Sunken Sub

“This is the USS *Livingston*. We’ve lost ballast and are going down! Our coordinates are . . . ” A submarine is sinking and you’re in charge of the rescue. Can you save the crew?

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Introduction

Whenever a child is reported missing in the woods, a skier is buried in an avalanche, or a ship is lost at sea, rescue teams scramble to find them. But locating someone wandering in the wilderness or adrift on the ocean can be a daunting task. To increase the odds of a successful rescue, searchers use probability to devise search patterns.

In this module, you follow the search for the USS *Livingston*, a fictional submarine lost during a training cruise. First, you use probability to determine where the submarine might have gone down; then, you design a search strategy. Finally, you determine the chances of finding the *Livingston* in time to save the crew.

History Note

The U.S. Navy prides itself on the safety of its ships. But going below the waves in a submarine is an inherently risky business. As the depth of the water increases, the pressure on the ship's hull also increases. Despite the Navy's stringent safety precautions, two nuclear submarines have been lost at sea.

On April 10, 1963, the USS *Thresher* was on a test cruise off Cape Cod after a long period in dry dock for repairs. At shallow depths, the *Thresher* performed perfectly. The captain then took the ship and its 129-member crew deeper.

Later, a surface vessel monitoring the *Thresher*'s dive received the message "attempting to blow." This meant that the *Thresher* was blowing out its ballast tanks and coming to the surface. But why did the radio message say "attempting"? As no further contact was ever made with the *Thresher*, the answer to this question remains unknown. The wreckage of the submarine was found on the ocean floor, 2500 m below the surface.

Five years after the *Thresher* disaster, the USS *Scorpion* with a crew of 99 was returning home to Norfolk Naval Base in Virginia. On May 21, the *Scorpion* made routine radio contact with Norfolk, reporting its position as 75 km south of the Azores Islands in the mid-Atlantic. Two days later, however, the Scorpion had neither arrived in port nor reported another position. The wreckage was eventually pinpointed by an anti-submarine listening post located off the Azores.

Submarines have improved since the losses of the *Thresher* and the *Scorpion*. They now have stronger hulls and can withstand greater underwater pressures. To help retrieve the crew of a disabled submarine, the Navy has developed a Deep Submergence Rescue Vehicle.

Activity 1

The USS *Livingston* left its base on a training mission at 0800 hours. At 1100 hours, the submarine sent the following message: "This is the USS *Livingston*. We're going down! The crew is secure in the forward compartments. Our coordinates are . . .". Here the message was interrupted by the crackle of static. The radio operator immediately called the base commander.

The base commander ordered all officers to the control center. They began reviewing the facts and working on a rescue plan.

The base is located on the northern shore of an island in the Pacific Ocean. The island is roughly circular, with a radius of about 10 km. Within 50 km of the shoreline, the water is relatively shallow. Beyond 50 km, the ocean bottom drops off to a depth of 1000 m. During training missions, submarines can go no farther than 75 km from the base.

To attempt to locate the *Livingston*, the rescue team decided to send out submarines equipped with special sonar devices. A sonar technician informed the officers that if the submarine went down in shallow water and the entire region is searched, they have about a 70% chance of detecting it. If the *Livingston* went down in deep water, the rescuers have only about a 10% chance of detecting it.

Exploration

a.

1. Draw a sketch of the island, the submarine base, and the training area.

2. Determine the surface area of the region in which the submarine may have gone down, and record this value on your sketch.

b.

1. Draw the region of relatively shallow water around the island. Determine the surface area of this region and record it on your sketch.

2. Determine the surface area of the region of deep water that lies within the training area and record it on your sketch.

Mathematics Note

When the outcomes in an event can be modeled geometrically, the theoretical probability of the event may be found using the ratio below:

$$P(\text{E}) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

Some examples of geometric measures are lengths, angle measures, areas, and volumes.

For example, suppose that you have lost a gold earring on a lawn measuring 10 m by 15 m. You plan to search the lawn with an instrument that can detect metal within a radius of 0.5 m. As shown in Figure **1**, one pass with the metal detector along the width of the lawn will cover an area of approximately 10 m^2.

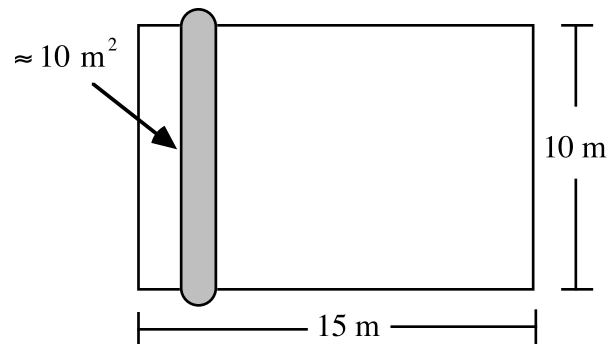


Figure 1: Area covered by metal detector on lawn

The entire lawn has an area of $10 \bullet 15 = 150 \text{ m}^2$. Assuming that the earring will be found if the metal detector passes over it, the probability that you will find it on one pass is approximately equal to the ratio of areas:

$$\frac{10}{150} \approx 0.067 = 6.7\%$$

c. Assume that the chances of the *Livingston* sinking in any one point in the training area are uniform.

1. Determine the probability that it went down in relatively shallow water.
2. Determine the probability that it went down in deep water.

d. Recall that **conditional probability** is the probability of an event occurring given that an initial event, or **condition** occurs. The probability that event B occurs given that event A occurs is denoted by the expression $P(\mathrm{B}|\mathrm{A})$.

According to the sonar technician, the conditional probability of finding the *Livingston*, given that it went down in deep water and the entire region is searched, is 10%. What is the probability that the *Livingston* will not be found after one search of the area, given that it went down in deep water?

Discussion

a. Describe how you used a geometric model to determine the probability that the *Livingston* went down in shallow water.

b. Describe two ways to determine the probability that the *Livingston* went down in deep water.

c. In a **multistage experiment**, one event is followed by one or more other events. In a multistage experiment involving conditional probabilities, the probability of event A followed by event B is found by multiplying the probability of A by the conditional probability of B, given A has already occurred. This can be denoted mathematically as shown below:

$$P(\mathrm{A \text{ and } B}) = P(\mathrm{A}) \bullet P(\mathrm{B}|\mathrm{A})$$

Describe the differences that exist between the following two probabilities:

1. the probability of finding the *Livingston*, given that it went down in deep water
2. the probability that the *Livingston* went down in deep water and will be found.

d. Without knowing where the sub went down, should the rescue team begin their search in shallow water or in deep water? Explain your response.

Assignment

1.1 If the *Livingston* went down in shallow water and the entire region is searched one time, the rescuers have about a 70% chance of detecting it. If the *Livingston* went down in deep water, the rescuers have about a 10% chance of detecting it with one complete search.

a. Create a tree diagram to show the possible outcomes of the search for the *Livingston*. Label each branch with the appropriate probability.

b. Describe how to use the tree diagram to determine the probability that the *Livingston* went down in deep water and will be found.

1.2 Use your tree diagram from Problem **1.1** to determine each of the following:

a. the probability that the *Livingston* went down in shallow water and will be found

b. the probability that the *Livingston* went down in deep water and will not be found

c. the total probability that the *Livingston* will be found.

1.3 Jamal and Laurie are enjoying a cross-country ski trip in Yellowstone National Park. As they pass under a cornice of snow, a small avalanche occurs. Laurie is swept down the hill and buried.

Jamal rushes to search for his friend. The debris from the avalanche is spread over a rectangular region 12 m by 10 m. Laurie could be buried anywhere in this area. Jamal estimates that the snow is no more than 2 m deep. Using the cord from his hood, he lashes two ski poles together to form a pole about 2.5 m long. This enables him to probe the full depth of the snow.

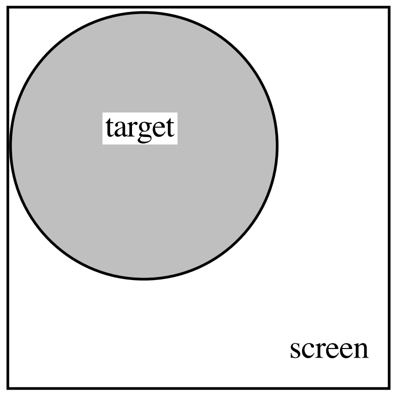
a. Assume that Laurie is lying face down under the snow. Using your own estimate of Laurie's body size, determine the probability that Jamal will find Laurie on his first probe with the pole.

b. If Jamal can search an area of 6 m^2 every minute, how long will it take him to search the entire region of the avalanche?

c. Assume that if Jamal probes the location in which Laurie is buried, he will find her. If Jamal finds Laurie within the first 10 min of the search, there is an 80% chance that she will survive. If he finds her from 10 to 20 min after the avalanche, her chances of survival fall to about 55%. What is the probability that Laurie will survive her ordeal?

1.4 In one game at the local arcade, a player selects a circular target area on a square video screen. Once the target zone has been chosen, the player presses a button that randomly illuminates one pixel on the screen. If the illuminated pixel is in the target zone, the player wins a prize.

The video screen measures 800 pixels by 800 pixels, while the target has a radius of 280 pixels. The diagram below shows one possible position for the target on the screen.



a. What is the probability that the lighted pixel is inside the target area?

b. Are the chances of winning improved by moving the target to another location? Explain your response.

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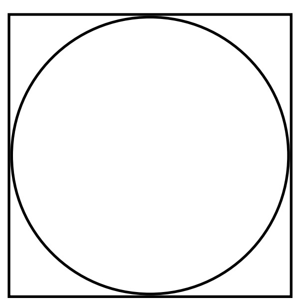
1.5 Laurie challenged Jamal to a game similar to the one described in Problem **1.4**. As shown below, she drew a circle inscribed in a square on a sheet of paper. She then placed the paper on the floor, blindfolded Jamal, and asked him to drop a dart onto this target. If the dart falls inside the square, but outside the circle, Jamal wins.

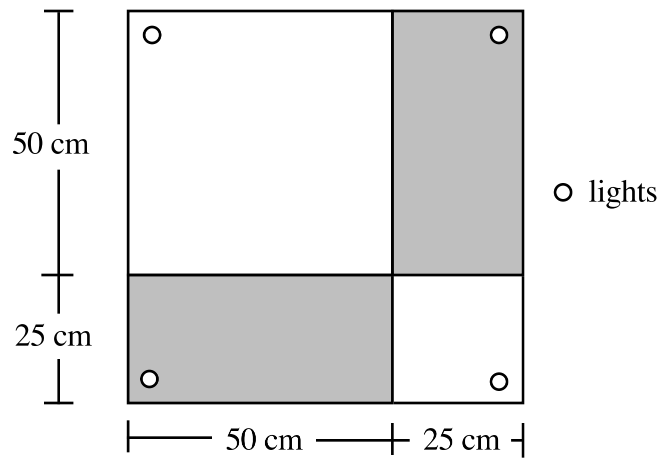


a. Given that the dart lands inside the square, what is the probability that it lands inside the circle? Explain your response.

b. Given that the dart lands inside the square, what is the probability that it lands exactly on the circle? Explain your response.

c. Given that the dart lands inside the square, what is the probability that Jamal wins the game?

1.6 In one game at the school carnival, a player tosses a spinning top onto a board. As shown below, the board is divided into four regions, two of which are shaded. Each region has a light beneath it. These lights are turned on and off at random so that only one light is lit at any one time. For a player to win the game, the top must stop in a shaded region and a light must be on in a shaded region.



a. What is the probability that the top lands in a shaded region?

b. What is the probability that a light is on in a shaded region??

c. What is the probability of winning the game?

d. In a situation involving conditional probabilities, $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$. If A and B are independent events, however, then $P(\text{A and B}) = P(\text{A}) \bullet P(\text{B})$. (In other words, when A and B are independent, $P(\text{B}|\text{A}) = P(\text{B})$.)

1. Consider the event that the top lands in a shaded region and the event that a light is on in a shaded region. Are these independent events? Explain your response.

2. Consider the event that the top lands in a shaded region and the event that a player wins the game. Are these independent events? Explain your response.

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Activity 2

Unfortunately for the crew of the *Livingston*, the search is proceeding at a painfully slow pace. Two other submarines from the base are combing the training region using sonar. As shown in Figure **2**, each sub is able to scan a circular region on the ocean floor 2 km in diameter. In order to search the bottom effectively, however, the two subs are forced to limit their cruising speed to 10 km/hr .

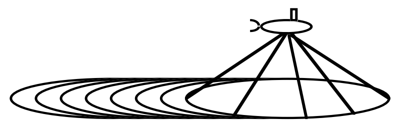

Figure 2: Rescue submarine searching for the USS *Livingston*

The crew of the *Livingston* was equipped with enough oxygen and supplies to last five days. Can the rescuers search the entire area before time runs out?

Exploration

The region of shallow water around the submarine base has a surface area of about 10,996 km^2 . Within a 75-km radius of the base, the region of deep water has a surface area of 6361 km^2 . If the rescue team passes over the lost submarine in shallow water, there is a 70% chance they will detect it. If they pass over the lost submarine in deep water, there is only a 10% chance they will detect it.

a. Find the surface area of the region that the two rescue submarines can cover if they search without a break for five days. (Recall that their cruising speed is limited to 10 km/hr .)

b. **1.** Assume that the USS *Livingston* went down in shallow water. If the subs confine their search to this region, determine the probability that the rescue submarines pass over it in five days.

2. Assume that the USS *Livingston* went down in deep water. If the subs confine their search to this region, determine the probability that the rescue submarines pass over it in five days.

c. **1.** Create a tree diagram to show the possible outcomes of the search for the *Livingston* if both subs search the shallow water. Label each branch with the appropriate probability.

Your diagram should show the probabilities that the sub went down in deep water or shallow water, the probabilities that the rescue team will pass over the lost submarine in the five-day period, and the probabilities that the lost submarine will be detected.

2. Create a tree diagram to show the possible outcomes of the search for the *Livingston* if both subs search the deep water. Label each branch with the appropriate probability. **Note:** Save both tree diagrams for use in the assignment.

Discussion

a. One officer predicted that the two rescue submarines could make one complete search of the region of shallow water in five days. Do you agree with this estimate?

b. How many days would it take the two submarines to search the entire training region?

c. **1.** How many submarines would be needed to search the entire training region in five days?

2. Would involving this many subs in the search guarantee that the *Livingston* will be found?

d. Suppose that the base commander is able to obtain enough submarines to search the entire training region two times in five days. Compare the probability of finding the *Livingston* after one complete search with the probability of finding it after two complete searches.

Assignment

2.1 In order to find the *Livingston*, the team must pass over the lost submarine and detect it. Use your tree diagrams from Part **c** of the exploration to complete Parts **a–c** below.

a. If both submarines search the shallow water, determine the probability that the *Livingston* will be found in time to save the crew.

b. If both submarines search the shallow water, determine the probability that the *Livingston* will not be found in time to save the crew.

c. If both submarines search the deep water, determine the probability that the *Livingston* will not be found in time to save the crew.

2.2 At the beginning of the search operation's third day, three more submarines arrive. The commander assigns them to search the deep water around the island. (Recall that if a rescue team passes over the *Livingston* in deep water, there is only a 10% chance they will detect it.)

a. What percentage of the region of deep water can the three submarines cover in the remaining three days?

b. Create a new tree diagram to show the possible outcomes of the search for the *Livingston*, given that two submarines search in shallow water for five days and three submarines search in deep water for three days.

c. What is the probability that the three new submarines will find the *Livingston*?

d. What is the total probability that *Livingston* will be found within five days?

2.3 Given that two submarines search in shallow water for five days and three submarines search in deep water for three days, what is the probability that the rescue team will not scan the area where the *Livingston* went down? Justify your response.

2.4 What is the probability that the five submarines will pass over the *Livingston* but not detect it? Justify your response.

2.5 Imagine that the first two submarines were assigned to search the region of deep water, while the three submarines that arrived on the third day were assigned to search the shallow water. What is the probability that the *Livingston* will be found using this strategy? Justify your response.

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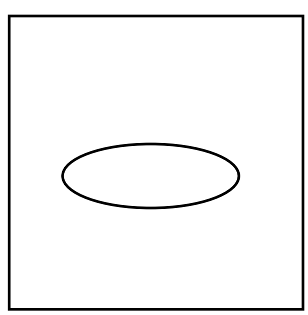
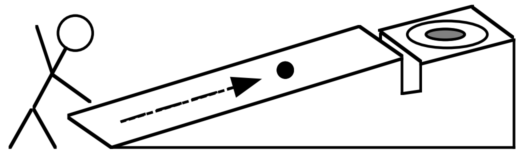
2.6 Laurie and Jamal are playing another game like the one described in Problem **1.6**. In this game, the target is an ellipse on a square sheet of paper 30 cm on each side. While blindfolded, Laurie drops a dart onto the paper. If the dart lands inside the ellipse, she wins.



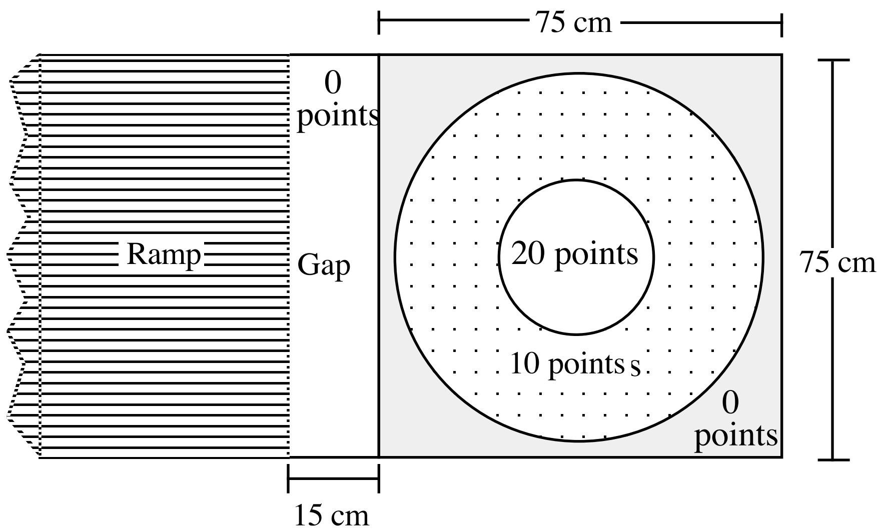
30 cm

After dropping a dart onto the paper 100 times, Laurie has a total of 12 wins. Use this information to estimate the area of the ellipse.

2.7 At the school carnival, one of the most popular games is called "Roller Ball." As shown in the diagram below, players try to roll a small ball up a ramp, over a gap, and into a circular hole. Points are awarded depending on the region in which the ball lands.



The following diagram shows a top view of the game and the number of points awarded for each region. The radius of the larger circle is 30 cm. The radius of the inner circle is 10 cm.



a. Each player receives two rolls. After two rolls, what total scores are possible?

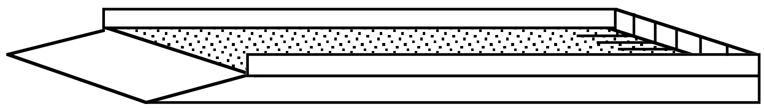
b. In how many different ways can each of the scores in Part **a** be earned?

c. Assuming that the ball is equally like to land at any given point, determine the probability that a ball rolled up the ramp will land in each region shown above.

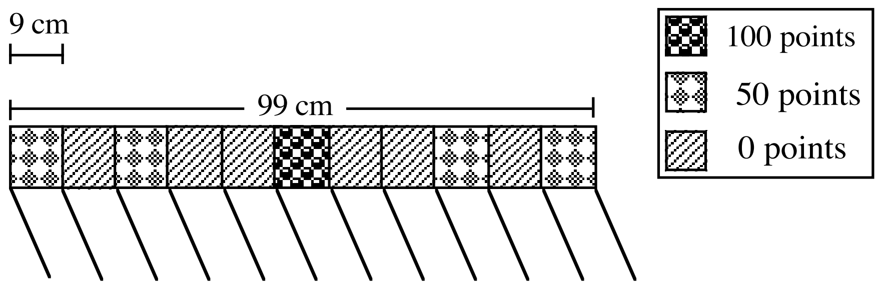
d. Using your responses to Parts **a–c**, determine the probability that a player will win each prize in the table below.

Total Points	Prize
40	large stuffed animal
20–30	poster
10 or less	piece of candy

2.8 Another game at the school carnival is called "Golf-O-Rama." In this game, players hit a golf ball up a ramp and into 1 of 11 slots at the end of a runway. Each slot is worth a different number of points. A side view of the game is shown below.



The following figure shows the dimensions of the end of the runway, along with the number of points awarded for each slot.



Each player receives two balls. Assuming that a ball is equally likely to land in any slot, determine the probability that a player's total score is:

a. exactly 100

b. 150 or more

c. 50 or less.

2.9 Rafael has misplaced his car keys. He estimates that the probability that they are somewhere on his cluttered desk is 0.7. However, he might also have left them in a more obvious place. The probability that they are in his coat pocket is 0.2, while the probability that they are in the glove compartment is 0.1.

If the keys are in his coat pocket or the glove compartment, he will find them on the first search of either location. If they are on the desk, the probability of finding them after one complete search is 0.7.

Design a search strategy for Rafael that minimizes the expected number of searches.

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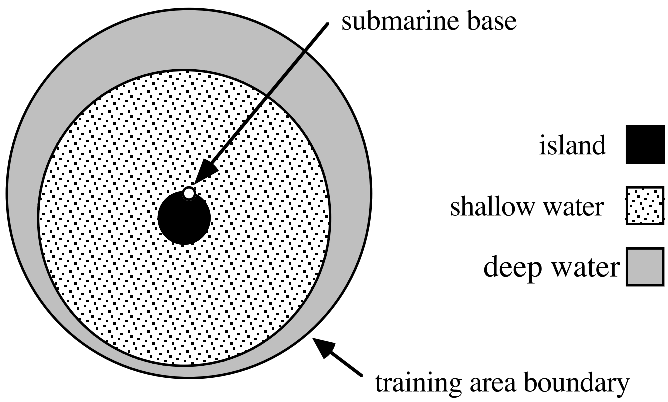
Summary Assessment

A few months after the successful rescue of the *Livingston*, another training mission loses contact with the submarine base.

As shown in the diagram below, the base is located on an island surrounded by a roughly circular region of shallow water. The total area of the training region is $17,357 \text{ km}^2$. The area of the region of shallow water is $10,996 \text{ km}^2$, while the area of the region of deep water is 6361 km^2.



The base commander orders eight submarines to participate in the search. Each is equipped with a sonar device capable of scanning a circular area 2 km in diameter. In order to search the bottom effectively, the subs must limit their cruising speed to 10 km/hr. If the searchers pass over the lost submarine in shallow water, there is a 70% chance it will be detected. If they pass over the lost submarine in deep water, there is only a 10% chance it will be detected.

Devise an effective search strategy for the rescue of the missing submarine. Your response should include the calculations of all probabilities used to determine the strategy.

Module Summary

- When the outcomes in an event can be modeled geometrically, the theoretical probability of the event may be found using the ratio below:

$$P(\text{E}) = \frac{\text{measure of geometric model representing outcomes in the event}}{\text{measure of geometric model representing all outcomes in the sample space}}$$

- **Conditional probability** is the probability of an event occurring given that an initial event, or **condition**, has already occurred. The probability that event B occurs given that event A has occurred is denoted by the expression $P(\text{B}|\text{A})$.

- In a **multistage experiment**, one event is followed by one or more other events. In a multistage experiment involving conditional probabilities, the probability of event A followed by event B is found by multiplying the probability of A by the conditional probability of B, given A has already occurred. This can be denoted mathematically as shown below:

$$P(\text{A and B}) = P(\text{A}) \bullet P(\text{B}|\text{A})$$

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