A Ride with Markov

At Boards Incorporated, the first priority is giving customers a quality ride—on their skateboards, of course. In this module, you examine how a process developed by Russian mathematician Andrei Markov can help the company control quality.

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Introduction

Welcome to Boards Incorporated, skateboard manufacturer. The company makes about 1000 skateboards every week. As with any production process, skateboard manufacturing has its ups and downs. Not every board that rolls off the assembly line is good enough to carry the company name. In the past, the defective rate has been approximately 20%.

Currently, the quality-control department is trying to develop a sampling strategy to monitor the number of defective products. The sampling process must be cost effective. It must be repeated enough times to ensure quality, yet not so many times that the cost of sampling becomes too great.

What might an effective sampling strategy look like? How can the cost of the strategy be predicted? What factors will serve to indicate that the number of products sampled needs to be changed? In this module, you use the Markov process—named after Andrei Andreyevich Markov (1856–1922)—and binomial experiments to answer these questions.

Activity 1

You may recall from previous modules that a **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions (trials) of the same action.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure.
- The probability of a success remains the same from trial to trial.
- The total number of successes is observed.

The sampling done for quality control typically does not satisfy all the conditions for a binomial experiment, since a defective product is not returned to the population before sampling the next item. However, a binomial experiment approximates the process fairly well if the sample size is small compared to the total population.

Exploration

The quality-control process at Boards Incorporated consists of three states: "ordinary," "relaxed," and "heightened." In the ordinary state, a sample of 30 boards is examined once each week. If the number of defectives in the sample is too large, then the process moves into the heightened state. If the number of defectives is small enough, then the process moves into a relaxed state. One hundred boards per week are sampled in the heightened state, while only 5 boards per week are sampled in the relaxed state.

How should the company decide when to change the sample size? In this exploration, you use a probability distribution to examine this question.

a. At Boards Incorporated, a 20% defective rate is considered acceptable. To model this situation, create a population of 150 beans in which 20% have a distinguishing mark on them.

b. In the ordinary state, quality-control specialists sample 30 skateboards from the population. Given a 20% defective rate, how many defectives would you expect to find in each sample?

c.

1. Take a random sample of 30 beans from the population. Record the number of defectives (or marked beans) in the sample, then return the sample to the container and mix thoroughly.
2. Repeat the sampling process 49 more times, for a total of 50 samples.

d. Recall that the **frequency** of an item in a data set is the number of times that item is observed. A **frequency histogram** consists of bars of equal width whose heights indicate the frequencies of items or intervals.

Create a frequency histogram of the numbers of defectives found in your samples. Represent frequency on the y-axis and numbers of marked beans on the x-axis.

e. Calculate the mean number of defectives for your 50 samples.

f.

1. Combine the results of your 50 samples with those of the rest of the class.
2. Determine the mean number of defectives for the class data.
3. Display the class data in a frequency histogram.
4. Connect the midpoints of the tops of the bars to create a **frequency polygon**, as shown in Figure **1**.

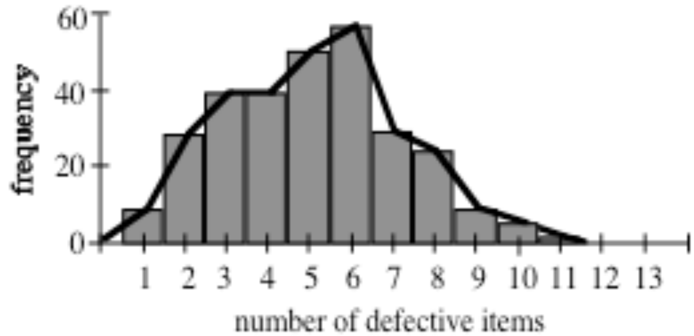

Figure 1: A frequency polygon

Mathematics Note

The probability distribution for a binomial experiment is a **binomial distribution**. The mean of a binomial distribution is the product of the number of trials and the probability of a success. In other words, the mean μ can be found as follows, where n is the number of trials and p is the probability of a success:

$$\mu = np$$

The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

For example, consider an experiment that consists of tossing a six-sided die 10 times and observing the number of times that a 6 appears. In this case, $n = 10$ and the probability of a success is $1/6$. Therefore, the mean of the corresponding binomial distribution is $10(1/6) = 10/6 \approx 1.67$. The standard deviation is $\sqrt{10(1/6)(5/6)} = \sqrt{50/36} \approx 1.18$.

g. Determine the mean μ and standard deviation σ for the binomial distribution that models the sampling of 30 skateboards from a population with a 20% defective rate.

h. On the histogram from Part **f**, draw and label vertical lines that represent each of the following:

1. the mean μ
2. values 1 standard deviation σ on either side of the mean
3. values 2 standard deviations on either side of the mean
4. values 3 standard deviations on either side of the mean.

i. Estimate the percentage of samples that lie within each of the following intervals on the histogram:

1. $[\mu + \sigma,\ \mu - \sigma]$

2. $[\mu + 2\sigma,\ \mu - 2\sigma]$

3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

Discussion

a. As noted previously, a binomial experiment provides a reasonable model for the sampling in the exploration if the number of items removed is small compared to the total population. Why is this requirement necessary?

b.
1. How does the mean of your 50 samples compare with the mean μ of the corresponding binomial distribution?

2. How does the mean of the class data compare with μ?

3. In general, how would you expect the mean of a very large set of samples to compare with μ?

c. What patterns did you observe in your histograms from the exploration?

d.
1. What would the histograms have looked like if the percentage of defectives had been 40% instead of 20%?

2. What would the histograms have looked like if twice as many samples were taken?

Mathematics Note

A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.

In this situation, the probabilities of these outcomes can be represented graphically by the area enclosed by the *x*-axis, a specific real-number interval, and a distribution curve. The sum of the non-overlapping areas that cover the entire interval is 1.

One continuous probability distribution is the **normal distribution**. As shown in Figure **2**, the graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.

As in all continuous probability distributions, the total area between the horizontal axis and a normal curve is 1. In a normal distribution, approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

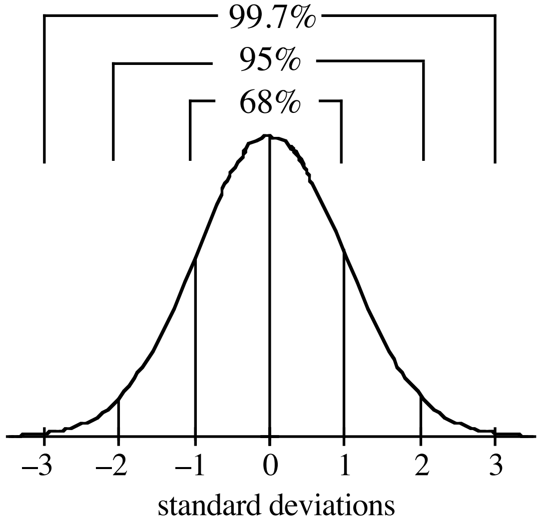

Figure 2: A normal curve and the 68–95–99.7 rule

Normal distributions can be used to model a wide variety of data sets. When this distribution provides a reasonable model, the 68–95–99.7 rule can help you characterize a population. For example, if a population appears to be normally distributed with a mean of 100 and a standard deviation of 10, then you would expect about 68% of the observations to fall between 90 and 110, 95% of the observations to fall between 80 and 120, and 99.7% of the observations to fall between 70 and 130.

e. How does the shape of the frequency histogram for the class data compare with a normal curve?

f. How does the percentage of samples that fell in each of the following intervals compare with the 68–95–99.7 rule?

1. $[\mu + \sigma,\ \mu - \sigma]$
2. $[\mu + 2\sigma,\ \mu - 2\sigma]$
3. $[\mu + 3\sigma,\ \mu - 3\sigma]$

g. What percentage of all samples of a given size would you expect to find in each of the following regions under a normal curve:

1. to the right of μ ?
2. to the right of $\mu + 1\sigma$?
3. to the left of $\mu - 1\sigma$?
4. to the right of $\mu - 3\sigma$?
5. to the left of $\mu + 2\sigma$?

Assignment

1.1 Determine whether or not each of the following procedures is a binomial experiment. If the procedure is not a binomial experiment, can it be reasonably modeled by one? Explain your responses.

a. You select a random sample of 6 computer chips from a batch of 20, replacing each chip before selecting the next one. The total number of defectives is recorded.

b. You select a random sample of 6 computer chips from a batch of 20, without replacement, and record the total number of defectives.

c. You select a random sample of 500 American teenagers and record their favorite brand of tennis shoes from a list of 10 brands.

d. You select a random sample of 5 electric motor shafts from an assembly line that produces 2000 shafts a day and determine if the shaft diameter is between 1.71 cm and 1.73 cm.

e. You roll a pair of dice 5 times and record the number of times their sum is greater than 8.

1.2 At Boards Incorporated, the ordinary quality-control process involves a sample of 30 boards per week. As long as the number of defectives is within 1 standard deviation of the mean, the quality-control process remains in the ordinary state.

a. Given a 20% defective rate, what interval for the number of defectives in a sample of 30 would keep the quality-control process in the ordinary state?

b. If the number of defectives is less than $\mu - 1\sigma$, the quality-control process moves to the relaxed state. What is the greatest number of defectives that would cause this move?

c. If the number of defectives is more than $\mu + 1\sigma$, the quality-control process moves to the heightened state. What is the least number of defectives that would cause this move?

d. Describe the numbers of defectives for which the quality-control process would stay in the ordinary state, move from ordinary to heightened, and move to relaxed from ordinary.

1.3 When the quality-control process moves into the heightened state at Boards Incorporated, the sample size increases to 100. If the number of defectives found is no more than 1 standard deviation above the mean, the quality-control process moves back to the ordinary state.

a. Assuming a defective rate of 20%, determine the mean number of defectives for samples of size 100.

b. What is the greatest number of defectives that would cause the quality-control process to move back into the ordinary state?

c. Given a population in which the actual defective rate is 20%, what percentage of samples of size 100 would keep the quality-control process in the heightened state?

d. Describe the numbers of defectives for which the quality-control process would stay in the heightened state or move from the heightened state to the ordinary state.

1.4 When the quality-control process moves into the relaxed state, the sample size decreases to 5. If the number of defectives found is greater than 1 standard deviation above the mean, the quality-control process moves back to the ordinary state.

a. Assuming a defective rate of 20%, determine the mean number of defectives for samples of size 5.

b. What is the least number of defectives that would cause the quality-control process to move back into the ordinary state?

c. Given a population in which the actual defective rate is 20%, what percentage of samples of size 5 would keep the quality-control process in the relaxed state?

d. Describe the numbers of defectives for which the quality-control process would stay in the relaxed state or move from the relaxed state to the ordinary state.

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1.5 Imagine that you are conducting an experiment using an unfair coin. The probability of obtaining a head on any one toss is 0.4.

a. Are the formulas for the mean and standard deviation of a binomial distribution appropriate in this situation? Explain your response.

b. Determine the mean and standard deviation for the number of heads given each of the following numbers of trials:

1. 20

2. 40

3. 60.

c. Repeat Part **b** when the probability of obtaining a head on any toss is 0.2.

1.6 The mean score on an examination was 72 with a standard deviation of 9. Students whose scores fall in the top 2.5% will receive a scholarship to the college of their choice. If the scores are normally distributed, what is the minimum score required to receive a scholarship?

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Activity 2

The potential cost of their quality-control strategy is important to Boards Incorporated. The company's managers realize that more sampling means higher costs. They also know that if the quality-control process remains in a heightened state for a long time, the assembly line must be shut down and the source of the increased defects identified and repaired.

To estimate costs, the quality-control specialist has been asked to predict what portion of time will be spent in each state. In this activity, you examine one method for making such predictions.

Exploration

In Activity **1**, you identified the numbers of defectives that cause the quality-control process to move from one state to another. For example, if the number of defectives found in the ordinary state is less than 4, the process moves to the relaxed state. If the number of defectives is greater than 8, the process moves to the heightened state.

In the relaxed state, if the number of defectives is greater than 1, the process returns to the ordinary state. The process returns to the ordinary state from the heightened state when 24 or fewer defectives are found.

In this exploration, you determine the probabilities of moving from one state to another, assuming a defective rate of 20%. Since the quality-control process can be modeled by a binomial experiment, these probabilities can be determined using the **binomial probability formula**.

Mathematics Note

The **binomial probability formula** can be used to determine the probability of obtaining *r* successes in *n* trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where *p* is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

For example, if 25% of a population of computer disks are defective, then $(1-25\%) = 75\%$ are not. The theoretical probability that exactly 4 defective disks will occur in a sample of 10 is:

$$\begin{aligned} P(4 \text{ successes in } 10 \text{ trials}) &= C(10,4) \bullet (0.25)^4 \bullet (0.75)^{10-4} \\ &= 210 \bullet (0.25)^4 \bullet (0.75)^6 \\ &\approx 0.15 \end{aligned}$$

a. Assuming that the defective rate is 20%, complete Table **1**. This is the probability distribution for the possible numbers of defective boards in a sample of size 30.

Table 1: Probability distribution

No. of Defectives (n)	$P(n)$
0	
1	
2	
$\vdots$	
29	
30	

b. Recall that given two events A and B, the theoretical probability of either A or B occurring can be found as follows:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

1. Use your probability distribution to determine the probability that the quality-control process will move from the ordinary state to the heightened state.
2. Determine the probability that the quality-control process will move from the ordinary state to the relaxed state.
3. Determine the probability that the quality-control process will remain in the ordinary state.

c.

1. Repeat Part **a** for the possible numbers of defective boards in a sample of size 5.
2. Determine the probability that the quality-control process will move from the relaxed state to the ordinary state.
3. Determine the probability that the quality-control process will move from the relaxed state to the heightened state.
4. Determine the probability that the quality-control process will remain in the relaxed state.

d. Table **2** below shows the probability distribution for the possible numbers of defective boards in a sample of size 100.

n	***P*(*n*)**	***n***	***P*(*n*)**
0	0.00000	22	0.08490
1	0.00000	23	0.07198
2	0.00000	24	0.05773
3	0.00000	25	0.04388
4	0.00000	26	0.03164
5	0.00001	27	0.02168
6	0.00006	28	0.01413
7	0.00020	29	0.00877
8	0.00058	30	0.00519
9	0.00148	31	0.00293
10	0.00336	32	0.00158
11	0.00688	33	0.00081
12	0.01275	34	0.00040
13	0.02158	35	0.00019
14	0.03353	36	0.00009
15	0.04806	37	0.00004
16	0.06383	38	0.00002
17	0.07885	39	0.00001
18	0.09090	40	0.00000
19	0.09807	41	0.00000
20	0.09930	⋮	⋮
21	0.09457	100	0.00000

1. Using Table **2**, determine the probability that the quality-control process will move from the heightened state to the ordinary state.
2. Determine the probability that the quality-control process will move from the heightened state to the relaxed state.
3. Determine the probability that the quality-control process will remain in the heightened state.

e. Complete Table **3** using the probabilities found in Parts **b–d**. Each entry represents the probability of moving from the present state to another state. These probabilities can be denoted as follows:

$$P(\text{new state}\,|\,\text{present state})$$

For example, the probability that the quality-control process will move to the relaxed state, given that it is presently in the relaxed state, is:

$$\begin{aligned} P(\mathrm{R}|\mathrm{R}) &= P(0) + P(1) \\ &= C(5,0) \bullet 0.2^0 \bullet 0.8^5 + C(5,1) \bullet 0.2^1 \bullet 0.8^4 \\ &\approx 0.7373 \end{aligned}$$

Round each probability to the nearest thousandth.

Table 3: Probabilities of quality-control states

	Next State			
Present State	relaxed	ordinary	heightened	**Total**
relaxed	0.737			
ordinary				
heightened				

Mathematics Note

The process of moving from one **state**, or outcome, to another is a **transition**.

For example, the weather may be described by one of three states: cloudy, rainy, or sunny. The process of moving from one state to another—from sunny to cloudy, for example—is a transition.

A **transition diagram** is a convenient way to display the possible changes among states. In such diagrams, each state is represented by a vertex, while each transition is represented by a directed edge labeled with a probability.

For example, Figure **3** shows a transition diagram for three states of weather: cloudy (C), sunny (S), and rainy (R).

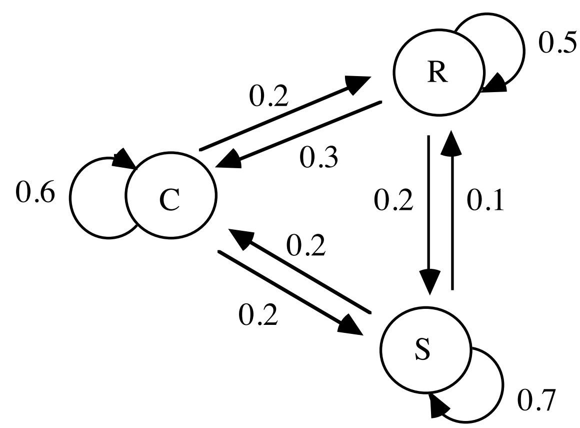

Figure 3: A transition diagram for the weather

The directed edge from R to C in Figure **3**, labeled 0.3, represents the probability of a transition from a rainy day to a cloudy day, or $P(\mathrm{C}|\mathrm{R}) = 0.3$. The loop about C, labeled 0.6, represents the probability of transition from a cloudy day to another cloudy day, or $P(\mathrm{C}|\mathrm{C}) = 0.6$.

f. Draw a transition diagram for the three states of the quality-control process, using the information in Table **3**. **Note:** Save a copy of Table **3** and the corresponding transition diagram for use later in the module.

Discussion

a. **1.** What does the entry in each cell in Table **3** represent?

2. How is the entry in each cell illustrated in the transition diagram you created in Part **f** of the exploration?

b. **1.** What does an entry of 0 in any cell of Table **3** represent?

2. How is an entry of 0 in a cell represented in a transition diagram?

c. **1.** What does the total for each row in Table **3** represent?

2. How is each row total represented in a transition diagram?

Assignment

2.1 According to the U.S. Bureau of Labor Statistics, 65% of the graduating class of 1996 attended college in the fall of that year. Consider a random sample of 25 students from this population.

a. Create a probability distribution for the number of students who attended college in a sample of 25 from this population.

b. What is the probability that more than half the students in a sample of 25 attended college?

2.2 **a.** For each row in Table **3** from the exploration, draw a tree diagram. Label each branch with the corresponding probability.

b. Do your tree diagrams from Part **a** represent transition diagrams? Explain your response.

2.3 The managers of Boards Incorporated suggest that the quality-control process use only two states: ordinary and heightened. If 0 to 8 defective boards are found in a sample of 30, the ordinary state would continue. If 9 or more are found, the quality-control process would move into the heightened state. The process would return to the ordinary state if 24 or fewer defective boards are found in a sample of 100.

a. Create a transition diagram to represent this situation.

b. Does it appear that this change will result in more frequent occurrences of the heightened state? Explain your response.

c. Because of the smaller sample size, the relaxed state costs less to administer than the ordinary state. Similarly, the ordinary state costs less than the heightened state. Given this fact, would you recommend that the company use a quality-control process with three states or with two states? Justify your response.

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2.4 The table below shows the probabilities of movement in the U.S. population from one region to another in 1991.

	To			
From	**Northeast**	**Midwest**	**South**	**West**
Northeast	0.9818	0.0034	0.0111	0.0037
Midwest	0.0013	0.9866	0.0075	0.0046
South	0.0022	0.0049	0.9886	0.0043
West	0.0014	0.0035	0.0077	0.9874

Source: U.S. Bureau of the Census, 1993.

a. Explain what is represented by the entry in the second row, third column of the table.

b. Describe the meaning of the entries in the principal diagonal (the diagonal from the upper left-hand corner to the lower right-hand corner).

c. Create a transition diagram for the information in this table.

2.5 Imagine that you own 50 shares of stock in a company that manufactures electric cars. The table below shows the movement in the stock's price for the past 30 business days. The entry for day 1, for example, indicates that the stock price went up during that day.

Day	Change	Day	Change	Day	Change
1	up	**11**	down	**21**	up
2	up	**12**	up	**22**	up
3	no change	**13**	up	**23**	down
4	down	**14**	down	**24**	down
5	no change	**15**	no change	**25**	no change
6	no change	**16**	no change	**26**	no change
7	up	**17**	up	**27**	up
8	up	**18**	up	**28**	up
9	down	**19**	down	**29**	no change
10	down	**20**	down	**30**	down

a. Using the information given, create a list of ordered pairs in which the first element indicates the movement in stock price for a given day and the second element represents its movement on the following day. For example, the first ordered pair in the list should be (U,U), where U represents up.

b. Use the ordered pairs in Part **a** to complete the following table of probabilities. For example, there are 12 days on which the stock price was up. Of the 12 following days, the stock price went up again on 6. Thus, $P(\text{U,U}) = 6/12 = 0.5$.

	Following Day		
Given Day	up	no change	down
up	0.500		
no change			
down			

c. Assuming that the data is representative of the stock's long-term behavior, create a transition diagram.

d. Use the information from Parts **a–c** to predict what will happen on day 32, given that the stock price is down on day 31.

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Activity 3

Overall, the managers of Boards Incorporated are satisfied with their quality-control process. However, they are concerned about what to expect in the future. Is there some way to predict how often the state will change in the long run, given that the process begins in the ordinary state?

Exploration

As you discovered in the previous activity, the probabilities for the states in the company's quality-control process can be represented in a transition diagram. This transition diagram is shown in Figure **4**.

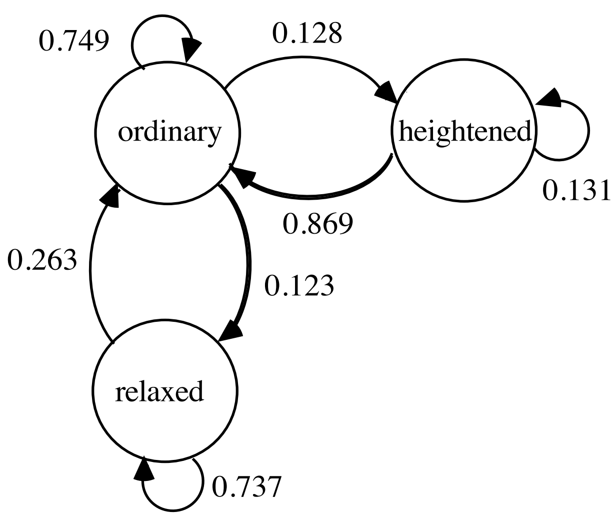


Figure 4: Transition diagram for quality control

In this exploration, you examine how transition diagrams can be used to predict future trends.

a. Assume that the quality-control process at Boards Incorporated is presently in the ordinary state. Draw a tree diagram that shows all the possible changes that can occur after one transition. Label each branch with the corresponding probability.

b. Extend this tree diagram, complete with probabilities, for a second transition. (Assume that the probabilities of moving from one state to another remain the same as in Figure **4**.)

c. Given that the quality-control process was originally in the ordinary state, calculate the probability that the process will be in each of the following states after two transitions:

1. the ordinary state
2. the relaxed state
3. the heightened state

Mathematics Note

A **Markov chain** is a model for predicting the probability of moving from one state (or outcome) to other states, given that there are a finite number of states and the probability of being in one state depends only on the state before the move.

The technique of predicting the probabilities of these transitions is the **Markov process**.

The probabilities of moving among states can be displayed in a **transition matrix**. A transition matrix **T** for a Markov chain has the following characteristics.

- The matrix is square with dimensions $m \times m$, where m represents the number of states.
- All the elements in the matrix are between 0 and 1, inclusive.
- The sum of the elements in any row is 1.

The element in row i, column j in the matrix $\mathbf{T}^n$ represents the probability of moving from state i to state j after n transitions.

For example, consider the transition diagram for the weather shown in Figure **5**.

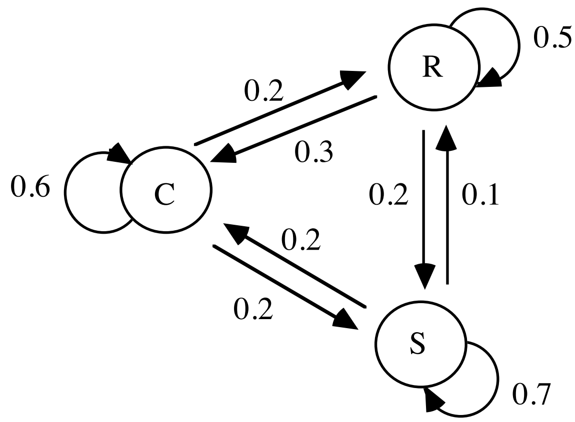


Figure 5: A transition diagram for the weather

The corresponding transition matrix, where C represents cloudy days, R represents rainy days, and S represents sunny days, is shown below.

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

This matrix has all the characteristics of a transition matrix for a Markov chain—it is square, has dimensions 3×3, and the sum of the elements in any row is 1. The element in row S, column R represents the probability of the transition from a day of sunny weather to a day of rainy weather (or the proportion of rainy days that follow sunny days).

d. Create the corresponding transition matrix **T** for the information in Figure **4**. **Note:** Save this matrix for use in the assignment.

e. Calculate the matrix $\mathbf{T}^2$.

f. Calculate each matrix $\mathbf{T}^n$, for values of n from 1 to 50. Describe any patterns you observe in these matrices.

Discussion

a. According to the previous mathematics note, a transition matrix must be square. Explain why this is true.

b. Why are the elements of a transition matrix always between 0 and 1, inclusive?

c. Why is the sum of the elements in a row of a transition matrix always 1?

d. Explain why each of the following matrices does not represent a Markov chain.

1. $\begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ **3.** $\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.2 & 0.9 \\ 0.3 & 0.7 & 0 \end{bmatrix}$

e. **1.** How do the probabilities you determined in Part **c** of the exploration compare with the elements in the second row of $\mathbf{T}^2$? Justify your response by comparing multistage probability to matrix multiplication.

2. What do the elements in the remaining rows of $\mathbf{T}^2$ represent?

f. What do the elements in each of the following matrices represent?

1. $\mathbf{T}^3$

2. $\mathbf{T}^n$

Mathematics Note

A transition matrix **T** is **regular** if for some n, all of the elements in the matrix $\mathbf{T}^n$ are positive. A Markov chain is **regular** if its transition matrix is regular.

For example, the transition matrix for the weather given in the previous mathematics note is regular because all of its elements are positive. Consequently, the Markov chain for that situation also is regular.

g. Does the information in matrix **T** in the exploration describe a regular Markov chain?

h. Describe what you observed about $\mathbf{T}^n$ in the exploration as n increased. Such matrices are referred to as **stable** (or **steady**) **state matrices**.

Mathematics Note

A **stable** (or **steady**) **state matrix** is formed by raising a transition matrix to some power such that the difference between any two elements in the same column is very small.

For example, consider the following transition matrix **T** for the weather:

$$\mathbf{T} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

When **T** is raised to a large power, such as 100, the matrix stabilizes to the steady state matrix **S**:

$$\mathbf{T}^{100} = \mathbf{S} = \begin{array}{c} \\ C \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & R & S \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

i. What information is provided by a stable state matrix?

Assignment

3.1 In Problem **2.3**, the managers of Boards Incorporated suggested a quality-control process using only two states: ordinary and heightened. The probability of moving from the ordinary state to the heightened state was 0.128. The probability of moving from the heightened state back to the ordinary state was 0.869.

a. Create the transition matrix for this situation.

b. Determine the probability that the quality-control process is in the heightened state after three transitions.

c. **1.** Does the matrix from Part **a** represent a Markov chain? Explain your response.

2. Does it represent a regular Markov chain? Explain your response.

3.2 The table below shows the movement in the U.S. population, by region, in 1991.

	To			
From	Northeast	Midwest	South	West
Northeast	0.9818	0.0034	0.0111	0.0037
Midwest	0.0013	0.9866	0.0075	0.0046
South	0.0022	0.0049	0.9886	0.0043
West	0.0014	0.0035	0.0077	0.9874

Source: U.S. Bureau of the Census, 1993.

a. Create a transition matrix for this information.

b. Does this transition matrix represent a regular Markov chain? Explain your response.

c. Assuming that changes in residence depend only on the present residence, determine the probability of each of the following:

1. living in the West after two moves, given that the person originally lived in the Northeast
2. living in the Northeast after five moves, given that the person originally lived in the Midwest
3. living in the South after three moves, given that the person originally lived in the West.

d. Do you think that the assumption made in Part **c** is a reasonable one? In other words, do an individual's future moves depend only on the present? Explain your response.

3.3 Assuming that the acceptable defective rate remains at 20%, the quality control specialist at Boards Incorporated suggests another sampling strategy consisting of three states. In the ordinary state, a sample of 20 skateboards is taken each week. The heightened state requires a sample of 30 boards per week, while the relaxed state requires a sample of 10 boards.

If 0 to 2 defectives are found when sampling in the ordinary state, the process is moved into the relaxed state. If 6 or more defectives are found, the process is moved into the heightened state. Otherwise the quality-control process continues as usual.

Four or more defectives found when sampling in the relaxed state moves the process back to the ordinary state. When sampling in the heightened state, 9 or fewer defectives moves the process back to the ordinary state. A transition between the heightened and relaxed states is not permitted.

a. Create a transition matrix for this information.

b. Can this matrix form a stable state matrix? Explain your response.

3.4 A polling organization compiles ratings for three television networks: CBA, CBN, and SBC. In its most recent survey, the organization found that after 1 hr of television, CBA viewers continued watching CBA 50% of the time, switched to CBN 20% of the time, and changed to SBC 30% of the time.

After the same period, CBN viewers continued watching CBN 30% of the time, switched to SBC 10% of the time, and turned to CBA 60% of the time. Viewers of SBC, on the other hand, continued watching SBC 10% of the time, changed to CBA 60% of the time, and switched to CBN 30% of the time.

Use this information to predict the probability of each of the following:

a. watching CBA after 4 hr, given that a viewer was originally tuned to SBC

b. watching SBC after 5 hr, given that a viewer was originally tuned to CBN

c. watching CBA after 10 hr, given that a viewer was originally tuned to CBA.

* * * * *

3.5 At the end of basketball practice, the coach asks a pair of players to stand at the free-throw line. One player shoots free throws, while the other rebounds. The first player gets to continue shooting until she misses. The players then change positions. The coach wants the pair to shoot a total of 150 free throws.

a. Suppose that the first player's free-throw percentage is 80%, while the second player's free-throw percentage is 60%. Create a transition matrix for this situation.

b. How many shots would you predict each player to take? Explain your reasoning.

c. How many free throws would you expect each player to make?

3.6 Consider the transition matrix **A** shown below, where a, b, and c are not equal to 0.

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Determine the values of a, b, and c after many transitions.

* * * * * * * * * *

Activity 4

Besides planning an overall quality-control process, the managers of Boards Incorporated also must consider day-to-day quality issues. Does knowing the probability of moving from today's state to another help predict what to expect tomorrow or next week? In this activity, you examine how to use the Markov process to predict the probability of future events.

Mathematics Note

The **initial state vector** $\mathbf{X}_0$ of a population is represented by a matrix with a single row. Each element of the matrix represents the probability of a state before any transitions.

The **state vector** after one transition, $\mathbf{X}_1$, is determined by multiplying the initial state vector $\mathbf{X}_0$ by the transition matrix **T**. In other words, $\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1$. The order of the states in $\mathbf{X}_0$ must match the order of the states in the corresponding transition matrix. Each element of the state vector $\mathbf{X}_1$ represents the probability of a state after one transition.

Similarly, when $\mathbf{X}_1$ is multiplied by **T**, the result is the state vector $\mathbf{X}_2$. Each of its elements represents the probability of a state after two transitions. In general, $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T}$ and each element in $\mathbf{X}_n$ represents the probability of a state after *n* transitions.

For example, consider the following initial state vector for the weather:

$$\begin{array}{cc} & \begin{array}{ccc} \mathrm{C} & \mathrm{R} & \mathrm{S} \end{array} \\ \mathbf{X}_0 = & \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

The elements in this matrix indicate a 20% chance of clouds, a 20% chance of rain, and a 60% chance of sunny weather, respectively, on a given day. To predict the weather for the next day, you could multiply $\mathbf{X}_0$ by the corresponding transition matrix **T** to obtain $\mathbf{X}_1$, as shown below.

$$\mathbf{X}_0 \bullet \mathbf{T} = \mathbf{X}_1$$

$$\begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \bullet \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \end{bmatrix}$$

The state vector $\mathbf{X}_1$ indicates that there is a 30% chance of clouds, a 20% chance of rain, and a 50% chance of sunshine for the next day. To predict the weather in two days, the state vector $\mathbf{X}_1$ can be multiplied by **T** to obtain $\mathbf{X}_2$, and so on.

Exploration

How does the current state affect the quality-control process in upcoming weeks? In this exploration, you examine how different initial state vectors affect predictions for future events.

a. As with any row of a transition matrix, the sum of the probabilities within an initial state vector must total 1. Suppose Boards Incorporated has decided to begin the sampling process in the ordinary state. Write an initial state vector that reflects this decision in the form below.

$$\begin{array}{c} \quad\ \ \text{R} \quad \text{O} \quad \text{H} \\ \mathbf{X}_0 = [\qquad\qquad\ ] \end{array}$$

b. The quality-control process under consideration allows transitions among three states. It is possible to represent this process using the following transition matrix:

$$\mathbf{T} = \begin{array}{c} \\ \text{R} \\ \text{O} \\ \text{H} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{O} & \text{H} \end{array} \\ \begin{bmatrix} 0.737 & 0.263 & 0.000 \\ 0.123 & 0.749 & 0.128 \\ 0.000 & 0.869 & 0.131 \end{bmatrix} \end{array}$$

Use this matrix to predict the probability of each state after the following numbers of transitions:

1. 1
2. 2
3. 3–30.

c.
1. Repeat Parts **a** and **b** assuming that the quality-control process began in the relaxed state.
2. Repeat Parts **a** and **b** assuming that the quality-control process began in the heightened state.

d. Create an initial state vector of your choice and repeat Part **b**.

e.
1. Multiply your initial state vector from Part **d** by $\mathbf{T}^{30}$.
2. Compare the resulting product matrix with the matrix for $\mathbf{X}_{30}$ obtained in Part **d**.

f. Using the symbols $\mathbf{X}_0$ and $\mathbf{T}^n$, write a general formula for finding $\mathbf{X}_n$.

Discussion

a. The sum of the elements in an initial state vector must be 1. Explain why this true.

b. Compare the distributions in the state vectors for 3–30 transitions. Describe any patterns you observe.

c. In general, $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T}$. How does this formula compare with the one you wrote in Part **f** of the exploration?

d. **1.** What impact does $\mathbf{X}_0$ have on the resulting state vector $\mathbf{X}_n$ after a large number of transitions?

2. In Part **e** of the exploration, how did the elements of $\mathbf{T}^{30}$ compare with the elements of $\mathbf{X}_{30}$?

Mathematics Note

Multiplying any state vector by the stable state matrix results in a **stable** (or **steady**) **state vector**. This vector consists of one row of the stable state matrix.

For example, consider the following state vector $\mathbf{X}_2$ for the weather:

$$\begin{array}{c c} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{X}_2 = & \begin{bmatrix} 0.3 & 0.1 & 0.6 \end{bmatrix} \end{array}$$

Multiplying $\mathbf{X}_2$ by the stable state matrix **S** results in the stable state vector **P**:

$$\begin{array}{c c} & \begin{array}{ccc} C & R & S \end{array} \\ \mathbf{P} = & \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

The entries in this stable state vector indicate that, over the long term, there is a 40% chance of clouds, a 20% chance of rain, and a 40% chance of sunshine on any given day.

All regular Markov chains have a stable state matrix and vector. Markov chains that are not regular may or may not have a stable state matrix and vector.

Assignment

4.1 Determine the stable state matrix, if it exists, for each of the following transition matrices.

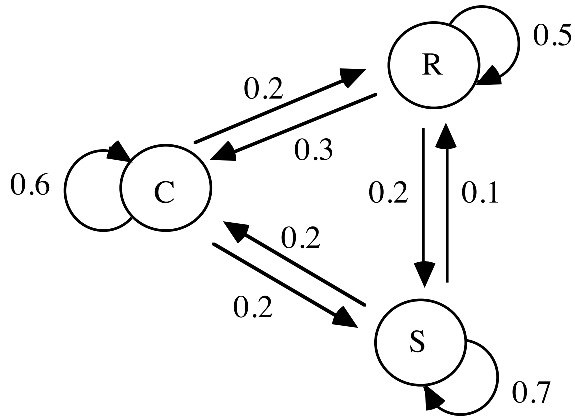
a. $\begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.9 & 0.0 & 0.1 \end{bmatrix}$

b. $\begin{bmatrix} 0.0 & 0.2 & 0.8 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.6 & 0.4 \end{bmatrix}$

c. $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.7 & 0.1 & 0.2 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$

d. $\begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.4 & 0.0 & 0.6 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}$

4.2 In the transition diagram below, C represents cloudy weather, R represents rainy weather, and S represents sunny weather. Use this diagram to predict the numbers of rainy, sunny, and cloudy days in any one year.



4.3 In Problem **2.3**, the managers of Boards Incorporated suggested a quality-control using only two states. The probability of moving from the ordinary state to the heightened state was 0.128. The probability of moving from the heightened state back to the ordinary state was 0.869.

a. Find the stable state matrix for this information, if one exists.

b. Find the stable state vector for this information, if it exists.

c. After 25 transitions, what proportion of the sampling would you predict will be done in each state?

4.4 In Problem **3.3**, a quality control specialist suggested another sampling strategy involving three states. In the ordinary state, a sample of 20 skateboards is taken each week. The heightened state requires a sample of 30 boards per week, while the relaxed state requires a sample of 10 boards. The transition matrix for this strategy is given below:

$$\mathbf{T} = \begin{array}{c} \\ R \\ O \\ H \end{array}\begin{array}{c} \begin{array}{ccc} R & O & H \end{array} \\ \begin{bmatrix} 0.879 & 0.121 & 0.000 \\ 0.206 & 0.598 & 0.196 \\ 0.000 & 0.129 & 0.871 \end{bmatrix} \end{array}$$

a. Determine the probability of each of the following:

1. being in the ordinary state after two transitions, given that the initial state was ordinary.
2. being in the ordinary state after five transitions, given that the initial state was relaxed.
3. being in a relaxed state after three transitions, given that the initial state was ordinary.

b. Determine the probability of each state after 52 weeks of sampling.

c. Describe the effects of the initial state vector on the resulting state vector as the number of transitions increases.

4.5 Consider a carnival game in which players toss a ball through a hole in a board. Players win prizes based on the number of successful tosses in a row: the more successful tosses in a row, the bigger the prize.

After any successful toss, a player can decide that the game is over and collect a prize. The player also may continue the game, as long as the previous toss was successful. After any unsuccessful toss, the game is over and no prize is awarded.

a. After making a successful toss, players make another successful toss 25% of the time, are unsuccessful 65% of the time, and quit 10% of the time. Create a transition matrix for this situation.

b. Assuming that the initial toss is successful, predict the probability that a player will make a successful sixth toss.

c. Predict the probability of each state over the long run.

d. This Markov chain has a state called the **absorbing state**. What do you think this term means?

* * * * *

4.6 A large ski resort has decided to upgrade the mass transit system it uses to bring skiers from a nearby city to the mountain. Currently, the resort uses school buses. They would like to change to motor coaches. At present, 28% of skiers take the mass transit system, while 72% drive their own vehicles.

To help predict the percentage of skiers who will use the upgraded transit system, the resort owners conducted a survey. The results of the survey are shown in the following transition matrix.

$$\begin{array}{cc} & \text{Next Year} \\ & \begin{array}{cc} \text{Mass Transit} & \text{Own Vehicle} \end{array} \\ \text{This Year} \begin{array}{r} \text{Mass Transit} \\ \text{Own Vehicle} \end{array} & \begin{bmatrix} 0.65 & 0.35 \\ 0.25 & 0.75 \end{bmatrix} \end{array}$$

a. Assuming that the total number of skiers remains constant, what percentage do you predict will use the new transit system after each of the following numbers of years?

1. 1

2. 2

3. 5

b. What percentage of skiers do you predict will use the mass transit system in the long run?

* * * * * * * * * *

Research Project

The game Tug O'Spot can be played by one or two people. The game is played by placing a marker on a board, then moving it according to the outcome of a roll of dice. As shown in Figure **6** below, there are five available starting positions. Depending on where you start, the probabilities of winning change.

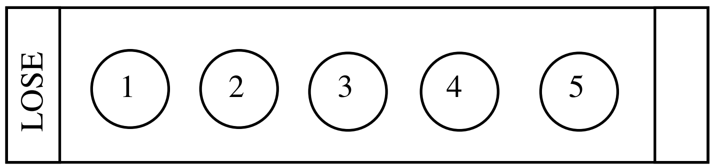


Figure 6: Tug O'Spot game board

To determine the marker's starting position in the two-person version of the game, one player rolls a die. If a 6 shows, the player rolls again. If any other number shows, the player places his marker on the starting position that corresponds to the number on the die.

Once the marker has been placed, each player tosses a die. If the two rolls are the same, the marker is moved two positions toward WIN. If the two rolls differ by 1, then the marker is moved one position toward WIN. In all other cases, the marker is moved one position toward LOSE. When the marker reaches either end, the game is over. The player who placed the marker is credited with either a win or a loss.

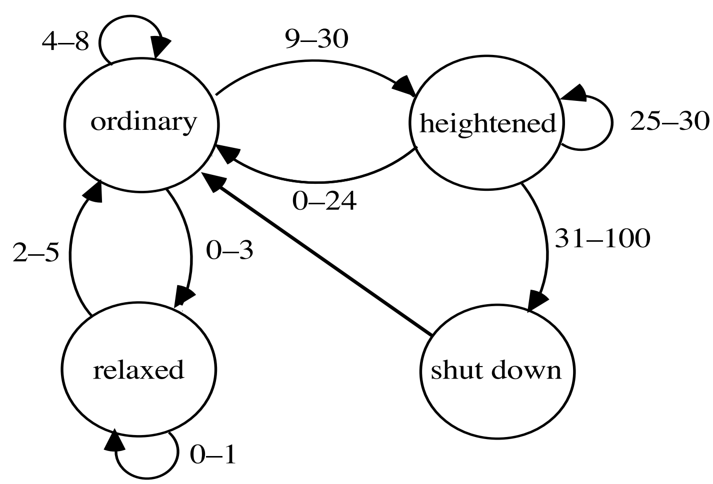
The one-person version is played in a similar manner, with the player tossing two dice each time.

Use your knowledge of Markov processes to determine the probabilities of winning from each of the five possible starting positions.

Summary Assessment

Boards Incorporated has decided to make one more adjustment to the company's quality-control process. If 31 or more defective skateboards are found while sampling in the heightened state, the plant will be shut down for one week while the source of the defects is identified. In the following week, sampling will continue in the ordinary state.

The quality-control process now contains four states. The diagram below shows how many defective items result in the various transitions in the process.



It costs the company about $15 to test each skateboard sampled. The estimated cost of a shut down is $2000 per week.

1. Predict how much it will cost to follow the approved quality-control process for one year. Your response should contain a detailed explanation of how you arrived at your estimate, including any assumptions you made.

2. The quality-control process described above is based on an assumed defective rate of 20%. Explain what you would expect to occur if the actual rate of defective items was greater than 20%.

Module Summary

- A **binomial experiment** has the following characteristics:
  1. It consists of a fixed number of repetitions (trials) of the same action.
  2. The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
  3. Each trial has only two possible outcomes: a success or a failure.
  4. The probability of a success remains the same from trial to trial.
  5. The total number of successes is observed.
- The probability distribution for a binomial experiment is a **binomial distribution**.
- The mean μ of a binomial distribution is the product of the number of trials and the probability of a success. In other words, $\mu = np$, where n is the number of trials and p is the probability of a success.
- The standard deviation σ of a binomial distribution is the square root of the product of the number of trials, the probability of a success, and the probability of a failure:

$$\sigma = \sqrt{np(1-p)}$$

- The **binomial probability formula** can be used to determine the probability of obtaining r successes in n trials in a binomial experiment. Symbolically, the binomial formula can be written as follows, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

- A **continuous probability distribution** results when the outcomes of an experiment can take on all possible real-number values within an interval.
- A **normal distribution** is a continuous probability distribution. The graph of a normal distribution is symmetric about the mean and tapers to the left and right like a bell. The curve that describes the shape of the graph is the **normal curve**. The equation of the normal curve that models a particular set of data depends on the mean and standard deviation of the data.

  As in all continuous probability distributions, the total area between the horizontal axis and a normal curve is 1. Approximately 68% of this area falls within 1 standard deviation of the mean, 95% within 2 standard deviations of the mean, and 99.7% within 3 standard deviations of the mean. This is the **68–95–99.7 rule**.

- The process of moving from one **state**, or outcome, to another is a **transition**.
- A **transition diagram** is a convenient way to display the possible changes among states. In such diagrams, each state is represented by a vertex, while each transition is represented by a directed edge labeled with a corresponding probability.
- A **Markov chain** is a model for predicting the probability of moving from one state (or outcome) to other states, given that there are a finite number of states and the probability of being in one state depends only on the state before the move.
- The technique of predicting the probabilities of these transitions is the **Markov process**.
- The probabilities of moving among states can be displayed in a **transition matrix**. A transition matrix **T** for a Markov chain has the following characteristics.

  1. The matrix is square with dimensions $m \times m$, where m represents the number of states.

  2. All the elements in the matrix are between 0 and 1, inclusive.

  3. The sum of the elements in any row is 1.

- The element in row i, column j in the matrix $\mathbf{T}^n$ represents the probability of moving from state i to state j after n transitions.
- A transition matrix **T** is **regular** if for some n, all of the elements in the matrix $\mathbf{T}^n$ are positive. A Markov chain is **regular** if its transition matrix is regular.
- The **initial state vector** $\mathbf{X}_0$ of a population is represented by a matrix with a single row. Each element of the matrix represents the probability of a state before any transitions.
- In general, the **state vector** $\mathbf{X}_n = \mathbf{X}_{n-1} \bullet \mathbf{T} = \mathbf{X}_0 \bullet \mathbf{T}^n$. Each element of $\mathbf{X}_n$ represents the probability of a state after n transitions.

  Each element of $\mathbf{T}^n$ represents the probability of moving from one state to another state after n transitions.

- A **stable** (or **steady**) **state matrix** is formed by raising a transition matrix to some power such that the difference between any two elements in the same column is very small.
- Multiplying any state vector by the stable state matrix results in a **stable** (or **steady**) **state vector**. This vector consists of one row of the stable state matrix.
- All regular Markov chains have a stable state matrix and vector. Markov chains that are not regular may or may not have a stable state matrix and vector.

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