $y = \cos x$ and $y = \cos(x + k)$ on the same set of axes for several different values of k. Describe the effect that k has on the graph of $y = \cos x$.

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Activity 4

The sounds involved in typical song are made up of millions of pressure waves, each with its own frequency and amplitude. In this activity, you create your own music and model it with sine curves.

Science Note

Each musical note has its own frequency. For example, the note A has a frequency of 440 Hz (440 cycles/sec). Figure **11** shows the changes in pressure produced by the note A over time.

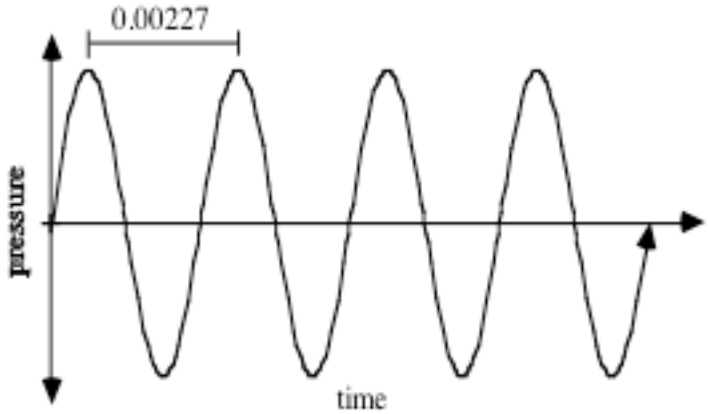


Figure 11: Sine curve of note A

The length of one period is approximately 0.00227 sec. Therefore, the frequency can be determined as follows:

$$\text{frequency} = \frac{1}{\text{period}} \approx \frac{1}{0.00227} \approx 440 \text{ Hz}$$

Table **4** shows some musical notes and their corresponding frequencies. Notice how the frequency of a note doubles when the next octave is reached. The notes on a piano range from low A (27.5 Hz) to high C (4186 Hz).

Table 4: Notes and their frequencies

Note	Frequency (Hz)
C	262
$C^{\#}$ (C-sharp)	277
D	294
$D^{\#}$	311
E	330
F	349
$F^{\#}$	370
G	392
$G^{\#}$	415
A	440
$A^{\#}$	446
B	494
C (next octave)	524
$C^{\#}$	555
D	588

Exploration

In this exploration, you produce sounds by blowing across the top of a bottle partially filled with water, then model the notes using a graphing utility.

a. Obtain a bottle from your teacher. Fill the bottle about half full of water.

b. Blow across the top of the bottle to produce a whistling sound.

c. Use technology to record the pressure waves produced by this sound.

d. Determine the period, frequency, and amplitude of your graph.

e. Find a function that models the curve formed by the sound.

f. Remove some water from the bottle and repeat Parts **a–e**.

Discussion

a. Describe how you determined the period, frequency, and amplitude in Part **d** of the exploration.

b. Using Table **4**, determine which musical note is closest to the first sound you produced with the bottle.

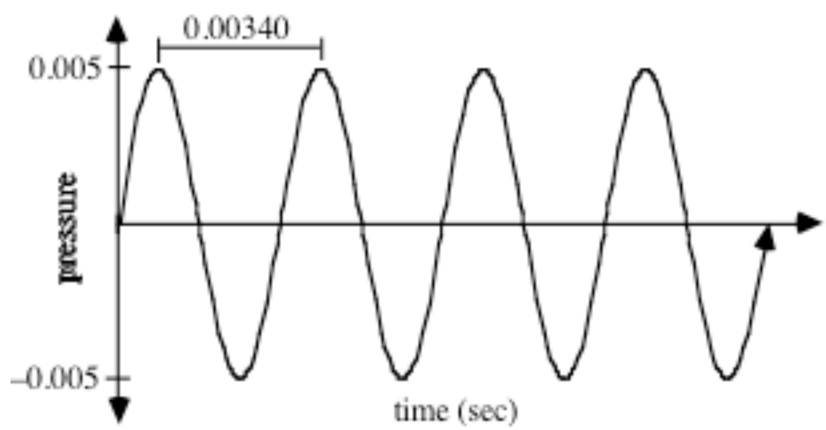
c. Describe the function that models this sound.

d. Compare the function for your note with those of others in the class.

e. How could you create a note with a higher frequency?

Assignment

4.1 The following graph shows the pressure variations produced by a musical note.



a. Determine the frequency of the graph.

b. Determine a function that models the curve.

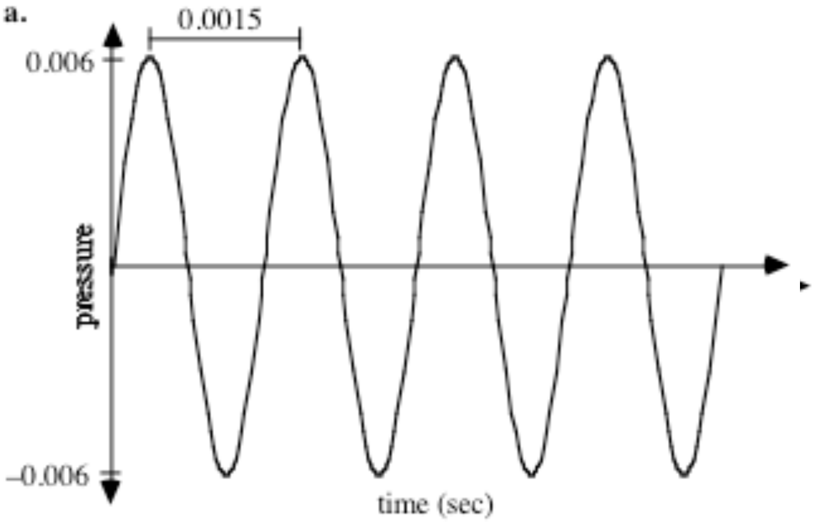
c. Use Table **4** to identify the note.

4.2 Describe the similarities and differences between the graphs of middle C, with a frequency of 262 Hz, and the note C one octave higher.

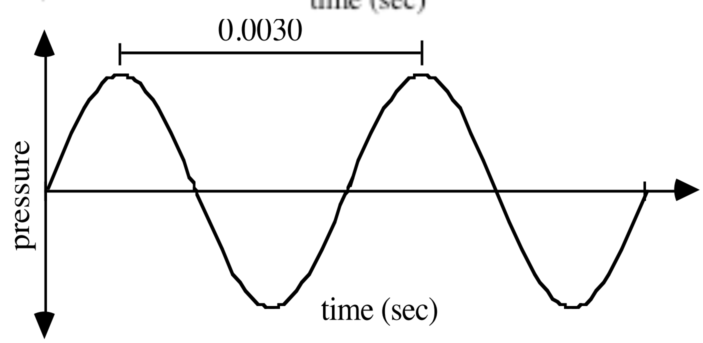
4.3 Write a function that could describe a graph of the note $F^{\#}$.

4.4 For each graph in Parts **a** and **b** below, identify the amplitude, period, frequency, and the name of the corresponding musical note.

a.

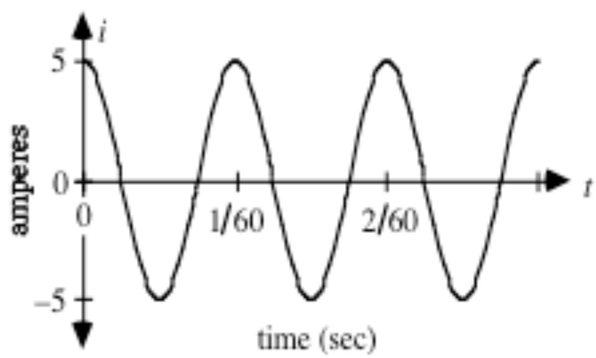


b.



* * * * *

4.5 The electrical power supplied to most households is called "alternating current." The magnitude of alternating current varies over time and can be modeled by a sine or cosine function, as shown in the following graph. The standard unit of current is the **ampere** (amp).



a. Describe the amplitude, period, and frequency of the graph.

b. Write an equation to model the variation in current shown in the graph.

c. Use the equation from Part **b** to determine the current, in amperes, at each of the following times:

1. 0.01 sec

2. 0.15 sec.

d. Identify the times when the current is:

1. 0 amp

2. –5 amp.

* * * * * * * * * *

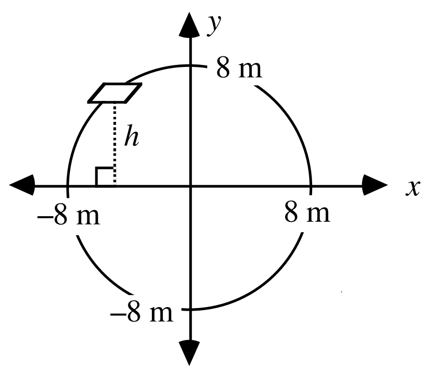
Research Project

Musicians often play musical notes in combinations called **chords**. One chord consists of the notes C, E, and G played simultaneously. Use technology to collect sound-wave data when this chord is played on a musical instrument such as a guitar or piano. Also collect data for the individual notes played on the same instrument.

Report on how the graphs of individual notes compare to that of the chord and describe how the combination of notes affects the graph of the sound.

Summary Assessment

1. Imagine that you are riding the Ferris wheel at a fair. This Ferris wheel has a radius of 8 m and moves at a constant rate of 2 revolutions per minute in a counterclockwise direction. The diagram below shows the path of a chair on the Ferris wheel, with the center of the wheel located at the origin of a two-dimensional coordinate system.



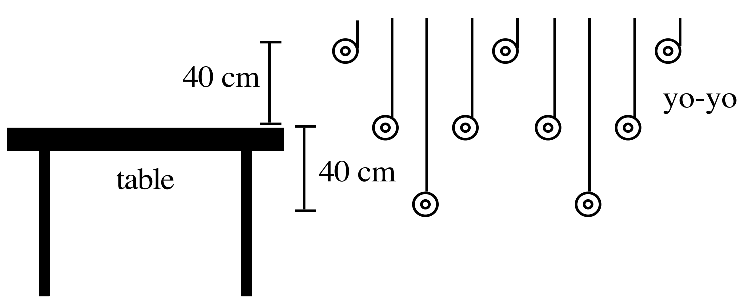
a. Assume that the coordinates of the chair's initial position are (8,0). As the chair travels around the wheel, determine the chair's vertical distance from the *x*-axis at 10 different instants in time.

b. Create a scatterplot of the data from Part **a**.

c. Write the equation of a sine curve that models the scatterplot and graph the curve on the same coordinate system as in Part **b**.

d. Identify the amplitude, period, and frequency of the graph from Part **c**.

2. The diagram below shows the positions of a yo-yo with respect to a table top at regular time intervals.



a. Describe how you could make the vertical distance from a yo-yo to a table top vary periodically with time.

b. Let the x-axis represent the level of the table top in the diagram. Assuming that it takes 2.4 sec for the yo-yo to complete all the cycles illustrated, find a periodic function that models the motion of this yo-yo and graph it over a domain of 0 to 3.6 sec.

c. Identify the amplitude, period, and frequency of the graph from Part **b**.

d. Use your model from Part **b** to describe the position of the yo-yo at each of the following times:

1. 1.5 sec
2. 1.9 sec
3. 3.4 sec.

Module Summary

- A **periodic function** is a function in which values repeat at constant intervals. The **period** is the smallest constant interval of the domain over which the function repeats.
- A **unit circle** is a circle with a radius of 1.
- On a unit circle, the measure of a central angle whose sides intercept an arc with a length of 1 unit is 1 **radian**.
- The set of real numbers can be associated with the points on a unit circle by placing a number line so that 0 on the number line is tangent to the circle at the point (1,0). When the positive portion of the number line is wrapped around the unit circle in a counterclockwise direction, each positive number is paired with exactly one point, (a,b), on the circle. Positive numbers that differ by 2π units are paired with the same point on the circle.

  There is a similar correspondence between the negative real numbers and points on the unit circle, found by wrapping the negative portion of the number line in a clockwise direction. Negative numbers that differ by 2π units are paired with the same point on the circle.
- The pairing of each real number with exactly one point on the unit circle is the basis for the **circular functions**.

  Two of the circular functions, **sine** and **cosine**, can be defined as follows: For any real number x, where x is paired with the point (a,b) on a unit circle with center at the origin, $\sin x = b$ and $\cos x = a$.
- The **tangent function** is equivalent to the ratio of the sine function to the cosine function:

$$\tan x = \frac{\sin x}{\cos x}$$

- If a periodic function has a maximum M and a minimum m, its **amplitude** is defined as:

$$\frac{M - m}{2}$$

- A **cycle** is the portion of a periodic function included in one period.
- The **frequency** of a periodic function is the number of cycles per unit on the x-axis. The frequency is the reciprocal of the period.
- The frequency of waves is typically measured in **hertz (Hz)**, where 1 Hz represents one cycle per second.

Selected References

Gross, E. E., Jr., and A. P. G. Peterson. *Handbook of Noise Measurement*. Concord, MA: General Radio Company, 1972.

Texas Instruments. *CBL™ System Experiment Workbook*. Dallas, TX: Texas Instruments Inc., 1994.