Let There Be Light

Why are circles, ellipses, parabolas, and hyperbolas so important in our technological world? In this module, you answer this question by studying a group of shapes called conics.

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Introduction

At a stadium in Pasadena, California, two soccer teams stand ready. In Rome, Italy, fans anxiously await the beginning of the match. It's the World Cup final—the world's most watched sporting event! Millions of soccer fans expect to view every second of the game, live on television. But how can a television signal be received at countless locations, all over the globe, all at the same time?

The answer involves parabolic reflectors. First, a reflector in the shape of a parabola focuses a signal from a broadcasting station near the stadium, transmitting it to a satellite 36,000 km above the earth's surface. After receiving this signal, the satellite broadcasts it back towards earth. Parabolic dishes throughout the world receive and concentrate the signal. Local networks then transmit it to the television screens of eager fans.

In this module, you investigate the reflective qualities of parabolas and three other unique shapes, which together are known as the **conic sections**.

Mathematics Note

A **conic section** can be formed by the intersection of a plane with a double-napped cone. Depending on the slope of the plane, the intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.

Figure **1** shows the four conic sections and the plane-cone intersections that produce them. Note that circles and ellipses are closed figures, while parabolas and hyperbolas are not closed. Notice also that a hyperbola has two parts or **branches**.

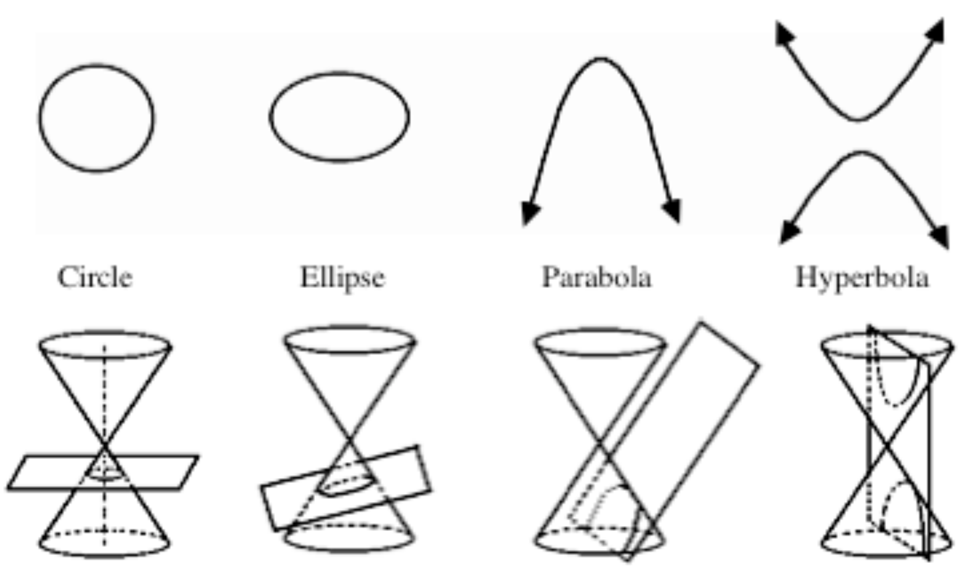


Figure 1: The conic sections

Activity 1

A satellite dish reflects television signals, a concert hall or band shell reflects sound, and a space heater reflects light and warmth. Each of these objects, shaped like a different conic section, reflects in a different way. The reflective properties of each conic are closely linked to a point (or points) known as the **focus** (or **foci**).

Mathematics Note

Figure **2** shows a diagram of the four conics. Each dot in the diagram indicates the location of a **focus** (plural **foci**) for a particular conic. Hyperbolas and ellipses each have two foci. Circles and parabolas each have one focus.

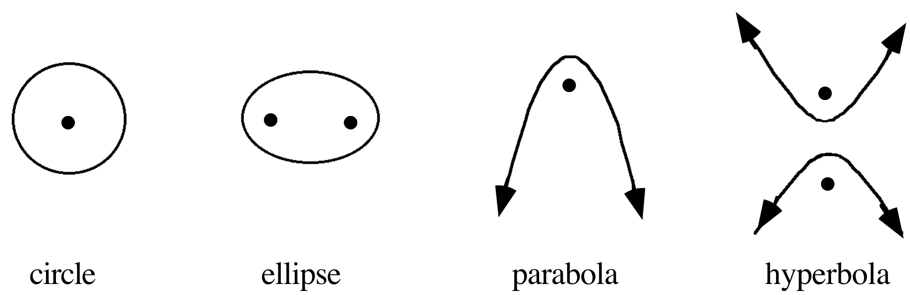


Figure 2: The foci of the conic sections

Each conic divides the plane that contains it into regions. The **interior** of a conic is the region (or regions) that contains at least one focus. The **exterior** of a conic is the region that does not contain a focus.

Exploration

In this exploration, you experiment with the reflective properties of conics by passing light rays through their foci.

a. Partially cover a flashlight lens with black electrical tape, leaving a slit 1–2 mm wide, as shown in Figure **3**.

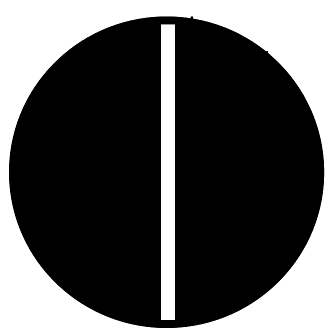


Figure 3: Flashlight lens partially covered with tape

b. Obtain an ellipse template, a sheet of cardboard, and some reflective material from your teacher.

1. Tape or glue the template to the sheet of cardboard.
2. To create a slot for the reflective material, cut through the cardboard along approximately $2/3$ of the outline of the ellipse.
3. As shown in Figure **4** below, insert a strip of reflective material in the slot from Step **2**.

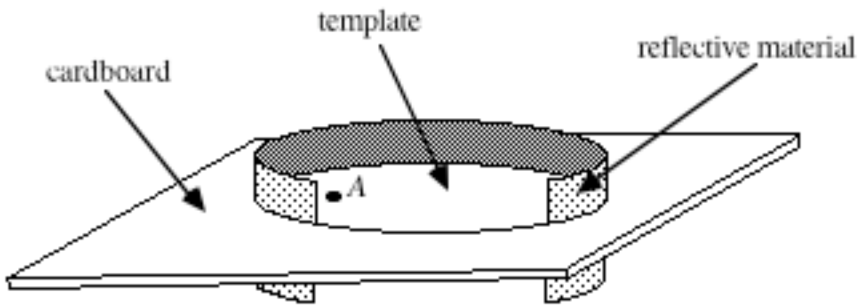


Figure 4: Model of ellipse with reflective material

c.
1. Hold the flashlight so that the long axis of the slit you created in Part **a** is vertical. As shown in Figure **5**, shine the beam of light through a focus and onto the reflective material.

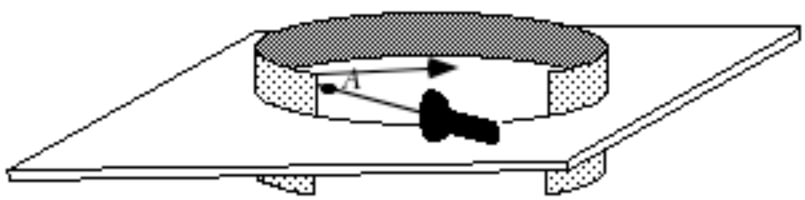


Figure 5: Flashlight beam passing through a focus

2. Use a pencil to trace the path of the reflected light.
3. Move the flashlight so that the beam of light, after passing through the focus, strikes the reflective material at several different locations. Trace the path of the reflected light in each case.
4. Move the flashlight so that the beam of light does not pass through the focus. Trace the path of reflected light. **Note:** Save the template for use in the assignment.

d. Repeat Parts **b** and **c** using the template of a circle.

e. Repeat Parts **b** and **c** using the template of a parabola.

Discussion

a. Each conic section has at least one axis of symmetry. Describe the locations of the axes for a circle, an ellipse, a parabola and a hyperbola.

b. Describe the reflective properties of each of the following conic sections:

1. an ellipse

2. a circle

3. a parabola.

Mathematics Note

The reflective properties of conics are used in many real-world applications. In most cases, the actual reflectors are the three-dimensional counterparts of the conic sections. Figure **6** shows one example of each of these shapes: a sphere, a paraboloid, a hyperboloid, and an ellipsoid.

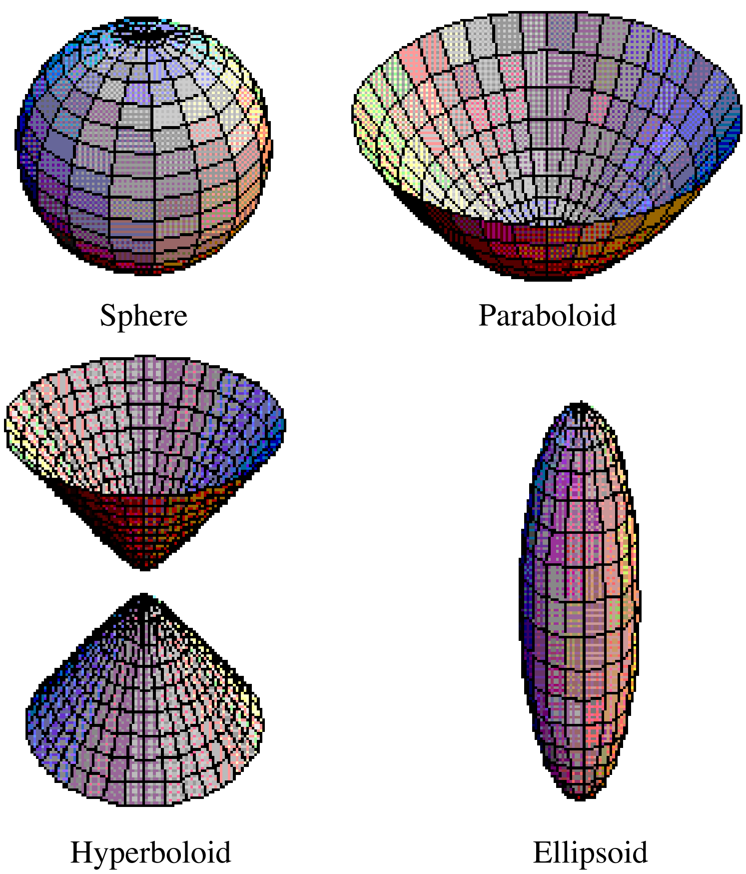


Figure 6: Three-dimensional conic reflectors

A **sphere** is generated by rotating a circle around any axis of symmetry.

A **paraboloid** is generated by rotating a parabola around its axis of symmetry.

A **hyperboloid** is generated by rotating a hyperbola around the axis of symmetry that contains the foci.

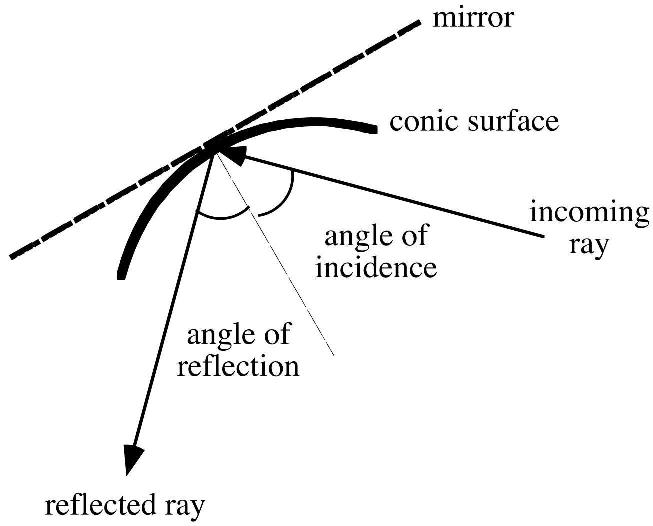
An **ellipsoid** is generated by rotating an ellipse around the axis of symmetry that contains the foci.

Light passing through a focus of one of these three-dimensional shapes is reflected in the same manner as light reflected from the corresponding two-dimensional conic. For example, light rays passing through one focus of an ellipsoid are reflected through the other focus.

c. Consider a mirror shaped like a paraboloid. If a ray of light enters the mirror parallel to the paraboloid's axis of symmetry, in what direction would you expect it to be reflected?

Assignment

1.1 Light rays reflecting off a flat mirror form an angle of incidence that is congruent to the angle of reflection. As shown in the diagram below, when light is reflected off a point on a conic (or any other curve), it behaves as if it were reflecting off a flat mirror tangent to the curve at the point of reflection.



a. **1.** On the template of the ellipse from the exploration, locate an angle formed by an incoming light ray and its reflected ray.

2. Bisect the angle to find the angle of incidence and the angle of reflection.

3. To draw the line represented by the flat mirror in the diagram above, construct a line perpendicular to the angle bisector at the point of reflection. This line is tangent to the curve at the point of reflection.

b. Consider a light ray traveling toward a focus of an ellipse from its exterior.

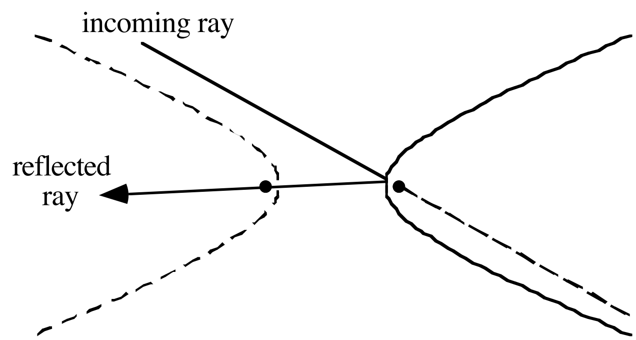
1. Sketch one such ray that intersects the ellipse at the point of reflection from Part **a**.

2. Draw the reflected ray for the incoming ray in Step **1**. (Recall that the angle of incidence and the angle of reflection are congruent.)

c. Describe the path that a ray of light traveling toward the focus of a circle would take if it were reflected off the exterior of the circle. Draw a diagram to illustrate your conclusions.

d. Repeat Part **c** for a parabola.

1.2 When light rays traveling toward a focus from the exterior of a hyperbolic mirror reflect off the surface, they are directed towards the other focus, as shown in the diagram below.



a. Obtain a copy of this diagram from your teacher. At the point of reflection, construct a line tangent to the hyperbola. (Recall that the angle of incidence is congruent to the angle of reflection.)

b. Consider a light ray traveling through the focus from the interior of the hyperbolic reflector towards the same point of reflection as Part **a**.

1. Sketch the paths of the incoming and reflected rays.

2. Draw the lines containing these rays.

c. Repeat Parts **a** and **b** for several other light rays.

d. In a brief paragraph, describe the reflective properties of a hyperbola.

1.3 **a.** Consider a paraboloid mirror with a light bulb placed at its focus. Make a sketch of a cross section of this paraboloid. On your sketch, show how a ray of light from the bulb would be reflected off the mirror.

b. A car headlight consists of a parabolic reflector with a bulb at the focus. Why is this an appropriate design?

c. The dish for a satellite television antenna is a paraboloid. The receiver for the television signal is located at the focus of the paraboloid. Why is this an appropriate design?

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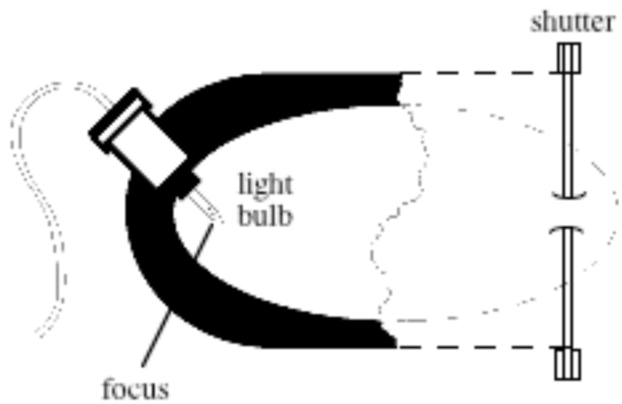
1.4 Why does cupping your hand around your ear make it easier to hear faint sounds?

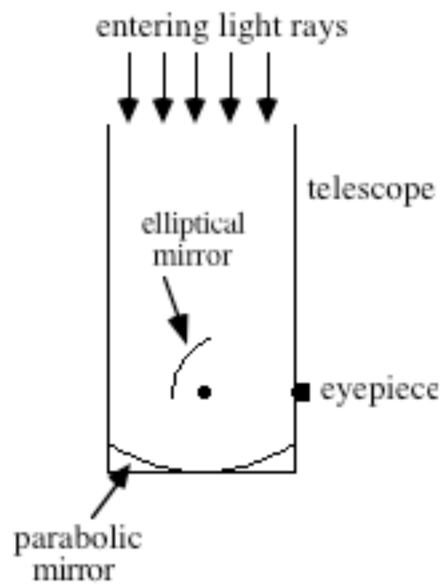
1.5 The Statuary Hall in the U.S. Capitol was built in the shape of an ellipse. By standing at a particular location, you can overhear the whispers of another person across the room. Explain why this is possible.

1.6 The following diagram shows a cross section of a spotlight with an ellipsoidal reflector. The light bulb is located at one focus of the ellipse.



On this type of spotlight, the shutter creates a circular beam of light. Through what point do all the light rays pass before continuing on through the hole in the shutter?

1.7 Some telescopes are constructed so that the focus of a parabolic mirror and the focus of an elliptical mirror occur at the same point. The eyepiece is located at the other focus of the ellipse. Describe what happens to a ray of light entering the telescope.



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Activity 2

When designing reflective surfaces, engineers often use the **standard form** of the equations for a conic sections. In this activity, you examine the standard form of the equation for an ellipse.

Exploration

a. Construct a rectangular coordinate system with its origin near the center of a sheet of graph paper.

1. Label the *x*- and *y*-axes and graph two points F_1 (–4,0) and F_2 (4,0). These points represent the foci of an ellipse.
2. On a string with a length of about 25 units (in your coordinate system), mark the endpoints of a segment 10 units long.

   When held straight, without stretching, your segment should extend from (–5,0) to (5,0) along the *x*-axis.

b. Place the two points marked on the string so that they coincide with the foci, F_1 and F_2. As shown in Figure **7**, use the edge of a ruler to hold these points firmly in position.

Trace the portion of an ellipse above the ruler with a pencil, making sure to keep the string taut. Reposition the string and ruler to complete the lower portion of the ellipse.

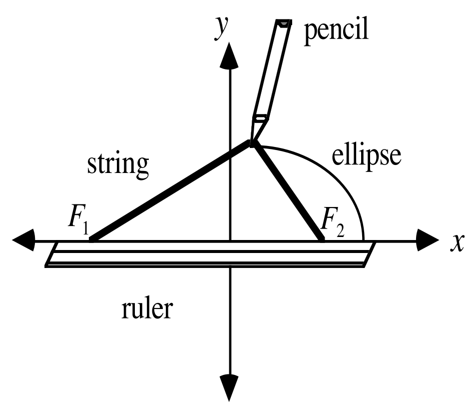


Figure 7: Drawing an ellipse

c. Use your graph from Part **b** to complete Table **1** for points on the ellipse. Estimate each coordinate as accurately as possible.

Table 1: Coordinates of points on an ellipse

	Foci: $F_1(-4,0)$ and $F_2(4,0)$											
x	0	0			1	1	2	2	–1	–1	–2	–2
y			0	0								

d. For each point P in Table **1**, find PF_1, the distance from point P to the focus F_1, and PF_2, the distance from P to F_2.

e. Calculate $PF_1 + PF_2$ for each point and describe any patterns you observe.

f. Determine the length of the segment that contains the foci and whose endpoints are on the ellipse.

g. Using the same sheet of graph paper, repeat Parts **b–f** for each of the following pairs of foci:

1. (–1,0) and (1,0)

2. (0,–4) and (0,4)

Discussion

a. Describe the symmetry of the three ellipses you drew in the exploration.

b. **1.** What happens to the shape of an ellipse as the distance between the foci decreases?

2. What shape would result if the distance between the foci was 0?

c. What happens to the shape of an ellipse as the distance between the foci increases?

d. How does an ellipse with its foci located on the *y*-axis compare with an ellipse with its foci located on the *x*-axis?

e. Suppose that the length of the segment marked on the string had been 12 units instead of 10 units. In this case, what would have been your response to Part **e** of the exploration?

f. In the exploration, you examined the sum of the distances from a point on the ellipse to the foci. You also found the length of the segment that contains the foci and whose endpoints are on the ellipse. What is the relationship between these two lengths?

Mathematics Note

An **ellipse** is a set of points in the plane such that the sum of the distances from each point to two foci is a constant. For example, Figure **8** shows an ellipse with foci at F_1 and F_2. For any point *P* on the ellipse, $PF_1 + PF_2$ is a constant.

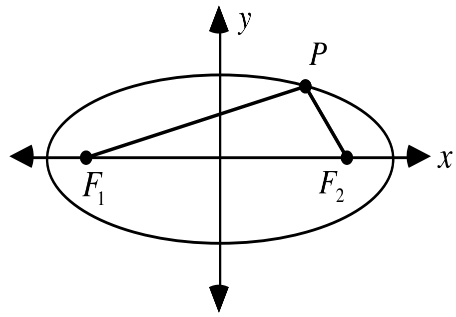


Figure 8: An ellipse with its foci on the *x*-axis

The **major axis** of an ellipse is the segment, with endpoints on the ellipse, that contains the foci. The **minor axis** is the segment, with endpoints on the ellipse, contained in the perpendicular bisector of the major axis. The major and minor axes intersect at the **center** of the ellipse. The endpoints of the major and minor axes are the **vertices** of the ellipse.

For example, Figure **9** shows an ellipse with foci at F_1 and F_2. In this case, $\overline{MN}$ is the major axis and $\overline{TW}$ is the minor axis. The origin *O* is the center of the ellipse. Points *M*, *N*, *W*, and *T* are the vertices of the ellipse.

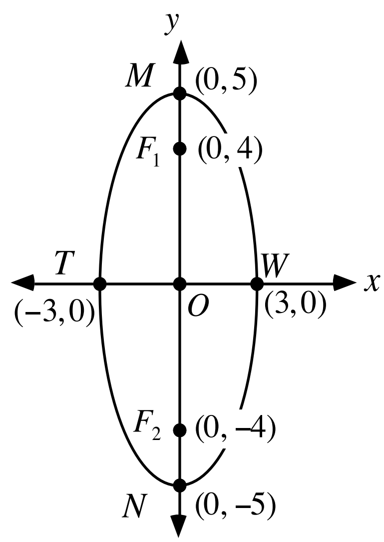


Figure 9: An ellipse with its foci on the *y*-axis

g. What are the vertices of the three ellipses you created in the exploration?

h. Figure **10** shows the graph of an ellipse with foci at $(-c,0)$ and $(c,0)$.

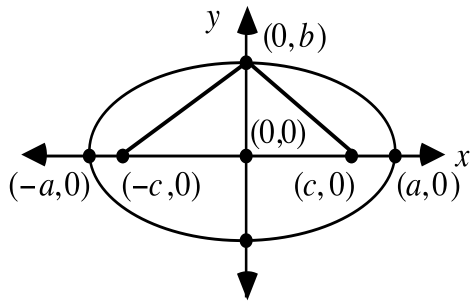


Figure 10: An ellipse with foci at $(-c,0)$ and $(c,0)$

1. Using this graph, describe how the length *s* of the segment marked on the string and the locations of the foci determine the locations of the vertices contained in the major axis. Hint: Examine the distances from the vertices to the center and from the vertices to the foci.
2. Describe how the length *s* and the locations of the foci determine the locations of the vertices contained in the minor axis.

Mathematics Note

The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *x*-axis is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where *a* is half the length of the major axis and *b* is half the length of the minor axis.

The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the *y*-axis is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

For example, the ellipse in Figure **9** has a major axis 10 units long and a minor axis 6 units long. The equation of this ellipse is shown below:

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

i. Describe the equation in standard form of each ellipse constructed in the exploration.

j. **1.** Figure **10** shows an ellipse with foci at $(-c, 0)$ and $(c, 0)$. The length of its major axis is $2a$. Use this figure to write an equation that describes the relationship among a, b, and c.

2. Given the equation of an ellipse with center at the origin, describe how you could use the relationship among a, b, and c to determine the coordinates of the foci.

k. Describe the shape of an ellipse in which $a = b$.

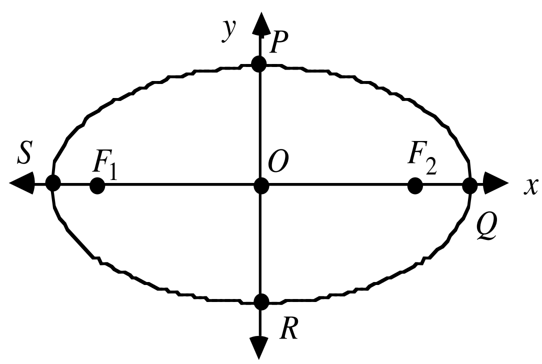
Assignment

2.1 Describe how the definition of an ellipse given in the mathematics note is illustrated by the string constructions you made in the exploration.

2.2 Describe a method for determining the intercepts of an ellipse with center at the origin, given its equation in the form below:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2.3 The figure below shows an ellipse with foci at F_1 and F_2 and center at the origin.



The equation of this ellipse is:

$$\frac{x^2}{10^2} + \frac{y^2}{8^2} = 1$$

a. Find the coordinates of the vertices P, Q, R, and S and the foci F_1 and F_2.

b. Determine each of the following distances: PR, SQ, PO, PF_2, and OQ.

c. Write an equation that describes the relationship among the length of the major axis, the length of the minor axis, and the distance between the foci.

2.4 In Part **c** of the exploration, you constructed an ellipse with foci at (–4,0) and (4,0) in which the length of the major axis was 10.

a. **1.** Write the equation of the ellipse in standard form.

2. Test your equation using the coordinates of points in Table **1**.

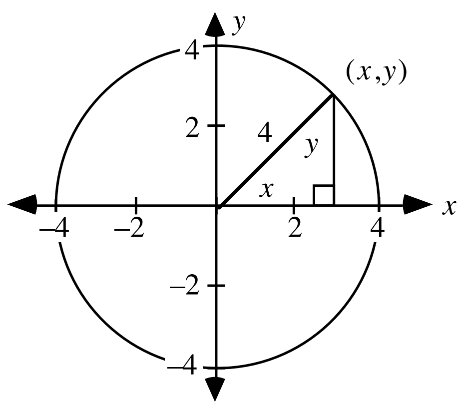
b. Consider a point P with coordinates (x,y) on an ellipse with center at the origin and foci at $(-c,0)$ and $(c,0)$. Using the definition of an ellipse and the distance formula, the length $(2a)$ of the major axis can be described as follows:

$$2a = \sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2}$$

1. Substitute the values for a and c from the ellipse in Part **a** into this equation.

2. Test the resulting equation using the coordinates of points in Table **1**.

2.5 The circle shown in the figure below is the set of all points whose distance from the origin is 4.



a. Given that (x,y) is any point on the circle, use the Pythagorean theorem to write an equation that shows the relationship among x, y, and 4. This is the standard form of the equation of a circle with center at the origin and radius 4.

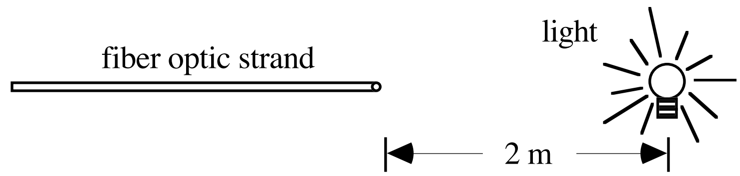
b. Use a symbolic manipulator to solve the following equation for y:

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} = 1$$

c. Use a graphing utility to graph the result in Part **b**.

d. Show algebraically that the equation you wrote in Part **a** is equivalent to the equation given in Part **b**.

2.6 Imagine that you are an engineer at a fiber optics company. As part of an ongoing project, you must design a reflector to focus the light from a bulb onto the end of a fiber optic strand 2 m away.



a. Design an appropriate reflector. Describe the locations of the bulb and the end of the fiber optic strand in relation to the reflector and explain why you chose these locations.

b. Make a scale drawing of the reflector and label its dimensions.

c. Write an equation that would allow a computer to draw a cross section of your reflector.

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2.7 The equation $9x^2 + 16y^2 = 144$ describes an ellipse.

a. Graph this equation.

b. Determine the *x*- and *y*-intercepts of the ellipse.

c. Find the coordinates of the foci.

2.8 Consider a satellite traveling in an elliptical orbit directly above the earth's equator. One focus of the ellipse is at the earth's center. The distance from each focus to the center of the ellipse is 500 km. The length of the major axis is 16,000 km.

a. Write an equation that describes the satellite's orbit.

b. How does the shape of the satellite's orbit compare to a circle?

c. The radius of the earth at the equator is about 6400 km.

1. How far is the satellite from the earth's surface when the orbit is nearest the earth ?

2. How far is the satellite from the earth's surface when the orbit is farthest from the earth?

2.9 **a.** Graph the following two ellipses on the same coordinate system.

$$16x^2 + 64y^2 = 1024$$

$$49x^2 + 25y^2 = 4900$$

b. The formula for the area of an ellipse is $A = \pi ab$, where a and b are half the lengths of the major and minor axes, respectively. Determine the areas of the two ellipses in Part **a**.

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Activity 3

Some applications of reflectors require the scattering of heat, light, or sound. For example, an effective wall heater disperses heat uniformly in a room, while a well-designed band shell reflects sound so that the entire audience can hear.

The conic whose reflective properties are well-suited for these applications is the hyperbola. Like an ellipse, a hyperbola can be defined as a set of points that satisfy a common rule.

Mathematics Note

A **hyperbola** is a set of points in the plane such that the difference of the distances from each point to two foci is a constant.

For example, Figure **11** shows a hyperbola with foci F_1 and F_2. For any point P on the hyperbola, the difference between PF_1 and PF_2 is a constant.

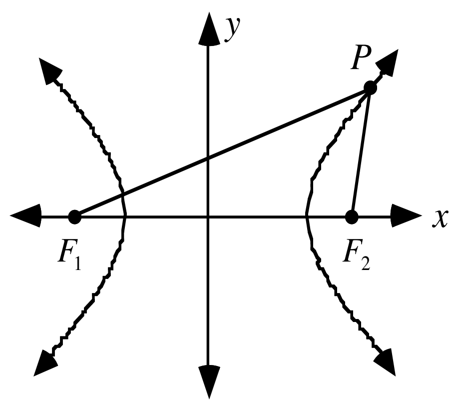


Figure 11: A hyperbola with center at the origin

The **vertices** of a hyperbola are the intersections of the hyperbola and the segment that joins the foci. The **transverse axis** is the line that passes through the vertices. The **conjugate axis** is perpendicular to the transverse axis and bisects the segment joining the vertices at the **center** of the hyperbola.

In Figure **12**, for example, the vertices of the hyperbola are $(-a,0)$ and $(a,0)$. The transverse axis is the *x*-axis and the conjugate axis is the *y*-axis. The center is at the origin, the midpoint of $(-a,0)$ and $(a,0)$.

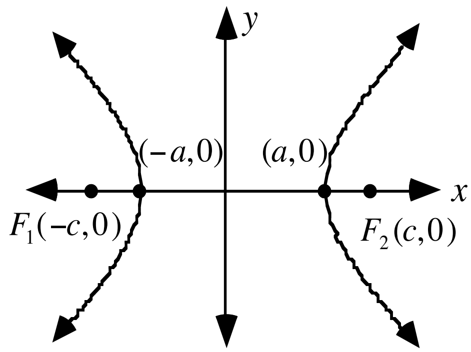


Figure 12: A hyperbola with foci on the *x*-axis

The **standard form** of the equation of a hyperbola with center at the origin and foci at $(-c,0)$ and $(c,0)$ is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a is the distance from the center to a vertex and $b^2 = c^2 - a^2$.

For example, consider a hyperbola with foci at (–5,0) and (5,0) and vertices at (–3,0) and (3,0). In this case, $c = 5$, $a = 3$, and $b = \sqrt{25-9} = 4$. The equation of this hyperbola, in standard form, is

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$$

Exploration

a. Select values for a and b in the following equation of a hyperbola with center at the origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

b.
1. Determine the coordinates of the vertices.
2. Determine the coordinates of the foci.
3. Determine the distance between each focus and the center.

c. Solve the equation in Part **a** for y and graph both parts of your solution.

d.
1. Select a point P on the hyperbola in the first quadrant. Find $|PF_1 - PF_2|$, where F_1 and F_2 are the foci.
2. Select a point Q on the hyperbola in the second quadrant. Verify that $|QF_1 - QF_2|$ equals the constant difference found in Step **1**.
3. Determine the distance between the vertices of the hyperbola.
4. Describe the relationship between the constant difference and the distance between the vertices.

e. The hyperbola in Part **a** is not a function because each x-value in the domain, other than the vertices, has two corresponding y-values.

Complete Table **2** for points on the hyperbola, using the given x-values. Describe any trends or patterns you observe.

Table 2: Coordinates of points on a hyperbola

x	10	10	100	100	1000	1000	10,000	10,000
y								
y/x								

f. Determine the values of b/a and $-b/a$ for the hyperbola. Compare these values with the ratios for y/x in Table **2**.

g. In Part **f**, you found that as the value of x increases, the ratio y/x appears to approach b/a or $-b/a$. If $b/a = y/x$, then $y = (b/a)x$.

1. Graph the lines $y = (b/a)x$ and $y = -(b/a)x$ on the same coordinate system as the hyperbola in Part **c**.
2. Compare the y-values for points on the hyperbola with those of the corresponding points on the lines.

h. On your graph from Part **g**, draw a rectangle with center at the origin, a horizontal length of $2a$, and a vertical length of $2b$. Draw the diagonals of the rectangle and note any relationships you observe between the rectangle and the hyperbola.

i.

1. Replot the graphs of $y = (b/a)x$, $y = -(b/a)x$, and the hyperbola using intervals for the domain and range that are 10 times those in Part **g**. Compare the y-values for points on the hyperbola with those of the corresponding points on the lines.
2. Repeat Step **1** using intervals for the domain and range that are 100 times those in Part **g**.

j. Using the values you chose for a and b in Part **a**, repeat Parts **b** and **c** for the following equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line approaches zero as the distance from P to the origin increases without bound, where P is on a suitable part of the curve.

The asymptotes for a hyperbola with an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are the lines $y = (b/a)x$ and $y = -(b/a)x$.

For example, Figure **13** shows a graph of the hyperbola

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$$

The equations of the asymptotes are $y = (4/3)x$ and $y = -(4/3)x$.

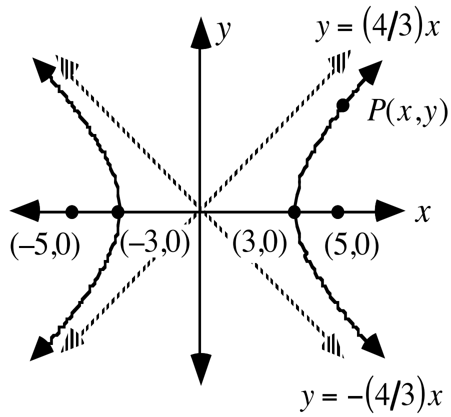


Figure 13: A hyperbola and its asymptotes

Discussion

a. Compare the graph of your hyperbola in Part **c** with those of others in the class. What role do the values of a and b in the equation below have in determining the shape of the hyperbola?

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

b. How are the values of a and b in the standard form of the equation of a hyperbola related to the distance between a focus and the center?

c. Given an equation of a hyperbola in standard form, how could you find the coordinates of the foci?

d. Consider a rectangle with center at the origin and side lengths of $2a$ and $2b$. Describe how this rectangle can be used to quickly sketch a graph of the hyperbola with the equation below:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

e. Compare the graphs of the hyperbolas in Parts **c** and **j** of the exploration.

f. Describe the equation, in standard form, of a hyperbola with vertices located on the *y*-axis.

g. How could you determine the equations of the asymptotes for a hyperbola with its foci on the *y*-axis?

Assignment

3.1 **a.** **1.** Write an equation in standard form for a hyperbola with center at the origin and foci on the *x*-axis where $a = 10$ and $b = 5$.

2. Determine the coordinates of the foci.

3. Create a graph of the hyperbola.

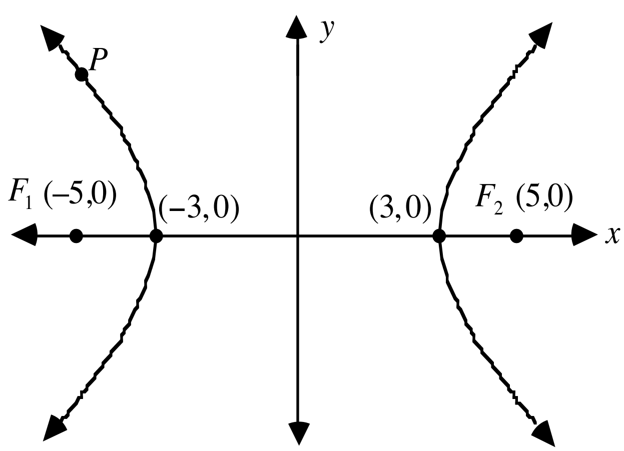
b. Repeat Part **a** for $a = 4$ and $b = 5$.

3.2 **a.** Graph the hyperbola with the equation below.

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

b. Write the equations of its asymptotes and determine the coordinates of its foci and vertices.

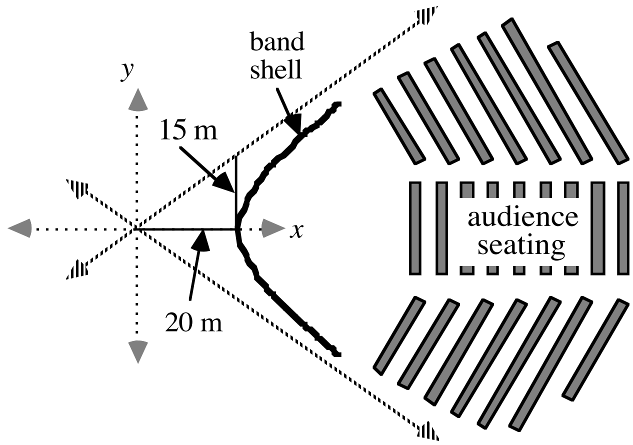
3.3 **a.** Determine the equation, in standard form, of the hyperbola shown in the following graph.



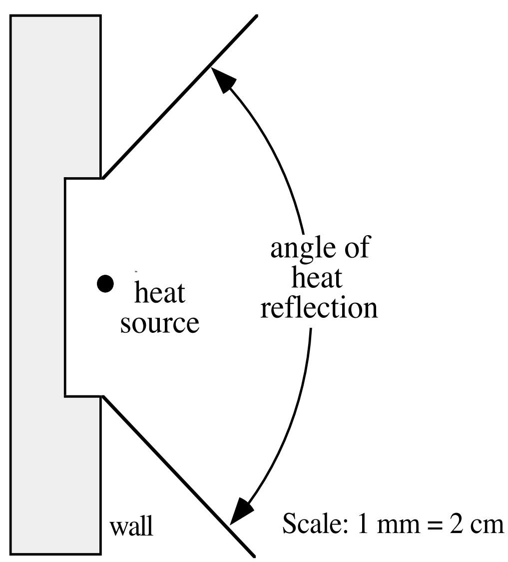
b. Consider a light ray that passes through F_1 and reflects off the hyperbola at *P*. Describe the direction in which the ray is reflected.

3.4 The conductor of a community band is designing a band shell shaped like one branch of a hyperbola. In this application, the asymptotes of the hyperbola describe the area within which sound waves will scatter when reflected off the band shell.

Write an equation, in standard form, for the hyperbolic band shell shown in the diagram below.



3.5 An engineer is designing a wall-mounted heater. To help disperse heat around the room, she wants to place a hyperbolic reflector behind the heat source. The diagram below shows a preliminary sketch of her design.



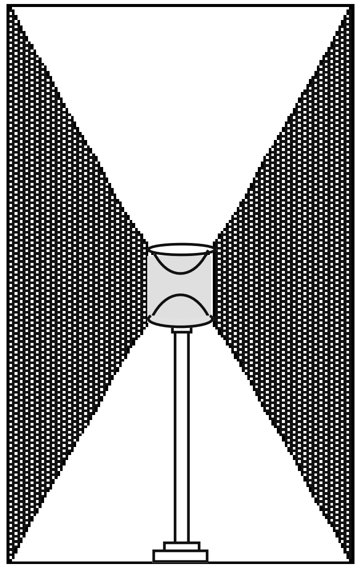
a. On a copy of this diagram, sketch the shape of a hyperbolic reflector that would help disperse heat.

b. Use your sketch to determine the lengths of *a* and *b* in centimeters.

c. Write an equation, in standard form, for your hyperbolic reflector.

* * * * *

3.6 The light cast on the wall by the lamp in the illustration below forms a hyperbola.



Trace a copy of this hyperbola onto a sheet of graph paper. Assuming that the center is located at the origin, write an equation that approximately describes the hyperbola.

3.7 Two hyperbolas are **conjugates** if they have the same asymptotes. The following hyperbolas are conjugates of each other.

$$\frac{x^2}{25}-\frac{y^2}{144}=-1 \text{ and } \frac{x^2}{25}-\frac{y^2}{144}=1$$

a. Graph these hyperbolas on the same coordinate system.

b. Compare the coordinates of the foci of the conjugate hyperbolas.

3.8 The equation $9x^2-16y^2=144$ describes a hyperbola.

a. Graph this equation.

b. Determine the x- and y-intercepts of the hyperbola.

c. Find the coordinates of the foci.

d. Rewrite the equation of this hyperbola in standard form.

* * * * * * * * * *

Activity 4

In Activity **1**, you examined the reflective properties of the parabola. In this activity, you investigate how these properties can be used to design a radio telescope.

Mathematics Note

A **parabola** is the set of all points in a plane that are the same distance from a fixed line and a fixed point not on the line. The line is the parabola's **directrix** and the point is its **focus**.

A parabola is symmetric about the line perpendicular to the directrix and passing through the focus. The point where the axis of symmetry intersects the parabola is the **vertex**.

For example, Figure **14** shows a parabola with its focus at point F and vertex at the origin. Its axis of symmetry is the y-axis. For any point P on the parabola, $PQ = PF$.

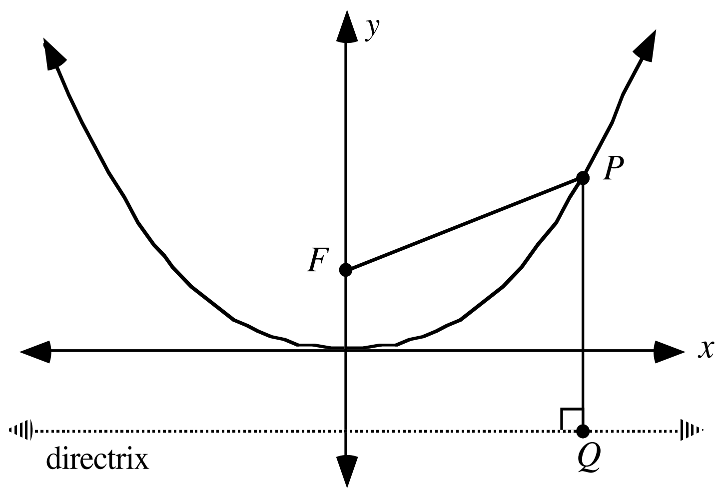


Figure 14: Parabola with vertex at origin

The general equation for a parabola with its vertex at the origin and the y-axis as its axis of symmetry is $y = ax^2$.

Exploration

Imagine that you are an astronomer. Your observatory has just received a grant to build a new radio telescope. A radio telescope consists of a collecting antenna and a radio receiver. Because of the reflective properties of a parabola, you have decided to use a huge collecting dish, 30 m across, in the shape of a paraboloid.

To determine where to place the radio receiver, you must locate the focus of the paraboloid. In this exploration, you investigate a method for relating the general equation of a parabola to the location of its focus.

a. Figure **15** shows a parabola with focus at (0,2) and vertex at the origin.

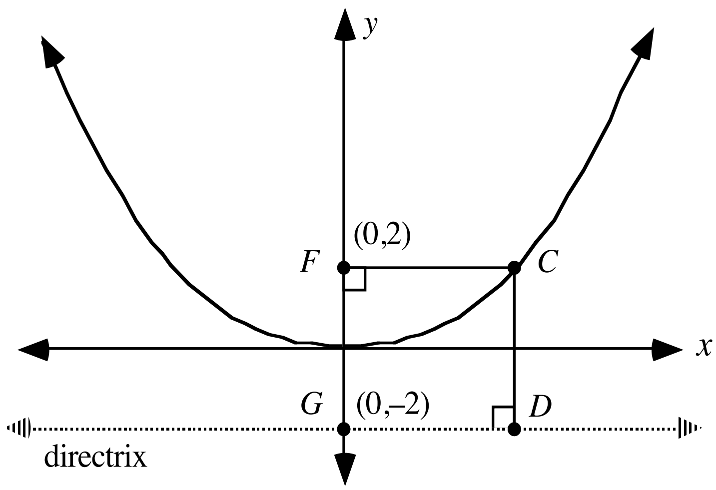


Figure 15: Parabola with focus at (0,2)

1. Given that the coordinates of F are (0,2) and the coordinates of G are (0,–2), determine the coordinates of point C.
2. As mentioned in the previous mathematics note, the general equation for a parabola with its vertex at the origin and the y-axis as its axis of symmetry is $y = ax^2$.

   Substitute the coordinates of point C into the equation $y = ax^2$ and solve for a.
3. Write an equation for the parabola in Figure **15**.

b. Repeat Part **a** given that the coordinates of F are $(0, p)$ and the coordinates of G are $(0, -p)$.

Discussion

a. Given a parabola described by an equation of the form $y = ax^2$, describe how to find the coordinates of the focus.

b. If the coordinates of the focus are $(0, p)$, what happens to the value of p as the value of a changes?

c. Describe what happens to the value of p as the distance between the focus and the vertex increases.

d. Describe what happens to the shape of the parabola as the distance between the focus and the vertex increases.

e. The equation of a parabola is often a function. What type of function defines a parabola?

Assignment

4.1. **a.** Create graphs of the equation $y = ax^2$ for four different values of a. Include two negative and two positive values for a.

b. How does the value of a affect the shape of a parabola?

4.2 Determine the coordinates of the focus for each of the following parabolas.

a. $y = 2x^2$

b. $y = 0.5x^2$

c. $y = -10x^2$

4.3 The equation $x = 3y^2$ describes a parabola that opens to the right.

a. Sketch a graph of this parabola.

b. Determine the coordinates of the focus.

c. Find the equation of the directrix.

d. Is this equation a function? Explain your response.

4.4 Imagine that you are building a radio telescope with a parabolic dish 10 m deep and 30 m in diameter. The radio receiver will be located at the focus of the paraboloid.

a. To identify the receiver's position, complete Steps **1–5** below.

1. Make a scale drawing of a cross section of the dish on a two-dimensional coordinate system. Place the vertex at the origin.
2. Label a point B on the edge of the dish and identify its coordinates.
3. Substitute the coordinates of B into the following equation and solve for p.

$$y = \frac{1}{4p}x^2$$

4. Write an equation, in standard form, for the parabola that describes the cross section of the dish.
5. Determine the coordinates of the focus.

b. During construction of the telescope, cost overruns require you to change the size of the collecting dish. To save money, you must reduce the depth of the dish to 5 m. The diameter of 30 m remains unchanged. Repeat Part **a** for this new design.

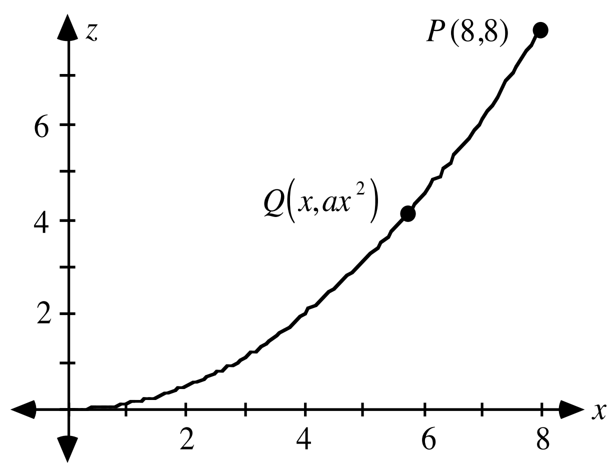
4.5 When soccer fans watch a game on television, they hear the referee's whistles on the field—but not the cries of popcorn vendors in the stands. This is because the camera crew uses directional microphones to pick up the sounds of the game.

a. A directional microphone uses a paraboloid dish to collect sounds at a focus. Sketch a cross section of a paraboloid dish. Identify the location of the microphone. Use your diagram to show how the sound waves are collected.

b. The directional microphones used at sporting events are often small enough to be carried by hand. On your sketch from Part **a**, suggest an appropriate width and depth for the paraboloid dish.

c. Using the width and depth suggested in Part **b**, determine the distance from the vertex to the focus.

4.6 The following graph shows a portion of the parabola $z = ax^2$, where $0 \le x \le 8$.



a. **1.** Determine the value of a.

2. Identify the coordinates of the focus.

b. When this curve is rotated about the z-axis, its path describes a paraboloid. Determine the coordinates, in the form (x,y,z), of the vertex and focus of this paraboloid.

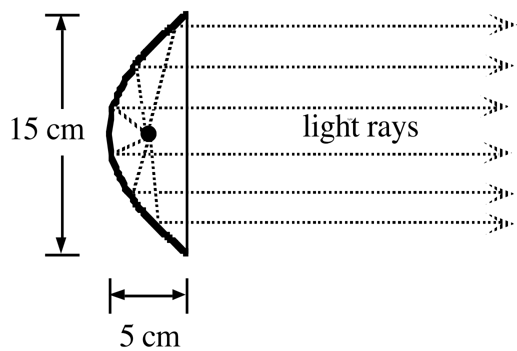
c. As the curve is rotated about the z-axis, the rim of the paraboloid is described by the path of point P. Given that the coordinates of P are (8,8), what is the radius of the rim?

d. The coordinates of point Q are $\left(x, ax^2\right)$. Describe the set of points defined by the path of Q as the curve is rotated about the z-axis.

e. Use a three-dimensional graphing utility to graph the equation $z = a\left(x^2 + y^2\right)$, where a is the value determined in Part **a**. How does this graph appear to be related to the paraboloid in Part **b**?

* * * * *

4.7 You have been asked to design a headlight for a new car. As shown in the diagram below, the headlight consists of a halogen bulb mounted in a parabolic reflector. The reflector should be 5 cm deep, and must produce a beam of parallel light rays 15 cm wide.



a. When the base of the bulb is mounted on the surface of the reflector, the light-producing filament is located 2 cm from the reflective surface. Explain why this location will not satisfy the manufacturer's constraints.

b. How would you redesign the headlight to satisfy the manufacturer's requirements?

4.8 Imagine that you are an engineer working on a new research project. Your job involves gathering radio waves from deep space in an attempt to find signs of other intelligent life. To collect radio waves, you have built a paraboloid dish 80 m in diameter and 25 m deep.

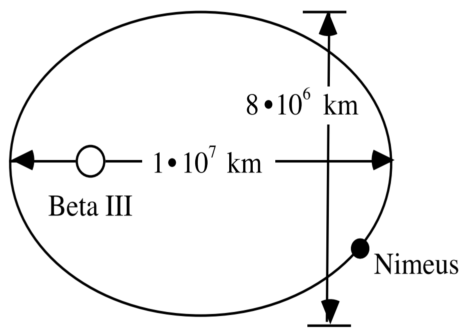
a. The radio receiver must be located at the focus of the dish. How far from the vertex of the paraboloid should you place the receiver?

b. Write an equation, in standard form, for the parabola that describes a cross section of the dish.

* * * * * * * * * *

Summary Assessment

1. The distant star Beta III is orbited by a single planet, Nimeus. Like most planets in the universe, Nimeus follows an elliptical orbit, with its star located at one focus of the ellipse. The Beta III system is shown in the diagram below.



a. Describe the location at which Nimeus is farthest from Beta III.

b. What is the maximum distance between Nimeus and Beta III? What is the minimum distance?

2. In this age of worldwide telecommunications, orbiting satellites reflect television signals back to large sections of the earth. Many of these satellites are located about 36,000 km above the equator, in what is known as the Clarke Belt. This region is named for Arthur C. Clarke, author of *2001: A Space Odyssey*. Written in 1945, this popular novel described television signals bouncing off satellites and back to earth more than 10 years before any country launched an object into space.

a. The antennas that broadcast signals from earth to a satellite are shaped like paraboloids. This allows the signal to be transmitted in focused parallel waves. Explain why this is true.

b. What disadvantages would there be to using a paraboloid reflector to direct television signals from the satellite back to earth?

c. What conic section would you choose to reflect the television signal back to earth? Defend your response.

d. Assume that the focus of the satellite's reflector is 3 m from its vertex. If television signals can be received at any point on the earth with an unobstructed line of sight to the satellite, determine an equation that describes the curve of the reflector. (The earth's diameter at the equator is about 13,000 km.)

Module Summary

- A **conic section** can be formed by the intersection of a plane with a double-napped cone. Depending on the slope of the plane, the intersection may result in a **circle**, an **ellipse**, a **parabola**, or a **hyperbola**.
- Each conic divides the plane into regions. The **interior** of a conic is the region (or regions) that contains at least one focus. The **exterior** of a conic is the region that does not contain a focus.
- **Spheres**, **paraboloids**, **hyperboloids**, and **ellipsoids** are the three-dimensional counterparts of the conic sections.
- An **ellipse** is a set of points in the plane such that the sum of the distances from each point to two foci is a constant.
- The **major axis** of an ellipse is the segment, with endpoints on the ellipse, that contains the foci. The **minor axis** is the segment, with endpoints on the ellipse, contained in the perpendicular bisector of the major axis. The major and minor axes intersect at the **center** of the ellipse. The endpoints of the major and minor axes are the **vertices** of the ellipse.
- The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the x-axis is:

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

  where a is half the length of the major axis and b is half the length of the minor axis.
- The **standard form** of the equation of an ellipse with its center at the origin and its major axis contained in the y-axis is:

  $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

  where a is half the length of the major axis and b is half the length of the minor axis.
- The **standard form** of the equation of a circle with center at the origin and radius r is $x^2 + y^2 = r^2$.
- A **hyperbola** is a set of points in the plane such that the difference of the distances from each point to two foci is a constant.

- The **vertices** of a hyperbola are the intersections of the hyperbola and the segment that joins the foci. The **transverse axis** is the line that passes through the vertices. The **conjugate axis** is perpendicular to the transverse axis and bisects the segment joining the vertices at the **center** of the hyperbola.
- The **standard form** of the equation of a hyperbola with center at the origin and foci at $(-c, 0)$ and $(c, 0)$ is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

  where a is the distance from the center to a vertex and $b^2 = c^2 - a^2$.
- An **asymptote** to a curve is a line such that the distance from a point P on the curve to the line decreases to zero as the distance from P to the origin increases without bound, where P is on a suitable part of the curve.
- The asymptotes for a hyperbola with an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

  are the lines $y = (b/a)x$ and $y = -(b/a)x$.
- A **parabola** is the set of points in a plane that are the same distance from a fixed line and a fixed point not on the line. The line is the parabola's **directrix** and the point is its **focus**.
- A parabola is symmetric about the line perpendicular to the directrix and passing through the focus. The point where the axis of symmetry intersects the parabola is the **vertex**.
- The **standard form** of the equation of a parabola with vertex at the origin and focus on the y-axis is:

$$y = \frac{1}{4p}x^2$$

  where p is the directed distance from the vertex to the focus.

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