

Wilderness Wanderings



What do a hike through the forest, a ride down a country road, and the thrust of a jet engine have in common? They all can be modeled by vectors. In this module, you learn how to use vectors to represent displacement, velocity, and force.

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Introduction

Imagine that you are a member of the Wilderness Club. The club is planning a backpacking trip in the Northwest Territories, a remote region of Canada between the Yukon and Hudson Bay.

Since the forest in the Northwest Territories is dense, with few distinguishable landmarks, you must follow a planned route. The leaders of a previous trip have recorded this route using **displacement vectors**.

Mathematics Note

A **vector** is a quantity with both magnitude and direction. A vector can be represented by an arrow as shown in Figure 1. The vector's length from tip to tail indicates its magnitude. The orientation of the arrowhead indicates its direction.

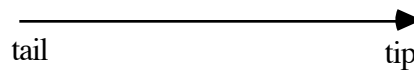


Figure 1: A vector

A vector is typically symbolized using a bold lowercase letter (\mathbf{r}), or in handwritten work, by an arrow over a letter (\vec{r}). The magnitude of a vector \mathbf{r} is represented by $|\mathbf{r}|$.

A **displacement vector** indicates both the distance moved by an object and the direction of its change in position. For example, the vectors in Figure 2 represent changes in position of 20 cm and 38 cm at azimuths of 250° and 75° , respectively, where the **azimuth** is an angle measured clockwise from north.

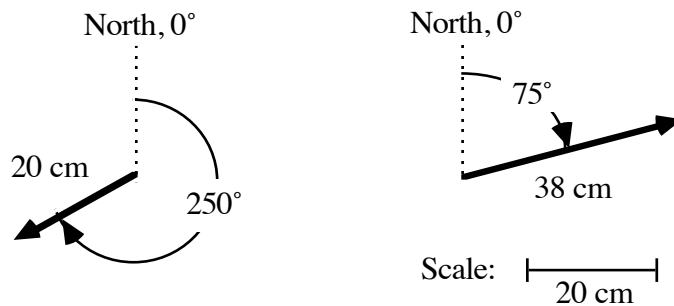


Figure 2: Displacement vectors and azimuths

Vectors can be represented using ordered pairs of numbers in many different ways. For example, the first number could represent the vector's magnitude, while the second represents its azimuth. Using this method, the vectors in Figure 2 can be represented by the ordered pairs $(20, 250)$ and $(38, 75)$, respectively.

Discussion

- a. Describe how a displacement vector could be used to identify the straight-line path from your desk to your teacher's desk.
- b. What other measurements, besides the azimuth, could be used to represent a vector's direction?

Activity 1

Your planned route to the final campsite has five sections. Table 1 describes each section as a displacement vector written as an ordered pair. The magnitudes are given in kilometers and the azimuths in degrees. In the following exploration, you use these vectors to model your route.

Table 1: Five sections of the route written as ordered pairs

Section of Hike	Vector (km,degrees)
1	(4,90)
2	(2,215)
3	(3,150)
4	(3,200)
5	(5,80)

Exploration

- a. The first section of your route begins at the base camp. To represent this camp's location, draw a point near the upper left-hand corner of a sheet of centimeter grid paper.
- b. Using an appropriate scale (and with true north located at the top of the page), construct a vector to represent the first section of the route.
- c. Starting at the tip of the first vector, construct a vector to represent the second section of the route. **Note:** Make sure to use the same scale.
- d. Continue this process until you have drawn a vector to represent each section of the route.

Mathematics Note

Equal vectors have the same magnitude and direction. **Opposite vectors** have the same magnitude but opposite directions. In Figure 3, for example, vectors **a** and **c** are equal vectors, while vector **b** is the opposite of both **a** and **c**.

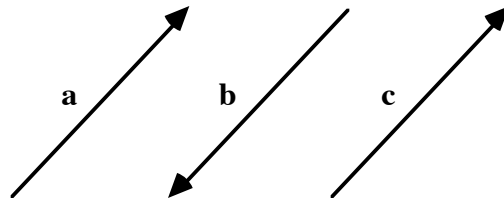


Figure 3: Three vectors

A **zero vector** has a magnitude of 0.

Vector addition can be performed geometrically by the **tip-to-tail** method. Using this method, each vector to be added is drawn so that its tail coincides with the tip of the previous vector.

The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum. Figure 4 shows the addition of vectors **a** and **b**, along with their resultant vector **r**.

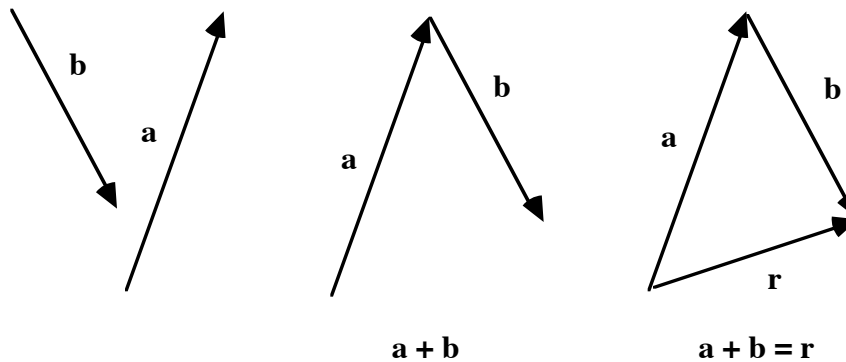


Figure 4: The tip-to-tail method of vector addition

- e. Use the tip-to-tail method to add the five vectors you drew in Parts **b–d**. Write the resultant vector as an ordered pair. This vector models the straight-line path from the base camp to the final campsite.
- f.
 1. Construct the five vectors in Table 1 in an order other than the one given in Parts **b–d**. For example, you could begin at the base camp with section 5 and end with section 1.
 2. Use the tip-to-tail method to find the resultant vector.

- g. After reaching your final campsite, you decide to fill your canteens at a nearby spring. Select a point somewhere on your diagram to represent the spring's location.
1. Draw one vector to represent the route from the campsite to the spring, and a second vector to represent the route from the spring to the campsite.
 2. Write an ordered pair to describe each vector in Step 1.
 3. Add the two vectors you drew in Step 1.

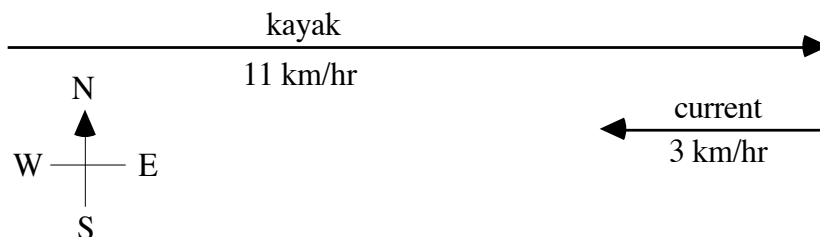
Discussion

- a.
 1. Compare the resultant vectors you identified in Parts e and f.
 2. Do you think that you would have found the same final campsite regardless of the order in which you hiked the five sections of the route? Explain your response.
- b.
 1. Do you think that vector addition is commutative? In other words, does $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$?
 2. Do you think that vector addition is associative? In other words, does $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$?
- c.
 1. Compare the ordered pairs you wrote for the two opposite vectors in Part g of the exploration.
 2. Describe the resultant vector.

Assignment

- 1.1 Consider a kayaker who can paddle at a speed of 11 km/hr on still water. How fast can she move on a river flowing to the west with a current of 3 km/hr? That depends on the direction in which she paddles.

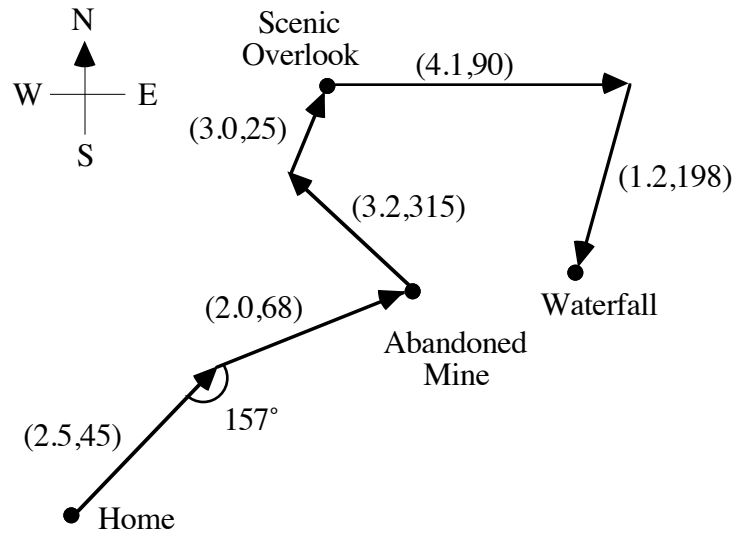
Since velocity has both magnitude and direction, it can be represented by vectors. For example, the vectors in the following diagram represent the velocities of the kayaker and the current described above when the kayaker paddles directly upstream.



- a. Describe the directions of the kayaker and the current in terms of azimuths measured clockwise from north.
- b. The sum of the two vectors in the diagram describes the kayaker's velocity with respect to the river bank. Determine this sum.

- c. What would the kayaker's resultant velocity be if she turned around and paddled downstream? Explain your response.
 - d. Describe the kayaker's resultant velocity if she paddles in the current at a speed of 11 km/hr at an azimuth of 235° .
 - e. Describe the kayaker's resultant velocity if she paddles in the current at a speed of 1 km/hr at an azimuth of 235° .
- 1.2** As part of his daily workout, a jogger runs the following routes, in any order: 400 m due east, 100 m due north, and 400 m due west.
- a. List all the different orders possible for running these routes.
 - b. Sketch a vector diagram for each possible order.
 - c. For each vector diagram in Part **b**, write a displacement vector that describes the jogger's change in position from the beginning of the run to the end of the run.
 - d. What do your results in Part **c** demonstrate about vector addition?
- 1.3** Janine loves to sail on Crystal Lake. One afternoon, as she finished her lunch on Pelican Island, a strong wind began to blow from the north. To sail back to her parents' cabin, she had to follow a zigzag course. This procedure is called tacking. On her trip home, Janine had to tack four times. Her first tack covered 3.0 km at an azimuth of 45° , her second covered 5.5 km at 300° , and her third 2.5 km at 55° .
- a. Obtain a map of Crystal Lake from your teacher. Use vectors to make a scale drawing of the displacements for Janine's first three tacks.
 - b. Draw the resultant vector that represents her change in position after the first three tacks. Write this vector as an ordered pair (distance,azimuth).
 - c. Draw the vector that represents the final tack Janine must sail to return to the dock. Write this vector as an ordered pair.
- 1.4** Using a map of the United States, determine the displacement vectors that describe a direct flight from your home to Seattle, Washington, followed by a direct flight to San Francisco, California, and a final direct flight back to your home.
- * * * * *
- 1.5** During a hike, Daniel used displacement vectors to keep track of his route. He wrote the following vectors, recording the magnitudes in kilometers and the degrees azimuth: (2,65), (1.5,0), (2.4,300), and (4,205).
- a. Make a scale drawing of Daniel's hike.
 - b. Write a resultant vector that describes his change in position.
 - c. Write a displacement vector that describes the straight-line route needed to return Daniel to his starting point.
 - d. What is the relationship between the vectors you described in Parts **b** and **c**? Explain your response.

- 1.6 After spending a day exploring in a park near his home, Manuel returned with the following map.



Manuel plans to take some friends to visit the mine, the overlook, and the waterfall. To determine shorter routes to each site, he wants to find a direct path from home to the mine, from the mine to the overlook, and from the overlook to the waterfall.

Since he has not drawn his map to scale, Manuel cannot draw and measure the appropriate resultant vectors. Instead, he decides to use his knowledge of the relationships among the lengths of the sides and the measures of the angles in a triangle.

Recall that in a triangle ABC , the law of sines states that

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}$$

while the law of cosines states that

$$c^2 = a^2 + b^2 - 2ab \cos \angle C$$

- Manuel begins his calculations by identifying the angle between the first two sections of his old route: 157° . Describe how he could find this angle measure.
- Determine a displacement vector that represents each of the following:
 - a direct route from Manuel's house to the mine
 - a direct route from the mine to the scenic overlook
 - a direct route from the scenic overlook to the waterfall.

Activity 2

In Activity 1, you used the tip-to-tail method to add vectors geometrically. In this activity, you investigate a method for adding vectors algebraically.

Exploration 1

- Draw a two-dimensional coordinate system on a sheet of centimeter grid paper, with the origin located near the center of the sheet. Tape the paper to a table.
- Force** is a physical quantity that can affect the motion of an object. Like displacement and velocity, force has both magnitude and direction. The metric unit of force is the newton (N). The magnitude of a force can be measured using spring scales.

Hook two spring scales to a ring and pull them in opposite directions until each scale reads 12 N. Position the ring over the origin of your coordinate system, with the scales extending into the first and third quadrants, as shown in Figure 5.

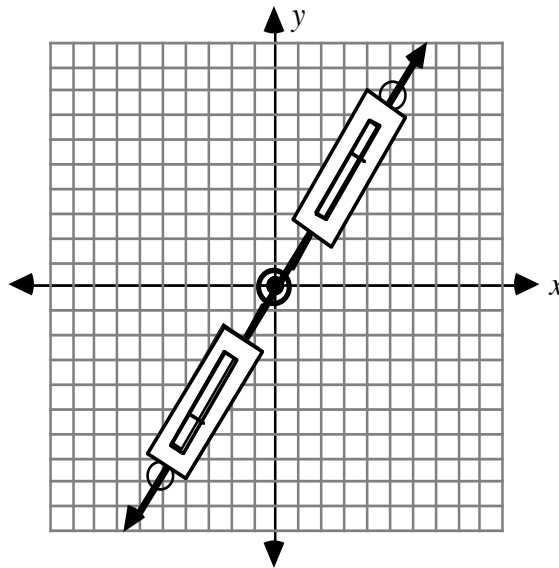


Figure 5: Scales and ring over graph paper

- Mark a line on the grid paper that shows the angle at which the scales cross the x -axis. Measure and record this angle.

- d. Draw a vector on the grid paper to represent the force on each scale, letting 1 cm represent 1 N. Locate the tail of each vector at the origin and label the magnitudes. Your grid paper should now resemble Figure 6. Note that vectors **a** and **b** are opposite vectors.

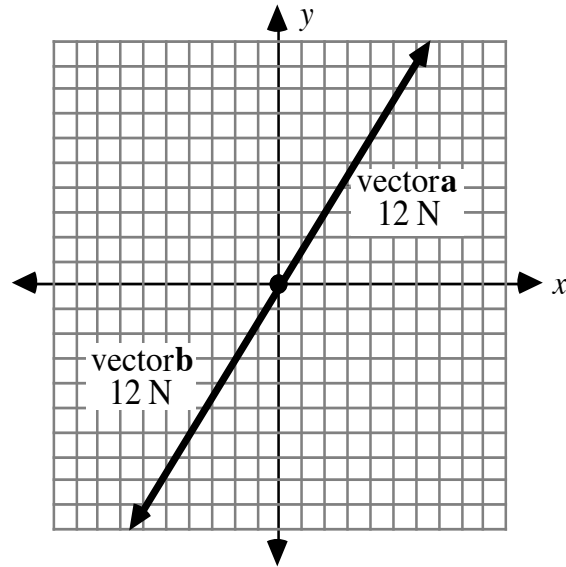


Figure 6: Graph paper with vectors

- e. Hook a third scale to the ring. Position one scale directly over vector **b**. Position the other two scales so that they are directly over the axes, as shown in Figure 7. Keeping the ring centered over the origin, pull on the three scales until the scale over vector **b** reads 12 N.

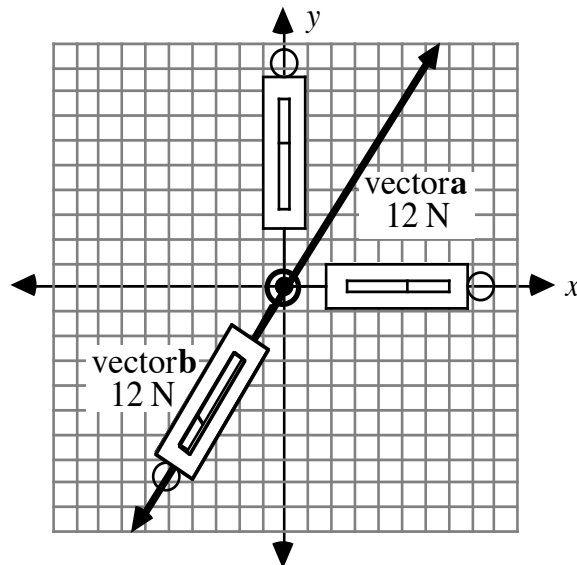


Figure 7: Positioning three scales

- f. Record the readings of the scales positioned over the x - and y -axes. These values are the magnitudes, in newtons, of the corresponding force vectors.

- g. To represent the forces on these scales, draw vectors **c** and **d** on the grid paper. The grid paper should now resemble Figure 8.

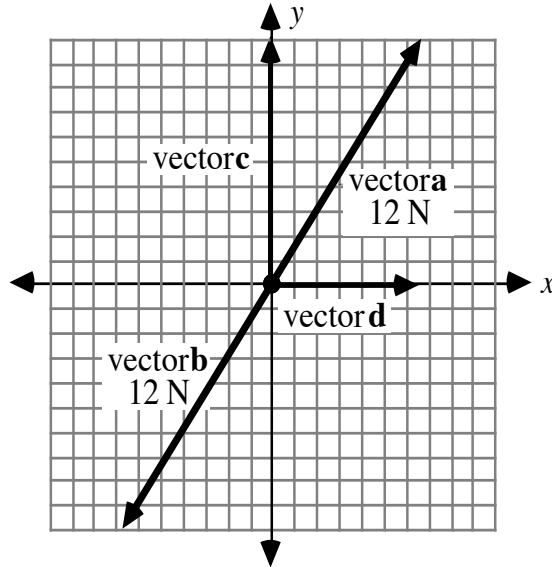


Figure 8: Graph paper with force vectors

- h. Sketch a vector diagram representing the sum of vectors **c** and **d**. Compare the resultant vector with vector **a**.
- i. Use right-triangle trigonometry, the magnitude of vector **a**, and the angle that vector **a** forms with the horizontal vector **d** to determine the magnitudes of vectors **c** and **d**.
- j. Repeat Parts **a–i** one more time, varying the angle formed by the two scales in Part **b** and the x -axis.

Mathematics Note

The **components** of a vector are the pair of horizontal and vertical vectors that, when added, result in the given vector. The horizontal component of vector **r** is denoted by r_x (read “**r** sub x ”), while its vertical component is r_y .

The angle measure between a vector **r** and its horizontal component r_x is the **reference angle**.

For example, Figure 9 shows a graph of vector **r** with its tail located at the origin, along with its components. In this case, the reference angle is θ .

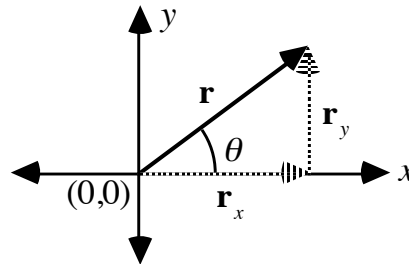


Figure 9: A vector and its components

Since a vector and its components form a right triangle, the magnitudes of the components can be found using right-triangle trigonometry. By convention, the direction of a component vector is indicated by its sign. A positive r_x is in the direction of the positive x -axis, while a negative r_x is in the direction of the negative x -axis. A similar rule applies to the vertical component r_y .

For example, consider vector s shown in Figure 10 below.

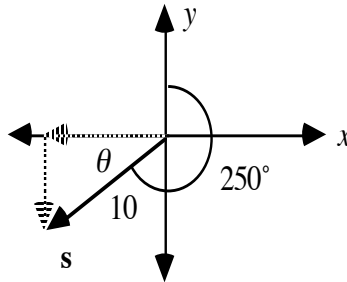


Figure 10: A vector and its components

Vector s has a magnitude of 10 and a direction of 250° , measured clockwise from the positive y -axis. To express s in terms of its components, you might complete the following steps.

- First, determine the measure of the reference angle. In this case, $\theta = 20^\circ$.
- Next, find the magnitude of the horizontal component s_x :

$$\begin{aligned} |s_x| &= |s| \cdot \cos \theta \\ &= 10 \cdot \cos 20^\circ \\ &\approx 9.4 \end{aligned}$$

Since s_x is in the direction of the negative x -axis, its sign is negative. As a result, $s_x \approx -9.4$.

- Next, find the magnitude of the vertical component s_y :

$$\begin{aligned} |s_y| &= |s| \cdot \sin \theta \\ &= 10 \cdot \sin 20^\circ \\ &\approx 3.4 \end{aligned}$$

Since s_y is in the direction of the negative y -axis, its sign also is negative. As a result, $s_y \approx -3.4$.

- Vector s now can be expressed as the sum of the following components:
 $s_x \approx -9.4$ and $s_y \approx -3.4$.

Discussion 1

- a. In Part e of Exploration 1, how does the total force exerted on the two scales positioned along the axes compare with the force exerted on the third scale? Explain your response.
- b. How can you find the magnitude of a resultant given its horizontal and vertical components?
- c. How can you find the measure of the reference angle given the horizontal and vertical components?

Exploration 2

In this exploration, you express the displacement vector for each section of the hike in Activity 1 in terms of its components. Assume that north is in direction of the positive y -axis and that east is in the direction of the positive x -axis.

- a. The displacement vectors for the route from the base to the campsite, in order, were $(4,90)$, $(2,215)$, $(3,150)$, $(3,200)$, and $(5,80)$, where the magnitudes are given in kilometers and the azimuths in degrees. Draw a vector diagram of this route.
- b. Draw the horizontal and vertical components of each vector on the diagram, with the tail of each horizontal component at the tail of the corresponding vector.
 1. Use trigonometry to determine the magnitude of each component.
 2. Determine the appropriate sign for each component.
- c. The displacement vector that describes a direct route from the base to the camp is $(10.3,127)$. Determine the horizontal and vertical components for this resultant vector.
- d.
 1. Compare the horizontal component of the resultant vector to the sum of the horizontal components of the five individual vectors.
 2. Compare the vertical component of the resultant vector to the sum of the vertical components of the five individual vectors.

Discussion 2

- a. In Exploration 2, you assumed that north is in the direction of the positive y -axis and that east is in the direction of the positive x -axis.
 1. For which interval of azimuths is a vector's vertical component negative and its horizontal component positive?
 2. For which interval of azimuths is a vector's vertical component positive and its horizontal component negative?
- b. Describe how you determined the magnitude of the components of each vector in Exploration 2.

- c. Could you have determined the components of these vectors without drawing them? Explain your response.
- d. Describe how you could determine the horizontal and vertical components of a resultant vector using the horizontal and vertical components of each vector in the sum.
- e. Given the reference angle for a vector, how can you determine its azimuth?
- f. What are the advantages of expressing a vector in terms of its components?

Assignment

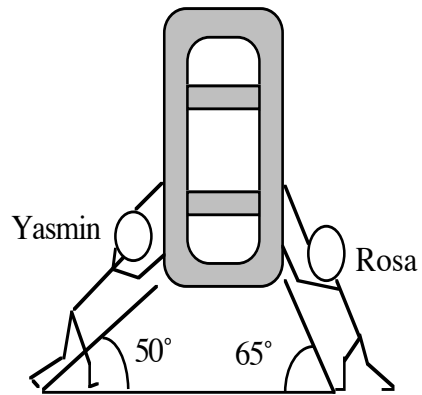
- 2.1** To help locate some interesting sights near your camp in the Northwest Territories, a helicopter flies over the area. The pilot locates a waterfall 10 km from camp at an azimuth of 320° .
- a. Make a diagram of the area around your camp, using a displacement vector to show the direct route to the waterfall.
 - b. Determine the reference angle of the displacement vector.
 - c. Determine the vector's horizontal and vertical components.
- 2.2** On your second day at the campsite, you and some friends go on a hike. At first, you walk 2.5 km at an azimuth of 60° . Next, you walk 6.3 km at an azimuth of 85° . Because dusk is only two hours away, you would like to hike straight back to camp.
- a. Using the tip-to-tail method, determine the displacement vector that represents a direct route back to camp.
 - b. Express each vector in component form, then use these components to justify your response to Part a algebraically.
 - c. If you walk at approximately 4 km/hr, will you make it back to camp before it gets dark?
- 2.3** The speed of a plane or helicopter with respect to the air is its airspeed. But pilots seldom fly in still air. When determining the proper airspeed and azimuth for their destination, they must consider the effect that wind will have on their movement with respect to the ground.

In terms of vectors, $\mathbf{a} + \mathbf{w} = \mathbf{v}$, where \mathbf{a} is the velocity with respect to the air, \mathbf{w} is the wind velocity, and \mathbf{v} is the velocity with respect to the ground.

Suppose that a helicopter leaves the base camp to pick up supplies. To reach the nearest town, the pilot would like to fly with a groundspeed of 200 km/hr at an azimuth of 336° . If the wind is blowing to the north at a speed of 12 km/hr, at what airspeed and azimuth should the pilot fly? Justify your response algebraically.

- 2.4** While gathering berries near the campsite, you leave camp and walk 400 m at an azimuth of 315° . You then turn due east and walk 320 m.
- How far are you from camp?
 - What azimuth should you take to return directly to camp?
- 2.5** The velocity of a plane can be represented by the vector $(800, 203)$, where the magnitude is given in kilometers per hour and the azimuth in degrees.
- Draw a diagram of this vector, including its horizontal and vertical components.
 - Determine the magnitudes and signs of the horizontal and vertical components. Justify your answers algebraically.
- 2.6** A plane leaves the airport with an airspeed of 480 km/hr at an azimuth of 210° . The wind is blowing to the east, resulting in an actual azimuth of 205° .
- What is the speed of the plane with respect to the ground?
 - What is the speed of the wind?
- * * * * *
- 2.7** A cruise ship travels at 30 km/hr and an azimuth of 303° for 10 hr. It then turns and travels at 35 km/hr and an azimuth of 229° for another 10 hr.
- Sketch the displacement vectors that correspond to this situation.
 - Determine the horizontal and vertical components of the resultant displacement vector.
 - Determine the magnitude and azimuth of the resultant displacement vector.
- 2.8** A ferry leaves the dock at a speed of 28 km/hr and an azimuth of 20° . Just off shore, it encounters a current flowing to the east, which results in an actual azimuth of 40° . Determine the speed of the current, as well as the actual speed of the ferry with respect to the shore.

2.9 Yasmin and Rosa are pushing their inflatable raft off a sandbar. Yasmin pushes with a force of 90 N, while Rosa pushes with a force of 80 N. These forces are directed as shown in the diagram below.

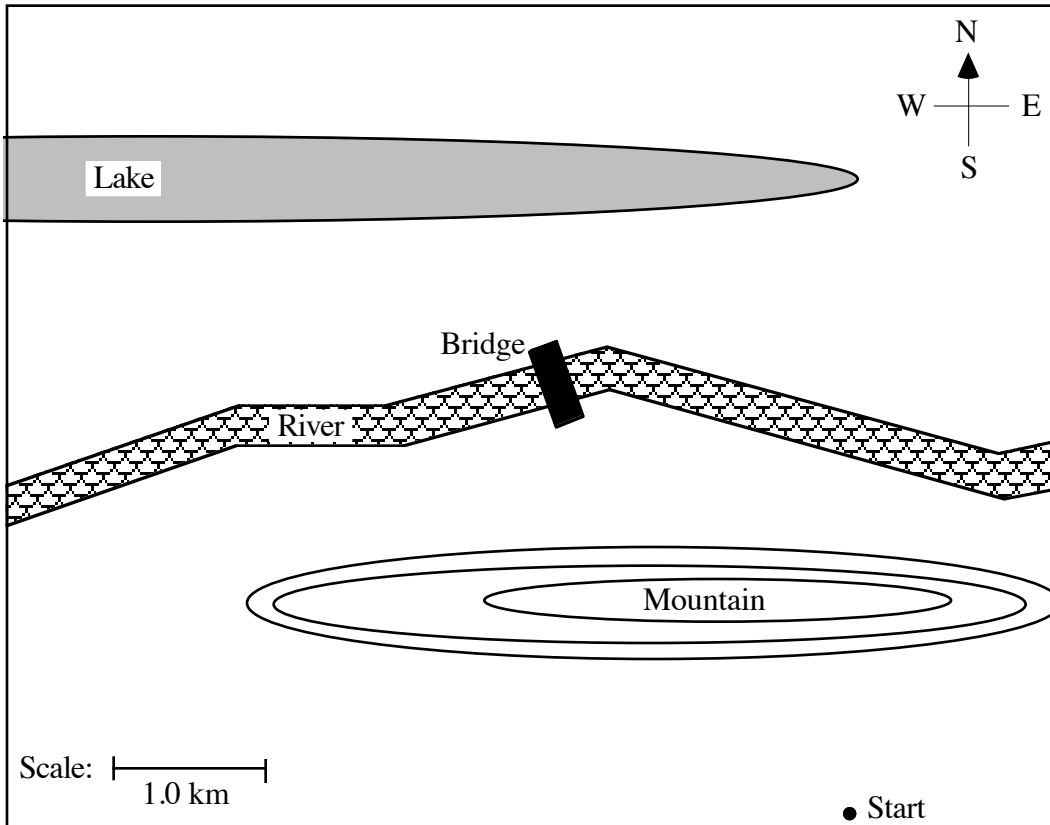


Determine the magnitude and direction of the resultant force.

* * * * *

Summary Assessment

As a participant in a route-finding contest, you are given the following map, which identifies the starting position and several natural obstacles.



1. Your destination for the contest can be described by the displacement vector $(5.9, 332)$, where the magnitude is in kilometers and the azimuth in degrees. The tail of this vector is on the starting position and its tip is on the destination point. Find the destination point on the map.
2. Your route must travel west of the mountain, then across the bridge, and then east of the lake. Draw a series of displacement vectors to model your route from the start to the destination. Write an ordered pair to represent each of these vectors.
3. Use trigonometry to find the horizontal and vertical components of each vector you drew in Problem 2. Verify algebraically that the sum of these vectors equals the displacement vector $(5.9, 332)$.
4. How long will it take you to walk the route at a rate of 3 km/hr?

Module Summary

- A **vector** is a quantity with both magnitude and direction. A vector can be represented graphically by an arrow. The vector's length from tip to tail indicates its magnitude. The orientation of the arrowhead indicates its direction.
- A vector is typically symbolized using a bold lowercase letter (**r**), or in handwritten work, by an arrow over a letter (\vec{r}). The magnitude of a vector **r** is represented by $|\mathbf{r}|$.
- An **azimuth** is an angle measured clockwise from north.
- **Displacement** is a change in position in a particular direction. A displacement vector indicates both the distance moved and the direction.
- **Equal vectors** have the same magnitude and direction.
- **Opposite vectors** have the same magnitude but opposite directions.
- A **zero vector** has a magnitude of 0.
- Vector addition can be performed geometrically by the **tip-to-tail** method. Using this method, each vector is drawn so that its tail coincides with the tip of the previous vector.
- The sum of any number of vectors is a **resultant vector**. In the tip-to-tail method, the resultant vector joins the tail of the first vector to the tip of the last vector in the sum.
- The **velocity** of an object is its speed in a specific direction. Like displacement, velocity has both magnitude and direction.
- **Force** is a physical quantity that can affect the motion of an object. Like displacement and velocity, force has both magnitude and direction. The metric unit of force is the newton (N).
- The **components** of a vector are the pair of horizontal and vertical vectors that, when added, result in the given vector. The horizontal component of vector **r** is denoted by \mathbf{r}_x (read "r sub x"), while its vertical component is \mathbf{r}_y .
- The angle measure between a vector **r** and its horizontal component \mathbf{r}_x is the **reference angle**.

- Since a vector and its components form a right triangle, the magnitudes of the components can be found using right-triangle trigonometry. If θ is the reference angle, then $|\mathbf{r}_x| = |\mathbf{r}| \cdot \cos \theta$ and $|\mathbf{r}_y| = |\mathbf{r}| \cdot \sin \theta$.

By convention, the direction of a component vector is indicated by its sign. A positive \mathbf{r}_x is in the direction of the positive x -axis, while a negative \mathbf{r}_x is in the direction of the negative x -axis. A similar rule applies to the vertical component \mathbf{r}_y .

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